Nonlinear Resonance 1980-1995 with Eli Isaacson (Invited by D. Marchesin)

Blake Temple, UC-Davis

National Meeting of the
Brazilian Mathematical Society
July 25-29, 2015

This Talk is to Honor

Eli Isaacson

and

Our 15-Year Collaboration

(…and many good times in Rio!)

Alternative Title: How to make a talk on old work interesting…

Tell a Story

My story begins in 1980, the first year the NSF offered their NSF Postdoctoral Research Fellowships

In 1980 I finished my Ph.D. thesis on the Glimm Scheme under Joel Smoller University of Michigan

At that time, hardly anyone understood the technicalities of the Glimm Scheme…

Peter Lax, Joel Smoller, Ron Diperna, Tai-ping Liu …a few others and…

Glimm Himself

At Joel Smoller's urging I applied for an NSF Postdoc to work with Glimm at Rockefeller University

Years later Igor Stackgold told me the committee was dominated by TOPOLOGISTS…

…and it was a fight.

—Ivar Stackgold

But I won anyway…
and went to Rockefeller University, NYC, 1980
on an
NSF Postdoc
which payed $13,000/yr

I did not care about money, tenure, or security then…
I just wanted to see if I could make an important contribution to Mathematical Physics…

The first day at Rockefeller I met Eli Isaacson, my office-mate

And many other postdocs working in Glimm's group at Rockefeller University…

(No photos allowed because they're not dead yet!!!)

The first day Eli told me he (working with others) had solved the Riemann problem for the Polymer Equations…

References

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7. T. P. Liu and C. H. Wang, On a hyperbolic system of conservation laws which is not strictly hyperbolic, MRC Technical Summary Report 2184 March, 1981.
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He told me they had a ``global solution", and it looked very different from gas dynamics, exhibiting…

- Non-uniqueness for RP
- Non-Lax shocks
- A nonlinear field with coinciding shock and rarefaction curves
- A ``transition curve'' of coinciding wave speeds

The only other global Riemann problem I knew of then was the Nishida System, the topic of my Thesis…

I immediately set out to prove an existence theorem by Glimm's method, and succeeded in the fall of 1980…

ADVANCES IN APPLIED MATHEMATICS 3, 335–375 (1982)

Global Solution of the Cauchy Problem for a Class of 2×2 Nonstrictly Hyperbolic Conservation Laws*

BLAKE TEMPLE

Department of Mathematical Physics,
The Rockefeller University,
New York, New York 10021

We prove the existence of a global weak solution to the Cauchy problem for a class of 2×2 equations which model one-dimensional multiphase flow, and which represent a natural generalization of the scalar Buckley–Leverett equation. Loss of strict hyperbolicity (coinciding wave speeds with a $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ normal form) occurs on a curve in state space, and waves in a neighborhood of this curve contribute unbounded variation to the approximate Glimm scheme solutions. The unbounded variation is handled by means of a singular transformation; in the transformed variables, the variation is bounded. Glimm's argument must be modified to handle the unbounded variation that appears in the statement of the weak conditions, and this requires that the random choice variable be random in space as well as time.

- After that, Eli and I extended the ideas in RP and Glimm analysis in numerous papers…

JOURNAL OF DIFFERENTIAL EQUATIONS **65**, 250–268 (1986)

Analysis of a Singular Hyperbolic System of Conservation Laws

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AND

J. BLAKE TEMPLE *,†

Mathematics Research Center, Madison, Wisconsin

Received March 8, 1985

SIAM J. APPL. MATH.
Vol. 48, No. 5, October 1988

003

THE RIEMANN PROBLEM NEAR A HYPERBOLIC SINGULARITY: THE CLASSIFICATION OF SOLUTIONS OF QUADRATIC RIEMANN PROBLEMS I*

E. ISAACSON†, D. MARCHESIN‡, B. PLOHR§, AND B. TEMPLE¶

Abstract. The purpose of this paper is to classify the solutions of Riemann problems near a hyperbolic singularity in a nonlinear system of conservation laws. Hyperbolic singularities play the role in the theory of Riemann problems that rest points play in the theory of ordinary differential equations: Indeed, generically, only a finite number of structures can appear in a neighborhood of such a singularity. In this, the first of three papers, the program of classification is discussed in general and the simplest structure that occurs is characterized.

Key words. nonlinear hyperbolic conservation laws, Riemann problems, hyperbolic singularities

AMS(MOS) subject classifications. 35L65, 35L67, 35L80

SIAM J. APPL. MATH.
Vol. 48, No. 6, December 1988

006

THE RIEMANN PROBLEM NEAR A HYPERBOLIC SINGULARITY II*

E. ISAACSON† AND B. TEMPLE‡

Abstract. This paper is interested in classifying the solutions of Riemann problems for the 2×2 conservation laws that have homogeneous quadratic flux functions. Such flux functions approximate an arbitrary 2×2 system in a neighborhood of an isolated point where strict hyperbolicity fails. Here the solution for the symmetric systems in Region III of the four region classification of Schaeffer and Shearer is given. The solution is based on the qualitative shape of the integral curves described by Schaeffer and Shearer and a numerical calculation of the Hugoniot loci and their shock types.

Key words. Riemann problem, nonstrictly hyperbolic conservation laws, umbilic points

AMS(MOS) subject classifications. 65M10, 76N99, 35L65, 35L67

SIAM J. APPL. MATH.
Vol. 48, No. 6, December 1988

007

THE RIEMANN PROBLEM NEAR A HYPERBOLIC SINGULARITY III*

E. ISAACSON† AND B. TEMPLE‡

Abstract. This paper is interested in classifying the solutions of Riemann problems for the 2×2 conservation laws that have homogeneous quadratic flux functions. Such flux functions approximate an arbitrary 2×2 system in a neighborhood of an isolated point where strict hyperbolicity fails. Here the solution for the symmetric systems in Region II of the four region classification of Schaeffer and Shearer is given. The solution is based on the qualitative shape of the integral curves described by Schaeffer and Shearer and a numerical calculation of the Hugoniot loci and their shock types.

Key words. Riemann problem, nonstrictly hyperbolic conservation laws, umbilic points

AMS(MOS) subject classifications. 65M10, 76N99, 35L65, 35L67

ADVANCES IN APPLIED MATHEMATICS 11, 205–219 (1990)

The Structure of Asymptotic States in a Singular System of Conservation Laws

ELI ISAACSON*

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AND

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Department of Mathematics, University of California, Davis, Davis, California 95616

Contemporary Mathematics
Volume 108, 1990

NONLINEAR RESONANCE IN INHOMOGENEOUS SYSTEMS OF CONSERVATION LAWS[1,2]

Eli Isaacson[3] and Blake Temple[4]

ABSTRACT: We solve the Riemann problem for a general inhomogeneous system of conservation laws in a region where one of the nonlinear waves in the problem takes on a zero speed. We state generic conditions on the fluxes that guarantee the solvability of the Riemann problem, and these conditions determine a unique underlying structure to the solutions. The inhomogeneity is modeled by a linearly degenerate field. Our analysis thus provides a general framework for studying (what we are calling) resonance between a linear and a nonlinear family of waves in a system of conservation laws. Special cases of this phenomenon are observed in model problems for gas dynamical flow in a variable area duct and in Buckley-Leverett type systems that model multiphase flow in a porous medium.

Mat. Aplic. Comp., V. 11, no. 2, pp.147-166, 1992

MULTIPHASE FLOW MODELS WITH SINGULAR RIEMANN PROBLEMS*

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DAN MARCHESIN[2]
BRADLEY J. PLOHR[3]
J. BLAKE TEMPLE[4]

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SIAM J. APPL. MATH.
Vol. 52, No. 5, pp. 1260–1278, October 1992

003

NONLINEAR RESONANCE IN SYSTEMS OF CONSERVATION LAWS*

ELI ISAACSON† AND BLAKE TEMPLE‡

Abstract. The Riemann problem for a general inhomogeneous system of conservation laws is solved in a neighborhood of a state at which one of the nonlinear waves in the problem takes on a zero speed. The inhomogeneity is modeled by a linearly degenerate field. The solution of the Riemann problem determines the nature of wave interactions, and thus the Riemann problem serves as a canonical form for nonlinear systems of conservation laws. Generic conditions on the fluxes are stated and it is proved that under these conditions, the solution of the Riemann problem exists, is unique, and has a fixed structure; this demonstrates that, in the above sense, resonant inhomogeneous systems generically have the same canonical form. The wave curves for these systems are only Lipschitz continuous in a neighborhood of the states where the wave speeds coincide, and so, in contrast to strictly hyperbolic systems, the implicit function theorem cannot be applied directly to obtain existence and uniqueness. Here we show that existence and uniqueness for the Riemann problem is a consequence of the uniqueness of intersection points of Lipschitz continuous manifolds of complementary dimensions. These systems are resonant for two reasons: The linearized problem exhibits classical resonant behavior, while the nonlinear initial value problem exhibits a "nonlinear resonance" in the sense that wave speeds from different families of waves are not distinct; so the number of times a pair of waves can interact in a given solution cannot be bounded a priori. Since waves are reflected in other families every time a pair of waves interact, a proliferation of reflected waves can occur by the interaction of a single pair of waves. Examples of resonant inhomogeneous systems are observed in model problems for the flow of a gas in a variable area duct and in Buckley–Leverett systems that model multiphase flow in a porous medium.

Key words. Riemann problem, nonstrictly hyperbolic, resonance

AMS(MOS) subject classifications. 35L65, 35L67, 65M10, 76N99

This culminated in our last publication SIAM J. Appl. Math, 1995…

SIAM J. APPL. MATH.
Vol. 55, No. 3, pp. 625–640, June 1995

003

CONVERGENCE OF THE 2×2 GODUNOV METHOD FOR A GENERAL RESONANT NONLINEAR BALANCE LAW*

ELI ISAACSON† AND BLAKE TEMPLE‡

Abstract. We introduce a generalized solution of the Riemann problem for a general resonant nonlinear balance law, and we prove the convergence of the 2×2 Godunov numerical method based on these solutions. In particular, we obtain generic conditions that guarantee a canonical structure for the elementary waves in the solution of the Riemann problem, and an interesting multiplicity of time-asymptotic wave patterns is observed and characterized.

I now discuss our first and last papers in detail...

Our FIRST papers

E. ISAACSON, *Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery, Rockefeller University preprints (1980).*

. . . .

ADVANCES IN APPLIED MATHEMATICS **3**, 335–375 (1982)

Global Solution of the Cauchy Problem for a Class of 2 × 2 Nonstrictly Hyperbolic Conservation Laws*

BLAKE TEMPLE

*Department of Mathematical Physics,
The Rockefeller University,
New York, New York 10021*

Our LAST paper

SIAM J. APPL. MATH.
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ADVANCES IN APPLIED MATHEMATICS 3, 335–375 (1982)

Global Solution of the Cauchy Problem for a Class of 2 × 2 Nonstrictly Hyperbolic Conservation Laws*

BLAKE TEMPLE

Department of Mathematical Physics,
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I. The Polymer Equations

$$s_t + [sG(s,b)]_x = 0$$

$$b_t + [bG(s,b)]_x = 0$$

I. The Polymer Equations

$$s_t + [sG(s,b)]_x = 0$$

$$b_t + [bG(s,b)]_x = 0$$

$s = \text{saturation of } H_2O$

$c = \text{concentration of POLYMER in } H_2O$

$b = sc = \text{total concentration of POLYMER}$

$$0 \leq s \leq b \leq 1$$

I. The Polymer Equations

$$s_t + [sG(s,b)]_x = 0$$

$$b_t + [bG(s,b)]_x = 0$$

Basic Two Phase Flow Model
in secondary oil recovery,
modeling the flow of
Oil together with Polymer+Water
in varying concentrations

I. The Polymer Equations

$$s_t + [sG(s,b)]_x = 0$$

$$b_t + [bG(s,b)]_x = 0$$

- A Natural Generalization of the scalar Buckley-Leverett Equation…

I. The Polymer Equations

$$s_t + [sG(s,b)]_x = 0$$

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- A Natural Generalization of the scalar Buckley-Leverett Equation…
- Derivation based on Darcy's Law…

I. The Polymer Equations

$$s_t + [sG(s,b)]_x = 0$$

$$b_t + [bG(s,b)]_x = 0$$

- A Natural Generalization of the scalar Buckley-Leverett Equation…
- Derivation based on Darcy's Law…
- Reduces to the Buckley-Leverett when

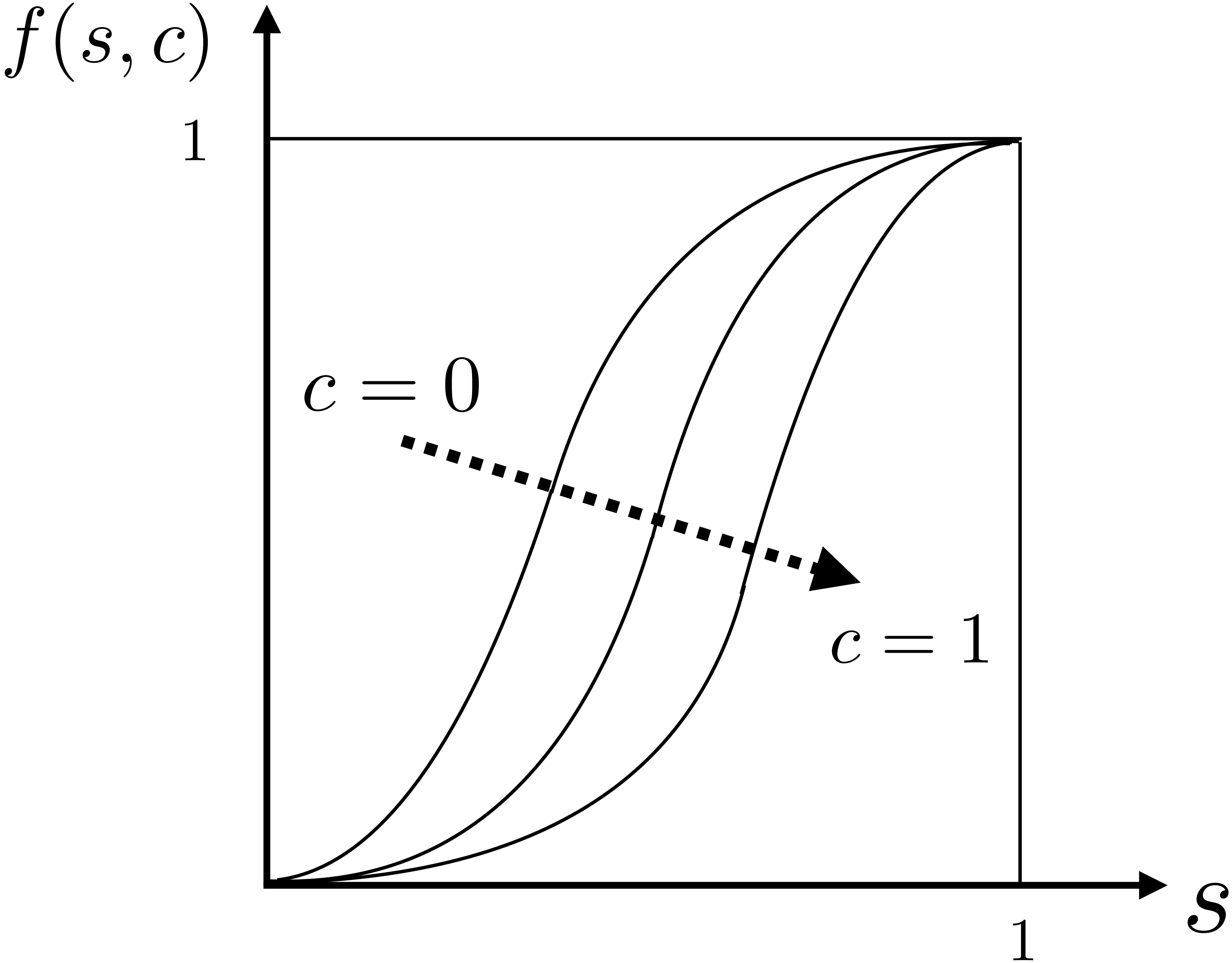
$$c = const.$$

The Natural Variables are (s, c):

$$s_t + f(s,c)_x = 0$$

$$(sc)_t + [f(s,c)c]_x = 0$$

Varying c changes the Buckley-Leverett S-shaped curve:



Buckley-Leverett at $c = const.$

The Eigen-Families:

$$s_t + [sG(s,b)]_x = 0$$

$$b_t + [bG(s,b)]_x = 0$$

$$\begin{pmatrix} s \\ b \end{pmatrix}_t + \begin{pmatrix} G + sG_s & sG_b \\ bG_s & G + bG_b \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix}_x = 0$$

$$U_t + dF(U)\,U_x = 0$$

The Eigen-Families:

$$dF = \begin{pmatrix} G + sG_s & sG_b \\ bG_s & G + bG_b \end{pmatrix}$$

$$\lambda_c = \frac{f}{s}$$

RI: $\dfrac{f}{s} = const.$

$$\lambda_s = f_s$$

RI: $c = const.$

The **s**-wave family (nonlinear):

$$\lambda_s = f_s$$

$$\text{RI: } c = const.$$

s-waves solve the Buckley-Leverett scalar equation at $c = const.$

The **c**-wave family (linear):

$$\boxed{\begin{aligned} \lambda_c &= \frac{f}{s} \\ \text{RI: } \frac{f}{s} &= const. \end{aligned}}$$

c-waves are contact-discontinuities at

$$g = \frac{f}{s} = const.$$

RI: Riemann Invariants

- Riemann Invariants are constant along Integral Curves of the Eigenvectors of DF, defining coordinate system of wave curves
- For Polymer Equations: solutions restricted to wave curves reduce to solutions of scalar cons. laws ``in the weak sense”
- This happens in Two Different Ways!

