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LINEAR WAVES THAT EXPRESS THE SIMPLEST POSSIBLE PERIODIC STRUCTURE OF THE COMPRESSIBLE EULER EQUATIONS*

Dedicated to Professor James Glimm on the occasion of his 75th birthday

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Abstract In this paper we show how the simplest wave structure that balances compression and rarefaction in the nonlinear compressible Euler equations can be represented in a solution of the linearized compressible Euler equations. Such waves are exact solutions of the equations obtained by linearizing the compressible Euler equations about the periodic extension of two constant states separated by entropy jumps. Conditions on the states and the periods are derived which allow for the existence of solutions in the Fourier 1-mode. In [3, 4, 5] it is shown that these are the simplest linearized waves such that, for almost every period, they are isolated in the kernel of the linearized operator that imposes periodicity, and such that they perturb to nearby nonlinear solutions of the compressible Euler equations that balance compression and rarefaction along characteristics in the formal sense described in [3]. Their fundamental nature thus makes them of interest in their own right.

Key words compressible Euler; periodic solutions; conservation laws

2000 MR Subject Classification 35L65; 76N10

1 Introduction

In this note, which is a self contained reduction of Sections 3 to 5 of [4], we outline the simplest route to a closed form expression for the simplest linearized solutions of the compressible Euler equations that formally balance compression and rarefaction along characteristics under infinitesimal perturbation. These solutions $V(t)$ solve the linearized eigenvalue problem $\{\mathcal{M}-\mathcal{I}\}\,V=0$ which expresses periodicity at the linearized level, (see below). The author's

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work in [3, 4, 5] is strong evidence that this wave pattern is physically realized in nearby time-periodic solutions of the Euler equations themselves, but a complete mathematical proof remains open.

In [3] the authors first derived the wave pattern as the simplest possible wave pattern that formally balances compression and rarefaction along characteristics. In [4] the authors showed that for almost every period, $\mathcal{M} - \mathcal{I}$ is non-resonant in the sense that it is invertible on the complement of the solution kernel, with algebraic bounds on eigenvalues. In [5] we showed that the problem is amenable to a Liapunov-Schmidt decomposition, thereby fully reducing the problem of existence of periodic solutions of compressible Euler to a new KAM type small divisor problem in bifurcation theory. A corollary of this provides a proof that periodic solutions of Euler exist up to an arbitrarily high Fourier cutoff. Given the many successes of KAM theory in similar problems, the authors believe one can only expect that the technical issues with small divisors can be overcome, and that nearby time-periodic solutions of the nonlinear compressible Euler equations exist having the same wave structure entailed by the linearized solutions derived here, cf. [1].

But these linearized time-periodic solutions have reason to be interesting in their own right, not the least of which is the discovery of new phenomena when the sound speed is incommensurate with the period. The consistency of this possibility at the linearized level comes naturally out of the analysis, and implies chaotic motion of sound wave trajectories relative to the solution periods. For example, the wave crests propagate at the speed of the period, not the sound speed. This then identifies a new ergodic way in which nearby nonlinear solutions can find a balance between compression and rarefaction along characteristics in the sense of [3]. Moreover, in [4] the authors discovered that the linearized operator is non-resonant only in the case when the wave speed is incommensurate with the period. Thus, for example, if the resonances are not just anomalies that go away at the next order of approximation of the nonlinear solution, it raises the possibility that this is prerequisite for nonlinear perturbation.

In Section 2 we review the compressible Euler equations and introduce symmetric coordinates. In Section 3 we introduce the simplest eigenvalue problem for periodicity that is consistent with the formal balancing of compression and rarefaction along characteristics. In Section 4 we derive a non-dimensional form for the compressible Euler equations amenable to the solution of the linearized eigenvalue problem. In Section 5, we reduce the linearized eigenvalue problem to a series of matrix problems, which we analyze in Section 6. Finally, in Section 6, we explicitly describe the simplest wave structure and briefly discuss the perturbation problem. One of our main objectives is to exhibit how the non-dimensional formulation of the problem allows us to represent linear evolution by pure rotation in each invariant subspace of the linear eigenvalue problem, thereby facilitating a very clean geometrical method for constructing the solution.

2 The Compressible Euler Equations

The compressible Euler equations describe the time evolution of a perfect fluid in the absence of dissipative effects. These are

$$\rho_t + \mathrm{div}[\rho\mathbf{u}] = 0, \tag{1}$$

$$(\rho u^i)_t + \mathrm{div}[\rho u^i \mathbf{u}] = -\nabla p, \tag{2}$$

$$E_t + \mathrm{div}[(E+p)\mathbf{u}] = 0, \tag{3}$$

describing conservation of mass, momentum and energy, respectively. Here the physical fields are density ρ, velocity $\mathbf{u} \in \mathbb{R}^3$ and energy density $E = \frac{1}{2}\rho\mathbf{u}^2 + \rho\varepsilon$, with ε the specific internal energy. To close the system, the pressure p is related to ε and ρ by an equation of state. We consider a polytropic gamma-law gas, described by

$$\varepsilon = c_\tau \, \tau^{-(\gamma-1)} \, \mathrm{e}^{S/c_\tau} \quad \text{and} \quad p = \frac{c_\tau}{\gamma - 1} \, \tau^{-\gamma} \, \mathrm{e}^{S/c_\tau}, \tag{4}$$

where $\tau = 1/\rho$ is the specific volume, S is the specific entropy, $\gamma > 1$ is the adiabatic gas constant, and c_τ the specific heat [2]. The (Lagrangian) sound speed is defined by

$$c(\tau, S) = \sqrt{-p_\tau(\tau, S)}. \tag{5}$$

For smooth solutions, the energy equation (3) is equivalent to the adiabatic constraint or entropy equation

$$(\rho S)_t + \mathrm{div}(\rho S \mathbf{u}) = 0, \tag{6}$$

which states that entropy is transported with the fluid [2].

For sound wave propagation in one direction x', the equations reduce to the system of 3×3 Euler equations

$$\begin{aligned} &\rho_t + (\rho\, u)_{x'} = 0, \\ &(\rho u)_t + (\rho\, u^2 + p)_{x'} = 0, \\ &E_t + [(E+p)\, u]_{x'} = 0. \end{aligned}$$

In a Lagrangian (co-moving) frame of reference, the equations are

$$\tau_t - u_x = 0, \tag{7}$$

$$u_t + p_x = 0, \tag{8}$$

$$E^*_t + (up)_x = 0, \tag{9}$$

where x is the (Lagrangian) material coordinate, given by

$$x = \int \rho \, \mathrm{d}x',$$

and x' is the (Eulerian) spatial coordinate, with $E^* = \frac{1}{2}u^2 + \varepsilon$. In this Lagrangian frame, the adiabatic constraint (6) takes on the particularly simple form

$$S_t = 0, \tag{10}$$

which can be used instead of (9) on smooth solutions [2].

