Causal Dissipation for the Relativistic Fluid Dynamics of Ideal Gases

Heinrich Freistühler
and
Blake Temple

Proceedings of the Royal Society-A
May 2017

Culmination of a 15 year project:

PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

Check for updates

Cite this article: Freistühler H, Temple B. 2017 Causal dissipation for the relativistic dynamics of ideal gases. *Proc. R. Soc. A* 20160729. http://dx.doi.org/10.1098/rspa.2016.0729

Received: 24 September 2016

Accepted: 19 April 2017

Subject Areas:
differential equations, fluid mechanics, relativity

Keywords:
causality, dissipation, relativistic, Navier–Stokes, ideal gas

Authors for correspondence:
Heinrich Freistühler
e-mail: heinrich.freistuehler@uni-konstanz.de
Blake Temple
e-mail: temple@math.ucdavis.edu

Causal dissipation for the relativistic dynamics of ideal gases

Heinrich Freistühler[1] and Blake Temple[2]

[1]Department of Mathematics, University of Konstanz, Konstanz, Germany

[2]Department of Mathematics, University of California, Davis, CA 95616, USA

BT, 0000-0002-6907-1101

We derive a general class of relativistic dissipation tensors by requiring that, combined with the relativistic Euler equations, they form a second order system of partial differential equations which is symmetric hyperbolic in a second order sense when written in the natural Godunov variables that make the Euler equations symmetric hyperbolic in the first order sense. We show that this class contains a unique element representing a causal formulation of relativistic dissipative fluid dynamics which (i) is equivalent to the classical descriptions by Eckart and Landau to first order in the coefficients of viscosity and heat conduction and (ii) has its signal speeds bounded sharply by the speed of light. Based on these properties, we propose this system as a natural candidate for the relativistic counterpart of the classical Navier–Stokes equations. Q1

1. Introduction

In the absence of dissipation, relativistic fluid dynamics is governed by the Euler equations

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta}=0, \quad \frac{\partial}{\partial x^\beta}N^\beta=0, \qquad (1.1)$$

In this we propose:

A Causal
Navier-Stokes Equation
consistent with
the principles of
Einstein's Theory of Relativity

In this we propose:

A Causal
Navier-Stokes Equation
consistent with
the principles of
Einstein's Theory of Relativity

Applicable to the general five field theory ideal gases: $(\rho, \mathbf{u}, s)$

Our initial publication (2014):

THE ROYAL SOCIETY
PUBLISHING

PROCEEDINGS
OF THE ROYAL SOCIETY A

MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES

search

Advanced

Home Content Information for About us Sign up Submit

CrossMark
click for updates

Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation

Heinrich Freistühler, Blake Temple

Published 19 March 2014. DOI: 10.1098/rspa.2014.0055

Article | Info & Metrics | eLetters | PDF

Abstract

Current theories of dissipation in the relativistic regime suffer from one of two deficits: either their dissipation is not causal or no profiles for strong shock waves exist. This paper proposes a relativistic Navier–Stokes–Fourier-type viscosity and heat conduction tensor such that the resulting second-order system of partial differential equations for the fluid dynamics of pure radiation is symmetric hyperbolic. This system has causal dissipation as well as the property that all shock waves of arbitrary strength have smooth profiles. Entropy production is positive both on gradients near those of solutions to the dissipation-free equations and on gradients of shock profiles. This shows that the new dissipation stress tensor complies to leading order with the

Abstract:

We derive a general class of relativistic dissipation tensors by requiring that, combined with the relativistic Euler equations, they form a second order system of partial differential equations which is symmetric hyperbolic in a second order sense when written in the natural Godunov variables that make the Euler equations symmetric hyperbolic in the first order sense. We show that this class contains a unique element representing a causal formulation of relativistic dissipative fluid dynamics which (i) is equivalent to the classical descriptions by Eckart and Landau to first order in the coefficients of viscosity and heat conduction and (ii) has its signal speeds bounded sharply by the speed of light. Based on these properties, we propose this system as a natural candidate for the relativistic counterpart of the classical Navier–Stokes equations.