(Surprising to me as equations are Nonlinear!!)

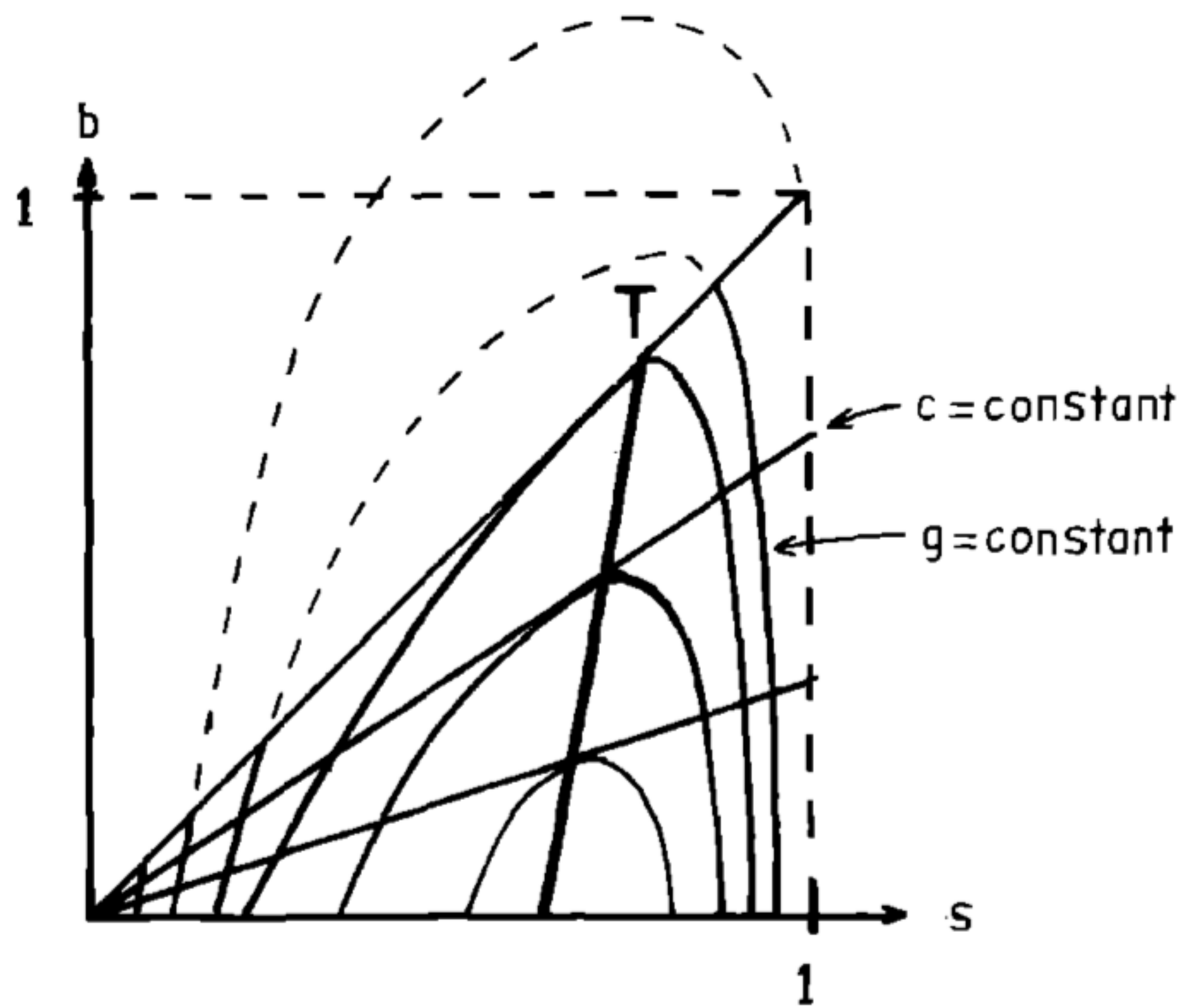
Scalar Equations 2 Different Ways

(1) When $g = \frac{f}{s} = const.$ nonlinear equations reduce to a scalar linear equation

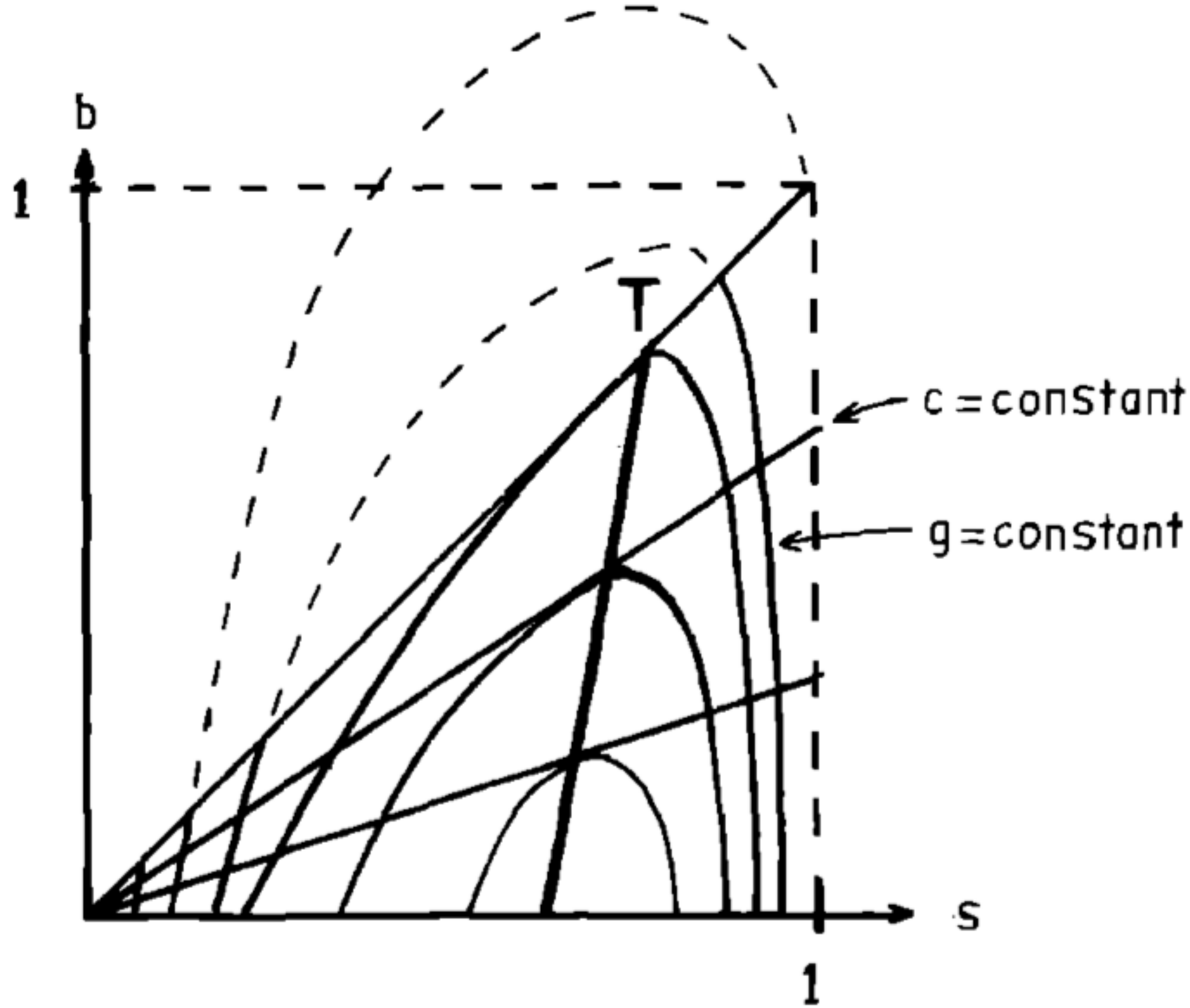
(Like the entropy waves in gas dynamics!)

(2) When $c = const.$ solutions reduce to a scalar non-linear equation.

The coordinate system of RI's

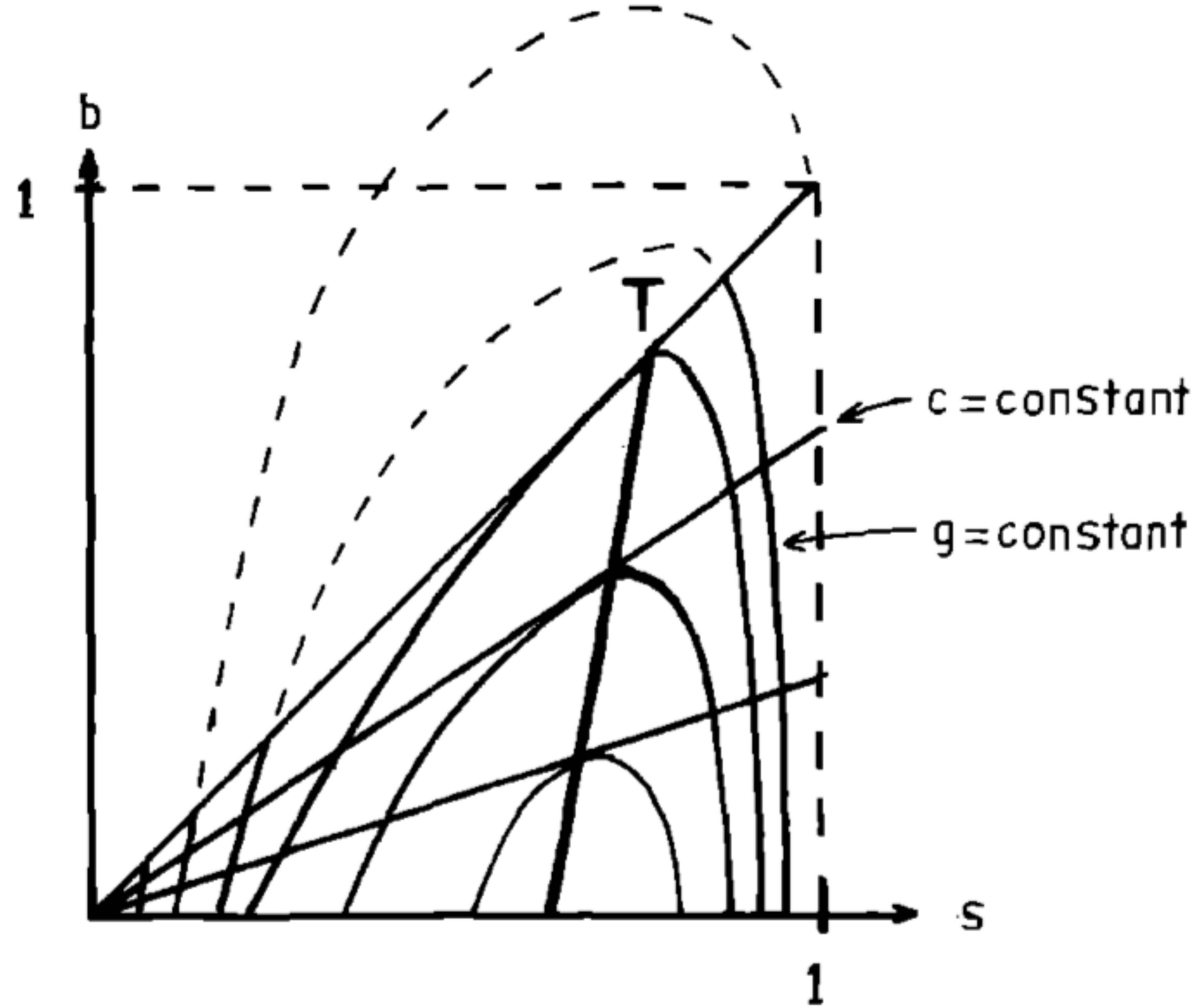


Conserved quantities: $U = (s, b)$

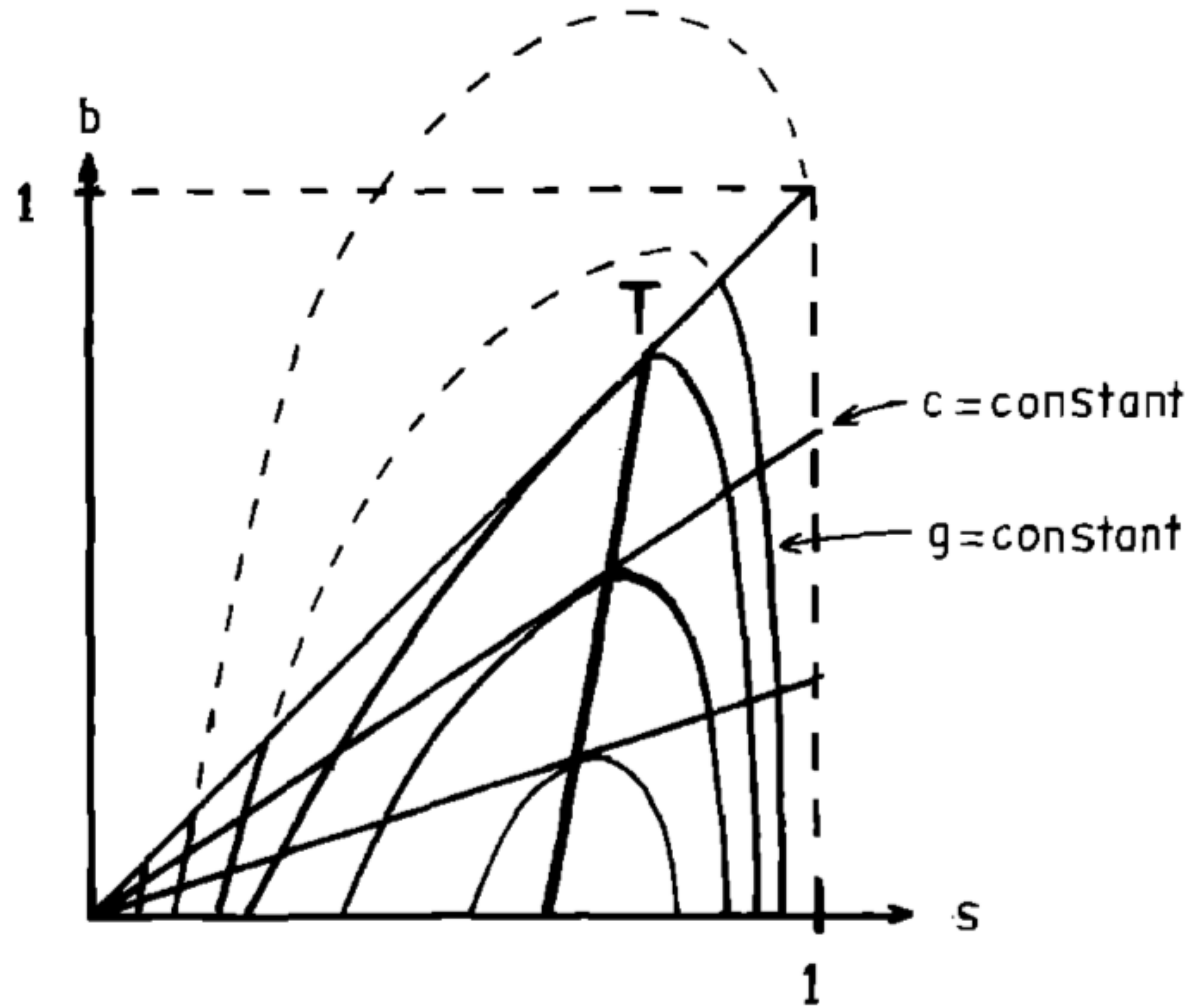


Wave curves are tangent when:

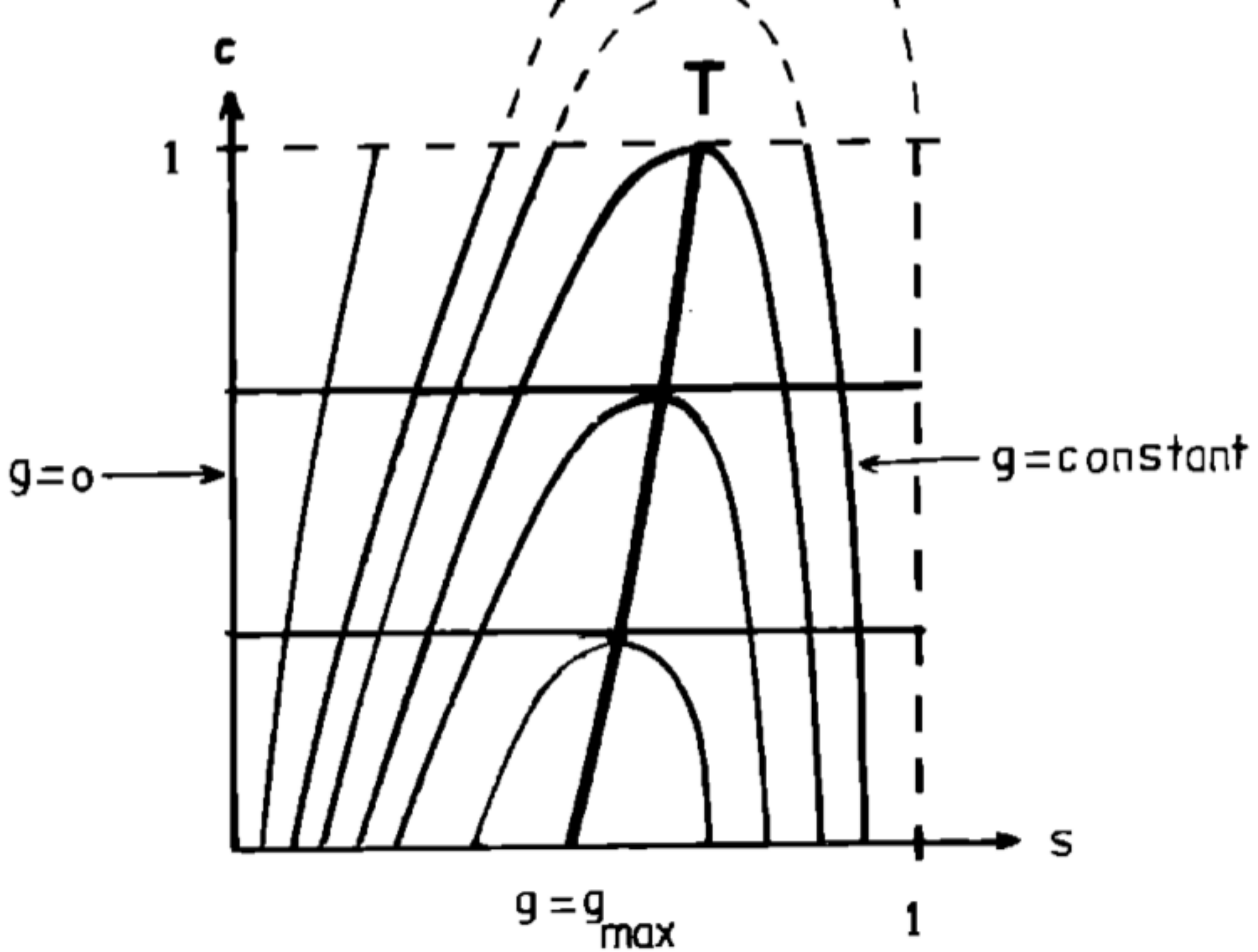
$$\lambda_s = f_s(s,c) = \frac{f(s,c)}{s} = g(s,c) = \lambda_c$$



Conclude: The coordinate system of wave curves is singular and wave speeds coincide on a co-dimension one curve T



T = Transition Curve



I.e., on $g(s,c) = const.,$ we have: $\dfrac{dc}{ds} = -\dfrac{g_s}{g_c}$

So on T: $\boxed{0 = \; g_s = \dfrac{f_s}{s} - \dfrac{f}{s^2} = \dfrac{1}{s}\left(\lambda_s - \lambda_c\right)}$

Turns Out:

The only way a system of CL's can reduce to a linear scalar equation is when the wave curves are level curves of an eigenvalue…

AND…

The only way a system of CL's can reduce to a nonlinear scalar equation is when the shock and rarefaction curves coincide…

…and this can only happen when the wave curves are straight lines in the plane of conserved quantities….

Contemporary Mathematics
Volume 17, 1983

SYSTEMS OF CONSERVATION LAWS WITH COINCIDING SHOCK AND RAREFACTION CURVES*

BLAKE TEMPLE

1. INTRODUCTION

Systems of conservation laws which have coinciding shock and rarefaction curves arise in the study of oil reservoir simulation, nonlinear wave motion in elastic strings, as well as in multicomponent chromatography [1, 4, 5, 6, 9, 11, 12]. These systems have many interesting features. The Riemann problem for these equations can be explicitly solved in the large, and wave interactions have a simplified structure, even in the presence of a nonconvex flux function. For this reason, these systems represent some of the few examples for which the Cauchy problem has been solved for arbitrary data of bounded variation. Also, hyperbolic degeneracies appear in each of these systems. In the present paper we are concerned with locating the class of equations that exhibit the phenomenon of coinciding shock and rarefaction curves. For $n \times n$ systems, we give necessary and sufficient conditions for a shock curve to coincide with a rarefaction curve. We use these general results to write down explicitly the class of 2×2 conservation laws which have shock and rarefaction curves that coincide.

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 280, Number 2, December 1983

SYSTEMS OF CONSERVATION LAWS WITH INVARIANT SUBMANIFOLDS

BY

BLAKE TEMPLE[1]

ABSTRACT. Systems of conservation laws with coinciding shock and rarefaction curves arise in the study of oil reservoir simulation, multicomponent chromatography, as well as in the study of nonlinear motion in elastic strings. Here we characterize this phenomenon by deriving necessary and sufficient conditions on the geometry of a wave curve in order that the shock wave curve coincide with its associated rarefaction wave curve for a system of conservation laws. This coincidence is the one dimensional case of a submanifold of the state variables being invariant for the system of equations, and the necessary and sufficient conditions are derived for invariant submanifolds of arbitrary dimension. In the case of 2×2 systems we derive explicit formulas for the class of flux functions that give rise to the coupled nonlinear conservation laws for which the shock and rarefaction wave curves coincide.

Line and Contact Families

Let (λ, R) be an eigenfamily: $DF \cdot R = \lambda R$

(1) Contact Family: The integral curves $\mathcal{R}(\xi)$ are level curves $\lambda = const.$

$$\boxed{u(\xi(x,t)) : \quad u_t + \lambda\, u_x = 0}$$ (linear)

(2) Line Family: The integral curves $\mathcal{R}(\lambda)$ are straight lines in $u-$space

$$\boxed{u(\lambda(x,t)) : \quad \lambda_t + \lambda\, \lambda_x = 0}$$ (nonlinear)

Turns out there are lots of systems of conservation laws with coinciding shock and rarefaction curves:

Chromatography: Aris and Amundson

$$u_t + \left\{ \frac{u}{1+u+v} \right\}_x = 0,$$

$$v_t + \left\{ \frac{\kappa v}{1+u+v} \right\}_x = 0.$$

Two Line fields

CL's with coinciding shock and rarefaction curves are highly nonlinear, but wave interactions are simpler, so the analysis of solutions by Glimm's method is easier…

…a good deal of interest in these systems followed…

Dennis Serre was very generous in citing this work by naming them…

The Polymer Equations are Canonical

in a mathematical sense because they represent the simplest system with coinciding shock & rarefaction curves with

BOTH:

Line Family and Contact Family

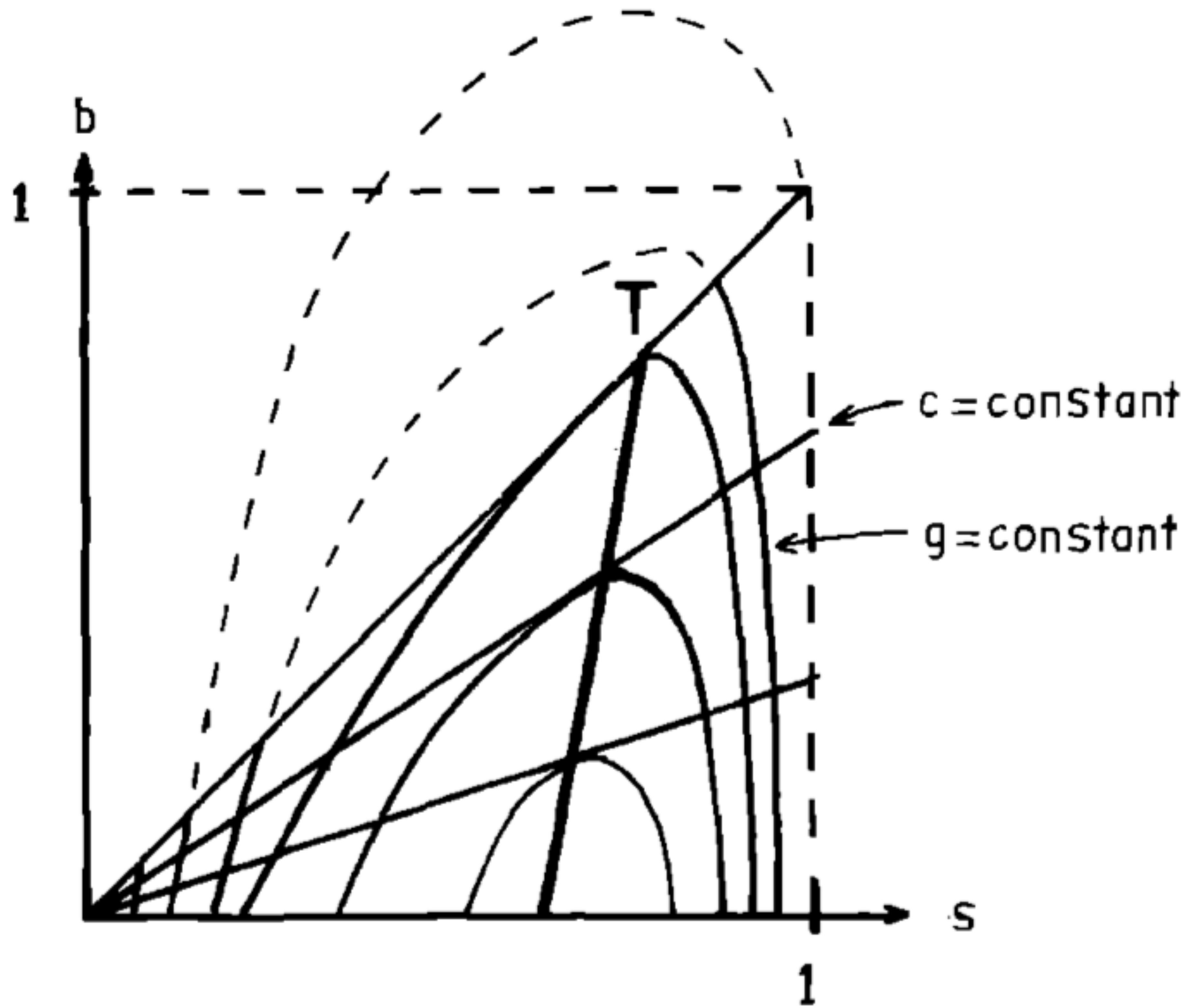
C.f. equivalent systems identified by Keyfitz and Kranzer from models in elasticity…