Because we are treating solutions which are (piecewise) smooth, we consider equations (7), (8) and (10). We recall from [3] the change of variables

$$m = \mathrm{e}^{S/2c_\tau} \quad \text{and} \quad z = K_z \, \tau^{-\frac{\gamma-1}{2}}, \tag{11}$$

so that (4) becomes

$$\varepsilon = K_\varepsilon\, m^2\, z^2 \quad \text{and} \quad p = K_p\, m^2\, z^{\frac{2\gamma}{\gamma-1}}, \tag{12}$$

and $K_{\cdot}$'s are appropriately given constants, while the sound speed (5) is given by

$$c(m,z) = K_c\, m\, z^{\frac{\gamma+1}{\gamma-1}}. \tag{13}$$

In these variables, for smooth solutions, equations (7)–(10) can be written

$$\begin{aligned} &z_t + \frac{c}{m} u_x = 0, \\ &u_t + mcz_x + 2\frac{p}{m} m_x = 0, \\ &m_t = 0, \end{aligned} \tag{14}$$

where we have used (10).

Recall that a strictly hyperbolic 3×3 system has three wave families, each corresponding to an eigenvalue or wavespeed of the system. In the Lagrangian frame, the wavespeeds of system (14) are $\pm c$ and 0. The nonlinear waves are the forward and backward waves, with respective wavespeeds $+c$ and $-c$, and stationary waves of zero wavespeed, which are contact or jump discontinuities and which are linearly degenerate.

We consider solutions in which the entropy m (or S) is piecewise constant, varying periodically in space but stationary in time. To resolve the jump in variables across the entropy jumps (stationary waves), we must apply the Rankine-Hugoniot conditions, which in Lagrangian coordinates are

$$\begin{aligned} &[u] = s[-\tau] \\ &[p] = s[u] \\ &[u\, p] = s\Big[\frac{1}{2}u^2 + \varepsilon\Big], \end{aligned} \tag{15}$$

where, as usual, s is the speed of propagation of the discontinuity and $[\cdot]$ is the jump. Since we are concerned only with contact discontinuities, we take $s = 0$, and the jump conditions become $[u] = [p] = 0$, or

$$u_L = u_R \quad \text{and} \quad m_L\, z_L^{\frac{\gamma-1}{\gamma}} = m_R\, z_R^{\frac{\gamma-1}{\gamma}}. \tag{16}$$

On regions where the entropy is constant, the 3×3 system reduces to the 2×2 quasilinear system

$$\begin{aligned} &z_t + \frac{c}{m} u_x = 0, \\ &u_t + mcz_x = 0, \end{aligned} \tag{17}$$

which is just the p-system using z as the thermodynamic coordinate [6].

3 The Eigenvalue Problem

We now reformulate the problem of existence of a periodic solution of the compressible Euler equations as a nonlinear eigenvalue (fixed point) problem. We will then nondimensionalize and find exact solutions of the corresponding linearized eigenvalue problem, and briefly discuss the perturbation problem of obtaining periodic solutions of the nonlinear problem from these exact solutions of the linearized problem.

The periodic structure described in [3] is supported on an entropy field that oscillates between two different values separated by contact discontinuities. We are looking for solutions which are periodic an space and time, and want to describe the problem in some convenient basis. Since we assume the solution is smooth in time but has jumps in space, in order to formulate the eigenvalue problem, we use the space variable x as the evolution variable. Following [3], the evolution problem then becomes: starting at $x = 0$, evolve across the first entropy level, then jump to the second entropy level, evolve at this level and jump back to the first entropy level, and continue. The simplest periodic configuration identified in [3] requires four such evolutions (two at each level), and we impose a further symmetry by replacing the second two by a half-period time shift.

So consider a smooth solution $U(x,t) = (z(x,t), u(x,t))$ of (17) evolving through an entropy field that oscillates between two values $\overline{m}$ and $\underline{m}$, with $\overline{m} > \underline{m}$, where we have used (11). Call the corresponding widths of the entropy levels $\overline{x}$ and $\underline{x}$, so that $m(x,t) = \overline{m}$ for $0 < x < \overline{x}$ and $m(x,t) = \underline{m}$ for $\overline{x} < x < \overline{x} + \underline{x}$, and continue this periodically in x. Similarly, denote the corresponding states by $\overline{U}(x,t) = (\overline{z}(x,t), \overline{u}(x,t))$ and $\underline{U}(x,t) = (\underline{z}(x,t), \underline{u}(x,t))$, respectively. Thus $\overline{U}(x,t)$ solves

$$\begin{aligned} &z_t + \frac{c(z)}{\overline{m}} u_x = 0, \\ &u_t + \overline{m}\, c(z) z_x = 0, \end{aligned}$$

for $0 < x < \overline{x}$, and $\underline{U}(x,t)$ solves the same system with $\overline{m}$ replaced by $\underline{m}$ for $\overline{x} < x < \overline{x} + \underline{x}$.

To ensure that we have an exact weak solution, we need to enforce the Rankine-Hugoniot conditions (16) at the entropy jumps at $x = \overline{x}$ and $x = 0$ (or $\overline{x} + \underline{x}$). Thus we must have

$$\overline{u} = \underline{u} \quad \text{and} \quad \underline{z} = \overline{z} \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{\gamma-1}{\gamma}}, \tag{18}$$

which also implies

$$\underline{m}\, \underline{z} = \overline{m}\, \overline{z} \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{-1}{\gamma}} \quad \text{and} \quad \underline{c} = \overline{c} \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{1}{\gamma}}.$$

Note that we can regard (18) as a linear operator, since $\overline{m}$ and $\underline{m}$ are fixed.

We now introduce convenient notation from [3, 4]. First, denote the limiting states at the entropy jumps by

$$\begin{aligned} &\dot{U}(t) = U(0+,t), \quad \check{U}(t) = U(\overline{x}-,t), \quad \tilde{U}(t) = U(\overline{x}+,t), \\ &\hat{U}(t) = U(\overline{x}+\underline{x}-,t) \quad \text{and} \quad U^*(t) = U(\overline{x}+\underline{x}+,t). \end{aligned} \tag{19}$$

Next, denote the nonlinear (spatial) evolution operators by $\overline{\mathcal{E}}$ and $\underline{\mathcal{E}}$, respectively, and the (linear) jump operators at $\overline{x}$ and $\overline{x} + \underline{x}$ by $\mathcal{J}$ and $\mathcal{J}^{-1}$, respectively. Then we have

$$\begin{aligned} &\check{U} = \overline{\mathcal{E}} \dot{U}, \\ &\tilde{U} = \mathcal{J} \check{U}, \\ &\hat{U} = \underline{\mathcal{E}} \tilde{U} \quad \text{and} \quad U^* = \mathcal{J}^{-1} \hat{U}. \end{aligned} \tag{20}$$

Finally, we restrict the "initial data" $\dot{U}(t)$ to 2π-periodic functions at $x=0$, and denote the half period shift operator by $\mathcal{S}$, so that $[\mathcal{S}U](t)=U(t+\pi)$.

