On the Essential Regularity of Singular Connections in Geometry

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UC-Davis

Nonlinear Analysis Seminar
Central Taiwan University
April 15, 2026

All Joint Work With: Moritz Reintjes

The question we address:

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"How do we determine, apriori, whether a singularity in a gravitational metric tensor in General Relativity is essential or removable by coordinate transformation?"

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Virtually "everyone" asks this question after reading the derivation of the Schwarzschild's solution in General Relativity…

Karl Schwarzschild (1873-1914):

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Singularity
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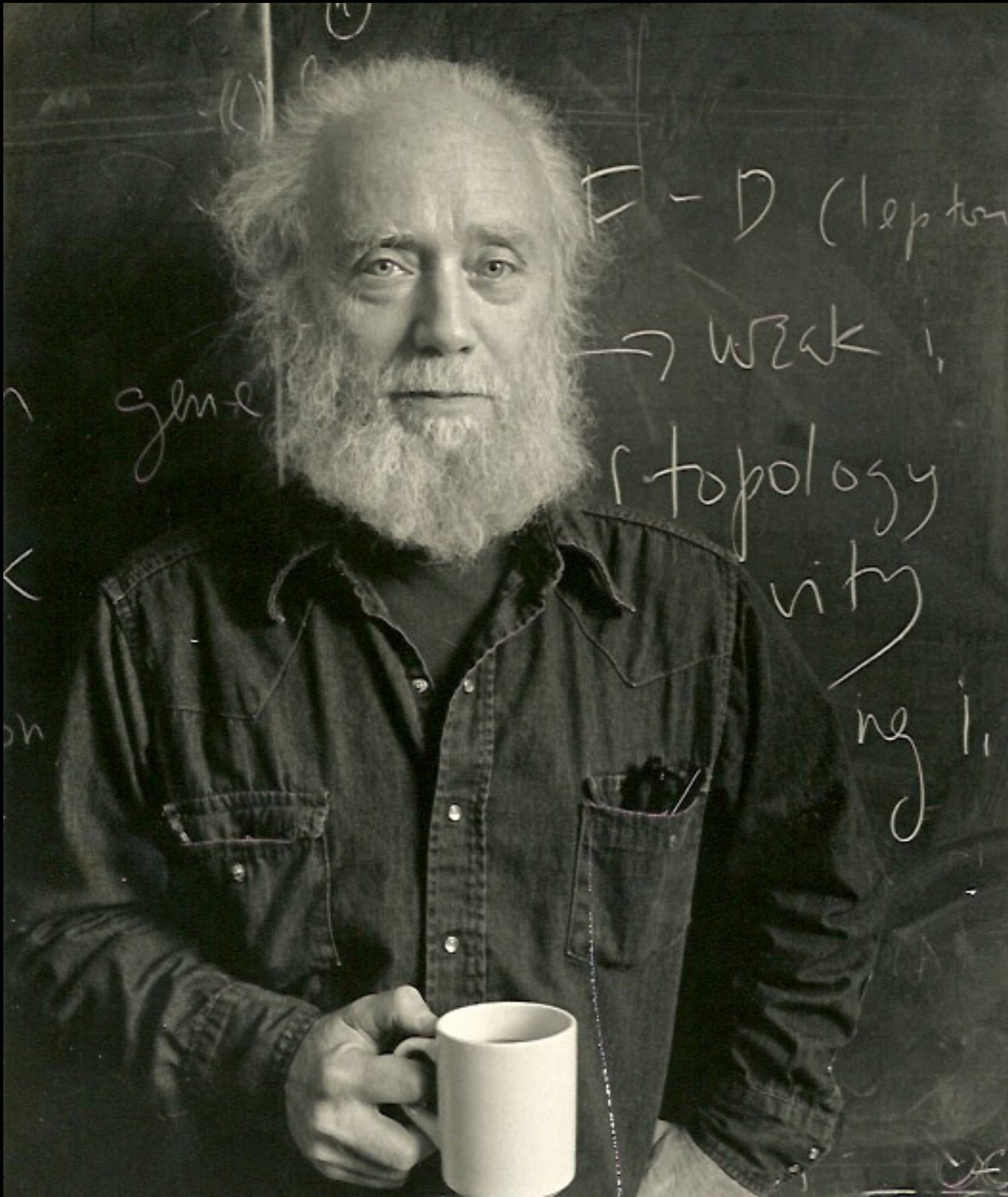
Question: Is this essential or removable by coordinate transformation?

Answer: Eddington-Finkelstein early 1920's:

Sir Arthur Eddington
(1882-1944):

David Finkelstein (1929-2016):



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No singularity in the new coordinates $r \neq 0$!

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Q: How could you tell ahead of time the Schwarzschild singularity is removable?

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…implying Black Holes are physically possible…

Q: How could you tell ahead of time the Schwarzschild singularity is removable?

And what procedure provides the regularizing coordinate transformations?

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(3) Does every metric/connection have an essential (highest possible) regularity to which it can be lifted?

Answer: Yes. By theory of the RT-equations

Our Question:

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Everything depends on regularity of the Riemann Curvature Tensor

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Our theory relies on an existence theory for the RT-equations

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…the unknowns are the Jacobians J of coordinate transformations which regularize the components of a connection Γ… coupled to the unknown regularized connection $\tilde{\Gamma}$.

A connection is the most general construct in geometry which has a Riemann Curvature Tensor..

The RT-equations:

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A),\\
\Delta J &= \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A,\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big),\\
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$$\Big(d\vec{J} = Curl(J) = \partial_j J^\mu_i - \partial_i J^\mu_j \Big)$$

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Key: " δ comes after d ":

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$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A),$$

This is required to get existence for $\Gamma \in L^p$, $p > n$

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They are constructed from the Euclidean metric of the coordinate system in which components are given…

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They are constructed from the Euclidean metric of the coordinate system in which components are given…

Hence they are elliptic independent of metric signature…

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(Locally, in a neighborhood of every point.)

Our existence theorem for the RT-equations is based on elliptic regularity in L^p-spaces:

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M. Reintjes and B. Temple, *On the optimal regularity implied by the assumptions of geometry I: Connections on tangent bundles*, Meth. Appl. Anal., Vol. 29, No. 4, 303-396, (2023)

Our existence theorem for the RT-equations is based on elliptic regularity in L^p–spaces:

Theorem (RT): Assume

$$\Gamma \in L^{2p} \quad \text{and} \quad Riem(\Gamma) \in L^p, \quad p > n/2$$

in a given coordinate system x. Then there always exist coordinate transformations

$$x \to y$$

such that in y-coordinates

$$\Gamma \in W^{1,p}, \quad Riem(\Gamma) \in L^p$$

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Connection one full derivative above curvature…

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The existence of such coordinate transformations rules out "regularity singularities" in General Relativity.

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(We called this optimal regularity because we did not realize that J could regularize the curvature as well...so "optimal regularity" was not the "essential regularity" of the connection...)

The extra derivative implies compactness for sequences $\{\Gamma_i\}_{k=1}^{\infty}$ of affine connections:

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Theorem (RT): Assume

$$\Gamma_i \in L^{\infty} \quad \text{and} \quad Riem(\Gamma_i) \in L^p, \quad p > n$$

with uniform bounds. Then there exist a convergent subsequence in y-coordinates such that

$$\Gamma_i \to \Gamma \quad \text{strongly in } L^p, \text{ weakly in } W^{1,p}$$

Same for smooth solutions:

If: $\Gamma, Riem(\Gamma) \in W^{m,p}, m \geq 1$.

