

**THE REGULARITY TRANSFORMATION EQUATIONS:
AN ELLIPTIC MECHANISM FOR SMOOTHING
GRAVITATIONAL METRICS IN GENERAL RELATIVITY**

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ABSTRACT. *Regularity singularities* are points in spacetime where the gravitational metric tensor of General Relativity fails to be at least two levels more regular than its curvature tensor. Whether regularity singularities exist for shock wave solutions constructed by the Glimm scheme in GR is an open problem. In this paper we pose the problem at the general level of connections $\Gamma \in W^{m,p}$ satisfying $d\Gamma \in W^{m,p}$ as well, and ask the question as to whether there always exists a coordinate transformation with Jacobian $J \in W^{m+1,p}$ which smooths the connection by one order. Introducing a new approach to this problem, we derive a system of nonlinear elliptic Poisson equations, which we call the *Regularity Transformation equations* (RT-equations), with matrix-valued differential forms as unknowns, and prove that the existence of solutions to these equations is equivalent to the Riemann-flat condition, which was shown by the authors to be equivalent to the existence of a coordinate transformation which smooths the connection by one order [22]. Different from earlier approaches to optimal metric regularity, our method does not employ any apriori coordinate ansatz. In a companion paper authors establish an existence theory for the RT-equations at the level of smoothness $m \geq 1$. In the final section of this paper we propose a mathematical framework for extending the existence theory to the L^∞ case of shock waves.

1. INTRODUCTION

Although the Einstein equations of General Relativity (GR) are covariant, solutions are constructed in coordinate systems in which the PDE's take on a solvable form. A very first question in GR is then, which properties of the spacetime represent the true geometry, and which are merely anomalies of the coordinate system? In particular, does a solution of the Einstein equations exhibit its optimal regularity in the coordinate system in which it was constructed? The most regular coordinate systems define the local properties of spacetime, and these determine the degree to which the physics

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in curved spacetime corresponds locally to the physics of Special Relativity, (Einstein's Correspondence Principle).¹

A particularly intriguing case is GR shock waves [25, 15, 20, 28, 2]. In [13], shock wave solutions of the Einstein equations generated by the Glimm scheme could only be constructed in coordinate systems in which the metric is only Lipschitz continuous ($C^{0,1}$) at shocks, even though both the connection and curvature tensor of such solutions stay bounded in L^∞ . The question as to whether a $C^{0,1}$ metric can always be smoothed one order to $C^{1,1}$ by coordinate transformation is intimately related to the existence of locally inertial coordinate systems, and to the local correspondence of GR with the physics of Special Relativity. In [20, 21] the authors conjectured that if such coordinate systems do not always exist, then shock wave interactions create a new kind of mild singularity which the authors termed *Regularity Singularities*. It remains an open problem whether regularity singularities exist at shock waves, or not. This question is central to numerical analysis in GR, as knowing whether a numerical singularity is geometric, or a coordinate anomaly, is essential to understanding the validity of the numerics.

The question as to the existence of such smoothing transformations is surprisingly subtle. Although the construction of locally inertial coordinate systems by the Riemann normal construction is straightforward at a fixed level of smoothness, this simple process is not sufficient to smooth a connection. At smooth, non-interacting shock surfaces, coordinate transformation to Gaussian normal coordinates at the surface, suffices to smooth an L^∞ gravitational connection by one order to $C^{0,1}$ at shocks, by a now classical result of Israel in 1966 [15]. But for more general shock wave interactions, the only result we have is due to Reintjes [19], who proved that the gravitational metric can always be smoothed one order to $C^{1,1}$ in a neighborhood of the interaction of two shock waves from different characteristic families, in spherically symmetric spacetimes. Reintjes' procedure for finding the local coordinate systems of optimal smoothness is *orders of magnitude* more complicated than the Riemann normal, or Gaussian normal construction process. The coordinate systems of optimal $C^{1,1}$ regularity are constructed in [19] by solving a complicated non-local PDE highly tuned to the structure of the interaction. Trying to guess the coordinate system of optimal smoothness a priori, for example harmonic, wave, or Gaussian normal coordinates

¹For example, assume one were to construct a solution to the Einstein equations $G = \kappa T$ in a given coordinate system x in which the equations produce a unique solution within a given smoothness class, say metric $g \in W^{m+2,p}$, connection $\Gamma \in W^{m+1,p}$, and $Riem(\Gamma) \in W^{m,p}$. Then application of a transformation $x \rightarrow y$ with Jacobian $J \in W^{m+1,p}$ will in general lower the regularity of the metric and its connection Γ by one order, but preserve $Riem(\Gamma) \in W^{m,p}$, because it is a tensor. Thus, if that existence theory were posed in y -coordinates, it would produce the unique transformed solution $g \in W^{m+1,p}$, $\Gamma \in W^{m,p}$, and $Riem(\Gamma) \in W^{m,p}$. In the latter case, we would not know that our unique solution was one order below optimal smoothness without knowing about the existence of the inverse transformation $y \rightarrow x$.

[3], is not successful. In Reintjes' construction, several apparent *miracles* happen in which the Rankine-Hugoniot jump conditions come in to make seemingly over-determined equations consistent, but, the principle behind what PDE's must be solved to smooth the metric in general, or when this is possible, appears entirely mysterious.

In this paper we introduce a new approach to optimal metric regularity which is different from earlier approaches in that it does not employ any apriori coordinate ansatz.² Rather than imposing an apriori coordinate ansatz, (like harmonic coordinates), and trying to establish regularity of solutions of the Einstein equations in those coordinates, the approach here begins with the belief that the coordinates of optimal regularity are too difficult to guess apriori, and one has to find and solve equations for the coordinates themselves. We address the problem at the general level of connections $\Gamma \in W^{m,p}$, under the assumption that $d\Gamma \in W^{m,p}$, so the curvature tensor $\text{Riem}(\Gamma) \in W^{m,p}$ as well, (i.e., component functions are in $W^{m,p}$ in some given coordinate system x)³, and ask the question as to whether there always exists a coordinate transformation $x \rightarrow y$ with Jacobian $J \in W^{m+1,p}$, such that in y -coordinates, the connection is one degree smoother, i.e., in $W^{m+1,p}$.

Treating the unknowns as matrix valued differential forms, we derive a system of nonlinear elliptic Poisson equations which we call the *Regularity Transformation equations* (RT-equations), and prove that the existence of solutions to these equations is equivalent to the Riemann-flat condition, which was shown in [22] to be equivalent to the existence of such a coordinate transformation smoothing the connection. The Riemann-flat condition is the condition that there should exist a tensor $\tilde{\Gamma}$, one order smoother than the connection Γ , such that $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$. The Riemann-flat condition gives an equation for $d\tilde{\Gamma}$ which can be augmented to a first order system of Cauchy-Riemann equations by addition of an equation for $\delta\tilde{\Gamma}$ with arbitrary right hand side. This system can then be converted into a second order semi-linear Poisson equation for $\tilde{\Gamma}$ by use of the identity $d\delta + \delta d = \Delta$, but this couples the right hand side of the equation to the Jacobian J of the unknown coordinate transformation which smooths the connection. Thus to close the equations, it is necessary to couple the Poisson equation in the unknown $\tilde{\Gamma}$, to an equation in J . Employing the identity $J^{-1}dJ = \Gamma - \tilde{\Gamma}$

²The problem of optimal regularity for Riemannian metrics with curvature tensors of low regularity was addressed in [8]. For Lorentzian metrics, this problem was addressed in [1] for GR vacuum spacetimes, and for non-vacuum spacetimes subject to additional assumptions. In the case of GR vacuum solutions results were improved in [4, 17]. These papers do not address GR shock waves, the case when the matter sources are non-zero and the Riemann curvature is in L^∞ [15, 25, 20]. Different from these approaches, we do not assume any apriori coordinate ansatz.

³Here $d\Gamma$ denotes the exterior derivative of Γ viewed as a matrix valued 1-form. By Gaffney's inequality, our assumption $\Gamma, d\Gamma \in W^{m,p}$ implies all of the loss of derivative in Γ occurs in $\delta\Gamma$, c.f. (2.32) below.

which comes from the Riemann-flat condition, together with the identity $d\delta + \delta d = \Delta$, we derive a semi-linear elliptic Poisson equation for J which closes the system in $(\tilde{\Gamma}, J)$. The RT-equations are then obtained by using the freedom in $\delta\tilde{\Gamma}$ to impose the integrability of the Jacobian by coupling the equations in $(\tilde{\Gamma}, J)$ to an additional Cauchy-Riemann equation in the auxiliary variable $A = J\delta\tilde{\Gamma}$. The resulting system of equations in $(\tilde{\Gamma}, J, A)$ is *elliptic*, each equation being of Poisson, or Cauchy-Riemann type. By a fortuitous identity, we show that all *bad terms* involving $\delta\Gamma$ can be re-expressed in terms of $d\Gamma$, leading to a gain of one derivative on the right hand side. By this, the RT-equations are formally correct at the levels of regularity sufficient to smooth the original connection by one order, consistent with known results on harmonic smoothing by the Poisson equation in L^p -spaces, [6, 10, 12]. This is established in a rigorous existence theory for the RT-equations in authors' companion paper [23].

The RT-equations involve matrix valued differential forms, and to derive them we develop a Euclidian Cartan algebra associated with the Riemann-flat condition. The derivation relies on special operations on matrix valued differential forms meaningful when the dimension of the matrices equals the dimension of the space, and these have no classical scalar analogue (c.f. [6]). This framework appears to be forced on us to close the equations, by bridging matrix and vector valued differential forms through special operations. Remarkable to us is that the RT-equations reduce the question of regularity singularities in *Lorentzian* spacetimes, to an existence problem for a system of *elliptic* Poisson equations. So the metric signature is of no relevance to the question of regularity singularities.

The problem of solving the RT-equations at the threshold low regularity of L^∞ connections, the setting of GR shock waves, is problematic, due mainly to the existence of Calderón-Zygmund singularities⁴ in solutions with L^∞ source terms on the right hand side of the Poisson equation [16]. The space $C^{0,\alpha}$ is a fractional level of smoothness above L^∞ , and the space $W^{m,p}$ naturally embeds in $C^{0,\alpha}$ for $m \geq 1, p > n$, by Sobolev embedding [10]. Thus we begin the study of the *RT*-equations here by addressing what we term the *smooth case*, the case when $\Gamma \in W^{m,p}$, $m \geq 1$, under the assumption that $p > n$ so that Sobolev embedding implies Hölder continuity, [10]. In this paper we establish the equivalence of the *RT*-equations with the Riemann-flat condition in the smooth case, and in [23] we prove existence in the smooth case by reducing the existence theory to known results on elliptic regularity for the Poisson equation. In Section 4, we propose a framework

⁴By Calderón-Zygmund singularities we mean counterexamples demonstrating that solutions of the linear Poisson equation are not always in $C^{1,1}$ when the sources are in L^∞ , (i.e., solutions do not always gain two derivatives above the sources, [16, 10]). Thus, by ellipticity of the RT-equations, the problem of regularity singularities at shock waves in GR is intimately connected to the existence of classical Calderón-Zygmund singularities for the Poisson equation.

for extending the existence theory to the lower regularity Γ , $\text{Riem}(\Gamma) \in L^\infty$, the setting of GR shock waves. This is the topic of authors' current research.

The main result of this paper is the following theorem which establishes the equivalence of the Riemann flat condition with the solvability of an explicit semi-linear elliptic system of Poisson equations, stated in the smooth case Γ , $d\Gamma \in W^{m,p}$, and hence $\text{Riem}(\Gamma) \in W^{m,p}$, for $m \geq 1$, $p > n$, by which we mean the component functions of Γ and $d\Gamma$ are in $W^{m,p}$ in some given, but otherwise arbitrary, coordinate system x . For the theorem, $\Gamma \equiv \Gamma_{\nu^k}^\mu dx^k$ is viewed as a matrix valued 1-form. The unknowns in the equations are $\tilde{\Gamma}$, J , A also viewed as matrix valued differential forms as follows: $J \equiv J_\nu^\mu$ is the Jacobian of the sought after coordinate transformation which smooths the connection, viewed as a matrix-valued 0-form; $\tilde{\Gamma} \equiv \tilde{\Gamma}_{\nu^k}^\mu dx^k$ is the unknown tensor one order smoother than Γ such that $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$, viewed as a matrix-valued 1-form; and $A \equiv A_\nu^\mu$ is an auxiliary matrix valued 0-form introduced to impose $\text{Curl}(J) = 0$, the integrability condition for the Jacobian.