With this notation, the (fully nonlinear) eigenvalue problem that captures the simplest periodic structure of [3] simply becomes the condition

$$\mathcal{S}\,U^{*}=\dot{U}. \tag{21}$$

Eliminating the intermediate states, we can write the nonlinear eigenvalue problem as

$$\mathcal{S}\cdot\mathcal{J}^{-1}\cdot\underline{\mathcal{E}}\cdot\mathcal{J}\cdot\overline{\mathcal{E}}\,\dot{U}=\dot{U}. \tag{22}$$

Note that the half period shift just imposes an extra symmetry, and we could also get the periodic structure of [3] by repeating the jumps and shifts. We conclude that a (smooth enough) solution to the eigenvalue problem (22) corresponds to an exact shock-free periodic solution to the compressible Euler equations.

4 Non-dimensionalization

We now further simplify (22) by recasting it in non-dimensional form. We do this by rescaling the state variables z and u and the spatial variable x. To this end, restrict first to smooth solutions $U(x,t)=(z(x,t),u(x,t))$ of the compressible Euler equations defined at constant entropy $m\equiv m_0$, in a region $x\geq x_0$. Let z_0 and u_0 be base states from which values of z and u are measured, respectively, and set $c_0=c(m_0,z_0)$ equal to the sound speed at (z_0,m_0), cf. (13). Give time and space the same dimension by defining y through the relation

$$y-y_0=\frac{x-x_0}{c_0}, \tag{23}$$

so that equations (17) take on the dimensionless form

$$\begin{aligned}\left(\frac{z}{z_0}\right)_t+\frac{c(z)}{c_0}\left(\frac{u}{m_0z_0}\right)_y&=0,\\ \left(\frac{u}{m_0z_0}\right)_t+\frac{c(z)}{c_0}\left(\frac{z}{z_0}\right)_y&=0,\end{aligned} \tag{24}$$

in the region $y\geq y_0$. Now define the dimensionless variables

$$w=\frac{z}{z_0}\quad\text{and}\quad v=\frac{u-u_0}{m_0z_0}, \tag{25}$$

and let

$$\sigma=\frac{c(z_0)}{c(z)}=\frac{z_0^d}{z^d}=w^{-d}\equiv\sigma(w), \tag{26}$$

where we have used (13), namely $c(m,z)=K_cmz^d$, with

$$d\equiv\frac{\gamma+1}{\gamma-1}>1, \tag{27}$$

and γ is the gas constant (4). Then the nonlinear equations (24) have the non-dimensional form

$$\begin{aligned}w_y+\sigma(w)\;v_t&=0,\\ v_y+\sigma(w)\;w_t&=0,\end{aligned} \tag{28}$$

and where we have explicitly made y the evolution variable.

Across an entropy jump between two constant values $\overline{m}$ and $\underline{m}$ with base states $\overline{U}_0$ and $\underline{U}_0$, respectively, the jump conditions are

$$[p] = \overline{m}^2\,\overline{z}^{d+1} - \underline{m}^2\,\underline{z}^{d+1} = 0, \quad \text{and} \quad [u] = \overline{u}_0 - \underline{u}_0 + \overline{m}\,\overline{z}_0\,\overline{v} - \underline{m}\,\underline{z}_0\,\underline{v} = 0. \tag{29}$$

Thus, provided the base states $\overline{U}_0$ and $\underline{U}_0$ satisfy the jump condition (29) as well, we find that the jump conditions in dimensionless variables take on the simple form

$$\overline{w} = \underline{w} \quad \text{and} \quad \overline{m}^{\frac{d-1}{d+1}}\,\overline{v} = \underline{m}^{\frac{d-1}{d+1}}\,\underline{v}, \tag{30}$$

the latter following by using the jump relation for the base states in the form

$$\frac{\overline{z}_0}{\underline{z}_0} = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{-2}{d+1}}.$$

In particular, (30) implies that the dimensionless wavespeed $\sigma(w)$ is continuous across entropy jumps.

The following observation allows us to break translation symmetry in t, further simplifying our representation of solutions. The proof is straightforward: see [4] for more details.

Lemma 1 The property that w be an even function and v an odd function of t is preserved under the evolution of equations (28). The same property is also preserved under the jumps (30) and the half-period shift at the entropy discontinuities.

Since our functions are 2π-periodic in t, the lemma implies that w and v can be described by Fourier cosine and sine series, respectively. Before explicitly doing so, we develop notation for the eigenvalue problem that expresses periodicity in dimensionless variables.

Consider the nonlinear problem described above for smooth solutions evolving through two entropy levels $\overline{m}$, $\underline{m}$ of widths $\overline{x}$, $\underline{x}$, with base states $\overline{U}_0$, $\underline{U}_0$, respectively, and extended periodically in x. Then define $y = y(x)$ as the unique piecewise linear Lipschitz continuous function of x such that $y(0) = 0$, and such that (23) holds at each entropy level. That is, $y(0) = 0$, $\mathrm{d}y/\mathrm{d}x = 1/\overline{c}$ in each $\overline{m}$ level, and $\mathrm{d}y/\mathrm{d}x = 1/\underline{c}$ in each $\underline{m}$ level. Assuming this, the entropy levels $0 < x < \overline{x}$ and $\overline{x} < x < \overline{x} + \underline{x}$ map to $0 < y < \overline{\theta}$ and $\overline{\theta} < y < \overline{\theta} + \underline{\theta}$, respectively, where

$$\overline{\theta} = \frac{\overline{x}}{c(\overline{z}_0)} \quad \text{and} \quad \underline{\theta} = \frac{\underline{x}}{c(\underline{z}_0)}. \tag{31}$$

The (different) nonlinear operators $\overline{\mathcal{E}}$ and $\underline{\mathcal{E}}$, expressed in dimensionless variables (w, v), reduce to evolution in y by the same system (28) for $\overline{\theta}$ and $\underline{\theta}$, respectively. We now make one more simplifying assumption: namely, we assume that the (dimensionless) widths of the different entropy levels are the same, $\overline{\theta} = \underline{\theta} \equiv \theta$. Let $V = (w, v)$ denote the dimensionless variables, and let $\mathcal{E}(\theta)$ be evolution by system (28) through a y-interval of length θ. That is,

$$\mathcal{E}(\theta)V(0,\cdot) = V(\theta,\cdot), \tag{32}$$

where $V(y,t)$ is the unique solution of the Cauchy problem for system (28) with Cauchy data $V(0,\cdot)$. Also define the entropy jump operator $\mathcal{J}$ acting on V pointwise by

$$\mathcal{J}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}\begin{bmatrix} w \\ v \end{bmatrix}, \tag{33}$$

where J is the scalar

$$J=\left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}}, \tag{34}$$

as in (30). Note that $\mathcal{E}(\theta)$ is nonlinear, but $\mathcal{J}$ and $\mathcal{S}$ are both linear operators when $\overline{m}$ and $\underline{m}$ are assumed fixed. We now show that each solution of the non-dimensional problem leads to a family of dimensional solutions.