Then: $x \to y$ gives

$$\Gamma \in W^{m+1,p}, Riem(\Gamma) \in W^{m,p}$$

"I.e., Γ one derivative above $Riem(\Gamma)$"

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Vector Bundle version of the RT-equations

"Same Theorems"

M. Reintjes and B. Temple, *Optimal regularity and Uhlenbeck compactness for General Relativity and Yang-Mills Theory*, Proc. R. Soc. A 479: 20220444 (2022)

M. Reintjes and B. Temple, *On the optimal regularity implied by the assumptions of geometry II: Connections on vector bundles*, Adv. Theo. Math. Phys. Volume 27, Number 3, 623–684, (2023)

Vector Bundle version of the RT-equations

⇒ Same Theorems

Both compact and non-compact Lie Groups:

$$\begin{aligned} \Delta\tilde{A} &= \delta d\mathcal{A} - \delta\left(dU^{-1}\wedge dU\right) \\ \Delta U &= U\delta\mathcal{A} - (U^{T}\eta)^{-1}\langle dU^{T};\eta dU\rangle \end{aligned}$$

$\mathcal{A} \equiv$ Non-optimal Connection

$U \equiv$ Gauge Transformation to optimal regularity...(we do case $SO(r,s)$)

This extends Uhlenbeck compactness to connections on vector bundles:

Theorem (RT): Assume

$(A_i, \Gamma_i) \in L^\infty$ and $dA \in L^1, d\Gamma \in L^p$, $p > n$

with uniform bounds. Then there exist a convergent subsequence such that under a gauge and coordinate transformation,

$(A_i, \Gamma_i) \to (A, \Gamma)$ strongly in L^p , weakly in $W^{1,p}$

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…to arbitrary connections on vector bundles allowing for compact and non-compact Lie groups…

…including the Lorentzian metrics and affine connections of General Relativity…

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…the starting Hypothesis of Geometry.

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Hence our titles…

"On the Regularity Implied by the Assumptions of Geometry"

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Riemann (1854): "On the Hypotheses which lie at the Foundation of Geometry "

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M. Reintjes, B. Temple, *On weak solutions to the geodesic equation in the presence of curvature bounds*, Jour. Diff Eqns, Vol. 392, 306-324 (2024)

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Geodesic equation: $$\ddot{\gamma}^k = \Gamma^k_{ij}\,\dot{\gamma}^i\dot{\gamma}^j$$

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Define weak solutions by "coordinate transformation" instead of "multiply by test function and integrate by parts"

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Penrose's Strong Cosmic Censorship Conjecture asserts the in-extendability of Cauchy developments with Lipschitz metrics and Riemann curvature bounded in L^p.

- The RT-equations imply that it suffices to establish this for metrics in $W^{1,p}$, $p > n$.

M. Reintjes, *Strong Cosmic Censorship with bounded curvature*, Class. and Quant. Grav., Volume 41, Number 17, (July 2024)

The application I discuss in this talk…

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- The essential regularity of singular connections in Geometry…

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M. Reintjes, B. Temple, *The essential regularity of singular connections in Geometry*, (under review)

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*Our theory relies on an existence theory for the RT-equations…

*…together with the fact that the RT-equations lift the regularity of the curvature along with the connection until the essential regularity of the connection is reached.

Plan for Remainder of Talk:

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Introduction

The Riemann Curvature Tensor

In Riemann's Theory of Curvature:

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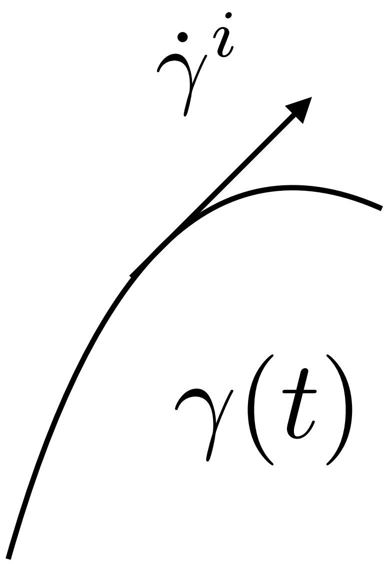

$$\|\dot{\gamma}\| = \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}$$

$$L = \int ds = \int_{t_0}^{t} \|\dot{\gamma}\| dt$$

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$$g_y = J^t g_x J$$ (nxn matrices)

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…we say geometry is locally flat.

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We say curved spacetime is locally Minkowski…

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— $Riem(g)$ measures second derivative Taylor errors but transforms by first derivative Jacobians…

Riemann Curvature Tensor:

—transforms by 1st derivative Jacobians

$$R^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{k}}{\partial y^{\beta}} \frac{\partial x^{l}}{\partial y^{\gamma}} \frac{\partial x^{j}}{\partial y^{\delta}}$$

(tensor)

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—But measures 2nd derivatives in the Taylor series

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Riemann Curvature Tensor:

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—Thm (Riemann): $R \equiv 0$ iff $O(|p - p_0|)^2 \equiv 0$

$$g_{ij}(p) = \delta_{ij} + O(|p - p_0|^2)$$

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Christoffel Symbols

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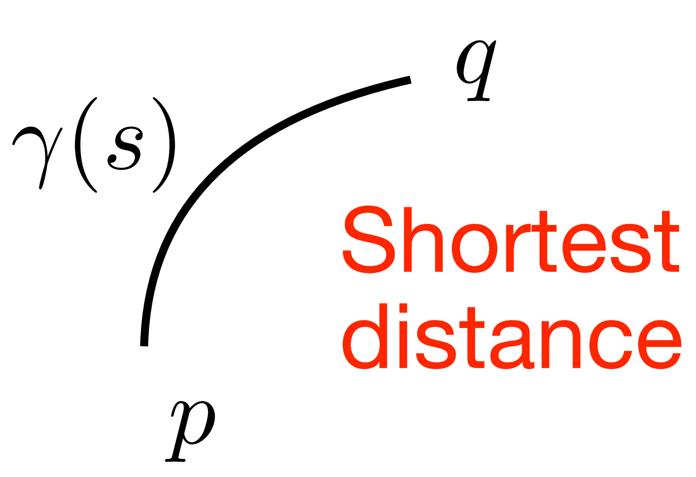
$$\frac{d^2}{ds^2} \gamma^k(s) = \Gamma^k_{ij} \gamma^i(s) \gamma^j(s)$$



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(pointwise)

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$Commutator$

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R does NOT bound ALL the derivatives of Γ !

