OPTIMAL METRIC REGULARITY IN GENERAL RELATIVITY FOLLOWS FROM THE RT-EQUATIONS BY ELLIPTIC REGULARITY THEORY IN L^p-SPACES

MORITZ REINTJES AND BLAKE TEMPLE

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Abstract. We prove existence of solutions to the Regularity Transformation equations (RT-equations), and thereby demonstrate that *regularity singularities* do not exist in General Relativity, when the connection Γ of the gravitational metric satisfies $\Gamma, \mathrm{Riem}(\Gamma) \in W^{m,p}$, $m \geq 1$, $p > n$. Authors proved previously that the existence of solutions of the RT-equations is equivalent to the Riemann-flat condition, which in turn is equivalent to the existence of a coordinate transformation which smooths Γ, (and hence the metric), by one order. From this we conclude that if the components of Γ and $\mathrm{Riem}(\Gamma)$ are in $W^{m,p}$ in a coordinate system x, then there always exists a coordinate transformation $x \to y$, such that the components of Γ in y-coordinates are in $W^{m+1,p}$. This demonstrates that the method of determining optimal metric regularity by the RT-equations works. So the problem of existence of coordinate systems in which non-optimal solutions of the *hyperbolic* Einstein equations exhibit optimal metric regularity is resolved by *elliptic* regularity theory in L^p-spaces applied to the RT-equations, equations which determine the coordinate transformations themselves.

1. Introduction

Existence theorems for the Einstein equations are established in coordinate systems in which the equations take a solvable form. In such coordinates the metric may not exhibit its optimal regularity, that is, two degrees smoother than its Riemann curvature tensor, or may lose its optimal regularity under time evolution [10]. In this paper we give the first proof of existence of solutions to the Regularity Transformation equations, (RT-equations), equations derived in [17] for the Jacobian of the coordinate transformations that map a gravitational metric in General Relativity (GR) to coordinates in which the metric displays its optimal regularity. This is a new approach to optimal metric regularity in GR because, rather than imposing an apriori coordinate ansatz, (like harmonic coordinates [6, 2] or Gaussian normal coordinates [12, 19]), and trying to establish regularity of solutions of the Einstein equations in those coordinates, the approach

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here begins with the belief that the coordinate systems of optimal regularity are in general too difficult to guess apriori, and to find them, one has to discover and solve equations for the coordinates themselves. In [17] the authors accomplished their goal of deriving such a system of equations, the *RT-equations*. The RT-equations are a system of elliptic PDE's derived from a geometric principle, the *Riemann-flat condition*, which the authors introduced in [16]. Our motivation in deriving the RT-equations came from the problem of optimal metric regularity at shock waves in GR, which is equivalent to the problem as to whether *regularity singularities* can be created by shock interaction in GR, [13, 14, 15, 17]. In this paper we apply elliptic regularity theory in L^p spaces to give the first proof of existence of solutions to the RT-equations. From this we deduce the following theorem for geometry:

Theorem 1.1. *Let Γ be a connection and $Riem(\Gamma)$ its curvature tensor given by components Γ^i_{jk} and R^i_{jkl} in some coordinate system x defined in an open set $\Omega \subset \mathbb{R}^n$. Assume all components satisfy $\Gamma^i_{jk}, R^i_{jkl} \in W^{m,p}(\Omega)$ for $m \geq 1$, $p > n \geq 2$. Then for each point $q \in \Omega$ there exists a neighborhood $\Omega_q \subset \Omega$ containing q, and a coordinate transformation $x \mapsto y$ with $J \equiv \frac{\partial y^\mu}{\partial x^i} \in W^{m+1,p}(\Omega_q)$, such that, in y-coordinates, the components of Γ are bounded in $W^{m+1,p}(\Omega_q)$.*

Theorem 1.1 applies to connections of arbitrary metric signature, applicable to solutions of the Einstein equations with arbitrary sources. The result does not rely on special properties of the Einstein equations. It establishes that no *regularity singularities* exist when the curvature is in $W^{1,p}$ (c.f. [14]). Authors' current research program is to use the RT-equations to resolve the problem as to whether regularity singularities can be created by shock wave interaction in General Relativity, the case Γ and $\mathrm{Riem}(\Gamma)$ in L^∞.

Theorem 1.1 introduces a new point of view on solutions of the Einstein equations of General Relativity: It tells us that it is sufficient to solve the Einstein equations in coordinates in which the metric is only *one order* smoother than the curvature, allowing for equations which are only *first order* in metric components, and by Theorem 1.1 we know local coordinate transformations always exist which smooth the metric by one order to optimal regularity, i.e., two derivatives smoother than the curvature. Since first order equations can be simpler than second order equations, Theorem 1.1 establishes that it is sufficient to solve the Einstein equations in coordinates in which the equations are simpler, and the solutions are weaker, and conclude in general that once the existence of weaker solutions is established, the stronger solutions with optimal regularity are guaranteed. Indeed, the Einstein equations naturally allow for solutions in which the metric regularity is only one level higher than that of its Riemann curvature, and hence not optimal. That is, given a solution of the Einstein equations of optimal regularity, say with metric in $W^{m+2,p}$ and its curvature in $W^{m,p}$, $m \geq 0$, then applying a coordinate transformation with Jacobian in $W^{m+1,p}$, the resulting metric is no longer optimal, being in $W^{m+1,p}$, with connection dropping to $W^{m,p}$, and curvature remaining in $W^{m,p}$, [17]. Theorem 1.1 establishes that this can always be reversed in the case $m \geq 1$, $p > n$, which is essentially one derivative above the GR shock wave case $\mathrm{Riem}(\Gamma) \in L^\infty$. Theorem 1.1 guarantees that if in the time evolution of any such GR solution of optimal smoothness, the regularity breaks down by the metric losing one derivative relative to its curvature tensor, then this is only a breakdown in the coordinate system, not in the geometry. This can be taken as a new regularity principle for the numerical simulation of solutions in GR. In particular, excluding non-optimal solutions from the initial value problem when they exist would lead to an incomplete picture of the solution space, and hence an incomplete picture of the physics.

As an application, Theorem 1.1 resolves the problem of optimal regularity for spherically symmetric solutions constructed in Standard Schwarzschild Coordinates (SSC) when $m \geq 1$, $p > n$ (c.f. Section 8). In future publications the authors will address the problem as to whether the RT-equations can always be solved, and whether solutions can always be smoothed by one order, in the presence of GR shock waves, when the curvature and connection are in L^∞. Such is the case for solutions of the Einstein equations constructed in SSC by the first order Glimm scheme, [10, 18]. Although the RT-equations give an explicit algorithm for constructing coordinate systems of optimal regularity, structural properties of the metric in the coordinate systems of optimal regularity could be very complicated. The example of SSC tells us that the spacetime metric can be simpler, and more comprehensible, in an atlas of coordinate systems in which the gravitational metric is one order less regular than optimal.

Theorem 1.1 resolves the problem of optimal metric regularity at the level of curvatures in $W^{m,p}$, $m \geq 1$, essentially one order larger than the case L^∞ (or L^p), applicable to shock wave theory in GR, [10, 13, 14, 15, 16, 17]. The case of L^∞ (or L^p) curvature is the threshold between weak and strong solutions of the Einstein equations, c.f. [14, 17]. However, even in the L^∞ case, the equivalence between the existence of coordinate systems of optimal metric regularity and the existence of solutions of the RT-equations still applies, [17]. There are two main obstacles to extending Theorem 1.1 to the case of L^∞ curvature. First is the problem of Calderon-Zygmund singularities, the central issue in the L^∞ case of elliptic regularity theory [17], and second, the problem of handling nonlinear products in L^p. Obstacles to solving the RT-equations in the case of GR shock waves could lead to the discovery of new kinds of *regularity singularities* in GR [14, 15]. The L^∞ case is the setting most intriguing to the authors, and the problem of extending solutions of the RT-equations to the lower regularity of L^∞ (and L^p) will be addressed in forthcoming publications. Even so, Theorem 1.1 demonstrates for the first time that determining optimal metric regularity by the RT-equations *works.* The RT-equations bring all the power of elliptic regularity theory to bear on the problem of optimal regularity in General

Relativity, available now to resolve the problem of regularity singularities at shock waves.

The point of departure for this paper is the following theorem, proven in [17], which establishes the equivalence of the Riemann-flat condition with the solvability of the RT-equations when Γ and $d\Gamma \in W^{m,p}$, for $m \geq 1$, $p > n$, (and hence $\mathrm{Riem}(\Gamma) \in W^{m,p}$ by Morrey's inequality (2.10), c.f. [17]). By this we mean the components of Γ and $d\Gamma$ are functions in $W^{m,p}$ in some given, but otherwise arbitrary, coordinate system x. The Riemann-flat condition was derived in [16] as a condition on a given connection Γ equivalent to the existence of a local coordinate transformation which smooths the connection by one order. The Riemann-flat condition states that there should exist a tensor $\tilde{\Gamma}$, one order smoother than Γ, such that $Riem(\Gamma - \tilde{\Gamma}) = 0$. It applies to connections down to the lowest regularity $\Gamma, d\Gamma \in L^\infty$, and in this case the theorem in [16] states that there exists a coordinate transformation which smooths the components of Γ to $C^{0,1}$ if and only if there exists a tensor $\tilde{\Gamma} \in C^{0,1}$ such that $Riem(\Gamma - \tilde{\Gamma}) = 0$. It turns out that $\tilde{\Gamma}$ agrees with the smoothed connections in the new coordinates. The RT-equations were derived in [17]. The J and $\tilde{\Gamma}$ components of the RT-equations come from two equivalent forms of the Riemann-flat condition, namely, $Riem(\Gamma - \tilde{\Gamma}) = 0$ and $dJ = J(\Gamma - \tilde{\Gamma})$. These two first order equations are then converted into the RT-equations by use of the identity $\Delta \equiv d\delta + \delta d$ to re-express the first order equations as second order Poisson equations, (here Δ is the Laplacian of the Euclidean metric in x-coordinates), and by augmenting the resulting system by equations which arrange for the integrability condition $Curl(J) \equiv \partial_j J^\mu_i - \partial_i J^\mu_j = 0$, c.f. [17]. The unknowns in the RT-equations are the matrix valued differential forms $(\tilde{\Gamma}, J, A)$ which have the following meaning: $J \equiv J^\mu_\nu$ is the Jacobian of the sought after coordinate transformation which smooths the connection, viewed as a matrix valued 0-form; $\tilde{\Gamma} \equiv \tilde{\Gamma}^\mu_{\nu k} dx^k$ is the unknown tensor one order smoother than Γ such that $Riem(\Gamma - \tilde{\Gamma}) = 0$, viewed as a matrix valued 1-form; and $A \equiv A^\mu_\nu$ is an auxiliary matrix valued 0-form introduced to impose $Curl(J) = 0$. Also, $\vec{A}, \vec{J}$ are vector valued 1-forms, the *vectorizations* of A and J, introduced so that $Curl(J) = d\vec{J}$ and the integrability condition takes the form $d\vec{J} = 0$, which allows us to augment the above two Riemann-flat conditions by an equation for A, resulting in the RT-equations, c.f. [17]. We find the interplay between the interpretation of the Jacobian as a matrix valued 0-form J, to re-express the Riemann-flat condition, and its interpretation as a vector valued 1-form $\vec{J}$, required to incorporate the integrability condition and close the RT-equations at the correct regularity, very interesting.

Theorem 1.2. *Assume Γ is defined in a fixed coordinate system x on Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. Assume that $\Gamma \in W^{m,p}(\Omega)$ and $d\Gamma \in W^{m,p}(\Omega)$ for $m \geq 1$, $p > n$. Then the following equivalence holds:*

If there exists a coordinate transformation $x \mapsto y$ with Jacobian $J = \frac{\partial y}{\partial x} \in W^{m+1,p}(\Omega)$ such that the components of Γ in y-coordinates are in $W^{m+1,p}(\Omega)$, then there exists $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ such that $(J, \tilde{\Gamma}, A)$ solve the elliptic system

$$\Delta \tilde{\Gamma} = \delta d\big(\Gamma - J^{-1} dJ\big) + d(J^{-1}A), \tag{1.1}$$

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \tag{1.2}$$

$$d\vec{A} = \overrightarrow{div}\big(dJ \wedge \Gamma\big) + \overrightarrow{div}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \tag{1.3}$$

$$\delta \vec{A} = v, \tag{1.4}$$

with boundary data

$$d\vec{J} = 0 \quad on\ \partial\Omega. \tag{1.5}$$

Here $v \in W^{m-1,p}(\Omega)$ is a vector valued 0-form free to be chosen.

Conversely, if there exists $J \in W^{m+1,p}(\Omega)$ invertible, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ which solve (1.1) - (1.5) in Ω, then for each $q \in \Omega$, there exists a neighborhood $\Omega_q \subset \Omega$ of q such that J is the Jacobian of a coordinate transformation $x \mapsto y$ on Ω_q, and the components of Γ in y-coordinates are in $W^{m+1,p}(\Omega_q)$.

We call system (1.1) - (1.4) the *Regularity Transformation equations*, or *RT-equations*. The principal parts are the Laplacian $\Delta = \partial^2_{x^1} + ... + \partial^2_{x^n}$, the exterior derivative d, and the co-derivative δ, all taken with respect to the Euclidean metric in x-coordinates. The operations $\vec{\cdot}$, $\overrightarrow{\mathrm{div}}$ and $\langle \cdot\, , \cdot\rangle$ are introduced in [17] as special operations on matrix valued differential forms meaningful when the dimension of the matrices equals the dimension of the physical space, c.f. (2.1) - (2.5) below. Note that the vector valued 0-form v (which is free to be chosen) has been introduced in (1.4) so that (1.3)-(1.4) takes the Cauchy-Riemann form $d\vec{A} = f$, $\delta \vec{A} = g$. The consistency condition $df = 0$ is met in (1.3) because the derivation shows the right hand side is exact, (equation (1.3) is obtained by setting d of the "vectorized" right hand side of (1.2) equal to zero, c.f. equation (3.40) in [17]), and $\delta g = 0$ in (1.4) because $\delta v = 0$ is an identity for vector valued 0-forms v.

The RT-equations apply to connections of arbitrary metric signature and to solutions of the Einstein equations with arbitrary sources. In this paper we establish the first existence theory for the (non-linear) RT-equations by proving existence of solutions when $\Gamma, d\Gamma \in W^{m,p}$, $m \geq 1$, $p > n$, referred to as the *smooth* case in [17]. The proof is based on a new iteration scheme. A key insight for the proof was to augment the RT-equations by ancillary elliptic equations in order to convert the non-standard boundary condition $Curl(J) = 0$, which is of neither Neumann nor Dirichlet type, into Dirichlet data for J at each stage of the iteration, c.f. Section 3. By this, each iterate can be constructed by applying standard existence theorems and elliptic regularity in L^p spaces for the (linear) Poisson equation. This is our main existence theorem:

Theorem 1.3. *Assume the components of* $\Gamma, d\Gamma \in W^{m,p}(\Omega)$ *for* $m \geq 1$, $p > n \geq 2$ *in some coordinate system* x. *Then for each* $q \in \Omega$ *there exists a solution* $(\tilde{\Gamma}, J, A)$ *of the RT-equations* (1.1) - (1.5) *defined in a neighborhood* Ω_q *of* q *such that* $\tilde{\Gamma} \in W^{m+1,p}(\Omega_q)$, $J \in W^{m+1,p}(\Omega_q)$, $A \in W^{m,p}(\Omega_q)$.

Theorem 1.1 follows directly from Theorem 1.2 together with Theorem 1.3, and requires no further proof. (Recall that $d\Gamma \in W^{m,p}$ is equivalent to $\mathrm{Riem}(\Gamma) \in W^{m,p}$, when $m \geq 1$, $p > n$ by Morrey's inequality (2.10), c.f. [17].) The proof of Theorem 1.3 is the subject of the remainder of this paper. This is a new application of the elliptic regularity theory in L^p spaces developed by Agmon, Nierenberg and others in the $'50$, at the time connecting the new theory of distributions to solutions of PDE's [1]. Interestingly, the analysis of the RT-equations requires L^p spaces, and this cannot be replaced by the simpler L^2 spaces because of non-linear products, nor by a Green's function approach which would require higher regularities. Most interesting to us is that one can address the problem of optimal regularity of solutions of the *hyperbolic* Einstein equations by *elliptic* regularity theory alone.

The structure of this paper is as follows: In Section 2 we give preliminaries and state the results we require from elliptic regularity theory in L^p spaces. In Section 3 we show how to augment the RT-equations by ancillary equations in order to reduce the boundary condition (1.5) to standard Dirichlet data. In Section 4 we set up the iteration scheme and we introduce a small parameter ϵ into the RT-equations to handle the non-linearities. In Section 5 we outline the proof of convergence of our iteration scheme for the ϵ rescaled RT-equations. Section 6 contains the detailed proofs of the technical lemmas stated in Section 5 from which the proof of convergence of the iteration scheme is deduced. In Section 7 we complete the proof of Theorem 1.3, by proving that the ϵ rescaled RT-equations can always be obtained by restricting to small neighborhoods. In Section 8 we discuss an application of Theorem 1.1 to spherically symmetric solutions of the Einstein equations in Standard Schwarzschild Coordinates.

2. Preliminaries

The point of departure for this paper is authors' prior paper [17], and we refer the reader to this for more details on notation, motivation, and background. We now recall several definitions and identities from Section 2.1 in [17]. To begin, recall that we work in a fixed (but arbitrary) coordinate system x defined on n-dimensional bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary. The unknowns in the Rt-equations are matrix valued differential forms. By a matrix valued differential k-form A we mean an $(n \times n)$-matrix whose components are k-forms, and we write

$$A = A_{[i_1 \ldots i_k]} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \equiv \sum_{i_1 < \ldots < i_k} A_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \qquad (2.1)$$

for $(n \times n)$-matrices $A_{i_1 \dots i_k}$ that are totally anti-symmetric in the indices $i_1, \dots, i_k \in \{1, \dots, n\}$. We define the wedge product of a matrix valued k-form A with a matrix valued l-form $B = B_{j_1 \dots j_l} dx^{j_1} \wedge \dots \wedge dx^{j_l}$ as

$$A \wedge B \equiv \frac{1}{l!k!} A_{i_1 \dots i_k} \cdot B_{j_1 \dots j_l} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}, \quad (2.2)$$

where the dot denotes standard matrix multiplication. Note the wedge product of a matrix valued k-form with itself is non-zero unless the component matrices commute, which is the main difference between matrix valued and scalar valued differential forms. The exterior derivative d and its co-derivative δ are defined component-wise on matrix-components, so all properties of d and δ on scalar forms carry over to matrix valued forms. In particular the Laplacian $\Delta \equiv d\delta + \delta d$ acts component-wise on matrix- and on k-form components and, in fact, Δ is identical to the Laplacian of the Euclidean metric in x-coordinates, $\Delta = \partial^2_{x^1} + \dots + \partial^2_{x^n}$. The exterior derivative satisfies the product rule

$$d(A \wedge B) = dA \wedge B + A \wedge dB, \quad (2.3)$$

where $A \in W^{1,p}(\Omega)$ is a matrix valued k-form and $B \in W^{1,p}(\Omega)$ is a matrix valued j-form, (c.f. Lemma 3.3 of [17]), which implies for a matrix valued 0-form J that

$$d\big(J^{-1} \cdot dJ\big) = J^{-1} dJ \wedge J^{-1} dJ. \quad (2.4)$$

Regarding the co-derivative δ, we require the following product rule

$$\delta(J \cdot w) = J \cdot \delta w + \langle dJ; w \rangle \quad (2.5)$$

where $J \in W^{2,p}(\Omega)$ is a matrix valued 0-form, $w \in W^{2,p}(\Omega)$ a matrix valued 1-form, and where $\langle \cdot\,; \cdot \rangle$ is the matrix valued inner product defined on matrix valued k-forms A and B by,

$$\langle A\,; B \rangle^\mu_\nu \equiv \sum_{i_1 < \dots < i_k} A^\mu_{\sigma\, i_1 \dots i_k} B^\sigma_{\nu\, i_1 \dots i_k}. \quad (2.6)$$

So $\langle A\,; B \rangle$ converts two matrix valued k-forms into a matrix valued 0-form.