Solving the Riemann Problem for the Polymer Equations

(1) The Line Field

(2) The Contact Field

(1) RP for the Line Field (s,b)-plane



(1) RP for the Line Field (s,c)-plane

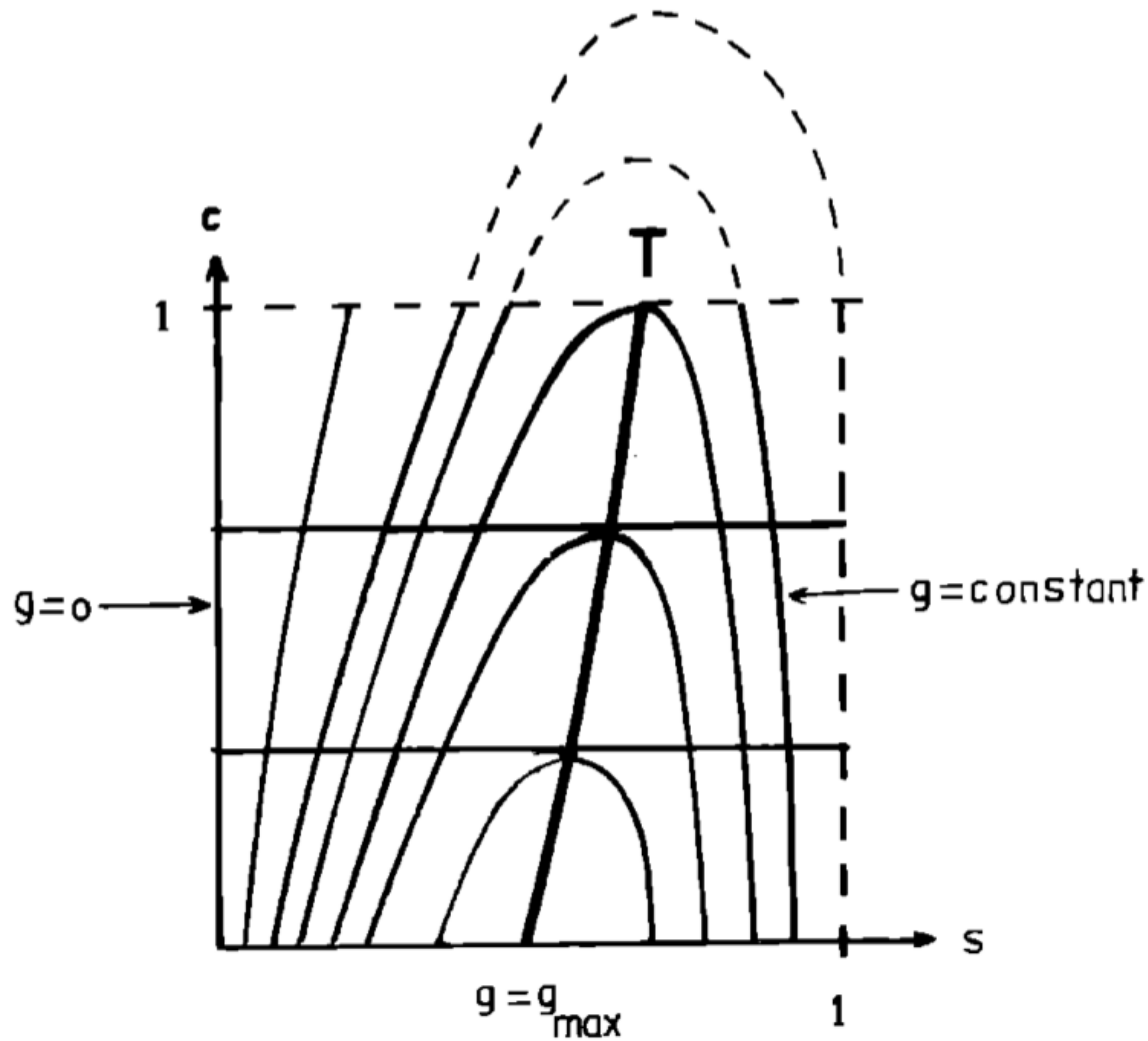


FIGURE 4

(1) Line Field-Buckley Leverett at fixed c

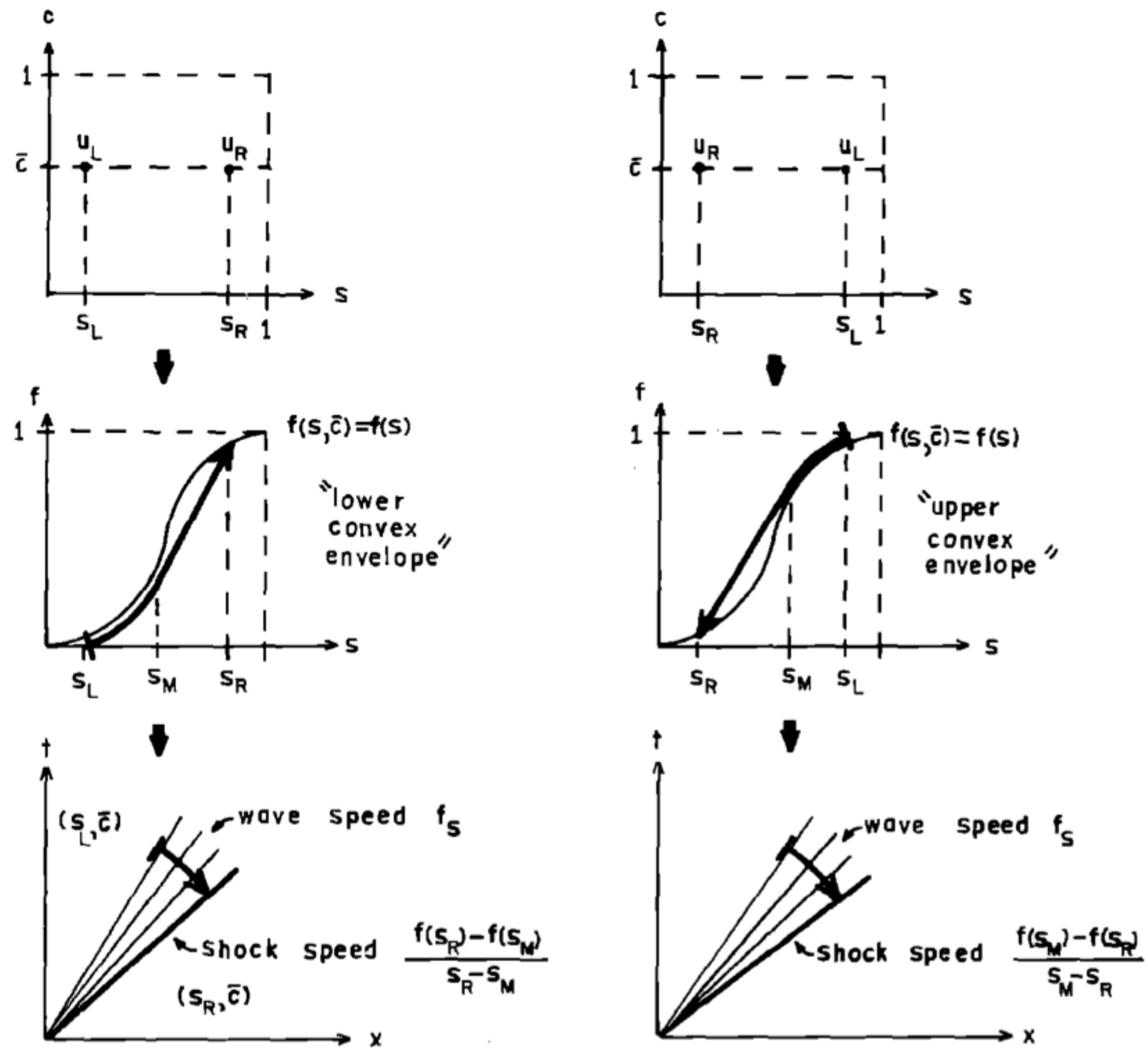


FIG. 6. Riemann problem for nonconvex scalar equation (s-waves).

(2) RP for the Contact Field (s,c)-plane

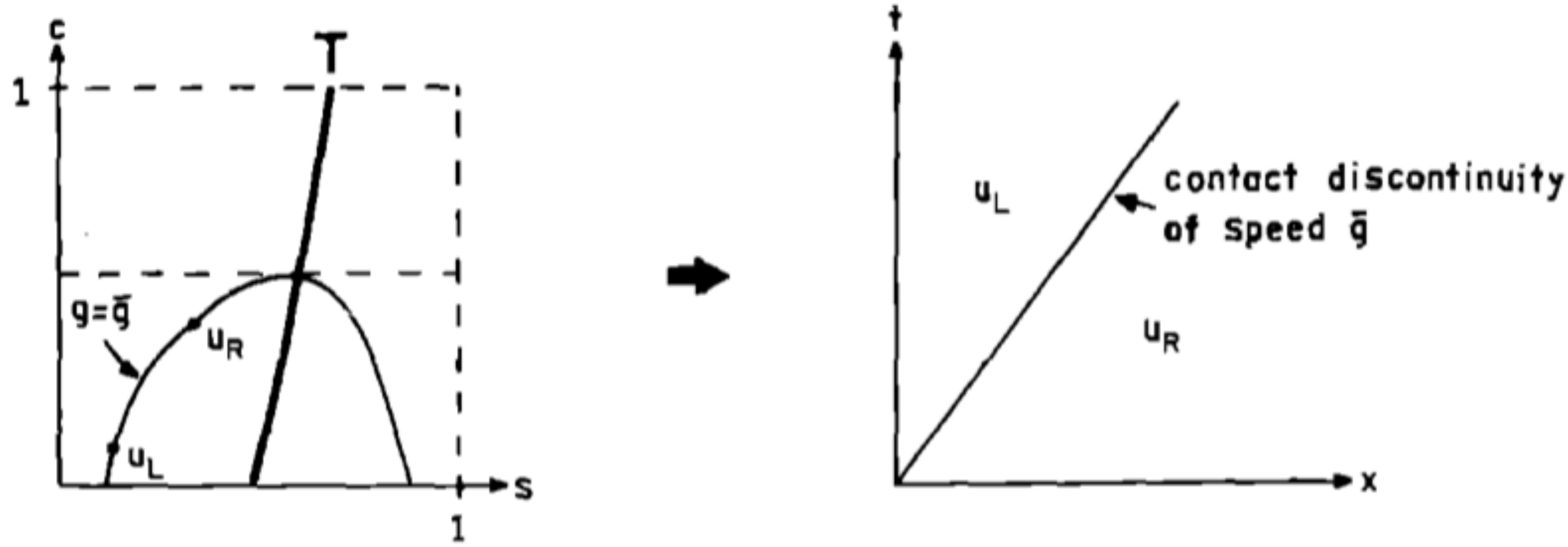


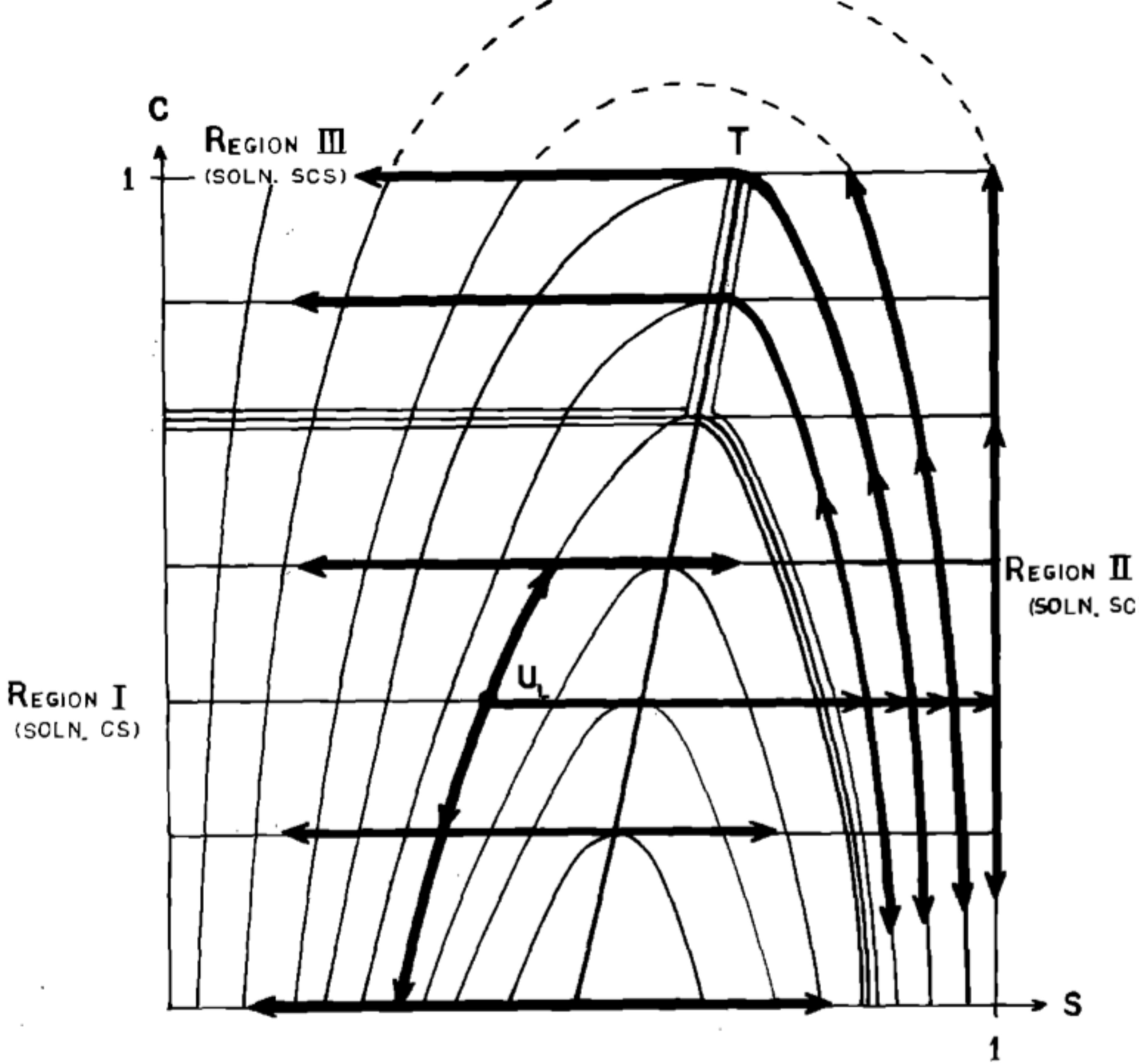
FIG. 7. *c*-wave solutions.

Theorem: There is a unique solution of the RP involving s-wave and c-waves…

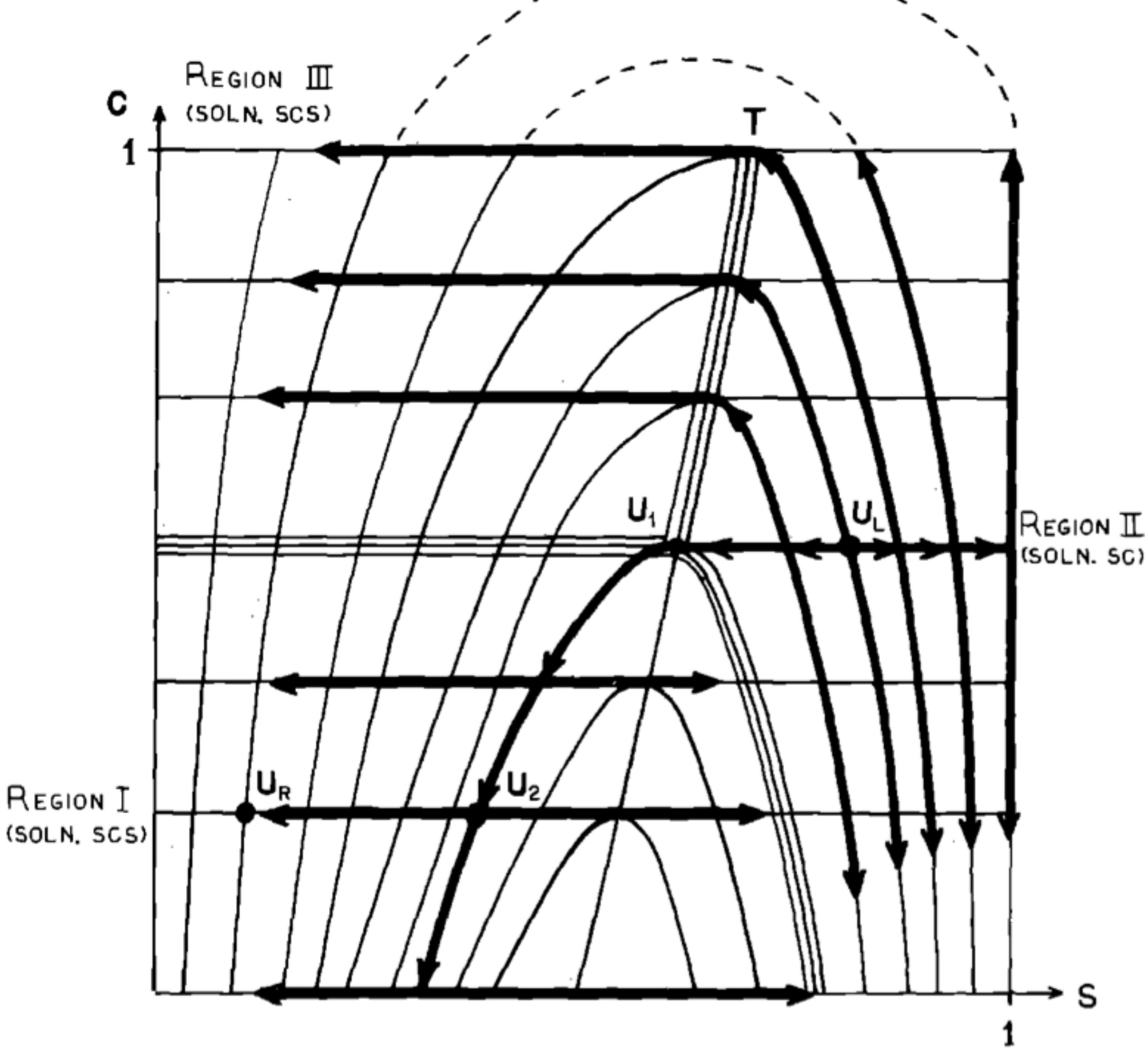
…subject to the Entropy Condition that c-waves are admissible iff they do not cross the Transition curve.