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Co-derivatives $\delta\Gamma$ are uncontrolled (pointwise)

View Γ as a matrix valued 1-form:

$$\Gamma \equiv \Gamma_k dx^k \equiv \left(\Gamma^i_j\right)_k dx^k$$

Then: $$R = d\Gamma + \Gamma \wedge \Gamma$$

R is a "Curl" plus a "Commutator"

as nxn matrices expressed as wedge product

RT-Equations: Equations for Jacobians that lift the regularity of connection Γ to one derivative above $d\Gamma$

$$\begin{aligned} \Delta J &= \delta(J\cdot\Gamma) - B, \\ d\vec{B} &= \overrightarrow{\text{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\text{div}}\big(J\,d\Gamma\big), \\ \delta\vec{B} &= v'. \end{aligned}$$

(Reduced RT-equations)

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A),$$

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(Note: the metric is ALWAYS one derivative more regular than its connection…no wiggle room there…)

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The problem is: Riem involves second derivatives of the metric, but transforms by first derivative Jacobians

“Proof:”

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Transform to y-coordinates: $x \rightarrow y$

Assume: $$J \equiv \frac{\partial x}{\partial y} \in C^{k+1}$$

In y-coordinates:

$$\bar{g}_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}$$

C^{k+1} $\quad$ C^{k+2} $\quad$ C^{k+1} $\quad$ C^{k+1}

In y-coordinates:

$$\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$

$$\underset{C^{k+1}}{\bar{g}_{\alpha\beta}} = \underset{C^{k+2}}{g_{ij}} \; \underset{C^{k+1}}{\frac{\partial x^i}{\partial y^\alpha}} \; \underset{C^{k+1}}{\frac{\partial x^j}{\partial y^\beta}}$$

Metric in y-coordinates: $\bar{g} \in C^{k+1}$

In y-coordinates:

$$\bar{R}^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{k}}{\partial y^{\beta}} \frac{\partial x^{l}}{\partial y^{\gamma}} \frac{\partial x^{j}}{\partial y^{\delta}}$$

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C^k $\qquad$ C^k $\qquad$ C^{k+1}

Curvature in y-coordinates: $\bar{R} \in C^k$

Conclude: Under $x \to y$

$$R \in C^k, \quad \Gamma \in C^{k+1}, \quad g \in C^{k+2}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\bar{R} \in C^k, \quad \bar{\Gamma} \in C^k, \quad \bar{g} \in C^{k+1}$$

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$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\bar{R} \in C^k, \quad \bar{\Gamma} \in C^k, \quad \bar{g} \in C^{k+1}$$

A coord trans $x \to y$ at regularity g lowers the regularity of g, Γ by one order, but preserves regularity of curvature R.

Our Question: Does the reverse hold?

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At the lowest regularity, points of non-optimality in metrics and connections look like singularities…

Our work began with GR shock wave solutions of the Einstein equations constructed by the Glimm Scheme:

For Shock-Waves in GR:

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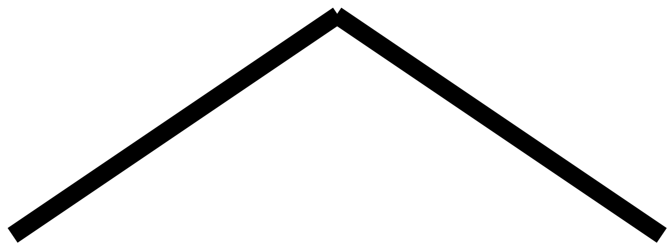
g is Lipschitz



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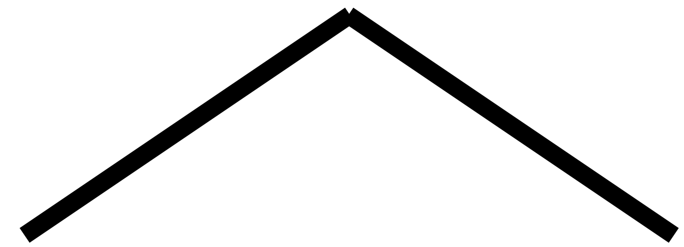

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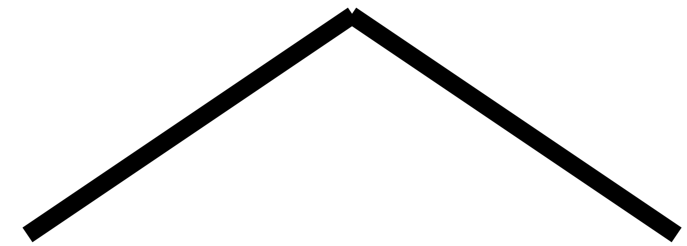

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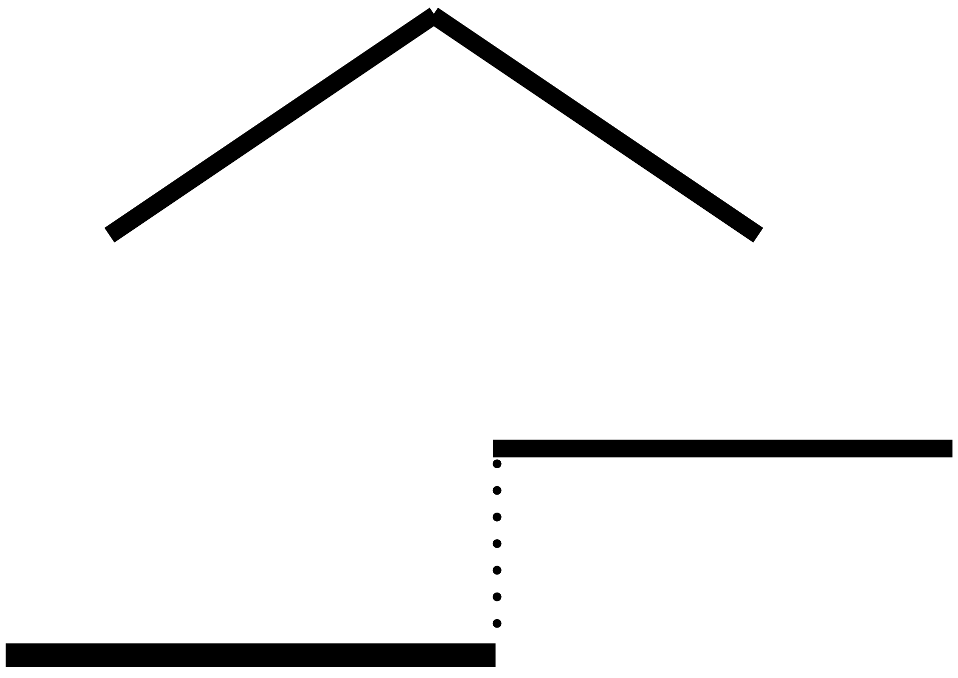

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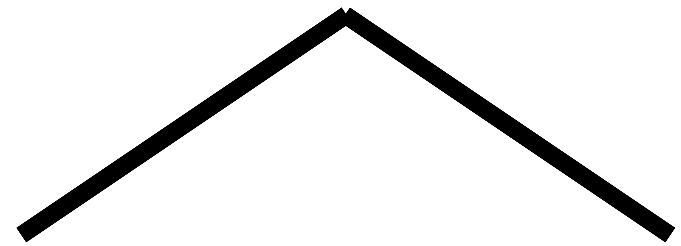