The two operations which convert matrix valued differential forms to vector valued forms on the right hand side of (1.4) are *vec* and *vec−divergence*. First, *vec* converts matrix valued 0-forms into vector valued 1-forms by the operation, (c.f. (2.20) of [17]),

$$\vec{A}^\mu = A^\mu_i dx^i. \quad (2.7)$$

The operation *vec−divergence* converts matrix valued k-forms A into vector valued k-forms $\overrightarrow{\text{div}}(A)$ by the operation

$$\overrightarrow{\text{div}}(A)^\alpha \equiv \sum_{l=1}^n \partial_l \big((A^\alpha_l)_{i_1 \dots i_k} \big) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

For a matrix valued 1-form w and a matrix valued 0-form J, Lemma 2.4 of [17] gives the important identity

$$d\big(\overrightarrow{\delta(J\cdot w)}\big) = \overrightarrow{\mathrm{div}}\big(dJ \wedge w\big) + \overrightarrow{\mathrm{div}}\big(J\cdot dw\big), \tag{2.8}$$

which is crucial for the regularity to close in the RT-equations, c.f. Section 1 in [17].

We denote by $\|\cdot\|_{W^{m,p}(\Omega)}$ the standard $W^{m,p}$-norm, defined as the sum of the L^p-norms of derivatives up to order m [7]. (We often write ∂^m for such derivatives in place of multi-index notation.) When applied to matrix valued differential forms ω, $\|\omega\|_{W^{m,p}(\Omega)}$ denotes the Hilbert-Schmidt matrix norm of the matrix whose entries are the $W^{m,p}$-norm of the components of ω, that is, the sum of the $W^{m,p}$-norm of all components. The L^2-inner product on matrix valued forms is given by

$$\langle\cdot,\cdot\rangle_{L^2} \equiv \int_\Omega \mathrm{tr}\,\big(\langle\cdot\,;\cdot\rangle\big), \tag{2.9}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix and $\langle\cdot;\cdot\rangle$ is the matrix valued inner product (2.6). Theorems 1.2 and 1.3 apply to connections in the space $W^{m,p}$ for $m \geq 1$, $p > n$, because for these parameter values, Sobolev's theorem implies that $W^{1,p}(\Omega)$ is embedded in the space of Hölder continuous functions $C^{0,\alpha}(\overline{\Omega})$. Namely, for $p > n$ Morrey's inequality gives

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_M \|f\|_{W^{1,p}(\Omega)}, \tag{2.10}$$

where $\alpha \equiv 1 - \frac{n}{p}$ and $C_M > 0$ is a constant only depending on n, p and Ω [7]. Morrey's inequality (2.10) extends unchanged to components of matrix valued differential forms.

We finally summarize the estimate we use from elliptic theory. We assume throughout that $m \geq 1$, $p > n \geq 2$ and that Ω is a bounded domain, simply connected and with smooth boundary. In fact, one could assume without loss of essential generality that Ω is a ball in $\mathbb{R}^n$. Our estimates are based on the following theorems, which extend to matrix valued and vector valued differential forms.

Theorem (Elliptic Regularity): *Let $f \in W^{m-1,p}(\Omega)$, for $m \geq 1$, and $u_0 \in W^{m+\frac{p-1}{p},p}(\partial\Omega)$ both be scalar functions. Assume $u \in W^{m+1,p}(\Omega)$ solves the Poisson equation $\Delta u = f$ with Dirichlet data $u|_{\partial\Omega} = u_0$. Then there exists a constant $C > 0$ depending only on Ω, m, n, p such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|f\|_{W^{m-1,p}(\Omega)} + \|u_0\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big). \tag{2.11}$$

Equation (2.11) is the basic estimate of elliptic regularity theory in L^p spaces, c.f. equation (14.4) in [1]. For completeness we show how to derive (2.11) for the critical case $m = 1$ in Section A from standard apriori estimates in [7, 8, 9, 22], assuming Hölder continuity to simplify the proof. Our analysis of the iteration scheme introduced in Section 4.2 below requires an existence theory for Dirichlet problems for the Poisson equations (1.1) - (1.2), applicable for 1-forms $\tilde{\Gamma}$ and 0-forms J. This is provided by the following existence theorem, c.f. Theorems 9.15 and 9.19 in [8].

Theorem 2.1. *Let $f \in W^{m-1,p}(\Omega)$, for $m \geq 1$, and $u_0 \in W^{m+\frac{p-1}{p},p}(\partial\Omega)$ both be scalar functions. Then there exists a unique $u \in W^{m+1,p}(\Omega)$ which solves the Poisson equation*

$$\Delta u = f \qquad in \quad \Omega,$$

with Dirichlet data $u|_{\partial\Omega} = u_0$.

The estimate corresponding to (2.11) for first order equations is given by Gaffney's inequality, (c.f. Theorem 5.21 in [5]).

Theorem (Gaffney Inequality): *Let $u \in W^{m+1,p}(\Omega)$ be a k-form for $m \geq 0$, $1 \leq k \leq n-1$ and (for simplicity) $n \geq 2$. Then there exists a constant $C > 0$ depending only on Ω, m, n, p, such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|du\|_{W^{m,p}(\Omega)} + \|\delta u\|_{W^{m,p}(\Omega)} + \|u\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big). \tag{2.12}$$

Our analysis of an iteration scheme below requires an existence theory for the first order Cauchy-Riemann type equations (1.3) and (1.4) of the RT-equations (1.1) - (1.4), the case when $\vec{A}$ is a 1-form. For this we are free to impose whatever boundary conditions are sufficient for a suitable existence theory. The following special case of Theorem 7.4 in [5], which gives a refinement of Gaffney's inequality (2.12) for 1-forms and 0-forms, with certain boundary conditions, provides the existence theorem sufficient for our purposes. (Again, assume throughout that $\partial\Omega$ is smooth and regular.)

Theorem 2.2. *(i) Let $f \in W^{m,p}(\Omega)$ be a 2-form with $df = 0$ and let $g \in W^{m,p}(\Omega)$ be a 0-form, so $\delta g = 0$, and assume $m \geq 0$, $n \geq 2$. Then there exists a 1-form $u = u_i\, dx^i \in W^{m+1,p}(\Omega)$ which satisfies*

$$du = f \quad and \quad \delta u = g \qquad in \quad \Omega,$$

together with the boundary condition

$$u \cdot N = 0 \qquad on \quad \partial\Omega,$$

where N is the unit normal on $\partial\Omega$ and $u \cdot N \equiv u_i N^i$. Moreover, there exists a constant $C > 0$ depending only on Ω, m, n, p, such that

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|f\|_{W^{m,p}(\Omega)} + \|g\|_{W^{m,p}(\Omega)}\Big). \tag{2.13}$$

(ii) Let $f \in W^{m,p}(\Omega)$ be a 1-form with $df = 0$ and let $q \in \overline{\Omega}$ be an arbitrary point. Then there exists a 0-form $u \in W^{m+1,p}(\Omega)$ such that $du = f$ and $u(q) = 0$, and estimate (2.13) holds with $g = 0$.[1]

Finally, we require the following standard trace theorem, c.f. Theorem 1.5.1.3 in [9].

Theorem 2.3. *Let $u \in W^{m+1,p}(\Omega) \cap C^0(\overline{\Omega})$ for $m \geq 0$, then there exists a constant $C > 0$ depending only on Ω, m, n, p such that*

$$\|u\|_{W^{m+1-\frac{1}{p},p}(\partial\Omega)} \leq C\|u\|_{W^{m+1,p}(\Omega)}.$$

We can apply Theorem 2.3 since the iterates we construct in Section 4.2 are Hölder continuous on the closure of Ω, by Morrey's inequality (2.10).

3. Reduction to standard Dirichlet boundary data

To prove Theorem 1.3 we introduce an iteration scheme to construct approximate solutions of the RT-equations (1.1)-(1.5), introduce a small parameter to handle the nonlinearities, and apply standard results on elliptic regularity in L^p spaces to obtain convergence together with the sought after levels of smoothness. One of the main technical issues is how to handle the non-standard boundary condition (1.5), which is neither standard Neumann nor Dirichlet data for the PDE (1.2) which determines J. We now introduce a reformulation of the boundary condition (1.5) for the J equation (1.2), (the only boundary condition specified by the RT-equations), as an equivalent implicit boundary condition, which has the advantage that it reduces to standard Dirichlet conditions for J at each level of our iteration scheme introduced in Section 4.2.

So assume $(\tilde{\Gamma}, J, A)$ is a solution of the RT-equations, and write (1.1) - (1.4) using the following compact notation:

$$\Delta\tilde{\Gamma} = \tilde{F}(\tilde{\Gamma}, J, A), \tag{3.1}$$

$$\Delta J = F(\tilde{\Gamma}, J) - A, \tag{3.2}$$

$$d\vec{A} = d\vec{F}(\tilde{\Gamma}, J) \tag{3.3}$$

$$\delta\vec{A} = v, \tag{3.4}$$

where $\vec{F}(\tilde{\Gamma}, J)$ is the vectorized version of $F(\tilde{\Gamma}, J)$, so that $d\vec{F}(\tilde{\Gamma}, J)$ is identical to the right hand side of (1.3), c.f. the derivation leading to equation (3.40) in [17]. Now (3.3) implies the consistency condition

$$d\big(\vec{F}(\tilde{\Gamma}, J) - \vec{A}\big) = 0,$$

so that we can solve

$$\begin{cases} d\Psi = \vec{F}(\tilde{\Gamma}, J) - \vec{A}, \\ \delta\Psi = 0, \end{cases} \tag{3.5}$$

[1] Note, $\delta u = 0$ holds for any 0-form as an identity. Also, for u a 0-form, one can prove existence by integrating the right hand side of the gradient equation $du = f$ in each direction in a suitable way, c.f. proof of Proposition 1 in [16].

for a vector valued function Ψ, (c.f. Theorem 7.4 in [5]). Let y then be any solution of

$$\Delta y = \Psi. \tag{3.6}$$

Now we claim that in place of the Poisson equation (1.2) for J with the boundary condition (1.5), it suffices to solve the boundary value problem

$$\Delta J = F(\tilde{\Gamma}, J) - A \quad \text{in} \quad \Omega, \tag{3.7}$$

$$\vec{J} = dy \quad \text{on} \quad \partial\Omega. \tag{3.8}$$

(Assigning $\vec{J}$ on $\partial\Omega$ is the same as assigning J on $\partial\Omega$ because both contain the same component functions.) To see this, write

$$\Delta dy = d\Delta y = d\Psi = \vec{F} - \vec{A} = \Delta\vec{J}, \tag{3.9}$$

which uses that, after taking *vec* on both sides of the J-equation (3.7), the operation *vec* commutes with Δ on the left hand side (3.7) because Laplacian acts component-wise. Thus,

$$\begin{aligned} \Delta(\vec{J} - dy) = 0 \quad &\text{in} \quad \Omega, \\ \vec{J} - dy = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{3.10}$$

which implies by uniqueness of solutions of the Laplace equation that $\vec{J} = dy$ in Ω. Since second derivatives commute, we conclude that

$$d\vec{J} = Curl(\vec{J}) = 0 \qquad \text{in} \quad \Omega, \tag{3.11}$$

on solutions of (3.7), (3.8), as claimed. The point of using (3.8) in place of (1.5) is that dy can be determined at the k-th step of an iteration scheme in which the $(k+1)$-st iterate is determined by (3.7), (3.8), c.f. Section 4. In this setting, (3.8) is standard Dirchlet data for J. The equivalence between the boundary conditions (1.5) and (3.8) is recorded in the following theorem.

Proposition 3.1. *Assume* $J \in W^{m+1,p}(\Omega)$ *is invertible, and assume* J, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ *and* $A \in W^{m,p}(\Omega)$ *solve* (1.1) - (1.4), *where* $m \geq 1$, $p > n$. *Then the boundary condition* (1.5) *holds if and only if*

$$\vec{J} = dy \quad \text{on} \quad \partial\Omega, \tag{3.12}$$

for some y *satisfying* (3.6).

Proof. The argument between equations (3.1) and (3.11) proves that the boundary data (3.12) implies that $d\vec{J} = 0$ holds everywhere in Ω. By Sobolev imbedding (for $p > n$), $d\vec{J}$ is Hölder continuous on the closure of Ω, (c.f. (2.10) below), so that we can restrict $d\vec{J}$ to the boundary $\partial\Omega$ which gives the sought after boundary condition (1.5).

To prove the inverse implication, assume that $(J, \tilde{\Gamma}, A)$ solves the RT-equations (1.1) - (1.4) with boundary data (1.5). Lemma 3.7 in [17] then implies that $d\vec{J} = 0$ in Ω so that one can integrate J to some coordinate function y, i.e. $dy = \vec{J}$. Defining $\Psi \equiv \Delta y$, it follows from J solving (3.7) that

$$d\Psi = d\Delta y = \Delta dy = \Delta\vec{J} = \overrightarrow{\Delta J} \overset{(3.7)}{=} \vec{F} - \vec{A}.$$

Thus Ψ satisfies (3.5), while (3.6) holds by the above definition of Ψ. So restriction of $dy = \vec{J}$ to $\partial\Omega$ gives the sought after boundary data (3.12). This completes the proof of Proposition 3.1. □

4. The iteration scheme

In this section we introduce our iteration scheme for approximating solutions of the RT-equations. We begin by setting up our iteration scheme in terms of the extended RT-equations (1.1) - (1.4) and (3.5) - (3.6) with standard Dirichlet data (3.8) in a non-technical way. In Section 4.1, we introduce a small parameter $\epsilon > 0$ into the RT-equations, (by smallness of the coordinate neighborhood), which allows us to estimate the non-linearities on the right hand side of the RT-equations and prove convergence of the iterates for sufficiently small $\epsilon > 0$ in Section 5. In Section 4.2, we introduce our iteration scheme in terms of the ϵ-rescaled RT-equations and prove its well-posedness. Throughout the remainder of this paper we take, in (1.4),

$$v \equiv 0,$$

to fix the freedom to choose $v \in W^{m-1,p}(\Omega)$. We assume a given connection Γ of suitable regularity, defined in a given coordinate system x in an open and bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary. To define the iteration by induction, it suffices to start with given $(\tilde{\Gamma}_0, J_0)$, show how to construct A_1, $\tilde{\Gamma}_1$ and J_1 from $(\tilde{\Gamma}_0, J_0)$. This then tells us how to construct $(A_{k+1}, \tilde{\Gamma}_{k+1}, J_{k+1})$ from $(\tilde{\Gamma}_k, J_k)$ for each $k \geq 1$ by recursion.

So assume $\tilde{\Gamma}_k$ and J_k are given for some $k \geq 0$. Define A_{k+1} as the solution of

$$\begin{cases} d\vec{A}_{k+1} = d\vec{F}(\tilde{\Gamma}_k, J_k), \\ \delta\vec{A}_{k+1} = 0, \end{cases} \tag{4.1}$$

for $A_{k+1}\cdot N = 0$ on $\partial\Omega$, where N is the unit normal vector of $\partial\Omega$ which is multiplied to the matrix A_{k+1}. Our iteration does not require to assume A_k, since A_{k+1} is defined in terms of $\tilde{\Gamma}_k$ and J_k alone. Note, our choice of boundary data in (4.1) and $v = 0$ was made so that Theorem 2.2 applies to give existence.

To introduce the Dirichlet data for J_{k+1}, we first define the auxiliary variables ψ_{k+1} and y_{k+1}, for which we again do not require the previous iterates ψ_k and y_k. So use the identity $d(\vec{F}(\tilde{\Gamma}_k, J_k) - \overrightarrow{A_{k+1}}) = 0$ of (4.1) to solve

$$\begin{cases} d\Psi_{k+1} = \vec{F}(\tilde{\Gamma}_k, J_k) - \overrightarrow{A_{k+1}}, \\ \delta\Psi_{k+1} = 0, \end{cases} \tag{4.2}$$

and then solve

$$\Delta y_{k+1} = \Psi_{k+1}, \tag{4.3}$$

where for (4.2) and (4.3) any convenient boundary condition can be implemented.

Now, define J_{k+1} to be the solution of the following *standard Dirichlet* boundary problem:

$$\begin{aligned} \Delta J_{k+1} &= F(\tilde{\Gamma}_k, J_k) - \overrightarrow{A_{k+1}}, && (4.4)\\ \overrightarrow{J_{k+1}} &= dy_{k+1} \ \text{ on } \ \partial\Omega, && (4.5) \end{aligned}$$

and, to obtain $\tilde{\Gamma}_{k+1}$, solve

$$\Delta\tilde{\Gamma}_{k+1} = \tilde{F}(\tilde{\Gamma}_k, J_k, A_{k+1}), \tag{4.6}$$

where the boundary data for $\tilde{\Gamma}_{k+1}$ is free to be chosen.

The implicit boundary condition (3.12) reduced to (4.5), which is standard Dirichlet data at each step of the iteration. As in Proposition 3.1, one can show that the iterates so defined imply $Curl(J_{k+1}) = 0$ for each $k \geq 0$, which we will prove in Lemma 4.3 for the iterates of the rescaled RT-equations. We demonstrated here that one can define an iteration scheme for the RT-equations in terms of solutions of the Dirichlet problem for the linear Poisson equation.

4.1. **The rescaled equations.** We now introduce a small parameter $\epsilon > 0$ and derive an ϵ-rescaled version of the RT-equations, which allows us to handle the non-linearities. To introduce a small parameter ϵ, assume the components of Γ are given in x-coordinates in an open and bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary, and assume Γ and $d\Gamma$ are both bounded in $W^{m,p}(\Omega)$. Let Γ^* be a connection in x-coordinates satisfying

$$\|\Gamma^*\|_{W^{m,p}(\Omega)} \ + \ \|d\Gamma^*\|_{W^{m,p}(\Omega)} < C_0, \tag{4.7}$$

for $m \geq 1$ and C_0 a fixed constant. Now, we assume without loss of generality that Γ scales with $\epsilon > 0$ according to the definition

$$\Gamma = \epsilon\Gamma^*. \tag{4.8}$$

The assumptions (4.7) and (4.8) can be made without loss of generality regarding the *local* problem of optimal metric regularity. Namely, given any connection $\Gamma' \in W^{m,p}(\Omega)$ with $d\Gamma'$ bounded in $W^{m,p}(\Omega)$, we define Γ^* as the restriction of Γ' to the ball of radius ϵ with its components transformed as scalars to the ball or radius 1 (which we take to be Ω), while Γ is taken to be the connection resulting from transforming Γ' *as a connection* under the same coordinate transformation, c.f. the proof of Theorem 1.3 in Section 7.

We further assume the scaling ansatz

$$J = I + \epsilon\, J^*, \qquad \tilde{\Gamma} = \epsilon\, \tilde{\Gamma}^*, \qquad A = \epsilon A^*. \tag{4.9}$$

Since we only need to prove *existence* of a solution for our purposes, assumption 4.9 is again made without loss of generality for the problem of optimal metric regularity. Now, to derive the RT-equations tuned to the ϵ-scaling, substitute (4.8) and (4.9) into the RT-equations (1.1) - (1.4) for $v \equiv 0$ and divide by ϵ, we then obtain an equivalent set of equations as recorded in the following lemma.

