On the Regularity Implied by the Hypotheses of Geometry

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Colloquium Talk
UC-Berkeley
October 28, 2021

All Joint Work With: Moritz Reintjes

The title is intended to conjure up Riemann's Habilitation Lecture of 1862, usually translated as:

"On the Hypotheses which Lie at the Foundation of Geometry"

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—in which Riemann introduced the Riemann curvature tensor.

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—Although Riemann based his theory on Riemannian metrics, Riemann's curvature is now viewed more generally as a property of a connection Γ.

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—This extends important work of Kazden-DeTurck and Uhlenbeck from (positive definite) Riemannian metrics to Lorentzian metrics of General Relativity

Our theory includes the discovery of the Regularity Transformation equations (RT-equations)

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…as well as a subsequent existence theory to establish optimal regularity and Uhlenbeck compactness for general connections.

I begin with a statement of our results:

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The RT-equations are a nonlinear elliptic system of equations with matrix valued differential forms as unknowns…

The RT-equations:

$$\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A), \\
\Delta J &= \delta(J\cdot\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \\
\delta\vec{A} &= v
\end{aligned}$$

$$d\vec{J} = 0 \quad \text{on } \partial\Omega.$$

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$$\Big(\, d\vec{J} = Curl(J) = \partial_j J^\mu_i - \partial_i J^\mu_j \,\Big)$$

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They are constructed from the Euclidean metric of the coordinate system in which components are given…

Hence they are elliptic independent of metric signature…

Existence Theorems for the RT-equations establish coordinate transformations

$$x \to y$$

sufficient to smooth an affine connection to optimal regularity…

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The existence of such coordinate transformations rules out "regularity singularities" in General Relativity.

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This is a fully multi-dimensional theory, requiring no symmetry assumptions…

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then there always exist local coord trans $x \to y$ such that in y-coordinates,

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Theorem (R-T 2021): If

$$\Gamma \in L^{2p} \text{ and } Riem(\Gamma) \in L^p, \;\; p > n/2$$

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$$\Gamma \in W^{1,p}, \;\; Riem(\Gamma) \in L^p$$

Extra derivative implies compactness
for sequences $\{\Gamma_i\}_{k=1}^{\infty}$ of affine connections:

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Theorem (R-T 2021): If

$\Gamma_i \in L^\infty$ and $Riem(\Gamma_i) \in L^p,\ p > n/2$

with uniform bounds, then there exists a convergent subsequence in y-coordinates:

$\Gamma_i \to \Gamma$ strongly in L^p, weakly in $W^{1,p}$

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"I.e., Γ one derivative above $Riem(\Gamma)$"

Vector Bundle version of the RT-equations

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$$\begin{aligned}\Delta \tilde{\mathcal{A}} &= \delta d\mathcal{A} - \delta\big(dU^{-1}\wedge dU\big)\\ \Delta U &= U\delta\mathcal{A} - (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle\end{aligned}$$

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$\mathcal{A} \equiv$ Non-optimal Connection

$U \equiv$ Gauge Transformation to optimal regularity…(we do case $SO(r,s)$)

Extends important results of Kazden-DeTurck and Uhlenbeck for (positive definite) Riemannian metrics and compact Lie groups…

…to arbitrary connections on vector bundles allowing for compact and non-compact Lie groups…

…including the Lorentzian metrics and affine connections of General Relativity…

By the RT-equations, optimal regularity and Uhlenbeck compactness follow from the transformation law for connections alone…

…the starting Hypothesis of Geometry

Hence our title…

Introduction

The Riemann Curvature Tensor

Thesis statement: The existence of non-optimal metrics is a direct consequence of Riemann's program to find a tensorial measure of curvature.

I.e., $Riem(\Gamma)$ involves second derivatives of the metric, but transforms by first derivative Jacobians

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Metrical properties of a space are given by a

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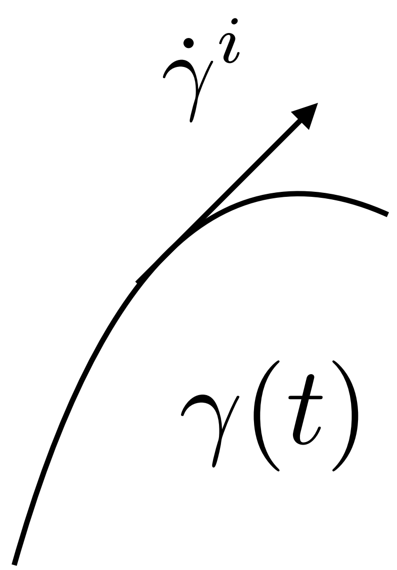

$$\|\dot{\gamma}\| = \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}$$

$$L = \int ds = \int_{t_0}^{t} \|\dot{\gamma}\| dt$$

g transforms like a bilinear form under $x \to y$

$$g_{\mu\nu} = g_{ij} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$$ (components)

$$g_y = J^t g_x J$$ (nxn matrices)

For (positive definite) Riemannian metrics, we recover flat Euclidean space locally…

$$g_{ij}(p) = \delta_{ij} + O(|p - p_0|^2)$$

I.e. $g_{ij} = \delta_{ij}$ Implies $ds^2 = dx_1^2 + \cdots + dx_n^2$

For metrics of signature $\delta(r, s)$, we locally recover flat (Minkowski) space…

$$g_{ij}(p) = \delta_{ij}(r, s) + O(|p - p_0|^2)$$

I.e. $g_{ij} = \delta_{ij}(r, s)$ Implies

$$ds^2 = -dx_1^2 - \cdots - dx_r^2 + dx_{r+1}^2 + \cdots dx_{r+s}^2$$

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— $Riem(\Gamma)$ measures second derivative Taylor errors but transforms by first derivative Jacobians…

Riemann Curvature Tensor:

—transforms by 1st derivative Jacobians

$$R^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial x^{k}}{\partial y^{\beta}}\frac{\partial x^{l}}{\partial y^{\gamma}}\frac{\partial x^{j}}{\partial y^{\delta}}$$ (tensor)

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—But measures 2nd derivatives in the Taylor series

$$g_{ij}(p) = \delta_{ij} + O(|p - p_0|^2)$$

Riemann Curvature Tensor:

—transforms by 1st derivative Jacobians

$$R^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{k}}{\partial y^{\beta}} \frac{\partial x^{l}}{\partial y^{\gamma}} \frac{\partial x^{j}}{\partial y^{\delta}} \qquad \text{(tensor)}$$

—Thm (Riemann): $R \equiv 0$ iff $O(|p - p_0|)^2 \equiv 0$

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Christoffel Symbols

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$$\Gamma^\alpha_{\beta\gamma} = \Gamma^i_{jk} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}$$

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(pointwise)

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Riemann Curvature: $$R^l_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{j\sigma}\Gamma^\sigma_{ik} - \Gamma^l_{k\sigma}\Gamma^\sigma_{ij}$$

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$Commutator$

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R is a "Curl" plus a "Commutator"

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Riemann Curvature: $$R = d\Gamma + [\Gamma, \Gamma]$$

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R does NOT bound ALL the derivatives of Γ !