Moreover: The entire entropy solution of the RP can be drawn in…

TWO DIAGRAMS



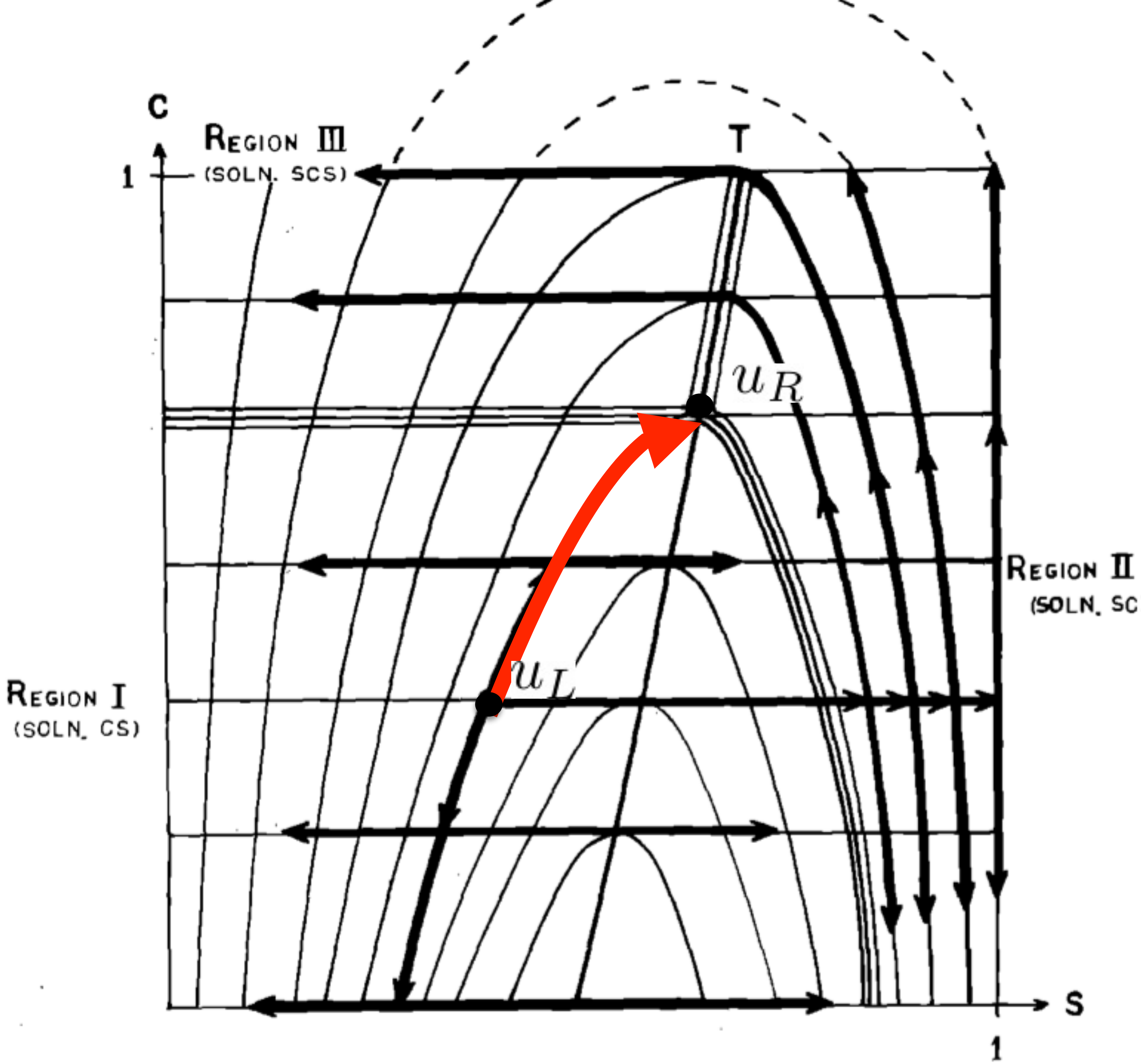
Solution of the RP for $u_L < \mathcal{T}$



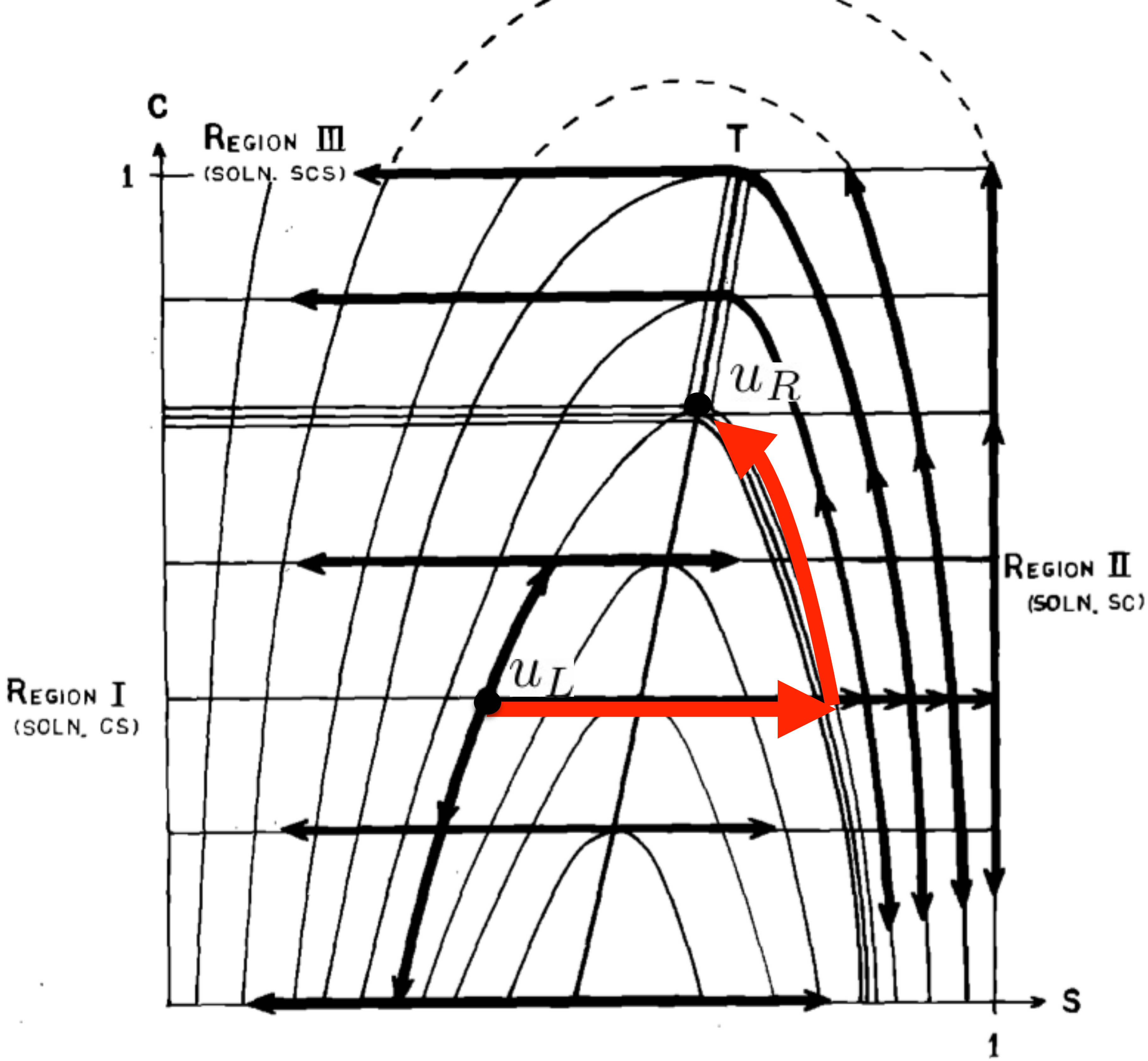
Solution of the RP for $u_L > \mathcal{T}$

Entropy Solution of the RP

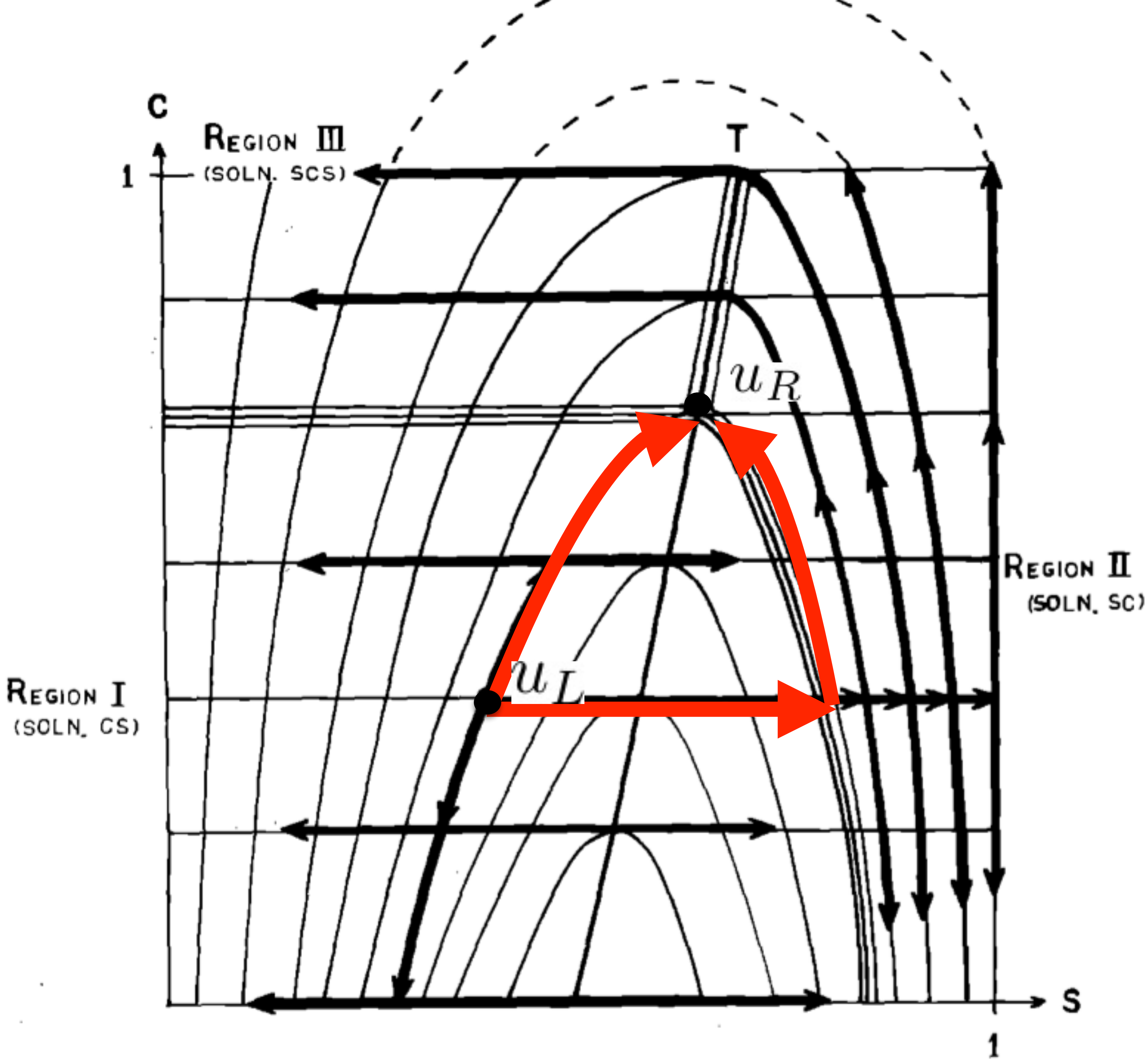
- RP can contain three waves, but only
  one (linear) c-wave
- Every solution of form scs
- Wave curves depend dis-continuously on u_L, u_R
- RP depends continuously in L^1 at each fixed time in the xt-plane.



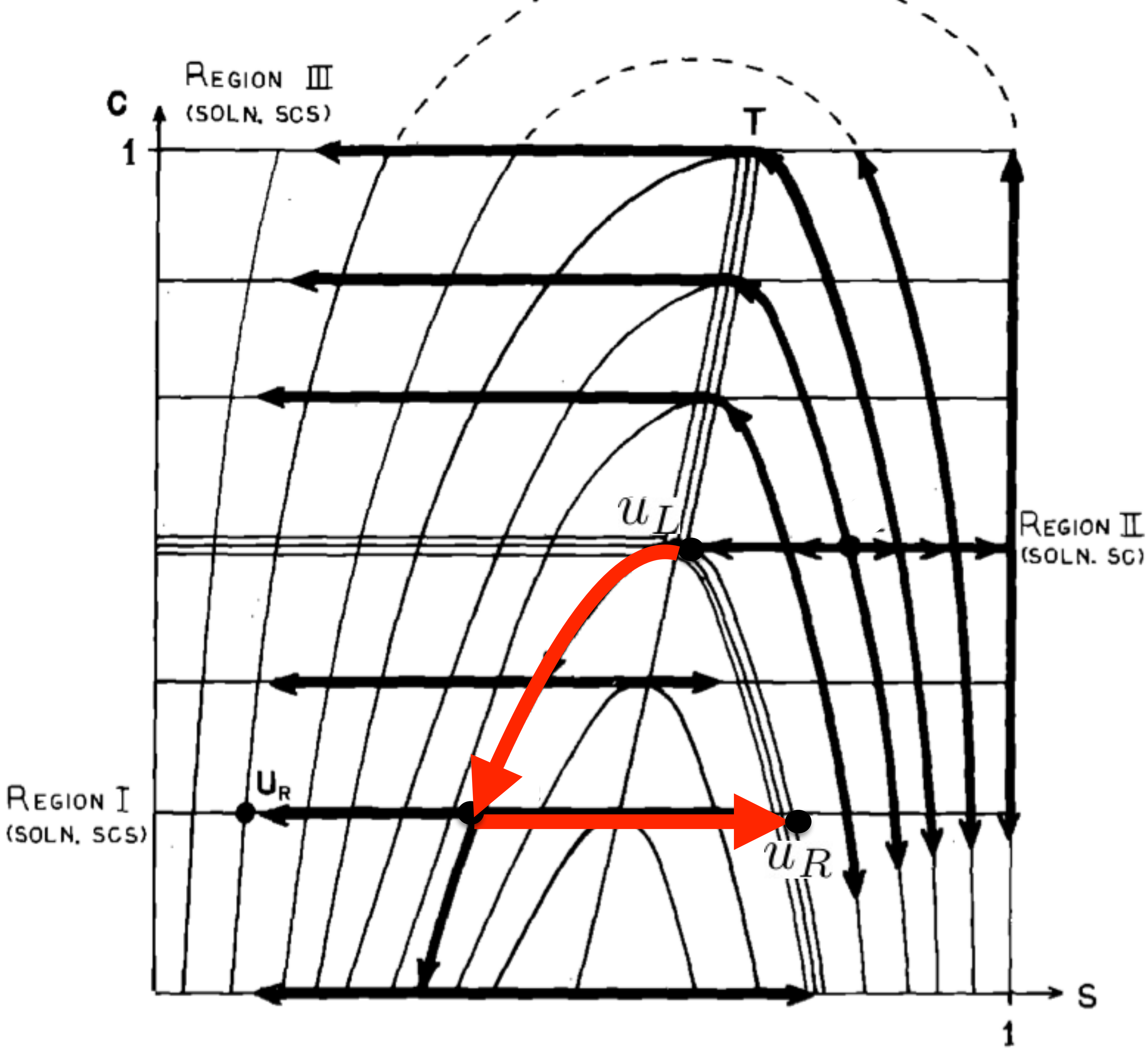
Dis-continuous waves when $\boxed{u_R \in \mathcal{T}}$