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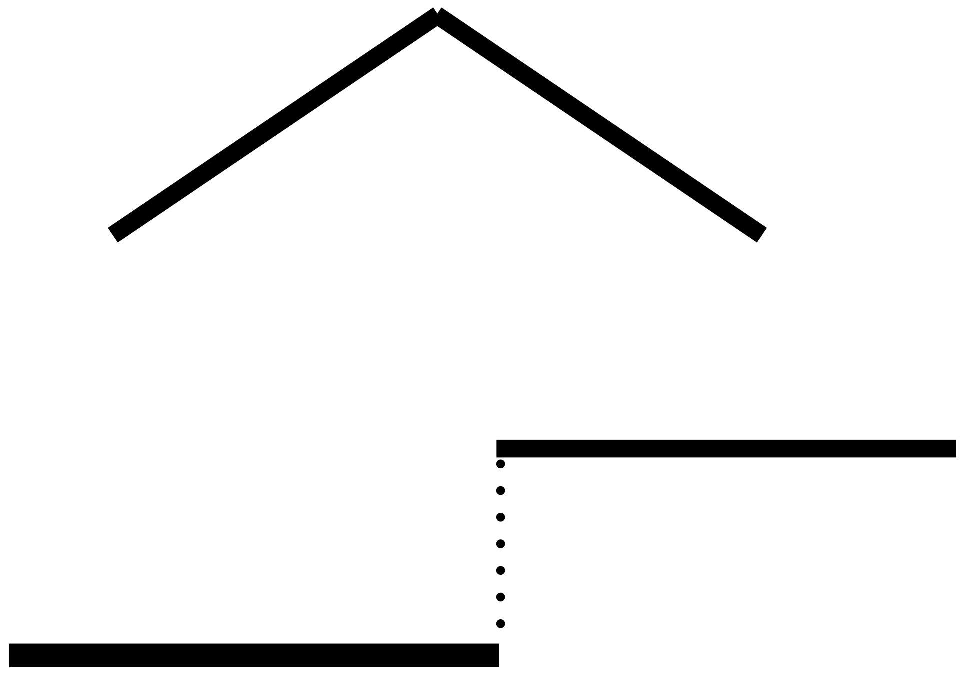

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$G = \kappa T$ only holds in the weak sense:

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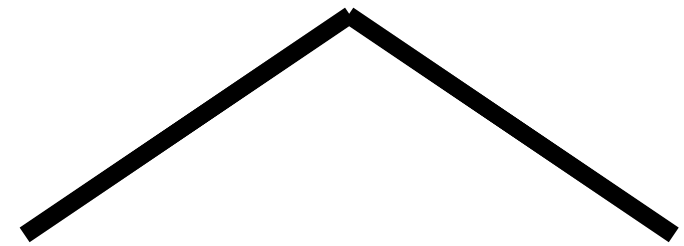

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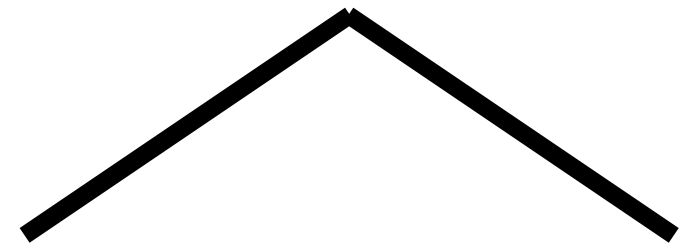
Geodesics don't exist…

For Shock-Waves in GR:

g is Lipschitz



$\Gamma \approx \partial g$ is L^∞

$R \approx \partial^2 g$ is L^∞

$(d\Gamma \in L^\infty)$

At this regularity shock waves look like singularities:

Geodesics don't exist… $\ddot{\gamma} = \Gamma^k_{ij}\, \dot{\gamma}^i \dot{\gamma}^j \qquad \Gamma^k_{ij}(x) \in L^\infty$

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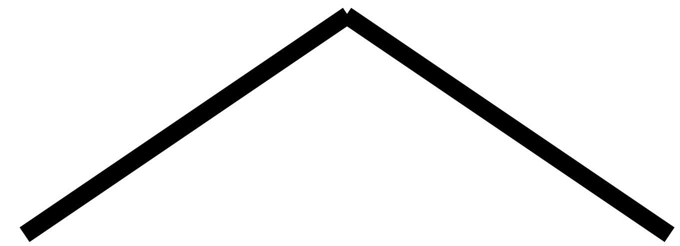

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Locally inertial coordinates don't exist

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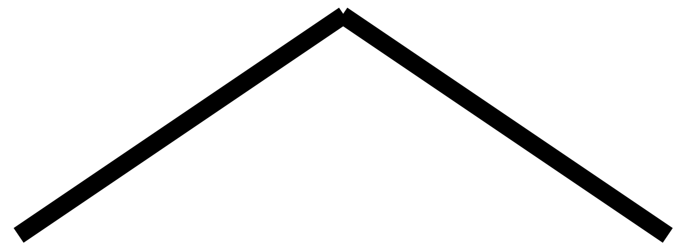

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At this regularity shock waves look like singularities:

Spacetime does not look "locally flat"...

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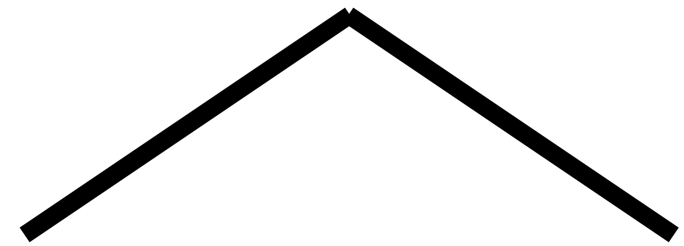

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The classical limit looks suspect…

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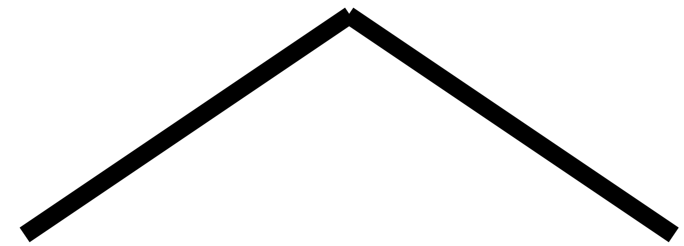

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At this regularity shock waves look like singularities:

Experts questioned the physical significance of such low regularity solutions of Einstein's equations…

GR-Shock Waves by the Glimm Scheme

Assume a gravitational metric ansatz of the SSC form:

$$ds^2 = -B(t,r)dt^2 + \frac{dr^2}{A(t,r)} + r^2 dr^2$$

Plug into the Einstein equations :

$$G = \kappa T$$

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}$$

Standard Schwarzschild Coordinates

Four PDE's

$$\left\{-r\frac{A_r}{A}+\frac{1-A}{A}\right\} = \frac{\kappa B}{A}r^2T^{00} \quad (1)$$

$$\frac{A_t}{A} = \frac{\kappa B}{A}rT^{01} \quad (2)$$

$$\left\{r\frac{B_r}{B}-\frac{1-A}{A}\right\} = \frac{\kappa}{A^2}r^2T^{11} \quad (3)$$

$$-\left\{\left(\frac{1}{A}\right)_{tt}-B_{rr}+\Phi\right\} = 2\frac{\kappa B}{A}r^2T^{22}, \quad (4)$$

where

$$\Phi = \frac{B_tA_t}{2A^2B}-\frac{1}{2A}\left(\frac{A_t}{A}\right)^2-\frac{B_r}{r}-\frac{BA_r}{rA} + \frac{B}{2}\left(\frac{B_r}{B}\right)^2-\frac{B}{2}\frac{B_r}{B}\frac{A_r}{A}.$$