It has long been thought that there is no relativistic version of the Navier-Stokes equations consistent with Einstein's principle of causality—that all signals should propagate at sub-luminal speeds $v \leq c$.

It has long been thought that there is no relativistic version of the Navier-Stokes equations consistent with Einstein's principle of causality—that all signals should propagate at sub-luminal speeds $v \leq c$.

By Navier-Stokes we mean equations in which dissipation is measured by bulk and shear viscosity, and heat conductivity alone. .

It has long been thought that there is no relativistic version of the Navier-Stokes equations consistent with Einstein's principle of causality—that all signals should propagate at sub-luminal speeds $v \leq c$.

By Navier-Stokes we mean equations in which dissipation is measured by bulk and shear viscosity, and heat conductivity alone. .

Navier-Stokes is a leading order theory of dissipation, correct to first order in viscosity and heat conduction.

The Classical Navier-Stokes equations:

The Classical Navier-Stokes equations:

$$\rho_t + div(\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E + p) \cdot \mathbf{u}) = div\,\{\eta \mathcal{S} \cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

The Classical Navier-Stokes equations:

$$\rho_t + div(\rho\mathbf{u}) = 0$$

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u}\otimes\mathbf{u} + pI) = div\left\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\right\}$$

$$E_t + div((E+p)\cdot\mathbf{u}) = div\left\{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\right\}$$

$$\rho = \mathrm{d}ensity$$

The Classical Navier-Stokes equations:

$$\rho_t + div(\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E + p) \cdot \mathbf{u}) = div\,\{\eta \mathcal{S} \cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

$\rho = \mathrm{d}ensity$ $\quad p = \mathrm{p}ressure$

The Classical Navier-Stokes equations:

$$\rho_t + div(\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E + p) \cdot \mathbf{u}) = div\,\{\eta \mathcal{S} \cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

$\rho = \mathrm{d}ensity \qquad p = \mathrm{p}ressure$

$\mathbf{u} = (u^1, u^2, u^3) = \mathrm{v}elocity$

The Classical Navier-Stokes equations:

$$\rho_t + div(\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E + p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

$\rho = \mathrm{d}ensity \qquad p = \mathrm{p}ressure$

$\mathbf{u} = (u^1, u^2, u^3) = \mathrm{v}elocity$

$E = \rho\epsilon + \frac{1}{2}\rho\mathbf{u}^2 = \mathrm{e}nergy\ density$

The Classical Navier-Stokes equations:

$$\rho_t + div(\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

$\rho = \mathrm{d}ensity$ $\quad$ $p = \mathrm{p}ressure$

$\mathbf{u} = (u^1, u^2, u^3) = \mathrm{v}elocity$

$E = \rho\epsilon + \frac{1}{2}\rho\mathbf{u}^2 = energy\ density$ $\quad$ $\epsilon = \mathrm{s}pecific\ energy$

Compressible Navier-Stokes Equations

$$\rho_t + div(\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E + p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

Compressible
Euler

Compressible Navier-Stokes Equations

$$\rho_t + div(\rho\mathbf{u}) = 0$$

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u}\otimes\mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E+p)\cdot\mathbf{u}) = div\,\{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

Compressible Euler

Navier-Stokes Dissipation

Compressible Navier-Stokes Equations

$$\rho_t + div(\rho\mathbf{u}) = 0$$

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u}\otimes\mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$\underbrace{E_t + div((E+p)\cdot\mathbf{u})}_{\text{Compressible Euler}} = \underbrace{div\,\{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}}_{\text{Navier-Stokes Dissipation}}$$

Three dissipation parameters: η, ζ, χ

Compressible Navier-Stokes Equations

$$\rho_t + div(\rho\mathbf{u}) = 0$$

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u}\otimes\mathbf{u} + pI) = div\{\eta\mathcal{S}(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$E_t + div((E+p)\cdot\mathbf{u}) = div\{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\nabla\theta\}$$

