UHLENBECK COMPACTNESS AND OPTIMAL REGULARITY FOR YANG-MILLS THEORY WITH LORENTZIAN GEOMETRY AND NON-COMPACT LIE GROUPS

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Abstract. We extend the authors' prior theory of the RT-equations from the setting of affine connections, to the general setting of connections defined on vector bundles over arbitrary manifolds, including Yang-Mills connections over Lorentzian manifolds in Physics. By this, our theory of the RT-equations extends optimal regularity and Uhlenbeck compactness from the case of vector bundles over Riemannian manifolds with compact Lie group, to vector bundles over *arbitrary* manifolds, allowing for *both* compact *and* non-compact Lie groups. Our results here apply to L^p connections, $p > n$.

Contents

1. Introduction

In prior work [11, 12, 13, 14] the authors derived the RT-equations, a Poisson-type nonlinear elliptic system of PDE's whose solutions determine coordinate transformations which lift a non-optimal affine connection Γ, (one for which the connection components are no more regular than the leading order term $d\Gamma$ in its Riemann curvature tensor $\mathrm{Riem}(\Gamma)$), to optimal regularity. Our prior results for the RT-equations [13, 14] based on

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the theory of elliptic regularity in L^p spaces then extend the theorems of Kazdan-DeTurck [1] and Uhlenbeck [18] on optimal regularity and Uhlenbeck compactness from the case of vector bundles with *compact* Lie group over *positive* definite Riemannian manifolds, to the case of general affine connections, i.e., arbitrary connections Γ defined on the tangent bundle of an arbitrary manifold (no metric required), including the Lorentzian metric connections of General Relativity– but our prior work was not general enough to handle the case of connections $\mathcal{A}$ on *vector bundles.* The existence theory for the case $(\Gamma, d\Gamma) \in W^{m,p}$, $m \geq 1$, $p > n$, was accomplished in [12], and existence for the case $(\Gamma, d\Gamma) \in L^\infty$ was accomplished in [14]. Although the case of L^∞ affine connections Γ was general enough to resolve the authors' original problem of establishing the optimal regularity of shock wave solutions of the Einstein equations in General Relativity[1] [4, 6, 8, 9, 10, 16], the vector bundle setting of Uhlenbeck's original work [18],with its numerous applications [2, 17, 19], naturally begs the question as to whether the theory of the RT-equations, including the optimal regularity and Uhlenbeck compactness they imply, might extend from the case of affine connections on an arbitrary manifold, to the case of connections $\mathcal{A}$ on vector bundles over such arbitrary manifolds.

In this paper we accomplish this extension of our theory to the case of vector bundles with non-optimal connections in L^p, $p > n$, (an improvement over L^∞). The theory is general enough to establish optimal regularity and Uhlenbeck compactness for connections on vector bundles associated with *both* compact and non-compact Lie groups, over arbitrary manifolds–a setting general enough to incorporate Yang Mills connections over Lorentzian manifolds of relativistic Physics.[2] Taken together with our earlier results, this establishes optimal regularity for connections $\mathcal{A} \in L^p$ on the fibres, and $\Gamma \in L^\infty$ on the base manifold.

We accomplish this by deriving a vector bundle version of the RT-equations whose solutions produce gauge transformations which lift the regularity of non-optimal connections up to optimal regularity, and by the extra derivative obtained, extend Uhlenbeck compactness to the general setting of connections $\mathcal{A}$ on vector bundles, with $(\mathcal{A}, d\mathcal{A}) \in L^p$, $p > n$. That is, by proving an existence theorem based on elliptic regularity in L^p-spaces for the vector bundle version of the RT-equations, analogous to our theory in [13, 14], we establish the existence of gauge transformations which lift connections of non-optimal regularity to optimal regularity[3] $\mathcal{A} \in W^{1,p/2}$. The

[1] I.e., the case of general Lipschitz continuous metrics which solve the Einstein equations in the sense of distributions.

[2] To be concrete we restrict our attention here to vector bundles with compact and non-compact gauge groups $SO(r, s)$, $r, s \geq 0$. Our results extend to complex vector bundles with Lie groups $U(r, s)$ and $SU(r, s)$, i.e. to the Lie groups important in Physics, by the same argument. Our methods of proof can be extended to more general matrix Lie groups.

[3] By *optimal regularity*, we mean that the connection components have one full derivative of regularity above the curvature tensor. Our result is sharp in the following sense.

RT-equations in the vector bundle case are actually simpler in the sense that they do not couple to a first order Cauchy-Riemann type equation required in [13, 14] to impose integrability of Jacobians to coordinates. Thus the reader trying to understand the theory of the RT-equations in detail might do well to consider the vector bundle case here first.

In contrast to [18], our method does not require compactness of the Lie group, nor positive definiteness of a metric on the base manifold. No metric structure at all, just an affine connection, need to be assumed on the base manifold; and while Uhlenbeck's argument is based on *geometric* objects, our argument is coordinate based.

Our proof in the case of non-compact Lie groups requires an interesting new twist to the analysis. A modification of the basic ideas for the existence theory already set out in [12] and [14] are sufficient to establish the existence of matrix valued solutions of the RT-equations in the vector bundle case. But we show the condition that the equations actually produce solutions which lie within the Lie group, requires proving that solutions generated by our iteration scheme produce only the trivial (zero) solution of an auxiliary elliptic equation in an auxiliary variable $w = U^T \eta U - \eta$, i.e., $w = 0$ imposes the condition that solutions U of the RT-equations lie within the Lie group $SO(r,s)$, (c.f. equations (2.1), (2.2) below).

The interesting point here is that, even though we were unable to prove $w \to 0$ by estimates based on the iteration scheme alone, we were able to derive $w = 0$ on the limit by an auxiliary elliptic equation which w only satisfies in the limit. In the case when the Lie group is *compact*, we show the auxiliary equation for w is strongly elliptic, and from this it is straightforward to prove that $w = 0$ is the *only* solution, implying that solutions of the RT-equations generated by our iteration scheme always lie within the Lie group $SO(N)$. But in the case of *non-compact* Lie groups $SO(r,s)$, the auxiliary equation for w needn't be strongly elliptic, and can have an associated non-trivial spectrum with non-zero eigenfunctions for special non-optimal connections $\mathcal{A}$ on the vector bundle. So one cannot guarantee directly from the auxiliary equation for w, that $w = 0$ is the only solution. We circumvent this problem by introducing an additional *spectral parameter* λ, and prove (by *Fredholm's alternative*) that the auxiliary equation for w has only the trivial solution $w = 0$ for *almost every* λ. We then argue that the uniform convergence of our iteration scheme by which we generate solutions of the RT-equations, implies the continuity of solutions w of the auxiliary equation with respect to λ. Thus, by continuity of w with respect to λ, we can extend $w = 0$ from almost every λ, to $w = 0$ for all λ.

If $(\mathcal{A}, d\mathcal{A}) \in L^p$, the Hölder inequality applied to the commutator term in the curvature 2-form places the curvature of $\mathcal{A}$ in $L^{p/2}$; and since the coordinate transformation to optimal regularity puts $\mathcal{A} \in W^{1,p/2}$ in the new gauge, while preserving the regularity of the curvature, it follows that the transformed connection is *precisely* one Sobolev derivative more regular than its curvature in the transformed gauge. Note, interestingly, the curvature enters our argument in this paper only through $d\mathcal{A}$.

We thereby conclude that the iteration scheme used to obtain existence of solutions to the RT-equations, always produces solutions $w = 0$, for every connection $\mathcal{A}$ on the underlying vector bundle. By this we establish that solutions of the RT-equations generated by our iteration scheme always lie within the Lie group, even when the Lie group is non-compact and the auxiliary equation in w admits non-trivial solutions.

Most interesting to us is that the RT-equations reproduce a fundamental cancellation observed in [11] in the case of affine connections, namely, the cancellation of the terms $d\delta\mathcal{A}$ on the right hand side of the first RT-equation (2.5), the "bad terms" which lie at a regularity too low for the Laplacian to lift them to optimal regularity. In the setting of vector bundles, this cancellation is due to an interesting interplay between the Lie group and Lie algebra of the fibres as expressed through the RT-equations. Said differently, that the assumed bound on $d\mathcal{A}$ alone is sufficient to dominate the uncontrolled derivatives $\delta\mathcal{A}$ appears to be a principle built into the RT-equations through the geometry of connections. Also, surprisingly, to get to the low level of L^∞ regularity in the case of affine connections [14], we needed to use the invariance transformations of the RT-equations to decouple the equation for the Jacobian from the equation for the regularized connection. Here, no such group of invariances for the RT-equation is required, and the equation for the regularizing gauge transformation almost automatically de-couples from the equation for the regularized connection. Another aspect to point out is that, in the vector bundle case here, we do not require an additional condition to impose the requirement that solutions of the RT-equations lie in the gauge group, (recall that in the affine connection case, we needed an auxiliary equation to impose the integrability of the Jacobians, [12, 13, 14]), but rather we build this into the RT-equations themselves by solving an equation for the value of a free parameter associated with the invariance transformations of the equations, (i.e., the choice of α in (4.7)). Finally, and also interestingly, note that we have no simple new derivation of the RT-equations for the vector bundle case addressed here, but obtained the equations purely by analogy with the RT-equations for affine connections. Our derivation of the RT-equations for affine connections is based on the Riemann-flat condition, [10].

2. Statement of Results

The problem of optimal regularity and Uhlenbeck compactness for connection components associated with the fibres of a vector bundle, uncouples from the problem of optimal regularity for affine connections on the base manifold. Both arguments are based on the Euclidean metric in an arbitrary but fixed coordinate system, and this auxiliary Riemannian structure acts independently of the affine connection on the base manifold.[4] For this

[4] The restriction $\mathcal{A}$ of a connection on a vector bundle to its fibres, and the affine connection Γ on the base manifold, are coupled through the curvature, [5].

reason, to extend our results to vector bundles, without loss of generality, it suffices to assume the base manifold is Euclidean. The extension to non-trivial non-optimal affine connections on the base manifold is then a straightforward application of our results in [14]. In the first subsection we state our results for vector bundles in the case of Euclidean base manifold, and in the second subsection we incorporate our earlier results on the optimal regularity of affine connections in [14] into the statements of the theorems.

2.1. **Connections on the fibre of a vector bundle.** Let $\mathcal{A}_{V\mathcal{M}}$ denote a given connection on a vector bundle $V\mathcal{M}$ of a n-dimensional differentiable base manifold $\mathcal{M}$ with N-dimensional real valued fibres on which the Lie group $SO(r,s)$ acts as the gauge group, $r+s=N$. Since our problem is local and can be considered in any fixed coordinate system on $\mathcal{M}$, we assume without loss of generality the vector bundle is trivial,

$$V\mathcal{M} \simeq \mathbb{R}^N \times \Omega,$$

for some $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary. (One can view Ω as the image of a coordinate patch $(x,\mathcal{U})$ of $\mathcal{M}$, i.e., $\Omega = x(\mathcal{U})$ with resulting Cartesian coordinates x on Ω.) The gauge group $SO(r,s)$ is defined by the condition that an $N\times N$ matrix U is an element of $SO(r,s)$ if and only if

$$U^T\eta U = \eta \qquad \text{and} \qquad \det(U) = 1, \tag{2.1}$$

where η is the diagonal matrix

$$\eta = \mathrm{diag}(1,...,1,-1,...,-1), \tag{2.2}$$

with r entries 1 and s entries -1, for non-negative integers r and s, $r+s=N$. We denote with $\mathcal{A}_{\mathbf{a}}$ the connection components of $\mathcal{A}_{V\mathcal{M}}$ on $V\mathcal{M}$ with respect to a choice of frame $\mathbf{a}$ on $V\mathcal{M}$, a so-called *gauge*, i.e. $\mathbf{a}$ assigns a basis of $\mathbb{R}^N$ at each point of Ω. Assume now $\mathbf{b}$ is another frame of $V\mathcal{M}$ resulting from $\mathbf{a}$ by a gauge transformation

$$U : \Omega \to SO(r,s), \tag{2.3}$$

that is, $\mathbf{b} = U\cdot\mathbf{a}$ in the sense of pointwise matrix multiplication with each basis vector in $\mathbf{a}$. The components of $\mathcal{A}_{V\mathcal{M}}$ in the new frame $\mathbf{b}$ are related to $\mathcal{A}_{\mathbf{a}}$ by the connection transformation law

$$\mathcal{A}_{\mathbf{a}} = U^{-1}dU + U^{-1}\mathcal{A}_{\mathbf{b}}U, \tag{2.4}$$

where d denotes the exterior derivative on the matrix valued differential 0-form U, i.e., $dU = \frac{\partial U}{\partial x^i}dx^i$ and we use Einstein's convention that we always sum over repeated upper and lower indices. An abstract connection $\mathcal{A}_{V\mathcal{M}}$, by *definition*, assigns connection components $\mathcal{A}_{\mathbf{a}}$ to every frame $\mathbf{a}$ such that (2.4) holds, (c.f. [5, 7] for an introduction to vector bundles and [17, 19] for more background on Uhlenbeck compactness).