(1)+(2)+(3)+(4) ⟺ (1)+(3)+div T=0

(weakly)

Theorem: (Te-Gr) The equations close in a "locally inertial" formulation of (1), (2) & Div T=0:

$$\begin{aligned}
\left\{T_M^{00}\right\}_{,0} + \left\{\sqrt{AB}T_M^{01}\right\}_{,1} &= -\frac{2}{r}\sqrt{AB}T_M^{01}, && (1)\\
\left\{T_M^{01}\right\}_{,0} + \left\{\sqrt{AB}T_M^{11}\right\}_{,1} &= -\frac{1}{2}\sqrt{AB}\left\{\frac{4}{r}T_M^{11} + \frac{(1-A)}{Ar}(T_M^{00} - T_M^{11})\right. && (2)\\
&\qquad \left. + \frac{2\kappa r}{A}(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4rT^{22}\right\}, &&\\
rA_r &= (1-A) - \kappa r^2 T_M^{00}, && (3)\\
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$$T_M^{00} = \frac{\rho c^2 + p}{1 - \left(\frac{v}{c}\right)^2} \qquad T_M^{01} = \frac{\rho c^2 + p}{1 - \left(\frac{v}{c}\right)^2}\,\frac{v}{c}$$

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Conservation Law

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Source Term (→ right-hand side of (1))

Conservation Law (↑ left-hand side)

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The metric components A,B are only one derivative smoother than the sources T…

$G = \kappa T$ → The metric is only one derivative smoother than curvature tensor

Conclude: For shock wave solutions of the Einstein equations $G = \kappa T$ generated by the Glimm Scheme:

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Conclude: These are non-optimal solutions which can be smoothed to optimal regularity by our THM…

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Eg, regularity too low for geodesics to exist…

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$$\bar{g}_{\alpha\beta} = g_{ij}\,\frac{\partial x^i}{\partial y^\alpha}\,\frac{\partial x^j}{\partial y^\beta}$$

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Continuous ⇒

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Continuous ⇨

"Discontinuities at shocks have to all miraculously cancel out in the Leibniz products!"

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Continuous

Nevertheless…Our theorem says such transformations always exist…

Theorem (R-T): If

$$\Gamma \in L^\infty \text{ and } Riem(\Gamma) \in L^\infty$$

in a given coordinate system x,

then there always exist local coord trans $x \to y$ such that in y-coordinates,

$$\Gamma \in W^{1,p}, \quad Riem(\Gamma) \in L^\infty$$

We get this by solving the RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\text{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\text{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

$$Curl(J) \equiv \partial_j J_i^{\mu} - \partial_i J_j^{\mu} = 0 \ \text{ on } \partial\Omega, \tag{5}$$

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Theorem (R-T 2021): If

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…by generalized Div-Curl Lemma

Compensated Compactness

$\|\Gamma_i\|_\infty \le C \Rightarrow \Gamma_i \to \Gamma$ weakly in L^∞

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I.e.

$$R_i = d\Gamma_i + \Gamma_i \wedge \Gamma_i \to d\Gamma + \Gamma \wedge \Gamma$$

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"Wedge products weakly continuous when derivative bounds are exterior derivative…"

TRANSACTIONS OF THE
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Volume 303, Number 2, October 1987

ON WEAK CONTINUITY AND THE HODGE DECOMPOSITION

JOEL W. ROBBIN, ROBERT C. ROGERS[1] AND BLAKE TEMPLE[2]

ABSTRACT. We address the problem of determining the weakly continuous polynomials for sequences of functions that satisfy general linear first-order differential constraints. We prove that wedge products are weakly continuous when the differential constraints are given by exterior derivatives. This is sufficient for reproducing the Div-Curl Lemma of Murat and Tartar, the null Lagrangians in the calculus of variations and the weakly continuous polynomials for Maxwell's equations. This result was derived independently by Tartar who stated it in a recent survey article [7]. Our proof is explicit and uses the Hodge decomposition.

1. Introduction. The characterization of weakly continuous functionals has been an important tool in some recent developments in partial differential equations. In particular, the Div-Curl Lemma was instrumental in the work of Tartar [6] and DiPerna [3] on conservation laws, and the characterization of the null La-

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linear

"Wedge products weakly continuous when derivative bounds are exterior derivative…"

Our Theorem-same assumptions...

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Our theorem implies not just weak convergence, but strong convergence in a transformed coordinate system!

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...in y-coordinates

RT-Equations: Equations for Jacobians that lift the regularity of connection Γ to one derivative above $d\Gamma$

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\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A),\\
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Metric:

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$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$$

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$g_{00}(x)$ $\qquad$ $g_{11}(x)$

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Curvature:

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So far we have a theory for $p > n$

The Essential Regularity of a Connection

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Essential regularity applies to ANY connection, independent of metric or metric signature!

The Essential Regularity of a Connection

We use the RT-equations to prove a connection has an inherent geometric level of regularity, even though a manifold by itself does not…

We define this to be the essential regularity of the connection…it applies both locally and globally…

We conclude: a connection is the point at which a geometric level of regularity enters the very subject of Geometry!

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To start, we find a necessary and sufficient condition for determining, apriori, whether a singularity in a metric or connection is essential or removable…

If it is removable, then we construct coordinate transformations which lift it all the way up to its essential regularity…

…the highest level of regularity reachable by coordinate transformation…

The Essential Regularity of a Connection

To globalize, start with a connection Γ defined by its components in an atlas $\mathcal{A}$ of coordinate charts $x : \Omega_x \to \mathbb{R}^n$ which in turn define manifold $\mathcal{M}$.

The regularity of an atlas:

Definition: An atlas $\mathcal{A}$ is said to have regularity $W^{s,p}$ if all of its transition maps on the overlaps of coordinate neighborhoods have regularity $W^{s,p}$.

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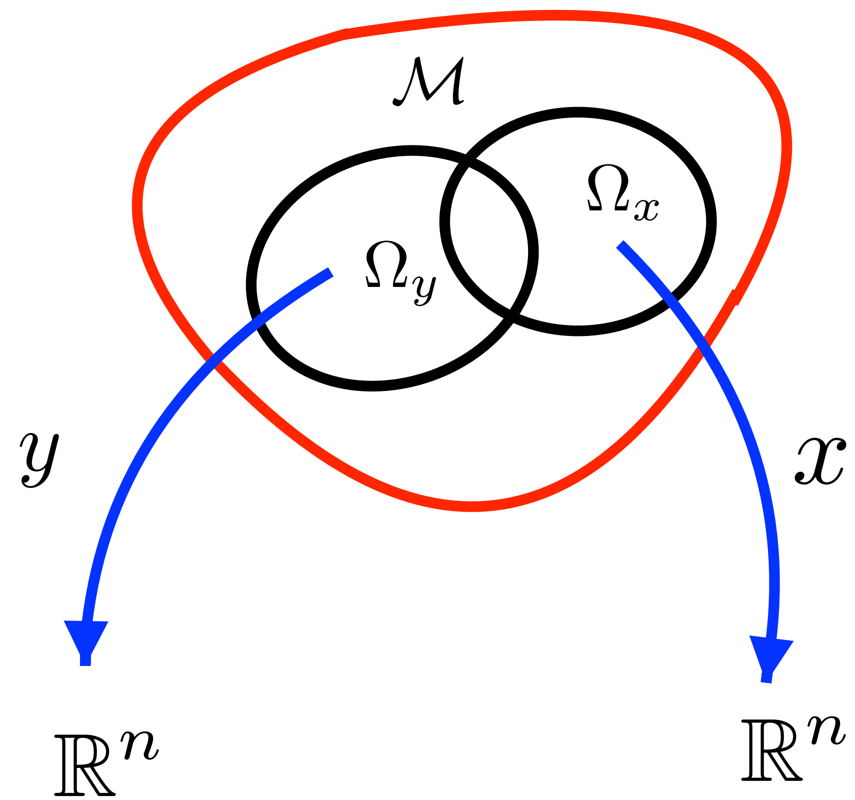