Theorem 1 For fixed positive real numbers θ and J, define the nonlinear operator $\mathcal{N}\equiv\mathcal{N}(\theta,J)$ by

$$\mathcal{N}\equiv\mathcal{S}\cdot\mathcal{J}^{-1}\cdot E(\theta)\cdot J\cdot E(\theta), \tag{35}$$

and let $V(t)=(w(t),v(t))$ denote any smooth solution of

$$\mathcal{N}V(\cdot)=V(\cdot), \tag{36}$$

that satisfies the average one and zero average conditions

$$\frac{1}{2\pi}\int_0^{2\pi}w(t)\mathrm{d}t=1,\quad\text{and}\quad\frac{1}{2\pi}\int_0^{2\pi}v(t)\mathrm{d}t=0, \tag{37}$$

respectively. Then, given *any* base state $U_0=(\overline{z}_0,\overline{u}_0)$ and entropy state $\overline{m}$, there is a periodic solution $U(x,t)=(z(x,t),u(x,t)$ of (7)–(9), determined uniquely by $V(t)$, with average values

$$\frac{1}{2\pi}\int_0^{2\pi}z(0,t)\mathrm{d}t=\overline{z}_0,\quad\text{and}\quad\frac{1}{2\pi}\int_0^{2\pi}u(0,t)\mathrm{d}t=\overline{u}_0.$$

Proof Given the states $\overline{z}_0$, $\overline{u}_0$ and $\overline{m}$, and the parameters θ and J, define

$$\underline{u}_0=\overline{u}_0,\quad\underline{m}=\overline{m}\,J^{-\frac{d+1}{d-1}},\quad\text{and}\quad\underline{z}_0=\left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{2}{d+1}}\overline{z}_0,$$

and set the values of $\overline{x}$ and $\underline{x}$ equal to

$$\overline{x}_0=c(\overline{z}_0)\,\theta,\quad\text{and}\quad\underline{x}_0=c(\underline{z}_0)\,\theta.$$

Now define the nonlinear evolution operator

$$V(y,\cdot)=\mathcal{N}(y)[V(\cdot)]\equiv\begin{cases}\mathcal{E}(y)[V(\cdot)], & 0\le y<\theta,\\ \mathcal{E}(y-\theta)\ \mathcal{J}\ \mathcal{E}(\theta)[V(\cdot)], & \theta<y<2\theta,\\ \mathcal{S}\ \mathcal{J}^{-1}\ \mathcal{E}(\theta)\ \mathcal{J}\ \mathcal{E}(\theta)[V(\cdot)], & y=2\theta.\end{cases} \tag{38}$$

By this definition, $\mathcal{N}(y)[V(\cdot)]$ defines the evolution of initial data $V(t)$ through interval $[0,y)$, for the dimensionless nonlinear problem consisting of entropy jumps at $y=\theta$ and $y=2\theta$. Thus it follows directly from (35) that $V(y,t)$ extends to a global periodic solution of the non-dimensionalized equations having periodic tile $0<t<2\pi$, $0<y<2\theta$, with periodic motion vector $\mathbf{X}=(2\theta,\pi)$. Thus defining $U(x,t)=(z(x,t),u(x,t))$ by

$$z=w\,z_0,\quad u=u_0+m\,z_0\,v,$$

from our constructions above, $U(x,t)$ is a global periodic solution of (7)–(9) having periodic tile $0<t<2\pi$, $0<x<\overline{x}+\underline{x}$, with periodic motion vector $\mathbf{X}=(\overline{x}+\underline{x},\pi)$. The correspondence of the averages follows directly.

Note that the periodic motion vector $\mathbf{X}$ determines the "group velocities" of the solution: these are the effective propagation speeds of global features (such as local extrema), given by

$$c_g = \pm\frac{\overline{x} + \underline{x}}{\pi}, \tag{39}$$

and these are different from the local characteristic speeds $\pm c$. Thus we have dispersive type behavior, with different group and phase velocities, in a purely hyperbolic nonlinear system. To our knowledge, this is the first discovery of such an effect.

In summary, our fundamental nonlinear problem is the dimensionless eigenvalue problem (36), where the operator $\mathcal{N}$ is defined in (35). Solutions of (36) correspond to periodic solutions of (7)–(9) in a neighborhood of any given state U_0, $\overline{m}$.

5 Periodic Solutions of the Linearized Problem

Since constant states are solutions of equations (28), and since $\mathcal{J}$ has the simple diagonal form (33), it is evident that the average conditions (37) are preserved by the fully nonlinear operator $\mathcal{N}$, so that $V_0 = (1, 0)$ is an exact solution of the nonlinear eigenvalue problem, corresponding to a stationary square wave in the entropy field in the Euler equations (7)–(9).

We now linearize about the (unique) constant solution satisfying (37), to obtain a linearized eigenvalue problem. Since $\mathcal{J}$ and $\mathcal{S}$ are already linear, we need only linearize the evolution: since $\sigma = w^{-d}$ by (26), linearizing (28) around $V_0 = (1, 0)$ gives the linear wave equation

$$\begin{aligned} w_y + v_t &= 0, \\ v_y + w_t &= 0. \end{aligned} \tag{40}$$

We denote evolution (in y) by this equation by $\mathcal{L}(y)$, so that

$$\mathcal{L}(y)V(0,\cdot) \equiv V(y,\cdot), \tag{41}$$

where $V(y,t)$ is the unique solution of (40).

Now to get the linearized eigenvalue problem, we simply replace the nonlinear evolution operator $\mathcal{E}(\theta)$ by the linear operator $\mathcal{L}(\theta)$. Then the linearized eigenvalue problem associated with (36) is the problem

$$\mathcal{M}\,V(\cdot) = V(\cdot), \tag{42}$$

where $\mathcal{M} \equiv \mathcal{M}(\theta, J)$ is the linear operator defined by

$$\mathcal{M} \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\theta) \cdot \mathcal{J} \cdot \mathcal{L}(\theta), \tag{43}$$

where again $\mathcal{J}$ is the linear jump operator (33) and $\mathcal{S}$ is the half-period shift.

In analogy with (19), we define the limiting states at the entropy jumps for the linear problem by

$$\begin{aligned} &\dot{V}(t) = V(0+,t), \quad \check{V}(t) = V(\theta-,t), \quad \tilde{V}(t) = V(\theta+,t), \\ &\hat{V}(t) = V(2\,\theta-,t) \quad \text{and} \quad V^*(t) = V(2\,\theta+,t). \end{aligned} \tag{44}$$

Then, as in (20) we have

$$\check{V} = \mathcal{L}(\theta)\,\dot{V}, \quad \tilde{V} = \mathcal{J}\,\check{V}, \quad \hat{V} = \mathcal{L}(\theta)\,\tilde{V} \quad \text{and} \quad V^* = \mathcal{J}^{-1}\,\hat{V}, \tag{45}$$

and the linearized eigenvalue problem is

$$\mathcal{S}\,V^*(t) = \dot{V}(t). \tag{46}$$

Also, as in (38), we let $V(y,\cdot)$ be the function obtained at $y \in (0, 2\theta)$ in the linear evolution: that is,

$$V(y,\cdot) = \begin{cases} \mathcal{L}(y)V_0, & \text{for} \quad 0 < y < \theta, \\ \mathcal{L}(y-\theta)\mathcal{J}\mathcal{L}(\theta)V_0, & \text{for} \quad \theta < y < 2\theta. \end{cases} \tag{47}$$

We will let V refer to $V(t)$ or $V(y,t)$ according to the context.