Compressible Euler

Navier-Stokes Dissipation

Three dissipation parameters: η, ζ, χ

η=shear viscosity

Compressible Navier-Stokes Equations

$$\rho_t + div(\rho\mathbf{u}) = 0$$

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u}\otimes\mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$\underbrace{E_t + div((E+p)\cdot\mathbf{u})}_{\text{Compressible Euler}} = \underbrace{div\,\{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}}_{\text{Navier-Stokes Dissipation}}$$

Three dissipation parameters: η, ζ, χ

η=shear viscosity $\quad \zeta$=bulk viscosity

Compressible Navier-Stokes Equations

$$\rho_t + div(\rho\mathbf{u}) = 0$$

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$\underbrace{E_t + div((E+p)\cdot\mathbf{u})}_{\text{Compressible Euler}} = \underbrace{div\,\{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\nabla\theta\}}_{\text{Navier-Stokes Dissipation}}$$

Three dissipation parameters: η, ζ, χ

η=shear viscosity ζ=bulk viscosity χ=heat conductivity

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$S(\mathbf{u})$ = 2x symmetric trace free part of velocity gradient

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$S(\mathbf{u})$ = 2x symmetric trace free part of velocity gradient

η measures viscosity due to shearing

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u}\otimes\mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$S(\mathbf{u}) = 2D - \frac{2}{3}(div\,\mathbf{u})I$$

$$D = \frac{u^i{}_{,j} + u^j{}_{,i}}{2} = \text{symmetric part of velocity gradient}$$

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$S(\mathbf{u}) = 2D - \frac{2}{3}(div\,\mathbf{u})I$$

$$D = \frac{{u^i}_{,j} + {u^j}_{,i}}{2}$$

$$div\,\mathbf{u} = Trace(D)$$

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$S\mathbf{u} = 2D - \tfrac{2}{3}(div\,\mathbf{u})\,I$$

$$D = \frac{u^i{}_{,j} + u^j{}_{,i}}{2}$$

$$div\,\mathbf{u} = Trace(D)$$

ζ measures viscosity due to compression and dilation

Compressible Navier-Stokes Equations

Consider the momentum equation:

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + pI) = div\,\{\eta S(\mathbf{u}) + \zeta(div\,\mathbf{u})I\}$$

$$S\mathbf{u} = 2D - \tfrac{2}{3}(div\,\mathbf{u})\,I$$

$$div\,\mathbf{u} = Trace(D)$$

$$D = \frac{u^i{}_{,j} + u^j{}_{,i}}{2}$$

$$\mathrm{div}\,\mathbf{u} = u^i{}_{,i}$$

ζ measures viscosity due to compression and dilation

Compressible Navier-Stokes Equations

Consider the energy equation:

Compressible Navier-Stokes Equations

Consider the energy equation:

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

Compressible Navier-Stokes Equations

Consider the energy equation:

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\underbrace{\eta\mathcal{S}\cdot\mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u}} + \chi\,\nabla\theta\}$$

Energy changes due to
shear and bulk viscosity

Compressible Navier-Stokes Equations

Consider the energy equation:

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

Heat Conduction

Compressible Navier-Stokes Equations

Consider the energy equation:

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

$$div(\kappa\nabla\theta) = \kappa\Delta\theta + l.o.t$$

Heat Conduction

Compressible Navier-Stokes Equations

Consider the energy equation:

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

$$div(\chi\nabla\theta) = \chi\Delta\theta + l.o.t.$$

Newton Law of Cooling

Heat Conduction

Compressible Navier-Stokes Equations

Consider the energy equation:

$$E_t + div((E+p)\cdot \mathbf{u}) = div\,\{\eta \mathcal{S}\cdot \mathbf{u} + \zeta(div\,\mathbf{u})\mathbf{u} + \chi\,\nabla\theta\}$$

Heat Conduction

$$div(\chi\nabla\theta) = \chi\Delta\theta + l.o.t.$$

Newton Law of Cooling

Note: η, ζ, χ can depend on temperature (and density), but this does not affect leading second order terms

The Classical Navier-Stokes equations consist of the first order compressible Euler equations together with second order dissipation terms linear in velocity and temperature gradients on the RHS.