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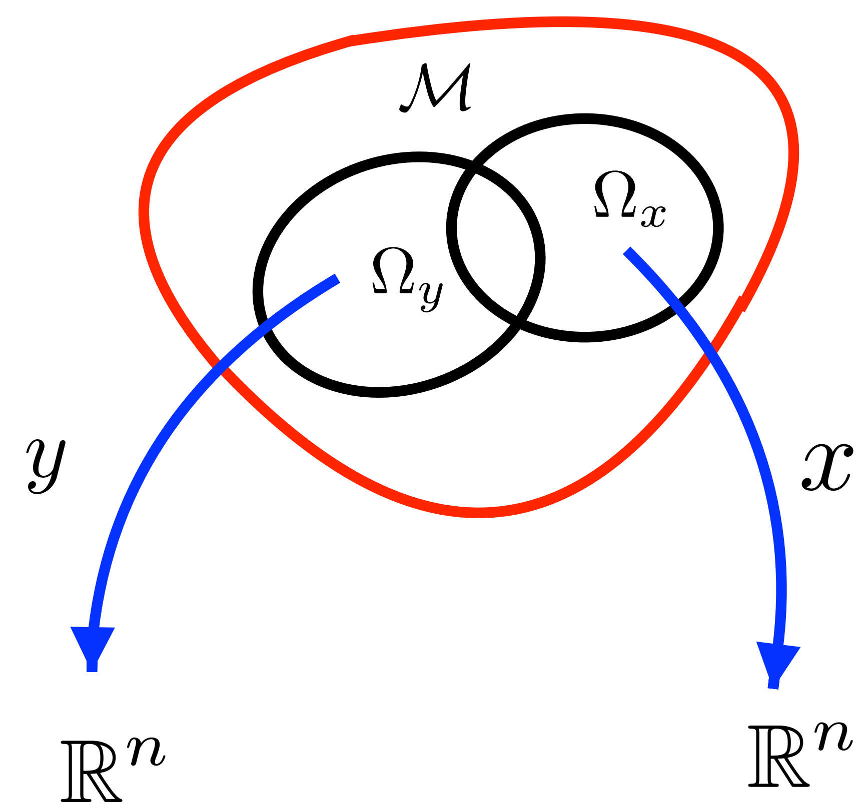

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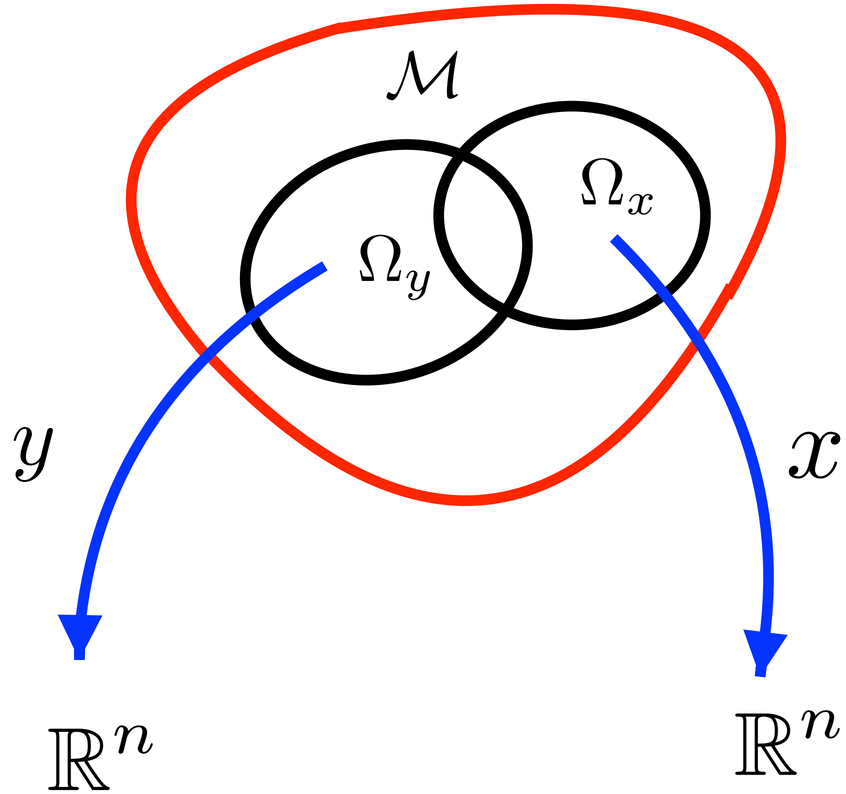

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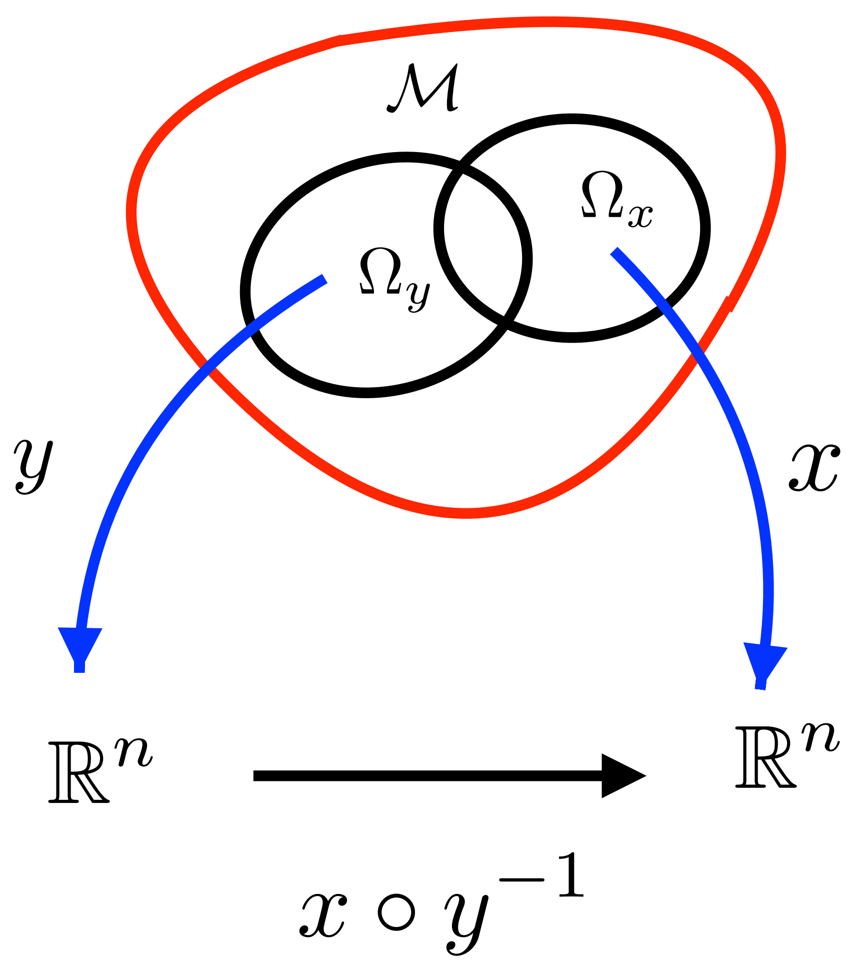