We now find elements of the kernel of the linear operator $\mathcal{M}-\mathcal{I}$, which yield solutions of the linearized eigenvalue problem (42). In the next section, we give conditions on the parameters that isolate these solutions in the kernel, which we expect will allow the kernel to perturb, generating periodic solutions of the nonlinear problem (36).

We note that Lemma 1 continues to hold for the linearized equation, so that $w(y,t)$ and $v(y,t)$ admit Fourier cosine and sine expansions, respectively. Thus we consider the set

$$\Sigma \equiv L^2_{\text{even}}[0,2\pi) \times L^2_{\text{odd}}[0,2\pi),$$

and we take $V(t) \in \Sigma$ to have the Fourier expansion

$$V(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} w_n \cos nt \\ v_n \sin nt \end{bmatrix} \in \Sigma, \tag{48}$$

where $V_n \equiv (w_n, v_n) \in \mathbb{R}^2$. The expression (48) gives an expression for $w(t)$ and $v(t)$ in terms of the orthonormal basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos nt, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin nt \right\}_{n=0}^{+\infty},$$

for the 2π-periodic, square integrable complex valued functions $(w(t), v(t))$, even in w odd in v, defined on $0 \le t < 2\pi$. That is, let

$$\Sigma_n \equiv \text{Span}\left\{ \begin{bmatrix} \cos nt \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sin nt \end{bmatrix} \right\}, \quad \text{so that} \quad \Sigma = \bigoplus \Sigma_n \tag{49}$$

gives an orthogonal decomposition of Σ with respect to the L^2-inner product

$$\left\langle \begin{bmatrix} w_1(\cdot) \\ v_1(\cdot) \end{bmatrix}, \begin{bmatrix} w_2(\cdot) \\ v_2(\cdot) \end{bmatrix} \right\rangle = \frac{1}{\pi}\int_0^{2\pi} [w_1(t)w_2(t) + v_1(t)v_2(t)]\,\mathrm{d}t.$$

We now show that $\mathcal{M}$ factors over the decomposition (49). To this end, let $\mathcal{T}_n : \mathbb{R}^2 \to \Sigma_n$ be the representation of Σ_n defined by

$$\mathcal{T}_n \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos nt \\ b\sin nt \end{bmatrix}, \tag{50}$$

and as in (48), write

$$\begin{bmatrix} w_0(t) \\ v_0(t) \end{bmatrix} = \sum_0^{+\infty} \mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix},$$

where $a_0 = 1$ and $b_0 = 0$.

Lemma 2 The linear operators $\mathcal{L}(\theta)$, $\mathcal{J}$ and $\mathcal{S}$ respect the orthogonal decomposition (49), and moreover, the following formulas hold:

$$\mathcal{L}(\theta)\,\mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \mathcal{T}_n\, R(n\theta) \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \tag{51}$$

$$\mathcal{J}\,\mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \mathcal{T}_n\, D \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \tag{52}$$

$$\mathcal{S}\,\mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \mathcal{T}_n\, (-1)^n \begin{bmatrix} a_n \\ b_n \end{bmatrix}. \tag{53}$$

Here $R(\alpha)$ denotes counterclockwise rotation in $\mathbb{R}^2$ through angle α,

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

and $D \equiv D(J)$ denotes the diagonal 2×2 real matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \quad \text{with} \quad J = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}} = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{1}{\gamma}}. \tag{54}$$

Proof Equation (51) follows by separation of variables for the linear wave equation (28), cf. [4]. Next, (52) follows directly from (33), and (53) follows from the definition of the shift operator $\mathcal{S}$.

Because each of these operators respects (49), the infinite dimensional linear operator $\mathcal{M}$ reduces to a 2×2 matrix on each subspace Σ_n.

Theorem 2 The linear operator $\mathcal{M}$ respects the orthogonal decomposition into Fourier modes (49). That is, we can write

$$\mathcal{M}\left(\sum \mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix}\right) = \sum \mathcal{T}_n\, M_n \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \tag{55}$$

where $M_n = M_n(\theta, J)$ is the real 2×2 matrix given by

$$M_n = (-1)^n M(n\theta, J), \tag{56}$$

with

$$M(\theta, J) \equiv D(J)^{-1}\, R(\theta)\, D(J)\, R(\theta), \tag{57}$$

and where R and D are given above.

Proof It follows from (51)–(53) that each Σ_n is an invariant subspace for $\mathcal{L}(y)$, $\mathcal{J}$ and $\mathcal{S}$, and hence is also for $\mathcal{M}$, a composition of these operators. Moreover, applying (51)–(53) we have

$$\begin{aligned}\mathcal{M}\left(\mathcal{T}_n \begin{bmatrix} a \\ b \end{bmatrix}\right) &= \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\theta) \cdot \mathcal{J} \cdot \mathcal{L}(\theta) \cdot \mathcal{T}_n \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \mathcal{T}_n (-1)^n\, D^{-1}\, R(n\theta)\, D\, R(n\theta) \begin{bmatrix} a \\ b \end{bmatrix} = \mathcal{T}_n\, M_n \begin{bmatrix} a \\ b \end{bmatrix},\end{aligned}$$

where M_n is the real 2×2 matrix

$$M_n = (-1)^n\, D^{-1}\, R(n\theta)\, D\, R(n\theta) = (-1)^n M(n\theta, J).$$

This verifies (55), (56) and (57), and completes the proof of the theorem. Note that $D(J)$ and $R(\theta)$ do not in general commute.

Consider now the linear eigenvalue problem (42) for a 2π-periodic function $V(t)$. Let

$$V = \sum_{n \geq 0} \mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \in \Sigma. \tag{58}$$

Theorem 3 The function $V(t)$ solves the linear eigenvalue problem (42) with (37) if and only if $a_0 = 1$, $b_0 = 0$ and

$$(-1)^n M(n\theta, J) \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \tag{59}$$

for every $n \geq 1$.

Proof The average conditions (37) immediately imply $a_0 = 1$ and $b_0 = 0$. Let V have the decomposition (58), and assume $\mathcal{M}V = V$. Then

$$\sum \mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} = V = \mathcal{M}V = \mathcal{M}\left(\sum \mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix}\right) = \sum \mathcal{T}_n M_n \begin{bmatrix} a_n \\ b_n \end{bmatrix},$$

where we have applied Theorem 2. Thus it follows that

$$\mathcal{T}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \mathcal{T}_n\, M_n \begin{bmatrix} a_n \\ b_n \end{bmatrix},$$

for every $n \geq 1$, so also

$$M_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_n \\ b_n \end{bmatrix},$$

for all $n \geq 1$, which is (59).