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$d\Gamma$ is pointwise bounded by R

Co-derivatives $\delta\Gamma$ are uncontrolled (pointwise)

View Γ as a matrix valued 1-form:

$$\Gamma \equiv \Gamma_k dx^k \equiv \left(\Gamma^i_j\right)_k dx^k$$

Then: $R = d\Gamma + \Gamma \wedge \Gamma$

R is a "Curl" plus a "Commutator"

as nxn matrices expressed as wedge product

Optimal Regularity

Definition: We say a metric g has optimal regularity if it is two derivatives more regular than $Riem(\Gamma)$

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For metric connections: $\Gamma \sim \partial g$

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Theorem: The existence of Non-optimal metrics (and connections) is a direct consequence of the tensorial nature of Riemann's curvature…

I.e., $Riem(\Gamma)$ involves second derivatives of the metric, but transforms by first derivative Jacobians

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Transform to y-coordinates: $x \to y$

Assume: $$J \equiv \frac{\partial x}{\partial y} \in C^{k+1}$$

In y-coordinates:

$$\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$

C^{k+1} $\quad$ C^{k+2} $\quad$ C^{k+1} $\quad$ C^{k+1}

Metric in y-coordinates: $\quad \bar{g} \in C^{k+1}$

In y-coordinates:

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{i}_{jk} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial y^{\beta}} \frac{\partial x^{k}}{\partial y^{\gamma}} + \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial^2 x^{i}}{\partial y^{\beta} \partial y^{\gamma}}$$

C^k $\quad$ C^{k+1} $\quad$ C^{k+1} $\quad$ C^{k+1} $\quad$ C^k

Connection in y-coordinates: $\bar{\Gamma} \in C^k$

In y-coordinates:

$$\bar{R}^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{k}}{\partial y^{\beta}} \frac{\partial x^{l}}{\partial y^{\gamma}} \frac{\partial x^{j}}{\partial y^{\delta}}$$

C^k $\qquad$ C^k $\qquad$ C^{k+1}

Curvature in y-coordinates: $\bar{R} \in C^k$

Conclude: Under $x \rightarrow y$

$$R \in C^k, \quad \Gamma \in C^{k+1}, \quad g \in C^{k+2}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\bar{R} \in C^k, \quad \bar{\Gamma} \in C^k, \quad \bar{g} \in C^{k+1}$$

A coord trans $x \rightarrow y$ at regularity g lowers the regularity of g, Γ by one order, but preserves regularity of curvature R.

Same result in Sobolev spaces $W^{m,p}$:

$$R \in W^{m,p}, \quad \Gamma \in W^{m+1,p}, \quad g \in W^{m+2,p}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

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A coord trans $x \rightarrow y$ at regularity g lowers the regularity of g, Γ by one order, but preserves regularity of curvature R.

Our Question: Does the reverse hold?

I.e., given non-optimal g, Γ, can you always find a coordinate transformation $x \to y$ which smooths them to optimal regularity?

At the lowest regularity, points of non-optimality in metrics and connections look like singularities…

Our work began with GR shock wave solutions of the Einstein equations constructed by the Glimm Scheme:

For Shock-Waves in GR:

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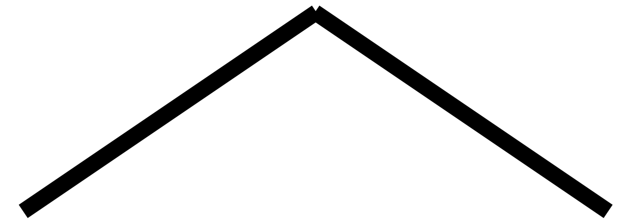
g is Lipschitz



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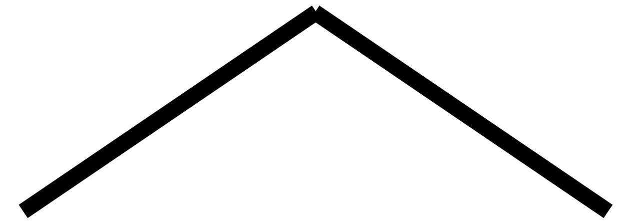

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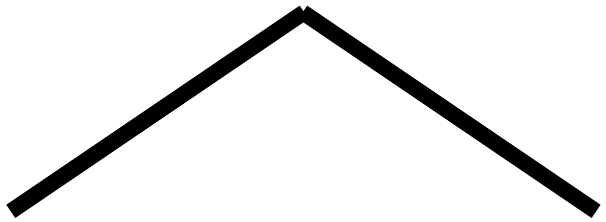

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$(d\Gamma \in L^\infty)$

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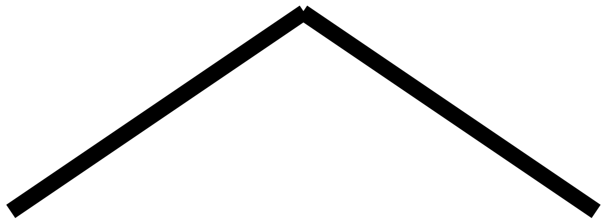

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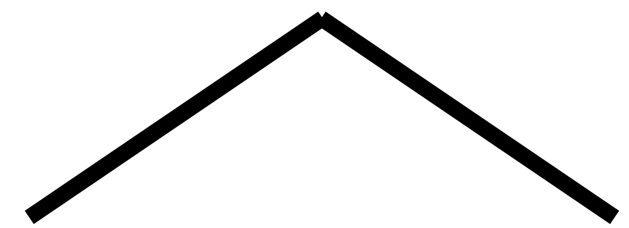

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"…but metric only one derivative above curvature"

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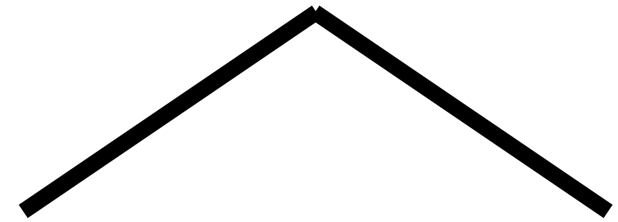