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Moreover, any such maximal atlas $\mathcal{A}^{\max}(s,p)$ contains subatlases of arbitrary higher regularity, including C^{∞} subatlases, and given any chart in $\mathcal{A}$ there exist a C^{∞} atlas which contains this chart.

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The largest relevant extension is thus… $\mathcal{A}^{\max}(2,p)$

This leads to the following definition of the essential (highest possible) regularity of a connection…

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Definition: Γ defined on $(\mathcal{M},\mathcal{A})$ has global essential regularity $m = ess_{\mathcal{M}}(\Gamma) \geq 0$ if there exists a sub-atlas $\mathcal{A}_m$ of $A^{\max}(2,p)$ of $\mathcal{A}$ such that $\Gamma \in W^{m,p}_{\mathcal{A}_m}$, and there does not exist a subatlas $\mathcal{A}_s$ of $A^{\max}(2,p)$ in which Γ, is more regular.

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Theorem (RT-2024): This provides a natural consistent geometric definition of the essential regularity of a connection.

Theorem (RT-2024): Assume $\Gamma \in W^{s,p}_{\mathcal{A}_s}$ is given on $(\mathcal{M}, \mathcal{A}_s)$, $p > n$, $s \geq 0$. Then:

(i) $ess_{\mathcal{M}}(\Gamma) = s$ if and only if $Riem(\Gamma) \in W^{s-1,p}_{\mathcal{A}}$ and $Riem(\Gamma) \notin W^{s',p}_{\mathcal{A}}$, any $s' \geq s$;

(ii) $ess_{\mathcal{M}}(\Gamma) = s + 1$ if and only if $Riem(\Gamma) \in W^{s,p}_{\mathcal{A}}$ and $Riem(\Gamma) \notin W^{s',p}_{\mathcal{A}}$, any $s' > s$;

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Moreover: RT-equations provide iterative method for lifting any connection below essential regularity...

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Note: Curvature is more regular than connection whenever you are two derivatives below essential.

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(This explains why the Schwarzshild curvature is one order more regular than the connection...)

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$Riem(\Gamma_y) \in W^{s,p}$

$J, J^{-1} \in W^{s+1,p}$

$Riem(\Gamma_x) \in W^{s,p}$

We didn't realize…

… J can lift the regularity of curvature as well!

To see this…

Apply a low regularity transformation to a connection starting at its essential regularity…

$$\Gamma \in W^{m,p}_{\mathcal{A}_m} \qquad Riem(\Gamma) \in W^{m-1,p}_{\mathcal{A}_m}$$

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$$\Rightarrow \quad \Gamma_y \in W^{s,p}_{\mathcal{A}} \qquad \mathcal{A} \in W^{s+2,p} \qquad R^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} \frac{\partial x^l}{\partial y^\gamma}$$

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$$s = m-1 \qquad Riem(\Gamma_y) \in W^{s,p}$$

$$s < m-1 \qquad Riem(\Gamma_y) \in W^{s+1,p}$$

To make this a proof, we need to know there do not exist Jacobians of regularity below $W^{s+1,p}$ which transform $\Gamma_y \to \Gamma_x \in W^{s,p}$

…i.e., by some weird cancellation of terms…

Lemma: The Jacobian J transforms $\Gamma_y \in W^{s,p}$ to $\Gamma_x \in W^{r,p}$, $0 \leq r \leq s$ if and only if the components of J satisfy $J \in W^{r+1,p}$.

"Proof": You can solve for the Jacobian in the connection transformation law…

$$(\Gamma_x)^{\mu}_{\rho\nu} = (J^{-1})^{\mu}_{\alpha}\Big(J^{\beta}_{\rho} J^{\gamma}_{\nu} (\Gamma_y)^{\alpha}_{\beta\gamma} + \tfrac{\partial}{\partial x^{\rho}} J^{\alpha}_{\nu}\Big)$$

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...i.e., by some weird cancellation of terms...

Lemma: The Jacobian J transforms $\Gamma_y \in W^{s,p}$ to $\Gamma_x \in W^{r,p}$, $0 \leq r \leq s$ if and only if the components of J satisfy $J \in W^{r+1,p}$.

"Proof": You can solve for the Jacobian in the connection transformation law...

$$(\Gamma_x)^\mu_{\rho\nu} = (J^{-1})^\mu_\alpha \left(J^\beta_\rho J^\gamma_\nu (\Gamma_y)^\alpha_{\beta\gamma} + \frac{\partial}{\partial x^\rho} J^\alpha_\nu \right)$$

$$\frac{\partial}{\partial x^\rho} J^\alpha_\nu = J^\alpha_\mu (\Gamma_x)^\mu_{\rho\nu} - J^\beta_\rho J^\gamma_\nu (\Gamma_y)^\alpha_{\beta\gamma}$$

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The regularity of Γ_x

determines regularity of $\frac{\partial}{\partial x^\rho} J^\alpha_\nu \in W^{s,p}$

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Summary: Everything works because the regularity of a connection controls the regularity of both the atlas… and the Jacobians which transform it…

Conclude:

- Essential regularity is characterized by the condition that the curvature is precisely one derivative below the regularity of its connection.

- Connection one derivative below essential regularity iff curvature is at precisely the same regularity as the connection.

- Connection two or more derivatives below essential regularity iff the curvature is precisely one derivative more regular than the connection.

- Theorem (RT-2024): Every connection can be lifted to essential regularity $m < \infty$ by a sequence of solutions of the RT-equations.

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Now we know a manifold together with any connection, always does…for $p > n$.

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Hirsch, Theorem 2.9: A C^1-manifold alone does not have enough structure to determine a geometric level of regularity.

Now we know a manifold together with any connection, always does…for $p > n$.

This would apply to Black Hole singularities if we extend our existence theory for the RT-equations to $p < n$.

The Essential Regularity of a Connection

(1) Given a metric or connection, is there a necessary and sufficient condition for determining, apriori, whether a singularity is essential or removable? YES

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(2) If so: Is there a canonical procedure for constructing coordinate transformations which remove it? YES

(3) Does every metric/connection have an essential (highest possible) regularity to which it can be lifted? YES

End

Thank you!!

Derivation of the RT-equations

M. Reintjes and B. Temple, *On the optimal regularity implied by the assumptions of geometry I: Connections on tangent bundles*, Meth. Appl. Anal., Vol. 29, No. 4, 303-396, (2023)

M. Reintjes and B. Temple, *The regularity transformation equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity*, Adv. Theor. Math. Phys 24.5, (2020), 1203-1245.

