The Riemann Problem for Systems of Conservation Laws

The *p*-System

Introduction

In this paper, we will be exploring ideas leading up to an analysis of the *p*-system, a system of hyperbolic conservation laws, of which we will show the complete solution. None of these ideas are my own, I read about them in Joel Smoller's book, Shock Waves and Reaction-Diffusion Equations, and the *p*-system analysis can be found in Chapter 17, §A. of his book. Many of the words in this paper are not my own, and are taken from Smoller's book stated above. It is my sole intention to compile notes from his book and organize them in a manner I thought would provide the best understanding to someone who is new to conservation laws (as I was myself).

This paper is meant to begin the study of equations of the form

$$u_t + f(u)_x = 0$$

where $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$, $n \geq 1$, and $(x,t) \in \mathbf{R} \times \mathbf{R}_+$, the subscripts $u_{t,x}$ denote partial derivatives, and we assume the vector-valued function f is C^2 (twice continuously differentiable) in some open subset $\Omega \subset \mathbf{R}^n$. More specifically, the *p*-system deals with a specific case when $n = 2$. These equations are commonly called *conservation laws* because many systems of equations in science arise that are of this form. We will show some background information in weak solutions, the jump condition, and characteristics, before showing the solution to the single conservation law, and finally the *p*-system.

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§A. Weak Solutions of Conservation Laws

Often times, solutions to conservation laws contain discontinuities in them, and thus pose a problem because discontinuous functions cannot be differentiated. Thus, we first begin by exploring exactly what it means for a function to be called a "solution."

Consider the initial-value problem

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x) \tag{1.1}$$

Let us suppose that u is a classical[1] solution to (1.1). Let ϕ be a C^1 (differentiable) function that vanishes outside of a compact subset in $t \geq 0$, i.e. $(spt\phi) \cap (t \geq 0) \subseteq D$, where D is a rectangle $0 \leq t \leq T$, $a \leq x \leq b$, chosen so that $\phi = 0$ outside of D and on the lines $t = T$, $x = a$, $x = b$.

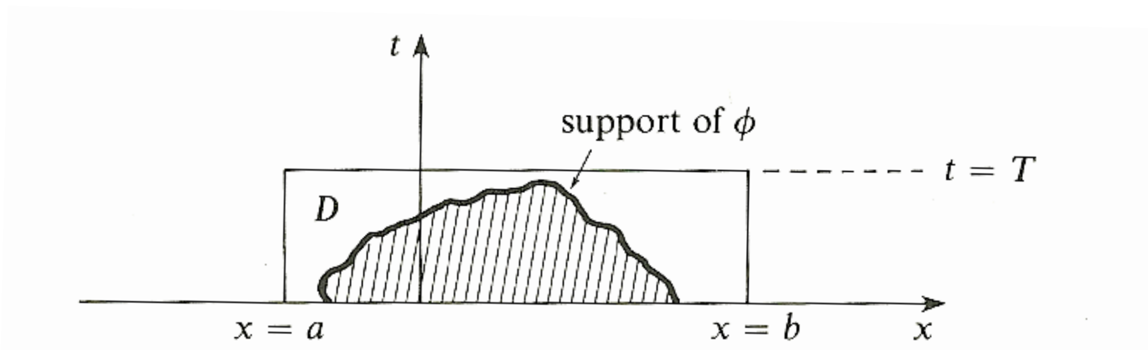

Figure 1.1

If we multiply (1.1) by ϕ and integrate over $t > 0$, to get

$$\iint_{t>0} (u_t + f_x)\phi dxdt = \iint_D (u_t + f_x)\phi dxdt = \int_a^b \int_0^T (u_t + f_x)\phi dxdt = 0$$

Now integrating by parts gives

$$\int_a^b \int_0^T (u_t + f_x)\phi dxdt = \int_a^b \int_0^T u_t\phi dxdt + \int_a^b \int_0^T f_x\phi dxdt$$

$$\int_a^b \int_0^T u_t\phi dxdt = \int_a^b u\phi\Big|_0^T dx - \int_a^b \int_0^T u\phi_t dxdt = -\int_a^b u_0(x)\phi(x,0)dx - \int_a^b \int_0^T u\phi_t dxdt$$

$$\int_a^b \int_0^T f_x\phi dxdt = \int_0^T f\phi\Big|_a^b dt - \int_a^b \int_0^T f\phi_x dxdt = -\int_a^b \int_0^T f\phi_x dxdt$$

[1] What exactly is a classical solution?

and finally

$$\iint_{t\geq 0}(u_t + f_x)\phi dxdt = -\int_a^b u_0(x)\phi(x,0) - \iint_{t\geq 0}(u\phi_t + f\phi_x)dxdt = 0$$

$$\iint_{t\geq 0}(u\phi_t + f(u)\phi_x)dxdt + \int_{t=0} u_0\phi dx = 0 \qquad (1.2)$$

Thus, if u is a classical solution to (1.1), then (1.2) holds for all $\phi \in C_0^1$. But (1.2) makes perfect sense if u, u_0 are merely bounded and measurable. We thus define:

Definition: A bounded, measurable function $u(x,t)$ is called a weak solution of the initial value problem (1.1) with bounded and measurable initial data u_0, provided (1.2) holds for all $\phi \in C_0^1$.

This may not have as big of a purpose in a practical sense, since when we solve the *p*-system we never actually check to see if our solution satisfies (1.2), however it does show us that discontinuities in a solution are natural and in fact possible to deal with.

§B. Jump Condition

We shall now explore what a discontinuity in a solution looks like. Not all discontinuities are acceptable, as we shall see.

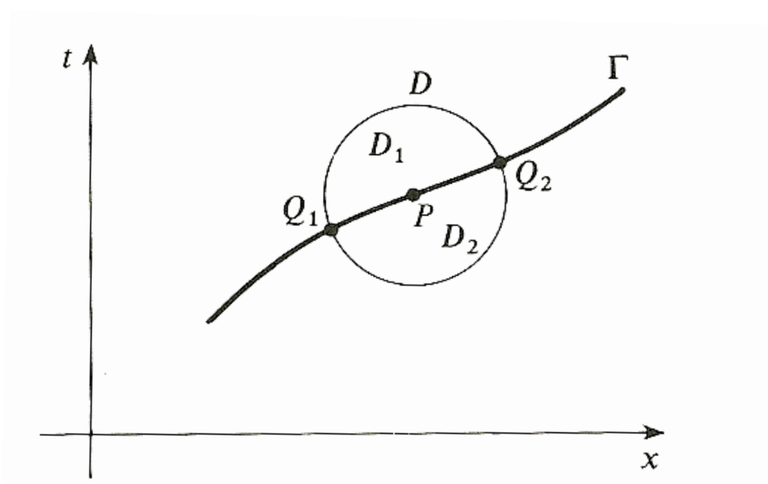

Figure 1.2

Let Γ be a smooth curve across which u has a jump discontinuity. i.e. u has well defined limits on both sides of Γ. Let P be any point on Γ, and let D be a small ball centered on P. We assume that in D, Γ is given by $x = x(t)$. Let D_1 and D_2 be the components of D, which are determined by Γ. Let $\phi \in C_0^1(D)$.