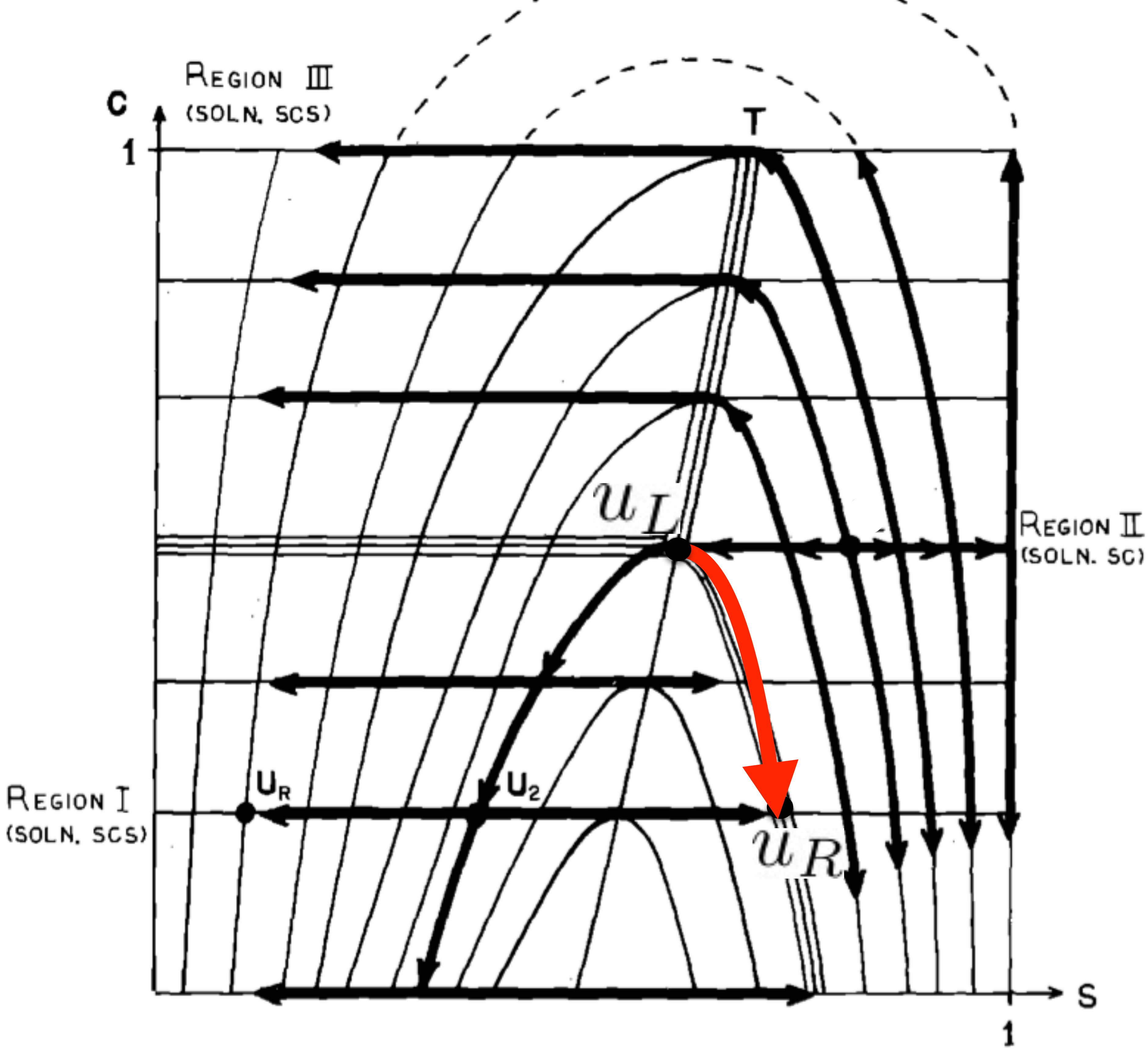
Dis-continuous waves when $u_R \in \mathcal{T}$



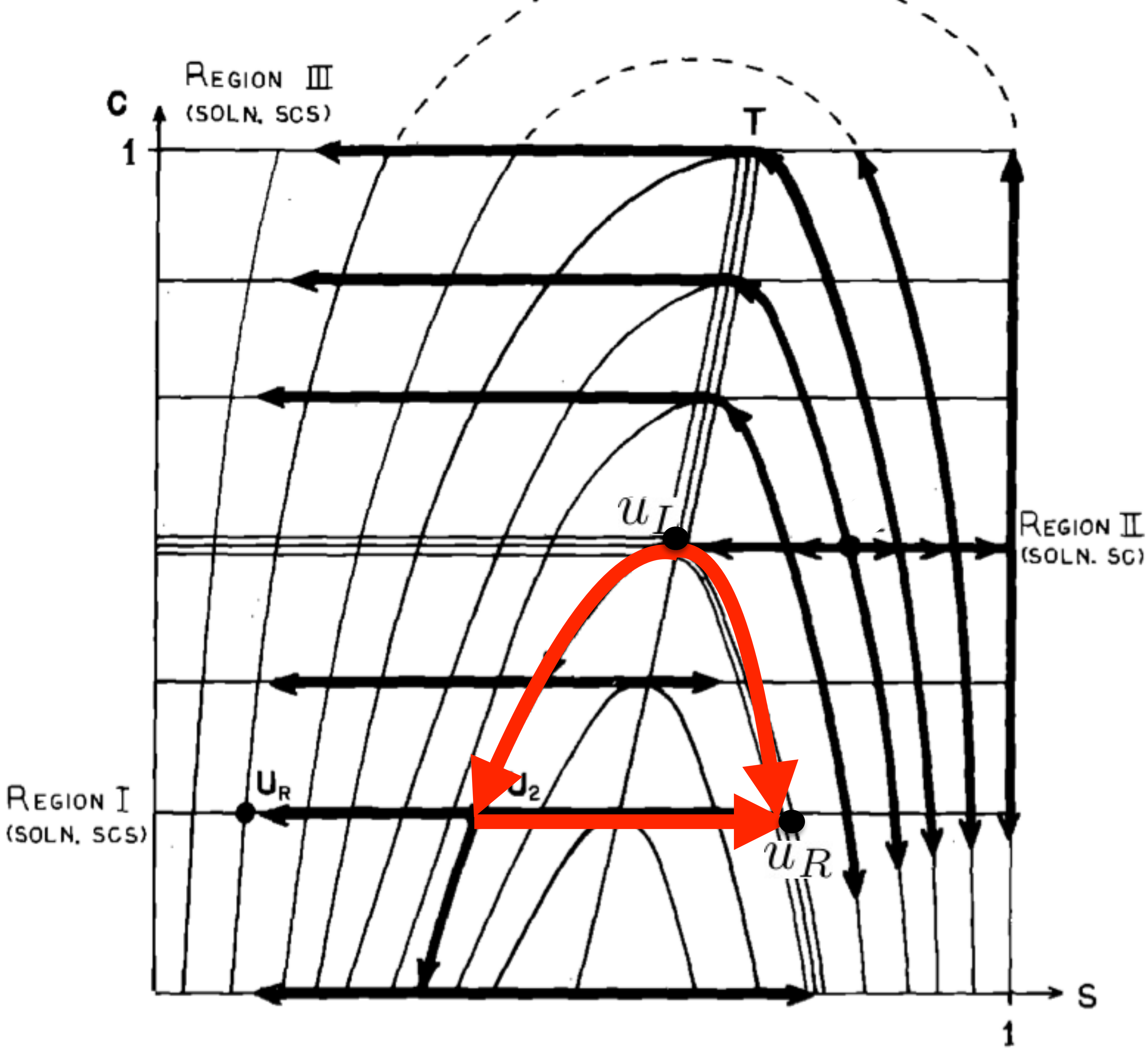
Dis-continuous waves when $\boxed{u_R \in \mathcal{T}}$



Dis-continuous waves when $\boxed{u_L \in \mathcal{T}}$



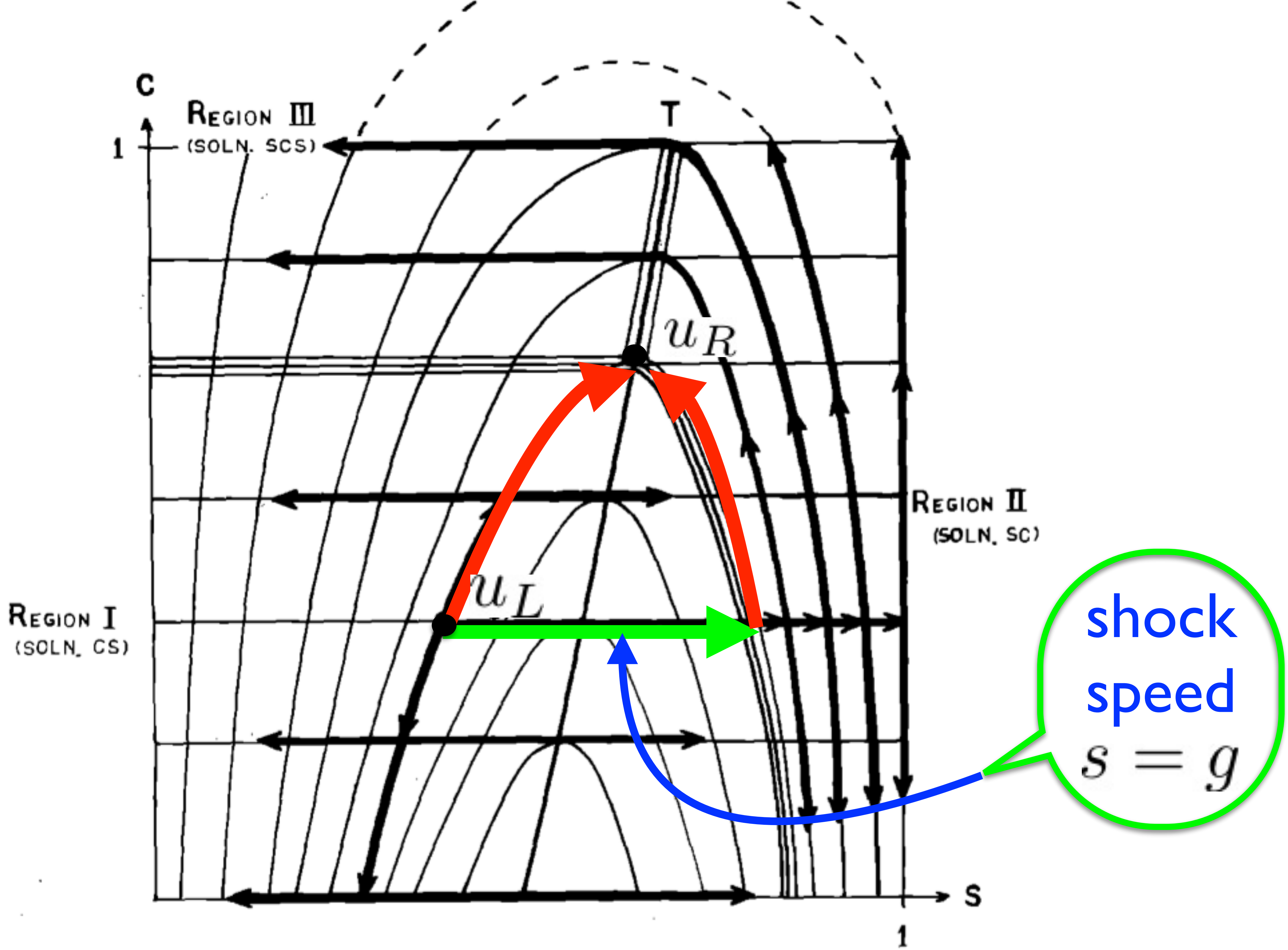
Dis-continuous waves when $u_L \in \mathcal{T}$