Lemma 4.1. *Let* $u \equiv \begin{pmatrix} \tilde{\Gamma}^* \\ J^* \end{pmatrix}$ *and* $a \equiv A^*$, *and define*

$$F_u(u,a) \equiv \begin{pmatrix} \delta d\Gamma^* - \delta d\big(J^{-1}\cdot dJ^*\big) + d(J^{-1}a) \\ \delta\Gamma^* + \epsilon\,\delta(J^*\cdot\Gamma^*) - \epsilon\,\langle dJ^*;\tilde{\Gamma}^*\rangle - a \end{pmatrix}, \tag{4.10}$$

$$F_a(u) \equiv \overrightarrow{div}\big(d\Gamma^*\big) + \epsilon\,\overrightarrow{div}\big(J^*\cdot d\Gamma^*\big) + \epsilon\,\overrightarrow{div}\big(dJ^*\wedge\Gamma^*\big) - \epsilon\, d\big(\overrightarrow{\langle dJ^*;\tilde{\Gamma}^*\rangle}\big). \tag{4.11}$$

Then, substituting (4.8) *and* (4.9) *into the RT-equations* (1.1) - (1.4) *for* $v \equiv 0$ *and dividing by* ϵ, *we obtain the equivalent set of equations*

$$\Delta u = F_u(u,a), \tag{4.12}$$

$$\begin{cases} d\vec{a} = F_a(u) \\ \delta\vec{a} = 0. \end{cases} \tag{4.13}$$

Proof. Substituting (4.8) and (4.9) into the RT-equations (1.1) - (1.4) with $v \equiv 0$, and dividing by ϵ, we obtain

$$\Delta\tilde{\Gamma}^* = \delta d\Gamma^* - \delta d\big(J^{-1}\cdot dJ^*\big) + d(J^{-1}A^*), \tag{4.14}$$

$$\Delta J^* = \delta(J\cdot\Gamma^*) - \epsilon\,\langle dJ^*;\tilde{\Gamma}^*\rangle - A^* \tag{4.15}$$

$$d\vec{A}^* = \epsilon\,\overrightarrow{\mathrm{div}}\big(dJ^*\wedge\Gamma^*\big) + \overrightarrow{\mathrm{div}}\big(J\cdot d\Gamma^*\big) - \epsilon\, d\big(\overrightarrow{\langle dJ^*;\tilde{\Gamma}^*\rangle}\big). \tag{4.16}$$

$$\delta\vec{A}^* = 0. \tag{4.17}$$

Now, equations (4.14) - (4.17) together with the definitions of u, a and (4.10) - (4.11) imply the sought after equations (4.12) - (4.13). □

We often refer to (4.12) - (4.13) as the "rescaled RT-equations". We further introduce the following useful notation,

$$\begin{aligned} F_{\tilde{\Gamma}}(u,a) &\equiv \delta d\Gamma^* + da - \delta d\big(J^{-1}\cdot dJ^*\big) + d(J^{-1}a) \\ F_J(u) &\equiv \delta\Gamma^* + \epsilon\,\delta(J^*\cdot\Gamma^*) - \epsilon\,\langle dJ^*;\tilde{\Gamma}^*\rangle, \end{aligned} \tag{4.18}$$

so that $F_u(u,a) = (F_{\tilde{\Gamma}}(u,a), F_J(u) - a)$ and by equation (3.40) in [17] we have $F_a(u) = d\overrightarrow{F_J}$. The rescaled RT-equations (4.12) - (4.13) can then be written equivalently as

$$\Delta u = \begin{pmatrix} F_{\tilde{\Gamma}}(u,a) \\ F_J(u) - a \end{pmatrix}, \tag{4.19}$$

$$\begin{cases} d\vec{a} = d\overrightarrow{F_J(u)} \\ \delta\vec{a} = 0. \end{cases} \tag{4.20}$$

We use the alternative form (4.19) - (4.20) to set up the iteration scheme below. As proven in Section 7, Theorem 1.3 now follows from the following theorem, the proof of which is the topic of Sections 4.2 - 6.

Theorem 4.2. *Let $\Gamma^*, d\Gamma^* \in W^{m,p}(\Omega)$ satisfy* (4.7) *and let $m \geq 1$, $p > n \geq 2$. Then there exists ϵ_* such that, if $\epsilon < \epsilon_*$, then there exists $u \in W^{m+1,p}(\Omega)$ and $a \in W^{m,p}(\Omega)$ which solve the RT-equation* (4.12) - (4.13) *with boundary data* (1.5).

In Section 5 we summarize the proof of Theorem 4.2 which is based on the iteration scheme in Section 4.2. The details of the proof are postponed to Section 6.

4.2. **The Iteration scheme for the rescaled equations.** In this section we define the iteration scheme (u_k, a_k), $k \geq 0$, for approximating solutions of (4.12)-(4.13), and set up the framework for proving convergence of the scheme in the appropriate Sobolev spaces for ϵ sufficiently small. Define (u_{k+1}, a_{k+1}) by induction as follows. Start the induction by assuming

$$u_0 = a_0 = 0.$$

Then, given $u_k \in W^{m+1,p}(\Omega)$ and $a_k \in W^{m,p}(\Omega)$ for $k \geq 0$, we define $a_{k+1} \in W^{m,p}(\Omega)$ by solving

$$\begin{cases} d(\overrightarrow{a_{k+1}}) = F_a(u_k), \\ \delta(\overrightarrow{a_{k+1}}) = 0, \end{cases} \tag{4.21}$$

with Dirichlet boundary data

$$a_{k+1} \cdot N = 0, \tag{4.22}$$

where N is the unit normal on the boundary $\partial\Omega$, and a_{k+1} is a matrix valued 0-form. (Our boundary data (4.22) and the equation $\delta(\overrightarrow{a_{k+1}}) = 0$ are chosen so that the existence theory in [5] applies.) Next, in order to arrange for the non-standard boundary condition (1.5), we introduce the vector valued 0-form $\psi_{k+1} \in W^{m,p}(\Omega)$ as the solution of

$$d\psi_{k+1} = \overrightarrow{F_J(u_k)} - \overrightarrow{a_{k+1}}, \tag{4.23}$$

with boundary data $\psi_{k+1}(q) = 0$ at some arbitrary point $q \in \Omega$, fixed independent of k. Since d of a 0-form is the gradient, (4.23) determines the solution up to a constant, which we have chosen to be zero so that we can estimate ψ_{k+1} by the right hand side of (4.23) by Poincaré's inequality. Recall that $d\overrightarrow{F_J(u_k)} = F_a(u_k)$, as explained below (4.18). In terms of ψ_{k+1} we define the vector valued function $y_{k+1} \in W^{m+2,p}(\Omega)$ as the solution of

$$\begin{cases} \Delta y_{k+1} = \psi_{k+1}, \\ y_{k+1}\big|_{\partial\Omega} = 0. \end{cases} \tag{4.24}$$

The vector valued functions ψ_{k+1} and y_{k+1} are auxiliary variables which we introduce so that we can assign standard Dirichlet data for the Poisson equation which defines $u_{k+1} = (J^*_{k+1}, \tilde{\Gamma}_{k+1})$. Namely, we define $u_{k+1} \in W^{m+1,p}(\Omega)$ as the solution of

$$\Delta u_{k+1} = F_u(u_k, a_{k+1}), \tag{4.25}$$

with Dirichlet boundary data

$$\tilde{\Gamma}^*_{k+1}|_{\partial\Omega} = 0, \tag{4.26}$$

$$J^*_{k+1}|_{\partial\Omega} = dy_{k+1}|_{\partial\Omega}. \tag{4.27}$$

The next lemma shows that this standard Dirichlet data suffices to impose the non-standard boundary condition (1.5).

Lemma 4.3. *Any solution $u_{k+1} = (J^*_{k+1}, \tilde{\Gamma}^*_{k+1}) \in W^{m+1,p}(\Omega)$ of (4.25) with boundary data (4.26) - (4.27) satisfies*

$$d\overrightarrow{J^*_{k+1}} \equiv Curl(J^*_{k+1}) = 0 \tag{4.28}$$

in Ω, which automatically implies the boundary condition (1.5). Then $J_{k+1} \equiv I + \epsilon J^*_{k+1}$ is integrable and defines the Jacobian of the coordinate transformation $x \mapsto x + \epsilon y_{k+1}(x)$, where y_{k+1} is defined in (4.24).

Proof. We compute that

$$\Delta(dy_{k+1}) = d(\Delta y_{k+1}) \overset{(4.24)}{=} d\psi_{k+1} \overset{(4.23)}{=} F_J(u_k) - a_{k+1} \overset{(4.25)}{=} \Delta J^*_{k+1},$$

which implies that

$$\Delta\big(J^*_{k+1} - dy_{k+1}\big) = 0. \tag{4.29}$$

Now, since $J^*_{k+1} - dy_{k+1}$ vanishes on $\partial\Omega$ by (4.27), we conclude that $J^*_{k+1} = dy_{k+1}$ in Ω. This implies (4.28), and since

$$d(x + \epsilon y_{k+1}) = I + \epsilon J^*_{k+1} = J_{k+1},$$

we conclude that J_{k+1} is the Jacobian of the coordinate transformation $x \mapsto x + \epsilon y_{k+1}(x)$. This completes the proof. □

Our strategy for completing the proof of Theorem 4.2 is to first state the main technical lemmas in Lemmas 4.4 - 5.2 together with Proposition 5.3 to follow, use them to prove Theorem 4.2, and postpone the proofs of these lemmas to Sections 6.1 - 6.5. We end this section by stating the first technical lemma which addresses the well-posedness of the iteration scheme (4.21) - (4.27).

Lemma 4.4. *Assume $u_k \in W^{m+1,p}(\Omega)$ is given, for $m \geq 1$, $p > n \geq 2$, and that $\epsilon > 0$ satisfies*

$$\epsilon \leq \epsilon(k) \equiv \frac{1}{4C_M \|u_k\|_{W^{m+1,p}}}, \tag{4.30}$$

where $C_M > 0$ is the constant from Morrey's inequality (2.10). Then there exists $a_{k+1} \in W^{m,p}(\Omega)$ which solves (4.21) - (4.22), there exists the auxilliary iterates $\psi_{k+1} \in W^{m,p}(\Omega)$ and $y_{k+1} \in W^{m+2,p}(\Omega)$ which solve (4.23) - (4.24), and there exists $u_{k+1} \in W^{m+1,p}(\Omega)$ which solves (4.25) with boundary data (4.26) - (4.27). Moreover, these iterates satisfy the elliptic estimates

$$\|a_{k+1}\|_{W^{m,p}(\Omega)} \leq C_e \, \|F_a(u_k)\|_{W^{m-1,p}(\Omega)}, \tag{4.31}$$

$$\|u_{k+1}\|_{W^{m+1,p}(\Omega)} \leq C_e \, \|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)}, \tag{4.32}$$

and the auxiliary iterates satisfy

$$\begin{aligned} \|\psi_{k+1}\|_{W^{m,p}(\Omega)} &\leq C_e \, \|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)}, \quad (4.33) \\ \|y_{k+1}\|_{W^{m+2,p}(\Omega)} &\leq C_e \, \|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)}, \quad (4.34) \end{aligned}$$

for some constant $C_e > 0$ depending only on m, n, p and Ω.

The proof of Lemma 4.4, given in Section 6.2, is based solely on the L^p elliptic estimate (2.11) and Gaffney's inequality (2.12).

5. Convergence of Iterates and Proof of Theorem 4.2

In this section we state the main lemmas and propositions required for the proof of Theorem 4.2, and assuming these, give the proof of Theorem 4.2. Proofs of the supporting lemmas and propositions are postponed until Section 6 below. The proof of Theorem 4.2 follows directly from the existence result of Lemma 4.4 together with Proposition 5.3 alone, the latter providing estimates for the differences between subsequent iterates. The main steps in the proof of Proposition 5.3 are contained in Lemmas 5.1 and 5.2. To outline the proof here we state these lemmas in this section, and their proofs are given in Sections 6.3 - 6.4.

To begin, observe that Lemma 4.4 yields a sequence of iterates $(u_k, a_k)_{k\in\mathbb{N}}$. In order to establish convergence of this sequence in $W^{m+1,p}(\Omega) \times W^{m,p}(\Omega)$, we require estimates on the differences

$$\begin{aligned} \overline{a_k} &\equiv a_k - a_{k-1}, \\ \overline{u_k} &\equiv u_k - u_{k-1}, \end{aligned} \quad (5.1)$$

in terms of the corresponding differences of source terms,

$$\begin{aligned} \overline{F_a(u_k)} &\equiv F_a(u_k) - F_a(u_{k-1}), \\ \overline{F_u(u_k, a_{k+1})} &\equiv F_u(u_k, a_{k+1}) - F_u(u_{k-1}, a_k). \end{aligned} \quad (5.2)$$

The next technical lemma provides estimates of (5.1) in terms of (5.2). The proof of Lemma 5.1 is given in Section 6.3.

Lemma 5.1. *Assume $0 < \epsilon \leq \min\big(\epsilon(k), \epsilon(k-1)\big)$, that is, ϵ satisfies (4.30) in terms of u_k and u_{k-1}. Then*

$$\begin{aligned} \|\overline{F_u(u_k, a_{k+1})}\|_{W^{m-1,p}} &\leq C_u(k)\Big(\epsilon \, \|\overline{u_k}\|_{W^{m+1,p}} + \|\overline{a_{k+1}}\|_{W^{m,p}}\Big), \quad (5.3) \\ \|\overline{F_a(u_k)}\|_{W^{m-1,p}} &\leq \epsilon \, C_a(k) \, \|\overline{u_k}\|_{W^{m+1,p}}, \quad (5.4) \end{aligned}$$

where

$$\begin{aligned} C_u(k) &\equiv C_s\big(1 + \|u_k\|_{W^{m+1,p}} + \|u_{k-1}\|_{W^{m+1,p}} + \|a_{k+1}\|_{W^{m,p}}\big), \quad (5.5) \\ C_a(k) &\equiv C_s\big(1 + \|u_k\|_{W^{m+1,p}} + \|u_{k-1}\|_{W^{m+1,p}}\big), \quad (5.6) \end{aligned}$$

where C_s is a constant that only depends on m, n, p, Ω and the constant C_0 of (4.7).

The next lemma establishes the induction step for our proof that the iteration scheme converges in the appropriate spaces, by bounding C_u and C_a independent of k for $\epsilon > 0$ sufficiently small. Recall, C_0 is the constant bounding Γ^* and $d\Gamma^*$ in (4.7) and C_e is the constant introduced in Lemma 4.4. We assume from now on and without loss of generality that $C_e > 1$, which allows us to simplify the ϵ-bound (5.8) below. The proof of Lemma 5.2 is given in Section 6.4.

Lemma 5.2. *Assume the induction hypothesis*

$$\|u_k\|_{W^{m+1,p}(\Omega)} \leq 4\, C_0 C_e^2, \tag{5.7}$$

for some $k \in \mathbb{N}$ and let $C_e > 1$. If

$$\epsilon \leq \epsilon_1 \equiv \min\left(\tfrac{1}{4C_e^2 C_s(1+2C_eC_0+4C_e^2C_0)}, \tfrac{1}{16 C_M C_0 C_e^2} \right), \tag{5.8}$$

then $0 < \epsilon_1 \leq \epsilon(k+l)$ for all $l \in \mathbb{N}$, (c.f. (4.30)), and the subsequent iterates satisfy the bounds

$$\begin{aligned} \|a_{k+l}\|_{W^{m,p}} &\leq 2C_0C_e, \qquad \forall\, l \in \mathbb{N}, && (5.9)\\ \|u_{k+l}\|_{W^{m+1,p}} &\leq 4C_0C_e^2, \qquad \forall\, l \in \mathbb{N}. && (5.10) \end{aligned}$$

In Section 6.5, we prove the following proposition, which is based on combining Lemmas 5.1 and 5.2 together with the elliptic estimates (4.31) - (4.32). This is the main step needed to prove convergence of the iteration scheme.

Proposition 5.3. *Assume the induction hypothesis* (5.7) *and $C_e > 1$. If $0 < \epsilon \leq \epsilon_1$, so ϵ satisfies* (5.8)*, then there exists a constant $C_d > 0$ such that*

$$\begin{aligned} \|\overline{a_{k+1}}\|_{W^{m,p}} &\leq \epsilon\, C_d\, \|\overline{u_k}\|_{W^{m+1,p}}, && (5.11)\\ \|\overline{u_{k+1}}\|_{W^{m+1,p}} &\leq \epsilon\, C_d\, \|\overline{u_k}\|_{W^{m+1,p}}, && (5.12) \end{aligned}$$

and $C_d > 0$ depends only on m, n, p, Ω and C_0.

At this stage of the argument it is important to note that the auxiliary iterates ψ_k and y_k are not coupled to the equations for a_k and u_k, except through the boundary data (4.27). The only purpose of the auxiliary variables ψ_k and y_k is to impose that the Jacobian $J_k = I + \epsilon J_k^*$ be curl free in each step of the iteration, and this only requires that dy_k converges on the boundary. Since differences between boundary data (4.27) can be estimated using the trace theorem, Theorem 2.3 namely

$$\begin{aligned} \|dy_{k+1} - dy_k\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} &\leq C\|y_{k+1} - y_k\|_{W^{m+2,p}(\Omega)} \\ &\leq C\|\overline{F_u(u_k, a_{k+1})}\|_{W^{m-1,p}(\Omega)}, \end{aligned}$$

where $C > 0$ depends only on m, n, p, Ω, (c.f. proof of Proposition 5.3 below), the convergence of dy_k follows directly from the convergence of a_k and u_k. Since this is all we need, we do not address the convergence of ψ_k and y_k in the proof of Theorem 4.2, (although one could easily prove their convergence with our methods).

Assuming Lemmas 4.4 - 5.2 and Proposition 5.3, we now prove the following theorem which gives convergence of the iteration scheme. This directly implies, and hence completes, the proof of Theorem 4.2.

Theorem 5.4. *Let* $\Gamma^*, d\Gamma^* \in W^{m,p}(\Omega)$ *satisfy* (4.7) *for* $m \geq 1$*,* $p > n \geq 2$*. Assume* $\epsilon > 0$ *satisfies*

$$\epsilon < \epsilon_2 \equiv \min\Big(\epsilon_1, \frac{1}{C_d}\Big), \tag{5.13}$$

where ϵ_1 *is defined in* (5.8) *and* $C_d > 0$ *is the constant in* (5.11) - (5.12)*. Then the sequence of iterates* $(u_k, a_k)_{k\in\mathbb{N}}$ *defined by* (4.21) - (4.27) *converges in* $W^{m+1,p}(\Omega) \times W^{m,p}(\Omega)$*, and the corresponding limits*

$$\begin{aligned} u \equiv \lim_{k\to\infty} u_k \ &\in W^{m+1,p}(\Omega), \\ a \equiv \lim_{k\to\infty} a_k \ &\in W^{m,p}(\Omega), \end{aligned}$$

solve the RT-equations (4.12) - (4.13) *with boundary data* (1.5)*.*

Proof. Assume Lemma 4.4 and Proposition 5.3 hold. Then, given two iterates $u_k, u_l \in W^{m+1,p}(\Omega)$, $(k \geq l)$, estimate (5.12) implies

$$\|u_k - u_l\|_{W^{m+1,p}} \ \leq \ \sum_{j=l+1}^{k} \|\overline{u_j}\|_{W^{m+1,p}} \leq \sum_{j=l+1}^{k} (\epsilon C)^j. \tag{5.14}$$

By (5.13), the above geometric series converges as $k \to \infty$. This implies that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $W^{m+1,p}(\Omega)$. Therefore, $(u_k)_{k\in\mathbb{N}}$ converges to some u in $W^{m+1,p}(\Omega)$. Similarly, (5.11) implies

$$\|a_k - a_l\|_{W^{m,p}} \ \leq \ \sum_{j=l+1}^{k} \|\overline{a_j}\|_{W^{m,p}} \leq \sum_{j=l+1}^{k} (\epsilon C)^j, \tag{5.15}$$

which in light of (5.13) is also a convergent geometric series. This implies convergence of $(a_k)_{k\in\mathbb{N}}$ to some a in $W^{m,p}(\Omega)$.

Now the limit (u, a) solves (4.12) and (4.13) because each term in the equations (4.21) and (4.25) converge to the corresponding terms in (4.12) and (4.13) in the L^p-norm on Ω. By Lemma 4.3, (u, a) satisfies the boundary condition (1.5), since $Curl(J_k^*) = 0$ in Ω for all $k \in \mathbb{N}$, and this property is maintain under the limit. Thus the limit satisfies $Curl(J^*) = 0$ in Ω which implies the sought after boundary condition (1.5) by restriction to the boundary. □

Theorem 5.4 is a refined restatement of Theorem 4.2, so this completes the proof of Theorem 4.2. It remains to give the proofs of Lemmas 4.4 - 5.2 and Proposition 5.3, which is accomplished in Sections 6.1 - 6.3.