The Classical Navier-Stokes equations consist of the first order compressible Euler equations together with second order dissipation terms linear in velocity and temperature gradients on the RHS.

NS is a parabolic-like system like the heat equation — all linearized modes decay, but have speeds that tend to infinity with wave number…

The Classical Navier-Stokes equations consist of the first order compressible Euler equations together with second order dissipation terms linear in velocity and temperature gradients on the RHS.

NS is a parabolic-like system like the heat equation — all linearized modes decay, but have speeds that tend to infinity with wave number…

…the NS equations are not causal…

Question 1: Is there a relativistic version of Navier-Stokes that is causal?

Question 1: Is there a relativistic version of Navier-Stokes that is causal?

Question 2: Could the classical NS dissipation terms actually be the limit of a causal relativistic dissipation based on the wave equation?

Question 1: Is there a relativistic version of Navier-Stokes that is causal?

Question 2: Could the classical NS dissipation terms actually be the limit of a causal relativistic dissipation based on the wave equation?

Question 3: What would constitute a derivation of a relativistic Navier-Stokes equation?

Following the work of Eckhart (1940s), L. Landau and S. Weinberg, math physics essentially gave up looking for a causal version of NS, and this led to the Israel-Stewart and Mueller-Ruggeri theories of dissipation, based on kinetic theory following Grad’s classical Theory of Moments.

Following the work of Eckhart (1940s), L. Landau and S. Weinberg, math physics essentially gave up looking for a causal version of NS, and this led to the Israel-Stewart and Mueller-Ruggeri theories of dissipation, based on kinetic theory following Grad's classical Theory of Moments.

These theories are causal, but much more complicated, and known to be incorrect far from equilibrium. For example, IS equations do not admit shock profiles for strong shocks.

Following the work of Eckhart (1940s), L. Landau and S. Weinberg, math physics essentially gave up looking for a causal version of NS, and this led to the Israel-Stewart and Mueller-Ruggeri theories of dissipation, based on kinetic theory following Grad's classical Theory of Moments.

W. Israel: {\it Nonstationary irreversible thermodynamics: a causal relativistic theory}, Annals Phys.\ 100 (1976), 310-331.

W. Israel and J. M. Stewart: {\it Transient relativistic thermodynamics and kinetic theory}, Annals Phys. 118 (1979), 341-372.

Following the work of Eckhart (1940s), L. Landau and S. Weinberg, math physics essentially gave up looking for a causal version of NS, and this led to the Israel-Stewart and Mueller-Ruggeri theories of dissipation, based on kinetic theory following Grad's classical Theory of Moments.

I. Liu, I. M\"uller, T. Ruggeri: {\it Relativistic thermodynamics of gases},Ann.\ Physics 169 (1986), 191-219.

I. M\"uller and T. Ruggeri: Rational Extended Thermodynamics. Second edition. Springer Tracts in Natural Philosophy, 37. Springer-Verlag, New York, 1998.

Reference: We began by reading work of Jin and Xin (See also Liu, Levermore, et al) in which conservation laws with parabolic terms are derived by a Chapman-Enskog type expansion from relaxation models based on the wave equation.

S. Jin and Z. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, Comm. Pure Appl. Math., Vol. XLVIII, 235-276 (1995).

Reference: See also an causal models introduced by Lichnerowicz/Choquet-Bruhat and studied recently by Disconzi…

Y. Choquet-Bruhat: General Relativity and the Einstein Equations. Oxford University Press, Oxford, 2009.

A.Lichnerowicz: Th\'eories relativistes de la gravitation et de l'\'electromagn\'etisme. Relativit\'e g\'en\'erale et th \'eories unitaires. Masson et Cie., Paris, 1955.