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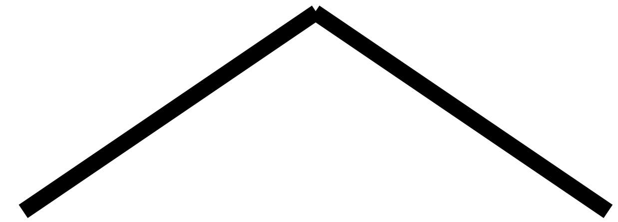

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At this regularity shock waves look like singularities:

$G = \kappa T$ only holds in the weak sense:

For Shock-Waves in GR:

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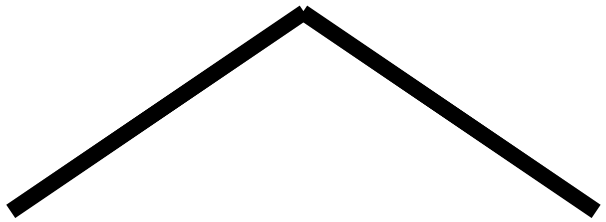

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At this regularity shock waves look like singularities:

Locally inertial coordinates don't exist

For Shock-Waves in GR:

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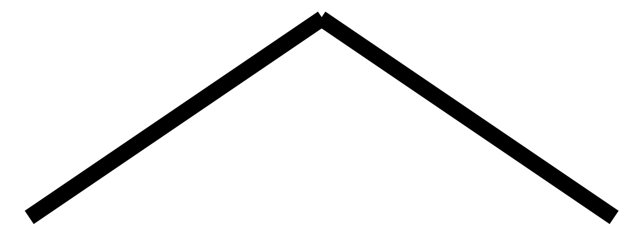

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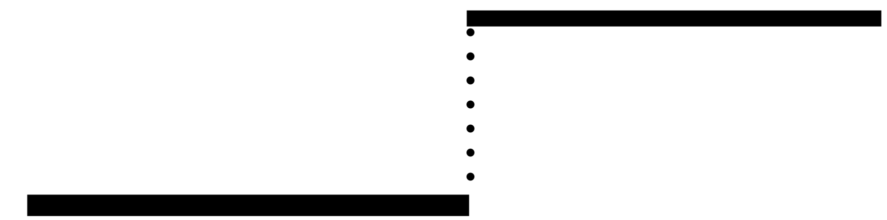

$R \approx \partial^2 g$ is L^∞

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At this regularity shock waves look like singularities:

Spacetime does not look "locally flat"

For Shock-Waves in GR:

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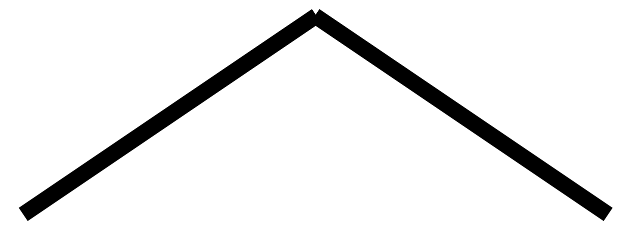

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At this regularity shock waves look like singularities:

⇨ The classical limit is suspect

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(1) There exist coordinate transformations which give the metric one more derivative…

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OR

(2) GR shock waves represent a new kind of

Regularity Singularity

The problem of constructing such a coordinate transformation $x \rightarrow y$ directly looks impossible…

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Continuous

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Continuous ⇨

"Discontinuities at shocks have to all miraculously cancel out in the Leibniz products!"

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Continuous

Nevertheless…Our theorem says such transformations always exists…

Theorem (R-T): If

$$\Gamma \in L^\infty \text{ and } Riem(\Gamma) \in L^\infty$$

in a given coordinate system x,

then there always exist local coord trans

$x \to y$ such that in y-coordinates,

$$\Gamma \in W^{1,p}, \quad Riem(\Gamma) \in L^\infty$$

We get this by solving the RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \quad (1)$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \quad (2)$$

$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \quad (3)$$

$$\delta\vec{A} = v, \quad (4)$$

$$Curl(J) \equiv \partial_j J^\mu_i - \partial_i J^\mu_j = 0 \ \text{ on } \partial\Omega, \quad (5)$$

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Key: " δ comes after d ":

$$Curl(J) \equiv \partial_j J_i^{\mu} - \partial_i J_j^{\mu} = 0 \;\text{ on } \partial\Omega, \tag{5}$$

Our compactness result can be viewed as a refinement of compensated compactness:

Theorem (R-T 2021): If

$\Gamma_i \in L^\infty$ and $Riem(\Gamma_i) \in L^p, \; p > n/2$

with uniform bounds, then there exists a convergent subsequence in y-coordinates:

$\Gamma_i \to \Gamma$ strongly in L^p, weakly in $W^{1,p}$

Our compactness result can be viewed as a refinement of Div-Curl lemma:

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…by generalized Div-Curl Lemma

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…by generalized Div-Curl Lemma

I.e.

$$R_i = d\Gamma_i + \Gamma_i \wedge \Gamma_i \to d\Gamma + \Gamma \wedge \Gamma$$

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I.e.

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I.e.

$$R_i = \underbrace{d\Gamma_i}_{\text{linear}} + \underbrace{\Gamma_i \wedge \Gamma_i} \to d\Gamma + \Gamma \wedge \Gamma$$

"Wedge products weakly continuous when derivative bounds are exterior derivative…"

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 303, Number 2, October 1987

ON WEAK CONTINUITY AND THE HODGE DECOMPOSITION

JOEL W. ROBBIN, ROBERT C. ROGERS[1] AND BLAKE TEMPLE[2]

ABSTRACT. We address the problem of determining the weakly continuous polynomials for sequences of functions that satisfy general linear first-order differential constraints. We prove that wedge products are weakly continuous when the differential constraints are given by exterior derivatives. This is sufficient for reproducing the Div-Curl Lemma of Murat and Tartar, the null Lagrangians in the calculus of variations and the weakly continuous polynomials for Maxwell's equations. This result was derived independently by Tartar who stated it in a recent survey article [**7**]. Our proof is explicit and uses the Hodge decomposition.