6. Proofs of technical Lemmas and Propositions

6.1. **Estimates on the non-linear sources.** In this section we prove the basic estimates for the non-linear sources on the right hand side of equations (4.13) - (4.12), which are required for the proofs of Lemmas 4.4 and 5.2. Our main tool is Morrey's inequality (2.10), which allows us to bound the supremum norm of (scalar) functions $f \in W^{1,p}(\Omega,\mathbb{R})$ by

$$\|f\|_{L^\infty(\Omega)} \leq C_M \|f\|_{W^{1,p}(\Omega)}, \tag{6.1}$$

when $p > n$. Below, we use (6.1) together with the boundedness of Ω to estimate L^p-norms of products of functions $f \in W^{1,p}(\Omega,\mathbb{R})$ and $g \in L^p(\Omega,\mathbb{R})$ by

$$\|fg\|_{L^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|g\|_{L^p(\Omega)} \leq C_M \|f\|_{W^{1,p}(\Omega)} \|g\|_{L^p(\Omega)}.$$

In fact, $W^{1,p}(\Omega)$ is closed under multiplication for $p > n$. That is,

$$\begin{aligned} \|fg\|_{W^{1,p}(\Omega)} &\leq \|fg\|_{L^p} + \|gDf\|_{L^p} + \|fDg\|_{L^p} \\ &\leq \|fg\|_{L^p} + \|Df\|_{L^p}\|g\|_{L^\infty} + \|Dg\|_{L^p}\|g\|_{L^\infty} \\ &\leq 3C_M \|f\|_{W^{1,p}(\Omega)} \|g\|_{W^{1,p}(\Omega)}, \end{aligned}$$

where $f, g \in W^{1,p}(\Omega,\mathbb{R})$. Before we derive the basic source estimates in Lemma 6.2, we establish bounds on the inverse Jacobian for $\epsilon > 0$ sufficiently small.

Lemma 6.1. *Let $J = I + \epsilon J^*$ for some $J^* \in W^{m+1,p}(\Omega)$, where $m \geq 0$ and $p > n$. Assume $\epsilon > 0$ satisfies the bound*

$$\epsilon \leq \frac{1}{2C_M \|J^*\|_{W^{m+1,p}}}, \tag{6.2}$$

where $C_M > 0$ is the constant from Morrey's inequality (6.1)*. Then J is invertible and there exists a matrix valued 0-form $J^{-*} \in W^{m+1,p}(\Omega)$ such that*

$$J^{-1} = I + \epsilon J^{-*} \tag{6.3}$$

and such that

$$\|J^{-*}\|_{W^{m+1,p}} \leq \; C_- \, \|J^*\|_{W^{m+1,p}}, \tag{6.4}$$

where $C_- > 0$ is a constant depending only on m, n, p, and Ω.

Note that in our iteration scheme $\epsilon \leq \epsilon(k)$ always guarantees for (6.2), because $\|J^*_k\|_{W^{m+1,p}(\Omega)} \leq \|u_k\|_{W^{m+1,p}(\Omega)}$, c.f. (4.30).

Proof. The ϵ-bound (6.2) implies that $J = I + \epsilon J^*$ is invertible, since the supremum-norm of the Hilbert Schmidt norm of J (taken point-wise) is bounded below by

$$\|J\|_{L^\infty} = \|I + \epsilon J^*\|_{L^\infty} \geq \|I\|_{L^\infty} - \epsilon \|J^*\|_{L^\infty} \overset{(6.1)}{\geq} 1 - \epsilon \, C_M \|J^*\|_{W^{1,p}} \overset{(6.2)}{>} \frac{1}{2},$$

keeping in mind that $\|I\|_{L^\infty} = 1$. Now, since $J \in W^{m+1,p}(\Omega)$ for $p > n$, Morrey's inequality implies that J is Hölder continuous, so J^{-1} is Hölder continuous as well. Substituting ansatz (6.3) into $JJ^{-1} = I$ and solving for J^{-*}, we obtain that

$$J^{-*} = -J^{-1}J^{*}, \tag{6.5}$$

which implies existence and continuity of J^{-*}.

To prove estimate (6.4), that $J^{-*} \in W^{m+1,p}(\Omega)$, we proceed by induction in $m \geq 0$. To derive (6.4) in the case $m = 0$, we first use (6.3) to write (6.5) equivalently as

$$J^{-*} = -J^{*} - \epsilon J^{-*}J^{*}.$$

We now apply Morrey's inequality (6.1) and the ϵ-bound (6.2) to estimate

$$\begin{aligned}
\|J^{-*}\|_{L^\infty} &\leq \|J^*\|_{L^\infty} + \epsilon\, \|J^*\|_{L^\infty} \|J^{-*}\|_{L^\infty} \\
&\overset{(6.1)}{\leq} \|J^*\|_{L^\infty} + \epsilon\, C_M \|J^*\|_{W^{1,p}} \|J^{-*}\|_{L^\infty} \\
&\overset{(6.2)}{\leq} \|J^*\|_{L^\infty} + \frac{1}{2}\|J^{-*}\|_{L^\infty}.
\end{aligned}$$

Subtraction of the last term gives

$$\|J^{-*}\|_{L^\infty} \ \leq\ 2\|J^*\|_{L^\infty} \ \leq\ 2\, C_M \|J^*\|_{W^{1,p}}, \tag{6.6}$$

where we used Morrey's inequality (6.1) in the last step. Since Ω is bounded, we conclude with the estimate

$$\|J^{-*}\|_{L^p} \leq \|J^{-*}\|_{L^\infty} \mathrm{vol}(\Omega) \overset{(6.6)}{\leq} 2C_M \mathrm{vol}(\Omega)\, \|J^*\|_{W^{1,p}}, \tag{6.7}$$

which proves (6.4) for the case $m = 0$ for $C_- = 2\,\mathrm{vol}(\Omega) C_M$.

We now show that $J^{-*} \in W^{1,p}(\Omega)$ and derive estimate (6.4) for $m = 1$. To begin, let D_h denote the difference quotient in x^j-direction, (so that $D_h(f)$ converges to $\partial_j f$ as $h \to 0$ for $f \in W^{1,p}(\Omega)$). Now, since

$$0 \ =\ D_h(J^{-1}J)|_x \ =\ D_h(J^{-1})|_x \cdot J(x+h) + J^{-1}(x) \cdot D_h(J)|_x, \tag{6.8}$$

we have

$$D_h(J^{-1})|_x = -J^{-1}(x) \cdot D_h(J)|_x \cdot J^{-1}(x+h). \tag{6.9}$$

The right hand side of (6.9) converges to $\partial_j f$ in $L^p(\Omega)$ as $h \to 0$, since

$$\big\|J^{-1}\big(D_h(J) - \partial_j J\big)J(\cdot + h)\big\|_{L^p} \ \leq\ \|J^{-1}\|^2_{L^\infty} \big\|D_h(J) - \partial_j J\big\|_{L^p}$$

converges to zero as $h \to 0$ by L^p-convergence of $D_h(J)$ to $\partial_j J$ for $J \in W^{1,p}(\Omega)$ and by boundedness of $\|J^{-1}\|_{L^\infty}$ independent of h in light of (6.6). Thus the left hand side of (6.9) converge in $L^p(\Omega)$ and the limit function is indeed the weak derivative of J^{-1}, which is given explicitly by

$$\partial_j J^{-1} = -J^{-1} \cdot \partial_j(J) \cdot J^{-1}. \tag{6.10}$$

This implies that $J^{-1} \in W^{1,p}(\Omega)$ and, in light of (6.3), $J^{-*} \in W^{1,p}(\Omega)$.

To derive estimate (6.4) for $m = 1$, substitute $J^{-1} = I + \epsilon J^{-*}$ on the left hand side of (6.10) and $J = I + \epsilon J^*$ on the right hand side, which gives

$$\epsilon \partial_j J^{-*} \ =\ \epsilon J^{-1} \partial_j(J^*)\, J^{-1},$$

so dividing by ϵ and substituting $J^{-1} = I + \epsilon J^{-*}$ yields

$$\partial_j J^{-*} = \partial_j J^* + \epsilon\, J^{-*}\partial_j(J^*) + \epsilon\, \partial_j(J^*)\, J^{-*} + \epsilon^2 J^{-*}\partial_j(J^*)\, J^{-*}. \tag{6.11}$$

This leads to the estimate

$$\begin{aligned}\|\partial_j J^{-*}\|_{L^p} &\leq \big(1+\epsilon\,\|J^{-*}\|_{L^\infty}\big)^2\|\partial_j J^*\|_{L^p}\\ &\overset{(6.6)}{\leq} \big(1+2\epsilon\,\|J^{*}\|_{L^\infty}\big)^2\|\partial_j J^*\|_{L^p}\\ &\overset{(6.1)}{\leq} \big(1+2C_M\epsilon\,\|J^{*}\|_{W^{1,p}}\big)^2\|\partial_j J^*\|_{L^p}\\ &\overset{(6.2)}{\leq} 4\,\|\partial_j J^*\|_{L^p},\end{aligned}$$

which in combination with (6.7) implies the sought after estimate

$$\|J^{-*}\|_{W^{1,p}} \leq C_- \|J^*\|_{W^{1,p}},$$

where $C_- \equiv 4 + \mathrm{vol}(\Omega) C_M$. This proves (6.4) in the case $m = 1$.

To prove the general case, let $k \geq 1$, and assume that $J \in W^{k+1,p}(\Omega)$, $J^{-*} \in W^{k,p}(\Omega)$ and that (6.4) holds for $m = k-1$, i.e.,

$$\|J^{-*}\|_{W^{k,p}} \leq C'_-\, \|J^*\|_{W^{k,p}}. \tag{6.12}$$

For the induction step, we need to show that (6.4) holds for $k+1$. For this, we take k-th order derivatives of (6.11) and find that

$$\begin{aligned}\partial^{k+1}(J^{-*}) \;&= \partial^{k+1}(J^*) + \epsilon\,\partial^k\Big(J^{-*}\partial(J^*) + \partial(J^*)\,J^{-*}\Big)\\ &\quad + \epsilon^2\partial^k\big(J^{-*}\partial(J^*)\,J^{-*}\big),\end{aligned} \tag{6.13}$$

where ∂^k denotes $k-th$ order partial derivatives, not necessarily all in the same direction. (Note that the right hand side of (6.11) contains no derivatives of J^{-*} so that we do not need to use difference quotients in (6.13).) From (6.13), it follows that $\|\partial^{k+1}(J^{-*})\|_{L^p}$ is bounded by $\|\partial^{k+1}(J^*)\|_{L^p}$, by terms linear in ϵ which are of the form

$$\begin{aligned}\epsilon\|\partial^k(J^{-*})\partial(J^*)\|_{L^p} &\leq \epsilon\|\partial^k(J^{-*})\|_{L^p}\|\partial(J^*)\|_{L^\infty}\\ &\overset{(6.12)}{\leq} C'_-\epsilon\|(J^*)\|_{W^{k,p}}\|\partial(J^*)\|_{L^\infty}\\ &\overset{(6.2)}{\leq} C'_-\frac{1}{2C_M}\|\partial(J^*)\|_{L^\infty}\\ &\overset{(6.1)}{\leq} \frac{1}{2}C'_-\|J^*\|_{W^{k+1,p}},\end{aligned}$$

or of the form

$$\begin{aligned}\epsilon\|J^{-*}\partial^{k+1}(J^*)\|_{L^p} &\leq \epsilon\|J^{-*}\|_{L^\infty}\|\partial^{k+1}(J^*)\|_{L^p}\\ &\overset{(6.6)}{\leq} \epsilon\|J^{*}\|_{L^\infty}\|\partial^{k+1}(J^*)\|_{L^p}\\ &\overset{(6.1)}{\leq} \epsilon\, C_M\|(J^*)\|_{W^{1,p}}\|\partial^{k+1}(J^*)\|_{L^p}\\ &\overset{(6.2)}{\leq} \frac{1}{2}\,\|J^*\|_{W^{k+1,p}},\end{aligned}$$

or the more regular terms containing mixed derivatives, and by ϵ^2-term in (6.13) which can be bounded in a similar fashion. Namely, denoting with

$\mathcal{L}$ the L^p-norm of terms not containing the critical derivative $\partial^k J^{-*}$, the ϵ^2-term in (6.13) can be estimated by

$$\begin{aligned}
&\epsilon^2 \|\partial^k \big(J^{-*}\partial(J^*)\, J^{-*}\big)\|_{L^p(\Omega)} \\
&\qquad \le \epsilon^2 \|\partial^k J^{-*}\|_{L^p(\Omega)} \|\partial(J^*)\|_{L^\infty(\Omega)} \|J^{-*}\|_{L^\infty(\Omega)} + \mathcal{L} \\
&\qquad \overset{(6.1)}{\le} \epsilon^2 C_M^2 \|\partial^k J^{-*}\|_{L^p(\Omega)} \|\partial(J^*)\|_{W^{1,p}(\Omega)} \|J^{-*}\|_{W^{1,p}(\Omega)} + \mathcal{L} \\
&\qquad \overset{(6.12)}{\le} \epsilon^2 C_M^2 \|J^*\|^3_{W^{k,p}(\Omega)} + \mathcal{L} \\
&\qquad \overset{(6.2)}{\le} \|J^*\|_{W^{k,p}(\Omega)} + \mathcal{L},
\end{aligned}$$

while the term $\mathcal{L}$ can be estimated similarly by using (6.1), (6.12) and (6.2). In summary, we showed that

$$\|\partial^{k+1}(J^{-*})\|_{L^p} \le C_- \|J^*\|_{W^{k+1,p}(\Omega)},$$

from which we conclude that (6.4) holds for $k+1$, taking C_- as the largest constant that appears in the above estimates (for $m \ge 1$ fixed). Recursion of the above argument proves (6.4) in the general case $m \ge 1$. This completes the proof of Lemma (6.1). $\square$

We now prove the basic estimates for the non-linear source terms on the right hand side of equations (4.12) - (4.13), which are required for the proofs of Lemmas 4.4 and 5.2.

Lemma 6.2. *Let $\Gamma^*, d\Gamma^* \in W^{m,p}(\Omega)$ for $m \ge 1$ and $p > n$, bounded by C_0 as in* (4.7)*, and assume $u \in W^{m+1,p}(\Omega)$ and $a \in W^{m,p}(\Omega)$. Then, if $\epsilon > 0$ satisfies the bound* (6.2)*, then there exists a constant $C_s > 0$ depending only on C_0, m, p and Ω, such that*

$$\begin{aligned}
\|F_u(u,a)\|_{W^{m-1,p}} &\le \epsilon\, C_s\big(1 + \|a\|_{W^{m,p}} + \|u\|_{W^{m+1,p}}\big)\|u\|_{W^{m+1,p}} \\
&\quad + C_0 + \|a\|_{W^{m,p}} \qquad (6.14)\\
\|F_a(u)\|_{W^{m-1,p}} &\le C_0 + \epsilon\, C_s\big(1 + \|u\|_{W^{m+1,p}}\big)\|u\|_{W^{m+1,p}}. \qquad (6.15)
\end{aligned}$$

Proof. We focus on proving the lemma in the case $m = 1$, since higher derivative estimates for $m > 1$ then follow by an analogous argument. Note that, because $\epsilon > 0$ is assumed to satisfy (6.2), Lemma 6.1 applies and yields the existence of the inverse $J^{-1} = I + \epsilon J^{-*}$ together with the estimate (6.4) on J^{-*}.

We first derive (6.14) in the case $m = 1$. From (4.10) we find that

$$\begin{aligned}
\|F_u(u,a)\|_{L^p} \le\ & \|\delta d\Gamma^*\|_{L^p} + \|\delta\Gamma^*\|_{L^p} + \|a\|_{L^p} + \|da\|_{L^p} + \epsilon\|\delta(J^*\cdot\Gamma^*)\|_{L^p} \\
& + \epsilon\,\|d(J^{-*}a)\|_{L^p} + \epsilon\|\langle dJ^*;\tilde\Gamma^*\rangle\|_{L^p} + \|\delta d\big(J^{-1}\cdot dJ^*\big)\|_{L^p}, \qquad (6.16)
\end{aligned}$$

where we used that $d(J^{-1}a) = \epsilon d(J^{-*}a) + da$ by (6.3). We now estimate the right hand side term by term. By our incoming assumption (4.7) on the spacetime connection we have

$$\|\delta d\Gamma^*\|_{L^p} + \|\delta\Gamma^*\|_{L^p} \leq \|d\Gamma^*\|_{W^{1,p}} + \|\Gamma^*\|_{W^{1,p}} \leq C_0, \tag{6.17}$$

and clearly we have

$$\|a\|_{L^p} + \|da\|_{L^p} \leq \|a\|_{W^{1,p}}. \tag{6.18}$$

Applying the Leibniz-rule (2.3), we find that

$$\delta(J^*\cdot\Gamma^*) = \langle dJ^*;\Gamma^*\rangle + J\cdot\delta\Gamma^*,$$

which leads to the bound

$$\begin{aligned}\|\delta(J^*\cdot\Gamma^*)\|_{L^p} &\leq \|dJ^*\|_{L^\infty}\|\Gamma^*\|_{L^p} + \|J\|_{L^\infty}\|\delta\Gamma^*\|_{L^p} \\ &\overset{(6.1)}{\leq} C_M\Big(\|dJ^*\|_{W^{1,p}}\|\Gamma^*\|_{L^p} + \|J\|_{W^{1,p}}\|\delta\Gamma^*\|_{L^p}\Big) \\ &\leq C_M\|\Gamma^*\|_{W^{1,p}}\|J^*\|_{W^{2,p}},\end{aligned} \tag{6.19}$$

where the Hölder continuity of $J^* \in W^{2,p}(\Omega)$ allowed us to estimate the L^p-norm of products in terms of the L^∞-norm on dJ^* and J^*, which we further estimated using Morrey's inequality (6.1). Similarly, the Hölder continuity of $a \in W^{1,p}(\Omega)$ and of J^{-*} together with the bound (6.4) on J^{-*} lead to

$$\begin{aligned}\|d(J^{-*}a)\|_{L^p} &\leq \|d(J^{-*})\|_{L^\infty}\|a\|_{L^p} + \|J^{-*}\|_{L^\infty}\|da\|_{L^p} \\ &\overset{(6.1)}{\leq} C_M\,\|a\|_{W^{1,p}}\Big(\|d(J^{-*})\|_{W^{1,p}} + \|J^{-*}\|_{W^{1,p}}\Big) \\ &\overset{(6.4)}{\leq} C_-\,C_M\,\|a\|_{W^{1,p}}\,\|J^*\|_{W^{2,p}}.\end{aligned} \tag{6.20}$$

In a similar fashion, we obtain

$$\|\langle dJ^*;\tilde{\Gamma}^*\rangle\|_{L^p} \leq \|dJ^*\|_{L^p}\|\tilde{\Gamma}^*\|_{L^\infty} \leq C_M\,\|J^*\|_{W^{1,p}}\|\tilde{\Gamma}^*\|_{W^{1,p}}, \tag{6.21}$$

where we applied again Morrey's inequality (6.1). For the last term in (6.16), use the Leibniz rule (2.4) together with $d^2 = 0$ and formula (6.3) for J^{-1}, to compute

$$d\big(J^{-1}\cdot dJ^*\big) = \epsilon\, dJ^{-*}\wedge dJ^*, \tag{6.22}$$

which leads to the estimate

$$\begin{aligned}\|\delta d\big(J^{-1}\cdot dJ^*\big)\|_{L^p} &\leq \epsilon\,\|\delta(dJ^{-*}\wedge dJ^*)\|_{L^p} \\ &\leq \epsilon\,\|J^{-*}\|_{W^{2,p}}\|dJ^*\|_{L^\infty} + \epsilon\,\|dJ^{-*}\|_{L^\infty}\|J^*\|_{W^{2,p}} \\ &\leq \epsilon\,2C_-C_M\,\|J^*\|_{W^{2,p}}\|J^*\|_{W^{2,p}},\end{aligned} \tag{6.23}$$

where we applied Morrey's inequality (6.1) together with the bound (6.4) on J^{-*} in the last step. Combing now the estimates (6.17) - (6.23) to bound the right hand side in (6.16), we obtain the sought after estimate (6.14).