M. Disconzi: On the well-posedness of relativistic viscous fluids, Nonlinearity 27, no.8, pp. 1915-1935 (2014)

Our Derivation: Starting with a general relativistic dissipation tensor linear in temp and velocity gradients, we impose three selection criteria…

Our Derivation: Starting with a general relativistic dissipation tensor linear in temp and velocity gradients, we impose three selection criteria…

(1) The resulting system should be symmetric hyperbolic in a second order sense (HKM) in the same variables that make the compressible Euler equations symmetric hyperbolic as a first order system (Godunov variables).

Our Derivation: Starting with a general relativistic dissipation tensor linear in temp and velocity gradients, we impose three selection criteria…

(1) The resulting system should be symmetric hyperbolic in a second order sense (HKM) in the same variables that make the compressible Euler equations symmetric hyperbolic as a first order system (Godunov variables).

(…that physical equations should be symmetric hyperbolic is a longstanding principle of Applied Mathematics-we ask the system be symmetric hyperbolic in the simplest possible sense…)

Our Derivation: Starting with a general relativistic dissipation tensor linear in temp and velocity gradients, we impose three selection criteria…

(1) The resulting system should be symmetric hyperbolic in a second order sense (HKM) in the same variables that make the compressible Euler equations symmetric hyperbolic as a first order system (Godunov variables).

(2) The resulting system should be equivalent to the Eckhart and Landau equations to leading order in viscosity and heat conduction.

Our Derivation: Starting with a general relativistic dissipation tensor linear in temp and velocity gradients, we impose three selection criteria…

(1) The resulting system should be symmetric hyperbolic in a second order sense (HKM) in the same variables that make the compressible Euler equations symmetric hyperbolic as a first order system (Godunov variables).

(2) The resulting system should be equivalent to the Eckhart and Landau equations to leading order in viscosity and heat conduction.

(3) The system should be sharply causal.

Theorem (Freist-Temp): There exists a unique relativistic dissipation tensor ΔT that meets conditions (1)-(3).

Equations:

$$Div(T + \Delta T) = 0$$

$$Div(N + \Delta N) = 0$$

$$T^{ij} = (\rho + p)u^i u^j + p g^{ij}$$

Stress tensor for a perfect fluid

I have to define ΔT and ΔN ...

Assume perfect fluid:

$\rho = n(m + e(n, s)) = \text{energy density}$ $\quad$ $m = \text{particle mass}$

$e = \text{internal energy}$

$s = \text{specific entropy}$

$$e = kn^{\gamma-1} exp\left(\frac{s}{c_v}\right) = c_v \theta$$ (polytropic equation of state)

$\theta = \text{temperature}$ $\quad$ $c_v = \text{specific heat}$

$$de = \theta\, ds - p\, d\left(\frac{1}{n}\right)$$ (2nd law of thermodynamics)

$p = n^2 e_n(n, s)$ $\quad$ $\theta = e_s(n, s)$

Then in the particle frame:

$$-\Delta T_0 = \begin{pmatrix} \sigma \nabla \cdot \mathbf{u} & -\sigma \dot{\mathbf{u}} \\ -\sigma \dot{\mathbf{u}} & \eta \mathbf{S}\mathbf{u} + \tilde{\zeta} \nabla \cdot \mathbf{u}\, \mathbf{I} \end{pmatrix}$$

$$u = (u^0, \mathbf{u}) = 4 - \text{velocity}$$

$$\sigma = \frac{\frac{4}{3}\eta + \zeta + (\gamma - 1)^2 \left(\frac{m^2}{h\theta}\right)\kappa}{1 - (\gamma - 1)\left(1 - \frac{m}{h}\right)}$$

$$h = \frac{\rho + p}{n} = \text{specific enthalpy}$$

$$\eta = \text{shear viscosity}$$

$$\zeta = \text{bulk viscosity}$$

$$\chi = \text{heat conductivity}$$

$$\kappa = \frac{\chi \theta^2}{n}$$

$$\tilde{\zeta} = \zeta + \tilde{\zeta}_1 + \tilde{\zeta}_2 \qquad \tilde{\zeta}_1 = (\gamma - 1)^2 \left(\frac{m^2}{h\theta}\right) \qquad \tilde{\zeta}_2 = (\gamma - 1)^2 \left(1 - \frac{m}{h}\right)\sigma$$