From (1.2), we see that

$$0 = \iint_D (u\phi_t + f\phi_x)dxdt = \iint_{D_1}(u\phi_t + f\phi_x)dxdt + \iint_{D_2}(u\phi_t + f\phi_x)dxdt$$

Now using the fact that u is differentiable in D_i, the divergence theorem gives

$$\iint_{D_i} (u\phi_t + f\phi_x)dxdt = \iint_{D_i} (u\phi)_t + (f\phi)_x dxdt = \int_{\partial D_i} \phi(-udx + fdt)$$

Since $\phi = 0$ on ∂D, these line integrals are nonzero only along Γ. Thus if $u_l = u(x(t) - 0, t)$ and $u_r = u(x(t) + 0, t)$, then we have

$$\int_{\partial D_1} \phi(-udx + fdt) = \int_{Q_1}^{Q_2} \phi(-u_l dx + f(u_l)dt)$$
$$\int_{\partial D_2} \phi(-udx + fdt) = -\int_{Q_1}^{Q_2} \phi(-u_r dx + f(u_r)dt)$$

Therefore

$$0 = \int_{Q_1}^{Q_2} \phi(-u_l dx + f(u_l)dt) - \int_{Q_1}^{Q_2} \phi(-u_r dx + f(u_r)dt) = \int_\Gamma \phi(-(u_l - u_r)dx + (f(u_l) - f(u_r))dt)$$
$$= \int_\Gamma \phi(-[u]dx + [f(u)]dt)$$

where $[u] = u_l - u_r$ and $[f(u)] = f(u_l) - f(u_r)$. Now from the previous equality,

$$0 = -[u]dx + [f(u)]dt$$
$$[u]dx = [f(u)]dt$$
$$\frac{dx}{dt}[u] = [f(u)] \qquad (1.3)$$

Thus, since ϕ was arbitrary, we conclude that $s[u] = [f(u)]$ at each point on Γ, where $s = \frac{dx}{dt}$. We call s the speed of the discontinuity (the reciprocal of the slope of Γ). This is called the "jump condition," and if all discontinuities in our solution satisfy this condition, than the discontinuities are acceptable and our solution is valid.

§C. Characteristics

Characteristic curves are essentially curves of the form $x = x(t)$ where the partial differential equations become a system of ordinary differential equations. We change coordinates from (x,t) to (x_0, s), along which we can integrate the solution from some initial data to obtain a specific solution to our original PDE. We compute

$$\frac{d}{ds}(u(x(t),t)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} \qquad (1.4)$$

For example, consider the advection equation

$$u_t + au_x = 0 \qquad x \in \mathbf{R},\ t \geq 0 \qquad (1.5)$$

If we compare (1.4) to (1.5), we equate (what are called the characteristics)

$$\frac{dx}{ds} = a,\ \frac{dt}{ds} = 1$$

Then, assuming $t(0) = 0$, we see that $t = s$ and $x(t) = at + const$. Moreover, we note that along the curves $x(t) - at = const$, u stays constant because

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + au_x = 0$$

Now, for example, if we consider Figure 1.3(b), we can see that if we have specified the initial condition $u(x,0) = u_0(x)$ then the solution will be $u(x,t) = u_0(x - at)$, and the values of u at each point along the x-axis will propagate along the characteristic lines. Thus if we know the initial condition, we know the solution for all of time.

Now if we consider the following boundary problem

$$u_t + au_x = 0, \quad x > 0,\ t > 0,\ u(x,0) = u_0(x) \text{ in } x \geq 0,\ u(0,t) \text{ in } t \geq 0$$

where u is defined on the positive quarter plane, then considering Figure 1.3(b) (when $a > 0$, thus the solution propagates to the right), we can see that we would have to specify not only the initial condition along $t = 0$, but the boundary condition along $x = 0$. If only the initial condition is specified, we only have the solution for half of the quarter plane, since the information is moving to the right. However, if we

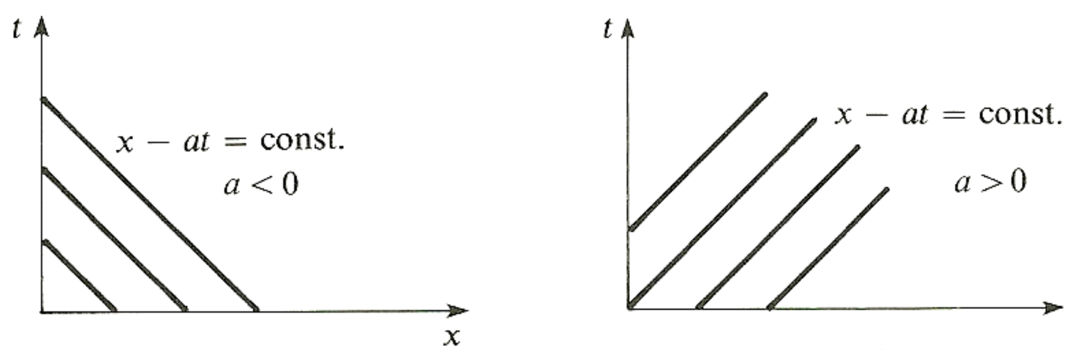

Figure 1.3 (a) (b)

specify the boundary condition, then the entire quarter plane will be filled. On the other hand, considering Figure 1.3(a) (when $a < 0$ and the solution propagates to the left), we see that only the initial condition along $t = 0$ is necessary to solve for u in the entire plane. This idea will prove useful in the next section.

§D. Shock Waves Inequalities

We will now take a moment to consider a general discontinuity in a solution, typically called a "shock wave".

Now suppose we have a hyperbolic[2] system (defined on the positive quarter plane)

$$u_t + Au_x = 0, \quad u \in \mathbf{R}^n \qquad (1.6)$$

where A is a constant $n \times n$ matrix with eigenvalues $\lambda_1 < \ldots < \lambda_k < 0 < \lambda_{k+1} < \ldots < \lambda_n$. Let P be a nonsingular matrix such that $P^{-1}AP = diag(\lambda_1, \ldots, \lambda_n) \equiv \Lambda$. If we define $u = Pv$, then $u_{x,t} = Pv_{x,t}$, and (1.6) becomes

$$Pv_t + APv_x = 0$$
$$P^{-1}(Pv_t + APv_x = 0)$$
$$v_t + \Lambda v_x = 0$$

and the system decouples into n scalar equations

$$v_t^i + \lambda_i v_x^i = 0, \; i = 1,2,\ldots,n$$

Thus, if $i \le k$ and $\lambda_i < 0$, then $v^i(0,t)$ (that is, the solution along the line x=0) is determined by the initial data (similar to the previous example in §C if $a < 0$, since the information will propagate into the line x=0). However, if $i > k$ and $\lambda_i > 0$, we must specify $v^i(0,t), \; i = k+1, \ldots, n$ (similar to §C if $a > 0$, since the information propagates away from the line x=0).

Now each v^i is a linear combination of the u_i's, so we see we must specify $(n-k)$ conditions on the components of u on the boundary $x = 0$.

More generally, if we don't have a quarter plane problem but a boundary that moves with speed s, and if $\lambda_1 < \ldots < \lambda_k < s < \lambda_{k+1} < \ldots < \lambda_n$, then we need to specify $(n-k)$ boundary conditions in order to specify the solution in the region $x - st > 0, \; t > 0$.

Now if the boundary is a discontinuity of the hyperbolic system of conservation laws, these remarks can easily be extended.

Thus, let $\lambda_1(u) < \ldots < \lambda_n(u)$ be the eigenvalues of df, and let $u_l, \; u_r$ respectively, be the values of u on the left and the right sides of the discontinuity, which moves with

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[2] A "hyperbolic" system is a system where the Jacobian matrix has only real and distinct eigenvalues.

speed s. Suppose that $\lambda_k(u_r) < s < \lambda_{k+1}(u_r)$. Then from the above reasoning, we should specify $(n-k)$ conditions on the right boundary of the discontinuity. Similarly, on the left, if $\lambda_j(u_l) < s < \lambda_{j+1}(u_l)$, we must specify j conditions on the left boundary.