Theorem 1.1. *Assume Γ is defined in a fixed coordinate system x on Ω , for $\Omega \subset \mathbb{R}^n$ open and with smooth boundary. Assume that $\Gamma \in W^{m,p}(\Omega)$ and $d\Gamma \in W^{m,p}(\Omega)$ for $m \geq 1$, $p > n$. Then the following equivalence holds:*

If there exists a coordinate transformation $x \mapsto y$ with Jacobian $J = \frac{\partial y}{\partial x} \in W^{m+1,p}(\Omega)$ such that the components of Γ in y -coordinates are in $W^{m+1,p}(\Omega)$, then there exists $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ such that $(J, \tilde{\Gamma}, A)$ solve the elliptic system

$$\Delta \tilde{\Gamma} = \delta d(\Gamma - J^{-1}dJ) + d(J^{-1}A), \quad (1.1)$$

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \quad (1.2)$$

$$d\vec{A} = \overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J d\Gamma) - d(\langle dJ; \tilde{\Gamma} \rangle), \quad (1.3)$$

$$\delta \vec{A} = v, \quad (1.4)$$

with boundary data

$$\text{Curl}(J) \equiv \partial_j J_i^\mu - \partial_i J_j^\mu = 0 \quad \text{on } \partial\Omega, \quad (1.5)$$

where $v \in W^{m-1,p}(\Omega)$ is a vector valued 0-form free to be chosen.

Conversely, if there exists $J \in W^{m+1,p}(\Omega)$ invertible, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ which solve (1.1) - (1.5) in Ω , then for each $p \in \Omega$, there exists a neighborhood $\Omega' \subset \Omega$ of p such that J is the Jacobian of a coordinate transformation $x \mapsto y$ on Ω' , and the components of Γ in y -coordinates are in $W^{m+1,p}(\Omega')$.

We call system (1.1)-(1.4) the *Regularity Transformation equations*, or *RT-equations*. Here \vec{A} is the vector valued 1-form defined by $\vec{A} = A_i^\mu dx^i$, so $d\vec{A} \equiv \text{Curl}(A)$. The operations $\vec{\cdot}$, $\overrightarrow{\text{div}}$ and $\langle \cdot, \cdot \rangle$, introduced in Sections 2.1 below, are special operations on matrix valued differential forms meaningful

when the dimension of the matrices equals the dimension of the physical space. New features arise in the auxiliary Euclidean Cartan algebra essentially because we view J both as a matrix valued zero form and a vector valued 1-form at different stages of the argument.

The free vector valued 0-form v has been introduced in (1.4) so that (1.3), (1.4) takes the Cauchy-Riemann form $d\vec{A} = f$, $\delta\vec{A} = g$. Such systems require the consistency conditions $df = 0$, $\delta g = 0$, (c.f. Section 3.1 below). Condition $df = 0$ is met in (1.3) because the derivation shows the right hand side is exact, (equation (1.3) is obtained by setting d of the right hand side of (1.2) equal to zero), and $\delta g = 0$ in (1.4) because $\delta v = 0$ is an identity for vector valued 0-forms v .

The second derivative term $d(J^{-1}dJ)$ appearing on the right hand side of (1.1) has the same regularity as the first order term dJ by the important identity $d(J^{-1}dJ) = (J^{-1}dJ) \wedge (J^{-1}dJ)$ established in Lemma 3.3 below. Surprisingly, terms involving $\delta\Gamma$ which initially appear to be one order of smoothness too low on the right hand side of (1.3) for the solution A to be of the sought after regularity, can be re-expressed in terms of $d\Gamma$ via identities involving the operation $\overrightarrow{\text{div}}$ in (1.3). Thus, all such terms are one order smoother than they initially appear to be, because $d\Gamma$, (having the same regularity as $\text{Riem}(\Gamma)$), is assumed to be one order smoother than $\delta\Gamma$. This extra derivative “miracle”, necessary for the consistency of (1.1) - (1.4), appears to be analogous to the “miraculous” cancellation of terms by the Rankine-Hugoniot jump conditions in [19].

As a final comment, Theorem 1.1 states that there always exists a modification $\tilde{\Gamma}'$ of $\tilde{\Gamma}$ such that $\tilde{\Gamma}'$, not $\tilde{\Gamma}$, solves the Riemann-flat condition. The need for a modification is an interesting subtlety. Namely, the second order system (1.1) is derived from the first order Cauchy-Riemann system $d\tilde{\Gamma} = d(\Gamma - J^{-1}dJ)$, $\delta\tilde{\Gamma} = J^{-1}A$ by application of the identity $\Delta = d\delta + \delta d$, the first order system coming directly from the Riemann-flat condition $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$. But the advantage of the second order system over the first order system is that it allows us to impose more general boundary data, in particular, the simple boundary data (1.5), instead of the non-linear data (3.6) required for the standard equivalence between solutions of Cauchy-Riemann and Poisson type equations, c.f. Section 3.1. The resulting freedom to solve (1.1) with arbitrary boundary conditions is a propitious feature of the RT-equations.

In Section 1.1 we review the problem of regularity singularities arising from GR shock wave theory. In Section 2 we introduce the auxiliary Euclidean Cartan algebra of matrix-valued k -forms, express the Riemann-flat condition within this framework, and define the operations $\overrightarrow{\text{div}}$ and $\langle \cdot ; \cdot \rangle$, (c.f. (1.3) and (1.2), respectively). In Section 3 we clarify the connection between the first order Cauchy-Riemann equations and the Poisson equation in the setting of matrix valued differential forms, derive the RT-equations (1.1)-(1.4) together with an alternative formulation (in Section 3.4), and

prove our main result, Theorem 1.1. In Section 4, we outline a program for extending the existence theory to the L^∞ case of shock waves.

1.1. Background on GR Shock Waves. Shock waves form in solutions of the Einstein equations $G = \kappa T$ for a perfect fluid whenever the solution is sufficiently compressive, [5, 18]. At shock waves the Einstein curvature tensor G is bounded and discontinuous because the stress energy T for the matter sources is bounded and discontinuous due to discontinuities in the fluid density, velocity and pressure at the shocks, [25]. While a general existence theory for GR shock waves is still not known except in spherical symmetry [13], a natural setting for GR shock waves would be solutions assumed to have the curvature, (like the sources T), bounded in L^∞ . A theory of classical solutions of $G = \kappa T$ with curvature in L^∞ would then naturally follow from the assumption that the gravitational metric g is in $C^{1,1}$, so derivatives of the metric, and hence the connection Γ , would be Lipschitz continuous. But this optimal level of smoothness is not known for shock wave solutions in GR. The only general existence theory for shock waves in spherical symmetry, based on the Glimm method [13], produces gravitational metrics which are only Lipschitz continuous at shocks, so Γ is only known to be in L^∞ . For GR shock waves at the low regularity of Lipschitz continuous metric and L^∞ connection, the Einstein equations $G = \kappa T$ still imply that the curvature tensor is in L^∞ , and hence the implication would be that second derivatives of the metric contain delta function sources, (i.e., the metric is only Lipschitz), but these cancel out in the curvature tensor. By Christoffel's formula the connection bounds all first order metric derivatives, thus a most basic question for the subject of GR shock waves is the question as to whether a weak solution of the Einstein equations $G = \kappa T$ with its connection $\Gamma \in L^\infty$, can be smoothed by one order to Lipschitz continuous by coordinate transformation, under the assumption that the Riemann curvature tensor of the connection is bounded in L^∞ .

Starting with this open problem, the authors proposed that if shock wave solutions with $\Gamma \in L^\infty$ could *not* be smoothed to Lipschitz continuous by coordinate transformation, then such solutions would represent a new kind of mild singularity for General Relativity, which they named a *regularity singularity*. In [21] authors showed that at a regularity singularity, the spacetime would not be smooth enough to admit locally inertial frames, and such a singularity would change the character of the Newtonian limit and the local scattering of gravitational radiation.

The general open problem of regularity singularities then, is, given an L^∞ connection Γ with L^∞ curvature tensor, does there exist a $C^{1,1}$ coordinate transformation which smooths the connection by one order to Lipschitz continuous? And if not, under what further conditions on Γ does such a coordinate transformation exist? By the transformation laws for a connection, a coordinate transformation can smooth an L^∞ connection by one order only

if it has discontinuities in second derivatives of the transformation, so $C^{1,1}$, the space of functions whose derivatives are Lipschitz, is the function space in which such coordinate transformation would lie, [25, 20]. This is a natural question for the Einstein equations because any coordinate transformation with $J \in C^{0,1}$ automatically transforms a connection of regularity $C^{0,1}$ into a connection in $W^{m,p}$, (one loss of derivative), while preserving the regularity of $d\Gamma \in L^\infty$, and hence also $\text{Riem}(\Gamma) \in L^\infty$, because the symmetric second order derivative term in the transformation law for a connection cancels out in $d\Gamma$. Thus the question of Regularity Singularities is the question as to whether this operation can always be inverted.

In pursuit of a general theory of metric smoothing at shock waves, in [22] the authors proved that the problem of metric smoothing for symmetric $\Gamma \in L^\infty$ with $d\Gamma \in L^\infty$ is equivalent to the *Riemann-flat condition*, namely, that the connection can be smoothed by a $C^{1,1}$ coordinate transformation in a neighborhood of a point p if and only if there exists a Lipschitz continuous symmetric $(1,2)$ -tensor $\tilde{\Gamma}$ such that $\Gamma - \tilde{\Gamma}$ is *Riemann flat* in a neighborhood of p , i.e., $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$, (c.f. Theorem 1 of [22]). It is straightforward to extend the Riemann flat condition to connections which are Hölder continuous, so by Sobolev embedding, the Riemann flat condition applies to connections in $W^{m,p}$, $m \geq 1$, $p > n$, [10]. This is formulated in Theorem 2.5 below. We refer to $\Gamma \in L^\infty$, $d\Gamma \in L^\infty$ relevant for GR shock waves as the *L^∞ case*, and we refer to $\Gamma \in W^{m,p}$, $d\Gamma \in W^{m,p}$, $k \geq 1$, $p > n$, as the *smooth case*.

The Riemann-flat condition, the point of departure for this paper, gave us the idea that a closed set of elliptic equations for metric smoothing might be expressed within a Cartan type framework of matrix valued differential forms. Standard PDE methods do not apply to the Riemann flat condition by itself, but in this paper we show that the equations close when the Riemann-flat conditions is coupled to a similar equation for the Jacobian of the sought after coordinate transformation which smooths the connection. The final system (1.1)-(1.5) is obtained by coupling this to an auxiliary equation expressing the integrability of the Jacobian.

2. MATRIX VALUED DIFFERENTIAL FORMS

In this section we develop a theory of matrix valued differential forms in the special case when the dimension of the matrix components agrees with the dimension of the space, n . The exterior derivative d and its co-derivative δ operate on matrix valued k -forms component-wise, and the wedge product introduces the matrix commutator, both of which are independent of the size of the matrices. However, to close the equations, we need to introduce two new operations, c.f. (1.4). The first operation maps matrix valued 0-forms A to vector valued 1-forms \vec{A} via contraction of one matrix indices with dx^i . The second is a vectorized divergence \vec{div} which maps matrix valued k -forms to vector valued k -forms by taking the divergence with respect to the lower

matrix index. These vectorizing operations are meaningful only for matrix valued forms in which the matrices and the dimension of the space are both equal.