We now fix an arbitrary gauge $\mathbf{a}$ and assume the connection components $\mathcal{A}_{\mathbf{a}}$ of $\mathcal{A}_{V\mathcal{M}}$ are non-optimal, i.e., $\mathcal{A}_{\mathbf{a}} \in L^p(\Omega)$ and $d\mathcal{A}_{\mathbf{a}} \in L^p(\Omega)$ in the sense that the components of $\mathcal{A}_{\mathbf{a}}$ have this regularity with respect to fixed Cartesian coordinates on Ω, which we view as the coordinates resulting from a coordinate patch $(x,\mathcal{U})$ on the base manifold $\mathcal{M}$ such that $\Omega = x(\mathcal{U})$. The goal is then to find a gauge transformation

$$U : \Omega \to SO(r,s)$$

such that the connection components $\mathcal{A}_{\mathbf{b}}$ in the resulting gauge $\mathbf{b} = U \cdot \mathbf{a}$ exhibit optimal regularity, by which we mean $\mathcal{A}_{\mathbf{b}} \in W^{1,q}(\Omega)$ and $d\mathcal{A}_{\mathbf{b}} \in L^p(\Omega)$ for some $q \geq 1$. Here we establish optimal regularity with $q = \frac{p}{2}$ whenever $p > n$.[5] We subsequently always denote the connection components $\mathcal{A}_{\mathbf{a}}$ in the fixed incoming gauge $\mathbf{a}$ simply by $\mathcal{A}$, and refer to the collection of connection components $\mathcal{A}$ simply as a connection.

In this paper, we begin by deriving the following system of elliptic, Poisson type PDE's for the gauge transformations U in $SO(r,s)$ which transform a non-optimal connection $\mathcal{A} \equiv \mathcal{A}_{\mathbf{a}}$ to a connection $\mathcal{A}_{\mathbf{b}}$ of optimal regularity[6]:

$$\Delta \tilde{\mathcal{A}} = \delta d\mathcal{A} - \delta\big(dU^{-1} \wedge dU\big) \tag{2.5}$$

$$\Delta U = U\delta\mathcal{A} - (U^T\eta)^{-1}\langle dU^T; \eta dU\rangle. \tag{2.6}$$

We interpret equations (2.5) - (2.6) as the vector bundle version of the RT-equations introduced in [11], and refer to them here again as the *RT-equations*. The unknowns in equations (2.5) - (2.6) are $(\tilde{\mathcal{A}}, U)$, where $\tilde{\mathcal{A}}$ is a matrix valued 1-form associated to $\mathcal{A}_{\mathbf{b}}$, and U is the sought after gauge transformation interpreted as a matrix valued 0-form. Equations (2.5) - (2.6) are elliptic, with Δ being the standard Laplacian in $\mathbb{R}^n$ acting componentwise, $\Delta \equiv d\delta + \delta d = \partial^2_{x^1} + ... + \partial^2_{x^n}$, and d is the exterior derivative and δ its co-derivative based on the Euclidean metric in Cartesian x-coordinates. The matrix valued "inner product" $\langle \cdot\, ; \cdot \rangle$ is introduced in (3.7) in Section 3 below. Equation (2.6) is what we interpret as the vector bundle version of the *reduced* RT-equations, an equation decoupled from (2.5), for which solutions U are the gauge transformations in $SO(r,s)$ which map the connection $\mathcal{A}$ to optimal regularity.

That is, starting with a non-optimal connection $\mathcal{A}$, solutions of the reduced RT-equations (2.6) with Dirichlet boundary data $U = U_0$ on $\partial\Omega$ for some U_0 in $SO(r,s)$, yield a gauge transformation U in $SO(r,s)$, i.e. $U(x) \in SO(r,s)$ for every $x \in \Omega$. We prove below that a solution U of the reduced RT-equations then gives rise to a solution of (2.5), defined by

$$\tilde{\mathcal{A}}' = \mathcal{A} - U^{-1}dU, \tag{2.7}$$

that is, $\tilde{\mathcal{A}}'$ satisfies

$$\Delta \tilde{\mathcal{A}}' = \delta d\mathcal{A} - \delta(dU^{-1} \wedge dU). \tag{2.8}$$

[5] Our main concern here is the gain of one derivative, not the precise value of the exponent q. However, the result for $q = p/2$ is *sharp* because it places $\mathcal{A}_{\mathbf{b}}$ precisely one full Sobolev derivative above the regularity of its curvature $F \equiv d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$, i.e., $\mathcal{A}_{\mathbf{b}} \in W^{1,p/2}$, and the curvature $F \in L^{p/2}$ in both the $\mathbf{a}$ and $\mathbf{b}$ gauges, c.f. [5, 7].

[6] By U in $SO(r,s)$ we mean $U(x) \in SO(r,s)$ pointwise for every $x \in \Omega$.

The connection of optimal regularity in the gauge $\mathbf{b} = U \cdot \mathbf{a}$ is then given by

$$\mathcal{A}_{\mathbf{b}} = U\tilde{\mathcal{A}}'U^{-1}. \tag{2.9}$$

Since the right hand side of (2.8) has regularity just one derivative *below* $\mathcal{A}$, elliptic regularity will imply that $\tilde{\mathcal{A}}'$, and then by (2.9) also $\mathcal{A}_{\mathbf{b}}$, are one derivative of regularity *above* $\mathcal{A}$, thus establishing the optimal regularity of the connection $\mathcal{A}_{\mathbf{b}}$ in the resulting gauge $\mathbf{b} = U \cdot \mathbf{a}$. Thus the role of (2.5) is to raise the regularity of $\tilde{\mathcal{A}}'$, and hence $\mathcal{A}_{\mathbf{b}}$, by one derivative, to optimal regularity. This result is stated precisely in the following theorem. The idea of proof together with the derivation of the RT-equations (2.5) - (2.6) is the subject of Section 4.

Theorem 2.1. *Let $\mathcal{A} \equiv \mathcal{A}_{\boldsymbol{a}}$ be the connection components in a gauge $\boldsymbol{a}$ of a connection $\mathcal{A}_{V\mathcal{M}}$ on an $SO(r,s)$ vector bundle $V\mathcal{M}$ with base manifold $\mathcal{M} \equiv \Omega \subset \mathbb{R}^n$ open and bounded. Assume $\mathcal{A} \in L^p(\Omega)$ with $d\mathcal{A} \in L^p(\Omega)$, for $p > n$ with $p < \infty$.*[7] *Then the following equivalence holds:*

(i) *If there exists a solution $U \in W^{1,p}(\Omega)$ pointwise in $SO(r,s)$ of the reduced RT-equations* (2.6)*, then the gauge transformed connection $\mathcal{A}_{\boldsymbol{b}}$ in* (2.4) *has optimal regularity $\mathcal{A}_{\boldsymbol{b}} \in W^{1,p/2}(\Omega)$.*

(ii) *Conversely, if there exists a gauge transformation $U \in W^{1,p}(\Omega)$ pointwise in $SO(r,s)$, such that the gauge transformed connection $\mathcal{A}_{\boldsymbol{b}}$ in* (2.4) *has optimal regularity $\mathcal{A}_{\boldsymbol{b}} \in W^{1,p/2}(\Omega)$, then $\tilde{\mathcal{A}} \equiv U^{-1}\mathcal{A}_{\boldsymbol{b}}U \in W^{1,p/2}$ and U solve the RT-equations* (2.5) *and* (2.6)*, respectively.*

Theorem 2.1 also applies at higher levels of non-optimal regularity $\mathcal{A}, d\mathcal{A} \in W^{m,p}$, $m \geq 1$, $p > n$, $p < \infty$, in which case it gives the equivalence between the existence of $SO(r,s)$ gauge transformations $U \in W^{m+1,p}$ which smooth a connection to optimal regularity $\mathcal{A}_{\mathbf{b}} \in W^{m+1,p}$, and the existence of solutions $(U, \tilde{\mathcal{A}}) \in W^{m+1,p}$ of the RT-equations. Theorem 2.1 reduces the proof of our main results on optimal regularity and Uhlenbeck compactness, Theorems 2.3 and 2.4 below, to the following existence theorem, whose proof is the main technical effort of this paper.

Theorem 2.2. *Assume $\mathcal{A}, d\mathcal{A} \in L^p(\Omega)$, $n < p < \infty$, as in Theorem 2.1, and let $M > 0$ be a constant such that*

$$\|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega)} \equiv \|\mathcal{A}\|_{L^p(\Omega)} + \|d\mathcal{A}\|_{L^p(\Omega)} \ \leq \ M. \tag{2.10}$$

Then for any point in Ω there exists a neighborhood $\Omega' \subset \Omega$ of that point, (depending only on Ω, p, n and M), and there exists a solution $U \in W^{1,p}(\Omega')$ of the reduced RT-equations (2.6)*, such that $U(x) \in SO(r,s)$ for any $x \in \Omega'$ and such that U satisfies*

$$\|U\|_{W^{1,p}(\Omega')} \ \leq \ C(M) \ \|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega')}, \tag{2.11}$$

[7] Note that since Ω is bounded, $L^\infty(\Omega) \subset L^p(\Omega)$ for every $p < \infty$, and when $p = \infty$, Theorem 2.1 asserts optimal regularity $W^{1,q}$ for any $q < \infty$. This is the result we obtained in our prior work [14] for affine connections. The more general regularity $p < \infty$ was not addressed in [14].

for some constant $C(M) > 0$ depending only on Ω', p, n and M.

The existence result of Theorem 2.2 also applies at higher levels of non-optimal regularity $\mathcal{A}, d\mathcal{A} \in W^{m,p}$, $m \geq 1$, $n < p < \infty$, yielding a solution $U \in W^{m+1,p}(\Omega)$ in $SO(r,s)$ of the RT-equation satisfying the estimate

$$\|U\|_{W^{m+1,p}(\Omega')} \;\leq\; C(M)\;\|(\mathcal{A}, d\mathcal{A})\|_{W^{m,p}(\Omega')}, \tag{2.12}$$

under the assumption

$$\|(\mathcal{A}, d\mathcal{A})\|_{W^{m,p}(\Omega)} \equiv \|\mathcal{A}\|_{L^p(\Omega)} + \|d\mathcal{A}\|_{L^p(\Omega)} \;\leq\; M. \tag{2.13}$$

Note, interestingly, that Theorem 2.2, which only addresses the reduced RT-equation (2.6), actually holds without assuming the bound on $\|d\mathcal{A}\|_{L^p}$ in (2.10), (and we could omit $\|d\mathcal{A}\|_{L^p}$ on the right hand side of (2.11)). But our result on optimal regularity is based on applying Theorem 2.2 to construct a solution of the first RT-equation (2.5), and this argument *does* require the bound on $\|d\mathcal{A}\|_{L^p}$. Thus for simplicity, we assume the bound (2.10) from the start.

By the existence result in Theorem 2.2, in combination with part (i) of Theorem 2.1, we obtain our first main conclusion which gives optimal regularity of connections on $SO(r,s)$ vector bundles.

Theorem 2.3. *Assume $\mathcal{A}, d\mathcal{A} \in L^p(\Omega)$, $n < p < \infty$, as in Theorem 2.1, satisfies the bound* (2.10)*. Then for any point in Ω there exists a neighborhood $\Omega' \subset \Omega$ of that point, (depending only on Ω, p, n and M), and there exists a gauge transformation $U \in W^{1,p}(\Omega')$ in $SO(r,s)$, such that the connection components $\mathcal{A}_{\boldsymbol{b}}$ of the resulting gauge $\boldsymbol{b} = U \cdot \boldsymbol{a}$ in* (2.4) *have optimal regularity*

$$\mathcal{A}_{\boldsymbol{b}} \in W^{1,p/2}(\Omega'') \tag{2.14}$$

on every open set Ω'' compactly contained in Ω', and

$$\|\mathcal{A}_{\boldsymbol{b}}\|_{W^{1,p/2}(\Omega'')} \;\leq\; C(M)\;\|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega')} \tag{2.15}$$

for some constant $C(M) > 0$ depending only on Ω'', Ω', p, n and M.

Theorem 2.3 establishes the existence of local gauge transformations which transform non-optimal connections to optimal regularity in a neighborhood of each point in Ω, that is, $\mathcal{A}_{\mathbf{b}}$ is one Sobolev derivative more regular than $d\mathcal{A}_{\mathbf{b}}$ (and $d\mathcal{A}$) in Ω'', but measured with respect to the larger space $L^{p/2}$, the regularity of the curvature, c.f. Footnotes 3 and 5. Estimate (2.15) implies

$$\|\mathcal{A}_{\mathbf{b}}\|_{W^{1,p/2}(\Omega'')} \;\leq\; C(M)\|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega)} \;\leq\; C(M)M, \tag{2.16}$$

which provides the uniform bound required to prove Uhlenbeck compactness. That is, the extra full derivative of regularity for the connection provided by the bound (2.16) on $\mathcal{A}_{\mathbf{b}}$, together with the bound (2.11) on U, implies the following version of Uhlenbeck compactness:

Theorem 2.4. *Let $(\mathcal{A}_i)_{i\in\mathbb{N}}$ be a sequence of connections on $V\mathcal{M}$ in fixed gauge $\boldsymbol{a}$, satisfying the uniform bound*

$$\|(\mathcal{A}_i, d\mathcal{A}_i)\|_{L^p(\Omega)} \equiv \|\mathcal{A}_i\|_{L^p(\Omega)} + \|d\mathcal{A}_i\|_{L^p(\Omega)} \;\leq\; M \tag{2.17}$$

for some constant $M > 0$. Then for any point in Ω there exists a neighborhood $\Omega'' \subset \Omega$ of that point, (any Ω'' characterized in Theorem 2.3), such that the following holds:
(i) *There exists for each $\mathcal{A}_i$ a gauge transformation $U_i \in W^{1,p}(\Omega'', SO(r,s))$ to a gauge $\boldsymbol{b}_i = U_i \cdot \boldsymbol{a}$, such that the components $\mathcal{A}_{\boldsymbol{b}_i}$ of $\mathcal{A}_i$ in the gauge $\boldsymbol{b}_i$ have optimal regularity $\mathcal{A}_{\boldsymbol{b}_i} \in W^{1,p/2}(\Omega'')$, with uniform bound*

$$\|\mathcal{A}_{\boldsymbol{b}_i}\|_{W^{1,p/2}(\Omega'')} \;\leq\; C(M)M, \tag{2.18}$$

for some constant $C(M) > 0$ depending only on Ω'', Ω, p, n and M.
(ii) *The sequence of gauge transformations U_i is uniformly bounded in $W^{1,p}(\Omega'')$ by (2.11), and a subsequence of this sequence converges weakly in $W^{1,p}$ to some $U \in W^{1,p}(\Omega'')$ in $SO(r,s)$.*
(iii) Main Conclusion: *There exists a subsequence of $\mathcal{A}_i$, (denoted again by $\mathcal{A}_i$), such that the components of $\mathcal{A}_{\boldsymbol{b}_i}$ converge to some $\mathcal{A}_{\boldsymbol{b}}$ weakly in $W^{1,p/2}(\Omega'')$, strongly in $L^{p/2}(\Omega'')$, and $\mathcal{A}_{\boldsymbol{b}}$ are the connection coefficients of $\mathcal{A}$ in the gauge $\boldsymbol{b} = U \cdot \boldsymbol{a}$, where $\mathcal{A}$ is the weak limit of $\mathcal{A}_i$ in $L^p(\Omega'')$ in fixed gauge $\boldsymbol{a}$.*

The weak convergence in $W^{1,p/2}(\Omega'')$ asserted by Theorem 2.4 actually implies strong convergence in $L^{p/2}(\Omega'')$, and this is convergence strong enough to pass limits through non-linear products, a property inherently useful for non-linear analysis. Theorem 2.4 is our version of Uhlenbeck compactness which follows from the curvature bound (2.17) alone.