6 Matrix Eigenvalue Problems

Theorem 3 reduces the problem of finding solutions of (42) to the problem of finding solutions $(a, b) \in \mathbb{R}^2$ of the linear eigenvalue problems

$$M_n \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \tag{60}$$

where M_n is the 2×2 matrix $M_n = (-1)^n M(n\theta, J)$. As a first step, we characterize the eigenvalues of $M = M(\theta, J)$ in terms of (θ, J).

Lemma 3 The eigenvalues of $M(\theta, J)$ are given by

$$\lambda = \beta \pm \sqrt{\beta^2 - 1}, \tag{61}$$

where

$$\beta = \cos^2(\theta) - \frac{J^2 + 1}{2J} \sin^2(\theta). \tag{62}$$

Proof Note that $\det M = 1$, so the eigenvalues λ of M satisfy

$$\lambda^2 - 2\beta\lambda + 1 = 0,$$

where $\beta = \frac{1}{2}\mathrm{tr} M$. Solutions are given by (61), and writing

$$M = D^{-1}\, R(\theta)\, D\, R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta)/J & \cos(\theta)/J \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ J\sin(\theta) & J\cos(\theta) \end{pmatrix},$$

and calculating the trace yields (62).

We now construct a solution of (42) in the subspace Σ_1, that represents a periodic solution of the linearized Euler equations having the periodic structure described in [3]. For $n = 1$, the problem (60) becomes

$$M(\theta, J)\, q = D^{-1}(J)\, R(\theta)\, D(J)\, R(\theta)\, q = -q, \tag{63}$$

and a solution $q \in \mathbb{R}^2$ yields the solution $V = \mathcal{T}_1\, q \in \Sigma_1$ of (42).

We use the following notation: let $q = (a, b)^{\mathrm{tr}}$, let $\|q\| = \sqrt{a^2 + b^2}$ denote Euclidean norm, and let $\mu \in [0, 2\pi)$ be the angle q makes with the a-axis, so that $\tan\mu = b/a$. Now, in analogy with (19) and (44), let $\dot{q}$, $\check{q}$, $\tilde{q}$, $\hat{q}$, and q^* denote the vector states determined by the decomposition (57) of M, so that

$$\dot{q} = q, \quad \check{q} = R(\theta)\, \dot{q}, \quad \tilde{q} = D\, \check{q}, \quad \hat{q} = R(\theta)\, \tilde{q} \quad \text{and} \quad q^* = D^{-1}\, \hat{q}. \tag{64}$$

Also, let $\dot{\mu}$, $\check{\mu}$, $\tilde{\mu}$, $\hat{\mu}$ and μ^* denote the angles these vectors make with the positive x-axis, respectively.

Our problem then is to characterize solutions q of (63) in terms of the parameters (θ, J). We regard this as a problem of non-commuting transformations. Since $R(\theta)$ is rotation through angle θ and $D(J)$ just scales the second coordinate b by factor J, (63) places very tight constraints on the possible angles $\dot{\mu}$, $\check{\mu}$, $\tilde{\mu}$, $\hat{\mu}$ and μ^*.

Theorem 4 Assume that $J > 1$ and $0 < \theta < \pi/2$. Then q is a solution of (63) if and only if

$$J = \cot^2(\theta/2), \tag{65}$$

and $q \in \mathrm{Span}\,\{\mathbf{q}\}$, where

$$\mathbf{q} \equiv (\cos(\theta/2),\ -\sin(\theta/2)).$$

Furthermore, if $\dot{q} = \mathbf{q}$, then also

$$\begin{aligned}
\check{q} &= (\cos(\theta/2),\ \sin(\theta/2)) \\
\tilde{q} &= (\cos(\theta/2),\ J\sin(\theta/2)) = \rho\,(\cos(\pi/2-\theta/2),\ \sin(\pi/2-\theta/2)), \\
\hat{q} &= (-\cos(\theta/2),\ J\sin(\theta/2)) = \rho\,(-\cos(\pi/2-\theta/2),\ \sin(\pi/2-\theta/2)), \\
q^* &= (-\cos(\theta/2),\ \sin(\theta/2) = -q,
\end{aligned} \tag{66}$$

where we have set $\rho = \|\tilde{q}\|$. These states are diagrammed in Figure 1.

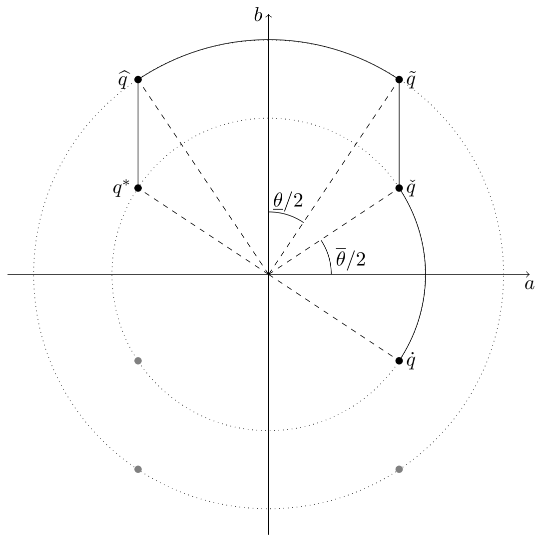

Fig.1 The states $\dot{q}, \check{q}, \tilde{q}, \hat{q}, q^*$ for $\mathbf{q} \in \Sigma_1$.

Since the problem we work with is symmetric, assuming $J > 1$, or equivalently $\overline{m} > \underline{m}$ by (54), is no restriction (we can replace J by J^{-1} by relabeling). On the other hand, for physical reasons described in [3], the assumption $\theta < \pi/2$ results in consistent widths of entropy levels consistent with $J > 1$. In particular, according to (39), this imposes a group velocity $2\,\theta/\pi$, slower than the characteristic wavespeed (phase velocity) $\sigma_0 = 1$, which is what would be expected. There are solutions of (63) for larger values of θ, but these do not correspond to the simplest periodic structure identified in [3].

For the proof of Theorem 4 we use the following lemma:

Lemma 4 Let $q = (a, b)$, $J \neq \pm 1$. If $\|q\| = \|q'\|$ and $\|Dq\| = \|Dq'\|$, then $q' = (\pm a, \pm b)$.

This is immediate from $a^2 + b^2 = (a')^2 + (b')^2$ and $a^2 + J^2 b^2 = (a')^2 + J^2 (b')^2$.

We can now prove the theorem.

Proof Assume $M\,q = -q$ for $q \in \mathrm{Span}\{\dot{q} = (\dot{a}, \dot{b})\}$, where $\dot{q}$ is a representative of the subspace satisfying $\|\dot{q}\| = 1$, and $-\pi/2 \le \dot{\mu} < \pi/2$.