Keep in mind, this is a *Euclidean* framework because we only consider matrix valued differential forms in the fixed coordinate system x in which our connection $\Gamma \equiv \Gamma_{ij}^k$ is originally assumed to be given, and we take the auxiliary metric on x to be Euclidean. Since x is assumed fixed, the covariance properties of these differential forms is not an issue.

2.1. Euclidean Cartan calculus for Lorenzian connections. To start, we interpret the connection Γ as a matrix valued 1-form $\Gamma_\nu^\mu \equiv \Gamma_{\nu i}^\mu dx^i$, in which case the Riemann curvature tensor of Γ can be written as the matrix valued 2-form

$$\text{Riem}(\Gamma) = d\Gamma + \Gamma \wedge \Gamma, \quad (2.1)$$

c.f., Lemma 2.1. By a matrix valued differential k -form A we mean an $(n \times n)$ -matrix whose components are k -forms over n -dimensional base space $\Omega \subset \mathbb{R}^n$, and we write

$$A = A_{[i_1 \dots i_k]} dx^{i_1} \wedge \dots \wedge dx^{i_k} \equiv \sum_{i_1 < \dots < i_k} A_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

for $(n \times n)$ -matrices $A_{i_1 \dots i_k}$ that are totally anti-symmetric in the indices $i_1, \dots, i_k \in \{1, \dots, n\}$. As is standard, we always indicate an increasing ordering of indices by a square bracket around the indices and we set

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} \equiv \sum_{\pi \in S_k} \text{sgn}(\pi) dx^{i_{\pi(1)}} \otimes \dots \otimes dx^{i_{\pi(k)}}, \quad (2.2)$$

where S_k denotes the set of all permutations of $\{1, \dots, k\}$. We define the exterior derivative of a matrix valued k -form by

$$\begin{aligned} dA &\equiv d(A_{[i_1 \dots i_k]}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ &= \partial_l A_{[i_1 \dots i_k]} dx^l \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}, \end{aligned} \quad (2.3)$$

and we define the wedge product of a matrix valued k -form A with a matrix valued l -form $B = B_{j_1 \dots j_l} dx^{j_1} \wedge \dots \wedge dx^{j_l}$ as

$$A \wedge B \equiv \frac{1}{l!k!} A_{i_1 \dots i_k} \cdot B_{j_1 \dots j_l} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}, \quad (2.4)$$

where the dot denotes standard matrix multiplication. The wedge product of a matrix valued k -form with itself is non-zero unless the component matrices commute, which we now illustrate for a matrix valued 1-form $A = A_i dx^i$ by computing

$$\begin{aligned} (A_i dx^i) \wedge (A_j dx^j) &\equiv A_i \cdot A_j dx^i \wedge dx^j \\ &= A_i \cdot A_j (dx^i \otimes dx^j - dx^j \otimes dx^i) \\ &= (A_i \cdot A_j - A_j \cdot A_i) dx^i \otimes dx^j \end{aligned} \quad (2.5)$$

and this vanishes if and only if $A_i A_j - A_j A_i = 0$. Thus, $\Gamma \wedge \Gamma$ in (2.1) is in general non-vanishing.

To define the co-derivative δ and the Laplace operator Δ for matrix valued k -forms, define the Hodge star operator $*$ by

$$A \wedge (*B) \equiv \langle A ; B \rangle dx^1 \wedge \dots \wedge dx^n, \quad (2.6)$$

for matrix valued k -forms A and B , where we define the matrix valued inner product as

$$\langle A ; B \rangle_\nu^\mu \equiv \sum_{i_1 < \dots < i_k} A_{\sigma i_1 \dots i_k}^\mu B_{\nu i_1 \dots i_k}^\sigma, \quad (2.7)$$

which is the Euclidean inner product on the components of k -forms. The Hodge-star operator $*$ maps k -forms linearly to $(n - k)$ -forms and (2.6) is equivalent to the orthogonality condition (for increasing indices)

$$dx^{[i_1 \wedge \dots \wedge dx^{i_k}] \wedge * (dx^{[j_1 \wedge \dots \wedge dx^{j_k}]}) = \begin{cases} dx^1 \wedge \dots \wedge dx^n, & \text{if } i_1 = j_1, \dots, i_k = j_k, \\ 0 & \text{otherwise,} \end{cases} \quad (2.8)$$

since we find from the definition of the wedge product (2.4) and (2.8) that

$$\begin{aligned} (A \wedge (*B))_\nu^\mu &= A_{\sigma [i_1 \dots i_k]}^\mu B_{\nu [j_1 \dots j_k]}^\sigma dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge * (dx^{j_1} \wedge \dots \wedge dx^{j_k}) \\ &= \langle A ; B \rangle_\nu^\mu dx^1 \wedge \dots \wedge dx^n. \end{aligned}$$

By (2.8), the Hodge star maps a basis element to its complementary element, from which we find that $**A = (-1)^{k(n-k)}A$, (where the factor $(-1)^{k(n-k)}$ appears when passing the dual basis element to the left hand side), and so

$$*^{-1} = (-1)^{k(n-k)} *.$$

The co-derivative of a k -form A is now defined (in the standard way) as the $(k - 1)$ -form

$$\delta A \equiv (-1)^{n-k} * (d(*^{-1}A)) \quad (2.9)$$

and the Laplace operator as

$$\Delta \equiv \delta d + d\delta. \quad (2.10)$$

The Laplacian acts on each component of A as the scalar Laplacian,

$$(\Delta A)_{\nu i_1 \dots i_k}^\mu = \Delta (A_{\nu i_1 \dots i_k}^\mu) = \sum_{j=1}^n \partial_j \partial_j (A_{\nu i_1 \dots i_k}^\mu), \quad (2.11)$$

c.f. Theorem 3.7 in [6], (where the last identity in (2.11) holds when x^i are Euclidean coordinates, the case we have here). A straightforward computation shows that $\delta A = 0$ for 0-forms A , and if $k = 1$, then the co-derivative is the divergence,

$$(\delta A)_\nu^\mu = \sum_{i=1}^n \partial_i A_{\nu i}^\mu. \quad (2.12)$$

With the exception of property (2.5) of the wedge product, matrix valued differential forms behave like standard scalar differential forms with scalar multiplication replaced by matrix multiplication whenever components are multiplied. In particular, the derivative operations (2.3), (2.9) and (2.10)

simply act component-wise on matrix components. We now prove that (2.1) holds for the Riemann curvature tensor

$$\text{Riem}(\Gamma)_\nu^\mu \equiv R_{\nu ij}^\mu dx^i \otimes dx^j,$$

the components of which are given by

$$\text{Riem}(\Gamma)_{\nu ij}^\mu \equiv R_{\nu ij}^\mu \equiv \Gamma_{\nu j, i}^\mu - \Gamma_{\nu i, j}^\mu + \Gamma_{\sigma i}^\mu \Gamma_{\nu j}^\sigma - \Gamma_{\sigma j}^\mu \Gamma_{\nu i}^\sigma \quad (2.13)$$

and where we interpret μ and ν as matrix indices.

Lemma 2.1. *In fixed coordinates x^i , the Riemann curvature tensor is the matrix-valued 2-form (2.1) with matrix components*

$$\text{Riem}(\Gamma)_\nu^\mu = R_{\nu [ij]}^\mu dx^i \wedge dx^j = d(\Gamma_{\nu i}^\mu dx^i) + \Gamma_{\sigma i}^\mu dx^i \wedge \Gamma_{\nu j}^\sigma dx^j. \quad (2.14)$$

Proof. We use (2.2) and the antisymmetry of $R_{\nu ij}^\mu$ in i and j to write

$$\begin{aligned} R_{\nu [ij]}^\mu dx^i \wedge dx^j &= R_{\nu [ij]}^\mu (dx^i \otimes dx^j - dx^j \otimes dx^i) \\ &= \sum_{i < j} R_{\nu ij}^\mu dx^i \otimes dx^j + \sum_{i < j} R_{\nu ji}^\mu dx^j \otimes dx^i \\ &= R_{\nu ij}^\mu dx^i \otimes dx^j, \end{aligned}$$

without losing any information of the curvature tensor, which turns $\text{Riem}(\Gamma)$ into a matrix valued 2-form. To prove the second equality in (2.14), use (2.3) to compute

$$\begin{aligned} d(\Gamma_{\nu i}^\mu dx^i) &= \Gamma_{\nu i, j}^\mu dx^j \wedge dx^i = \Gamma_{\nu i, j}^\mu (dx^j \otimes dx^i - dx^i \otimes dx^j) \\ &= (\Gamma_{\nu j, i}^\mu - \Gamma_{\nu i, j}^\mu) dx^i \otimes dx^j \end{aligned}$$

and use (2.4) to compute

$$\begin{aligned} \Gamma_{\sigma i}^\mu dx^i \wedge \Gamma_{\nu j}^\sigma dx^j &= \Gamma_{\sigma i}^\mu \Gamma_{\nu j}^\sigma dx^i \wedge dx^j = \Gamma_{\sigma i}^\mu \Gamma_{\nu j}^\sigma (dx^i \otimes dx^j - dx^j \otimes dx^i) \\ &= (\Gamma_{\sigma i}^\mu \Gamma_{\nu j}^\sigma - \Gamma_{\sigma j}^\mu \Gamma_{\nu i}^\sigma) dx^i \otimes dx^j \end{aligned}$$

which combined yields the sought after second equality in (2.14). \square

To proceed, let $W^{m,p}(\Omega)$ be the Sobolev space of functions with partial derivatives up to m -th order in L^p . We say that a matrix valued k -form w is in $W^{m,p}(\Omega)$ if its components are functions in $W^{m,p}(\Omega)$, with respect to the fixed coordinate system x . Assume now that $m \geq 1$ and $p > n$, so that the Sobolev embedding theorem implies functions in $W^{1,p}$ are Hölder continuous, c.f. Morrey's inequality in [10]. The following Leibnitz rule holds.

Lemma 2.2. *Let $A \in W^{1,p}(\Omega)$ be a matrix valued k -form and let $B \in W^{1,p}(\Omega)$ be a matrix valued j -form, and assume $p > n$, then*

$$d(A \wedge B) = dA \wedge B + A \wedge dB \in L^p(\Omega). \quad (2.15)$$

Proof. Assuming first that A and B are smooth, a straightforward computation gives

$$d(A \cdot B)_\nu^\mu = \frac{1}{l!k!} d(A_{\sigma i_1 \dots i_k}^\mu B_{\nu j_1 \dots j_l}^\sigma dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l})$$

$$\begin{aligned}
&= \frac{1}{l!k!} \partial_l (A_{\sigma i_1 \dots i_k}^\mu B_{\nu j_1 \dots j_l}^\sigma) dx^l \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} \\
&= \partial_l A_{\sigma [i_1 \dots i_k]}^\mu dx^l \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge B_\nu^\sigma \\
&\quad + A_\sigma^\mu \wedge \partial_l B_{\nu [j_1 \dots j_l]}^\sigma dx^l \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} \\
&= dA_\sigma^\mu \wedge B_\nu^\sigma + A_\sigma^\mu dB_\nu^\sigma,
\end{aligned} \tag{2.16}$$

which is the sought after identity (2.15). To extend the above computation to $W^{1,p}$, we use that the difference quotient (along the j -th coordinate axis) $D_h f$ of a function $f \in W^{1,p}(\Omega)$ converges to its weak derivative $\partial_j f$ in L^1 as $h \rightarrow 0$. It follows that for the product of two functions $f, g \in W^{1,p}(\Omega)$ we have at $x \in \Omega$

$$D_h(fg)|_x = D_h(f)|_x g(x+h) + f(x)D_h(g)|_x. \tag{2.17}$$

Now, since $p > n$, we know by the Sobolev embedding theorem that g and f are Hölder continuous, so that the right hand side in (2.17) converges in L^1 as $h \rightarrow 0$ and implies

$$\lim_{h \rightarrow 0} D_h(fg) = g \partial_j f + f \partial_j g \in L^p(\Omega).$$

Thus, since $D_h(fg)$ converges to the weak derivative $\partial_j(fg)$ in L^1 as $h \rightarrow 0$, we conclude that

$$\partial_j(fg) = g \partial_j f + f \partial_j g \in L^p(\Omega) \tag{2.18}$$

and thus $fg \in W^{1,p}(\Omega)$. Applying (2.18) component-wise for the third equality in (2.16) leads to the sought after equation (2.15). \square

We also require the following Leibnitz rule for the co-derivative.