The RT-equations:

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A),\\
\Delta J &= \delta(J\cdot\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A,\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big),\\
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$$d\vec{J} = 0 \ \text{ on } \partial\Omega.$$

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\end{aligned}
$$

free to be chosen

Here: $\tilde{\Gamma}$ is a matrix valued 1-form, J and A are matrix valued 0-forms, and $\vec{J}$, $\vec{A}$ are vector valued 1-forms as follows:

$$\tilde{\Gamma} \equiv \tilde{\Gamma}^{\mu}_{\nu i} dx^i$$

$$J \equiv J^{\mu}_{\nu} \qquad \vec{J} \equiv J^{\mu}_{i} dx^i \qquad d\vec{J} = Curl(J)$$

$$A \equiv A^{\mu}_{\nu} \qquad \vec{A} \equiv A^{\mu}_{i} dx^i \qquad d\vec{A} = Curl(A)$$

The integrability condition for J is: $Curl(J) = 0$

Two operations on matrix valued forms:

$$\overrightarrow{\text{div}}(\omega)^{\alpha} \equiv \sum_{l=1}^{n} \partial_l \Big((\omega_l^{\alpha})_{i_1,,,i_k} \Big) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

("take divergence in lower matrix component")

$$\langle A\,;B\rangle^{\mu}_{\nu} \equiv \sum_{i_1<\ldots<i_k} A^{\mu}_{\sigma\;i_1\ldots i_k} B^{\sigma}_{\nu\;i_1\ldots i_k}$$

("matrix valued inner product")

To derive the RT-equations…

The first breakthrough was the Riemann-flat condition…

The Riemann-flat Condition

M. Reintjes and B. Temple, *Shock Wave Interactions and the Riemann-flat Condition: The Geometry behind Metric Smoothing and the Existence of Locally Inertial Frames in General Relativity*, Arch. Rat. Mech. Anal. 235 (2020), 1873-1904.

The Riemann-flat condition:

Assume $\Gamma, R \in L^\infty$.

Then: There exists a $C^{1,1}$ coordinate transformation which smooths Γ to $C^{0,1}$ if and only if there exists a tensor $\tilde{\Gamma} \in C^{0,1}$ st

$$Riem(\Gamma + \tilde{\Gamma}) = 0.$$

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The theorem applies at other orders of smoothness, for example: $\Gamma, \tilde{\Gamma} \in W^{m,p}$

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Geometric & independent of metric signature…

“Proof”: Assume $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

and

$$\Gamma^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{i}_{jk}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial x^{j}}{\partial y^{\beta}}\frac{\partial x^{k}}{\partial y^{\gamma}} + \frac{\partial^2 y^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\frac{\partial x^{\sigma}}{\partial y^{\beta}}\frac{\partial x^{\tau}}{\partial y^{\gamma}}$$

L^{∞}

$C^{0,1}$ $\quad$ $C^{0,1}$

$-\tilde{\Gamma}^{\alpha}_{\beta\gamma}$

Riemann-flat

so

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

The "hard" part is: If

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

...then a smoothing transformation exists. ■

$\tilde{\Gamma}$ continuous implies $\Gamma + \tilde{\Gamma}$ has the same jump discontinuities (shock set) as Γ

First idea: Find Nash-type embedding theorem to extend the shock set to a flat connection

Better Idea: Use the Riemann-flat condition to derive a system of elliptic equations in $\tilde{\Gamma}, J$

"The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity"

Moritz Reintjes, Blake Temple

Arch. Rat. Mech. Anal. 235 (2020), 1873-1904

Derivation of the RT-equations from Riemann-Flat Condition

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Then: Connection Transformation Law gives:

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Then: $\Gamma^k_{ij} = (J^{-1})^k_{\alpha} J^{\beta}_i J^{\gamma}_j \Gamma^{\alpha}_{\beta\gamma} + (J^{-1})^k_{\alpha} \partial_j J^{\alpha}_i$

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$$\underbrace{\Gamma^k_{ij}}_{\Gamma} = \underbrace{(J^{-1})^k_\alpha J^\beta_i J^\gamma_j \Gamma^\alpha_{\beta\gamma}}_{\tilde{\Gamma}} + \underbrace{(J^{-1})^k_\alpha \partial_j J^\alpha_i}_{J^{-1}dJ}$$

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Flat

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$$\Gamma - \tilde{\Gamma} = J^{-1} dJ$$

$$Riem(\Gamma - \tilde{\Gamma}) = 0$$

Theorem (Riemann-flat): These are equivalent!

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Augment to first order system...

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$\delta =$ co-derivative of Euclidean coord metric

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"gauge freedom"

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We look to use the connection transformation law to couple this to an equation for J ...

$$J^{-1} dJ = \Gamma - \tilde{\Gamma} \iff dJ = J(\Gamma - \tilde{\Gamma})$$

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We now construct a closed system in $(\tilde{\Gamma}, J)$

from these 2 forms of Riemann-flat condition

(They start out as equivalent!)

To break the equivalence, apply $\Delta = d\delta + \delta d$ to

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to get Poisson equations…

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1} dJ\big) + d(J^{-1}A),$$

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where $h = J^{-1}A$ is free…

We still need a condition which guarantees J is the Jacobian of a coordinate transformation…

This is the condition:

$$d\vec{J} \equiv Curl(J) = 0$$

Since J is a 0-form, we need to "vectorize" it
To turn "Curl" into exterior derivative…

To impose $d\vec{J} \equiv Curl(J) = 0$ ...

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$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big)$$

This leads to the RT-equations:

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Another "miraculous cancellation" occurs in in A-equation

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Consider the A-equation:

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\end{aligned}
$$

Lemma: (for smooth Γ):

$$d\left(\overrightarrow{\delta(J\,\Gamma)}\right) = \underbrace{\overrightarrow{\mathrm{div}}\left(dJ \wedge \Gamma\right)}_{W^{m-2,p}} + \underbrace{\overrightarrow{\mathrm{div}}\left(J\, d\Gamma\right)}_{W^{m-1,p}}$$

$d\Gamma$ one derivative smoother than $\delta\Gamma$!

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Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations iff $(\tilde{\Gamma}', J)$ does...

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

Finally, $\tilde{\Gamma}$ which solves RT-equations may not be the actual connection transformed by J…

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$$\tilde{\Gamma}' \in W^{m+1,p} \qquad \Gamma \in W^{m,p} \qquad dJ \in W^{m,p}$$

$\tilde{\Gamma}'$ has the same regularity as $\tilde{\Gamma}$!

Theorem: $(\tilde{\Gamma}, J, A)$ is a solution of the RT-equations

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

$$Curl(J) \equiv \partial_j J_i^\mu - \partial_i J_j^\mu = 0 \;\; \text{on } \partial\Omega,$$

Summary: We start with two first order equations both equivalent to the Riemann-flat condition…

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Using the co-derivative of the coordinate Euclidean metric, we convert these in to a coupled system of Poisson equations…

…and by miraculous cancellations on the RHS, the solutions provide Jacobians which lift the connection to optimal regularity…

End

Thank you!!