To make the connection with our previous work in [14] which applies to non-optimal affine connections in L^∞, the following corollary states the conclusions of Theorems 2.3 and 2.4 for L^∞ connections on vector bundles.

Corollary 2.5. *(i) Assume $\mathcal{A}, d\mathcal{A} \in L^\infty(\Omega)$. Then in a neighborhood of every point in Ω there exists a local gauge transformation U which lifts the connection to optimal regularity $W^{1,q}$, any $1 < q < \infty$.*
(ii) Assume a given sequence of connections $\mathcal{A}_i$ satisfies the uniform bound (2.17) *with $p = \infty$. Then for any $1 < q < \infty$, assertions (i) - (iii) of Theorem 2.4 hold with $p/2$ replaced by q.*

Note, finally, that at higher regularities $\mathcal{A}, d\mathcal{A} \in W^{m,p}$, $m \geq 1$, $p > n$, $p < \infty$, the method of Theorem 2.3 establishes optimal regularity in $W^{m+1,p}$; and the method of Theorem 2.4 establishes the compactness of sequences of connections $\mathcal{A}_i$ weakly in $W^{m+1,p}$, strongly in $W^{m,p}$. The proofs of Theorems 2.1 - 2.4 are given in Sections 5 - 7. The (simpler) proofs[8] for the more regular case $\mathcal{A}, d\mathcal{A} \in W^{m,p}$ follow by a modification of the methods in [12], and are omitted.

2.2. **Incorporating non-trivial affine connections Γ into the base manifold.** Using our results on affine connections Γ on the tangent bundle of arbitrary base manifolds $\mathcal{M}$, established in prior publications [14], Theorems 2.3 and 2.4 on optimal regularity and Uhlenbeck compactness for vector bundles $V\mathcal{M}$ extend directly to the case of non-trivial Γ defined on the base manifold $\mathcal{M}$. For example, assume Γ is a non-optimal affine connection on $\mathcal{M}$ of an $SO(r,s)$ vector bundle $V\mathcal{M}$ with connection $\mathcal{A}$ on the fibres, and assume (2.10) holds together with the following L^∞ bound on Γ assumed in [14],

$$\|\Gamma, d\Gamma\|_{L^\infty} \quad = \quad \|\Gamma\|_{L^\infty} + \|d\Gamma\|_{L^\infty} < M. \tag{2.19}$$

Now by the results in [14], there exist coordinate transformations $x \to y$ which locally lift the regularity of the components of Γ to optimal regularity, $\Gamma \in W^{1,p}$, any $p < \infty$. But the Jacobians which accomplish this are regular enough so that in y-coordinates, the estimate (2.10) on the components of $\mathcal{A}$ continues to hold in y-coordinates with a modified upper bound M. Thus, since the arguments establishing Theorems 2.3 and 2.4 in the prior subsection are based on the (auxilliary) Euclidean coordinate metric, we can apply the same arguments in y-coordinates to conclude the existence of a gauge transformations $U : \mathbf{a} \to \mathbf{b} = U \cdot \mathbf{a}$, such that, in the transformed gauge $\mathbf{b}$, the connection $\mathcal{A}$ has optimal regularity, $\mathcal{A} \in W^{1,p}$. Since the gauge transformation U does not affect the connection Γ, Theorems 2.3 and 2.4 extend in a straightforward way to arbitrary non-optimal affine connections Γ on the base manifold $\mathcal{M}$ which satisfy the L^∞ bound (2.19), and to sequences of non-optimal connections on $\mathcal{M}$ in the case of Uhlenbeck compactness. Since the Theorems in [14] regarding affine connections Γ, and Theorems 2.3 and 2.4 regarding connections $\mathcal{A}$ on the fibres, act independently in this sense, it is straightforward to combine them into a single general theorem, and it suffices to state them separately.

3. Preliminaries

We now introduce the Cartan Calculus for matrix valued differential forms on the vector bundle required to formulate the RT-equations, which we introduced in [11, Sec. 3] for tangent bundles. We again consider a trivialization of a vector bundle $V\mathcal{M} \simeq \mathbb{R}^N \times \Omega$ with N-dimensional fibres and base space $\Omega \subset \mathbb{R}^n$ open and bounded. We continue to assume fixed Cartesian coordinates x on Ω. By a matrix valued differential k-form ω on $V\mathcal{M}$ we mean a k-form over Ω with $(N \times N)$-matrix components,

$$\omega = \omega_{[i_1 \dots i_k]} dx^{i_1} \wedge \dots \wedge dx^{i_k} \equiv \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \tag{3.1}$$

[8] "Simpler" essentially because $W^{1,p}$ is closed under nonlinear products by Morrey's inequality.

for $(N \times N)$-matrices $\omega_{i_1...i_k}$ such that total anti-symmetry holds in the indices $i_1, ..., i_k \in \{1, ..., n\}$, and we follow Einstein's convention of summing over repeated upper and lower indices, (but we never "raise" or "lower" indices). We define the wedge product of a matrix valued k-form ω with a matrix valued l-form $u = u_{j_1...j_l} dx^{j_1} \wedge ... \wedge dx^{j_l}$ as

$$\omega \wedge u \equiv \frac{1}{l!k!} \omega_{i_1...i_k} \cdot u_{j_1...j_l} \, dx^{i_1} \wedge ... \wedge dx^{i_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_l}, \tag{3.2}$$

where "$\cdot$" denotes matrix multiplication, (so $\omega \wedge \omega \neq 0$ is possible).

The exterior derivative d is defined component-wise on matrix components,

$$d\omega \equiv \partial_l \omega_{[i_1...i_k]} dx^l \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}, \tag{3.3}$$

and we define the co-derivative δ on a matrix valued k-form ω as

$$\delta\omega \equiv (-1)^{(k+1)(n-k)} * d * \omega,$$

where $*$ is the Hodge star introduced in terms of the Euclidean metric in x-coordinates on $\Omega \subset \mathbb{R}^n$. Both d and δ act component-wise on matrix components, and all properties of d and δ for scalar valued differential forms carry over to matrix valued forms. Note, d requires no metric, while δ is defined via the Euclidean metric in x-coordinates. As a result, the Laplacian $\Delta \equiv d\delta + \delta d$ is given by the standard Euclidean Laplacian

$$\Delta = \partial^2_{x^1} + ... + \partial^2_{x^n},$$

is hence elliptic, and acts component-wise on matrix components and differential form components.

The exterior derivative satisfies the product rule

$$d(\omega \wedge u) = d\omega \wedge u + (-1)^k \omega \wedge du, \tag{3.4}$$

where $\omega \in W^{1,p}(\Omega)$ is a matrix valued k-form and $u \in W^{1,p}(\Omega)$ is a matrix valued j-form, and if $p > n$, then the right hand side of (3.4) lies in $W^{1,p}(\Omega)$. For invertible matrix valued 0-forms $U \in W^{1,p}(\Omega)$, using $dU^{-1} = -U^{-1} \cdot dU \cdot U^{-1}$, (3.4) implies

$$d\big(U^{-1} \cdot dU\big) = d(U^{-1}) \wedge dU = -U^{-1} dU \wedge U^{-1} dU \in W^{1,p}(\Omega). \tag{3.5}$$

The co-derivative δ satisfies the product rule

$$\delta(U \cdot w) = U \cdot \delta w + \langle dU ; w \rangle, \tag{3.6}$$

where $U \in W^{1,p}(\Omega)$ is a matrix valued 0-form and $w \in W^{1,p}(\Omega)$ a matrix valued 1-form, and if $p > n$, then the right hand side of (3.6) lies in $W^{1,p}(\Omega)$. Here $\langle \cdot \, ; \cdot \rangle$ is the matrix valued inner product defined on matrix valued k-forms ω and u by

$$\langle \omega \, ; u \rangle^\mu_\nu \equiv \sum_{\sigma=1}^{n} \sum_{i_1 < ... < i_k} \omega^\mu_{\sigma \, i_1...i_k} u^\sigma_{\nu \, i_1...i_k}. \tag{3.7}$$

Note, $\langle \omega \,; u\rangle$ converts two matrix valued k-forms into a matrix valued 0-form, and $\langle \omega \,; u\rangle$ satisfies

$$U\cdot\langle \omega \,; u\rangle = \langle U\cdot\omega \,; u\rangle, \quad \langle \omega\cdot U \,; u\rangle = \langle \omega \,; U\cdot u\rangle, \quad \langle \omega \,; u\cdot U\rangle = \langle \omega \,; u\rangle\cdot U, \tag{3.8}$$

for multiplication by matrix valued 0-forms U; see [11, Sec. 3] for further details and proofs in the case of tangent bundles (when $N = n$), which all extend to the setting of vector bundles here in a straightforward way.

We define the L^2-inner product on matrix valued differential forms by

$$\langle \omega, u\rangle_{L^2} \equiv \int_\Omega \mathrm{tr}\langle \omega^T; u\rangle dx \;=\; \sum_{\nu,\sigma=1}^{N} \sum_{i_1<\ldots<i_k} \int_\Omega \omega^\nu_{\sigma\, i_1\ldots i_k} u^\nu_{\sigma\, i_1\ldots i_k} dx, \tag{3.9}$$

where dx is the Lebesgue measure on $\mathbb{R}^n$ and tr denotes the matrix trace. By the method of proof of Lemma 8.1 in [14], one can easily show the following integration by parts formula,

$$\langle du, \omega\rangle_{L^2} + \langle u, \delta\omega\rangle_{L^2} = 0, \tag{3.10}$$

where u is a matrix valued k-form and ω a matrix valued $(k+1)$-form, $k \geq 0$, such that $u \in W^{1,p}(\Omega)$ and $\omega \in W^{1,p^*}(\Omega)$, where $\frac{1}{p} + \frac{1}{p^*} = 1$, and where we assume either u or ω vanishes on the boundary $\partial\Omega$. As in [14], all Sobolev norms are taken component-wise on matrix valued differential forms, using mostly the notation $\|\cdot\|_{m,p} \equiv \|\cdot\|_{W^{m,p}(\Omega)}$, where $m \geq -1$.

4. The RT-equations associated to vector bundles

The RT-equations associated to vector bundles (2.5) - (2.6) lie at the heart of our proof of the main results, Theorems 2.3 and 2.4. Recall, we fix an arbitrary gauge $\mathbf{a}$ and assume the connection components $\mathcal{A} \equiv \mathcal{A}_{\mathbf{a}}$ of $\mathcal{A}_{V\mathcal{M}}$ in this gauge are non-optimal, i.e. $\mathcal{A} \in L^p(\Omega)$ and $d\mathcal{A} \in L^p(\Omega)$. In this section we first derive the RT-equations (2.5) - (2.6) from the assumption that a gauge transformation $U \in W^{1,p}(\Omega)$ exists which maps the connection to optimal regularity $W^{1,p/2}(\Omega)$. We then prove that solutions U of the reduced RT-equation (2.6) yields connections of optimal regularity and lie in $SO(r,s)$. The computations in this section are presented in a formal manner, and a discussion of the weak formalism required to differentiate connections in L^p is postponed to Section 6.