Dis-continuous waves when $\boxed{u_L \in \mathcal{T}}$

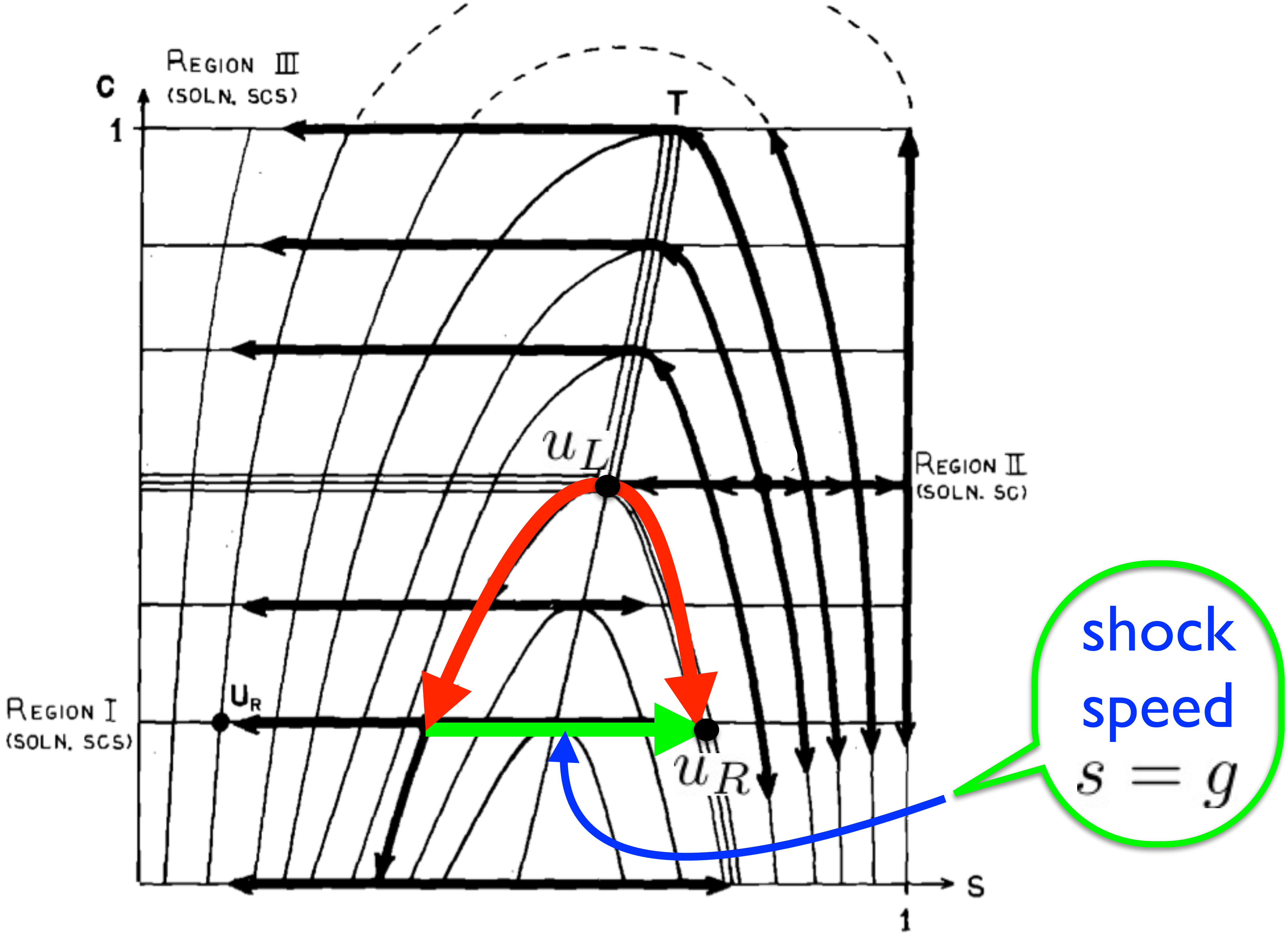
When wave curves are dis-continuous…

…solutions are L^1-continuous in the xt-plane



All three waves have speed $s = \lambda_c = g$

IMPLIES: L^1-continuity in xt-plane

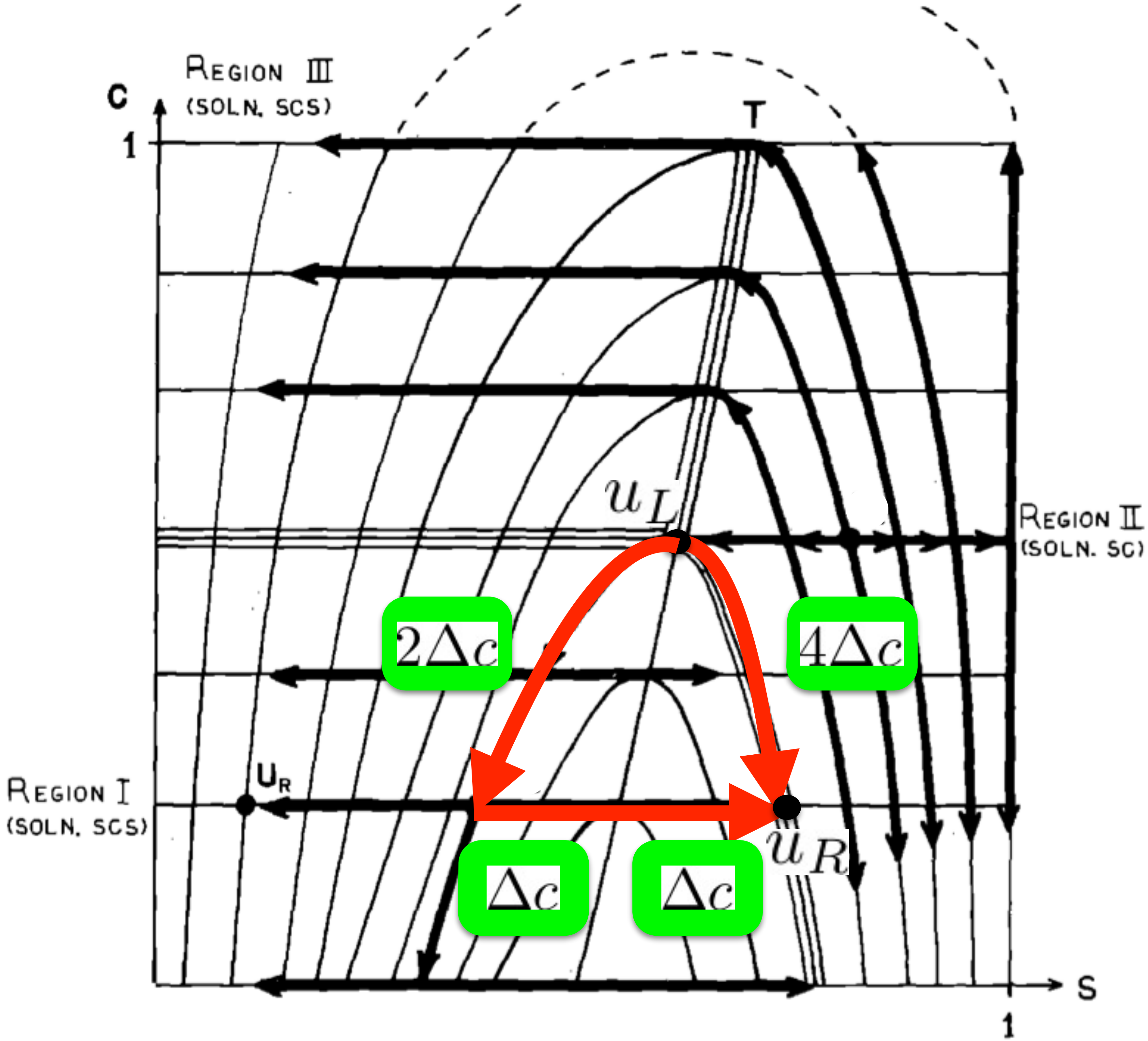


All three waves have same speed

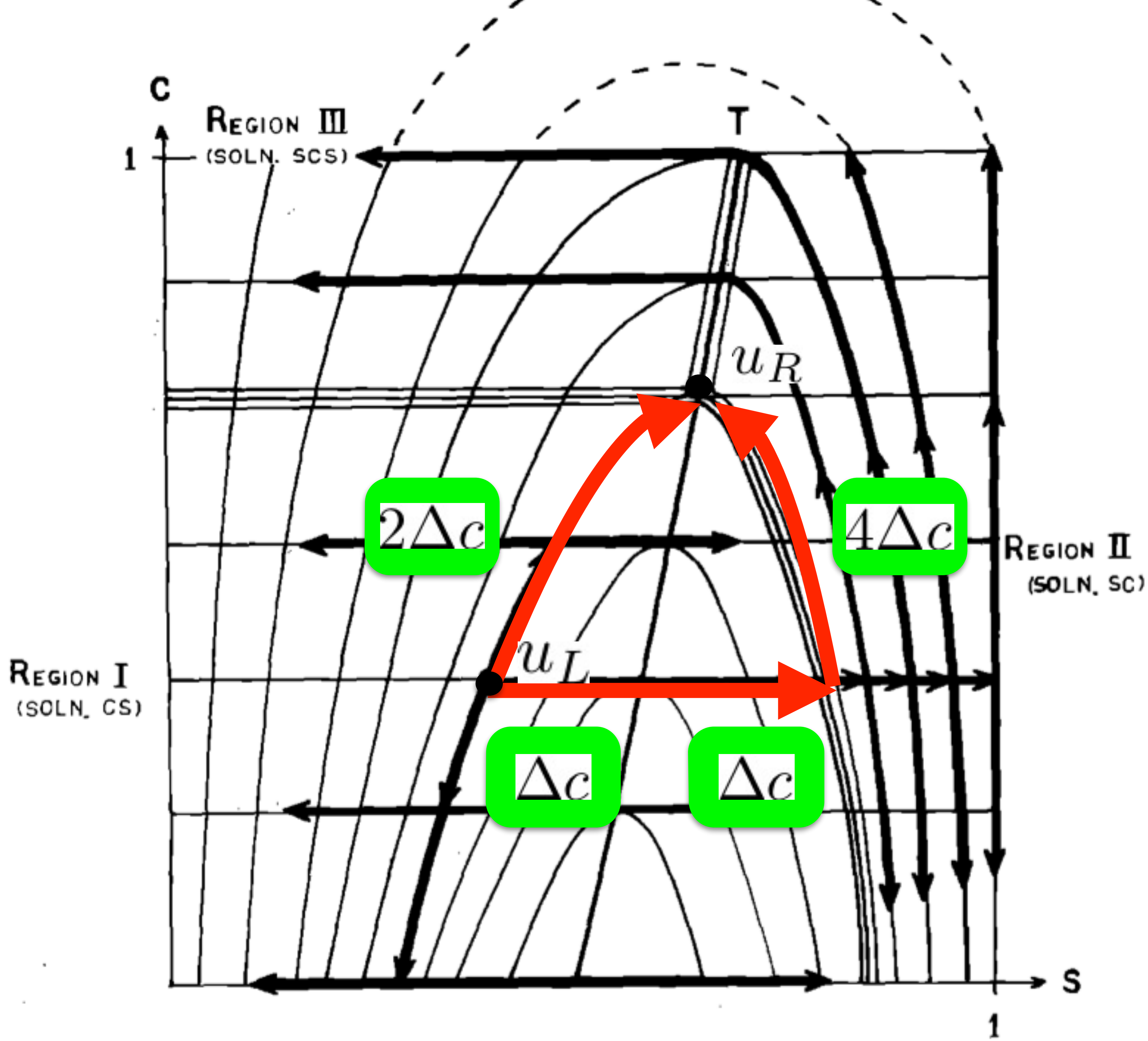
BOTH WAYS!

When wave curves are dis-continuous…

…there is a 1-2-4 weighting principle that makes wave strengths continuous as well…



The 1-2-4 principle balances wave strengths



The 1-2-4 principle balances wave strengths

BOTH WAYS!

Convergence of the Glimm Scheme

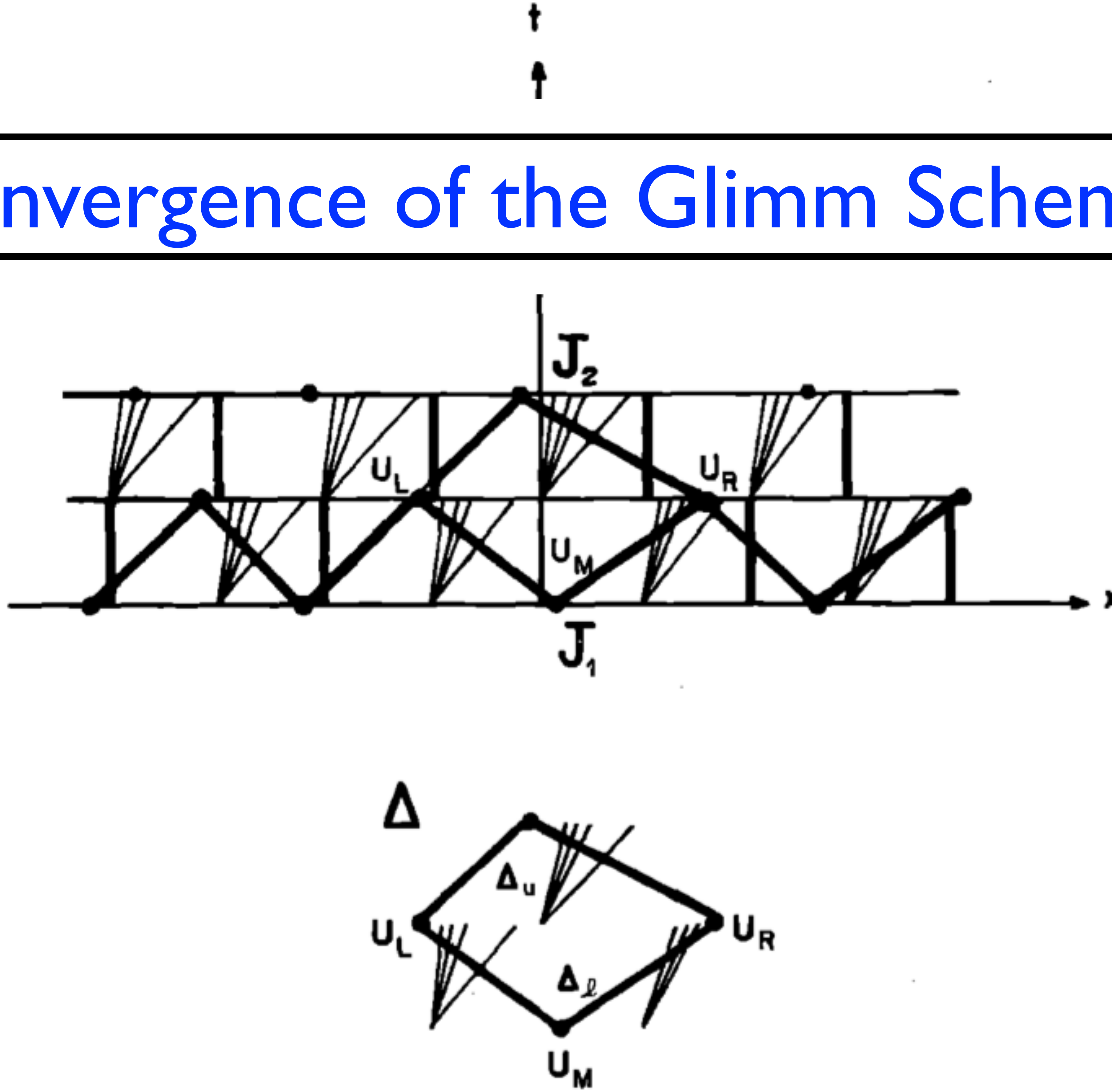


FIGURE 11

The Glimm Scheme

The Glimm Scheme

- Define a staggered grid… $(mr, (n-1)s)$.

The Glimm Scheme

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$$\{a_n\} \in \mathcal{A} = \Pi_n[0,1]$$

The Glimm Scheme

- Define a staggered grid… $(mr, (n-1)s)$.
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$$\{a_n\} \in \mathcal{A} = \Pi_n[0,1]$$
- Find a functional F that bounds TV and decreases across interaction diamonds

The Glimm Scheme

- Define a staggered grid… $(mr, (n-1)\sigma)$.
- Solve RP in each grid rectangle
- Continue according to a sample sequence

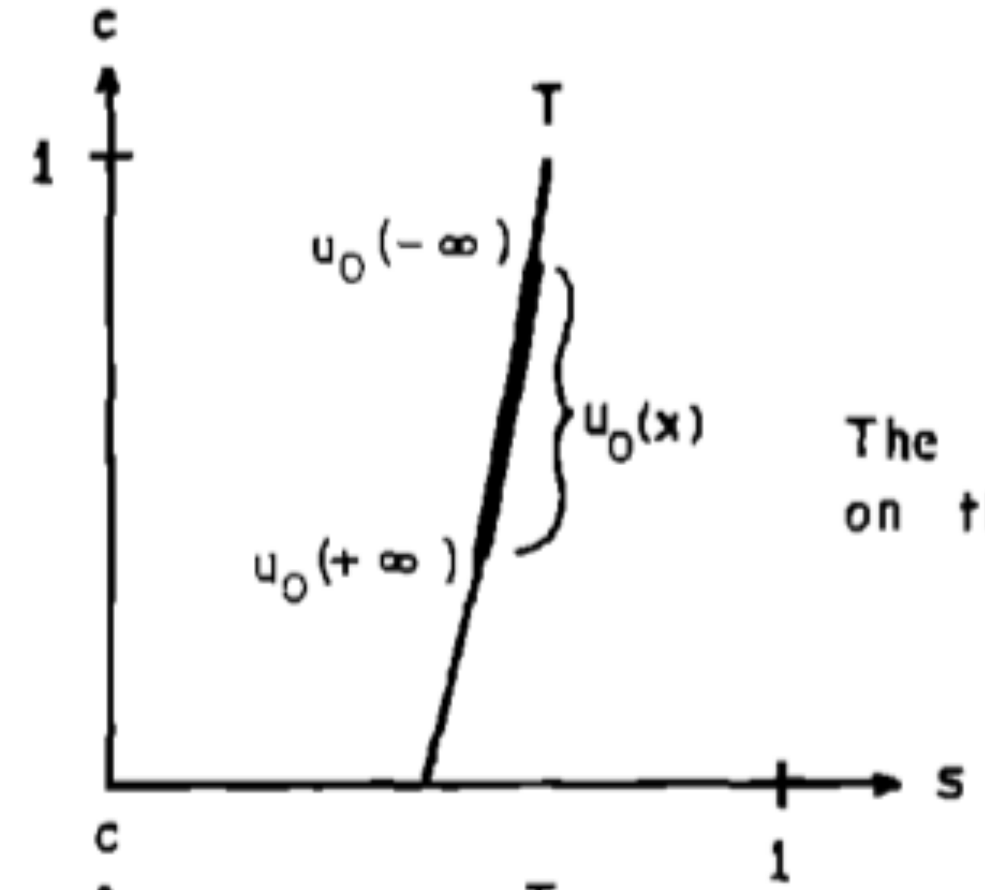
$$\{a_n\} \in \mathcal{A} = \Pi_n[0,1]$$

- Find a functional F that bounds TV and decreases across interaction diamonds
- Bound the TV at each time to prove convergence by Helly's Theorem

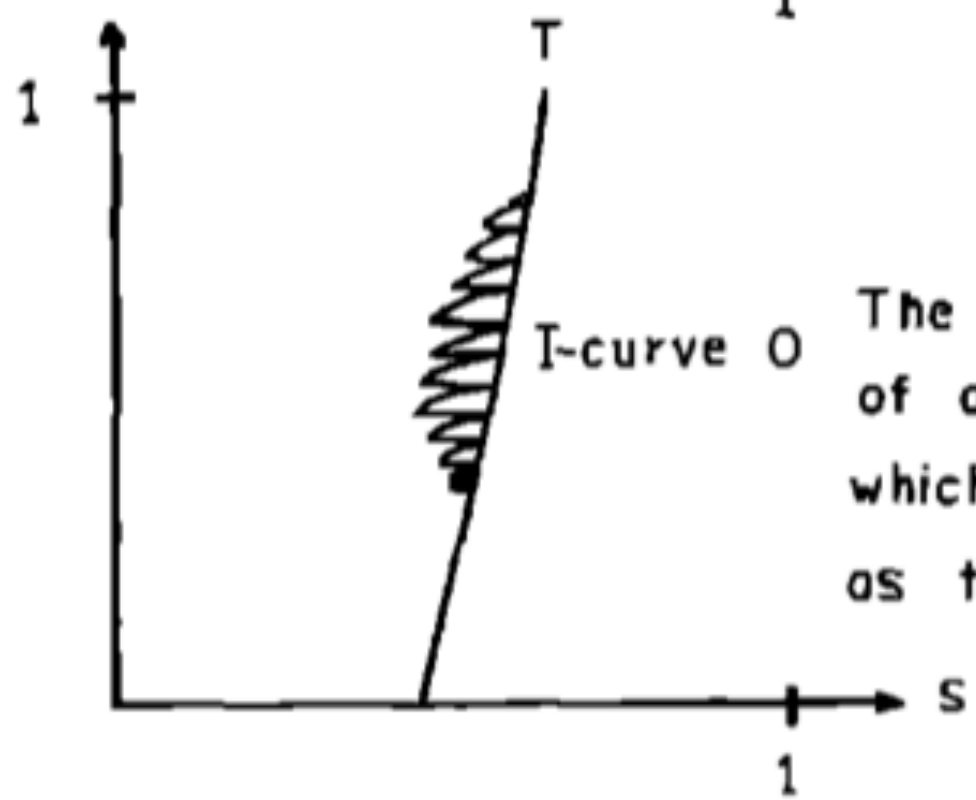
The Glimm Scheme

THEN: Prove the Residual tends to zero for almost every choice of sampling sequence.