Now the jump conditions $s(u_l - u_r) = f(u_l) - f(u_r)$ are n equations connecting the values on both sides of the discontinuity with s. But since $(u_l - u_r) \neq 0$, we can eliminate s from these equations to get $(n-1)$ equations (or conditions) between u_l and u_r.

Thus we require that $(n-k) + j = n - 1$, or $j = k - 1$.

We can thus conclude that, in view of these considerations, a discontinuity $(u_l, u_r; s)$ is permissible provided that for some index k, $1 \leq k \leq n$, the following inequalities hold:

$$\begin{aligned} &\lambda_k(u_r) < s < \lambda_{k+1}(u_r) \\ &\lambda_{k-1}(u_l) < s < \lambda_k(u_l) \end{aligned} \qquad (1.7)$$

We call such a discontinuity a k-shock wave, or a k-shock. The inequalities (1.7) are called the *entropy inequalities*, or the *(Lax) shock conditions*.

We can go one step further and rewrite (1.7) in the form

$$\begin{aligned} &\lambda_k(u_r) < s < \lambda_k(u_l) \\ &\lambda_{k-1}(u_l) < s < \lambda_{k+1}(u_r) \end{aligned}$$

Moreover, if $n = 1$, then the shock conditions become simply $\lambda(u_r) < s < \lambda(u_l)$. But since $u_t + f(u)_x = 0$, then $u_t + f'(u)u_x = 0$ and $\lambda(u) = f'(u)$, then the inequality becomes $f'(u_l) > s > f'(u_r)$. In other words, the characteristic speeds on either side of the discontinuity are moving *into* the discontinuity, and the value of the speeds must jump down across the boundary from left to right.

These shock inequalities will be very useful in solving the Riemann problem, both for a single conservation law and for the *p*-system.

§E. The Riemann Problem for a Single Conservation Law

Considering the Riemann problem for a single conservation law will give us a brief idea of what solutions to the *p*-system could look like. Consider the following

$$u_t + f(u)_x = 0, \qquad t > 0,\ x \in \mathbf{R} \qquad (1.8)$$

With initial data given by

$$u_0 = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases}$$

Where u_l and u_r are constants and $f'' > 0$. Our goal is to explore all possible solutions to this problem.

If $u(x,t)$ is a solution of (1.8), then for every constant $\lambda > 0$, the function

$$u_\lambda(x,t) = u(\lambda x, \lambda t)$$

is also a solution, as we can see by the following calculation:

Let $\phi = \lambda x$ and $\theta = \lambda t$. Then the function $u(\phi,\theta)$ solves (1.8) because

$$u_t + f(u)_x = \frac{\partial u}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial \phi}\frac{d\phi}{dx} = u_\theta \lambda + f(u)_\phi \lambda = \lambda(u_\theta + f(u)_\phi) = 0$$

Thus, since we seek the unique solution, it is natural to consider only solutions that depend on the ratio x/t.

The solutions $u = u(\xi)$, $\xi = x/t$, will have three types of solutions.

(a) Constant States: i.e. $u(\xi) = const$. These are genuine (classical) solutions.[3]
(b) Shock Waves: i.e. solutions of the form

$$u(x,t) = \begin{cases} u_0, & x < st, \\ u_1, & x > st, \end{cases}$$

Where, of course, $s(u_0 - u_1) = f(u_0) - f(u_1)$. In addition, we require that the entropy inequality holds, $f'(u_0) > s > f'(u_1)$ (see the end of §D).
(c) Rarefaction Waves: These are continuous solutions $u = u(\xi)$, $\xi = x/t$ of (1.8). Hence they must satisfy the ordinary differential equation

$$-\xi u_\xi + f(u)_\xi = 0$$

due to the following calculation:

$$u_t + f(u)_x = u_\xi \frac{\partial \xi}{\partial t} + f(u)_\xi \frac{\partial \xi}{\partial x} = \left(\frac{-x}{t^2}\right)u_\xi + \left(\frac{1}{t}\right)f(u)_\xi = \left(\frac{1}{t}\right)\left(-\xi u_\xi + f(u)_\xi\right) = 0$$

[3] Meaning they simply propagate through time, there are no discontinuities or rarefaction waves.

The ODE can be rewritten as $\left(f'(u)-\xi\right)u_\xi = 0$.

Thus, if $u_\xi \neq 0$, then $f'(u(\xi)) = \xi$ (or vice versa). We observe that the equation $f'(u(\xi)) = \xi$ defines a unique function $u(\xi)$, since $f''>0$.

We say that u_1 is connected to u_0 on the right by a rarefaction wave if $f'(u_1) > f'(u_0)$ and $f'(u(\xi)) = \xi$ if $f'(u_1) > \xi > f'(u_0)$.

Now, to completely solve the problem, there are only two cases to consider: (i) $u_l < u_r$ and (ii) $u_l > u_r$.

(i) Suppose $u_l < u_r$; then since $f''>0$, we have $f'(u_r) > f'(u_l)$ and the equation $f'(u(\xi)) = \xi$ has a solution $u(\xi)$ where $f'(u_r) > \xi > f'(u_l)$. Assuming that $f''(u_l) > 0$, the solution can be depicted as in Figure 1.4:

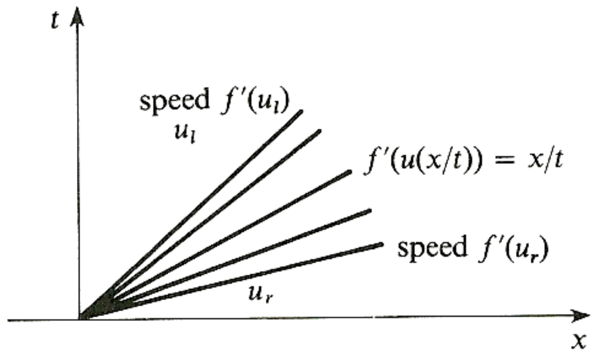

Figure 1.4

In the "fan-like" region, $f'(u_r) > x/t > f'(u_l)$ and the solution is given explicitly by solving the equation $f'(u(x/t)) = x/t$ for u (which is possible since $f''>0$).

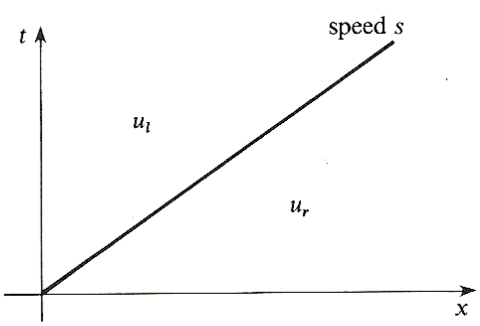

Figure 1.5

(ii) Now if $u_l > u_r$, we set $s = \dfrac{f(u_l) - f(u_r)}{u_l - u_r}$, then $f'(u_l) > s > f'(u_r)$, so the solution is a shock wave of speed s, connecting the two states u_l, u_r (see Figure 1.5).

These are the only two situations, and thus we have completely solved the problem.

§F. The *p*-system

The *p*-system is an important class of equations, especially in gas dynamics. It includes, as a special case, the equations of isentropic and isothermal gas dynamics, as seen in the following equations

$$v_t - u_x = 0, \qquad u_t + \left(\frac{k}{v^\gamma}\right)_x = 0, \qquad t > 0,\ x \in \mathbf{R}.$$

We define the *p*-system as the following generalized system of equations:

$$v_t - u_x = 0, \qquad u_t + p(v)_x = 0, \qquad t > 0,\ x \in \mathbf{R} \qquad (1.9)$$

where $p' < 0$, and $p'' > 0$.