4.1. **Derivation of the RT-equations.** Our strategy for deriving the RT-equations associated to connections on vector bundles parallels the one for affine connections in [11, Ch. 3 - 5], (see also [13, Ch. 5]). To begin, assume there exists a gauge transformation $U \in W^{1,p}(\Omega)$ in $SO(r,s)$ which maps $\mathcal{A} \equiv \mathcal{A}_{\mathbf{a}}$ to a connection $\mathcal{A}_{\mathbf{b}}$ which has optimal regularity $W^{1,p/2}(\Omega)$, $\mathbf{b} = U\cdot\mathbf{a}$. Define the matrix valued 1-form $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}} \equiv U^{-1}\mathcal{A}_{\mathbf{b}} U \quad \in\; W^{1,p/2}(\Omega), \tag{4.1}$$

then the connection transformation law (2.4) reads

$$\mathcal{A} = U^{-1}dU + \tilde{\mathcal{A}}. \tag{4.2}$$

Starting from the transformation law (4.2), we now derive solvable elliptic equations in U and $\tilde{\mathcal{A}}$. Taking the exterior derivative d of (4.2) yields

$$d\tilde{\mathcal{A}} = d\mathcal{A} - dU^{-1} \wedge dU, \tag{4.3}$$

where the last term follows by the Leibnitz rule (3.5). On the other hand, multiplying (4.2) by U and taking the co-derivative yields

$$\Delta U = U \cdot \big(\delta\mathcal{A} - \delta\tilde{\mathcal{A}}\big) + \langle dU; \mathcal{A} - \tilde{\mathcal{A}}\rangle, \tag{4.4}$$

where we used that $\Delta U = \delta dU$ for the matrix valued 0-form U, ($\delta U = 0$ for all 0-forms), and the Leibnitz-rule (3.6) for δ to derive the right hand side. The matrix valued "inner product" $\langle\cdot\,;\cdot\rangle$ was defined in (3.7). Observe now that the connection transformation law (4.2) leaves the co-derivative $\delta\tilde{\mathcal{A}}$ undetermined, so we are free to choose a matrix valued 0-form $\alpha \in L^{p/2}(\Omega)$ and set

$$\delta\tilde{\mathcal{A}} = U^{-1}\alpha. \tag{4.5}$$

System (4.3) and (4.5) is of Cauchy-Riemann type and would in principal determine $\tilde{\mathcal{A}}$, but in analogy to [11], we prefer to write this system as the Poisson type equation

$$\Delta\tilde{\mathcal{A}} = \delta d\mathcal{A} - \delta\big(dU^{-1} \wedge dU\big) + d(U^{-1}\alpha), \tag{4.6}$$

which results from taking d of (4.5) and δ of (4.3), adding the resulting equations, and using $\Delta = d\delta + \delta d$. Moreover, substituting (4.5) for $\delta\tilde{\mathcal{A}}$ in (4.4), we obtain

$$\Delta U = U\delta\mathcal{A} + \langle dU; \mathcal{A} - \tilde{\mathcal{A}}\rangle - \alpha. \tag{4.7}$$

Note, it will turn out that the loss of information in going from the first order system (4.3) and (4.5) to the Poisson type equation (4.6) is not relevant for our method of establishing optimal regularity, c.f. Lemma 4.1 below. The coupled system formed by (4.6) and (4.7) is what we take to be the preliminary version of the RT-equations. From these we can now derive the final version (2.5), (2.6), equations which, surprisingly, decouple. At this stage α is some matrix valued 0-form which is free to be chosen.

To continue the derivation, note that solutions of (4.6) - (4.7) could in general allow for arbitrary matrix valued solutions. The critical step now is to obtain solutions of (4.6) - (4.7) for which the solutions determine gauge transformations U which actually lie pointwise in $SO(r,s)$. That is, to show that solutions satisfy $U^T\cdot\eta\cdot U = \eta$ and $\det(U) = 1$. The condition $\det(U) = 1$ is met by solving the RT-equation with U close to the identity $\mathbb{1}$, and can therefore be neglected throughout this section. To arrange for $U^T\eta U = \eta$ we now derive an equation for α. To begin, define

$$w \equiv U^T\eta U - \eta, \tag{4.8}$$

so $w = 0$ is equivalent to $U \in SO(r,s)$ pointwise. We now assume $w = 0$, which implies $\Delta w = 0$. Then applying the Leibnitz rule to $\Delta w = 0$ gives

$$0 \;=\; \Delta w = \Delta\big(U^T\eta U\big) = (\Delta U)^T\eta U + 2\langle dU^T; \eta dU\rangle + U^T\eta\,\Delta U, \tag{4.9}$$

and substituting (4.7) for ΔU, we obtain

$$\begin{aligned} \alpha^T \cdot \eta U + U^T\eta \cdot \alpha &= (\delta\mathcal{A})^T \cdot U^T\eta U + U^T\eta U \cdot \delta\mathcal{A} + 2\langle dU^T;\eta dU\rangle \\ &\quad + \langle dU;\mathcal{A}-\tilde{\mathcal{A}}\rangle^T \cdot \eta U + U^T\eta \cdot \langle dU;\mathcal{A}-\tilde{\mathcal{A}}\rangle \end{aligned} \tag{4.10}$$

Using that $\delta\mathcal{A}$ lies pointwise in the Lie algebra of $SO(r,s)$, (the linear space of anti-symmetric matrices X with respect to η, i.e. $X^T\eta+\eta X=0$), together with $U^T\eta U=\eta$, we obtain the cancellation

$$(\delta\mathcal{A})^T \cdot U^T\eta U + U^T\eta U \cdot \delta\mathcal{A} = 0. \tag{4.11}$$

This cancellation is crucial for the regularities in the RT-equations to close and for the whole strategy to work out. From (4.11) we now conclude with

$$\alpha^T \cdot \eta U + U^T\eta \cdot \alpha = 2\langle dU^T;\eta dU\rangle + \big(U^T\eta\langle dU;\mathcal{A}-\tilde{\mathcal{A}}\rangle\big)^T + U^T\eta\langle dU;\mathcal{A}-\tilde{\mathcal{A}}\rangle \tag{4.12}$$

as the sought after equation for α. Since $\langle dU^T;\eta dU\rangle^T = \langle dU^T;\eta dU\rangle$ and $\eta^T=\eta$, the solutions of (4.12) are given by

$$\alpha \equiv (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle + \langle dU;\mathcal{A}-\tilde{\mathcal{A}}\rangle + UX, \tag{4.13}$$

where X can be any element in the Lie algebra with the same regularity as $\mathcal{A}$. Substitution of (4.13) into the preliminary version (4.6) - (4.7) of the RT-equations concludes the derivation as follows.

Substituting the expression (4.13) for α back into (4.7), we obtain

$$\Delta U = U\delta\mathcal{A} - (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle - UX, \tag{4.14}$$

the sought after RT-equation (2.6). The Lie algebra element X is free to be chosen and for our purposes here one can set $X=0$. For $U\in SO(r,s)$ we have $(U^T\eta)^{-1}=U^T\eta$, however, this substituting must not be made in (4.14), because the matching to $(U^T\eta)^{-1}$ instead of $U^T\eta$ in (4.14) appears to be an essential part of what guarantees solutions U of (4.14) to actually lie in $SO(r,s)$. In Section 4.3 below we show that solutions U to the RT-equations (4.14) always lie in $SO(r,s)$, as long that they lie in $SO(r,s)$ on the boundary of Ω.

To complete the derivation of the RT-equation in $\tilde{\mathcal{A}}$, (2.5), we substitute the expression for α in (4.13) into equation (4.6), and assume $U^T\eta U=\eta$, (so $U^{-1}=\eta^{-1}U^T\eta$)—anticipating that solutions U of (4.14) will lie in $SO(r,s)$. This substitution leads to the equation

$$\Delta\tilde{\mathcal{A}} = \delta d\mathcal{A} - \delta\big(dU^T\wedge dU\big) + d\big\langle dU^T; dU - U\cdot(\mathcal{A}-\tilde{\mathcal{A}})\big\rangle + dX, \tag{4.15}$$

which, by the connection transformation law (4.2), simplifies to the sought after RT-equation

$$\Delta\tilde{\mathcal{A}} = \delta d\mathcal{A} - \delta\big(dU^T\wedge dU\big) + dX. \tag{4.16}$$

Subsequently we set $X=0$, as this simplification is wholly sufficient for our theory.

4.2. **How the RT-equations yield optimal regularity.** A solution U to the reduced RT-equation (2.6) gives rise to a solution $\tilde{\mathcal{A}}'$ of the first RT-equation (2.5), as shown in the next lemma. By this, $\tilde{\mathcal{A}}'$ gains one derivative of regularity, from which optimal regularity will be proven for the connection

$$\mathcal{A}_{\mathbf{b}} = U\tilde{\mathcal{A}}'U^{-1} \tag{4.17}$$

in gauge $\mathbf{b} = U \cdot \mathbf{a}$ in Section 7.1 below, (c.f. (2.7) - (2.9) above).

Lemma 4.1. *Assume U solves the reduced RT-equation* (2.6) *such that U in $SO(r,s)$ pointwise, then*

$$\tilde{\mathcal{A}}' \equiv \mathcal{A} - U^{-1}dU \tag{4.18}$$

solves the first RT-equation (2.5) *with $\tilde{\mathcal{A}}$ replaced by $\tilde{\mathcal{A}}'$.*

Proof. From $\Delta = d\delta + \delta d$, a direct computation using the product rule (3.5) gives us

$$\begin{aligned} \Delta\tilde{\mathcal{A}}' &= \Delta\mathcal{A} - \Delta(U^{-1}dU) \\ &= \delta d\mathcal{A} - \delta(dU^{-1}\wedge dU) + d\Big(\delta\mathcal{A} - \delta(U^{-1}dU)\Big). \end{aligned} \tag{4.19}$$

Since U satisfies $U^T\eta U = \eta$ and since $\eta^{-1} = \eta$, we have $U^{-1} = \eta^{-1}U^T\eta$. Employing this, we find from the product rule (3.6) that

$$\delta(U^{-1}dU) = \eta^{-1}\langle dU^T;\eta dU\rangle + \eta^{-1}U^T\eta\cdot\Delta U, \tag{4.20}$$

where we used that and $\Delta U \equiv d\delta U$ since $\delta U = 0$ for 0-forms. Then, substituting the RT-equation (2.6) for ΔU, i.e. $\Delta U = U\delta\mathcal{A} - (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle$, and using $U^T\eta U = \eta$, we find that the last term in (4.19) vanishes,

$$\delta\mathcal{A} - \delta(U^{-1}dU) = 0. \tag{4.21}$$

Substitution of (4.21) into (4.19) then gives

$$\Delta\tilde{\mathcal{A}}' = \delta d\mathcal{A} - \delta(dU^{-1}\wedge dU), \tag{4.22}$$

which is the sought after RT-equation (2.5). This completes the proof. □

We prove in Section 7.1 below, that elliptic regularity theory implies that $\tilde{\mathcal{A}}'$ is two derivatives more regular than the right hand side of (2.5), which lies in $W^{-1,p/2}(\Omega)$ for a non-optimal connection $\mathcal{A}, d\mathcal{A} \in L^p(\Omega)$. This gives, after establishing the above lemma at the weak level to account for the low regularity in Section 6, the sought after optimal regularity $W^{1,p/2}$ for the connection $\mathcal{A}_{\mathbf{b}} = U\tilde{\mathcal{A}}'U^{-1}$ in the new gauge $\mathbf{b} = U \cdot \mathbf{a}$, and this forms the basis of the proof of Theorems 2.1 and 2.2, once existence of solutions to the RT-equations is shown.

4.3. **How the RT-equations yield solutions in $SO(r,s)$.** We now show that a solution U to (4.14) always lies in $SO(r,s)$, as long that it lies in $SO(r,s)$ on the boundary of Ω. We will see that the argument for compact groups $SO(N)$ is significantly simpler than the case for non-compact groups $SO(r,s)$, when $r>0$ and $s>0$.

For this recall that $U\in SO(r,s)$ is equivalent to $w=0$, because $w\equiv U^T\eta U-\eta$. We now derive an equation w satisfies whenever U is a solution of the reduced RT-equation (2.6). As in (4.9) the Leibniz rule gives

$$\Delta w=(\Delta U)^T\eta U+2\langle dU^T;\eta dU\rangle+U^T\eta\,\Delta U, \tag{4.23}$$

and substitution of the RT-equation (2.6), $\Delta U=U\delta\mathcal{A}-(U^T\eta)^{-1}\langle dU^T;\eta dU\rangle$, for ΔU and $(\Delta U)^T$ yields

$$\Delta w=(\delta\mathcal{A})^T\cdot U^T\eta U+U^T\eta U\cdot\delta\mathcal{A}, \tag{4.24}$$

using the symmetry $\langle dU^T;\eta dU\rangle^T=\langle dU^T;\eta dU\rangle$. Since $\delta\mathcal{A}$ is Lie algebra valued, we have $\delta\mathcal{A}^T\cdot\eta+\eta\cdot\delta\mathcal{A}=0$, and subtraction of this equation from the right hand side of (4.24) gives

$$\Delta w=(\delta\mathcal{A})^T\cdot w+w\cdot\delta\mathcal{A}, \tag{4.25}$$

which is the sought after equation in w. Equation (4.25) is a linear system of elliptic PDE's, and since w vanishes on $\partial\Omega$ by assumption, we conclude that $w=0$ is a solution of (4.25) in Ω. But to prove that $U\in SO(r,s)$, we have to show that $w=0$ is the only solution to (4.25) with zero boundary data, a non-trivial problem in view of the Fredholm alternative. We establish the desired uniqueness in the case of the compact group $SO(N)$ in the following lemma which is based on the fact that the effect of the right hand side of (4.25) cancels in the bilinear form associated with the elliptic equation (4.25) when $\eta=id$. A more subtle argument is developed below the Lemma to establish $w=0$ in the case of non-compact groups $SO(r,s)$. This is done rigorously in Section 5.3.

Lemma 4.2. *Assume $U\in W^{1,p}(\Omega)$ is a solution of (4.14) such that U lies in $SO(N)$ on the boundary $\partial\Omega$. Then $U(x)\in SO(N)$ for any $x\in\Omega$.*

Proof. We here address the special case $\eta=\mathbb{1}$. Define the bilinear functional $B(\cdot,\cdot)$ on matrix valued 0-forms $w,v\in H^1(\Omega)$ as

$$B(w,v)\equiv\int_\Omega\mathrm{tr}\langle dw^T;dv\rangle dx+\int_\Omega\mathrm{tr}\big(w^T\delta\mathcal{A}\,v+(w\delta\mathcal{A})^Tv\big)dx. \tag{4.26}$$

By integration by parts, it follows that B vanishes on solutions w of (4.25) for any test form $v\in H^1(\Omega)$. Thus, for any symmetric solution w of (4.25), using cyclic commutativity of the trace, we find that

$$\begin{aligned}0=B(w,w)&=\|dw\|_{L^2}+\int_\Omega\mathrm{tr}\big((\delta\mathcal{A}+\delta\mathcal{A}^T)w^2\big)dx\\&=\|dw\|_{L^2},\end{aligned} \tag{4.27}$$

where we used that $\delta\mathcal{A}+\delta\mathcal{A}^T=0$ for the last inequality. Applying next the Poincaré inequality, using that $w=0$ on $\partial\Omega$, gives

$$0=B(w,w)\geq c\|w\|_{H^1}, \tag{4.28}$$

which implies that any symmetric w solving (4.25) is the trivial solution $w=0$. (Note, the proof applies to the regularity $\delta\mathcal{A}\in W^{-1,p}(\Omega)$ by use of a standard mollification of $\delta\mathcal{A}$.) This completes the proof. □

Note that for $\eta\neq\mathbb{1}$, the cancellation in (4.27) does not take place and one gets the expression $\delta\mathcal{A}+\delta\mathcal{A}^T=\delta\mathcal{A}-\eta\cdot\delta\mathcal{A}\cdot\eta$ which is nonzero in general. So a more subtle argument is required to handle the non-compact case $SO(r,s)$.