Lemma 2.3. *Let $J \in W^{2,p}(\Omega)$ be a matrix valued 0-form and $w \in W^{2,p}(\Omega)$ a matrix valued 1-form, then*

$$\delta(J \cdot w) = J \cdot \delta w + \langle dJ; w \rangle \tag{2.19}$$

where $\langle \cdot; \cdot \rangle$ is the matrix valued inner product defined in (2.7).

Proof. Using that δ of a 1-form is the divergence (2.12), we find that

$$(\delta(J \cdot w))_i^\alpha = \delta(J_k^\alpha w_{ij}^k dx^j) = \sum_{j=1}^n \partial_j (J_k^\alpha w_{ij}^k) = \sum_{j=1}^n J_{k,j}^\alpha w_{ij}^k + J_k^\alpha (\delta w)_i^k$$

and this proves the lemma. \square

We close this section by introducing the two operations we require to close the equations, which relate matrix valued to vector valued differential forms. Note, a matrix valued 0-form J_i^α turns into a vector valued 1-form $J_i^\alpha dx^i$ by contracting the lower matrix index with a Cartan basis element, (where α labels the components of the vector). To start, let an arrow over a matrix valued 0-form A convert A to its equivalent vector valued 1-form, i.e.,

$$\vec{A} \equiv A_i^\alpha dx^i. \tag{2.20}$$

By this, we can express the integrability of the Jacobian J , (c.f., Frobenius Theorem, [27]), as

$$d\vec{J} = 0, \quad (2.21)$$

since

$$Curl(J) \equiv \frac{1}{2}(J_{i,j}^\alpha - J_{j,i}^\alpha)dx^j \otimes dx^i = J_{i,j}^\alpha dx^j \wedge dx^i = d(J_i^\alpha dx^i) \equiv d\vec{J}^\alpha.$$

For our elliptic system to close, we need one more operation to convert matrix valued to vector valued differential forms. Namely, for $\omega \in \Lambda_k^{1,p}(\Omega)$, we define

$$\overrightarrow{\text{div}}(\omega)^\alpha \equiv \sum_{l=1}^n \partial_l((\omega_l^\alpha)_{i_1, \dots, i_k}) dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad (2.22)$$

which is the divergence with respect to the lower matrix index, thus creating a vector valued k -form out of a matrix valued k -form. We close this subsection with the following intriguing identity for commuting d and δ which has no analogue for classical scalar valued differential forms and is the key identity for the regularity of the final elliptic system to close.

Lemma 2.4. *Let $\Gamma \in W^{m,p}(\Omega)$ and $J \in W^{m+1,p}(\Omega)$ for $p > n$ and $m \geq 1$, then*

$$d(\overrightarrow{\delta(J \cdot \Gamma)}) = \overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J \cdot d\Gamma). \quad (2.23)$$

Proof. Since δ of a matrix valued 1-form is the divergence (for fixed matrix components), we have

$$(\delta(J\Gamma))_j^\alpha = \delta(J_k^\alpha \Gamma_{ji}^k dx^i) = \sum_{l=1}^n \partial_l(J_k^\alpha \Gamma_{jl}^k)$$

and thus

$$(\overrightarrow{\delta(J\Gamma)})^\alpha = (\delta(J\Gamma))_j^\alpha dx^j = \sum_{l=1}^n \partial_l(J_k^\alpha \Gamma_{jl}^k) dx^j,$$

from which we find that

$$d(\overrightarrow{\delta(J\Gamma)})^\alpha = \partial_i(\delta(J\Gamma))_j^\alpha dx^i \wedge dx^j = \sum_{l=1}^n \partial_i \partial_l(J_k^\alpha \Gamma_{jl}^k) dx^i \wedge dx^j,$$

where in the case $m = 1$ second derivatives are taken in a distributional sense. Now, since weak derivatives commute, we obtain from the product rule (which applies since Γ and derivatives of J are Hölder continuous) that

$$\begin{aligned} d(\overrightarrow{\delta(J\Gamma)})^\alpha &= \sum_{l=1}^n \partial_l \partial_i(J_k^\alpha \Gamma_{jl}^k) dx^i \wedge dx^j \\ &= \sum_{l=1}^n \partial_l(J_{k,i}^\alpha \Gamma_{jl}^k) dx^i \wedge dx^j + \sum_{l=1}^n \partial_l(J_k^\alpha \Gamma_{l,j,i}^k) dx^i \wedge dx^j \\ &= \overrightarrow{\text{div}}(dJ \wedge \Gamma)^\alpha + \overrightarrow{\text{div}}(J \cdot d\Gamma)^\alpha. \end{aligned}$$

This completes the proof. \square

2.2. The Riemann-flat condition in terms of matrix valued differential forms. Consider the transformation law for a connection

$$(J^{-1})_{\alpha}^k (\partial_j J_i^{\alpha} + J_i^{\beta} J_j^{\gamma} \Gamma_{\beta\gamma}^{\alpha}) = \Gamma_{ij}^k, \quad (2.24)$$

where Γ_{ij}^k denotes the components of the connection in x^i -coordinates, $\Gamma_{\gamma\beta}^{\alpha}$ denotes its components in y^{α} -coordinates, and where $J_i^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^i}$ denotes the Jacobian of the coordinate transformation. Assume now that $\Gamma_{ij}^k \in W^{m,p}(\Omega)$, $J_i^{\alpha} \in W^{m+1,p}(\Omega)$ and $\Gamma_{\beta\gamma}^{\alpha} \in W^{m+1,p}(\Omega)$, for $m \geq 1$, so the Jacobian J smooths the connection Γ_{ij}^k by one order. For these given coordinates x and y , define

$$\tilde{\Gamma}_{ij}^k \equiv (J^{-1})_{\alpha}^k J_i^{\beta} J_j^{\gamma} \Gamma_{\beta\gamma}^{\alpha}. \quad (2.25)$$

Then requiring $\tilde{\Gamma}$ to transform as a $(1,2)$ -tensor, (2.25) defines the *tensor* $\tilde{\Gamma}$. By this, (2.24) can be written equivalently as

$$(J^{-1})_{\alpha}^k \partial_j J_i^{\alpha} = (\Gamma - \tilde{\Gamma})_{ij}^k. \quad (2.26)$$

Now, since adding a tensor to a connection yields another connection, (2.26) is just the condition that J transforms the connection $\Gamma - \tilde{\Gamma}$ to zero. This implies $\Gamma - \tilde{\Gamma}$ is a Riemann-flat connection, $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$. In the language of matrix valued differential forms (2.26) reads

$$J^{-1}dJ = \Gamma - \tilde{\Gamma},$$

where J is a matrix valued 0-form and Γ and $\tilde{\Gamma}$ are matrix valued 1-forms.

Note (2.24) - (2.26) apply to $\Gamma_{ij}^k \in L^{\infty}(\Omega)$ and $\Gamma_{\beta\gamma}^{\alpha} \in C^{0,1}(\Omega)$, and it is proven in [22] that the reverse implication is also true, even at this low level of regularity of $\Gamma \in L^{\infty}$. The equivalence is this: One can smooth an L^{∞} connection Γ one order to $C^{0,1}$ by a $C^{0,1}$ coordinate transformation if and only if the ‘‘Riemann-flat condition’’ holds, and we say that the Riemann-flat condition holds if there exists a Lipschitz continuous rank $(1,2)$ -tensor $\tilde{\Gamma}_{ij}^k$ with symmetry $\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k$ such that $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$ holds. Based on this, we now record the following version of the Riemann flat condition and related equivalencies applicable to the smoothness classes $\Gamma \in W^{m,p}$ relevant for this paper.

Theorem 2.5. *Let Γ_{ij}^k be a symmetric connection in $W^{m,p}(\Omega)$ for $m \geq 1$ and $p > n$ (in coordinates x^i). Then the following points are equivalent:*

- (i) *There exists a coordinate transformation $x^i \mapsto y^{\alpha}$ with Jacobian $J \in W^{m+1,p}(\Omega)$ such that $\Gamma_{\beta\gamma}^{\alpha} \in W^{m+1,p}(\Omega)$ in y -coordinates.*
- (ii) *There exists a symmetric $(1,2)$ -tensor $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and a matrix field $J \in W^{m+1,p}(\Omega)$ which solve*

$$J^{-1}dJ = \Gamma - \tilde{\Gamma}, \quad (2.27)$$

$$J_{i,j}^{\alpha} - J_{j,i}^{\alpha} = 0. \quad (2.28)$$

(iii) *There exists a symmetric (1,2) tensor $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ such that $\Gamma - \tilde{\Gamma}$ is Riemann-flat,*

$$\text{Riem}(\Gamma - \tilde{\Gamma}) = 0. \quad (2.29)$$

(iv) *There exists a symmetric (1,2) tensor $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ which, when viewed as a matrix valued 1-form in x -coordinates, solves*

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma}). \quad (2.30)$$

Proof. The equivalence of (i) and (ii) follows from (2.24) - (2.26), where (2.28) is the Frobenius integrability condition. That (ii) implies (iii) follows because (2.26) implies the Riemann-flat condition (2.29). The equivalence of (iii) and (iv) follows from Lemma 2.1. Finally, the implication (iii) to (i) is established in \square in the case of the lower regularity class $\Gamma \in L^\infty$, $\tilde{\Gamma}, J \in C^{0,1}$. The more regular case of $\Gamma \in W^{m,p}$, $\tilde{\Gamma}, J \in W^{m+1,p}$ here, follows by a similar argument using compactness in $W^{m,p}$ of the unit ball in $W^{m+1,p}$, in place of the Arzela-Ascoli theorem. (Details omitted.) \square

We close this section by proving the following lemma, relating symmetry of $\tilde{\Gamma}$ to integrability of J for solutions of the Riemann-flat condition.

Lemma 2.6. *Let $\Gamma \in W^{m,p}(\Omega)$ be symmetric. Assume $(J, \tilde{\Gamma}) \in W^{m+1,p}(\Omega)$ solve the Riemann-flat condition (2.27). Then J satisfies the integrability condition (2.28) if and only if $\tilde{\Gamma}$ is symmetric, $\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k$.*

Proof. In components (2.27) reads $(J^{-1})_\sigma^\mu \partial_i J_j^\sigma = \Gamma_{ij}^\mu - \tilde{\Gamma}_{ij}^\mu$, so clearly, symmetry on the right hand side is equivalent to symmetry on the left hand side. \square

2.3. Review of elliptic regularity estimates. We review basic elliptic regularity results relevant for the RT-equations (1.1) - (1.4). Note, the Laplacian $\Delta = d\delta + \delta d$ acts component-wise on differential forms, so regularity estimates for the scalar Poisson equation extend directly to matrix valued differential forms. We assume from now on that Ω is a bounded open set in \mathbb{R}^n with smooth boundary, (at least $C^{1,1}$).

Theorem (Elliptic Regularity): *Let $u \in W^{2,p}(\Omega)$ be a scalar, $\infty > p > 1$. Then there exists a constant $C > 0$ depending only on Ω , m, n, p , such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} \right). \quad (2.31)$$

Estimate (2.31) is equation (2,3,3,1) in [12]. Estimates for the regularity of the first order equations (3.1) that parallel the estimates for the classical Poisson equation (2.31) are given by the Gaffney inequality, which we now state, (c.f. Theorem 5.21 in [6]).