Estimate (6.14) for the general case $m \geq 1$ follows by a straightforward adaptation of the argument (6.16) - (6.23) to the $W^{m-1,p}$-norm, using Hölder continuity of $m-1$-derivatives of u, a, Γ^* or $d\Gamma^*$ to estimate products in terms of products of the L^p-norm and the L^∞-norm of such derivatives. For instance, estimate (6.23) extends as follows:

$$
\begin{aligned}
\|\delta d\big(J^{-1}\!\cdot dJ^*\big)\|_{W^{m-1,p}} &\overset{(6.22)}{\leq} \epsilon\,\|\delta(dJ^{-*}\wedge dJ^*)\|_{W^{m-1,p}} \\
&\overset{(*)}{\leq} \epsilon\,\|dJ^{-*}\|_{W^{m,p}}\, C_M\|dJ^*\|_{W^{m,p}} + \epsilon\, C_M\|dJ^{-*}\|_{W^{m,p}}\,\|dJ^*\|_{W^{m,p}} \\
&\overset{(6.4)}{\leq} \epsilon\, 2C_- C_M\,\|J^*\|_{W^{m+1,p}}\|J^*\|_{W^{m+1,p}}, \qquad (6.24)
\end{aligned}
$$

where in the first term in $(*)$ results from applying Morrey's inequality (6.1) to estimate derivatives of order less than $m-1$ of dJ^* (which are Hölder continuous), while the second term in $(*)$ results from applying (6.1) to derivatives of order less than $m-1$ of dJ^{-*}. Extending (6.16) - (6.21) analogously to (6.24) proves the sought after estimate (6.14) for the general case $m \geq 1$.

We now prove (6.15) in the case $m = 1$. From our definition of F_a in (4.11) we find that

$$
\begin{aligned}
\|F_a(u)\|_{L^p} &\leq \|\overrightarrow{\text{div}}\big(d\Gamma^*\big)\|_{L^p} + \epsilon\,\|\overrightarrow{\text{div}}\big(J^*\!\cdot d\Gamma^*\big)\|_{L^p} \\
&+ \epsilon\,\|\overrightarrow{\text{div}}\big(dJ^*\wedge\Gamma^*\big)\|_{L^p} + \epsilon\,\|d\big(\overrightarrow{\langle dJ^*;\tilde\Gamma^*\rangle}\big)\|_{L^p}. \qquad (6.25)
\end{aligned}
$$

We now estimate each term on the right hand side of (6.25) separately. Our incoming assumption (4.7) immediately gives

$$
\|\overrightarrow{\text{div}}\big(d\Gamma^*\big)\|_{L^p} \leq \|d\Gamma^*\|_{W^{1,p}} \leq C_0. \qquad (6.26)
$$

Applying Morrey's inequality (6.1) to bound the supremum-norm of J^* and $d\Gamma^*$ leads to

$$
\begin{aligned}
\|\overrightarrow{\text{div}}\big(J^*\!\cdot d\Gamma^*\big)\|_{L^p} &\leq \|J^*\|_{W^{1,p}}\|d\Gamma^*\|_{L^\infty} + \|J^*\|_{L^\infty}\|d\Gamma^*\|_{W^{1,p}} \\
&\overset{(4.7)}{\leq} 2C_M\, C_0\,\|J^*\|_{W^{1,p}} \\
&\leq 2C_M\, C_0\,\|J^*\|_{W^{2,p}}. \qquad (6.27)
\end{aligned}
$$

Likewise, using (6.1) to bound the supremum-norm of dJ^* and Γ^*, we obtain

$$
\begin{aligned}
\|\overrightarrow{\text{div}}\big(dJ^*\wedge\Gamma^*\big)\|_{L^p} &\leq \|dJ^*\|_{W^{1,p}}\|\Gamma^*\|_{L^\infty} + \|dJ^*\|_{L^\infty}\|\Gamma^*\|_{W^{1,p}} \\
&\overset{(6.1)}{\leq} C_M\big(\|dJ^*\|_{W^{1,p}}\|\Gamma^*\|_{W^{1,p}} + \|dJ^*\|_{W^{1,p}}\|\Gamma^*\|_{W^{1,p}}\big) \\
&\overset{(4.7)}{\leq} 2\,C_M\, C_0\,\|J^*\|_{W^{2,p}}. \qquad (6.28)
\end{aligned}
$$

Finally, we estimate the non-linear term by

$$
\begin{aligned}
\|d\big(\overrightarrow{\langle dJ^*;\tilde\Gamma^*\rangle}\big)\|_{L^p} &\leq \|dJ^*\|_{W^{1,p}}\|\tilde\Gamma^*\|_{L^\infty} + \|dJ^*\|_{L^\infty}\|\tilde\Gamma^*\|_{W^{1,p}} \\
&\overset{(6.1)}{\leq} C_M\big(\|dJ^*\|_{W^{1,p}}\|\tilde\Gamma^*\|_{W^{1,p}} + \|dJ^*\|_{W^{1,p}}\|\tilde\Gamma^*\|_{W^{1,p}}\big) \\
&\leq 2\,C_M\|u\|^2_{W^{2,p}}, \qquad (6.29)
\end{aligned}
$$

recalling that $u \equiv (J^*, \tilde\Gamma^*)$. Combing (6.26) - (6.29) to bound the right hand side in (6.25) we obtain the sought after estimate (6.15) in the case $m = 1$.

Estimate (6.15) for the general case $m \geq 1$ follows by extending (6.25) - (6.29) to the $W^{m-1,p}$-norm in a fashion similar to (6.24). Taking $C_s > 0$ as the maximum over all constants in (6.17) - (6.29) and the constants arising from higher derivatives estimates completes the proof. □

6.2. **Well-posedness of iteration scheme - Proof of Lemma 4.4.** We now proof Lemma 4.4, which gives well-posedness of the iteration scheme and the basic elliptic estimates (4.31) - (4.34). For this, assume $u_k \in W^{m+1,p}(\Omega)$ is given, for $m \geq 1$, $p > n \geq 2$, and assume ϵ satisfies (4.30), that is $0 < \epsilon \leq \epsilon(k)$. Lemma 4.4 states that there exists $a_{k+1} \in W^{m,p}(\Omega)$ which solves (4.21) - (4.22), there exists $\psi_{k+1} \in W^{m,p}(\Omega)$ and $y_{k+1} \in W^{m+2,p}(\Omega)$ which solve (4.23) - (4.24), and there exists $u_{k+1} \in W^{m+1,p}(\Omega)$ which solves (4.25) with boundary data (4.26) - (4.27), and these solutions satisfy the elliptic estimates (4.31) - (4.34).

Proof of Lemma 4.4. First note that assumption (4.30), that $0 < \epsilon \leq \epsilon(k)$ implies that ϵ satisfies the bound (6.2), so that the source estimates of Lemma 6.2 apply and yield $F_a(u_k) \in W^{m-1,p}(\Omega)$ and $F_u(u_k, a_{k+1}) \in W^{m-1,p}(\Omega)$.

We begin the proof by proving existence of a solution a_{k+1} to the first order system (4.21) - (4.22) by applying Theorem 2.2. For this, first note that the conditions of Theorem 2.2 (i) are met, since $g = 0$ in (4.21) (so $\delta g = 0$) and since the condition of zero boundary data is assumed in (4.22). Moreover, the condition $df = 0$ of Theorem 2.2 is satisfied by (4.21), since $F_a(u_k)$ is the exterior derivative d of a vector valued differential form, namely,

$$F_a(u) = d\big(\overrightarrow{\delta(J{\cdot}\Gamma)}\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{6.30}$$

c.f. equation (3.40) in [17]. Regarding regularity, our incoming assumption $u_k \in W^{m+1,p}(\Omega)$ together with the source estimates of Lemma 6.2 show that $F_a(u_k) \in W^{m-1,p}(\Omega)$. We conclude that Theorem 2.2 applies to (4.21) - (4.22) and yields the existence of a solution $a_{k+1} \in W^{m,p}(\Omega)$. Moreover, by Gaffney's inequality (2.13), this solution satisfies

$$\|\vec{a}_{k+1}\|_{W^{m,p}(\Omega)} \leq C\, \|F_a(u_k)\|_{W^{m-1,p}(\Omega)},$$

for some constant $C > 0$ depending only on Ω, m, n, p, which is the sought after estimate (4.31).

To prove the existence of a ψ_{k+1} solving (4.23), we first show the consistency condition that the exterior derivative on the right hand side of (4.23), interpreted as a vector valued 1-form, vanishes. For this recall from (4.18) that $F_a(u) = d\overrightarrow{F_J}$, so that equation (4.21) for a_{k+1} implies the sought after consistency condition

$$d\big(\overrightarrow{F_J(u_k)} - \overrightarrow{a_{k+1}}\big) = F_a(u) - d\overrightarrow{a_{k+1}} \overset{(4.21)}{=} 0. \tag{6.31}$$

Thus, (ii) of Theorem 2.2 yields existence of a vector valued 0-form ψ_{k+1} solving (4.23) such that $\psi_{k+1}(q) = 0$ for the $q \in \Omega$ fixed initially in the iteration scheme. Moreover, the source estimate (6.14) in combination with $a_{k+1} \in W^{m,p}(\Omega)$ imply that $\overrightarrow{F_J(u_k)} - \overrightarrow{a_{k+1}} \in W^{m-1,p}(\Omega)$, so that the regularity $\psi_{k+1} \in W^{m,p}(\Omega)$ follows. Since the left hand side of (4.23) is the gradient of ψ_{k+1}, estimate (4.33) directly follows by integration starting from the point $q \in \Omega$ where ψ_{k+1} is assumed to vanish.

The existence of a solution $y_{k+1} \in W^{m+2,p}(\Omega)$ to (4.24) follows from the existence theorem for the Dirichlet problem of the Poisson equation with L^p sources, Theorem 2.1, keeping in mind that $F_u(u_k, a_{k+1}) \in W^{m-1,p}(\Omega)$ by Lemma 6.2. We now prove estimate (4.34). Applying the elliptic estimate (2.11) component-wise, $\Delta y_{k+1} = \psi_{k+1}$ and $y_{k+1} = 0$ on $\partial\Omega$, c.f. (4.24), we obtain

$$\begin{aligned} \|y_{k+1}\|_{W^{m+2,p}(\Omega)} &\leq C\Big(\|\Delta y_{k+1}\|_{W^{m,p}(\Omega)} + \|y_{k+1}\|_{W^{m+2-1/p,p}(\partial\Omega)}\Big) \\ &\overset{(4.24)}{=} C\,\|\psi_{k+1}\|_{W^{m,p}(\Omega)} \\ &\overset{(4.33)}{\leq} C\,\|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)}, \end{aligned} \tag{6.32}$$

where we absorbed the constant from the estimate on $\|\psi_{k+1}\|_{W^{m,p}}$ into the universal constant $C > 0$. This is the sought after estimate (4.34).

Finally, we prove existence of a solution $u_{k+1} \in W^{m+1,p}(\Omega)$ of (4.25) with boundary data (4.26) - (4.27). Since $F_u(u_k, a_{k+1}) \in W^{m-1,p}(\Omega)$, existence of a solution $u_{k+1} \in W^{m+1,p}(\Omega)$ of the Poisson equation (4.25) with the Dirichlet boundary data (4.26) - (4.27) follows from Theorem 2.1. To prove estimate (4.32), we apply again the elliptic estimate (2.11) to obtain

$$\begin{aligned} \|u_{k+1}\|_{W^{m+1,p}(\Omega)} &\leq C\Big(\|\Delta u_{k+1}\|_{W^{m-1,p}(\Omega)} + \|u_{k+1}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big) \\ &\overset{(4.25)}{=} C\Big(\|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)} + \|u_{k+1}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big). \end{aligned} \tag{6.33}$$

Substituting now the boundary condition (4.26) - (4.27) and using Theorem (2.3), we estimate the above boundary term by

$$\begin{aligned} \|u_{k+1}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} &= \|dy_{k+1}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} \\ &\leq C\,\|dy_{k+1}\|_{W^{m+1,p}(\Omega)}, \end{aligned} \tag{6.34}$$

so that applying estimate (6.32) to bound $\|dy_{k+1}\|_{W^{m+1,p}(\Omega)}$ yields

$$\|u_{k+1}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} \leq C\,\|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)}.$$

Substituting the previous inequality back into (6.33) we obtain

$$\|u_{k+1}\|_{W^{m+1,p}(\Omega)} \leq C_e\,\|F_u(u_k, a_{k+1})\|_{W^{m-1,p}(\Omega)},$$

which is the sought after estimate (4.32), where we take C_e as the maximum over all constants in the above estimates. This completes the proof. $\square$

6.3. **Estimates on Differences of Iterates - Proof of Lemma 5.1.** We introduce the notation

$$\begin{aligned}\overline{\tilde{\Gamma}^*_k} &\equiv \tilde{\Gamma}^*_k - \tilde{\Gamma}^*_{k-1},\\ \overline{J^*_k} &\equiv J^*_k - J^*_{k-1},\end{aligned}$$

so $\overline{u_k} = (\overline{J^*_k}, \overline{\tilde{\Gamma}^*_k})$. Let J_k^{-1} be the inverse of $J_k \equiv I + \epsilon J^*_k$ and let J_{k-1}^{-1} be the inverse of $J_{k-1} \equiv I + \epsilon J^*_{k-1}$, and denote with J_k^{-*} the matrix valued 0-form that satisfies $J_k^{-1} = I + \epsilon J_k^{-*}$ and likewise $J_{k-1}^{-1} = I + \epsilon J_{k-1}^{-*}$. We begin by deriving a bound on $\overline{J_k^{-*}} \equiv J_k^{-*} - J_{k-1}^{-*}$.

Lemma 6.3. *Assume $u_k, u_{k-1} \in W^{m+1,p}(\Omega)$ for $m \geq 0$, $p > n$, and assume $0 < \epsilon \leq \min\big(\epsilon(k), \epsilon(k-1)\big)$, so ϵ satisfies (4.30) in terms of u_k and u_{k-1}. Then J_k and J_{k-1} are invertible with $J_k^{-1} = I + \epsilon J_k^{-*} \in W^{m+1,p}(\Omega)$ and $J_{k-1}^{-1} = I + \epsilon J_{k-1}^{-*} \in W^{m+1,p}(\Omega)$, and there exists a constant $C'_- > 0$ depending only on m, n, p, Ω, such that*

$$\|\overline{J_k^{-*}}\|_{W^{m+1,p}} \leq C'_- \|\overline{J^*_k}\|_{W^{m+1,p}}. \tag{6.35}$$

Proof. To begin, note that the ϵ-bound (4.30), $0 < \epsilon \leq \epsilon(k)$, implies that ϵ satisfies (6.2) for $J^* = J^*_k$, so that Lemma 6.1 implies that J_k is invertible with $J_k^{-1} = I + \epsilon J_k^{-*}$ and $J_k^{-*} \in W^{m+1,p}(\Omega)$. Likewise, the ϵ-bound (4.30) for u_{k-1} implies that J_{k-1} is invertible with $J_{k-1}^{-1} = I + \epsilon J_{k-1}^{-*} \in W^{m+1,p}(\Omega)$.

Now, substituting $J_k = I + \epsilon J^*_k$ and $J_k^{-1} = I + \epsilon J_k^{-*}$ into the identity

$$0 = J_k J_k^{-1} - J_{k-1} J_{k-1}^{-1},$$

and solving for $\overline{J_k^{-*}} \equiv J_k^{-*} - J_{k-1}^{-*}$, we find after dividing by ϵ that

$$\begin{aligned}\overline{J_k^{-*}} &= -\overline{J^*_k} - \epsilon\left(J^*_k J_k^{-*} - J^*_{k-1} J_{k-1}^{-*}\right)\\ &= -\overline{J^*_k} - \epsilon\left(\overline{J^*_k} \cdot J_k^{-*} + J^*_{k-1} \cdot \overline{J_k^{-*}}\right).\end{aligned} \tag{6.36}$$

Thus, taking the L^p norm of (6.36) and applying Morrey's inequality (6.1), gives

$$\begin{aligned}\big\|\overline{J_k^{-*}}\big\|_{L^p} &\leq \big\|\overline{J^*_k}\big\|_{L^p} + \epsilon\,\big\|\overline{J^*_k} J_k^{-*}\big\|_{L^p} + \epsilon\,\big\|J^*_{k-1}\overline{J_k^{-*}}\big\|_{L^p}\\ &\leq \big\|\overline{J^*_k}\big\|_{L^p} + \epsilon\,\big\|\overline{J^*_k}\big\|_{L^p}\|J_k^{-*}\|_{L^\infty} + \epsilon\,\|J^*_{k-1}\|_{L^\infty}\big\|\overline{J_k^{-*}}\big\|_{L^p}\\ &\overset{(6.1)}{\leq} \big\|\overline{J^*_k}\big\|_{L^p} + \epsilon\, C_M\big\|\overline{J^*_k}\big\|_{L^p}\|J_k^{-*}\|_{W^{1,p}} + \epsilon\, C_M\|J^*_{k-1}\|_{W^{1,p}}\big\|\overline{J_k^{-*}}\big\|_{L^p}.\end{aligned}$$

Using for the last term that the ϵ-bound (4.30) for u_{k-1} implies that

$$\epsilon\, C_M\|J^*_{k-1}\|_{W^{1,p}} \leq \frac{1}{2},$$

we find after subtraction of $\frac{1}{2}\big\|\overline{J_k^{-*}}\big\|_{L^p}$ that

$$\begin{aligned}\frac{1}{2}\big\|\overline{J_k^{-*}}\big\|_{L^p} &\leq \big(1 + \epsilon\, C_M\|J_k^{-*}\|_{W^{1,p}}\big)\big\|\overline{J^*_k}\big\|_{L^p}\\ &\overset{(6.4)}{\leq} \big(1 + \epsilon\, C_- C_M\|J^*_k\|_{W^{1,p}}\big)\big\|\overline{J^*_k}\big\|_{L^p}.\end{aligned} \tag{6.37}$$

Now, using the ϵ-bound (4.30) for u_k, we find that

$$\epsilon\, C_M \|J_k^*\|_{W^{1,p}} \leq \frac{1}{2},$$

which in light of (6.37) gives

$$\big\|\overline{J_k^{-*}}\big\|_{L^p} \leq (2C_M + C_-)\big\|\overline{J_k^*}\big\|_{L^p}. \tag{6.38}$$

To prove (6.35) for $m = 1$, we first differentiate (6.36) and find

$$\partial_j \overline{J_k^{-*}} = -\partial_j \overline{J_k^*} - \epsilon\Big(\partial_j \overline{J_k^*} J_k^{-*} + \partial_j J_{k-1}^* \overline{J_k^{-*}} + \overline{J_k^*}\, \partial_j J_k^{-*} + J_{k-1}^* \partial_j \overline{J_k^{-*}}\Big), \tag{6.39}$$

which implies for the gradient $d\overline{J_k^{-*}}$ the estimate

$$\begin{aligned}\|d\overline{J_k^{-*}}\|_{L^p} \quad\leq\quad & \|d\overline{J_k^*}\|_{L^p} + \epsilon\, \|J_k^{-*}\|_{L^\infty} \|\overline{J_k^*}\|_{W^{1,p}} + \epsilon\, \|J_{k-1}^*\|_{W^{1,p}} \|\overline{J_k^{-*}}\|_{L^\infty} \\ & + \epsilon\, \|J_k^{-*}\|_{W^{1,p}} \|\overline{J_k^*}\|_{L^\infty} + \epsilon\, \|J_{k-1}^*\|_{L^\infty} \|\overline{J_k^{-*}}\|_{W^{1,p}},\end{aligned}$$

where we bounded undifferentiated terms by their L^∞-norm and differentiated terms by their $W^{1,p}$-norm. Applying Morrey's inequality (6.1) we obtain the further estimate

$$\|d\overline{J_k^{-*}}\|_{L^p} \leq \|d\overline{J_k^*}\|_{L^p} + \epsilon\, 2C_M \Big(\|J_k^{-*}\|_{W^{1,p}} \|\overline{J_k^*}\|_{W^{1,p}} + \|J_{k-1}^*\|_{W^{1,p}} \|\overline{J_k^{-*}}\|_{W^{1,p}}\Big),$$

and applying the bound (6.4) on J_k^{-*} and J_{k-1}^{-*} yields

$$\begin{aligned}\|d\overline{J_k^{-*}}\|_{L^p} \quad\leq\quad & \|d\overline{J_k^*}\|_{L^p} + 2\epsilon\, C_M\, C_- \|J_k^*\|_{W^{1,p}} \|\overline{J_k^*}\|_{W^{1,p}} \\ & + 2\epsilon\, C_M \|J_{k-1}^*\|_{W^{1,p}} \|\overline{J_k^{-*}}\|_{W^{1,p}}\end{aligned}$$

so that the ϵ-bound (4.30) for u_k and u_{k-1} implies

$$\|d\overline{J_k^{-*}}\|_{L^p} \quad\leq\quad \|d\overline{J_k^*}\|_{L^p} + \frac{1}{2} C_- \|\overline{J_k^*}\|_{W^{1,p}} + \frac{1}{2} \|\overline{J_k^{-*}}\|_{W^{1,p}}. \tag{6.40}$$

Adding $\|\overline{J_k^{-*}}\|_{L^p}$ to both sides of (6.40) and using estimate (6.38) to bound $\|\overline{J_k^{-*}}\|_{L^p}$ on the right hand side, we find

$$\|\overline{J_k^{-*}}\|_{W^{1,p}} \quad\leq\quad 3(C_M + C_- + 1)\|\overline{J_k^*}\|_{W^{1,p}} + \frac{1}{2}\|\overline{J_k^{-*}}\|_{W^{1,p}}.$$

So subtraction of the second term on the right hand side finally yields

$$\|\overline{J_k^{-*}}\|_{W^{1,p}} \quad\leq\quad 6(C_M + C_- + 1)\|\overline{J_k^*}\|_{W^{1,p}},$$

which is the sought after bound (6.35) for $m = 1$ and $C'_- = 6(C_M + C_- + 1)$.