We now develop a different, more general argument sufficient to handle the case of non-compact groups $SO(r,s)$. The complication in the non-compact case over the compact case is that the spectrum associated with equation (4.25) can be non-trivial, allowing for non-zero eigenfunctions, for special $\mathcal{A}$. To prove that solutions U of (2.6) lie pointwise in $SO(r,s)$ in Ω assuming $U\in SO(r,s)$ on $\partial\Omega$ for every $\mathcal{A}$, we fix $\mathcal{A}$ and replace $\mathcal{A}$ in (2.6) by $\lambda\mathcal{A}$ for λ in a neighborhood of 1, say, $\lambda\in(0,1]$; we then show that for almost every $\lambda\in(0,1]$, the solution U^λ of (2.6) lies in $SO(r,s)$ because the equation for w has no non-trivial eigenvectors, analogous to the case $SO(N)$. We extend this to every λ by showing that our iteration scheme converges uniformly in λ, which implies continuity of U^λ with respect to λ, which in turn implies the continuity of w with respect to λ as well. Since $w^\lambda=0$ a.e. in λ, continuity implies $w\equiv 0$, and we have a proof that U^λ lies in $SO(r,s)$ for all $\lambda\in(0,1]$.

More precisely, the key observation in our argument is that (4.25) can be written as an eigenvalue problem when replacing $\mathcal{A}$ by $\lambda\mathcal{A}$ as follows: To begin, write the right hand side of (4.25) as

$$M(w)\equiv(\delta\mathcal{A})^T\cdot w+w\cdot\delta\mathcal{A}, \tag{4.29}$$

which is a linear mapping in w pointwise at each $x\in\Omega$. We then write (4.25) as

$$w=\Delta^{-1}M(w). \tag{4.30}$$

Since $M:H^1\to H^{-1}$ is a bounded linear operator, $K\equiv\Delta^{-1}M:H^{-1}\to H^1$ is bounded and compact, so it has a countable spectrum which we denote by Σ, [3].

Now for each fixed $\mathcal{A}$, consider the family of solutions U_λ of the RT-equation (4.14) with $\mathcal{A}$ replaced by $\lambda\mathcal{A}$ for $\lambda\in(0,1]$ such that U_λ lies in $SO(r,s)$ on the boundary of Ω for each $\lambda\in(0,1]$. For such a U_λ, the eigenvalue problem (4.30) turns into

$$\frac{1}{\lambda}w_\lambda=K(w_\lambda), \tag{4.31}$$

where

$$w_\lambda\equiv U_\lambda^T\eta U_\lambda-\eta. \tag{4.32}$$

But since $K \equiv \Delta^{-1} M$ is a compact operator, it has a countable spectrum, so the solution w^λ in (4.32) cannot be nonzero for almost every λ. To guarantee the solution U satisfies $w = 0$ at $\lambda = 1$ when $\lambda = 1$ is in the spectrum of $K \equiv \Delta^{-1} M$, we show that the uniform convergence of the iteration scheme implies that U^λ is a continuous function of λ in $W^{1,p}$. Thus, since $w^\lambda = 0$ for almost every λ, it follows by continuity that $w^\lambda = 0$ for all λ including $\lambda = 1$. This argument here will be incorporated into our existence theory in Section 5 in order to prove rigorously that U lies in $SO(r,s)$.

5. Existence theory of the RT-equations - Proof of Theorem 2.2

We now prove Theorem 2.2, our existence result for the reduced RT-equation (2.6). So assume $\mathcal{A}, d\mathcal{A} \in L^p(\Omega)$, $n < p < \infty$, and let $M > 0$ be a constant such that

$$\|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega)} \equiv \|\mathcal{A}\|_{L^p(\Omega)} + \|d\mathcal{A}\|_{L^p(\Omega)} \ \leq \ M. \tag{5.1}$$

Theorem 2.2 then states that for any point in Ω there exists a neighborhood $\Omega' \subset \Omega$ of that point, (depending only on Ω, p, n and M), and there exists a solution $U \in W^{1,p}(\Omega')$ of the reduced RT-equations (2.6), such that $U(x) \in SO(r,s)$ for any $x \in \Omega'$ and such that U satisfies

$$\|U\|_{W^{1,p}(\Omega')} \ \leq \ C(M) \ \|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega')}, \tag{5.2}$$

for some constant $C(M) > 0$ depending only on Ω', p, n and M.

The proof of Theorem 2.2, is based on the following iteration scheme. The initial iterate is $U_1 = \mathbb{1}$, so $U_1 \in SO(r,s)$. Assuming then that $U_k \in W^{1,p}(\Omega)$ is given, define the subsequent iterate $U_{k+1} \in W^{1,p}(\Omega)$ as the solution of

$$\begin{cases} \Delta U_{k+1} = U_k \delta \mathcal{A} - (U_k^T \eta)^{-1} \langle dU_k^T ; \eta dU_k \rangle \\ U_{k+1} = \mathbb{1} \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

The iterates U_k will in general not lie in the group $SO(r,s)$, but once we show that their limit U is a solution of the reduced RT-equation (2.6), one can prove that U lies in $SO(r,s)$ by the spectral argument in Sections 4.3 and 5.3. To handle the non-linearities and prove convergence of the iteration scheme, we need to introduce a small parameter $\epsilon > 0$ which is accomplished in the next subsection.

5.1. The ϵ-rescaled equations.

To handle the non-linearity of the RT-equation (2.6) and prove convergence of our iteration scheme, we now introduce a small parameter $\epsilon > 0$ into (2.6). For this, consider (2.6) to be given in some (arbitrary) coordinate system x on the base space Ω,

$$\Delta_x U = U \delta_x \mathcal{A} - (U^T \eta)^{-1} \langle d_x U^T ; \eta d_x U \rangle, \tag{5.4}$$

where the index x denotes the coordinate dependence of the derivatives. To start, assume without loss of generality that $\Omega = B_1(0)$. Now restrict $\mathcal{A}$ to the ball of radius ϵ with respect to the Euclidean coordinate norm in $\mathbb{R}^n$,

$\Omega = B_\epsilon(0) \equiv \{x \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 < 1\}$ for some $0 < \epsilon < 1$. We now introduce a change to coordinates

$$x \to x' \equiv x/\epsilon. \tag{5.5}$$

Note, coordinate transformations in the base space do not affect the objects on the fibres (U, η and $\mathcal{A}$), only their coordinate derivatives, i.e., the way their rate of change is measured. Under the coordinate transformation $x \to x'$, (5.4) transforms as

$$\frac{1}{\epsilon^2}\Delta_{x'} U = \frac{1}{\epsilon} U \delta_{x'} \mathcal{A} - \frac{1}{\epsilon^2}(U^T\eta)^{-1}\langle d_{x'}U^T; \eta d_{x'}U\rangle, \tag{5.6}$$

and is defined on the ball of radius 1 in x'-coordinates. We now make the ansatz that U is a small perturbation of the identity,

$$U(x') = \mathbb{1} + \epsilon v(x'). \tag{5.7}$$

Substituting (5.7) into (5.6) and multiplying by ϵ, we write (5.6) equivalently as

$$\Delta v = U\delta\mathcal{A} - \epsilon\,(U^T\eta)^{-1}\langle dv^T; \eta dv\rangle, \tag{5.8}$$

expressed in x'-coordinates on $\Omega' = B_1(0)$. Equation (5.8) is the rescaled RT-equation which we solve via the iteration scheme introduced below.

Now, to show that the initial bound (5.1) is maintained, observe the following difference in the scaling behavior of $\mathcal{A}$ and $d\mathcal{A}$,

$$\begin{aligned} \|\mathcal{A}(x')\|_{L^p(B_1(0))} &= \epsilon^{-\frac{n}{p}}\|\mathcal{A}(x)\|_{L^p(B_\epsilon(0))}, && (5.9)\\ \|d_{x'}\mathcal{A}\|_{L^p(B_1(0))} &= \epsilon^{1-\frac{n}{p}}\|d_x\mathcal{A}\|_{L^p(B_\epsilon(0))}. && (5.10)\end{aligned}$$

Since $p > n$, the scaling for $d\mathcal{A}$ preserves the initial bound (since $\epsilon < 1$), but the scaling of $\mathcal{A}$, in general, gives an L^p norm which grows as $\epsilon \to 0$. To address this problem, note that since optimal regularity and Uhlenbeck compactness are local properties, it suffices to restrict to arbitrarily small neighborhoods $B_\delta(0)$, and

$$\|(\mathcal{A}(x), d_x\mathcal{A})\|_{L^p(B_\delta(0))} \to 0,$$

as $\delta \to 0$. Thus, at the start, for each ϵ, choose $\delta = \delta(\epsilon)$ depending on ϵ such that

$$\|(\mathcal{A}(x), d_x\mathcal{A})\|_{L^p(B_\delta(0))} \leq \epsilon M.$$

Starting in $B_\delta(0)$, and doing the above scaling of coordinates in (5.5), gives equation (5.6) in $B_\delta(0)$, where $\delta > 0$, and where δ depends on ϵ. Then, working in x' coordinates in $B_\delta(0)$, we maintain by (5.9) the bound

$$\|(\mathcal{A}(x'), d_{x'}\mathcal{A})\|_{L^p(B_\delta(0))} \leq M.$$

Since the estimates used below, (namely, Morrey's inequality, Sobolev embedding, the Poincaré inequality and elliptic regularity), hold uniformly inside the ball of radius one,[9] instead of carrying δ along throughout the argument, we assume without loss of generality that $\delta = 1$.

From now on we denote the coordinates x' again as x-coordinates, and treat these x-coordinates as well as $\Omega \equiv \Omega' = B_1(0)$ fixed, while $\epsilon > 0$ in (5.8) can be varied, and must be chosen sufficiently small (bounded away from 0) for our iteration scheme below to converge.

5.2. **The iteration scheme.** Start with the iterate $v_1 = 0$. Then $U_1 \equiv \mathbb{1} + \epsilon v_1 = \mathbb{1}$ lies in $SO(r,s)$. Assume now that $v_k \in W^{1,p}(\Omega)$ is given with $v_k = 0$ on $\partial\Omega$, $p > n$, so $U_k \equiv \mathbb{1} + \epsilon v_k$ lies in $SO(r,s)$ on $\partial\Omega$, but not necessarily everywhere in Ω. Define the next iterate $U_{k+1} = \mathbb{1} + \epsilon v_{k+1} \in W^{1,p}(\Omega)$ by solving

$$\Delta v_{k+1} = U_k \delta\mathcal{A} - \epsilon\, (U_k^T \eta)^{-1} \langle dv_k^T ; \eta dv_k \rangle, \tag{5.11}$$

with Dirichlet boundary data $v_{k+1} = 0$ on $\partial\Omega$. As shown in Lemma 5.2 below, this defines a sequence of iterates $(v_k)_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega)$, lying in $SO(r,s)$ on $\partial\Omega$, but not necessarily everywhere in Ω.

We now derive estimates in terms of our incoming curvature bound (5.1), for the iterates v_k and differences of iterates $\overline{v_{k+1}} \equiv v_{k+1} - v_k$. We begin by clarifying the existence of the inverse of $U_k = \mathbb{1} + \epsilon v_k$ when $\epsilon > 0$ sufficiently small.

Lemma 5.1. *Let $0 < \epsilon < \epsilon_k$, for*

$$\epsilon_k \equiv \frac{1}{2C_0 \|v_k\|_{1,p}} \tag{5.12}$$

where $C_0 > 0$ is the constant from the Morrey inequality (A.4)*. Then the iterate $U_k \equiv \mathbb{1} + \epsilon v_k$ is invertible with inverse $U_k^{-1} = \mathbb{1} - \epsilon u_k^-$, and there exists a constant $C_{-1} > 0$ depending only on p, n, Ω and C_0 such that*

$$\|u_k^-\|_{W^{1,p}} \leq C_{-1} \|v_k\|_{W^{1,p}}; \tag{5.13}$$

and for $0 < \epsilon < \min(\epsilon_k, \epsilon_{k-1})$, the difference $\overline{u_k^-} \equiv u_k^- - u_{k-1}^-$ satisfies the estimate

$$\|\overline{u_k^-}\|_{W^{1,p}} \leq C_{-1} \|\overline{v_k}\|_{W^{1,p}}. \tag{5.14}$$

Proof. This is proven in [14], Lemmas 6.1 and 6.3. □

We now prove the existence of the iterates v_{k+1} in $W^{1,p}$ for $\epsilon > 0$ sufficiently small.