First, we prove that $\dot{q}$ must have a non-positive slope, $\dot{b}/\dot{a} \le 0$. Assume for contradiction that $\dot{b}/\dot{a} > 0$, so that $\dot{q}$ lies in the first quadrant. Then $q^* = -\dot{q}$ lies in the third quadrant, and $\hat{q} = Dq^*$ must also lie in the third quadrant on a circle of radius $\rho \equiv \|\hat{q}\| > 1$ because $J > 1$, see Figure 2. Then $\tilde{q} = R(-\theta)\hat{q}$ must lie on the circle of radius ρ, and $\check{q} = D^{-1}\tilde{q}$ lies on the circle of radius 1. But by Lemma 4, the only such point on the circle of radius ρ that lies within an angle of π from $\hat{q}$, and is mapped by D to a point on the circle of radius 1, is the point on the circle of radius ρ directly above the point $\hat{q}$, as in Figure 2. Now $\check{q} = D^{-1}\tilde{q} = R(\theta)$ must thus lie on the intersection of the vertical line through $\check{q}$ and the unit circle. The only way this can happen is if $\tilde{q}, \hat{q}, \tilde{q}$ and $\check{q}$ all lie on the same vertical line, passing through the circles $\|q\| = \rho$ and $\|q\| = 1$, see Figure 2.

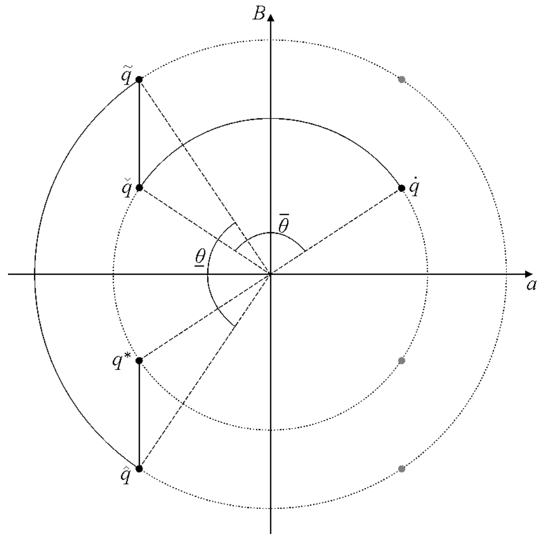

Fig.2 The states with $\dot{\mu} > 0$, showing $\underline{\theta} + \overline{\theta} > \pi$.

That is, $\tilde{q}$ is the only point on the circle of radius ρ within an angle of π from $\hat{q}$ that is mapped by D to a point on the circle of radius 1. From these considerations it follows that

$$q^* = (-\dot{a},\ -\dot{b}), \quad \hat{q} = (-\dot{a},\ -J\,\dot{b}), \quad \tilde{q} = (-\dot{a},\ J\,\dot{b}) \quad \text{and} \quad \check{q} = (-\dot{a},\ \dot{b}).$$

From $\check{q} = R(\theta)\dot{q}$, it follows that $\dot{\mu} = \pi/2 - \theta/2$ and $\check{\mu} = \pi/2 + \theta/2$, and it follows from $\hat{q} = R(\theta)\tilde{q}$ that $\tilde{\mu} = \pi - \theta/2$, and $\hat{\mu} = \pi + \theta/2$, as in Figure 2. But it is easily seen from Figure 2 that this configuration of angles implies that $\theta > \pi/2$, a contradiction. Thus we must have $-\pi/2 \le \dot{\mu} \le 0$.

Consider next the case $-\pi/2 < \dot{\mu} < 0$. In this case $\dot{b}/\dot{a} < 0$, so that $\dot{q}$ lies in the fourth quadrant. Then $q^* = -\dot{q}$ must lie in the second quadrant, and $\hat{q} = Dq^*$ must also lie in the second quadrant on a circle of radius $\rho = \|\hat{q}\| > 1$ because $J > 1$. Then $\tilde{q} = R(-\theta)\,\hat{q}$ must lie on the circle of radius ρ, and $\check{q} = D^{-1}\,\tilde{q}$ then lies on the circle of radius 1. But by Lemma 4, the only such point on the circle of radius ρ that lies within an angle of π from $\hat{q}$ and is mapped by D to a point on the circle of radius 1, is the point on the circle of radius ρ directly to the right of, and on the same horizontal line as the point $\hat{q}$, as diagrammed in Figure 1. Thus $\check{q} = D^{-1}\,\tilde{q} = R(\theta)\,\dot{q}$ must lie on the intersection of the vertical line through $\check{q}$ and the unit circle. The only way this can happen is if $\dot{q}, \check{q}$, and $\tilde{q}$ all lie on the same vertical line, with $\|q\| = \rho$ and $\|q\| = 1$, see Figure 1. It then follows that

$$q^* = (-\dot{a},\ -\dot{b}), \quad \hat{q} = (-\dot{a},\ -J\dot{b}), \quad \tilde{q} = (\dot{a},\ -J\dot{b}), \quad \check{q} = (\dot{a},\ -\dot{b}),$$

with $\dot{a} > 0$, $\dot{b} < 0$. Next, since $\check{q} = R(\theta)\,\dot{q}$, it follows that $\dot{\mu} = -\theta/2$ and $\check{\mu} = \theta/2$, and it follows from $\hat{q} = R(\theta)\tilde{q}$ that $\tilde{\mu} = \pi/2 - \theta/2$, and $\hat{\mu} = \pi/2 + \theta/2$, see Figure 1. It is easily seen from Figure 1 that this configuration of angles satisfies the condition $\theta < \pi/2$, and this condition makes the solution unique. This verifies equations (66).

Finally, to prove (65), we have the two expressions

$$\begin{aligned}&\tilde{q} = (\cos(\theta/2),\ J\,\sin(\theta/2)) \quad \text{and}\\ &\tilde{q} = \rho\,(\cos(\pi/2-\theta/2),\ \sin(\pi/2-\theta/2)) = \rho\,(\sin(\theta/2)),\ \cos(\theta/2)),\end{aligned} \tag{67}$$

where $\rho = \|\tilde{q}\|$, so that

$$\rho^2 = \cos^2(\theta/2) + J^2\sin^2(\theta/2).$$

Thus comparing first components of (67), we get

$$\cos^2(\theta/2) = \big(\cos^2(\theta/2) + J^2\sin^2(\theta/2)\big)\sin^2(\theta/2),$$

and solving for J leads directly to (65).

7 Wave Structure of the Linearized Solutions

In this section we reconstruct and discuss the linear periodic solution associated with the solution

$$V_0(t) = \mathcal{T}_1\,\dot{q} \in \Sigma_1$$

of (42), where $\dot{q}$ is the solution of (63) obtained in Theorem 4. That is, we use (47) to explicitly describe the pointwise values $V(y,t)$ of the periodic solution of the linearized Euler equations. Having described this solution in dimensionless variables, it is then routine to use Theorem 1 to give periodic solutions in physical (Lagrangian) variables.