Theorem (Gaffney Inequality): *Let $u \in W^{m+1,p}(\Omega)$ be a k -form for $m \geq 0$, $p \in (1, \infty)$, $1 \leq k \leq n - 1$ and (for simplicity) $n \geq 2$. Then there exists a constant $C > 0$ depending only on Ω , m, n, p , such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C \left(\|du\|_{W^{m,p}(\Omega)} + \|\delta u\|_{W^{m,p}(\Omega)} + \|u\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} \right). \quad (2.32)$$

Again, estimate (2.32) for scalar valued differential forms extend to matrix valued differential forms. In this paper we only rely on the following elliptic estimate with respect to the $W^{m+1,p}$ -norm, for $m \geq 1$, on compactly contained subsets of Ω , which we prove here for completeness.

Lemma 2.7. *Let $f \in W^{m-1,p}(\Omega)$ be a scalar, where $m \geq 1$, $p \in (1, \infty)$. Assume the scalar $u \in W^{m+1,p}(\Omega)$ solves $\Delta u = f$. Then, for any open set Ω' compactly contained in Ω , there exists a constant $C > 0$, depending only on Ω' , Ω , m, n, p , such that*

$$\|u\|_{W^{m+1,p}(\Omega')} \leq C (\|f\|_{W^{m-1,p}(\Omega)} + \|u\|_{W^{m-1,p}(\Omega)}). \quad (2.33)$$

Proof. Let Ω' be an open set that is compactly contained in Ω . Equation (9.36) of Theorem 9.11 in [11] gives estimate (2.33) in the case $m = 1$, that is,

$$\|u\|_{W^{2,p}(\Omega')} \leq C (\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}). \quad (2.34)$$

Now, from the definition of the $W^{m+1,p}$ -norm, we find that

$$\|u\|_{W^{m+1,p}(\Omega')} = \sum_{|\alpha| \leq m-1} \|D^\alpha u\|_{W^{2,p}(\Omega')}, \quad (2.35)$$

where α denotes a standard multindex and D^α the corresponding combination of partial derivatives, c.f. [10]. Differentiating $\Delta u = f$ by D^α and applying (2.34) to each term on the right hand side of (2.35) then yields

$$\begin{aligned} \|u\|_{W^{m+1,p}(\Omega')} &\leq C \sum_{|\alpha| \leq m-1} (\|D^\alpha f\|_{L^p(\Omega)} + \|D^\alpha u\|_{L^p(\Omega)}) \\ &\leq C (\|f\|_{W^{m-1,p}(\Omega)} + \|u\|_{W^{m-1,p}(\Omega)}), \end{aligned}$$

which is the sought after estimate (2.33). \square

3. A SYSTEM OF ELLIPTIC PDE'S EQUIVALENT TO THE RIEMANN FLAT CONDITION

In this section we derive a pair of nonlinear Poisson equations equivalent to the Riemann Flat Condition in the form (2.27), such that it closes up in the unknowns $(J, \tilde{\Gamma})$, with regularity in each term formally consistent with $\Gamma \in W^{m,p}$, but $\tilde{\Gamma}, J \in W^{m+1,p}$. This accomplishes the first step in our program to apply elliptic regularity results to solve the problem of regularity singularities. To start, observe that equations (2.27) - (2.28) are under-determined for unknowns $(J, \tilde{\Gamma})$. On the other hand, (2.30) is a system of equations for $\tilde{\Gamma}$ alone which is consistent with $\tilde{\Gamma}$ being one degree more

regular than Γ , $\tilde{\Gamma} \in W^{m+1,p}$, but a necessary condition to solve them is that the exterior derivative of the right hand side must vanish. The latter imposes additional conditions on $\tilde{\Gamma}$ that must be satisfied. The objective of this section is to derive equations (1.1) - (1.4) from (2.27) - (2.28), a system of elliptic PDE's which closes up in $(J, \tilde{\Gamma})$, and prove that finding solutions of this PDE is equivalent to solving the Riemann-flat condition (2.27) - (2.28).

3.1. Cauchy Riemann systems and Poisson equations. In this subsection we get started by briefly reviewing the classical equivalence between Poisson equations and Cauchy Riemann type equations for matrix valued differential forms at the level of smoothness we are dealing with. This is the starting point for our derivation of the elliptic system (1.1) - (1.4) in Sections 3.2 and 3.3. The Riemann-flat condition is stated in terms of exterior derivatives, and we apply the ideas in this section to replace the J equation, which as a first order equations is formally unsolvable, into a second order Poisson equation which is solvable. The starting point is the following classical result for scalar valued differential forms, c.f. [6]. (We prove a generalization of this in Lemma 3.1 below.)

Theorem: *Assume f is a smooth $(k+1)$ -form and g is a smooth $(k-1)$ -form such that $df = 0$ and $\delta g = 0$. Then a k -form u solves*

$$du = f \quad \text{and} \quad \delta u = g \quad (3.1)$$

if and only if u solves

$$\Delta u = \delta f + dg \quad (3.2)$$

with boundary data $du = f$ and $\delta u = g$ on $\partial\Omega$.

To introduce some ideas underlying Theorem 1.1, we now state and record the proof of a version of the classical result regarding the equivalence of (3.1) - (3.2), which applies to solutions of nonlinear PDE's involving matrix valued differential forms which more closely model (1.1) - (1.4). For this, assume f maps k -forms to $(k+1)$ -forms and g maps k -forms to $(k-1)$ -forms, let $\Lambda_k^{m,p}(\Omega)$ denote the space of matrix-valued k -forms with components in $W^{m,p}(\Omega)$, and assume that

$$\begin{aligned} f : \Lambda_k^{m+1,p}(\Omega) &\longrightarrow \Lambda_{k+1}^{m,p}(\Omega), \\ g : \Lambda_k^{m+1,p}(\Omega) &\longrightarrow \Lambda_{k-1}^{m,p}(\Omega). \end{aligned} \quad (3.3)$$

The loss and gain of derivatives in f and g are introduced to model the right hand side of (1.1) - (1.4).

Lemma 3.1. *Assume f and g as in (3.3), and assume $m \geq 2$, $1 < p < \infty$, such that $d(f(w)) = 0$ and $\delta(g(w)) = 0$ for any $w \in \Lambda_k^{m+1,p}(\Omega)$. Then $u \in \Lambda_k^{m+1,p}(\Omega)$ solves*

$$du = f(u) \quad \text{and} \quad \delta u = g(u), \quad (3.4)$$

if and only if u solves

$$\Delta u = \delta(f(u)) + d(g(u)) \quad (3.5)$$

with boundary data

$$du = f \quad \text{and} \quad \delta u = g \quad \text{on} \quad \partial\Omega. \quad (3.6)$$

Proof. To prove that (3.4) implies (3.5), recall that $\Delta \equiv d\delta + \delta d$ by (2.10). Taking δ of $du = f(u)$ and d of $\delta u = g(u)$ and adding the resulting equations, gives (3.5), and restricting (3.4) to $\partial\Omega$ gives (3.6). This proves the forward implications.

For the backward implication, assume (3.5) and (3.6). To show that u solves $du = f(u)$, take the exterior derivative d of the Poisson equation (3.5). Observing that $\Delta \equiv d\delta + \delta d$ commutes with d (and δ) and using $d^2 = 0$ and $df(u) = 0$, we obtain

$$\Delta(du - f(u)) = 0. \quad (3.7)$$

Thus each component of $du - f(u)$ is a harmonic function in Ω . Moreover, by (3.6), each component of $du - f(u)$ vanishes on the boundary, implying $du - f(u) = 0$ in Ω , thereby establishing the first equation in (3.4). Similarly, taking δ of (3.5), using $\delta^2 = 0$ and $\delta g(u) = 0$, we find

$$\Delta(\delta u - g(u)) = 0, \quad (3.8)$$

which combined with boundary data (3.6) implies $\delta u - g(u) = 0$ in Ω , so the second equation in (3.4) also holds. This proves the backward implication. \square

The above theorem and proof are correct at the level of classical derivatives, but for the A equation in system (1.1) - (1.4) we need to see that Lemma 3.1 holds for solutions one degree less regular. We state this as a lemma:

Lemma 3.2. *Lemma 3.1 is also true for $m \geq 1$, $1 < p < \infty$.*

Proof. The forward implication follows as in Lemma 3.1 because the boundary data makes sense in L^p by the trace theorem, [10].

For the backward implication at the lower regularity $m = 1$, we must take derivatives in a distributional sense. For this, take the L^2 inner product on matrix valued forms to be

$$\langle \cdot, \cdot \rangle_{L^2} \equiv \int_{\Omega} \text{tr}(\langle \cdot ; \cdot \rangle), \quad (3.9)$$

where $\text{tr}(\cdot)$ denotes the trace of a matrix and $\langle \cdot ; \cdot \rangle$ is the matrix valued inner product defined in (2.7). Using the definition of Hodge star (2.6), the product rule (2.15) for matrix value forms, and Stokes Theorem, it's easy to see that the standard integration by parts formula for k -forms extends to matrix valued forms,

$$\langle dw, v \rangle_{L^2} + \langle w, \delta v \rangle_{L^2} = 0, \quad (3.10)$$

where w is a matrix valued k -form and v a matrix valued $k + 1$ -form, both differentiable and at least one of them vanishing on $\partial\Omega$, (c.f. [6, Theorem 1.11]).

Now assume $\Delta u = \delta f + dg$ holds in Ω , $du = f$, $\delta u = g$ on $\partial\Omega$, and $u \in W^{2,p}$. We show $du = f(u)$ holds in the L^p sense. By Riesz representation, it suffices to show that

$$\langle (du - f), \phi \rangle_{L^2} = 0, \quad (3.11)$$

for all $\phi \in L^{p^*}(\Omega)$, where $\frac{1}{p^*} + \frac{1}{p} = 1$. Since the Laplacian is invertible, there exist a $\psi \in W^{2,p^*}(\Omega)$ such that $\Delta\psi = \phi$, and $\psi = 0$ on $\partial\Omega$. Since by assumption, $du - f(u) = 0$ on $\partial\Omega$, we can apply the integration by parts formula (3.10) and compute

$$\begin{aligned} \langle (du - f), \phi \rangle_{L^2} &= \langle (du - f), \Delta\psi \rangle_{L^2} \\ &= -\langle \delta(du - f), \delta\psi \rangle_{L^2} - \langle d(du - f), d\psi \rangle_{L^2} \\ &= -\langle \delta(du - f), \delta\psi \rangle_{L^2}, \end{aligned} \quad (3.12)$$

where in the last equality we use $d^2u = df = 0$. Since $\delta^2 = 0$ and $\delta u - g(u) = 0$ on $\partial\Omega$, we have

$$0 = \langle (\delta u - g), \delta^2\psi \rangle_{L^2} = -\langle d(\delta u - g), \delta\psi \rangle_{L^2}.$$

Adding this to (3.12), gives

$$\begin{aligned} \langle (du - f), \phi \rangle_{L^2} &= -\langle \delta(du - f), \delta\psi \rangle_{L^2} - \langle d(\delta u - g), \delta\psi \rangle_{L^2} \\ &= \langle (\Delta u - \delta f - dg), \delta\psi \rangle_{L^2} = 0, \end{aligned}$$

which proves $du - f(u) = 0$ in Ω . A similar reasoning proves that $\delta u = g(u)$ holds as well. This completes the proof. \square

3.2. A first equivalence to an elliptic system. In this section we derive a system of nonlinear Poisson equations equivalent to the Riemann-flat condition in the case Γ and $\text{Riem}(\Gamma) \in W^{1,p}(\Omega)$, $p > n$. When $\Gamma, \text{Riem}(\Gamma) \in W^{1,p}(\Omega)$, solutions are regular enough to impose boundary conditions, (c.f. Lemma 3.2), and $p > n$ guarantees $W^{m,p}$ is closed under taking products.⁵ Assuming $\text{Riem}(\Gamma) \in W^{1,p}(\Omega)$ is equivalent to assuming $d\Gamma \in W^{1,p}(\Omega)$, so only $\delta\Gamma$ is free to be one level less smooth than Γ and $d\Gamma$. We begin with the following two equivalent expressions of the Riemann-flat condition,

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma}) \quad , \quad (3.13)$$

$$J^{-1}dJ = (\Gamma - \tilde{\Gamma}), \quad (3.14)$$

c.f. (2.30) and (2.27). Recall, J , the Jacobian of the sought after coordinate transformation, is taken as a matrix valued 0-form, and $\tilde{\Gamma}$, the sought after correction to the connection, is taken as a matrix valued 1-form. The following lemma displays the remarkable connection between (3.13) and (3.14):

⁵Since the nonlinearities in the equations involve products of functions in L^p , (and more generally in $W^{m,p}$), and products of L^p functions are not generally in L^p , we assume $p > n$ so the Sobolev embedding implies L^p functions are Hölder continuous. Then we can estimate $\|fg\|_p \leq \|f\|_{sup}\|g\|_{L^p}$ for $f, g \in W^{1,p}$, and similarly for f, g in $W^{m,p}$.