Lemma 5.2. *For $0 < \epsilon < \epsilon_k$, with ϵ_k given in* (5.12)*, there exists a solution $v_{k+1} \in W^{1,p}(\Omega)$ of* (5.11) *with boundary data $v_{k+1} = 0$ on $\partial\Omega$, and there*

[9]Note, smooth functions of compact support (a dense subset) in the small ball can be view as functions of compact support in a fixed larger domain where the uniform estimates hold.

exists some constant $C_1 > 0$ depending only on p, n, Ω, M, C_0 and C_{-1} such that the following elliptic estimate holds,

$$\|v_{k+1}\|_{W^{1,p}} \;\leq\; C_1\Big(\|\mathcal{A}\|_{L^p} + \epsilon\|\mathcal{A}\|_{L^p}\|v_k\|_{1,p} + \epsilon\big(1+\epsilon\|v_k\|_{1,p}\big)\|v_k\|^2_{1,p}\Big). \tag{5.15}$$

Proof. The existence of a solution v_{k+1} to (5.11) together with estimate (5.15) for any $k \in \mathbb{N}$ follows from Theorem A.1, (a standard result from elliptic PDE theory included with references in the appendix). To apply Theorem A.1, we need to show that the right hand side of (5.11) is in $W^{-1,p}(\Omega)$, the space of functionals over $W^{1,p^*}(\Omega)$, with conjugate exponent p^* satisfying $\frac{1}{p} + \frac{1}{p^*} = 1$.

For this we estimate the $W^{-1,p}$-norm of the right hand side of (5.11) to show that it is bounded. Note first that $\|\delta\mathcal{A}\|_{-1,p} \leq \|\mathcal{A}\|_{L^p}$ by definition of the operator norm, so the zero order term on the right hand side of (5.11), resulting from substitution of $U_k = \mathbb{1} + \epsilon v_k$, is bounded. Moreover, $v_k\delta\mathcal{A}$ is in $W^{-1,p}(\Omega)$, since by (3.10) and cyclic commutativity we find for any matrix valued 0-form $\phi \in W^{1,p^*}_0(\Omega)$ that $\langle U_k\delta\mathcal{A}, \phi\rangle_{L^2} = -\langle \mathcal{A}, d(U_k^T\phi)\rangle_{L^2}$, and thus

$$\|v_k\delta\mathcal{A}\|_{-1,p} \leq C\|\mathcal{A}\|_{L^p}\|v_k\|_{1,p} \tag{5.16}$$

by the Hölder inequality for a generic constant $C > 0$ and using that $U\phi \in W^{1,p^*}(\Omega)$ for any $U \in W^{1,p}(\Omega)$ for $p > n$, by the Morrey inequality. By this, we conclude that the first source term in (5.11) is bounded by

$$\|U_k\delta\mathcal{A}\|_{-1,p} \leq \|\mathcal{A}\|_{L^p} + \epsilon\, C\|\mathcal{A}\|_{L^p}\|v_k\|_{1,p}. \tag{5.17}$$

To handle the non-linear term in (5.11), which is one derivative more regular than $\delta\mathcal{A}$, we use that Sobolev embedding gives $L^{p/2}(\Omega) \subset W^{-1,p}(\Omega)$ for $p > n$, from which we derive the estimate[10]

$$\begin{aligned} \|(U_k^T\eta)^{-1}\langle dv_k^T; \eta dv_k\rangle\|_{-1,p} &\leq C\|(U_k^T\eta)^{-1}\langle dv_k^T;\eta dv_k\rangle\|_{L^{\frac{p}{2}}} \\ &\leq C\|(U_k^T\eta)^{-1}\|_{1,p}\,\|dv_k\|^2_{L^p} \end{aligned} \tag{5.18}$$

in terms of a generic constant $C > 0$, where the first inequality follows from Sobolev embedding and the second one from the Morrey inequality (A.4), used to bound the supremums norm of $(U_k^T\eta)^{-1}$, in combination with the Hölder inequality, used to bound the inner product of dv_k. Using finally that the bound (5.13) of Lemma 5.1 on $U_k^{-1} = \mathbb{1} - \epsilon u_k^-$, we obtain

$$\|(U_k^T\eta)^{-1}\langle dv_k^T; \eta dv_k\rangle\|_{-1,p} \;\leq\; C\big(1 + \epsilon\|v_k\|_{1,p}\big)\|v_k\|^2_{1,p}. \tag{5.19}$$

In summary, the bounds (5.17) and (5.19) show that the right hand side of (6.4) defines a functional over W^{1,p^*}.

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[10]To be more precise, the embedding $L^{p/2}(\Omega) \subset W^{-1,p}(\Omega)$ follows from applying Sobolev embedding to show that $\phi \in W^{1,p^*} \subset L^{(p/2)^*}$. Thus the dual paring of $(U_k^T\eta)^{-1}\langle dv_k^T; \eta dv_k\rangle$ with $\phi \in W^{1,p^*}$ is finite for $p > n$, which gives the desired embedding. (Note, Sobolev embedding states that $\phi \in L^q$ for $\frac{1}{q} = \frac{1}{p^*} - \frac{1}{n}$. Now the required embedding $L^q \subset L^{(p/2)^*}$ holds if and only if $q \geq (p/2)^*$, which in turn is equivalent to the condition $p \geq n$.)

Theorem A.1 now implies the existence of $v_{k+1} \in W^{1,p}(\Omega)$ which solves (5.11) with Dirichlet data $v_{k+1} = 0$ on $\partial\Omega$. The standard elliptic estimate (A.3) of Theorem A.1 in the appendix yields that the $W^{1,p}$-norm of v_{k+1} is bounded by the $W^{-1,p}$-norm of the right hand side of (5.11), which in turn is bounded by (5.17) and ϵ times (5.19). This proves the sought after estimate (5.15) with a generic constant $C_1 > 0$ depending only on p, n, Ω and M. □

We now introduce an induction assumption which is maintained by our iteration scheme and which provides uniform bounds.

Lemma 5.3. *Let $k \in \mathbb{N}$ and assume*

$$\epsilon < \min\Big(\frac{1}{4MC_1C_0}, \frac{1}{4C_1^2M}, \frac{1}{4C_1M(1+C_1)}\Big). \tag{5.20}$$

Then, if

$$\|v_k\|_{1,p} \leq 2C_1M, \tag{5.21}$$

the subsequent iterate satisfies

$$\|v_{k+1}\|_{1,p} \leq 2C_1M. \tag{5.22}$$

Proof. First note that for any ϵ satisfying (5.20), the induction assumption $\|v_k\|_{1,p} \leq 2C_1M$ implies

$$\epsilon < \frac{1}{2C_0}\frac{1}{2MC_1} \leq \frac{1}{2C_0}\frac{1}{\|v_k\|_{1,p}} = \epsilon_k.$$

Thus, the elliptic estimate (5.15) applies and yields

$$\begin{aligned} \|v_{k+1}\|_{W^{1,p}} &\leq C_1M + \epsilon\, 2C_1^2M^2\big(1 + 2C_1 + \epsilon\, 4C_1^2M\big) \\ &\leq C_1M + \epsilon\, 2C_1^2M^2(2 + 2C_1) \\ &\leq 2C_1M, \end{aligned} \tag{5.23}$$

where the last two inequalities follow from the second and third ϵ-bound in (5.20). □

The uniform bound (5.22) provided by Lemma 5.3 already implies weak $W^{1,p}$-convergence of the iterates, and hence strong L^p-convergence, but this is not sufficient to prove the limit function solves the rescaled RT-equations (5.8). For this, we now establish strong convergences in the $W^{1,p}$-norm by deriving estimates on differences of iterates which are of order ϵ. From this a geometric series argument then implies that the sequence of iterates is Cauchy in $W^{1,p}(\Omega)$, hence converging strongly in $W^{1,p}(\Omega)$.

Lemma 5.4. *For $0 < \epsilon < \min(\epsilon_k, \epsilon_{k-1})$, there exists a constant $C_2 > 0$ depending only on $p, n, \Omega, C_0, C_{-1}$, such that difference of iterates $\overline{v_{k+1}} \equiv v_{k+1} - v_k$ satisfies the estimate*

$$\|\overline{v_{k+1}}\|_{W^{1,p}} \;\leq\; \epsilon\, C_2\Big(\|\mathcal{A}\|_{L^p} + C(k)\Big)\|\overline{v_k}\|_{W^{1,p}}, \tag{5.24}$$

where

$$C(k) \equiv \big(1 + \epsilon\|v_{k-1}\|_{1,p}\big)\big(\|v_k\|_{1,p} + \|v_{k-1}\|_{1,p}\big) + \epsilon\|v_k\|_{1,p}. \tag{5.25}$$

Proof. From (5.11), we find that

$$\Delta\overline{v_{k+1}} = \epsilon\, \overline{v_k} \cdot \delta\mathcal{A} - \epsilon\, \eta^{-1} N_k \tag{5.26}$$

where

$$\begin{aligned} N_k \;&\equiv\; (U_k^T)^{-1}\langle dv_k^T; \eta dv_k\rangle - (U_{k-1}^T)^{-1}\langle dv_{k-1}^T; \eta dv_{k-1}\rangle \\ &=\; \big(1-\epsilon(u_{k-1}^-)^T\big)\Big(\langle d\overline{v_k}^T; \eta dv_k\rangle - \langle dv_{k-1}^T; \eta d\overline{v_k}\rangle\Big) - \epsilon\overline{u_k^-}^T \langle dv_k^T; \eta dv_k\rangle, \end{aligned}$$

which is linear in differences of iterates and their inverses. The elliptic estimate (A.3) now yields

$$\|\overline{v_{k+1}}\|_{W^{1,p}} \leq \epsilon\, C\big(\|\mathcal{A}\|_{L^p}\, \|\overline{v_k}\|_{1,p} + \|N_k\|_{L^{\frac{p}{2}}}\big), \tag{5.27}$$

where we used that $\|N_k\|_{-1,p} \leq C\|N_k\|_{L^{\frac{p}{2}}}$ for $p > n$ by Sobolev embedding, where $C > 0$ is some constant depending only on p, n, Ω. Using now the Morrey inequality (A.4) as well as estimates (5.13) and (5.14) on inverses, we bound $\|N_k\|_{L^{\frac{p}{2}}}$ as

$$\|N_k\|_{L^{\frac{p}{2}}} \leq C\Big(\big(1+\epsilon\|v_{k-1}\|_{1,p}\big)\big(\|v_k\|_{1,p} + \|v_{k-1}\|_{1,p}\big) + \epsilon\|v_k\|_{1,p}\Big)\|\overline{v_k}\|_{1,p}, \tag{5.28}$$

where $C > 0$ is some constant depending only on p, n, Ω. This established (5.24) and proves the lemma. □

We now prove convergence of the iteration scheme, assuming without loss of generality that $C_0 \geq 1$, $C_1 \geq 1$, $C_2 \geq 1$ and $M \geq 1$.

Proposition 5.5. *There exist an $\bar{\epsilon} > 0$, such that for any $0 < \epsilon < \bar{\epsilon}$ the iterates v_k converge to some v strongly in $W^{1,p}(\Omega)$. Moreover, v solves the rescaled RT-equation* (5.8) *with boundary data $v = 0$ on $\partial\Omega$ and $U \equiv \mathbb{1} + \epsilon v$ is invertible.*

Proof. Choose $\epsilon > 0$ small enough to meet the ϵ-bound (5.20). Since $v_1 = 0$ meets the induction assumption (5.21), Lemma 5.3 implies that

$$\|v_k\|_{1,p} \leq 2C_1 M \qquad \forall\; k \in \mathbb{N}. \tag{5.29}$$

Combining (5.29) with the difference estimate (5.24) of Lemma 5.4 then gives

$$\|\overline{v_{k+1}}\|_{W^{1,p}} \;\leq\; \epsilon\, C_2(M + 10C_1M)\|\overline{v_k}\|_{W^{1,p}}, \tag{5.30}$$

since

$$C(k) \leq \big(1 + \epsilon 2C_1M\big)4C_1M + \epsilon 2C_1M \leq 10C_1M. \tag{5.31}$$

Restricting ϵ further to

$$\epsilon < \frac{1}{C_2(M + 10C_1M)}, \tag{5.32}$$

estimate (5.30) implies strong convergence in $W^{1,p}(\Omega)$ by the standard geometric series argument. The limit v therefore also solves (5.8) with boundary data $v = 0$ on $\partial\Omega$. □

Clearly, $U \equiv \mathbb{1}+\epsilon v \in W^{1,p}(\Omega)$ solves the RT-equation (2.6) with Dirichlet data $U = \mathbb{1}$ on $\partial\Omega$, and satisfies the sought after elliptic estimate (2.11). To complete the proof of Theorem 2.2, we next show that U does indeed lie pointwise in $SO(r,s)$.

5.3. **Proof that our iteration scheme generates solutions in $SO(r,s)$.** We now prove that the solution $U = \mathbb{1} + \epsilon v$ constructed in Proposition 5.5 through our iteration scheme does indeed lie pointwise in $SO(r,s)$. For this recall that $U \in SO(r,s)$ is equivalent to $w = 0$, for $w \equiv U^T \eta U - \eta = 0$. The property $\det(U) = 1$ is already satisfied by construction of $U = \mathbb{1} + \epsilon v$ close to the identity (for $\epsilon > 0$ sufficiently small). Recall further that we showed in Section 4.3 that any solution U of the RT-equation (2.6) satisfies (4.25), that is,

$$\Delta w = (\delta\mathcal{A})^T \cdot w + w \cdot \delta\mathcal{A}, \tag{5.33}$$

with Dirichlet boundary data

$$w = 0 \qquad \text{on} \qquad \partial\Omega. \tag{5.34}$$

Clearly $w = 0$ is a solution of (5.33). However, in light of the Fredholm alternative, solutions to (5.33) might not be unique, and our iteration scheme above provides no information whether $w_k \equiv U_k^T \eta U_k - \eta$ converges to zero. To prove that $w = 0$ does indeed hold, we need to incorporate the spectral argument outlined in Section 4.3 into the framework of the rescaled RT-equations (5.8).