1. Introduction. The characterization of weakly continuous functionals has been an important tool in some recent developments in partial differential equations. In particular, the Div-Curl Lemma was instrumental in the work of Tartar [**6**] and DiPerna [**3**] on conservation laws, and the characterization of the null La-

Our motivation: Shock waves in GR constructed by the Glimm scheme

GR-Shock Waves

In Einstein's theory of General Relativity:

The Einstein equations $G = \kappa T$

are equns for the gravitational metric $g = g_{ij}$

coupled to the fluid sources ρ, p, u

$$G_{ij}[g_{ij}] = \kappa T_{ij}(\rho, p, u)$$

$$DivT = T^{\sigma}_{j;\sigma} = 0$$

In Einstein's theory of General Relativity:

The gravitational metric tensor g determines the properties of spacetime…

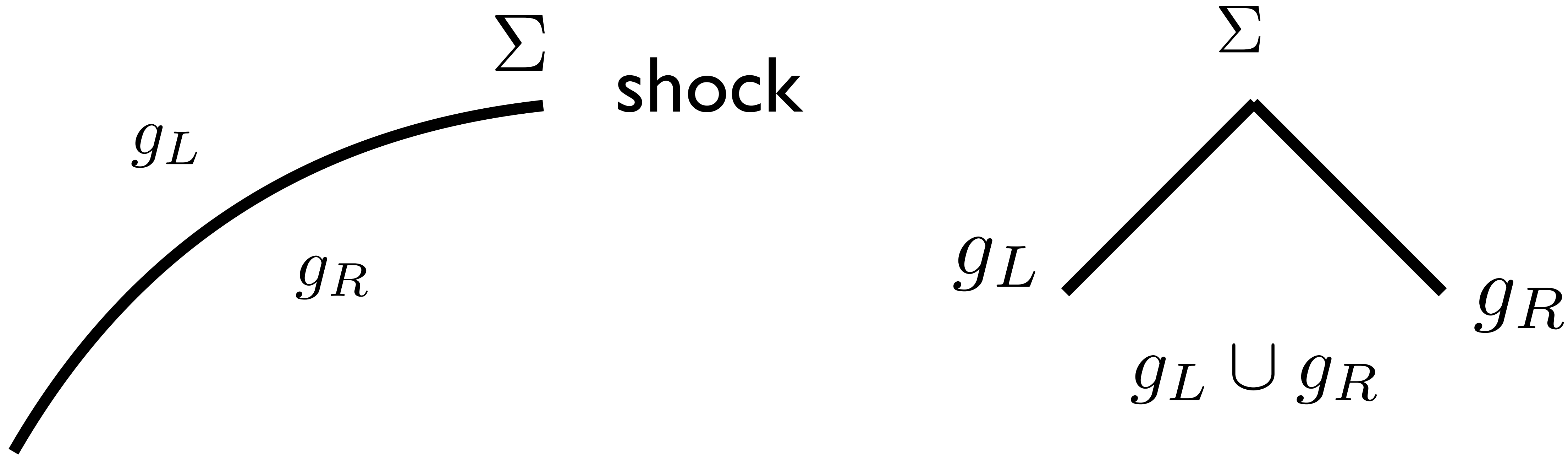
geodesics, parallel translation, time dilation, arc length…
…as well as the connection Γ and curvature R

Connection: $$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma} \left\{ -g_{jk,i} + g_{i,jk} + g_{ki,j} \right\}$$

Riemann Curvature: $$R^l_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{j\sigma}\Gamma^\sigma_{ik} - \Gamma^l_{k\sigma}\Gamma^\sigma_{ij}$$

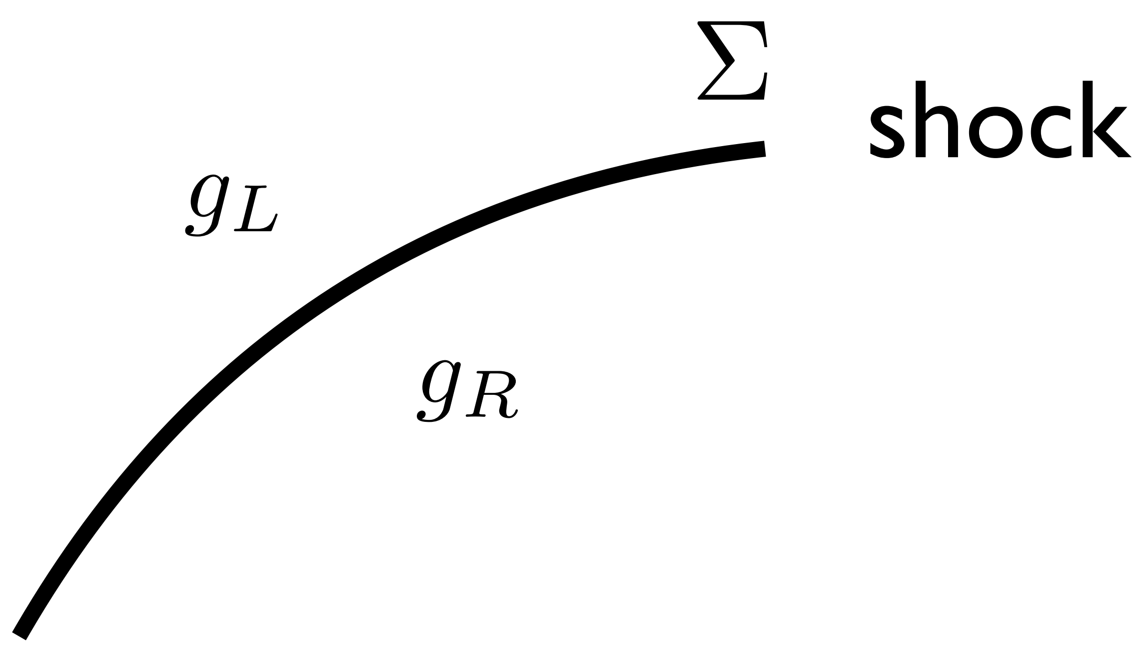
Israel (1960s) resolves issue for smooth shock surfaces