Lemma 3.3. *Any matrix valued 0-form $J \in W^{2,p}(\Omega)$ satisfies*

$$d(J^{-1}dJ) = -(J^{-1}dJ) \wedge (J^{-1}dJ). \quad (3.15)$$

That is, (3.15) explains why the exterior derivative of the right hand side of (3.13) vanishes, (required to be consistent with the left hand side), since (3.15) together with (3.13) implies

$$\text{Riem}(\Gamma - \tilde{\Gamma}) = d\Gamma - d\tilde{\Gamma} - d(J^{-1}dJ). \quad (3.16)$$

Also, taking the exterior derivative of (3.14) gives precisely $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$, thereby showing directly how information in (3.13) is encoded in (3.14). (Deriving (3.14) from (3.13) is not so straightforward, c.f. [22].) Our goal now is to derive our equations directly from (3.14) alone.

Proof of Lemma 3.3: We first show identity (3.15). Since the exterior derivative defined in (2.3) acts component-wise on matrix valued forms, it follows that $d^2J = 0$. Moreover, using the Leibniz rule we compute

$$0 = d(J^{-1}J) = \partial_i(J^{-1}J)dx^i = \partial_i J^{-1}dx^i J + J^{-1}\partial_i J dx^i = d(J^{-1})J + J^{-1}dJ,$$

from which we obtain that

$$d(J^{-1}) = -J^{-1} \cdot dJ \cdot J^{-1}.$$

Thus, since $d^2J = 0$, we find from the Leibniz rule for k -forms in (2.15) that

$$\begin{aligned} d(J^{-1}dJ) &= d(J^{-1}) \wedge dJ + J^{-1}d^2J \\ &= -J^{-1}dJ J^{-1} \wedge dJ \\ &= -J^{-1}dJ \wedge J^{-1}dJ, \end{aligned}$$

where we used for the last equality that matrices commute with basis elements of k -forms to commute the wedge product with J^{-1} . This establishes identity (3.15). \square

We now derive a set of equations in $(\tilde{\Gamma}, J)$ which is consistent and closes. For the $\tilde{\Gamma}$ equations, in light of (3.16), we take the Riemann-flat condition (2.29) as

$$d\tilde{\Gamma} = d\Gamma - d(J^{-1}dJ). \quad (3.17)$$

The right hand side is consistent with the left hand side since both are exterior derivatives. Motivated by the fact that only $d\tilde{\Gamma}$ appears in the curvature, we allow $\delta\tilde{\Gamma}$ to be a free function, and set

$$\delta\tilde{\Gamma} = h, \quad (3.18)$$

where $h \in W^{1,p}$ is an arbitrary matrix valued 0-form.⁶ For fixed function J , one could solve (3.17) - (3.18) for $\tilde{\Gamma}$ by the existence theory in [6], (the Poincaré Lemma), since the consistency condition is that the exterior derivative of the right hand side of (3.17) vanishes, and $\delta h = 0$ for matrix

⁶The freedom in choosing h reflects the freedom in choosing smooth coordinate transformations to maintain the smoothness of a spacetime connection.

valued 0-forms. Alternatively, adding δ of (3.17) and d of (3.18) produces the second order Poisson equation

$$\Delta \tilde{\Gamma} = \delta d(\Gamma - J^{-1}dJ) + dh. \quad (3.19)$$

By Lemma 3.1, it follows that any solution of (3.19) which satisfies (3.17) - (3.18) on $\partial\Omega$, is also a solution of the Cauchy-Riemann system (3.17) - (3.18) in Ω .

The problem of deriving the J equation is not so simple. It turns out we need a second order equation, since the consistency condition that the right hand side of the first order equation (3.14) have a vanishing exterior derivative, leads to circularity. To see this, we introduce the following lemma.

Lemma 3.4. *Assume (3.17) holds for $J, \tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $\Gamma \in W^{m,p}(\Omega)$ for $m \geq 1$, then*

$$d(J \cdot (\Gamma - \tilde{\Gamma})) = dJ \wedge ((\Gamma - \tilde{\Gamma}) - J^{-1}dJ). \quad (3.20)$$

Proof. A straightforward computation using the Leibniz rule for k -forms (2.15) gives

$$\begin{aligned} d(J \cdot (\Gamma - \tilde{\Gamma})) &= dJ \wedge (\Gamma - \tilde{\Gamma}) + J \cdot (d\Gamma - d\tilde{\Gamma}) \\ &= dJ \wedge (\Gamma - \tilde{\Gamma}) + J \cdot d(J^{-1}dJ), \end{aligned} \quad (3.21)$$

where we used (3.17) for the last equality, and substituting (3.15) for $d(J^{-1}dJ)$ yields

$$d(J \cdot (\Gamma - \tilde{\Gamma})) = dJ \wedge (\Gamma - \tilde{\Gamma}) - dJ \wedge J^{-1}dJ,$$

which is the sought after equation (3.20). \square

To see the circularity in the first order equation for J , note that one can solve the Riemann-flat condition (3.14) for J only under the consistency condition that the exterior derivative of J times its right hand side should vanish. By (3.20), the exterior derivative of the right hand side vanishes if either either $dJ = 0$, (in which case $\Gamma = \tilde{\Gamma}$, and $\tilde{\Gamma}$ does not have the regularity we seek), or if (3.14) holds, which just reproduces the equation for J we started with, which is circular; or else the right hand side of (3.20) produces a nonlinear PDE in J more complicated than the one we started with.

Thus, differently from the case of $\tilde{\Gamma}$, we need a second order equation in J in order to obtain a solvable PDE. The second order equation for J obtained from (3.14) is again a non-linear Poisson equation which does not require the constraint that the right hand side of (3.20) should vanish. To obtain this, again use $\Delta = \delta d + d\delta$, and note that for 0-forms J , $\Delta J = \delta dJ$. Thus taking δ of equation (3.14), we obtain

$$\Delta J = \delta(J \cdot (\Gamma - \tilde{\Gamma})). \quad (3.22)$$

Applying (2.19) gives

$$\delta(J \cdot \tilde{\Gamma}) = J \cdot \delta \tilde{\Gamma} + \langle dJ; \tilde{\Gamma} \rangle.$$

Thus, replacing $\delta\tilde{\Gamma} = h$ by (3.18) yields equation (3.22) in its final form,

$$\Delta J = \delta(J \cdot \Gamma) - J \cdot h - \langle dJ; \tilde{\Gamma} \rangle, \quad (3.23)$$

where again h is a free function. In contrast to the first order equation (3.14), solving the nonlinear Poisson equations (3.23) allows for more general boundary data and does not require the right hand side to have a vanishing exterior derivative.

To summarize, every solution $(J, \tilde{\Gamma})$ of the Riemann-flat condition (3.14) also solves the second order equations (3.19) and (3.23). In the next theorem we prove equivalence of (3.19) and (3.23) with the Riemann-flat condition (3.14), in the sense that a solution $(J, \tilde{\Gamma})$ of the Poisson system (3.19) and (3.23) gives rise to a solution of the original Riemann-flat condition (3.14) after suitable modification of $\tilde{\Gamma}$. Remarkably, in contrast to Lemma 3.1, the second order system (3.19) and (3.23) generate solutions of the first order system without requiring any boundary conditions.

Theorem 3.5. *Let $\Gamma, d\Gamma \in W^{m,p}(\Omega)$, and assume $m \geq 1$, $p > n$. Then if $(J, \tilde{\Gamma})$ solves the Riemann-flat condition (3.14) for matrix-valued 0-form $J \in W^{m+1,p}(\Omega)$ and matrix valued 1-form $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, then $(J, \tilde{\Gamma})$ also solves*

$$\Delta \tilde{\Gamma} = \delta d(\Gamma - J^{-1}dJ) + dh, \quad (3.24)$$

$$\Delta J = \delta(J \cdot \Gamma) - J \cdot h - \langle dJ; \tilde{\Gamma} \rangle, \quad (3.25)$$

for $h \equiv \delta\tilde{\Gamma} \in W^{m,p}(\Omega)$. Conversely, if $(J, \tilde{\Gamma}) \in W^{m+1,p}(\Omega)$ solves (3.24)-(3.25) for some matrix-valued 0-form $h \in W^{m,p}(\Omega)$, then the modified pair $(J, \tilde{\Gamma}')$ solves the Riemann-flat condition (3.14) in each open set $\Omega' \subset \subset \Omega$, (i.e., Ω' is compactly contained in Ω), with

$$\tilde{\Gamma}' = -J^{-1}dJ + \Gamma \in W^{m+1,p}(\Omega'). \quad (3.26)$$

Proof. For the forward implication, assume that J and $\tilde{\Gamma}$ satisfy the Riemann-flat condition (3.14) in Ω . Taking the exterior derivative d of (3.14) implies (3.17), while (3.18) follows by definition of h . Adding now δ of (3.17) and d of (3.18) gives the sought after Poisson equation (3.24) on $\tilde{\Gamma}$. Moreover, the argument between (3.22) and (3.23) shows that any solution of (3.14) also solves the Poisson equation (3.25) for J . This proves the forward implication.

To prove the backward implication, assume $\tilde{\Gamma}$ and J solve (3.24) - (3.25) for some matrix-valued 0-form $h \in W^{m,p}(\Omega)$. Define the matrix-valued 1-form

$$w \equiv J^{-1}dJ - (\Gamma - \tilde{\Gamma}) \in W^{m,p}(\Omega). \quad (3.27)$$

(Note, if we had $w = 0$, then $(J, \tilde{\Gamma})$ would already solve (3.14).) Clearly, setting

$$\tilde{\Gamma}' = \tilde{\Gamma} - w = -J^{-1}dJ + \Gamma, \quad (3.28)$$

the pair $(J, \tilde{\Gamma}')$ solves the Riemann-flat condition (3.14). The nontrivial part of the proof is to show that $\tilde{\Gamma}'$ has the sought after regularity, $\tilde{\Gamma}' \in$

$W^{m+1,p}(\Omega)$. By (3.28), since $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, it suffices to show that $w \in W^{m,p}(\Omega)$ is one level more regular, $w \in W^{m+1,p}(\Omega')$, on each compactly contained subset $\Omega' \subset\subset \Omega$. To show this, we derive below that (3.24) and (3.25) imply w satisfies the Poisson equation

$$\Delta w = -d(J^{-1}\langle dJ; w \rangle). \quad (3.29)$$

Assume for the moment w satisfies (3.29). Since $w, h \in W^{m,p}(\Omega)$ and $J \in W^{m+1,p}(\Omega)$ and since products of functions in $W^{m,p}(\Omega)$ are again in $W^{m,p}(\Omega)$ when $p > n$, c.f. Lemma 2.2, it follows that $J^{-1}\langle dJ; w \rangle \in W^{m,p}(\Omega)$, so that the right hand side of (3.29) is in $W^{m-1,p}(\Omega)$. Now, let $f \in W^{m-1,p}(\Omega)$ be a scalar and assume the scalar $u \in W^{m+1,p}(\Omega)$ solves $\Delta u = f$. Then, by the elliptic estimate (2.33) of Lemma 2.7, there exists for each $\Omega' \subset\subset \Omega$ a constant $C > 0$ (depending only on Ω', Ω, m, n, p), such that

$$\|u\|_{W^{m+1,p}(\Omega')} \leq C' (\|f\|_{W^{m-1,p}(\Omega)} + \|u\|_{W^{m-1,p}(\Omega)}). \quad (3.30)$$

Estimate (3.30) extends to any $u \in W^{m,p}(\Omega)$ with $\Delta u = f$, as can be shown, for instance, by a mollification argument. In more detail, let $f^\epsilon \in C^\infty(\Omega)$ be the standard mollifier of f and assume u^ϵ solves $\Delta u^\epsilon = f^\epsilon$, then $u^\epsilon \in C^\infty(\Omega)$, f^ϵ converges to f in $W^{m-1,p}(\Omega)$ and one can show that u^ϵ converges to u in $W^{m,p}(\Omega)$. Thus, applying (3.30) to u^ϵ and f^ϵ and using $W^{m-1,p}$ -convergence of f^ϵ and u^ϵ , we find that the right hand side of (3.29) gives an ϵ -independent bound on $\|u^\epsilon\|_{W^{m+1,p}(\Omega')}$ which implies convergence of u^ϵ to u in $W^{m+1,p}(\Omega)$ and shows that (3.30) holds as well. Now, since $w \in W^{m,p}(\Omega)$ and

$$f \equiv -d(J^{-1}\langle dJ; w \rangle) \in W^{m-1,p}(\Omega),$$

estimate (3.30) applies component-wise to $\Delta w = f$ and yields the sought after regularity $w \in W^{m+1,p}(\Omega')$.