ΔT_0 determined by η, ζ, χ

$$-\Delta N_0 = \left(-\tilde{\kappa}\dot{\psi} + \tilde{\sigma}\nabla\cdot\mathbf{u}, \tilde{\kappa}\nabla\psi - \tilde{\sigma}\dot{\mathbf{u}}\right)$$

$$N^\alpha = nu^\alpha$$ (number current density)

$$\psi^\alpha = \frac{u^\alpha}{\theta}$$ (classical Godunov variables)

$$\psi = \frac{h}{\theta} - s$$ (generalized Godunov variable)

$$\tilde{\kappa} = \frac{\kappa}{h} = \frac{\chi\theta^2}{h^2} \qquad \tilde{\sigma} = \frac{\sigma}{h}$$

ΔN_0 determined by η, ζ, χ

Our proposal for relativistic Navier-Stokes:

$$\frac{\partial}{\partial x^\alpha}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}\right) = 0$$

$$\frac{\partial}{\partial x^\alpha}\left(N^{\alpha} + \Delta N^{\alpha}\right) = 0$$

$$-\Delta T_0 = \begin{pmatrix} \sigma \nabla \cdot \mathbf{u} & -\sigma \dot{\mathbf{u}} \\ -\sigma \dot{\mathbf{u}} & \eta \mathbf{S}\mathbf{u} + \tilde{\zeta} \nabla \cdot \mathbf{u}\, \mathbf{I} \end{pmatrix}$$

$$-\Delta N_0 = \left(-\tilde{\kappa}\dot{\psi} + \tilde{\sigma}\nabla \cdot \mathbf{u}, \tilde{\kappa}\nabla\psi - \tilde{\sigma}\dot{\mathbf{u}}\right)$$

(rest frame expressions)

ΔT and ΔN can be written invariantly:

$$-\Delta T = \eta \Pi^{\alpha\gamma}\Pi^{\beta\delta}\left(\frac{\partial u_\alpha}{\partial x^\beta} + \frac{\partial u_\beta}{\partial x^\alpha} - \frac{2}{3} g_{\alpha\beta}\frac{\partial u^\gamma}{\partial x^\gamma}\right) + \tilde{\zeta}\Pi^{\alpha\beta}\frac{\partial u^\gamma}{\partial x^\gamma}$$

$$+\sigma\left(u^\alpha u^\beta \frac{\partial u^\gamma}{\partial x^\gamma} - u^\alpha u^\delta \frac{\partial u^\beta}{\partial x^\delta} - u^\beta u^\delta \frac{\partial u^\alpha}{\partial x^\delta}\right)$$

$$-\Delta N^\beta = \tilde{\kappa} g^{\beta\gamma}\frac{\partial \psi}{\partial x^\gamma} + \tilde{\sigma}\left(u^\beta \Pi^{\gamma\delta} - u^\delta \Pi^{\beta\gamma}\right)\frac{\partial u_\gamma}{\partial x^\delta}$$

$$\Pi^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$$ (projection)

These equation have many remarkable properties:

Theorem 1: The equations are symmetric hyperbolic in the second order sense of Hughs-Kato-Marsden (HKM) in the Godunov variable ψ^{α}, ψ which make the first order Euler equations symmetric hyperbolic as a first order system.

In fact, its the limit of a 5x5 symmetric hyperbolic system as a parameter tends to zero.

COR: The 5x5 system is well-posed.

Theorem 2: The linearized Fourier-Laplace modes are dissipative and sharply causal.

All Fourier modes, both transverse and longitudinal, are bounded by the speed of light, and the longitudinal mode speeds tend to the speed of light in the short wave length limit.