To derive (6.35) for $m \geq 2$, we proceed by induction. For this, assume (6.35) holds for some $1 \leq l \leq m$, i.e.

$$\|\overline{J_k^{-*}}\|_{W^{l,p}} \leq C'_- \|\overline{J_k^*}\|_{W^{l,p}}, \tag{6.41}$$

and assume $J_k^{-1} = I + \epsilon J_k^{-*} \in W^{m+1,p}(\Omega)$ and $J_{k-1}^{-1} = I + \epsilon J_{k-1}^{-*} \in W^{m+1,p}(\Omega)$, (c.f. Lemma 6.1). We need to show that (6.41) holds for $l + 1$. For this, denote with ∂^{l+1} a combination of partial derivatives of $l + 1$-st order (not necessarily in the same direction), i.e. ∂^{l+1} denotes partial differentiation corresponding to a specific multi-index. Now, taking ∂^{l+1} of (6.36), we obtain

$$\partial^{l+1}\overline{J_k^{-*}} = -\partial^{l+1}\overline{J_k^*} - \epsilon\, \partial^{l+1}\big(\overline{J_k^*}\cdot J_k^{-*}\big) - \epsilon\, \partial^{l+1}\big(J_{k-1}^*\cdot \overline{J_k^{-*}}\big),$$

which gives the estimate

$$\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p} \leq \|\partial^{l+1}\overline{J_k^*}\|_{L^p} + \epsilon\|\partial^{l+1}\big(\overline{J_k^*}\cdot J_k^{-*}\big)\|_{L^p} + \epsilon\|\partial^{l+1}\big(J_{k-1}^*\cdot\overline{J_k^{-*}}\big)\|_{L^p}. \tag{6.42}$$

The first term on the right hand side is bounded by the $W^{l+1,p}$-norm of $\overline{J_k^*}$. Using Morrey's inequality (6.1) to estimate product terms and using (6.4) to bound the $W^{l+1,p}$-norm of J_k^{-*}, we estimate the second term on the right hand side of (6.42) by

$$\begin{aligned}
\epsilon\,\|\partial^{l+1}\big(\overline{J_k^*}\cdot J_k^{-*}\big)\|_{L^p} &\overset{(6.1)}{\leq} \epsilon C_M\,(l+1)!\,\|\overline{J_k^*}\|_{W^{l+1,p}}\|J_k^{-*}\|_{W^{l+1,p}} \\
&\overset{(6.4)}{\leq} \epsilon C_M C_-\,(l+1)!\,\|\overline{J_k^*}\|_{W^{l+1,p}}\|J_k^{*}\|_{W^{l+1,p}} \\
&\overset{(4.30)}{\leq} C_-\,(l+1)!\,\|\overline{J_k^*}\|_{W^{l+1,p}},
\end{aligned} \tag{6.43}$$

where the factor $(l+1)!$ takes account for repeated lower derivative terms resulting form the product rule on the left hand side and is non-optimal. Similarly, using in addition the induction assumption (6.41), we obtain

$$\begin{aligned}
&\epsilon\|\partial^{l+1}\big(J_{k-1}^*\cdot\overline{J_k^{-*}}\big)\|_{L^p} \\
&\overset{(*)}{\leq} \epsilon C_M\,(l+1)!\,\|J_{k-1}^*\|_{W^{l+1,p}}\|\overline{J_k^{-*}}\|_{W^{l,p}} + \epsilon\|J_{k-1}^*\|_{L^\infty}\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p} \\
&\overset{(6.1)}{\leq} \epsilon C_M\,(l+1)!\,\|J_{k-1}^*\|_{W^{l+1,p}}\|\overline{J_k^{-*}}\|_{W^{l,p}} + \epsilon C_M\|J_{k-1}^*\|_{W^{1,p}}\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p} \\
&\overset{(4.30)}{\leq} (l+1)!\,\|\overline{J_k^{-*}}\|_{W^{l,p}} + \frac{1}{2}\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p} \\
&\overset{(6.41)}{\leq} (l+1)!C_-'\,\|\overline{J_k^*}\|_{W^{l,p}} + \frac{1}{2}\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p}
\end{aligned} \tag{6.44}$$

where the second term in $(*)$ results form the contribution of $(l+1)$-st order derivatives on $\overline{J_k^{-*}}$. Now, estimating the right hand side in (6.42) by (6.43) and (6.44), we find

$$\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p} \leq \|\overline{J_k^*}\|_{W^{l+1,p}} + 2C_-\,(l+1)!\,\|\overline{J_k^*}\|_{W^{l+1,p}} + \frac{1}{2}\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p}$$

so that subtraction of the last term yields

$$\|\partial^{l+1}\overline{J_k^{-*}}\|_{L^p} \leq 2\|\overline{J_k^*}\|_{W^{l+1,p}} + 4C_-\,(l+1)!\,\|\overline{J_k^*}\|_{W^{l+1,p}}. \tag{6.45}$$

Repeating the argument (6.42) - (6.45) for each multi-index ∂^{l+1} gives a suitable estimate on the L^p-norm of all combinations of $(l+1)$-st order derivatives. Adding then the $W^{l,p}$-norm of $\overline{J_k^{-*}}$ to both sides of that estimate, and applying the induction assumption (6.41) to bound the $W^{l,p}$-norm of $\overline{J_k^{-*}}$ on the resulting right hand side, the sought after estimate (6.35) for $l+1$ follows. This completes the induction and the proof of Lemma 6.3. $\square$

Proof of Lemma 5.1. We now estimate the difference of the source functions and thereby prove Lemma 5.1, which states that, if

$$0 < \epsilon \leq \min\big(\epsilon(k), \epsilon(k-1)\big),$$

(that is, (4.30) holds), then there exists a constant $C_s > 0$ depending only on m, n, p, Ω and $C_0 > 0$, such that (5.3) - (5.4) hold, i.e.

$$\begin{aligned}
\|\overline{F_u(u_k, a_{k+1})}\|_{W^{m-1,p}} &\leq C_u(k)\Big(\epsilon\, \|\overline{u_k}\|_{W^{m+1,p}} + \|\overline{a_{k+1}}\|_{W^{m,p}}\Big),\\
\|\overline{F_a(u_k)}\|_{W^{m-1,p}} &\leq \epsilon\, C_a(k)\, \|\overline{u_k}\|_{W^{m+1,p}},
\end{aligned}$$

where

$$\begin{aligned}
C_u(k) &\equiv C_s\big(1 + \|u_k\|_{W^{m+1,p}} + \|u_{k-1}\|_{W^{m+1,p}} + \|a_{k+1}\|_{W^{m,p}}\big),\\
C_a(k) &\equiv C_s\big(1 + \|u_k\|_{W^{m+1,p}} + \|u_{k-1}\|_{W^{m+1,p}}\big).
\end{aligned}$$

We only prove Lemma 5.1 for the critical case $m = 1$, since the cases $m \geq 2$ follow by an analogous reasoning, (see also (6.24) for an example of extending source estimate to higher derivatives). Note that, because $\epsilon > 0$ is assumed to satisfy (4.30), Lemma 6.3 applies and gives estimate (6.4) on $\overline{J_k^{-*}}$.

We begin by proving (5.4). From the definition of F_a in (4.11), using that the source term $d\Gamma^*$ cancels out in $\overline{F_a(u_k)}$, we obtain

$$\big\|\overline{F_a(u_k)}\big\|_{L^p} \leq \epsilon\big\|\overrightarrow{\text{div}}\big(d\overline{J_k^*} \wedge \Gamma^*\big)\big\|_{L^p} + \epsilon\big\|\overrightarrow{\text{div}}\big(\overline{J_k^*} \cdot d\Gamma^*\big)\big\|_{L^p} + \epsilon\big\|d\big(\overline{\overrightarrow{\langle dJ_k^*; \tilde{\Gamma}_k^*\rangle}}\big)\big\|_{L^p}. \tag{6.46}$$

We estimate the linear terms using Morrey's inequality (6.1) and resulting Hölder continuity, and obtain

$$\begin{aligned}
\big\|\overrightarrow{\text{div}}\big(d\overline{J_k^*} \wedge \Gamma^*\big)\big\|_{L^p} &\leq \|d\overline{J_k^*}\|_{W^{1,p}}\|\Gamma^*\|_{L^\infty} + \|d\overline{J_k^*}\|_{L^\infty}\|\Gamma^*\|_{W^{1,p}}\\
&\overset{(6.1)}{\leq} C_M\, \|\overline{J_k^*}\|_{W^{2,p}}\|\Gamma^*\|_{W^{1,p}}\\
&\overset{(4.7)}{\leq} C_M\, C_0\, \|\overline{u_k}\|_{W^{2,p}}
\end{aligned} \tag{6.47}$$

and

$$\begin{aligned}
\big\|\overrightarrow{\text{div}}\big(\overline{J_k^*} \cdot d\Gamma^*\big)\big\|_{L^p} &\leq \|\overline{J_k^*}\|_{W^{1,p}}\|d\Gamma^*\|_{L^\infty} + \|\overline{J_k^*}\|_{L^\infty}\|d\Gamma^*\|_{W^{1,p}}\\
&\overset{(6.1)}{\leq} C_M\, \|\overline{J_k^*}\|_{W^{1,p}}\|d\Gamma^*\|_{W^{1,p}}\\
&\overset{(4.7)}{\leq} C_M\, C_0\, \|\overline{u_k}\|_{W^{2,p}}.
\end{aligned} \tag{6.48}$$

For the non-linear term we first compute

$$\begin{aligned}
d\big(\overline{\overrightarrow{\langle dJ_k^*; \tilde{\Gamma}_k^*\rangle}}\big) &= d\big(\overrightarrow{\langle d(J_k^* - J_{k-1}^*); \tilde{\Gamma}_k^*\rangle}\big) + d\big(\overrightarrow{\langle dJ_{k-1}^*; (\tilde{\Gamma}_k^* - \tilde{\Gamma}_{k-1}^*)\rangle}\big)\\
&= d\big(\overrightarrow{\langle d\overline{J_k^*}; \tilde{\Gamma}_k^*\rangle}\big) + d\big(\overrightarrow{\langle dJ_{k-1}^*; \overline{\tilde{\Gamma}_k^*}\rangle}\big)
\end{aligned}$$

and then estimate

$$\begin{aligned}
\big\| d\big(\overrightarrow{\langle d\overline{J_k^*}; \tilde{\Gamma}_k^* \rangle}\big)\big\|_{L^p} &\leq \|d\overline{J_k^*}\|_{W^{1,p}} \|\tilde{\Gamma}_k^*\|_{L^\infty} + \|d\overline{J_k^*}\|_{L^\infty} \|\tilde{\Gamma}_k^*\|_{W^{1,p}} \\
&\overset{(6.1)}{\leq} C_M \, \|\tilde{\Gamma}_k^*\|_{W^{1,p}} \|\overline{J_k^*}\|_{W^{2,p}}, \\
\big\| d\big(\overrightarrow{\langle dJ_{k-1}^*; \overline{\tilde{\Gamma}_k^*} \rangle}\big)\big\|_{L^p} &\leq \|dJ_{k-1}^*\|_{W^{1,p}} \|\overline{\tilde{\Gamma}_k^*}\|_{L^\infty} + \|dJ_{k-1}^*\|_{L^\infty} \|\overline{\tilde{\Gamma}_k^*}\|_{W^{1,p}} \\
&\overset{(6.1)}{\leq} C_M \, \|J_{k-1}^*\|_{W^{2,p}} \|\overline{\tilde{\Gamma}_k^*}\|_{W^{1,p}},
\end{aligned}$$

which combined yields

$$\big\| d\big(\overrightarrow{\overline{\langle dJ_k^*; \tilde{\Gamma}_k^* \rangle}}\big)\big\|_{L^p} \leq \; C_M \, \big(2C_0 + \|u_{k-1}\|_{W^{2,p}} + \|u_k\|_{W^{2,p}} \big) \|\overline{u_k}\|_{W^{2,p}}. \quad (6.49)$$

Combining (6.47) - (6.49) with (6.46) yields the sought after bound (5.4).

We now prove (5.3). From definition (4.10), using that the source terms $\delta d\Gamma^*$ and $\delta\Gamma^*$ cancel and substituting $d(J_k^{-1} a_k) = da_k + \epsilon d(J_k^{-*} a_k)$, we find that

$$\begin{aligned}
\|\overline{F_u(u_k, a_{k+1})}\|_{L^p} \;\leq\; & \|\overline{a_{k+1}}\|_{W^{1,p}} + \epsilon \, \|\delta(\overline{J_k^*} \cdot \Gamma^*)\|_{L^p} + \epsilon \, \|d(\overline{J_k^{-*} a_{k+1}})\|_{L^p} \\
& + \epsilon \, \|\overline{\langle dJ_k^*; \tilde{\Gamma}_k^* \rangle}\|_{L^p} + \epsilon \, \|\delta d\big(\overline{J_k^{-*} \cdot dJ_k^*}\big)\|_{L^p},
\end{aligned} \quad (6.50)$$

where we used for the last term that $d^2 = 0$ gives

$$d\big(\overline{J_k^{-1} \cdot dJ_k^*}\big) = \epsilon \, d\big(\overline{J_k^{-*} \cdot dJ_k^*}\big).$$

Now, for the linear term in (6.50) we obtain

$$\begin{aligned}
\|\delta(\overline{J_k^*} \cdot \Gamma^*)\|_{L^p} &\leq \|\overline{J_k^*}\|_{W^{1,p}} \|\Gamma^*\|_{L^\infty} + \|\overline{J_k^*}\|_{L^\infty} \|\Gamma^*\|_{W^{1,p}} \\
&\overset{(6.1)}{\leq} C_M \, \|\overline{J_k^*}\|_{W^{1,p}} \|\Gamma^*\|_{W^{1,p}} \\
&\overset{(4.7)}{\leq} C_M \, C_0 \, \|\overline{u_k}\|_{W^{1,p}}.
\end{aligned} \quad (6.51)$$

For the first non-linear term we compute

$$\overline{J_k^{-*} \cdot a_{k+1}} = \overline{J_k^{-*}} \cdot a_{k+1} + J_{k-1}^{-*} \cdot \overline{a_{k+1}},$$

so that

$$d(\overline{J_k^{-*} \cdot a_{k+1}}) = d(\overline{J_k^{-*}}) \cdot a_{k+1} + \overline{J_k^{-*}} \cdot da_{k+1} + d(J_{k-1}^{-*}) \cdot \overline{a_{k+1}} + J_{k-1}^{-*} \cdot d(\overline{a_{k+1}}),$$

from which we obtain the estimate

$$\begin{aligned}
\|d(\overline{J_k^{-*} a_{k+1}})\|_{L^p} \leq & \|d(\overline{J_k^{-*}}) \cdot a_{k+1}\|_{L^p} + \|\overline{J_k^{-*}} \cdot d(a_{k+1})\|_{L^p} \\
& + \|d(J_{k-1}^{-*}) \cdot \overline{a_{k+1}}\|_{L^p} + \|J_{k-1}^{-*} \cdot d(\overline{a_{k+1}})\|_{L^p} \\
\leq & \|d(\overline{J_k^{-*}})\|_{L^p} \|a_{k+1}\|_{L^\infty} + \|\overline{J_k^{-*}}\|_{L^\infty} \|d(a_{k+1})\|_{L^p} \\
& + \|d(J_{k-1}^{-*})\|_{L^p} \|\overline{a_{k+1}}\|_{L^\infty} + \|J_{k-1}^{-*}\|_{L^\infty} \|d(\overline{a_{k+1}})\|_{L^p}, \quad (6.52)
\end{aligned}$$

so that Morrey's inequality (6.1) and (6.35), (the bound on $\overline{J_k^{-1}}$), yield

$$\begin{aligned}
\|d(\overline{J_k^{-*} a_{k+1}})\|_{L^p} &\overset{(6.1)}{\leq} C_M\big(\|\overline{J_k^{-*}}\|_{W^{1,p}}\|a_{k+1}\|_{W^{1,p}} + \|J_{k-1}^{-*}\|_{W^{1,p}}\|\overline{a_{k+1}}\|_{W^{1,p}}\big)\\
&\overset{(6.35)}{\leq} C_M\big(\|\overline{J_k^*}\|_{W^{1,p}}\|a_{k+1}\|_{W^{1,p}} + C\,\|J_{k-1}^{*}\|_{W^{1,p}}\|\overline{a_{k+1}}\|_{W^{1,p}}\big)\\
&\leq C_M\,\big(\|a_{k+1}\|_{W^{1,p}} + C\|u_{k-1}\|_{W^{1,p}}\big)\big(\|\overline{u_k}\|_{W^{1,p}} + \|\overline{a_{k+1}}\|_{W^{1,p}}\big).
\end{aligned} \tag{6.53}$$

For the second non-linear term, similar to the argument leading to (6.49), we first compute

$$\overline{\langle dJ_k^*;\tilde\Gamma_k^*\rangle} = \langle\overline{dJ_k^*};\tilde\Gamma_k^*\rangle + \langle dJ_{k-1}^*;\overline{\tilde\Gamma_k^*}\rangle$$

and then estimate

$$\begin{aligned}
\|\overline{\langle dJ_k^*;\tilde\Gamma_k^*\rangle}\|_{L^p} &\overset{(6.1)}{\leq} C_M\big(\|\overline{J_k^*}\|_{W^{1,p}}\|\tilde\Gamma_k^*\|_{W^{1,p}} + \|J_{k-1}^*\|_{W^{1,p}}\|\overline{\tilde\Gamma_k^*}\|_{W^{1,p}}\big)\\
&\leq C_M\,\big(\|u_k\|_{W^{1,p}} + \|u_{k-1}\|_{W^{1,p}}\big)\|\overline{u_k}\|_{W^{1,p}}.
\end{aligned} \tag{6.54}$$

For the last non-linear term we first compute

$$\overline{J_k^{-*}\cdot dJ_k^*} = \overline{J_k^{-*}}\cdot dJ_k^* + J_{k-1}^{-*}\cdot\overline{dJ_k^*},$$

so that the Leibniz rule (2.4) and $d^2=0$ yield

$$d\big(\overline{J_k^{-*}\cdot dJ_k^*}\big) \;=\; \overline{dJ_k^{-*}}\wedge dJ_k^* + dJ_{k-1}^{-*}\wedge\overline{dJ_k^*}.$$

From this, we obtain the (higher derivative) estimate

$$\begin{aligned}
\|\delta d\big(\overline{J_k^{-*}\cdot dJ_k^*}\big)\|_{L^p} &\leq \|\overline{dJ_k^{-*}}\|_{W^{1,p}}\|dJ_k^*\|_{L^\infty} + \|\overline{dJ_k^{-*}}\|_{L^\infty}\|dJ_k^*\|_{W^{1,p}}\\
&\quad + \|dJ_{k-1}^{-*}\|_{L^\infty}\|\overline{dJ_k^*}\|_{W^{1,p}} + \|dJ_{k-1}^{-*}\|_{W^{1,p}}\|\overline{dJ_k^*}\|_{L^\infty}\\
&\overset{(6.1)}{\leq} 2C_M\big(\|\overline{dJ_k^{-*}}\|_{W^{1,p}}\|dJ_k^*\|_{W^{1,p}} + \|dJ_{k-1}^{-*}\|_{W^{1,p}}\|\overline{dJ_k^*}\|_{W^{1,p}}\big)\\
&\leq 2C_M\big(\|\overline{J_k^{-*}}\|_{W^{2,p}}\|J_k^*\|_{W^{2,p}} + C\,\|J_{k-1}^{-*}\|_{W^{2,p}}\|\overline{J_k^*}\|_{W^{2,p}}\big)\\
&\overset{(6.35)}{\leq} 2C_M C\big(\|J_k^*\|_{W^{2,p}} + \|J_{k-1}^*\|_{W^{2,p}}\big)\|\overline{J_k^*}\|_{W^{2,p}}\\
&\leq 2C_M C\big(\|u_k\|_{W^{2,p}} + \|u_{k-1}\|_{W^{2,p}}\big)\|\overline{u_k}\|_{W^{2,p}}.
\end{aligned} \tag{6.55}$$

Combining (6.51) - (6.55) with (6.50) yields the sought after estimate (5.3). Taking $C_s > 0$ as the maximum over all constants (6.46) - (6.55) and the constant in (6.15) and (6.14), completes the proof of Lemma 5.1. $\square$

6.4. **Consistency of Induction Assumption - Proof of Lemma 5.2.** We now prove Lemma 5.2, which shows that the induction assumption (5.7) is maintained in each step of the iteration. Lemma 5.2 states that, if

$$0 < \epsilon \leq \epsilon_1,$$

i.e. (5.8) holds, and if the induction assumption (5.7) holds, namely

$$\|u_k\|_{W^{m+1,p}(\Omega)} \leq 4\,C_0 C_e^2,$$

then (5.9) - (5.10) holds, that is,

$$\|a_{k+l}\|_{W^{m,p}} \leq 2C_0C_e, \tag{6.56}$$

$$\|u_{k+l}\|_{W^{m+1,p}} \leq 4C_0C_e^2, \tag{6.57}$$

for all $l \in \mathbb{N}$, and the induction assumption (5.7) holds for each subsequent iterate.