It remains to prove that w satisfies (3.29). For this, we use definition (3.27) of w to compute $\Delta w \equiv \delta dw + d\delta w$. To begin, since J is a zero form, we have $\delta J = 0$, which implies $\Delta J = \delta dJ$, so we compute starting from (3.27) that

$$\begin{aligned} \delta(J \cdot w) &\stackrel{(3.27)}{=} \delta(dJ - J(\Gamma - \tilde{\Gamma})) \\ &= \Delta J - \delta(J \cdot \Gamma) + \delta(J \cdot \tilde{\Gamma}) \\ &\stackrel{(2.19)}{=} \Delta J - \delta(J \cdot \Gamma) + J \cdot \delta \tilde{\Gamma} + \langle dJ; \tilde{\Gamma} \rangle \\ &\stackrel{(3.25)}{=} \delta(J \cdot \Gamma) - J \cdot h - \langle dJ; \tilde{\Gamma} \rangle - \delta(J \cdot \Gamma) + J \cdot \delta \tilde{\Gamma} + \langle dJ; \tilde{\Gamma} \rangle, \end{aligned}$$

so that cancellation gives

$$\delta(J \cdot w) = J \cdot (\delta \tilde{\Gamma} - h). \quad (3.31)$$

Applying (2.19) of Lemma 2.3 to compute $\delta(J \cdot w)$, we write (3.31) as

$$\delta \tilde{\Gamma} - h = J^{-1} \delta(J \cdot w) = \delta w + J^{-1} \cdot \langle dJ; w \rangle,$$

or equivalently

$$\delta w = -J^{-1} \cdot \langle dJ; w \rangle + \delta \tilde{\Gamma} - h,$$

so that

$$d\delta w = -d(J^{-1} \cdot \langle dJ; w \rangle) + d(\delta\tilde{\Gamma} - h), \quad (3.32)$$

On the other hand,

$$\begin{aligned} \delta dw &\stackrel{(3.27)}{=} \delta d(J^{-1}dJ - \Gamma) + \delta d\tilde{\Gamma} \\ &= \delta d(J^{-1}dJ - \Gamma) + \Delta\tilde{\Gamma} - d\delta\tilde{\Gamma} \\ &\stackrel{(3.24)}{=} d(h - \delta\tilde{\Gamma}). \end{aligned} \quad (3.33)$$

Thus, taking d of (3.32) and adding the resulting equation to (3.33), we obtain (3.29). This completes the proof. \square

Note that system (3.24) - (3.25) can be solved for any $h \in W^{m,p}(\Omega)$, so, at this stage, h is a freely assignable matrix valued 0-form. The regularity $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ is consistent with equation (3.24), since (3.15) of Lemma 3.3 implies the right hand side of (3.24) to be in $W^{m-1,p}(\Omega)$. Let us finally remark that we could have established the equivalence of Theorem 3.5 for (3.25) together with (3.17) - (3.18), however, we find system (3.24) - (3.25) preferable, since the existence theory for the first order system (3.17) - (3.18) is more delicate than for (3.24), c.f. [6]. Note that the result of Theorem 3.5 holds with $\tilde{\Gamma}' \in W^{m+1,p}$ on all of Ω , under the additional regularity assumption (3.34) for w on the boundary, which we state as the following corollary.

Corollary 3.6. *Let $\Gamma, d\Gamma \in W^{m,p}(\Omega)$, and assume $(J, \tilde{\Gamma}) \in W^{m+1,p}(\Omega)$ solves (3.24)-(3.25), as in Theorem 3.5. Then, if $(J, \tilde{\Gamma})$ satisfy on the boundary the condition,*

$$w = J^{-1}dJ + \Gamma - \tilde{\Gamma} \in W^{m+\frac{p-1}{p},p}(\partial\Omega), \quad (3.34)$$

then the modified pair $(J, \tilde{\Gamma}')$ defined in (3.26) solves the Riemann-flat condition (3.14) in Ω and $\tilde{\Gamma}' \in W^{m+1,p}(\Omega)$.

Proof. The corollary follows from the proof of Theorem 3.5, using estimate (3.30) to conclude with the sought after regularity $\tilde{\Gamma}' \in W^{m+1,p}(\Omega)$. \square

3.3. The main equivalence theorem. We now consider the problem of imposing (2.28), that is, the condition that J be a true Jacobian, integrable to coordinates. The goal of this section is to augment system (3.24)-(3.25) with a first order PDE on the free function h in (3.24)-(3.25) to replace the integrability condition (2.28). Assume again throughout that $\Gamma, \text{Riem}(\Gamma) \in W^{m,p}(\Omega)$ for $m \geq 1$ and $p > n$.

The key idea to augment system (3.24) - (3.25) with an additional equation for the free function h which is equivalent to (2.28) expressed in terms of exterior derivatives. To accomplish this, note first that the integrability condition (2.28) is equivalent to

$$d\vec{J} = 0, \quad (3.35)$$

since

$$\text{Curl}(J)^\alpha \equiv \frac{1}{2}(J_{i,j}^\alpha - J_{j,i}^\alpha)dx^j \otimes dx^i = J_{i,j}^\alpha dx^j \wedge dx^i = d(J_i^\alpha dx^i) \equiv d\vec{J}^\alpha.$$

Now, to combine (3.35) with the Poisson equation (3.25), observe that

$$\overrightarrow{\Delta J} = (\Delta J_i^\alpha)dx^i = \Delta(J_i^\alpha dx^i) = \Delta\vec{J}, \quad (3.36)$$

since Δ acts component-wise on matrix valued k -forms by (2.11). Thus, interpreting the Poisson equation (3.25) in a vector sense, applying (3.36) and taking d of the resulting equation (3.25), we obtain

$$\Delta d\vec{J} = d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{J \cdot h}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}),$$

where we used that Δ and d commute. Therefore, if J solves (3.35) in addition to (3.25), then $A \equiv J \cdot h$ must satisfy the equation

$$d\vec{A} = d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}). \quad (3.37)$$

The right hand side of (3.37) is a vector valued 2-form and vanishes when taking its exterior derivative (since $d^2 = 0$) so that (3.37) is well-posed for A given J and $\tilde{\Gamma}$. Our next goal is to show the backward implication, that (3.37) together with the Poisson equation (3.25) on J imply (3.35).

Lemma 3.7. *Let $\Gamma \in W^{m,p}(\Omega)$ for $p > n$ and $m \geq 1$, and let $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, $J \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ be given. Assume J solves*

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \quad (3.38)$$

(the Poisson equation (3.25) with $h = J^{-1}A$). Then J satisfies the Curl-free condition (3.35), if and only if A solves

$$d\vec{A} = d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) \quad (3.39)$$

and

$$d\vec{J} = 0 \quad \text{on } \partial\Omega. \quad (3.40)$$

Proof. For the forward implication, assume J solves (3.35). Then $A \equiv J \cdot h$ solves (3.39) by the argument in (3.35) through (3.37). Moreover, (3.40) follows upon restriction of (3.35) to $\partial\Omega$, (using that derivatives of J are Hölder continuous because $p > n$). This proves the forward implication.

For the backward implication, assume A solves (3.39) and (3.40). Now, consider (3.38) as an equation on vector valued 1-forms and assume for the beginning that $m \geq 2$. Then, taking d of (3.38), we get

$$\Delta(d\vec{J}) = d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) - d\vec{A},$$

so that (3.39) implies

$$\Delta(d\vec{J}) = 0.$$

Therefore, since $d\vec{J}$ is assumed to vanish on $\partial\Omega$ as a Hölder continuous function, we conclude that (3.35) holds in Ω . This establishes the backward implication for $m \geq 2$.

Consider now the case that $m = 1$, then $\Delta J \in L^p(\Omega)$ and we need to take d in a distributional sense. For this, we proceed as in Lemma 3.2: By Riesz representation, it suffices to show that

$$\langle d\vec{J}, \phi \rangle_{L^2} = 0, \quad (3.41)$$

for all scalar valued 2-forms $\phi \in L^{p^*}(\Omega)$, where $\frac{1}{p^*} + \frac{1}{p} = 1$, and where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the standard L^2 inner product on differential forms which we apply component-wise to vector-valued forms. For each such ϕ , there exists a scalar valued 2-form $\psi \in W^{2,p^*}(\Omega)$ such that $\Delta\psi = \phi$, and $\psi = 0$ on $\partial\Omega$. Using the product rule (3.9) we compute

$$\begin{aligned} \langle d\vec{J}, \phi \rangle_{L^2} &= \langle d\vec{J}, \Delta\psi \rangle_{L^2} \\ &= -\langle \delta d\vec{J}, \delta\psi \rangle_{L^2} \\ &= -\langle \Delta\vec{J}, \delta\psi \rangle_{L^2}, \end{aligned} \quad (3.42)$$

where the last equality follows since

$$\langle \delta d\vec{J}, \delta\psi \rangle_{L^2} = \langle \delta\vec{J}, \delta^2\psi \rangle_{L^2} = 0.$$

Substituting now (3.38) for $\Delta\vec{J} = \overrightarrow{\Delta\vec{J}}$ in (3.41), we find

$$\begin{aligned} \langle \Delta\vec{J}, \delta\psi \rangle_{L^2} &= \left\langle \overrightarrow{\delta(J \cdot \Gamma)} - \overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}, \delta\psi \right\rangle_{L^2} - \langle \vec{A}, \delta\psi \rangle_{L^2} \\ &= \left\langle \overrightarrow{\delta(J \cdot \Gamma)} - \overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}, \delta\psi \right\rangle_{L^2} + \langle d\vec{A}, \psi \rangle_{L^2}. \end{aligned} \quad (3.43)$$

Substituting (3.39) for $d\vec{A}$ and using the product rule one more time gives

$$\begin{aligned} \langle d\vec{A}, \psi \rangle_{L^2} &= \left\langle d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}), \psi \right\rangle_{L^2} \\ &= -\left\langle \overrightarrow{\delta(J \cdot \Gamma)} - \overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}, \delta\psi \right\rangle_{L^2}, \end{aligned} \quad (3.44)$$

and substituting back into (3.43), a cancellation gives

$$\langle d\vec{J}, \phi \rangle_{L^2} = 0.$$

This completes the proof. \square

Before we state our main theorem, we discuss the regularity of A . Since we seek $\tilde{\Gamma} \in W^{m+1,p}$ and $dh = d(J^{-1}A)$ is a source term on the right hand side of the Poisson equation (3.24) for $\tilde{\Gamma}$, we need $A \in W^{m,p}$ (for $m \geq 1$) to be consistent with $\tilde{\Gamma} \in W^{m+1,p}$. But this appears to contradict the fact that the first term on the right hand side of (3.39) contains two derivatives on $\Gamma \in W^{m,p}$. Most remarkably, the consistency follows by our incoming assumption $d\Gamma \in W^{m,p}$ alone, in light of identity (2.23) of Lemma 2.4,

$$d(\overrightarrow{\delta(J \cdot \Gamma)}) = \overrightarrow{\operatorname{div}}(dJ \wedge \Gamma) + \overrightarrow{\operatorname{div}}(J \cdot d\Gamma),$$

where $\overrightarrow{\operatorname{div}}$ is defined in (2.22). Therefore, since we assume $d\Gamma \in W^{m,p}(\Omega)$, we find that

$$d(\overrightarrow{\delta(J \cdot \Gamma)}) \in W^{m-1,p}(\Omega)$$

and we conclude that the regularity of the right hand side of (3.39) is consistent with the regularity on the left hand side.