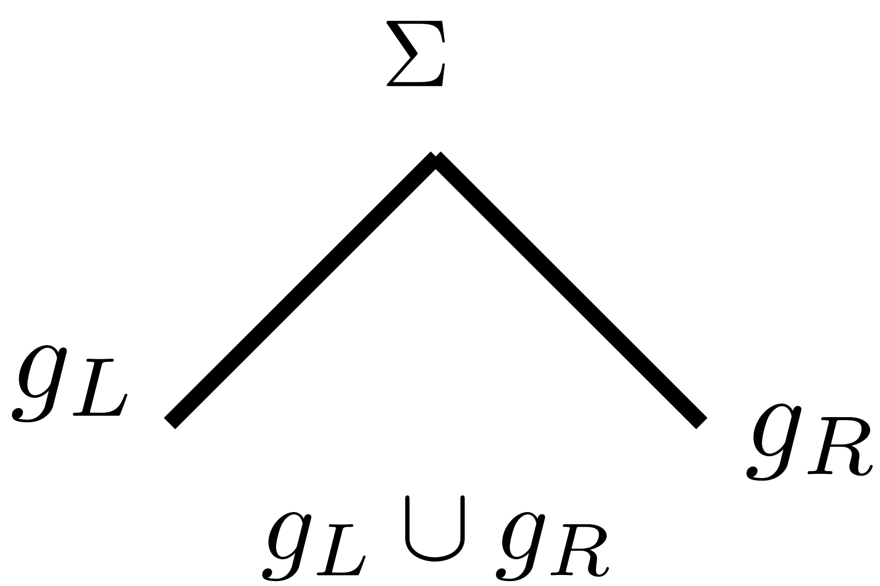
Assume g_L and g_R are smooth solutions of the Einstein equations which match Lipschitz continuously across a smooth shock surface Σ, and let... $g \equiv g_L \cup g_R$



Israel (1960s) resolves issue for smooth shock surfaces

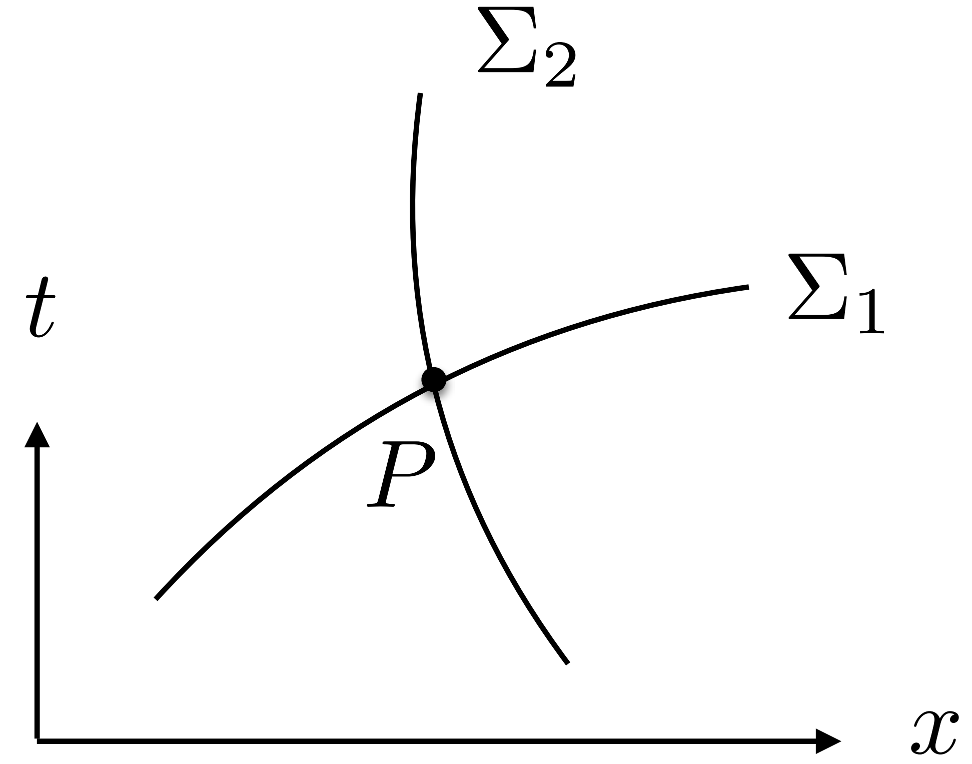
The map $x \to y$ to Gaussian normal coordinates regularizes the shock wave when the curvature and connection are in L^∞ ...





Theorem (Reintjes 2014): Extends Israel to regular shock-wave interaction with spherical symmetric

"Gaussian Normal Coordinates break down at P":



The coordinate transformations solve non-local PDE highly tuned to the structure of the interaction…

Trying to guess the coordinate system (eg harmonic or Gaussian normal) didn't work.

Rankine-Hugoniot jump conditions come in to make seemingly over-determined equations consistent…

… the general principle appeared entirely mysterious.

M. Reintjes, *Spacetime is Locally Inertial at Points of General Relativistic Shock Wave Interaction between Shocks from Different Characteristic Families*, Adv. Theor. Math. Phys., arXiv:1409.5060.

M. Reintjes and B. Temple, *No Regularity Singularities Exist at Points of General Relativistic Shock Wave Interaction between Shocks from Different Characteristic Families*,
Proc. R. Soc. A **471**:20140834.
http://dx.doi.org/10.1098/rspa.2014.0834

GR-Shock Waves by the Glimm Scheme

Assume a gravitational metric ansatz of the SSC form:

$$ds^2 = -B(t,r)dt^2 + \frac{dr^2}{A(t,r)} + r^2 dr^2$$

Plug into the Einstein equations :

$$G = \kappa T$$

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}$$

Standard Schwarzschild Coordinates

Four PDE's

$$\left\{-r\frac{A_r}{A}+\frac{1-A}{A}\right\} = \frac{\kappa B}{A}r^2T^{00} \qquad (1)$$

$$\frac{A_t}{A} = \frac{\kappa B}{A}rT^{01} \qquad (2)$$

$$\left\{r\frac{B_r}{B}-\frac{1-A}{A}\right\} = \frac{\kappa}{A^2}r^2T^{11} \qquad (3)$$

$$-\left\{\left(\frac{1}{A}\right)_{tt}-B_{rr}+\Phi\right\} = 2\frac{\kappa B}{A}r^2T^{22}, \qquad (4)$$

where

$$\Phi = \frac{B_tA_t}{2A^2B}-\frac{1}{2A}\left(\frac{A_t}{A}\right)^2-\frac{B_r}{r}-\frac{BA_r}{rA} + \frac{B}{2}\left(\frac{B_r}{B}\right)^2-\frac{B}{2}\frac{B_r}{B}\frac{A_r}{A}.$$

(1)+(2)+(3)+(4) ⟺ (1)+(3)+div T=0

(weakly)