Proof. To begin observe that the ϵ-bound (5.8), i.e.

$$0 < \epsilon \leq \epsilon_1 \equiv \min\left(\tfrac{1}{4C_e^2C_s(1+2C_eC_0+4C_e^2C_0)}, \tfrac{1}{16C_MC_0C_e^2}\right),$$

together with the induction assumption (5.7) imply

$$\epsilon \leq \frac{1}{4C_M \cdot 4C_0C_e^2} \overset{(5.7)}{\leq} \frac{1}{4C_M\|u_k\|_{W^{m+1,p}(\Omega)}} = \epsilon(k), \tag{6.58}$$

which is the ϵ bound (4.30) of Lemma 4.4, so that existence of iterates and the elliptic estimates (4.31) - (4.32) hold. Moreover, since $\|J_k^*\|_{W^{m+1,p}(\Omega)} \leq \|u_k\|_{W^{m+1,p}(\Omega)}$, (6.58) implies that

$$\epsilon \leq \epsilon(k) \leq \frac{1}{2C_M\|J_k^*\|_{W^{m+1,p}(\Omega)}}, \tag{6.59}$$

which is the ϵ-bound (6.2) of Lemma 6.2 in terms of $J^* = J_k^*$. Thus the source estimates (6.14) - (6.15) of Lemma 6.2 apply and yield that the right hand sides of the elliptic estimates (4.31) - (4.32) are indeed finite.

We now derive the uniform bound (6.56). From the elliptic estimate (4.31) together with the source estimate (6.15), we find that

$$\begin{aligned}\|a_{k+1}\|_{W^{m,p}} &\overset{(4.31)}{\leq} C_e\|F_a(u_k)\|_{W^{m-1,p}} \\ &\overset{(6.15)}{\leq} C_e\big(C_0 + \epsilon\, C_s(1 + \|u_k\|_{W^{m+1,p}})\|u_k\|_{W^{m+1,p}}\big),\end{aligned}$$

so application of the induction assumption (5.7) gives

$$\|a_{k+1}\|_{W^{m,p}} \leq C_eC_0 + \epsilon\, 4C_e^2C_s\big(1 + 4C_e^2C_0\big)C_eC_0. \tag{6.60}$$

Now, by the ϵ-bound (5.8), we have

$$\epsilon \leq \frac{1}{4C_e^2C_s(1+2C_eC_0+4C_e^2C_0)} \leq \frac{1}{4C_e^2C_s(1+4C_e^2C_0)},$$

so that substituting the above ϵ-bound into (6.60) yields

$$\|a_{k+1}\|_{W^{m,p}} \leq 2C_0C_e,$$

which is the sought after bound (6.56) for $l = 1$.

We now derive (5.10). From the elliptic estimate (4.32) together with the source estimate (6.14), we obtain that

$$\begin{aligned}\|u_{k+1}\|_{W^{m+1,p}} &\overset{(4.32)}{\leq} C_e\|F_u(u_k,a_{k+1})\|_{W^{m-1,p}}\\ &\overset{(6.14)}{\leq} C_eC_0 + C_e\|a_{k+1}\|_{W^{m,p}}\\ &\quad + \epsilon\, C_eC_s\big(1+\|a_{k+1}\|_{W^{m,p}}+\|u_k\|_{W^{m+1,p}}\big)\|u_k\|_{W^{m+1,p}}\\ &\overset{(5.7)}{\leq} C_eC_0 + 2C_e^2C_0 + \epsilon\, 4C_e^2C_s\big(1+2C_eC_0+4C_e^2C_0\big)C_eC_0,\end{aligned}\tag{6.61}$$

where we substituted $\|a_{k+1}\|_{W^{m,p}} \leq 2C_eC_0$ and the induction assumption $\|u_k\|_{W^{m+1,p}} \leq 4\,C_0C_e^2$ to obtain the last inequality. By (5.8), we have

$$\epsilon \leq \frac{1}{4C_e^2C_s(1+2C_eC_0+4C_e^2C_0)},$$

so that applying the above bound to ϵ in (6.61) gives

$$\|u_{k+1}\|_{W^{m+1,p}} \leq 2C_0C_e(1+C_e) \leq 4C_0C_e^2,$$

where the last inequality holds since we chose $C_e > 1$ initially. This is the sought after bound (6.57) for $l = 1$.

The bound $\epsilon \leq \epsilon(k+l)$ for $l \in \mathbb{N}$ together with (6.56) and (6.57) for $l \in \mathbb{N}$ follow now recursively, which completes the proof of Lemma 5.2. □

6.5. **Decay of the difference between iterates - Proof of Proposition 5.3.** We now prove Lemma 5.3, which completes the proof of Theorem 5.4. Lemma 5.3 states that, if $0 < \epsilon \leq \epsilon_1$, (i.e. (5.8) holds), then there exists a constant $C_d > 0$ depending only on m, n, p, Ω such that (5.11) and (5.12) hold, i.e.,

$$\begin{aligned}\|\overline{a_{k+1}}\|_{W^{m,p}} &\leq \epsilon\, C_d\, \|\overline{u_k}\|_{W^{m+1,p}},\\ \|\overline{u_{k+1}}\|_{W^{m+1,p}} &\leq \epsilon\, C_d\, \|\overline{u_k}\|_{W^{m+1,p}}.\end{aligned}$$

Proof of Lemma 5.3: We first establish estimate (5.11) on $\|\overline{a_{k+1}}\|_{W^{m,p}}$. By linearity of the Laplacian it is straightforward to extend the elliptic estimate (4.31) to $\overline{a_{k+1}}$ and obtain

$$\|\overline{a_{k+1}}\|_{W^{m,p}} \leq C_e\|\overline{F_a(u_k)}\|_{W^{m-1,p}},$$

for the same constant $C_e > 0$ as in (4.31). Applying the non-linear source estimate (5.4), we further find that

$$\|\overline{a_{k+1}}\|_{W^{m,p}} \leq \epsilon\, C_eC_a(k)\, \|\overline{u_k}\|_{W^{m+1,p}},\tag{6.62}$$

where $C_a(k)$ is defined in (5.6) as

$$C_a(k) = C_s\big(1+\|u_k\|_{W^{m+1,p}}+\|u_{k-1}\|_{W^{m+1,p}}\big).\tag{6.63}$$

Applying the induction hypothesis (5.7) we bound $C_a(k)$ by

$$C_a(k) \leq C_s\big(1+8C_0C_e^2\big),\tag{6.64}$$

which in combination with (6.62) implies the sought after estimate (5.11).

We now prove estimate (5.12) on $\|\overline{u_{k+1}}\|_{W^{m+1,p}}$. By linearity of the Laplacian, the elliptic estimate (4.32) extends to $\overline{u_{k+1}}$, so that the source estimate (5.3) implies

$$\begin{aligned}\|\overline{u_{k+1}}\|_{W^{m+1,p}} &\leq C_e\Big(\|\overline{F_u(u_k,a_{k+1})}\|_{W^{m-1,p}} + \|\overline{u_{k+1}}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big)\\ &\overset{(5.3)}{\leq} C_e C_u(k)\Big(\epsilon\,\|\overline{u_k}\|_{W^{m+1,p}} + \|\overline{a_{k+1}}\|_{W^{m,p}}\Big) + C_e\,\|\overline{dy_{k+1}}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)},\end{aligned} \tag{6.65}$$

where we substituted the boundary conditions (4.26) - (4.27) for the last step, and where $C_u(k)$ is defined in (5.5) as

$$C_u(k) = C_s\big(1 + \|u_k\|_{W^{m+1,p}} + \|u_{k-1}\|_{W^{m+1,p}} + \|a_{k+1}\|_{W^{m,p}}\big).$$

Using now the induction assumption (5.7) together with the bound $\|a_{k+1}\|_{W^{m,p}} \leq 2C_0C_s$ from Lemma 5.2, we obtain the uniform bound

$$C_u(k) \leq C_s\big(1 + 8C_0C_e^2 + 2C_0C_e\big). \tag{6.66}$$

Substituting (6.66) together with estimate (5.11) on $\|\overline{a_{k+1}}\|_{W^{m,p}}$ in (6.65) gives us

$$\|\overline{u_{k+1}}\|_{W^{m+1,p}} \quad\leq\quad \epsilon C\|\overline{u_k}\|_{W^{m+1,p}} + C_e\,\|\overline{dy_{k+1}}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}. \tag{6.67}$$

It remains to estimate the boundary term in (6.67). By Theorem 2.3, we obtain the trace estimate

$$\begin{aligned}\|\overline{dy_{k+1}}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} &\leq \|\overline{dy_{k+1}}\|_{W^{m+1,p}(\Omega)}\\ &\leq \|\overline{y_{k+1}}\|_{W^{m+2,p}(\Omega)}.\end{aligned} \tag{6.68}$$

By linearity of the Laplacian, (4.24) implies that $\overline{y_{k+1}}$ solves

$$\begin{cases}\Delta\overline{y_{k+1}} = \overline{\psi_{k+1}},\\ \overline{y_{k+1}}\big|_{\partial\Omega} = 0,\end{cases} \tag{6.69}$$

so that the elliptic estimate (2.11) yields

$$\|\overline{y_{k+1}}\|_{W^{m+2,p}(\Omega)} \leq C_e\|\overline{\psi_{k+1}}\|_{W^{m,p}(\Omega)}. \tag{6.70}$$

Since the elliptic estimate (4.33) extends to $\overline{\psi_{k+1}}$, we can bound the right hand side in (6.70) and find

$$\begin{aligned}\|\overline{y_{k+1}}\|_{W^{m+2,p}(\Omega)} &\overset{(4.33)}{\leq} C_e^2\|\overline{F_u(u_k,a_{k+1})}\|_{W^{m-1,p}(\Omega)}\\ &\overset{(5.3)}{\leq} C_e^2C_u(k)\Big(\epsilon\,\|\overline{u_k}\|_{W^{m+1,p}} + \|\overline{a_{k+1}}\|_{W^{m,p}}\Big).\end{aligned} \tag{6.71}$$

Substituting (6.71) back into (6.68), and bounding $C_u(k)$ by (6.66) and $\|\overline{a_{k+1}}\|_{W^{m,p}}$ by (5.11), we obtain the boundary estimate

$$\|\overline{dy_{k+1}}\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)} \leq \epsilon\, C\,\|\overline{u_k}\|_{W^{m+1,p}(\Omega)}, \tag{6.72}$$

for some suitable constant $C > 0$. Finally, using (6.72) to estimate the boundary contribution in (6.67), we obtain for some suitable constant $C_d > 0$ that

$$\|\overline{u_{k+1}}\|_{W^{m+1,p}(\Omega)} \;\; \leq \;\; \epsilon\, C_d\, \|\overline{u_k}\|_{W^{m+1,p}(\Omega)},$$

which is the sought after estimate (5.12). This completes the proof. $\square$

7. Proof of Theorem 1.3

The proofs in Sections 6.1 - 6.5 complete the proof of Theorem 5.4. We now prove our main theorem regarding existence of solutions of the RT-equations (1.1) - (1.5), Theorem 1.3, which follows from Theorem 5.4 together with a rescaling argument to arrange for the smallness assumption (4.8), that is, $\Gamma = \epsilon\, \Gamma^*$, and the uniform bound (4.7), i.e.

$$\|\Gamma^*\|_{W^{m,p}(\Omega)} \; + \; \|d\Gamma^*\|_{W^{m,p}(\Omega)} < C_0,$$

which are the incoming assumptions of Theorem 5.4. In more detail, given any connection $\Gamma' \in W^{m,p}(\Omega)$ with $d\Gamma'$ bounded in $W^{m,p}(\Omega)$, we define Γ^* as the restriction of Γ' to the ball of radius ϵ, but with its components transformed as scalars to the ball or radius 1 (which we take to be Ω), while Γ is taken to be the connection resulting from transforming Γ' as a connection. The proof below shows that this construction suffices to arrange for assumptions (4.7) and (4.8).

Proof of Theorem 1.3. By Theorem 5.4, for any connection Γ satisfying (4.8) for $\epsilon < \min(\epsilon_1, \epsilon_2)$ together with the $W^{m,p}$-bound (4.7), there exists $(\tilde{\Gamma}^*, J^*, A^*)$ which solve the rescaled RT-equations (4.12) - (4.13) with boundary data (1.5). Defining $(\tilde{\Gamma}, J, A)$ by (4.9) as

$$J = I + \epsilon\, J^*, \qquad \tilde{\Gamma} = \epsilon\, \tilde{\Gamma}^*, \qquad A = \epsilon A^*,$$

Lemma 4.1 implies that $(\tilde{\Gamma}, J, A)$ solves the RT-equations (1.1) - (1.4) with boundary data (1.5). It remains to show that, for any connection $\Gamma \in W^{m,p}(\Omega)$ with $d\Gamma \in W^{m,p}(\Omega)$, one can arrange for the hypotheses of Theorem 4.2, that is, the scaling $\Gamma = \epsilon\, \Gamma^*$ together with the uniform bound (4.7) on Γ^* as well as the ϵ-bounds (5.8) and (5.13), i.e., $\epsilon < \min(\epsilon_1, \epsilon_2)$.

We now show that, given a connection $\Gamma \in W^{m,p}(\Omega)$ with $d\Gamma \in W^{m,p}(\Omega)$, one can first restrict Γ to a small region and then scale the restriction of Γ to a large region (which we take to be Ω) such that the resulting Γ satisfies the hypotheses of Theorem 4.2. For this, we assume without loss of generality that $\Omega \equiv B_1(0)$ is the ball of radius 1 and we denote with $B_\epsilon(0)$ the ball of radius ϵ, for $0 < \epsilon \leq 1$. Under a coordinate transformation $x \mapsto y \equiv \epsilon x$, (which maps $B_1(0)$ in x-coordinates to $B_\epsilon(0)$ in y-coordinates), a connection $\Gamma(y)$ given in y-coordinates transforms as [11, 21]

$$\Gamma(x)^\sigma_{\mu\nu} \;\; = \;\; \frac{\partial x^\sigma}{\partial y^\gamma}\Big(\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\Gamma(y)^\gamma_{\alpha\beta} + \frac{\partial^2 y^\gamma}{\partial x^\mu \partial x^\nu}\Big)$$

which for the transformation $x \mapsto y \equiv \epsilon x$ reduces to the scaling

$$\Gamma(x)^\sigma_{\mu\nu} = \epsilon\, \Gamma(y)^\sigma_{\mu\nu}. \tag{7.1}$$

We now arrange for conditions (4.7) and (4.8) on $\Omega = B_1(0)$ in x-coordinates and we assume that the connection Γ we start with is given in coordinates y on Ω. For this, take $\Gamma(y)$ to be the restriction of Γ to $B_\epsilon(0)$ in y-coordinate, and define $\Gamma^*(x) \equiv \Gamma(y(x))$. That is, Γ^* is the connection $\Gamma(y)$ in x-coordinates, defined on $\Omega \equiv B_1(0)$, but with the components of $\Gamma(y)$ transformed as scalar functions—not as connection components. Moreover, the connection $\Gamma(x)$ that results from transforming the restriction of Γ to $B_\epsilon(0)$ in y-coordinate to x-coordinates according to the connection transformation law (7.1) satisfies the sought after scaling (4.8). Thus, taking $\Gamma(x)$ as the (initial) connection assumed in Theorem 5.4, we only need to show that Γ^* satisfies the uniform $W^{m,p}$-bound (4.7) in order to verify the hypotheses of Theorem 5.4.

To show that Γ^* satisfies (4.7) for the case $m = 1$, we now study the ϵ-scaling of the $W^{1,p}$-norm when the ball of radius ϵ is scaled up to the unit ball. For this, let $u \in W^{1,p}(B_\epsilon(0))$ be a scalar function, $p > n$. By Morrey's inequality (2.10), u is Hölder continuous, so the L^p-norm of u scales as

$$\begin{aligned} \|u\|_{L^p(B_\epsilon(0))} &\leq \|u\|_{L^\infty} \mathrm{vol}(B_\epsilon(0)) \\ &\leq \|u\|_{L^\infty} \mathrm{vol}(B_1(0))\, \epsilon^{\frac{n}{p}} \;=\; o\big(\epsilon^{\frac{n}{p}}\big), \end{aligned} \tag{7.2}$$

while we have by assumption

$$\|Du\|_{L^p(B_\epsilon(0))} = o(1), \tag{7.3}$$

i.e., bounded by a constant and tending to zero as $\epsilon \to 0$. Now under the transformation $y \mapsto x = \frac{y}{\epsilon}$, (which maps the ball of radius $\epsilon > 0$ in y-coordinates to the unit ball in x-coordinate), we have

$$\|u\|_{L^p(B_1(0))} = \epsilon^{-\frac{n}{p}} \|u\|_{L^p(B_\epsilon(0))} \overset{(7.2)}{\leq} \mathrm{vol}(B_1(0)) \|u\|_{L^\infty}, \tag{7.4}$$

and, since the scaling of first order derivatives cancels the scaling of the measure on $\mathbb{R}^n$ within an error of order ϵ^α, for $\alpha \equiv 1 - \frac{n}{p} > 0$, we further have

$$\begin{aligned} \|Du\|_{L^p(B_1(0))} &= \Big(\int_{B_1(0)} |D_x u|^p dx \Big)^{\frac{1}{p}} \\ &= \Big(\int_{B_1(0)} \epsilon^p |D_{\epsilon x} u(x)|^p \epsilon^{-n} d(\epsilon x) \Big)^{\frac{1}{p}} \\ &= \epsilon^{\frac{p-n}{p}} \|D_y u\|_{L^p(B_\epsilon(0))} \\ &\leq \|D_y u\|_{L^p(B_\epsilon(0))}, \end{aligned} \tag{7.5}$$

for all $0 < \epsilon \leq 1$, where D_x denotes differentiation with respect to x. Combining (7.4) - (7.5) we obtain

$$\|u\|_{W^{1,p}(B_1(0))} \;\leq\; C\big(\|u\|_{L^\infty} + \|D_y u\|_{L^p(B_\epsilon(0))} \big), \tag{7.6}$$

where $C > 0$ is a constant independent of ϵ.