To implement this spectral argument, write the rescaled RT-equation (5.8) as

$$\Delta v = (\mathbb{1} + \epsilon v)\delta\mathcal{A} - \epsilon H(v), \tag{5.35}$$

where for this argument

$$H(v) \equiv (U^T\eta)^{-1}\langle dv^T; \eta dv\rangle. \tag{5.36}$$

For $\lambda \in (0,1]$, consider the following modification

$$\Delta v^\lambda = (\mathbb{1} + \epsilon v^\lambda)\lambda\delta\mathcal{A} - \epsilon H(v^\lambda), \tag{5.37}$$

based on replacing $\mathcal{A}$ by $\lambda\mathcal{A}$. Proposition 5.5 applies for each $\lambda \in (0,1]$ and yields the existence of a solution $U^\lambda = \mathbb{1} + \epsilon v^\lambda \in W^{1,p}(\Omega)$ of (5.37). Clearly $w^\lambda \equiv (U^\lambda)^T \eta U^\lambda - \eta$ solves (5.33) with $\lambda\mathcal{A}$ in place of $\mathcal{A}$, and following the argument between (4.30) and (4.31) in Section 4.3, we write this equation as

$$\frac{1}{\lambda} w^\lambda = K w^\lambda \tag{5.38}$$

for the compact operator $Kw \equiv \Delta^{-1}\big(\delta\mathcal{A}^T \cdot w + w \cdot \delta\mathcal{A}\big)$. Thus, since the spectrum of K is countable, it follows that $w^\lambda = 0$ is the unique solution of (4.30) for *almost every* $\lambda \in (0,1]$, i.e., for every λ with $1/\lambda$ in the complement of the spectrum of K. We now prove continuity of the w^λ with respect to λ, for solutions w^λ generated by our iteration scheme. Continuity then implies $w^\lambda = 0$ for all $\lambda \in (0,1]$, and in particular at $\lambda = 1$, and this gives $w = 0$. The continuity of w^λ with respect to λ is a consequence of the following lemma, (which also implies that solution U^λ generated by our iteration scheme with $\mathcal{A}$ replaced by $\lambda\mathcal{A}$ converge uniformly in λ with respect to $W^{1,p}$).

Lemma 5.6. *Let $U^\lambda = \mathbb{1} + \epsilon v^\lambda$ and $U^{\lambda'} = \mathbb{1} + \epsilon v^{\lambda'}$ be solutions of the RT-equations* (5.37) *generated by our iteration scheme. Then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ we have the estimate*

$$\|U^\lambda - U^{\lambda'}\|_{W^{1,p}} \leq C_3|\lambda - \lambda'|, \tag{5.39}$$

for some constant $C_3 > 0$ depending only on p, n, Ω, C_0, C_1 and C_2.

Proof. By Lemma 5.3 there exists a $\epsilon_0' > 0$ such that for all $0 < \epsilon < \epsilon_0'$ any solution v^λ generated by the iteration scheme with $\lambda \in (0,1]$ satisfies the uniform bound

$$\|v^\lambda\| \leq 2C_1 M. \tag{5.40}$$

This provides a uniform bound on v^λ for $\lambda \in (0,1]$.

Next, we establish the Cauchy property (5.39) in the spirit of the proof of Lemma 5.4. Since v^λ and $v^{\lambda'}$ both satisfy the RT-equation (5.37), we have

$$\Delta(v^\lambda - v^{\lambda'}) = (\lambda - \lambda')\delta\mathcal{A} + \epsilon(\lambda v^\lambda - \lambda' v^{\lambda'})\delta\mathcal{A} - \epsilon\big(H(v^\lambda) - H(v^{\lambda'})\big), \tag{5.41}$$

and standard elliptic estimates (equation (A.3) with corresponding source estimates as in Section 5.2) now imply

$$\begin{aligned} \|v^\lambda - v^{\lambda'}\|_{1,p} \;\leq\; & |\lambda - \lambda'|\,\|\delta\mathcal{A}\|_{-1,p} + \epsilon\,\|\lambda v^\lambda - \lambda' v^{\lambda'}\|_{1,p}\,\|\delta\mathcal{A}\|_{-1,p} \\ & + \epsilon\,\|H(v^\lambda) - H(v^{\lambda'})\|_{-1,p}. \end{aligned} \tag{5.42}$$

As before $\|\delta\mathcal{A}\|_{-1,p} \leq \|\mathcal{A}\|_{L^p} \leq M$, while

$$\|\lambda v^\lambda - \lambda' v^{\lambda'}\|_{1,p} \leq |\lambda - \lambda'|\,\|v^\lambda\|_{1,p} + |\lambda'|\,\|v^\lambda - v^{\lambda'}\|_{1,p}; \tag{5.43}$$

and by the corresponding estimate in the proof of Lemma 5.4 on differences of iterates, we can bound the non-linear term by

$$\|H(v^\lambda) - H(v^{\lambda'})\|_{-1,p} \leq C_2\Big(\|\mathcal{A}\|_{L^p} + C(\lambda,\lambda')\Big)\|v^\lambda - v^{\lambda'}\|_{1,p}, \tag{5.44}$$

where

$$C(\lambda,\lambda') \equiv \big(1 + \epsilon\|v^{\lambda'}\|_{1,p}\big)\big(\|v^\lambda\|_{1,p} + \|v^{\lambda'}\|_{1,p}\big) + \epsilon\,\|v^\lambda\|_{1,p}. \tag{5.45}$$

Using now (5.40), in the form $\|v^\lambda\| \leq 2C_1 M$ and $\|v^{\lambda'}\| \leq 2C_1 M$, to further bound the right hand sides of (5.43) and (5.44), we find that (5.42) gives the bound

$$\|v^\lambda - v^{\lambda'}\|_{1,p} \;\leq\; (M + \epsilon\,2C_1 M)\,|\lambda - \lambda'| + \epsilon\, C_2'\|v^\lambda - v^{\lambda'}\|_{1,p}, \tag{5.46}$$

for some constant $C_2' > 0$ depending only on p, n, Ω, M, C_0, C_1 and C_2. Choosing now $0 < \epsilon_0 < \min(\epsilon_0', \frac{1}{2C_2'})$, subtraction of the last term in (5.46) from both sides of the equation yields

$$\frac{1}{2}\|v^\lambda - v^{\lambda'}\|_{1,p} \;\leq\; (M + \epsilon 2C_1 M)\,|\lambda - \lambda'|, \tag{5.47}$$

which implies the desired estimate (5.39). □

The continuity of U^λ and w^λ with respect to λ in the $W^{1,p}$-norm asserted by Lemma 5.4, implies that $w^\lambda = 0$ for all λ, including the original $\lambda = 1$. This proves that U lies in $SO(r,s)$ everywhere in Ω and completes the proof of Theorem 2.2.

6. Weak formalism

We finally address the weak formulation of the RT-equations (2.6) required for the low regularity $U \in W^{1,p}$ and $\mathcal{A}, d\mathcal{A} \in L^p(\Omega)$. Extending the proof of our existence result, Proposition 5.5 to this weak setting here is quite standard and should not require further explanation, (c.f., [14] for a detailed analysis of the weak setting in the case of tangent bundles). Only the derivation of equation (4.25) for $w = U^T\eta U - \eta$ in Section 4.3 as well as the proof of Lemma 4.1 require clarification, since both proofs involve substitutions of the RT-equation (2.6) in a pointwise sense.

We now introduce the weak form of the RT-equations (2.6) based on the integration by parts formula (3.10). To begin, we define the weak Laplace operator on $\tilde{\mathcal{A}}$ as

$$\Delta\tilde{\mathcal{A}}[\psi] \equiv -\langle d\tilde{\mathcal{A}}, d\psi\rangle_{L^2} - \langle \delta\tilde{\mathcal{A}}, \delta\psi\rangle_{L^2}, \tag{6.1}$$

acting on matrix valued 1-forms $\psi \in W_0^{1,(p/2)^*}(\Omega)$, and we define

$$\Delta U[\phi] \equiv -\langle dU, d\phi\rangle_{L^2}, \tag{6.2}$$

for matrix valued 0-forms $\phi \in W_0^{1,p^*}(\Omega)$, (note that $\delta\phi = 0$ for all 0-forms), where $\frac{1}{p/2} + \frac{1}{(p/2)^*} = 1$ and $\frac{1}{p} + \frac{1}{p^*} = 1$, and the L^2 inner product $\langle\cdot,\cdot\rangle_{L^2}$ defined in (3.9). Based on this, we define the weak form of the RT-equations (2.5) - (2.6) as

$$\Delta\tilde{\mathcal{A}}[\psi] = -\langle d\mathcal{A}, d\psi\rangle_{L^2} + \langle dU^{-1}\wedge dU, d\psi\rangle_{L^2} \tag{6.3}$$

$$\Delta U[\phi] = -\langle \mathcal{A}, d(U^T\phi)\rangle_{L^2} - \big\langle (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle, \phi\big\rangle_{L^2}. \tag{6.4}$$

From the Hölder inequality and Sobolev embedding it follows that the right hand side of (6.4) defines a functional in $W^{-1,p}(\Omega)$, (c.f. the proof of Lemma 5.2), and the right hand side of (6.4) defines a functional in $W^{1,p/2}(\Omega)$.

In the next lemma we extend the derivation of equation (4.25) for w in Section 4.3, i.e., $\Delta w = (\delta\mathcal{A})^T\cdot w + w\cdot\delta\mathcal{A}$, to the weak setting here.

Lemma 6.1. *Assume $U \in W^{1,p}(\Omega)$ is a weak solution of (6.4). Then $w \equiv U^T\eta U - \eta \in W^{1,p}(\Omega)$ solves (4.25) weakly, that is,*

$$\Delta w[\phi] = -\langle \mathcal{A}^T, d(\phi w)\rangle_{L^2} - \langle \mathcal{A}, d(w\phi)\rangle_{L^2} \tag{6.5}$$

holds for any matrix valued 0-forms $\phi \in W_0^{1,p^*}(\Omega)$.

Proof. Using that $dw = dU^T\cdot\eta U + U^T\eta\cdot dU$, a direct computation using cyclic commutativity of the matrix-trace shows that

$$\begin{aligned} -\Delta w[\phi] &\equiv \langle dw, d\phi\rangle \\ &= \langle dU^T, d\phi\cdot U^T\eta\rangle_{L^2} + \langle dU, \eta U\cdot d\phi\rangle_{L^2}. \end{aligned} \tag{6.6}$$

Writing $d\phi \cdot U^T\eta = d(\phi U^T\eta) - \phi \cdot d(U^T\eta)$ and $\eta U \cdot d\phi = d(\eta U\phi) - d(\eta U)\cdot\phi$ by the Leibnitz rule, and substituting this back into (6.6), we substitute for the terms containing total derivatives, (that is, $\langle dU, d(\eta U \cdot \phi)\rangle_{L^2}$ and $\langle dU^T, d(\phi U^T\eta)\rangle_{L^2} = \langle dU, d(\eta U\phi^T)\rangle_{L^2}$), the right hand side of the RT-equation (6.4).[11] A calculation then gives

$$-\Delta w[\phi] = \langle \mathcal{A}^T, d(\phi \cdot U^T\eta U)\rangle_{L^2} + \langle \mathcal{A}, d(U^T\eta U\cdot\phi)\rangle_{L^2}, \tag{6.7}$$

after several cancellations which become apparent by our definition of $\langle\cdot,\cdot\rangle_{L^2}$ in (3.9). In the last step, we now use that $\mathcal{A}^T\eta + \eta\mathcal{A}^T = 0$, (since $\mathcal{A}$ lies in the Lie algebra), to subtract from the right hand side of (6.7) the expression

$$0 = \langle(\mathcal{A}^T\eta + \eta\mathcal{A}^T), d\phi\rangle_{L^2} = \langle\mathcal{A}^T, d\phi\eta\rangle_{L^2} + \langle\mathcal{A}, \eta d\phi\rangle_{L^2}, \tag{6.8}$$

which yields the sought after equation (6.5). This completes the proof. □

By Lemma 6.1, it is straightforward to extend the argument in Section 5.3 to weak solutions of the RT-equation (6.4) and show that our iteration scheme in Section 5 produces solutions U which lie pointwise in $SO(r,s)$. Note, in order to define the compact operator $K \equiv \Delta^{-1}M$ in (4.30) at the weak level, it suffices to introduce M as the bilinear form on the right hand side of (6.5).

We next extend Lemma 4.1 to the weak setting, which is required for the proof of Theorem 2.1 in Section 7.1 below.

Lemma 6.2. *Assume $U \in W^{1,p}(\Omega)$ solves (6.3) such that U in $SO(r,s)$ pointwise, then $\tilde{\mathcal{A}}' \equiv \mathcal{A} - U^{-1}dU$ solves (6.3) with $\tilde{\mathcal{A}}$ replaced by $\tilde{\mathcal{A}}'$, and $\tilde{\mathcal{A}} \in W^{1,p/2}(\Omega)$.*

Proof. To begin observe the regularity $\tilde{\mathcal{A}}' \equiv \mathcal{A} - U^{-1}dU \in L^p(\Omega)$, and that by (3.5) $d\tilde{\mathcal{A}}' = d\mathcal{A} - dU^{-1}\wedge dU \in L^p(\Omega)$, which yields

$$-\langle d\tilde{\mathcal{A}}', d\psi\rangle_{L^2} = -\langle d\mathcal{A}, d\psi\rangle_{L^2} + \langle dU^{-1}\wedge dU, d\psi\rangle_{L^2}. \tag{6.9}$$

Thus, to show that $\tilde{\mathcal{A}}'$ solves the RT-equation (6.3) weakly, it remains to show that

$$-\langle\delta\tilde{\mathcal{A}}', \delta\psi\rangle_{L^2} = 0. \tag{6.10}$$

In analogy to the proof of Lemma 4.1, we accomplish this by showing that

$$\delta\tilde{\mathcal{A}}'[\phi] \equiv -\langle\tilde{\mathcal{A}}', d\phi\rangle_{L^2} = 0 \tag{6.11}$$

for any matrix valued 0-form $\phi \in W^{1,p^*}(\Omega)$. For this we use that $U^T\eta U = \eta$ implies $U^{-1} = \eta U^T\eta$, and compute

$$\begin{aligned} -\delta\tilde{\mathcal{A}}'[\phi] &= \langle\mathcal{A}, d\phi\rangle_{L^2} - \langle dU, \eta U\eta\cdot d\phi\rangle_{L^2} \\ &= \langle\mathcal{A}, d\phi\rangle_{L^2} + \langle dU, d(\eta U\eta)\phi\rangle_{L^2} - \langle dU, d(\eta U\eta\phi)\rangle_{L^2} \end{aligned} \tag{6.12}$$

and substitution of the weak RT-equation (6.4) for the last terms yields after a computation the sought after vanishing (6.12). This proves that $\tilde{\mathcal{A}}'$ solves the sought after RT-equation (6.3) in the weak sense.