According to (47), we have, for $0 < y < \theta$,

$$V(y,t) = \mathcal{L}(y)\,V_0 = \mathcal{L}(y)\,\mathcal{T}_1\,\dot{q} = \mathcal{T}_1\,R(y)\,\dot{q}, \tag{68}$$

where we have generalized (61) to any y instead of θ. Now, by (66), $\dot{q}$ is given by

$$\dot{q} = \begin{bmatrix}\cos(\theta/2)\\ -\sin(\theta/2)\end{bmatrix} = R(-\theta/2)\begin{bmatrix}1\\0\end{bmatrix},$$

so that

$$R(y)\,q = R(y)\,R(-\theta/2)\begin{bmatrix}1\\0\end{bmatrix} = R(y-\theta/2)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos(y-\theta/2)\\ \sin(y-\theta/2)\end{bmatrix}.$$

Thus by (50), (68) becomes

$$V(y,t) = \begin{bmatrix}\cos(y-\theta/2)\cos t\\ \sin(y-\theta/2)\sin t\end{bmatrix}, \quad \text{for} \quad 0 \le y < \theta. \tag{69}$$

Similarly, for $\theta \le y < 2\theta$ we have

$$V(y,t) = \mathcal{L}(y-\theta)\,\mathcal{J}\,\mathcal{L}(\theta)\,V_0 = \mathcal{L}(y-\theta)\,\mathcal{T}_1\,\tilde{q} = \mathcal{T}_1\,R(y-\theta)\,\tilde{q},$$

where, by (66) and Figure 1,

$$\tilde{q} = \rho\,R\Big(\frac{-\theta}{2}\Big)\begin{bmatrix}0\\1\end{bmatrix},$$

and $\rho = \|\tilde{q}\| = \cot(\theta/2)$. Thus for $\theta < y < 2\theta$,

$$V(y,t) = \mathcal{T}_1\,\rho\,R(y-\theta)\,R\Big(\frac{-\theta}{2}\Big)\begin{bmatrix}0\\1\end{bmatrix} = \mathcal{T}_1\,\rho\,R\Big(y-\theta-\frac{\theta}{2}\Big)\begin{bmatrix}0\\1\end{bmatrix},$$

which yields

$$V(y,t) = \frac{\cos(\theta/2)}{\sin(\theta/2)}\begin{bmatrix}-\sin(y-3\theta/2)\,\cos t\\ \cos(y-3\theta/2)\sin t\end{bmatrix}, \quad \text{for} \quad \theta < y < 2\theta. \tag{70}$$

Now scaling both (69) and (70) by $\sin(\theta/2)$, we have proved the following theorem:

Theorem 5 For each choice of parameter $0 < \theta < \pi/2$, the following closed form expression defines a linearized periodic solution $V(y,t)$ of the compressible Euler equations in dimensionless variables:

$$V(y,t) = \begin{cases} \sin(\theta/2)\begin{bmatrix}\cos(y-\theta/2)\,\cos t\\ \sin(y-\theta/2)\,\sin t\end{bmatrix}, & 0 < y < \theta. \\ \cos(\theta/2)\begin{bmatrix}-\sin(y-3\,\theta/2)\,\cos t\\ \cos(y-3\,\theta/2)\,\sin t\end{bmatrix}, & \theta < y < 2\theta. \end{cases} \tag{71}$$

Here, the (y,t)-region $\Omega \equiv [0,2\theta)\times[0,2\pi)$ defines one spacetime period, and the entire solution is obtained by mapping the solution in the period Ω to the (y,t)-plane via translation (72). In particular, the Rankine-Hugoniot jump conditions hold at the entropy jumps $y=\theta$ and $y=2\theta$, and the resulting solution has the property that nearby nonlinear solutions formally balance compression and rarefaction along characteristics in the sense of [3].

It is clear that $V(y,t)$ given by (71) is $2\,\pi$-periodic in time, and the conditions (42), (43), namely

$$\mathcal{M}\,V_0 = V_0 \quad \text{and} \quad \mathcal{M} = \mathcal{S}\,\mathcal{J}^{-1}\,\mathcal{L}\,\mathcal{J}\,\mathcal{L},$$

imply also that $V(y,t)$ is $2\,\theta$-periodic in space, after a $(t \to t+\pi)$ time translation. Since $\mathcal{S}^2 = \mathcal{I}$, it follows that $V(y,t)$ is exactly $4\,\theta$ space-periodic, but the rectangle $[0,2\theta)\times[0,2\pi)$ is a minimum periodic tile with period translation vector $\mathbf{X} = 2\,\theta\,\mathbf{e}_y + \pi\,\mathbf{e}_t$, which means that

$$V(y,t) = V((y,t)+\mathbf{X}) = V(y+2\,\theta, t+\pi). \tag{72}$$

In Figure 3, the characteristics of the dimensionless solution $V(y,t)$ in (y,t)-space is depicted for the case $\theta = \pi/(1+\sqrt{5})$, a case in which the sound speed is incommensurate, i.e., irrationally related to, the speed of the period. In this case the characteristics are quasi-periodic, and the balancing of rarefaction and compression along characteristics is achieved by ergodic motion through the period. In case the dimensionless sound speed is rational, so θ is a rational multiple of π, there is a resonance effect, shown by a characteristic eventually becoming periodic, see [4].

Taken all together, we interpret these results as supplying what one might call a physical proof that such periodic wave patterns exist. The focus of our current research is to perturb these linearized 1-mode solutions to the fully nonlinear problem, thus completing the rigorous proof that such periodic solutions exist in the fully nonlinear compressible Euler equations. In the nonresonant case, the periodic solutions constructed here are unique, but there are small divisors. In [5], we solve the associated bifurcation problem, and show that periodic solutions exist up to arbitrarily high Fourier mode cutoff. In the resonant case, the kernel of $\mathcal{M}-\mathcal{I}$ is larger, and different techniques must be used. This is the subject of the authors' ongoing research.

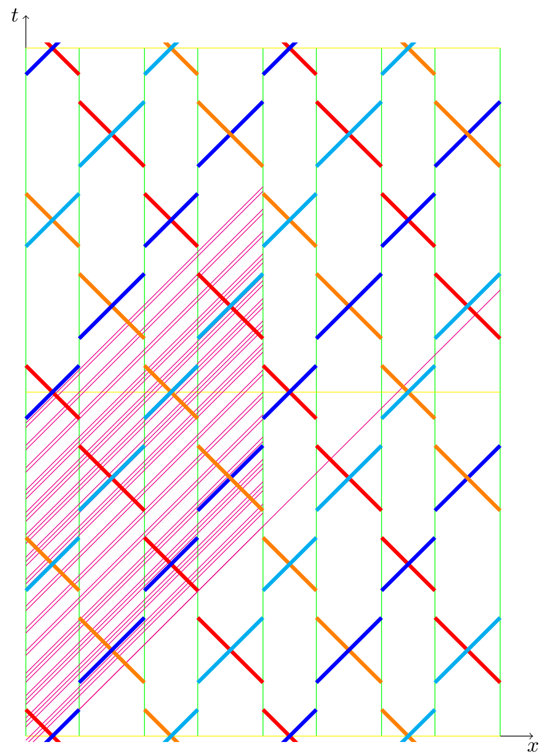

Fig.3 The dimensionless linearized periodic solution $V(y,t)$ showing incommensurability

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