Applying now (7.6) to Γ^* component-wise, we find for $\Gamma^*(x) = \Gamma(y(x))$, where $x \in \Omega = B_1(0)$, that

$$\begin{aligned}\|\Gamma^*\|_{W^{1,p}(\Omega)} &= \|\Gamma(y(\cdot))\|_{W^{1,p}(B_1(0))} \\ &\overset{(7.6)}{\leq} C\big(\|\Gamma(y)\|_{L^\infty} + \|D_y\Gamma(y)\|_{L^p(B_\epsilon(0))}\big)\end{aligned}$$

where $\|\Gamma(y)\|_{L^\infty}$ is the supremum of Γ in y-coordinates over $B_\epsilon(0)$, so that we can bound the right hand side further by taking the supremum and L^p-norm over Ω, namely,

$$\begin{aligned}\|\Gamma^*\|_{W^{1,p}(\Omega)} &\leq C\big(\|\Gamma(y)\|_{L^\infty(\Omega)} + \|D_y\Gamma(y)\|_{L^p(\Omega)}\big) \\ &\overset{(6.1)}{\leq} 2CC_M\|\Gamma(y)\|_{W^{1,p}(\Omega)}, \end{aligned} \tag{7.7}$$

by Morrey's inequality. Defining now C_0 in terms of the initial connection in y-coordinates as

$$C_0 \equiv 2CC_M\big(\|\Gamma(y)\|_{W^{1,p}(\Omega)} + \|d\Gamma(y)\|_{W^{1,p}(\Omega)}\big), \tag{7.8}$$

which is independent of ϵ, (7.7) implies that

$$\|\Gamma^*\|_{W^{1,p}(\Omega)} \leq C_0.$$

Likewise, applying (7.6) component-wise to $d\Gamma^*$, we obtain

$$\|d\Gamma^*\|_{W^{1,p}(\Omega)} \leq \|d\Gamma\|_{W^{1,p}(\Omega)} \leq C_0. \tag{7.9}$$

Combining (7.7) with (7.9) gives the sought after bound (4.7) for $m = 1$.

The general case $m \geq 1$, follows similarly by applying (7.6) component-wise to higher derivatives, $\partial^l\Gamma^*$ and $\partial^l(d\Gamma^*)$ for $l = 0, ..., m-1$, (keeping in mind that these terms are Hölder continuous, where ∂^l shall be understood as standard multi-index notation), and defining C_0 in (7.8) in terms of the $W^{m,p}$-norm of Γ and $d\Gamma$. To summarize, we proved that one can always arrange for the smallness assumption (4.8) - (4.7) required in Theorem 5.4, by first restricting a given connection to a ball of radius ϵ, and taking the transformation of this connection to the ball of radius 1 as the starting connection in Theorem 5.4, while taking for Γ^* the scalar transformed components of the restricted connection.

Finally, observe that the ϵ-bounds (5.8) and (5.13) depend only on the constants C_M, C_0, C_s and C_e, which in turn depend only on m, n, p and Ω. Since $\Omega = B_1(0)$ is kept fixed throughout the argument, we can first choose some ϵ small enough to satisfy the bounds (5.8) and (5.13), and then arrange for the scaling (4.8) for Γ by applying the argument (7.2) - (7.9). In summary, we proved that the hypotheses of Theorem 5.4 are satisfied, which completes the proof of Theorem 1.3. □

8. Applications to the Initial Value Problem in General Relativity

In this section we apply Theorem 1.1 to give a new theorem establishing the optimal smoothness of spherically symmetric solutions generated by the Einstein equations $G = \kappa T$ in Standard Schwarzschild Coordinates (SSC) with arbitrary source terms T. The issues around optimal regularity addressed by the RT-equations are represented nicely in SSC coordinates because three of the four Einstein equations $G = \kappa T$ are first order in the metric, and thus metric solutions are only one order smoother than the curvature tensor. We begin with a discussion of the central issue involved.

The Einstein equations $G = \kappa T$ of General Relativity are covariant tensorial equations defined independent of coordinates. The unknowns in the equations are the metric tensor g, and these are coupled to the variables which determine the sources in T. For example, in the case of a perfect fluid, the unknowns are g_{ij}, ρ, p, u_i, where [11, 4]

$$T = (\rho + p)u^i u^j + pg^{ij}.$$

The existence of solutions of the Einstein equations are established by PDE methods in coordinate systems in which the Einstein equations take on a solvable form. The coordinate systems are typically specified by an ansatz for the metric, for example, SSC coordinates for spherically symmetric spacetimes, or harmonic coordinates, wave-gauge coordinates, etc., for the general initial value problem in four dimensions, [11, 4]. Since solutions typically only exist locally in GR, it is important to know whether the breakdown is simply a breakdown of the coordinate system. This is important both to the theory of the initial value problem in GR, and to numerical relativity.

The question we ask here is–how do we know the gravitational metric, which is the solution of the equations in a given coordinate system, exhibits its optimal smoothness in the coordinate system in which it is constructed?

This is no mute point. Indeed, assume for example that one were to construct a solution to the Einstein equations $G = \kappa T$ in a given coordinate system x in which the equations produce unique solutions (locally) within a given smoothness class, starting from initial data. For example, assume the equations produce solutions of optimal smoothness with metric $g \in W^{m+2,p}$, connection $\Gamma \in W^{m+1,p}$, and $Riem(\Gamma) \in W^{m,p}$. Then application of a transformation $x \to y$ with Jacobian $J \in W^{m+1,p}$ will in general lower the regularity of the whole solution space, lowering the regularity of the metric and its connection Γ by one order, but the transformation will preserve the regularity of the curvature tensor $Riem(\Gamma) \in W^{m,p}$, because the connection involves *derivatives* of the Jacobian of the coordinate transformation, but the metric and Riemann curvature tensor, being tensors, involve only the undifferentiated Jacobian.[2] Therefore, if one were to then express the Einstein equations in the transformed coordinates y in which the metric is one order less smooth than optimal, the resulting existence theory posed in

[2]Alternatively, the anti-symmetric operator d applied to the symmetric leading order term in the formula for the transformed connection, kills the highest order derivatives in the formula for the transformed curvature tensor.

y-coordinates, *by construction*, would produce the unique transformed solution $g \in W^{m+1,p}$, $\Gamma \in W^{m,p}$, and $Riem(\Gamma) \in W^{m,p}$. Therefore, and this is the main point, *if* we were to construct our solutions in the y-coordinates *in the first place*, then we would not know that our unique solution was one order below optimal smoothness without knowing about the existence of the inverse transformation $y \to x$. It is precisely the existence of this transformation from y back to x that is guaranteed by Theorem 1.1, because its existence follows from existence for the RT-equations for $\Gamma \in W^{m,p}$, $d\Gamma \in W^{m,p}$, $m \geq 1$, $p > n$. Theorem 1.1 tells us that it is sufficient to solve the Einstein equations in a *weaker* sense than optimal, by stating that it is sufficient to solve a version of the Einstein equations which only produce metrics and connections one order less smooth than optimal. If Γ and $d\Gamma$ are in L^∞, then this is the difference between weak and strong solutions in the true sense of the theory of distributions, [14].

To amplify this point, the fact that the Einstein equations admit coordinate systems in which the metric is one degree less smooth than optimal, leads one to anticipate that the Einstein equations might be easier to solve at this lower level of smoothness.[3] In certain cases, the Einstein equations might actually take their simplest form in coordinate systems which produce only one metric derivative above the curvature tensor–because in coordinates where the metric is one order less smooth, the equations need impose *fewer constraints*. We now show that this is *precisely* what happens in spherically symmetric spacetimes in SSC, the example we now discuss in detail.

Consider then the case of time dependent spherically symmetric spacetimes in which the gravitational metric takes the general form

$$ds^2 = -B(t,r)dt^2 + \frac{dr^2}{A(t,r)} + E(t,r)dtdt + C(t,r)d\Omega^2, \tag{8.1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\theta^2$$

is the standard line element on the unit sphere. Then generically, when $C_r \neq 0$, (the most general case is not of interest here), there exists a coordinate transformation to coordinates in which the metric takes the Standard Schwarzschild Coordinate form [21]

$$ds^2 = -B(t,r)dt^2 + \frac{dr^2}{A(t,r)} + r^2 d\Omega^2, \tag{8.2}$$

and this represents the coordinates in which the Einstein equations (arguably) take their simplest form.[4] In SSC, the Einstein equations reduce

[3] Indeed, for elliptic equations, the Lax-Milgram Theorem is an example in which it is easier to establish the gain of one derivative in u over f in $\Delta u = f$, but the second derivative gain requires the development of elliptic regularity theory, [7].

[4] The authors invite the reader to put the metric ansatz into MAPLE to compute the Einstein equations in general case (8.1), to see that the equations are *significantly* more complicated in general coordinate systems than in SSC.

to a "locally inertial" formulation derived by Groah and Temple in [10] as follows.

According to [10], three of the four Einstein equations determined by $G=\kappa T$ are first order in A and B, and one is second order. The first order equations are equivalent to,[5]

$$\left\{-r\frac{A_r}{A}+\frac{1-A}{A}\right\} = \frac{\kappa B}{A}T^{00}r^2=\frac{\kappa}{A}T_M^{00}r^2 \tag{8.3}$$

$$\frac{A_t}{A} = \frac{\kappa B}{A}T^{01}r=\kappa\sqrt{\frac{B}{A}}T_M^{01}r \tag{8.4}$$

$$\left\{r\frac{B_r}{B}-\frac{1-A}{A}\right\} = \frac{\kappa}{A^2}T^{11}r^2=\frac{\kappa}{A}T_M^{11}r^2, \tag{8.5}$$

and the the two conservation laws $Div\,T=0$ are equivalent to

$$\{T_M^{00}\}_{,0}+\left\{\sqrt{AB}T_M^{01}\right\}_{,1}=-\frac{2}{r}\sqrt{AB}T_M^{01}, \tag{8.6}$$

$$\begin{aligned}\{T_M^{01}\}_{,0}+\left\{\sqrt{AB}T_M^{11}\right\}_{,1}=-\frac{1}{2}\sqrt{AB}\Bigg\{&\frac{4}{x}T_M^{11}+\frac{(\frac{1}{A}-1)}{r}(T_M^{00}-T_M^{11})\\ &+\frac{2\kappa r}{A}(T_M^{00}T_M^{11}-(T_M^{01})^2)-4rT^{22}\Bigg\},\end{aligned} \tag{8.7}$$

where $T_M^{\alpha\beta}$ is the Minkowski stress tensor defined by, (c.f. [10]),

$$T_M^{00}=BT^{00},\;\; T_M^{01}=\sqrt{\frac{B}{A}}T^{01},\;\; T_M^{11}=\frac{1}{A}T^{11},\;\; T_M^{22}=T^{22},$$

and we employ the standard notation

$$\frac{\partial}{\partial t}\{\cdot\}=\{\cdot\}_{,0}=\{\cdot\}_{,t}\,,\quad \frac{\partial}{\partial r}\{\cdot\}=\{\cdot\}_{,1}=\{\cdot\}_{,r}\,.$$

By the Bianchi identities, equations (8.3) - (8.7) follow from $Div\,T=0$ which follows as an identity from $G=\kappa T$. In [10] it was shown that the Einstein equations $G=\kappa\,T$ for metrics in SSC are equivalent to the system (8.3), (8.5), (8.6), (8.7), in the weak sense when $T\in L^\infty$. In addition, the system closes when an equation of state $p=p(\rho)$ is imposed, and the first order equation (8.4) follows as an identity, (c.f. [10]).

The SSC equations (8.3), (8.5), (8.6), (8.7) were introduced in [10] to prove the first existence theorem for shock wave solutions of the Einstein equations using the Glimm scheme, (c.f.[19, 12, 14, 20, 3]). Groah and Temple remarked that the equations could only be solved in coordinates in which the metric appeared to be singular at shock waves, (in the sense that, although no delta function sources appear in the L^∞ curvature tensor, the metric is only Lipschitz continuous, and this is only one derivative smoother

[5]In [10], the SSC metric ansatz is taken to be $ds^2=A(t,r)dt^2+B(t,r)dr^2+r^2d\Omega^2$, so to recover the formulas from [10], make the substitutions $A\to\frac{1}{B}$, $B\to A$

than the curvature). It is still an open question whether these $C^{0,1}$ metric solutions of $G = \kappa T$ can always be smoothed one order to $C^{1,1}$ by coordinate transformation, and based on this, authors in [14, 15], posed the problem of *Regularity Singularities.*

As an application of Theorem 1.1, note that if $T \in W^{m,p}$, $m \geq 1$, $p > n = 4$, then solutions of (8.3), (8.5), (8.6), (8.7), would in general have $(A, B) \in W^{m+1,p}$, $\Gamma \in W^{m,p}$, and since $G = \kappa T$, also $G \in W^{m,p}$. Putting the full Riemann curvature tensor into MAPLE one sees by inspection that the terms of lowest regularity in G match the terms of lowest regularity in $Riem(\Gamma)$, so in general, $Riem(\Gamma) \in W^{m,p}$. For such solutions of the SSC equations, we have that Γ and $d\Gamma$ have the same regularity $W^{m,p}$, and the metric $g \in W^{m+1,p}$ is only one derivative more regular. Thus solutions of the SSC equations with $T \in W^{m,p}$, $m \geq 1$, $p > 4$, is an example that fits the assumptions of Theorem 1.1. The result is a new regularity result for solutions of the SSC equations which we record in the following theorem:

Theorem 8.1. *Assume $T \in W^{m,p}$, $m \geq 1$, $p > 4$, and let $g \equiv (A, B)$ be a solution of the SSC equations (8.3), (8.5), (8.6), (8.7) satisfying*

$$g \in W^{m+1,p}, \quad \Gamma \in W^{m,p}, \quad d\Gamma \in W^{m,p},$$

in an open set Ω. Then for each $q \in \Omega$ there exists a coordinate transformation $x \to y$ defined in a neighborhood of q, such that, in y-coordinates, $g \in W^{m+2,p}$, $\Gamma \in W^{m+1,p}$, $Riem(\Gamma) \in W^{m,p}$.

Appendix A. Proof of elliptic estimate (2.11)

For completeness, we now give a simple proof of estimate (2.11) for the critical case $m = 1$, assuming Hölder continuity, starting from the fundamental elliptic estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{2-\frac{1}{p},p}(\partial\Omega)}\Big), \tag{A.1}$$

which applies to $u \in W^{2,p}(\Omega)$, c.f. equation (2,3,3,1) in [9]. Estimate (A.1) is the estimate stated in most text-books on elliptic regularity theory [7, 8, 9, 22].

Lemma A.1. *Let $f \in L^p(\Omega)$ and $u_0 \in W^{2-\frac{1}{p},p}(\partial\Omega)$, for $p > n$, be scalar valued functions. Assume the scalar $u \in W^{2,p}(\Omega)$ solves*

$$\begin{cases} \Delta u = f, \\ u|_{\partial\Omega} = u_0. \end{cases} \tag{A.2}$$

Then there exists a constant $C > 0$, depending only on Ω, n, p, such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|f\|_{L^p(\Omega)} + \|u_0\|_{W^{2-\frac{1}{p},p}(\partial\Omega)}\Big). \tag{A.3}$$

Proof. Assume (A.1) holds for $u \in W^{2,p}(\Omega)$. By interpolation, there exists a constant $K > 0$, depending only on n, p, Ω, such that

$$\|u\|_{W^{1,p}(\Omega)} \leq \epsilon \|u\|_{W^{2,p}(\Omega)} + \tfrac{K}{\epsilon} \|u\|_{L^p(\Omega)}, \tag{A.4}$$

for all $\epsilon > 0$, c.f. (2,3,3,8) in [9]. Combing (A.4) with (A.1) and choosing $\epsilon > 0$ small enough to subtract $\epsilon \|u\|_{W^{2,p}}$ from the resulting inequality, we obtain the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \|u\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} \Big), \tag{A.5}$$

after absorbing $\frac{K}{\epsilon}$ into the constant $C > 0$.

It remains to bound $\|u\|_{L^p(\Omega)}$ in terms of the boundary data and the source function. For this, assume that $w \in W^{2,p}(\Omega)$ solves

$$\begin{cases} \Delta w = f, \\ w|_{\partial\Omega} = 0. \end{cases} \tag{A.6}$$

Lemma 9.17 in [8] implies that[6]

$$\|w\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)} \tag{A.7}$$

for some constant $C > 0$ depending only on Ω, m, n, p. Moreover, let $v \in C^\infty(\Omega)$ solve

$$\begin{cases} \Delta v = 0, \\ v|_{\partial\Omega} = u_0, \end{cases} \tag{A.8}$$

then

$$\begin{cases} \Delta (w+v) = f \\ (w+v)|_{\partial\Omega} = u_0, \end{cases}$$

so we can conclude that

$$u = w + v, \tag{A.9}$$

by uniqueness of solutions of the Poisson equation. Now, from (A.7), we obtain the estimate

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\leq \|w\|_{W^{2,p}(\Omega)} + \|v\|_{W^{2,p}(\Omega)} \\ &\overset{(A.7)}{\leq} C\|f\|_{L^p(\Omega)} + \|v\|_{W^{2,p}(\Omega)}. \end{aligned} \tag{A.10}$$

Applying (A.5), we find that

$$\|v\|_{W^{2,p}(\Omega)} \leq C\Big(\|v\|_{L^p(\Omega)} + \|u_0\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} \Big). \tag{A.11}$$

[6]To clarify, Lemma 9.17 in [8] applies to $u \in W^{1,p}_0(\Omega)$, where $W^{1,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the $W^{1,p}$-norm. However, since u is Hölder continuous (for $p > n$ by our assumptions), it follows that $u \in W^{1,p}(\Omega)$ together with $u|_{\partial\Omega} = 0$ implies that $u \in W^{1,p}_0(\Omega)$, (see also Theorem 1.5.2 in [22] for the L^2 case). Also, the inverse implication is true by Hölder continuity, as can be shown by contradiction. So Lemma 9.17 in [8] applies to our setting in Lemma A.1.

We estimate further that

$$\|v\|_{L^p(\Omega)} \quad \leq \quad \mathrm{vol}(\Omega)\|v\|_{L^\infty(\Omega)},$$

and applying the maximums principle, (that is, harmonic functions attain their maximum on the boundary), we get

$$\begin{aligned} \|v\|_{L^p(\Omega)} \quad &\leq \quad \mathrm{vol}(\Omega)\|v\|_{L^\infty(\partial\Omega)} \\ &\overset{(6.1)}{\leq} \quad C_M \mathrm{vol}(\Omega)\|v\|_{W^{1,p}(\partial\Omega)}, \end{aligned} \tag{A.12}$$

where we applied Morrey's inequality (6.1) with respect to $\partial\Omega$ in the last step. Combining now (A.11) with (A.12) and substituting $v|_{\partial\Omega} = u_0$, we obtain

$$\|v\|_{W^{2,p}(\Omega)} \quad \leq \quad C_M \mathrm{vol}(\Omega)\|u_0\|_{W^{1,p}(\partial\Omega)}. \tag{A.13}$$

Substituting (A.13) into (A.10), we finally obtain

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|f\|_{L^p(\Omega)} + \|u_0\|_{W^{1,p}(\partial\Omega)}\Big), \tag{A.14}$$

which implies the sought after estimate (A.3). This completes the proof. $\square$

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Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisbon, Portugal
E-mail address: `moritzreintjes@gmail.com`

Department of Mathematics, University of California, Davis, CA 95616, USA
E-mail address: `temple@math.ucdavis.edu`