[11] Recall, $f\phi \in W^{1,p^*}(\Omega)$ for any $f \in W^{1,p}(\Omega)$ when $p > n$ by the Morrey inequality.

To prove the regularity gain from $\tilde{\mathcal{A}}' \in L^p(\Omega)$ to $\tilde{\mathcal{A}}' \in W^{1,p/2}(\Omega)$, observe that the right hand side in (6.3) defines a functionals over $W^{1,(p/2)^*}$. Namely, since $d\mathcal{A} \in L^p \subset L^{p/2}$ and $dU^{-1} \wedge dU \in L^{p/2}$, the L^2 inner products on the right hand side of (6.3) are both finite by Hölder inequality and indeed define functionals over $W^{1,(p/2)^*}$. This completes the proof. □

7. Proof of optimal regularity and Uhlenbeck compactness

Theorem 2.2, our fundamental existence result, was proven in Section 5. We now complete the proofs of Theorems 2.1, 2.3 and 2.4, which are based on Theorem 2.2, together with Lemmas 4.1 and 6.2 which give the regularity boost for $\tilde{\mathcal{A}}'$.

7.1. **Proof of Theorem 2.1.** Let $\mathcal{A} \equiv \mathcal{A}_{\mathbf{a}}$ be the connection components in a gauge $\mathbf{a}$ of a connection $\mathcal{A}_{V\mathcal{M}}$ on an $SO(r,s)$ vector bundle $V\mathcal{M}$ with base manifold $\mathcal{M} \equiv \Omega \subset \mathbb{R}^n$ open and bounded. Assume $\mathcal{A} \in L^p(\Omega)$ with $d\mathcal{A} \in L^p(\Omega)$, for $p > n$ with $p < \infty$.

Theorem 2.1, part (ii), then states that if there exists a gauge transformation $U \in W^{1,p}(\Omega)$ pointwise in $SO(r,s)$, such that the gauge transformed connection $\mathcal{A}_{\mathbf{b}}$ in (2.4) has optimal regularity $\mathcal{A}_{\mathbf{b}} \in W^{1,p/2}(\Omega)$, then $\tilde{\mathcal{A}} \equiv U^{-1}\mathcal{A}_{\mathbf{b}}U \in W^{1,p/2}$ together with U solve the RT-equations (2.5) and (2.6), respectively. The proof of this statement follows directly from the derivation of the RT-equation in Section 4.1, and it is straightforward to extend this derivation to the weak formalism of Section 6.

Theorem 2.1, part (i), states that if there exists a solution $U \in W^{1,p}(\Omega)$ in $SO(r,s)$ of the reduced RT-equations (2.6), then the gauge transformed connection $\mathcal{A}_{\mathbf{b}}$ in (2.4) has optimal regularity $\mathcal{A}_{\mathbf{b}} \in W^{1,p/2}(\Omega)$. The proof of this statement follows from Lemma 6.2. That is, Lemma 6.2 asserts that $\tilde{\mathcal{A}}' \equiv \mathcal{A} - U^{-1}dU$ has regularity $W^{1,p/2}(\Omega)$ as a result of solving the RT-equation (2.5). Thus, since the connection in the gauge $\mathbf{b} = U \cdot \mathbf{a}$ satisfies

$$\mathcal{A}_{\mathbf{b}} = U\tilde{\mathcal{A}}'U^{-1}, \tag{7.1}$$

it follows that $\mathcal{A}_{\mathbf{b}} \in W^{1,p/2}(\Omega)$, which is optimal regularity. This completes the proof of Theorem 2.1. □

7.2. **Optimal regularity - Proof of Theorem 2.3.** Assume $\mathcal{A}, d\mathcal{A} \in L^p(\Omega)$, $n < p < \infty$, as in Theorem 2.1, satisfy the bound (2.10), i.e. $\|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega)} \leq M$. Theorem 2.3 then asserts that for any point in Ω there exists a neighborhood $\Omega' \subset \Omega$ of that point, (depending only on Ω, p, n and M), and there exists a gauge transformation $U \in W^{1,p}(\Omega')$ in $SO(r,s)$, such that the connection components $\mathcal{A}_{\mathbf{b}}$ of the resulting gauge $\mathbf{b} = U \cdot \mathbf{a}$ in (2.4) have optimal regularity

$$\mathcal{A}_{\mathbf{b}} \in W^{1,p/2}(\Omega''), \tag{7.2}$$

on every open set Ω'' compactly contained in Ω', and such that

$$\|\mathcal{A}_{\mathbf{b}}\|_{W^{1,p/2}(\Omega'')} \;\leq\; C(M)\; \|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega')}, \tag{7.3}$$

for some constant $C(M) > 0$ depending only on Ω'', Ω', p, n and M.

To prove Theorem 2.3, observe that Theorem 2.2 yields the existence of the sought after gauge transformation $U \in W^{1,p}(\Omega')$ which, by Theorem 2.1 (i), lifts $\mathcal{A}$ to optimal regularity (7.2). It only remains to prove the elliptic estimate (7.3). For this, recall first that Theorem 2.2 asserts that U satisfies estimate (2.11), i.e.,

$$\|U\|_{W^{1,p}(\Omega')} \;\leq\; C(M)\; \|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega')}, \tag{7.4}$$

for some constant $C(M) > 0$ depending only on Ω', p, n and M. By Lemma 6.2, we know that $\tilde{\mathcal{A}} \in W^{1,p/2}$ solves the RT-equation (2.5). Now the interior elliptic estimate A.5 of Theorem A.2 implies, after bounding the right hand side of (2.5), by using the Hölder inequality and (7.4), that

$$\|\tilde{\mathcal{A}}'\|_{W^{1,p/2}(\Omega'')} \;\leq\; C(M)\; \|(\mathcal{A}, d\mathcal{A})\|_{L^p(\Omega')}, \tag{7.5}$$

for some constant $C(M) > 0$ depending only on Ω'', Ω', p, n and M. Combining now (7.4) with (7.5), it is straightforward to show that the connection of optimal regularity $\mathcal{A}_{\mathbf{b}} = U\tilde{\mathcal{A}}'U^{-1}$ satisfies the sought after bound (7.3). This completes the proof of Theorem 2.3. □

7.3. **Uhlenbeck compactness - Proof of Theorem 2.4.** Consider finally a sequence of connections $(\mathcal{A}_i)_{i\in\mathbb{N}}$ on $V\mathcal{M}$ in fixed gauge $\mathbf{a}$, satisfying the uniform bound

$$\|(\mathcal{A}_i, d\mathcal{A}_i)\|_{L^p(\Omega)} \equiv \|\mathcal{A}_i\|_{L^p(\Omega)} + \|d\mathcal{A}_i\|_{L^p(\Omega)} \;\leq\; M, \tag{7.6}$$

for some constant $M > 0$. Theorem 2.4 then states that for any point in Ω there exists a neighborhood $\Omega'' \subset \Omega$ of that point, (for which we can take any of the compactly contained neighborhoods asserted to exist by Theorem 2.3), such that statements (i) - (iii) hold.

(i) There exists for each $\mathcal{A}_i$ a gauge transformation $U_i \in W^{1,p}(\Omega'', SO(r,s))$ to a gauge $\mathbf{b}_i = U_i \cdot \mathbf{a}$, such that the components $\mathcal{A}_{\mathbf{b}_i}$ of $\mathcal{A}_i$ in the gauge $\mathbf{b}_i$ have optimal regularity $\mathcal{A}_{\mathbf{b}_i} \in W^{1,p/2}(\Omega'')$, with uniform bound

$$\|\mathcal{A}_{\mathbf{b}_i}\|_{W^{1,p/2}(\Omega'')} \;\leq\; C(M)M, \tag{7.7}$$

for some constant $C(M) > 0$ depending only on Ω'', Ω, p, n and M.

The proof of **(i)** is a direct consequence of the optimal regularity result of Theorem 2.3 applied to each connection, noting that the neighborhood Ω'' only depends on the upper bound M, and that the right hand side of (7.3) is uniformly bounded by $C(M)M > 0$.

(ii) The sequence of gauge transformations U_i is uniformly bounded in $W^{1,p}(\Omega'')$ by (2.11), and a subsequence of this sequence converges weakly in $W^{1,p}$ to some $U \in W^{1,p}(\Omega'')$ in $SO(r,s)$.

The proof of **(ii)** follows immediately from estimate (2.11) of Theorem 2.2, which holds on each U_i and turns into a uniform bound in light of (7.6). The Banach Alaouglu compactness theorem [3] then implies the sought after weak convergence of a subsequence of the U_i in $W^{1,p}(\Omega)$.

(iii) Main conclusion: There exists a subsequence of $\mathcal{A}_i$, (denoted again by $\mathcal{A}_i$), such that the components of $\mathcal{A}_{\mathbf{b}_i}$ converge to some $\mathcal{A}_{\mathbf{b}}$ weakly in $W^{1,p/2}(\Omega'')$, and $\mathcal{A}_{\mathbf{b}}$ are the connection coefficients of $\mathcal{A}$ in the gauge $\mathbf{b} = U \cdot \mathbf{a}$, where $\mathcal{A}$ is the weak limit of $\mathcal{A}_i$ in $L^p(\Omega'')$ in fixed gauge $\mathbf{a}$.

To prove **(iii)**, note that the existence of the converging subsequence follows from the uniform bound (7.7) with respect to the $W^{1,p}$-norm by the Banach Alaouglu compactness theorem. By first restricting only to those elements of the sequence $\mathcal{A}_i$ associated to the convergent subsequence of U_i, asserted by part (ii) with weak limit $U \in W^{1,p}$, it follows that the limit $\mathcal{A}_{\mathbf{b}}$ does indeed give the connection coefficients of $\mathcal{A}$ in the gauge $\mathbf{b} = U \cdot \mathbf{a}$. This completes the proof of Theorem 2.4, and gives our Uhlenbeck compactness result. □

Appendix A. Basic results from elliptic PDE theory

We now summarize the results we use from elliptic PDE theory. We assume throughout that $n < p < \infty$, $n \geq 2$ and that $\Omega \subset \mathbb{R}^n$ is a bounded open domain, simply connected and with smooth boundary. Our proofs in this paper use only the following two theorems of elliptic PDE Theory, which directly extend to matrix valued and vector valued differential forms because the Laplacian acts component-wise. Note, we take the weak Laplacian here as $\Delta u[\phi] = -\langle du, d\phi \rangle_{L^2}$ for scalar functions $u \in W^{1,p}(\Omega)$ and for test functions $\phi \in W_0^{1,p^*}(\Omega)$, where $W_0^{1,p^*}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the W^{1,p^*}-norm (so $\phi|_{\partial\Omega} = 0$). Our first theorem is based on Theorem 7.2 in [15], but adapted to the case of solutions to the Poisson equation with non-zero Dirichlet data, (see Appendix B in [14] for a proof).

Theorem A.1. *Let $\Omega \subset R^n$ be a bounded open set with smooth boundary $\partial\Omega$, assume $f \in W^{-1,p}(\Omega)$ and $u_0 \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ for $n < p < \infty$. Then the Dirichlet boundary value problem*

$$\Delta u[\phi] \;=\; f[\phi], \quad \text{in } \Omega \tag{A.1}$$
$$u \;=\; u_0 \quad \text{on } \partial\Omega, \tag{A.2}$$

for any $\phi \in W_0^{1,p^*}(\Omega)$, has a unique weak solution $u \in W^{1,p}(\Omega)$ with boundary data $u - u_0 \in W_0^{1,p}(\Omega)$. Moreover, any weak solution u of (A.1) - (A.2) *satisfies*

$$\|u\|_{W^{1,p}(\Omega)} \leq C \left(\|f\|_{W^{-1,p}(\Omega)} + \|u_0\|_{W^{1,p}(\Omega)} \right), \tag{A.3}$$

for some constant C depending only on Ω, n, p.

Note that $u \in W^{1,p}(\Omega)$ is Hölder continuous by Morrey's inequality (A.4), so the boundary data (A.2) can be assigned in the sense of continuous functions. Recall, that for $p > n$ Morrey's inequality gives

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_0 \|f\|_{W^{1,p}(\Omega)}, \tag{A.4}$$

where $\alpha \equiv 1 - \frac{n}{p}$ and $C_0 > 0$ is a constant depending only on n, p and Ω [3]. We also require the following interior elliptic estimates, which for completeness we derived from (A.3) in Appendix B in [14].

Theorem A.2. *Let $f \in W^{m-1,p}(\Omega)$, for $m \geq 0$ and $n < p < \infty$. Assume u is a weak solution of* (A.1). *Then $u \in W^{m+1,p}(\Omega')$ for any open set Ω' compactly contained in Ω and there exists a constant C depending only on Ω, Ω', m, n, p such that*

$$\|u\|_{W^{m+1,p}(\Omega')} \leq C\big(\|f\|_{W^{m-1,p}(\Omega)} + \|u\|_{W^{m,p}(\Omega)}\big). \tag{A.5}$$

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