If we let $U = \begin{pmatrix} v \\ u \end{pmatrix}$, $F(U) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}$, then (1.9) becomes

$$U_t + F(U)_x = 0 \qquad (1.10)$$

The Jacobian

$$dF = \frac{\partial(F_1, F_2)}{\partial(v, u)} = \begin{pmatrix} \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial u} \\ \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

has real and distinct eigenvalues

$$\det\begin{pmatrix} -\lambda & -1 \\ p'(v) & -\lambda \end{pmatrix} = \lambda^2 + p'(v) = 0, \text{ so } \lambda_1 \equiv -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} \equiv \lambda_2 \qquad (1.11)$$

Thus, the *p*-system is a hyperbolic system.

The Riemann Problem for (1.10) is the initial value problem with data

$$U(x,0) = U_0(x) = \begin{cases} U_l = (v_l, u_l), & x < 0, \\ U_r = (v_r, u_r), & x > 0. \end{cases} \qquad (1.12)$$

Now we will consider shock wave solutions to (1.10), (1.12). There are two types of shock waves for (1.11), namely 1-shocks and 2-shocks.

Recall (1.7)

$$\lambda_k(u_r) < s < \lambda_{k+1}(u_r)$$
$$\lambda_{k-1}(u_l) < s < \lambda_k(u_l)$$

Then the 1-shock must satisfy (when $k = 1$)

$$s < \lambda_1(u_l),\ \lambda_1(u_r) < s < \lambda_2(u_r) \quad (1.13)$$

while the 2-shocks satisfies

$$\lambda_1(u_l) < s < \lambda_2(u_l),\ \lambda_2(u_r) < s \quad (1.14)$$

since $\lambda_1 < 0 < \lambda_2$, we see that $s < 0$ for 1-shocks (also called back-shocks) and $s > 0$ for 2-shocks (also called front-shocks).

In view of (1.11) we see that (1.13) can be written

$$\lambda_1(u_r) < s < \lambda_1(u_l), \text{ so } -\sqrt{-p'(v_r)} < s < -\sqrt{-p'(v_l)} \quad \text{back} \quad (1.15)$$

and (1.14) can be written

$$\lambda_2(u_r) < s < \lambda_2(u_l), \text{ so } \sqrt{-p'(v_r)} < s < \sqrt{-p'(v_l)} \quad \text{front} \quad (1.16)$$

Now consider the following. Given a state $U_l = (v_l, u_l)$, what are the possible states $U = (v, u)$ that can be connected to U_l on the right by a back-shock (more specifically, a single back-shock)?

We know that they must satisfy the jump conditions

$$s(v_l - v) = (-u_l - (-u)) \qquad s(u_l - u) = p(v_l) - p(v),$$
$$-s(v - v_l) = -(-u + u_l) \qquad -s(u - u_l) = -p(v) + p(v_l), \text{ and finally}$$

$$s(v - v_l) = -(u - u_l) \qquad s(u - u_l) = p(v) - p(v_l) \quad (1.17)$$

Since $s \neq 0$, we can eliminate s from the equations and obtain

$$-\frac{(u - u_l)}{(v - v_l)} = \frac{p(v) - p(v_l)}{(u - u_l)},$$
$$(u - u_l)^2 = -(v - v_l)(p(v) - p(v_l)), \text{ and finally}$$

$$u - u_l = \pm\sqrt{(v - v_l)(p(v_l) - p(v))} \quad (1.18)$$

To determine the sign, we know that (1.15) must hold, so since $-\sqrt{-p'(v)} < -\sqrt{-p'(v_l)}$, we can conclude that $p'(v) < p'(v_l)$, and since $p'' > 0$, we have $v_l > v$. Since $s < 0$ (because it is a back-shock), the first equation of (1.17) implies that $u - u_l < 0$, so $u < u_l$. Thus we must take the minus sign for (1.18). Thus the set S of states that can be connected to U by a 1-shock (back-shock) on the right must lie on the curve

$$S_1 : u - u_l = -\sqrt{(v - v_l)(p(v_l) - p(v))} \equiv s_1(v;U_l),\ v_l > v \qquad (1.19)$$

This is called the back-shock curve, or the 1-shock curve.

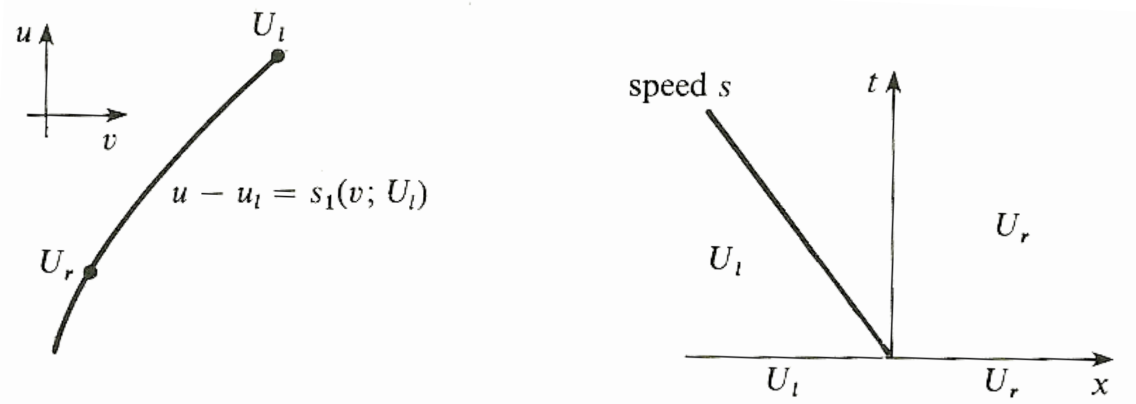

Figure 1.6 (a) (b)

Next we calculate $\frac{ds_1}{dv}$:

$$\begin{aligned}\frac{ds_1}{dv} &= -\frac{1}{2}((v - v_l)(p(v_l) - p(v)))^{-\frac{1}{2}}\left[p(v_l) - p(v) + (-p'(v)(v - v_l))\right] \\ &= -\frac{v - v_l}{2\sqrt{(v - v_l)(p(v_l) - p(v_l))}}\left[-p'(v) + \frac{p(v_l) - p(v)}{v - v_l}\right] \\ &= \frac{v - v_l}{2\sqrt{(v - v_l)(p(v_l) - p(v))}}\left[p'(v) + \frac{p(v_l) - p(v)}{v_l - v}\right] > 0\end{aligned}$$

A tedious, but straightforward calculation, shows that the curve $u - u_l = s_1(v;U_l)$ in the region $v_l > v$ is starlike with respect to the point U_l; i.e. any ray through U_l meets this curve in at most one point. Thus we can depict the back-shock curve as in Figure1.6(a). If U_r is any point on this curve, then the Riemann problem for (1.10) with initial data (1.12) can be solved by a back shock, as in Figure 1.6(b). The speed s of the shock can be obtained from the equation $s(v_r - v_l) = -(u_r - u_l)$, as follows from (1.17). Moreover, by construction, we know that (1.13) is valid for this solution.

With a similar analysis, we can construct the curve S_2, consisting of all those states that can be connected to the state U_l by a 2-shock (or front-shock) on the right. We find

$$S_2 : u - u_l = -\sqrt{(v - v_l)(p(v_l) - p(v))} \equiv s_2(v;U_l), \qquad v_l < v \qquad (1.20)$$

we call this the *front-shock curve*, or the *2-shock curve*. A calculation as above shows that $ds_2/dv<0$, and that the curve $u-u_l=s_2(v;U_l)$ is also starlike with respect to U_l in the region $v_l<v$. We can thus depict the front-shock curve as in Figure 1.7(a). As before, if U_r is any point on this curve, then the Riemann problem for (1.10) with data (1.12) is solvable by a 2-shock, as depicted in Figure 1.7(b). The speed of the shock is obtained as above from the equation $s(v_r-v_l)=-(u_r-u_l)$, and (1.14) is valid for this solution.