The amplitude of all nonzero modes decays in amplitude in forward time-like a parabolic system.

(Demonstration partly numerical in 3x3 case.)

Theorem 3: The equations are first order equivalent to the classical Eckhart and Landau equations.

Proof: The idea is that the conserved quantities can be replaced by anisotropic averages represented by correction terms linear in derivatives, on the order of the dissipation. These can then be incorporated into the dissipation tensor on the RHS if one neglects terms higher order in the dissipation.

That is…write

$$u^\alpha = \tilde{u}^\alpha + \epsilon \Delta \tilde{u}^\alpha$$

$$n = \tilde{n} + \epsilon \Delta \tilde{n}$$

$$n = \tilde{\rho} + \epsilon \Delta \tilde{\rho}$$

$$p = \tilde{p} + \epsilon \Delta \tilde{p}$$

where $\Delta \tilde{u}^\alpha, \Delta \tilde{n}, \Delta \tilde{\rho}, \Delta \tilde{p}$ are expressions linear

in the gradients $\dfrac{\partial \tilde{u}^\alpha}{\partial x^\beta}, \dfrac{\partial \tilde{n}}{\partial x^\beta}, \dfrac{\partial \tilde{\rho}}{\partial x^\beta}, \dfrac{\partial \tilde{p}}{\partial x^\beta}$

with coefficients depending on $\tilde{u}^\alpha, \tilde{n}, \tilde{\rho}, \tilde{p}$

Then writing…

$$\tilde{T}^{\alpha\beta} = (\tilde{\rho} + \tilde{p})\tilde{u}^{\alpha}\tilde{u}^{\beta} + \tilde{p}g^{\alpha\beta} \qquad \tilde{N}^{\alpha} = \tilde{n}\tilde{u}^{\alpha}$$

and substituting into

$$Div\,\{T + \Delta T\} = 0$$

$$Div\,N = 0$$

gives the equivalent system

$$Div\left\{\tilde{T} + \Delta T + \epsilon\delta\Delta\tilde{T}^{\alpha\beta} + O(\epsilon^2)\right\} = 0$$

$$Div\,\left\{N + \Delta N + \epsilon\delta\Delta\tilde{N}\right\} = 0$$

Defn: We say two systems

$$Div(T + \Delta T) = 0 \qquad Div\ \{N + \Delta N\} = 0$$

and

$$Div(\tilde{T} + \Delta\tilde{T}) = 0 \qquad Div\ \left\{\tilde{N} + \Delta\tilde{N}\right\} = 0$$

are first order equivalent if there exists a transformation

$$(u^\alpha, n, \rho, p) \to (\tilde{u}^\alpha, \tilde{n}, \tilde{\rho}, \tilde{p})$$

such that

$$\Delta\tilde{T} = \tilde{T} + \epsilon\delta\Delta\tilde{T} \qquad \Delta\tilde{N} = N + \epsilon\delta\Delta\tilde{N}$$

Lemma: Substituting inviscid gradient identities generates a first order equivalence (gradient re-expression)

That is, substituting into ΔT any identity derived from

$$\frac{\partial}{\partial x^\alpha} T^{\alpha\beta} = 0 \qquad \frac{\partial}{\partial x^\alpha} N^\alpha = 0$$

generates a first order equivalence.

To show our system is first order equivalent to Landau, we construct such an equivalence transformation:

$$\begin{pmatrix} 0 & 0 \\ 0 & \eta \mathbf{S}\mathbf{u} + \zeta \nabla \cdot \mathbf{u}\mathbf{I} \\ 0 & \left(\frac{\kappa}{h}\right) \nabla \psi \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -\sigma \cdot \mathbf{u}^t \\ -\sigma \cdot \mathbf{u} & \eta \mathbf{S}\mathbf{u} + \zeta \nabla \cdot \mathbf{u}\mathbf{I} \\ 0 & \left(\frac{\kappa}{h}\right) \nabla \psi - \left(\frac{\sigma}{h}\right) \dot{\mathbf{u}} \end{pmatrix}$$