Theorem: (Te-Gr) The equations close in a "locally inertial" formulation of (1), (2) & Div T=0:

$$\left\{T_M^{00}\right\}_{,0} + \left\{\sqrt{AB}T_M^{01}\right\}_{,1} = -\frac{2}{r}\sqrt{AB}T_M^{01}, \qquad (1)$$

$$\left\{T_M^{01}\right\}_{,0} + \left\{\sqrt{AB}T_M^{11}\right\}_{,1} = -\frac{1}{2}\sqrt{AB}\left\{\frac{4}{r}T_M^{11} + \frac{(1-A)}{Ar}(T_M^{00} - T_M^{11}) + \frac{2\kappa r}{A}(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4rT^{22}\right\}, \qquad (2)$$

$$rA_r = (1-A) - \kappa r^2 T_M^{00}, \qquad (3)$$

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$$T_M^{00} = \frac{\rho c^2 + p}{1 - \left(\frac{v}{c}\right)^2} \qquad T_M^{01} = \frac{\rho c^2 + p}{1 - \left(\frac{v}{c}\right)^2}\frac{v}{c}$$

$$T_M^{11} = \frac{p + \left(\frac{v}{c}\right)^2}{1 - \left(\frac{v}{c}\right)^2}\rho c^2 \qquad T^{22} = \frac{p}{r^2} \qquad v = \frac{1}{\sqrt{AB}}\frac{u^1}{u^0}$$

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Since $G = \kappa T$ → The metric is only one order derivative smoother than the curvature tensor…

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Conclude: These are non-optimal solutions which can be smoothed to optimal regularity by our THM...

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-Wave equation methods have not reproduced Glimm’s theorem, even for $1+1$ classical fluids

Comparison with other results:

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Prior results on optimal regularity GR were derived from 3+1 formulation

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Our theory is based on 4-d elliptic equations based on 4-d geometry of spacetime…

- Comparison with other results:

Positive Definite case: (Kazdan-DeTurck 1981)

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Some regularity theorems in riemannian geometry
Deturck, Dennis M. ; Kazdan, Jerry L.
Annales scientifiques de l'École Normale Supérieure, Série 4, Tome 14 (1981) no. 3, pp. 249-260.

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Elliptic Regularity lifts g 2-derivatives above R

- Lorentzian Case is Problematic…

Harmonic Coordinates: $\square_g x^i = 0$

$$\square_g g_{ij} = R_{ij} + l.o.t.'s$$

Regularity of g comes from the boundary

Results based on 3+1…

Anderson, Lefloch-Chen (vacuum)…

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Anderson, Lefloch-Chen (vacuum)…

(Not sufficient to regularize GR-shock waves)

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Results based on 3+1…

Anderson, Lefloch-Chen (vacuum)…

(Our theory removes all technical assumptions…)

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Results based on 3+1…

Anderson, Lefloch-Chen (vacuum)…

(…and applies to general connections.)

Overview of the proof of the Bounded L2 Curvature Conjecture

Sergiu Klainerman, Igor Rodnianski, Jeremie Szeftel

arXiv:1204.1772v2 [math.AP] 17 Jan 2013

Pages 1-149

THEOREM 1.6 (Main theorem). Let (M,g) an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that the Ricci curvature $Ric \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$, and Σ_0 has a strictly positive volume radius on scales ≤ 1, i.e. $r_{vol}(\Sigma_0, 1) > 0$. Then the following control holds on $0 \leq t \leq T$:

$$\|R\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C, \|\nabla k\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C$$

$$\inf_{0 \leq t \leq T} r_{vol}(\Sigma_t, 1) \geq 1.$$

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Applies to vacuum Einstein equations $G = 0$

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$$G = \kappa T$$

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The assumption on initial data:

$$Ric \in L^2(\Sigma_0), \nabla k \in L^2(\Sigma_0)$$

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Requires the second fundamental form be one degree smoother than curvature…

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Claim: Non-optimal solutions do not meet this condition in wave-gauge…

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holds ...

Eg: $\Gamma \in L^\infty$ **means** $k \in L^\infty$

on Cauchy hyper-surfaces…

so … $\nabla k \notin L^2$

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Conclude: The L2 theory does not apply when

$$\Gamma, Riem(\Gamma) \in L^\infty$$

Even for vacuum solutions…

Conclude: The Regularity Transformation equations apply at regularities too low to apply the L2-Theory…

$$\Gamma, Riem(\Gamma), k \in L^{\infty}$$

Summary (Our view): Regularizing coordinate transformations are highly tuned to a particular solution, (you have to solve the RT- equations), and there's no one ansatz that regularizes them "all at once".

C.f. Discussion on our webpage:

https://www.math.ucdavis.edu/~temple/WebPageRT-equations.pdf

Derivation of the RT-equations

"*The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity*"

Moritz Reintjes, Blake Temple
https://arxiv.org/abs/1805.01004

The RT-equations:

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A),\\
\Delta J &= \delta(J\cdot\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A,\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big),\\
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$$d\vec{J} = 0 \ \text{ on } \partial\Omega.$$

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\end{aligned}
$$

free to be chosen

Here: $\tilde{\Gamma}$ is a matrix valued 1-form, J and A are matrix valued 0-forms, and $\vec{J}$, $\vec{A}$ are vector valued 1-forms as follows:

$$\tilde{\Gamma} \equiv \tilde{\Gamma}^{\mu}_{\nu i} dx^i$$

$$J \equiv J^{\mu}_{\nu} \qquad \vec{J} \equiv J^{\mu}_{i} dx^i \qquad d\vec{J} = Curl(J)$$

$$A \equiv A^{\mu}_{\nu} \qquad \vec{A} \equiv A^{\mu}_{i} dx^i \qquad d\vec{A} = Curl(A)$$

The integrability condition for J is: $Curl(J) = 0$

Two operations on matrix valued forms:

$$\overrightarrow{\text{div}}(\omega)^{\alpha} \equiv \sum_{l=1}^{n} \partial_l \big((\omega_l^{\alpha})_{i_1,,,i_k} \big) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

("take divergence in lower matrix component")

$$\langle A\, ; B \rangle^{\mu}_{\nu} \equiv \sum_{i_1 < \ldots < i_k} A^{\mu}_{\sigma\; i_1 \ldots i_k} B^{\sigma}_{\nu\; i_1 \ldots i_k}$$

("matrix valued inner product")

To derive the RT-equations…

The first breakthrough was the Riemann-flat condition…

The Riemann-flat Condition

“Shock Wave Interactions in General Relativity: The Geometry behind Metric Smoothing and the Existence of Locally Inertial Frames”

Moritz Reintjes, Blake Temple

https://arxiv.org/abs/1610.02390

The Riemann-flat condition:

Assume $\Gamma, R \in L^\infty$.