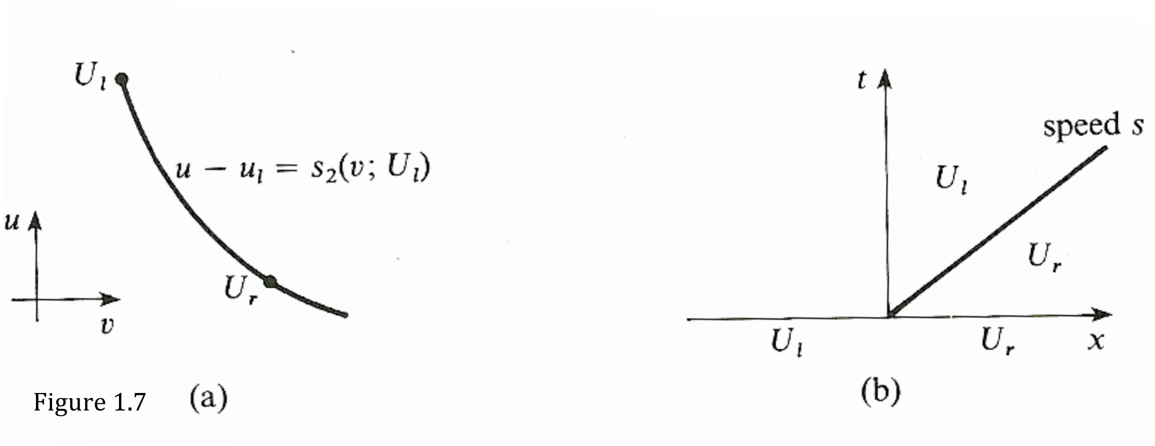

Figure 1.7

We will now consider the rarefaction-wave solutions of (1.10). We recall from §E that a *rarefaction wave* is a continuous solution of (1.10) of the form $U=U(x/t)$. There are two families of rarefaction waves, corresponding to either characteristic family λ_1 or λ_2. Thus a *k*-rarefaction wave must satisfy the additional condition that the *k*th characteristic speed increases as x/t increases, $k=1,2$. In other words, we require that $\lambda_k(U(x/t))$ increases as x/t increases.

Now if we let $\xi=x/t$, then we see that $U=U(x/t)$ satisfies the ordinary differential equation

$$-\xi U_\xi+F(U)_\xi=0 \qquad \text{(see §E.(c).)}$$

or

$$(dF-\xi I)U_\xi=0$$

If $U_\xi\neq 0$, then U_ξ is an eigenvector of dF for the eigenvalue ξ. Since dF has real and distinct eigenvalues $\lambda_1<\lambda_2$, there are two types of families.

1-rarefaction waves, or *back-rarefaction waves*, have the eigenvector $U_\xi=(v_\xi,u_\xi)^t$ which satisfies

$$\begin{pmatrix} -\lambda_1 & -1 \\ p'(v) & -\lambda_1 \end{pmatrix}\begin{pmatrix} v_\xi \\ u_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{which gives}$$

$-\lambda_1 v_\xi - u_\xi = 0$, or since $v_\xi \neq 0$, $\dfrac{u_\xi}{v_\xi} = \dfrac{du/d\xi}{dv/d\xi} = \dfrac{du}{dv} = -\lambda_1(v,u) = \sqrt{-p'(v)}$

We can integrate $\dfrac{du}{dv} = \sqrt{-p'(v)}$ to obtain

$$R_1 : u - u_l = \int_{v_l}^{v} \sqrt{-p'(y)}dy \equiv r_1(v;U_l), \quad v > v_l \qquad (1.21)$$

The requirement that $\lambda_1(U) > \lambda_1(U_l)$ (here's why: since $\lambda_1 < 0$, we know that this is a back-rarefaction, and therefore we need the left side to be moving faster than the right, but since all the speeds are negative, we want $\lambda_1(U)$ to be less negative, therefore, we want $\lambda_1(U) > \lambda_1(U_l)$) gives

$-\sqrt{-p'(v)} > -\sqrt{-p'(v_l)}$, so $p'(v) > p'(v_l)$ and $v > v_l$ since $p'' > 0$.

Finally by direct calculation,

$$\frac{dr_1}{dv} = \sqrt{-p'(v)} > 0 \text{ and } \frac{d^2 r_1}{dv^2} = \frac{-p''(v)}{2\sqrt{-p'(v)}} < 0$$

Thus, we can depict the curve $u - u_l = r_1(v;U_l)$ as in Figure 1.8(a). The solution varies

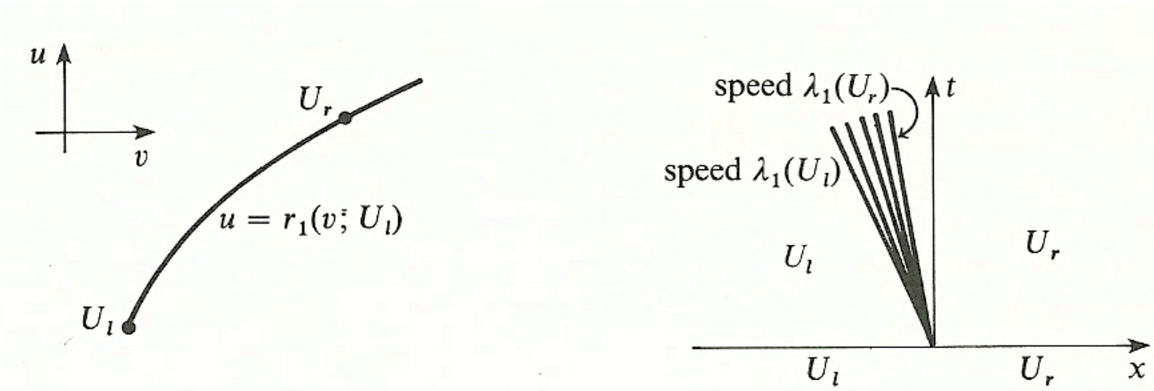

Figure 1.8 (a) (b)

smoothly in the "fan" area, and every value of U between U_l and U_r on the curve R_1 moves with speed $\lambda_1(U)$.

Finally, the *2-rarefaction wave* curve is given by

$$R_2 : u - u_l = -\int_{v_l}^{v} \sqrt{-p'(y)}dy \equiv r_2(v;U_l),\ v_l > v \tag{1.22}$$

and $dr_2/dv < 0$, $d^2r_2/dv^2 > 0$. This curve is depicted in Figure 1.9(a).

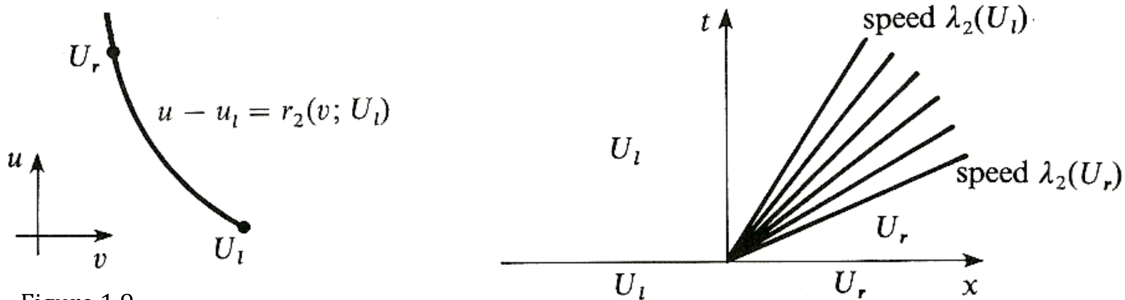

Figure 1.9

We can put all of these curves together in the $v-u$ plane to obtain a picture as in Figure 1.10. This shows that the $v-u$ plane is divided into four disjoint open regions **I**, **II**, **III**, and **IV** as depicted.