$$\Delta \mathbf{u} = -\sigma \dot{u}$$

Landau

$$\Delta \rho = -\sigma \nabla \cdot \mathbf{u}$$

$$\Delta n = \frac{\sigma}{n} \nabla \cdot \mathbf{u}$$

$$\Delta p = (\gamma - 1)\left(1 - \frac{m}{n}\right) \sigma \nabla \cdot \mathbf{u} = \tilde{\zeta}_2 \nabla \cdot \mathbf{u}$$

$$\longrightarrow \begin{pmatrix} \sigma \nabla \cdot \mathbf{u} & -\sigma \cdot \mathbf{u}^t \\ -\sigma \cdot \mathbf{u} & \eta \mathbf{S}\mathbf{u} + (\zeta + \tilde{\zeta}_2) \nabla \cdot \mathbf{u}\mathbf{I} \\ \left(\frac{\sigma}{h}\right) \nabla \cdot \mathbf{u} & \left(\frac{\kappa}{h}\right) \nabla \psi - \left(\frac{\sigma}{h}\right) \dot{\mathbf{u}} \end{pmatrix}$$

$$\Delta n = -\frac{k}{h}\dot{\psi}$$

$$\Delta p = (\gamma - 1)m\frac{k}{h}\dot{\psi}$$

→

$$\begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & -\sigma\cdot\mathbf{u}^t \\ -\sigma\cdot\mathbf{u} & \eta\mathbf{S}\mathbf{u} + \left\{(\zeta + \tilde{\zeta}_2) + (\gamma-1)m\left(\frac{\kappa}{h}\right)\dot{\psi}\right\}\nabla\cdot\mathbf{u}\mathbf{I} \\ -\left(\frac{\sigma}{h}\right)\dot{\psi} + \left(\frac{\sigma}{h}\right)\nabla\cdot\mathbf{u} & \left(\frac{\kappa}{h}\right)\nabla\psi - \left(\frac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix}$$

$$\dot{\psi} = (\gamma - 1)m\theta^{-1}\nabla\cdot\mathbf{u}$$

(gradiant re-expression)

→

$$\Delta T = \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & -\sigma\cdot\mathbf{u}^t \\ -\sigma\cdot\mathbf{u} & \eta\mathbf{S}\mathbf{u} + (\zeta + \tilde{\zeta}_1 + \tilde{\zeta}_2)\nabla\cdot\mathbf{u}\,\mathbf{I} \\ -\left(\frac{\sigma}{h}\right)\dot{\psi} + \left(\frac{\sigma}{h}\right)\nabla\cdot\mathbf{u} & \left(\frac{\kappa}{h}\right)\nabla\psi - \left(\frac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix}$$

■

Theorem 4: Entropy production is positive on Eulerian gradients (i.e., gradients of the inviscid equations sufficient for first order equivalence.)

Proof: Entropy production is given by the quadratic form

$$Q \equiv S^{\alpha}_{,\alpha} = -\left(\frac{u_\alpha}{\theta}\right)_{,\beta} \Delta T^{\alpha\beta} - \psi_{,\beta} \Delta N^{\beta}$$

Expressing Eulerian gradients in the particle frame and substituting into the RHS leads to:

$$Q = \epsilon \left(\frac{\eta}{2\theta} \|\mathbf{Su}\|^2 + \frac{\zeta}{\theta} (\nabla \cdot \mathbf{u})^2 + \frac{\kappa}{h} |\nabla \psi|^2 \right) + O(\epsilon^2)$$

Comment: The Landau and Eckart systems (which are first order related by a velocity transformation) are derived by the condition that entropy production be positive on all gradients. Our discussions began with the idea that this is too stringent a condition for a theory that is essentially only first order in dissipation.

Future work: Is our system, which is a limit of HKM symmetric hyperbolic systems, still well-posed?

Future work: Do these relativistic corrections change the theory of turbulence?

Thank You!