We now show that the existence of solutions $(J, \tilde{\Gamma}')$ of the Riemann-flat condition (3.14) together with the Curl-free condition (3.35) is equivalent to the existence of solutions $(J, \tilde{\Gamma}, A)$ to a coupled system of non-linear elliptic equations, system (1.1) - (1.4), and the equations are formally consistent at the levels of regularity we seek. This establishes Theorem 1.1.

Theorem 3.8. *Let Γ and $\text{Riem}(\Gamma)$ be in $W^{m,p}(\Omega)$ for $p > n$ and $m \geq 1$. Then the following equivalence holds:*

If there exists an invertible matrix-valued 0-form $J \in W^{m+1,p}(\Omega)$ and a matrix-valued 1-form $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ which solve

$$\begin{aligned} J^{-1}dJ &= \Gamma - \tilde{\Gamma}, \\ d\vec{J} &= 0, \end{aligned}$$

c.f. (3.14) and (3.35), then there exists $A \in W^{m,p}(\Omega)$ such that $(J, \tilde{\Gamma}, A)$ solve the elliptic system

$$\Delta \tilde{\Gamma} = \delta d(\Gamma - J^{-1}dJ) + d(J^{-1}A), \quad (3.45)$$

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \quad (3.46)$$

$$d\vec{A} = \overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J d\Gamma) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) \quad (3.47)$$

$$\delta \vec{A} = v \quad (3.48)$$

in Ω with boundary data

$$d\vec{J} = 0 \quad \text{on } \partial\Omega, \quad (3.49)$$

where $v \in W^{m-1,p}(\Omega)$ is a vector valued 0-form free to be chosen.

Conversely, if there exists $J \in W^{m+1,p}(\Omega)$ invertible, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ solving (3.45) - (3.49), then there exists a $\tilde{\Gamma}' \in W^{m,p}(\Omega)$ such that for every Ω' compactly contained in Ω we have $\tilde{\Gamma}' \in W^{m+1,p}(\Omega')$ and $(J, \tilde{\Gamma}')$ solve (3.14) and (3.35) in Ω' .

Proof. For the forward implication, assume there exists $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $J \in W^{m+1,p}(\Omega)$ which solve the Riemann-flat condition (3.14) together with the Curl-free condition (3.35). Theorem 3.5 implies that J and $\tilde{\Gamma}$ solve (3.24) - (3.25) for some $h \in W^{m,p}(\Omega)$, and setting $A = Jh$ it follows that $(J, \tilde{\Gamma})$ solve (3.45) - (3.46). Since J satisfies the Curl-free condition (3.35), which is equivalently to the integrability condition (2.28), Lemma 2.6 implies that $\tilde{\Gamma}$ is symmetric. Moreover, since J satisfies (3.35), Lemma 3.7 implies that $A \in W^{m,p}(\Omega)$ solves (3.47). This proves the forward implication.

For the backward implication, assume $J \in W^{m+1,p}(\Omega)$, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ solve the elliptic system (3.45) - (3.47), with $\tilde{\Gamma}$ symmetric and J invertible. Now, Theorem 3.5 implies that J and $\tilde{\Gamma}' \equiv J^{-1}dJ - \Gamma$ solve the Riemann-flat condition (3.14) in each Ω' compactly contained in

Ω , and $\tilde{\Gamma}' \in W^{m+1,p}(\Omega')$ has the required regularity. Moreover, since (3.46) and (3.47) hold together with the boundary condition (3.49), Lemma 3.7 applies and yields that J satisfies the integrability condition (3.35) in Ω and therefore also in $\Omega' \subset \Omega$. This completes the proof. \square

Equations (3.45)-(3.48) are the fundamental equations of this paper, the *RT-equations*. Theorem 3.8 establishes our main theorem, Theorem 1.1 of the Introduction, due to the equivalence of (i) and (ii) of Theorem 2.5. One can extend the result of Theorem 3.8 to all of Ω by assuming (3.34) at $\partial\Omega$, c.f. Corollary 3.6. Our program in [23] is to develop an existence theory for (3.45)-(3.49).

3.4. An alternative equivalent elliptic system. In this subsection, we prove the following proposition which shows that system (3.47) can also be written equivalently as a system of coupled semi-linear Poisson equations, but to assign classical boundary data for A we must assume one more order of smoothness than in Theorem 3.8.

Proposition 3.9. *Let $m \geq 2$ and assume that Γ and $d\Gamma$ are both in $W^{m,p}(\Omega)$ for $p > n$. Let $(J, \tilde{\Gamma}) \in W^{m+1,p}(\Omega)$ solve (3.45) - (3.46), where J is invertible. Then $A \in W^{m,p}(\Omega)$ solves (3.47) in Ω if and only if A solves*

$$\Delta \vec{A} = \delta \left(\overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J d\Gamma) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) \right) + dv, \quad (3.50)$$

in Ω with boundary data

$$d\vec{A} = \overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J d\Gamma) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}), \quad (3.51)$$

$$\delta \vec{A} = v \quad (3.52)$$

on $\partial\Omega$, where $v \in W^{m-1,p}(\Omega)$ is a vector valued 0-form free to be chosen.

Proof. This proposition is a consequence of Lemma 3.1 and 3.2. We summarize the argument here for completeness. To prove the forward implication and derive (3.50), add δ of (3.47) to d of the free vector valued function $\delta \vec{A} = v$. This gives (3.50). Restricting (3.47) and $\delta \vec{A} = v$ to the boundary gives (3.51) - (3.52).

To prove the backward implication assume first that $m \geq 3$, then take d of (3.50) to get

$$\begin{aligned} \Delta d\vec{A} &= d\delta \left(\overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J d\Gamma) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) \right) \\ &= d\delta \left(d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) \right) \\ &= \Delta \left(d(\overrightarrow{\delta(J \cdot \Gamma)}) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}) \right), \end{aligned}$$

which is equivalent to

$$\Delta w = 0,$$

with w defined by

$$w \equiv d\vec{A} - d(\overrightarrow{\delta(J \cdot \Gamma)}) + d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}).$$

Thus, since w vanishes on the boundary by (3.51), we conclude that $w = 0$ in Ω which is the sought after equation (3.47). The low regularity case $m = 2$ follows by Lemma 3.2. This completes the proof. \square

4. PROGRAM FOR SOLVING THE RT-EQUATIONS

We have reduced the problem of whether a connection $\Gamma \in W^{m,p}(\Omega)$ can be smoothed one order by coordinate transformation, under the assumption $d\Gamma \in W^{m,p}(\Omega)$, to the problem of finding solutions $(J, \tilde{\Gamma}, A)$ of the RT-equations (3.45)-(3.48) with boundary data (3.49) within the regularity class $J, \tilde{\Gamma} \in W^{m+1,p}(\Omega)$, $A \in W^{m,p}(\Omega)$. The main difficulty for constructing an appropriate existence theory for (3.45)-(3.49) is that the right hand sides are coupled nonlinearly, and (3.49) is not standard Dirichlet or Neumann boundary data. Existence for the case $\Gamma \in W^{m,p}(\Omega)$, for $p > n$, $m \geq 1$, will be established in authors' forthcoming paper [23].

The case $\Gamma, d\Gamma$ in L^∞ , relevant to regularity singularities in GR shock wave theory, is delicate, and is the topic of authors' current research. In particular, the condition (3.49) requires $\text{Curl}(J) = 0$ on the boundary of the domain, so Lipschitz continuity of J is a regularity too weak to assign boundary conditions in a classical (strong) sense. (The method of assigning Dirichlet data in our companion paper [23] is sufficient to resolve this problem, even in the case of L^∞ connections.) Moreover, the existence theory for the linear Poisson equation admits Calderon-Zygmund singularities when the source functions are in L^∞ , so solutions of the RT-equations can fail to be two levels more regular than the sources. Note that *consistency* of the RT-equations (3.45) - (3.49) is not an issue even in the L^∞ case, because any Lipschitz continuous connection can be transformed to a connection no smoother than L^∞ by application of a $C^{1,1}$ coordinate transformation, and reversing this, the inverse Jacobian together with $\tilde{\Gamma}$ will solve the Riemann-flat condition (3.14) for the transformed connection, where $\tilde{\Gamma}$ is the Lipschitz connection in the original coordinates we started with.

We can explore the possibility that Calderon-Zygmund singularities might be ruled out by imposing further conditions on Γ , for example assuming Γ lies in the space BMO (Bounded Mean Oscillation), a space containing L^∞ , or assuming Γ lies in BV (Bounded Variation), a subspace of L^∞ appropriate for shock wave theory, [24, 7]; or, since the problem is local, by modifying Γ off an arbitrarily small neighborhood of a given point. We also have the freedom to choose v in system (3.45) - (3.49).

Consider briefly the freedom to change Γ for the problem of regularity singularities. The problem is to establish the existence of a coordinate transformation $x \rightarrow y$ that smooths the connection in a neighborhood of any given point p . For this purpose, there is no loss of generality in taking Ω to be $B_\epsilon(p)$, the ball of radius ϵ centered at p in \mathbb{R}^n . Moreover, since the Riemann-flat condition is a point-wise condition, there is no loss of generality in replacing Γ by a connection Γ'_ϵ which agrees with Γ on $B_\epsilon(p)$, but

extends Γ beyond $B_\epsilon(p)$ by an auxiliary smooth connection. To make this precise, let $\Gamma_\infty \in C^\infty(\mathbb{R}^n)$ be such an auxiliary connection and define

$$\Gamma_\epsilon^* = (1 - \phi_r^\epsilon) \Gamma_\infty + \phi_r^\epsilon \Gamma,$$

where ϕ_r^ϵ is the standard smooth cutoff function satisfying $\phi_r^\epsilon(x) = 1$ if $x \in B_\epsilon(p)$ and $\phi_r^\epsilon(x) = 0$ if $x \in B_r(p)^c$, where $B_r(p)^c$ denotes the complement of $B_r(p)$ in \mathbb{R}^n , $r > \epsilon$. Clearly, $d\Gamma_\epsilon^* \in L^\infty(\mathbb{R}^n)$. Thus, if we can solve the RT-equations with Γ^* in place of Γ , we can employ Theorem 1.1 to conclude that the Riemann-flat condition holds for the original Γ , in a neighborhood of p . Note here that we have the freedom to choose Γ_∞ and $\tilde{\Gamma}_\infty$ to be a known solution of the Riemann-flat condition at the start, and can use ϵ as a small parameter in an existence theory. We conclude that there is *enormous* freedom, all the freedom to choose Γ_∞, v and ϵ, r , available to modify the sources in (3.45) - (3.48) in order to avoid Calderon-Zygmund singularities when the sources of the RT-equations are in L^∞ . Addressing the problem of regularity singularities for connections of regularity lower than $W^{1,p}$, $p > n$, is the topic of authors current research.

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