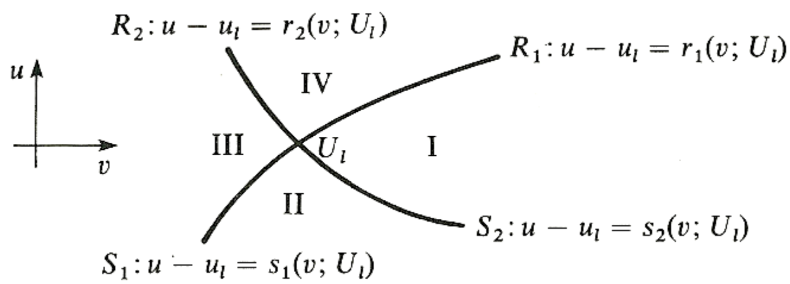

Figure 1.10

Now consider the general Riemann problem (1.10), (1.12). We consider U_l as being fixed, and allow U_r to vary. If U_r lies on any of the above curves, i.e. if $u_r - u_l = r_i(v_r;U_l)$ or $u_r - u_l = s_i(v_r;U_l)$, $i = 1,2$, then we have seen how to solve the problem. We thus assume that U_r lies in one of the four open regions **I**, **II**, **III**, or **IV**. We define, for $\overline{U} \in \mathbf{R}^2$,

$$S_i(\overline{U}) = \{(v,u) : u = s_i(v,\overline{U})\}, \qquad i = 1,2$$

$$R_i(\overline{U}) = \{(v,u) : u = r_i(v,\overline{U})\}, \qquad i = 1,2$$

and $\quad W_i(\overline{U}) = S_i(\overline{U}) \cup R_i(\overline{U}), \qquad i = 1,2.$

We consider the family of curves $\mathscr{F} = \{W_2(\overline{U}) : \overline{U} \in W_1(U_l)\}$. Let's assume for the

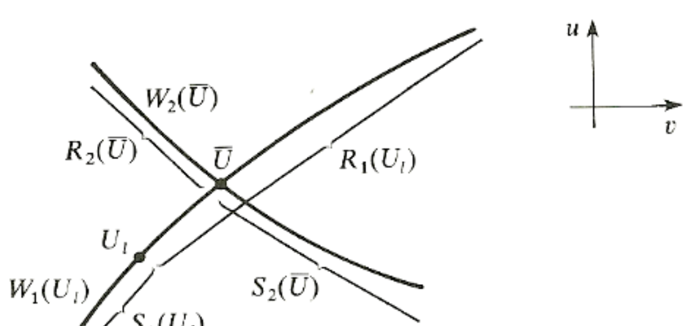

Figure 1.11

moment that the $v-u$ plane is covered univalently by the family of curves $\mathscr{F}$, i.e. through each point U_r there passes exactly one curve $W_2(\overline{U})$ of $\mathscr{F}$. Then the solution to the Riemann problem (1.10), (1.12) is given as follows: We connect $\overline{U}$ to U_l on the right by a backward (shock or rarefaction) wave, then connect U_r to $\overline{U}$ on the right by a forward (shock or rarefaction) wave. The type of wave depends, of course, on the position of U_r.

For example, if U_r lies in region **III**, then consider Figure 1.12(a). For each such U_r, there is a unique point $\overline{U}$, for which the curve $W_2(\overline{U})$ is in $\mathscr{F}$ and passes through U_r. Since $\overline{U} \in S_1(U_l)$, $\overline{U}$ is connected to U_l on the right by a back shock. Since $U_r \in R_2(\overline{U})$, U_r is connected to $\overline{U}$ on the right by a front rarefaction wave. See Figure 1.12(b).

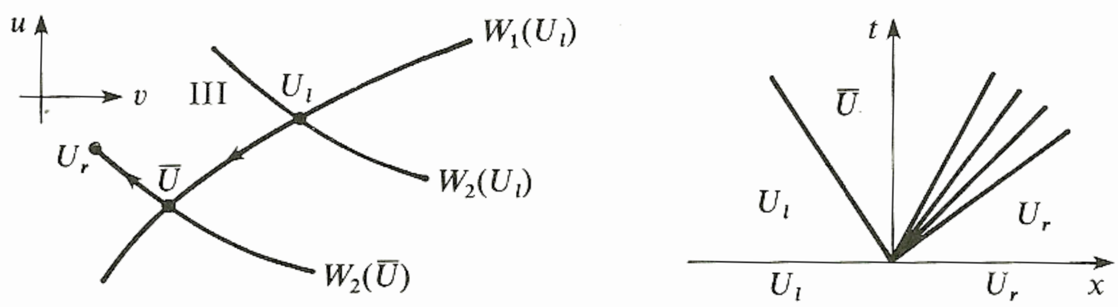

Figure 1.12 (a) (b)

Figure **17.8(not included yet)** illustrates all the possibilities. It remains to determine whether the curves of $\mathscr{F}$ cover the $v-u$ plane univalently. We'll split the proof up into two cases; in the first case we assume U_r is in the open regions **I**, **II**, or **III**; in the second case, we assume U_r lies in region **IV**.

Suppose that U_r lies in region **I**. Let the vertical line $v = v_r$ meet $W_1(U_l)$ at A, and let it meet $W_2(U_l)$ at B. We observe that the subfamily of curves in $\mathscr{F}$ consisting of the set $\{W_2(U) \equiv W_2(\bar{v},\bar{u}) : v_l \le \bar{v} \le v_0\}$, induces a continuous mapping $p \to \phi(p)$ from the arc U_lA to the line segment AB. This follows from transversality, since each of the

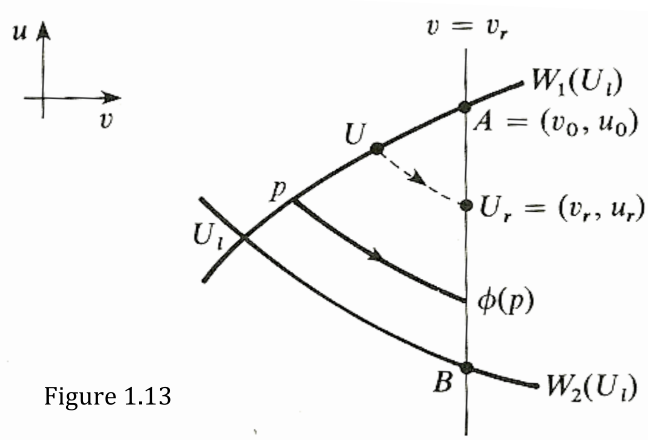

Figure 1.13

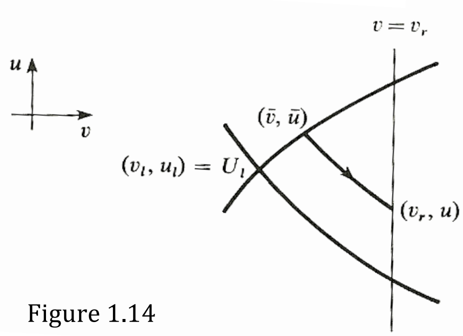

Figure 1.14

S_2 curves have negative slopes. Now in the region $U_l AB$, the slopes of these curves are bounded; thus we see that points P sufficiently close to A must map into points above U_r. Since U_l maps into B, which is below U_r, we see by continuity, that there must be a point U on the arc $U_l A$ which maps into U_r. This shows that region **I** is covered by curves in $\mathscr{F}$. Since a similar argument works if U_r lies in regions **II** or **III**, we see that regions **I**, **II**, and **III** are covered by members of $\mathscr{F}$. This proves the existence of a solution of the Riemann problem (1.10), (1.12), if U_r lies in regions **I**, **II**, or **III** with respect to U_l.