Then: There exists a $C^{1,1}$ coordinate transformation which smooths Γ to $C^{0,1}$ if and only if there exists a tensor $\tilde{\Gamma} \in C^{0,1}$ st

$$Riem(\Gamma + \tilde{\Gamma}) = 0.$$

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$\Gamma \in L^\infty$

$\tilde{\Gamma} \in C^{0,1}$

The theorem applies at other orders of smoothness, for example: $\Gamma, \tilde{\Gamma} \in W^{m,p}$

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$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

$\Gamma \in W^{m,p}$ $\qquad$ $\Gamma \in W^{m+1,p}$

Geometric & independent of metric signature…

“Proof”: Assume $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

and

$$\Gamma^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{i}_{jk} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial y^{\beta}} \frac{\partial x^{k}}{\partial y^{\gamma}} + \frac{\partial^2 y^{\alpha}}{\partial x^{\sigma} \partial x^{\tau}} \frac{\partial x^{\sigma}}{\partial y^{\beta}} \frac{\partial x^{\tau}}{\partial y^{\gamma}}$$

L^{∞}

$C^{0,1}$ $\quad$ $C^{0,1}$

$-\tilde{\Gamma}^{\alpha}_{\beta\gamma}$

Riemann-flat

so

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

The "hard" part is: If

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

…then a smoothing transformation exists. ■

$\tilde{\Gamma}$ continuous implies $\Gamma + \tilde{\Gamma}$ has the same jump discontinuities (shock set) as Γ

First idea: Find Nash-type embedding theorem to extend the shock set to a flat connection

Better Idea: Use the Riemann-flat condition to derive a system of elliptic equations in $\tilde{\Gamma}, J$

"The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity"

Moritz Reintjes, Blake Temple
https://arxiv.org/abs/1805.01004

Derivation of the RT-equations from Riemann-Flat Condition

Start with the Riemann-flat condition:

Assume: $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

Then
$$\underbrace{\Gamma^k_{ij}}_{\Gamma} = \underbrace{(J^{-1})^k_\alpha J^\beta_i J^\gamma_j \Gamma^\alpha_{\beta\gamma}}_{\tilde{\Gamma}} + \underbrace{(J^{-1})^k_\alpha \partial_j J^\alpha_i}_{J^{-1}dJ}$$

Riemann-flat

$$\Gamma - \tilde{\Gamma} = J^{-1}dJ$$
$$Riem(\Gamma - \tilde{\Gamma}) = 0$$

2-Equivalent forms of Riemann-flat condition

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…an equation for $d\tilde{\Gamma}$

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Augment to first order system...

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

$$\delta\tilde{\Gamma} = h$$

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$\delta =$ co-derivative of Euclidean coord metric

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"gauge freedom"

Yields 1st order Cauchy-Riemann system for $\tilde{\Gamma}$

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J

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This is not a solvable system!

We look to use our equivalent Riemann-flat condition to couple this to an equation for J

$$J^{-1}dJ = \Gamma - \tilde{\Gamma} \quad \Longleftrightarrow \quad dJ = J(\Gamma - \tilde{\Gamma})$$

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We now construct a closed system in $(\tilde{\Gamma}, J)$

from these 2 forms of Riemann-flat condition

(They start out as equivalent!)

To break the equivalence, apply $\Delta = d\delta + \delta d$ to

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to get Poisson equations…

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A),$$

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where $h = J^{-1}A$ is free…

To impose $d\vec{J} \equiv Curl(J) = 0$ . . .

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Require that d of the vectorized right hand side of the J–equation vanish, i.e.

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$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big)$$

This leads to the RT-equations:

$$\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A), \\
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Another "miraculous cancellation" occurs in in A-equation

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$$

$$
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$$

Consider the A-equation:

$$
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d\vec{A} &= \underbrace{\overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big)}_{d\big(\overrightarrow{\delta(J\,\Gamma)}\big)} - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle}\big),
\end{aligned}
$$

Lemma: (for smooth Γ):

$$d\left(\overrightarrow{\delta(J\,\Gamma)}\right) = \underbrace{\overrightarrow{\mathrm{div}}\left(dJ \wedge \Gamma\right)}_{} + \underbrace{\overrightarrow{\mathrm{div}}\left(J\,d\Gamma\right)}_{}$$

$W^{m-2,p}$ $\quad$ $W^{m-1,p}$

$d\Gamma$ one derivative smoother than $\delta\Gamma$!

Consider the A-equation:

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$$\underbrace{\overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big)}_{d\big(\overrightarrow{\delta(J\,\Gamma)}\big)}$$

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Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations iff $(\tilde{\Gamma}', J)$ does...

$$\tilde{\Gamma}' = \Gamma - J^{-1} dJ$$

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$$\tilde{\Gamma}' \in W^{m+1,p} \qquad \Gamma \in W^{m,p} \qquad dJ \in W^{m,p}$$

$\tilde{\Gamma}'$ has the same regularity as $\tilde{\Gamma}$!

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$\tilde{\Gamma}'$ has the same regularity as $\tilde{\Gamma}$!

Conclude: J transforms $\Gamma \in W^{m,p}$ to $\tilde{\Gamma}' \in W^{m+1,p}$

Theorem: $(\tilde{\Gamma}, J, A)$ is a solution of the RT-equations

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), && (1)\\
\Delta J &= \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, && (2)\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), && (3)\\
\delta\vec{A} &= v, && (4)
\end{aligned}
$$

$$
Curl(J) \equiv \partial_j J^\mu_i - \partial_i J^\mu_j = 0 \ \text{ on } \partial\Omega,
$$

Theorem: $(\tilde{\Gamma}, J, A)$ is a solution of the RT-equations

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\text{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\text{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

$$Curl(J) \equiv \partial_j J_i^{\mu} - \partial_i J_j^{\mu} = 0 \ \text{ on } \partial\Omega,$$

if and only if $(\tilde{\Gamma}', J, A')$ is a solution.