We shall now show that the curves in $\mathscr{F}$ cover regions **I**, **II**, and **III**, univalently; i.e. that through each point U_r belonging to any of the regions **I**, **II**, and **III**, there passes *exactly one* element of $\mathscr{F}$.

Again, let's suppose that U_r lies in region **I.** Referring to Figure 1.14, we se that it suffices to show that $\partial u / \partial \bar{v} > 0$. Now using the two equations

$$\bar{u} = u_l + \int_{v_l}^{\bar{v}} \sqrt{-p'(y)}dy \qquad \text{and} \qquad u = \bar{u} - \sqrt{(v_r - \bar{v})(p(\bar{v}) - p(v_r))},$$

we compute

$$\begin{aligned}\frac{\partial u}{\partial \bar{v}} &= \frac{\partial \bar{u}}{\partial \bar{v}} - \frac{1}{2\sqrt{(v_r - \bar{v})(p(\bar{v}) - p(v_r))}}\left\{(v_r - \bar{v})p'(\bar{v}) - (p(\bar{v}) - p(v_r))\right\} \\ &= \sqrt{-p'(\bar{v})} - \frac{(v_r - \bar{v})}{2\sqrt{(v_r - \bar{v})(p(\bar{v}) - p(v_r))}}\left\{p'(\bar{v}) + \frac{p(\bar{v}) - p(v_r)}{\bar{v} - v_r}\right\} > 0\end{aligned}$$

This implies the uniqueness in region **I**; the proofs for regions **II** and **III** are similar.

We turn our attention now to the case where U_r lies in region **IV.** It is perhaps surprising that not every point in this region can always be covered by an element of $\mathscr{F}$. For example, if

$$u_0 \equiv \int_{v_l}^{\infty} \sqrt{-p'(y)}dy < \infty, \qquad (1.23)$$

then it is easy to see that the curve

$$u = r_1(v; v_l, u_l) = u_l + \int_{v_l}^{v} \sqrt{-p'(y)}dy$$

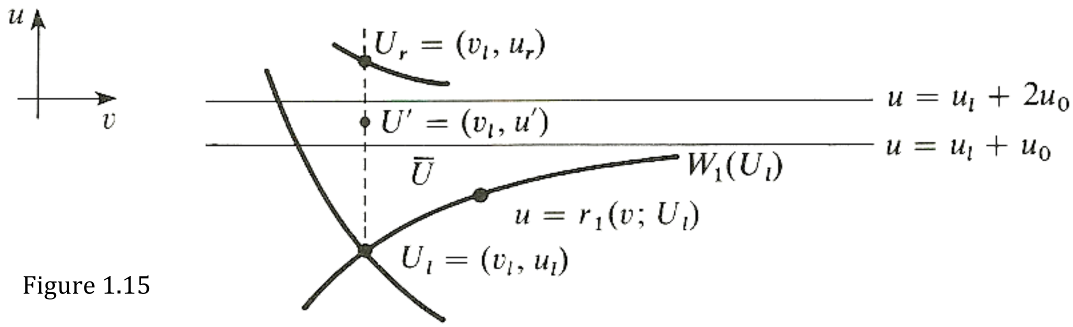

Figure 1.15

has a horizontal asymptote; namely, the line $u = u_l + u_0$. Referring to Figure 1.15, we take $U_r = (v_l, u_r)$ to be in region **IV**, where u_r is chosen so large that $u_r > u_l + 2u_0$. Let $\bar{U} = (\bar{v}, \bar{u})$ be any point on $W_1(U_l)$; then

$$\bar{u} = u_l + \int_{v_l}^{\bar{v}} \sqrt{-p'(y)}dy \le u_l + u_0.$$

The point U' on $W_2(\bar{U})$ with abscissa v_l, has ordinate

$$u' = \bar{u} - \int_{\bar{v}}^{v_l} \sqrt{-p'(y)}dy = \bar{u} + \int_{v_l}^{\bar{v}} \sqrt{-p'(y)}dy \le \bar{u} + \int_{v_l}^{\infty} \sqrt{-p'(y)}dy \le u_l + 2u_0 < u_r$$

Thus U_r cannot lie on any curve in $\mathscr{F}$!

We have shown that if (1.23) holds, then the region **IV** is not covered by curves in $\mathscr{F}$. Generally speaking however, (1.23) is true in the interesting examples. Thus, for the case where

$$p(v) = \frac{k}{v^\gamma}, \qquad \gamma \ge 1 \ (k = const. > 0)$$

an easy calculation shows that (1.23) holds if and only if $\gamma > 1$ (the physically relevant range).

The convergence of the integral (1.23) can be given a nice physical interpretation; namely, it corresponds to the appearance of a vacuum. For example, in the shock-tube problem, if the relative velocities on both sides of the membrane in the shock tube are sufficiently large, then a vacuum, or a region void of gas is formed. In this case, $\rho = 0$, or equivalently, $v = \rho^{-1}$ is infinite. To see this mathematically, note that the R_2 curve through U_r is given by

$$u - u_r = -\int_{v_l}^{v} \sqrt{-p'(y)}dy = r_2(v;U_r),$$

and, as above, u has a horizontal asymptote in the region $u < u_r$ (see the dotted curve in Figure 1.15). Thus, along this curve $-\sqrt{-p'(v)} = \lambda_1$, tends to zero as $v \to \infty$, since $p'' > 0$. Similarly, $\lambda_2 = \sqrt{-p'(v)} \to 0$ as $v \to \infty$, if v lies on the R_1-curve through U_l (see Figure 1.15). A "solution" of the problem (1.10), (1.12) in this case is given in Figure 1.16, where we connect U_l on the right by a *complete* back-rarefaction wave, and we connect U_r on the left by a *complete* front-rarefaction wave. The solution is undefined on the line $x = 0$, since $v = +\infty$ (i.e., $\rho = 0$) there, and, of course, u is undefined.

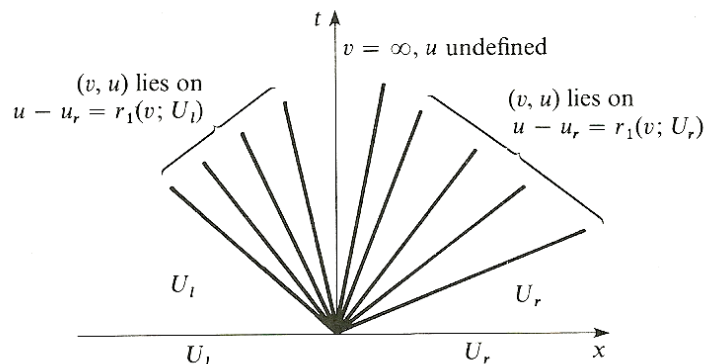

Figure 1.16

On the other hand, even if

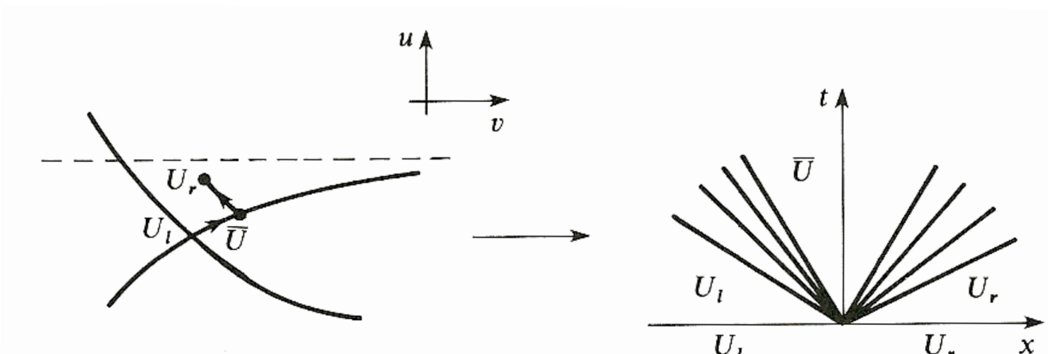

Figure 1.17

(1.23) holds, the vacuum does not necessarily appear, and it is possible to solve the Riemann problem if U_r is in region **IV**, provided that U_l and U_r are close; i.e., provided that $|U_l - U_r|$ is small. This solution is depicted in Figure 1.17.

We have shown how to solve the Riemann problem for the *p*-system (17.1), in the class of (at most three) constant states separated by shocks and rarefaction waves.

It is interesting to note that in the case where $p(v) = k / v^{\gamma}$, $\gamma = 1$; i.e., the isothermal gas case, the vacuum does not appear, since the integrals

$$\int_{v_l}^{\infty} \sqrt{-p'(y)}dy$$

all diverge.

If the solution of the Riemann problem does not assume the vacuum state, then an examination of all the possible cases shows that the intermediate state in the solutions lies between the rarefaction-wave curves determined by the initial states. That is, the solution satisfies the inequalities

$$r \geq \min(r_l, r_r) \equiv \min(r(v_l, u_l), r(v_r, u_r)),$$

$$s \leq \max(s_l, s_r) \equiv \max(s(v_l, u_l), s(v_r, v_r)),$$

where

$$r(v,u) = u - \int_{\bar{v}}^{v} \sqrt{-p'(s)}\,ds,$$

$$s(v,u) = u + \int_{\bar{v}}^{v} \sqrt{-p'(s)}\,ds.$$

Knowing how to solve the Riemann problem enables us to actually solve certain types of *interactions*. For example, suppose that we consider the equations (17.1) with the following initial data:

$$(v,u)(x,0) = \begin{cases} (v_l, u_l), & x < x_1, \\ (v_m, u_m), & x_1 \leq x \leq x_2, \\ (v_r, u_r), & x > x_2, \end{cases}$$

and suppose too that the discontinuity (v_l, u_l), (v_m, u_m) is resolved by a front shock S_1 of speed s_1, and that the discontinuity (v_m, u_m), (v_r, u_r) is also resolved by a front shock S_2 of speed s_2; see Figure 1.18. From (1.14), we find $0 < s_2 < \lambda_2(v_m, u_m) < s_1$, from which it follows that S_1 overtakes S_2 at some time $T > 0$. Notice that at $t = T$, we again have a Riemann problem with data (v_l, u_l), (v_r, u_r). In order to solve this problem, we must determine in what "quadrant" (v_r, u_r) lies in with respect to (v_l, u_l); see Figure 1.19.

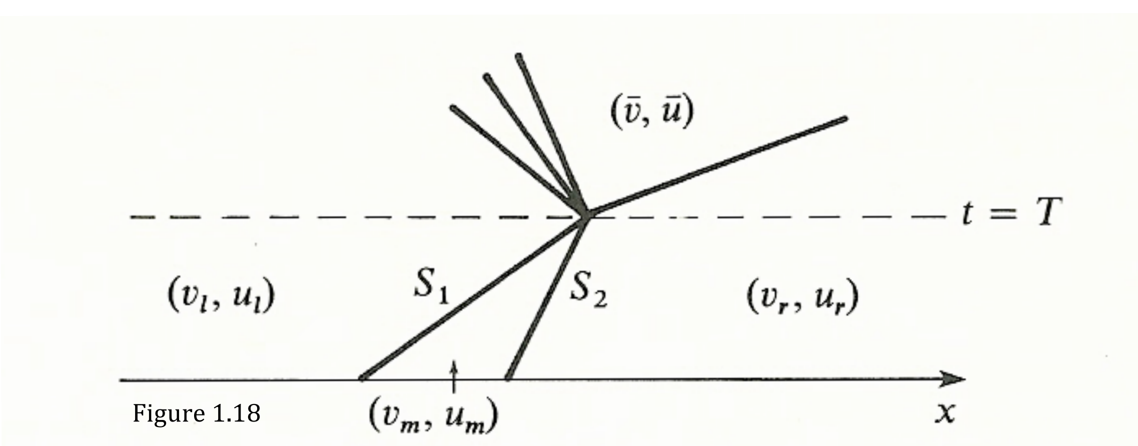

Figure 1.18

Our claim is that (v_r, u_r) lies in the first quadrant (not the second), so that the Riemann problem is resolved by a back-rarefaction wave and a front-shock, as depicted in Figure 1.18. To see this, note that (v_m, u_m) lies on the front-shock curve S_2 starting at (v_l, u_l), and (v_r, u_r) lies on the front-shock curve $\tilde{S}_2$ starting at (v_m, u_m). If we can show that $\tilde{S}_2$ always lies above S_2 (as depicted in Figure 1.19), we will have proved our claim. To this end, we shall prove that: (i) the slope of $\tilde{S}_2$ at (v_m, u_m) is greater than the slope of S_2 at (v_m, u_m), so that $\tilde{S}_2$ "breaks into" the depicted region; and (ii), $\tilde{S}_2$ never meets S_2 for $v > v_m$.

To show (i), note that S_2 is given by the equation

$$u - u_l = -\sqrt{(v - v_l)(p(v_l) - p(v))}$$

so that the slope of S_2 at (v_m, u_m) is

$$\frac{p'(v_m) + \dfrac{p(v_l) - p(v_m)}{v_l - v_m}}{2\sqrt{-\dfrac{p(v_l) - p(v_m)}{v_l - v_m}}} = \frac{p'(v_m) + p'(\xi)}{2\sqrt{-p'(\xi)}}, \qquad v_l < \xi < v_m$$

The slope of $\tilde{S}_2$ at (v_m, u_m) is $-\sqrt{-p'(v_m)}$ since the (normalized) right eigenvector at (v_m, u_m) is $(1, \sqrt{-p'(v_m)})^t$. Thus, since

$$-\sqrt{-p'(v_m)} > \frac{p'(v_m) + p'(\xi)}{2\sqrt{-p'(\xi)}},$$

statement (i) follows. To prove (ii), we define a new function $\phi(x,y) = \sqrt{(x - y)(p(y) - p(x))}$. It is not hard to show that if $x > y > z$, then $\phi(x,z) > \phi(x,y) + \phi(y,z)$. Thus, if there were a point $(v,u) \in S_2 \cap \tilde{S}_2$, with $v > v_m$, then $v > v_m > v_l$, and

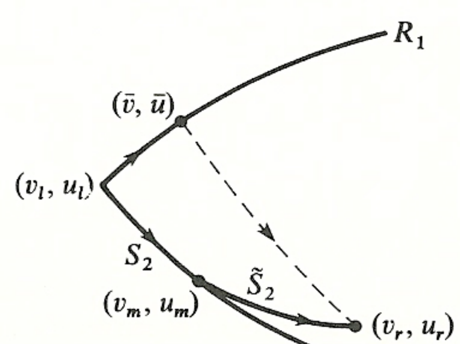

Figure 1.19

$$u - u_m = -\phi(v, v_m), \qquad u - u_l = -\phi(v, v_l).$$

This gives

$$\phi(v_l, v_m) = u_l - u_m = \phi(v, v_l) - \phi(v, v_m) > \phi(v_m, v_l)$$

which is a contradiction, and the proof is complete.