Summary: We start with two first order equations both equivalent to the Riemann-flat condition…

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Using the co-derivative of the coordinate Euclidean metric, we convert these in to a coupled system of Poisson equations…

…and by miraculous cancellations on the RHS, the solutions provide Jacobians which lift the connection to optimal regularity…

Steps in the existence proof for the RT-equations

The existence proof is based on an iteration scheme which applies the L^p theory of elliptic regularity at each stage…

The L^p-theory of derivatives is a linear theory, and the RT-equations are nonlinear, so an iteration scheme is required…

The proof that the iterates converge relies on only two theorems from classical elliptic PDE theory…

Theorem (Elliptic Regularity): *Let* $f \in W^{m-1,p}(\Omega)$, $m \geq 1$, *and* $u_0 \in W^{m+\frac{p-1}{p},p}(\partial\Omega)$ *both be scalar functions. Assume* $u \in W^{m+1,p}(\Omega)$ *solves the Poisson equation* $\Delta u = f$ *with Dirichlet data* $u|_{\partial\Omega} = u_0$. *Then there exists a constant* $C > 0$ *depending only on* Ω, m, n, p *such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|f\|_{W^{m-1,p}(\Omega)} + \|u_0\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$

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Theorem (Gaffney Inequality): *Let* $u \in W^{m+1,p}(\Omega)$ *be a* k*-form for* $m \geq 0$*,* $1 \leq k \leq n-1$ *and (for simplicity)* $n \geq 2$*. Then there exists a constant* $C > 0$ *depending only on* Ω*,* m, n, p*, such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|du\|_{W^{m,p}(\Omega)} + \|\delta u\|_{W^{m,p}(\Omega)} + \|u\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$

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L. C. Evans, *Partial Differential Equations*, Berkeley Mathematics Lecture Notes, **3A**, 1994.

G. Csató, B. Dacorogna and O. Kneuss, *The Pullback Equation for Differential Forms*, Birkhäuser, Progress in Nonlinear Differential Equations and Their Applications, Vol. 83, ISBN: 978-0-8176-8312-2.

D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 3rd edition, (1997), Springer Verlag, ISBN 3-540-41160-7.

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P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, SIAM ed. (2011), ISBN 978-1-611972-02-3.

S. Agmon, A.Douglas, L.Nirenberg, *Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions*, Comm. Pure Appl. Math, Vol **12**, 623-727 (1959)

- One of the main obstacles to overcome was how to reduce the existence theorem at each iterate to a problem with Dirchlet boundary conditions... so standard linear elliptic regularity applies...

For this we introduce ancillary variable

y so that $dy = \vec{J}$, $d^2y = d\vec{J} = Curl(J)$

...this requires coupling the RT-equations to additional equations in (y, Ψ)

…This produces a system of following form…

$$\Delta\tilde{\Gamma} = \tilde{F}(\tilde{\Gamma}, J, A),$$

$$\Delta J = F(\tilde{\Gamma}, J) - A,$$

$$d\vec{A} = d\vec{F}(\tilde{\Gamma}, J)$$

$$\delta\vec{A} = v,$$

$$\begin{cases} d\Psi = \vec{F}(\tilde{\Gamma}, J) - \vec{A}, \\ \delta\Psi = 0, \end{cases}$$

$$\Delta y = \Psi.$$

…we don't have to impose $d\vec{J} = 0 \quad \text{on } \partial\Omega$...

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$$\begin{cases} d\Psi = \vec{F}(\tilde{\Gamma}, J) - \vec{A}, \\ \delta\Psi = 0, \end{cases}$$

… $dy = J$ so any boundary condition is OK

Theorem: (smooth solutions):

If: $\Gamma, Riem(\Gamma) \in W^{m,p}, m \geq 1$.

Then: $x \to y$ gives

$$\Gamma \in W^{m+1,p}, Riem(\Gamma) \in W^{m,p}$$

"I.e., Γ one derivative above $Riem(\Gamma)$"

Reference: (update)

"Optimal metric regularity in General Relativity follows from the RT-equations by elliptic regularity theory in Lp-spaces"

Moritz Reintjes, Blake Temple
https://arxiv.org/abs/1808.06455

The low regularity case $\Gamma \in L^\infty, L^p$

is more problematic…

Nonlinear term not closed under iteration in L^∞ or L^p

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \boxed{\delta\big(dJ^{-1}\wedge dJ\big)} + d(J^{-1}A),\\
\Delta J &= \delta(J\cdot\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A,\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big),\\
\delta\vec{A} &= v
\end{aligned}
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d\vec{J} = 0 \quad \text{on } \partial\Omega.
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For this we use the gauge freedom in the RT-equations to uncouple the $\tilde{\Gamma}$ equation from the others:

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B &\equiv A + \langle dJ; \tilde{\Gamma} \rangle, \\
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Reduced RT-equations: (linear!)

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\Delta J &= \delta(J \cdot \Gamma) - B, \\
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Solve for J first…then solve nonlinear Poisson equation to get the regularity of $\tilde{\Gamma}'$

Theorem (R-T): If

$$\Gamma \in L^{\infty} \text{ and } \quad Riem(\Gamma) \in L^{\infty}$$

in a given coordinate system x ,

then there always exist local coord trans

$x \to y$ such that in y-coordinates,

$$\Gamma \in W^{1,p}, \quad Riem(\Gamma) \in L^{\infty}$$

Theorem (R-T 2021): If

$$\Gamma \in L^{2p} \text{ and } Riem(\Gamma) \in L^p, \; p > n/2$$

in a given coordinate system x ,

then there always exist local coord trans

$x \to y$ such that in y-coordinates,

$$\Gamma \in W^{1,p} \; , \; Riem(\Gamma) \in L^p$$

Conclusion

This theory is geometric, applies independent of matter sources or metric signature, requires no symmetry assumptions, and makes no apriori assumptions on the spacetime other than its regularity…

Optimal regularity is a consequence of the transformation law for connections alone…

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Non-optimal Γ exist because because $Riem(\Gamma)$ is second order, but transforms as a tensor…

$Riem(\Gamma)$ involves only $d\Gamma$, but controls ALL the derivatives of Γ.

Q: Might this refined compactness theorem find application for existence theories in GR?

Cor: GR-shock waves by the Glimm scheme in SSC are one derivative more regular than previously known…

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Q: What do the metrics look like in coordinates of optimal regularity?

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Q: What do the metrics look like in coordinates of optimal regularity?

Q: Could non-optimal coordinates play a role for existence theories for GR-shocks in multi-d?

Cor: It suffices to solve the Einstein equations in coordinates where the metric is non-optimal and be assured that the solutions exhibit optimal regularity in other coordinate systems…

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Cor: If a connection loses one derivative relative to the curvature in a numerical simulation, it is not a geometric problem…

The RT-equations are based on Einsteins 4-D theory of spacetime, not classical 3+1…

Q: Can this be done within the 3+1 framework of the initial value problem?

Ans: We don't know how to do this!

Open Problem...

Given $\Gamma, d\Gamma \in L^{\infty}$, does there exist coordinate transformation which achieves $\Gamma_y \in C^{0,1}$?

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Other applications in geometry??...

References (Reintjes Temple):

arXiv:1610.0239

arXiv:1808.06455

arXiv:1805.01004

Thank You!

END