TO CONCLUDE: Off a set of zero measure sampling sequences, the Glimm approximates converge to a weak solution.



The initial data $u_0(x)$ lies on the transition curve.



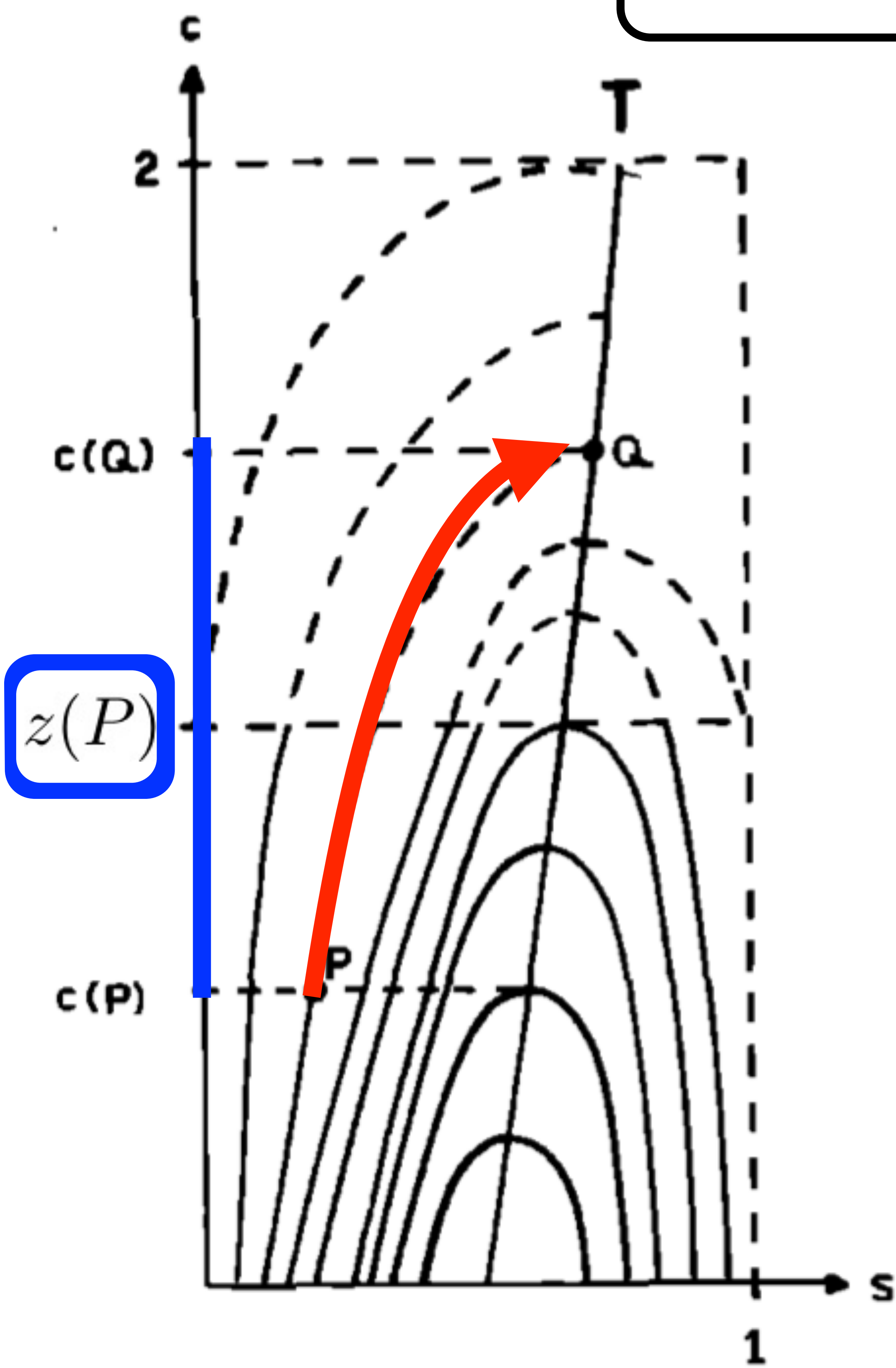
The I-curve 0 is a sequence of almost parallel s- and c-waves which aquires unbounded variation as the mesh length approaches zero.

Problem: The TV is un-bounded

Main Idea: Bound the Total Variation under a Singular Transformation of the Conserved Quantities…

$$(s, c) \to (z, c)$$

The variable **z**



$$\boxed{|z(P)| = |c(Q) - c(P)|}$$

$$z < \mathcal{T} \quad \text{sign } z = +$$

$$z > \mathcal{T} \quad \text{sign } z = -$$

Define wave strengths in terms of z:

$$|s| = |\Delta z|$$

$$|c| = \begin{cases} 2|\Delta c| & \text{if } s \nearrow \\ 4|\Delta c| & \text{if } s \searrow \end{cases}$$

I.e., by the 1-2-4 weight principle.

Define wave strengths in terms of z:

$$|s| = |\Delta z|$$

$$|c| = \begin{cases} 2|\Delta c| & \text{if } s \nearrow \\ 4|\Delta c| & \text{if } s \searrow \end{cases}$$

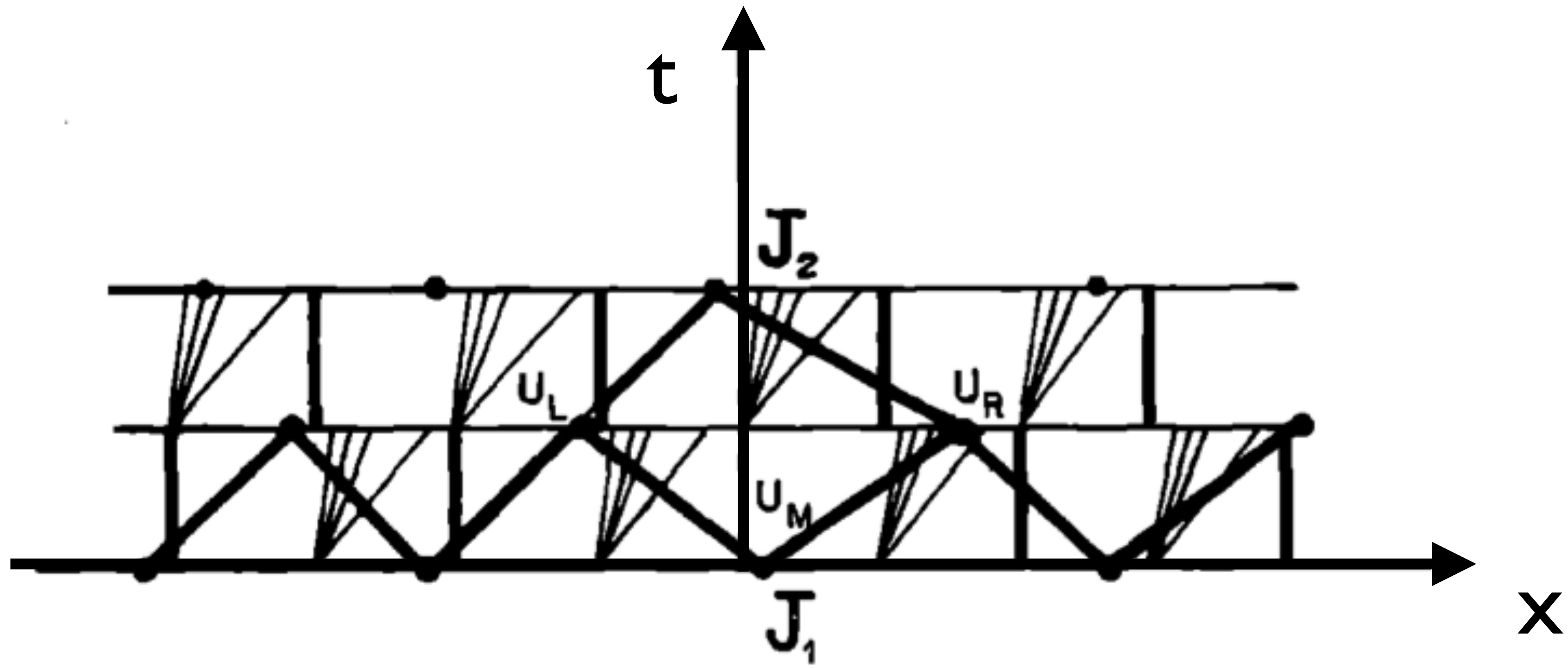
Define F in terms of wave strengths:

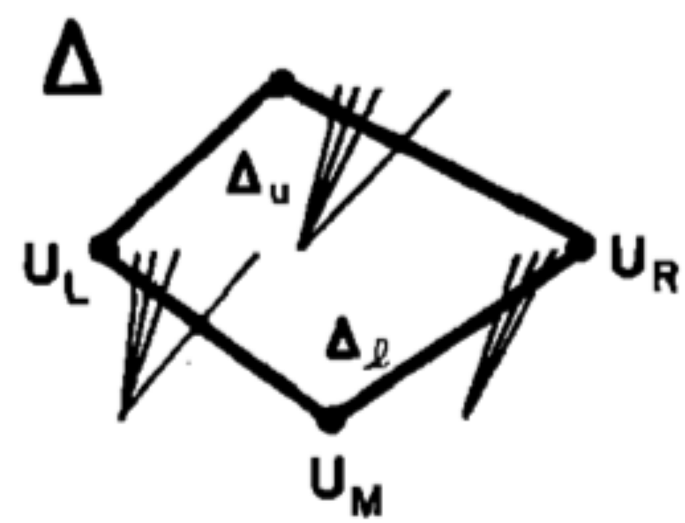
$$F(J) = \sum_J |s_i| + |c_i|$$

THEORM:

$$\boxed{F(J_2) \leq F(J_1)}$$

for J_2 a successor of J_1





$$F(J_2) \leq F(J_1)$$

THEOREM: TV bound for the Glimm Approximates:

$$TV_{zc}\{u_r(\cdot,t)\} \leq TV_{zc}\{u_0(\cdot)\}$$

Conclude:

- Helly Compactness ala Standard Glimm implies pointwise a.e, L^1-Lipschits in time convergence in the (z,c)-plane.

Conclude:

- Helly Compactness ala Standard Glimm implies pointwise a.e, L^1-Lipschits in time convergence in the (z,c)-plane.

- Uniform Continuity of the inverse map

$$(z,c) \to (s,c)$$

gives convergence in the (s,c)-plane

Theorem: For each choice of sampling, there exists a convergent subsequence of Glimm approximate solutions

$$\boxed{(s_r(x,t), c_r(x,t)) \underset{r\to 0}{\longrightarrow} (s(x,t), c(x,t))}$$

No bound on: $TV_{sc}\{(s_r, c_r)\}$

Uniform bound: $TV_{zc}\{(s_r, c_r)\}$

Because we have no bound on the Total Variation of the conserved quantities (s,b) or (s,c), the proof of convergence of the Residual must be modified…

Convergence of the Residual

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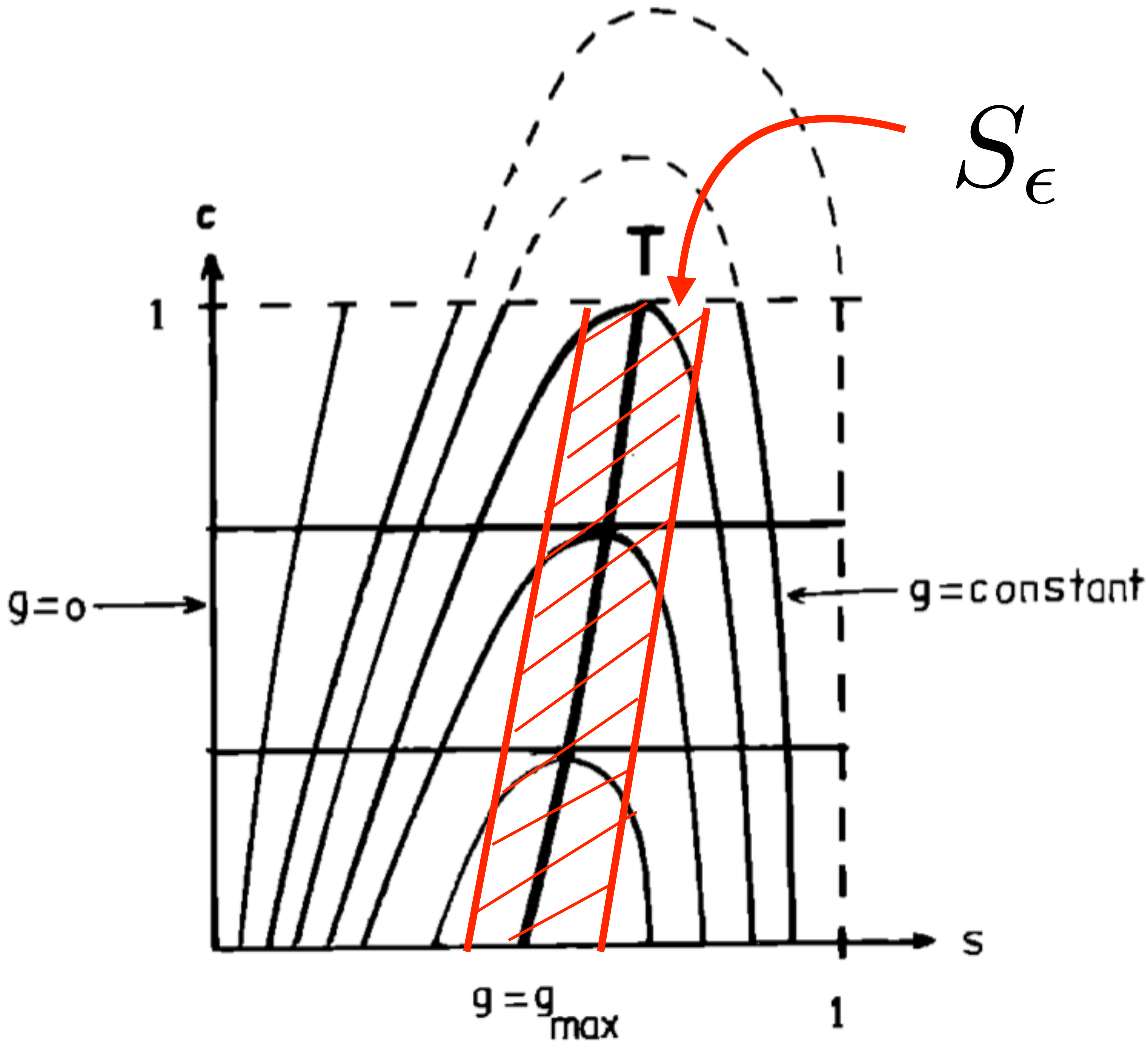
Convergence of the Residual

- Since the TV can be unbounded in plane of conserved quantities, Glimm's proof must be modified…

- To isolate the effect of the Transition Curve, one needs to equi-distribute in space and time.

- The idea is to partition the rectangles into those whose RP solutions intersect S_ϵ and those that do not.



S_ϵ
c
T
1
g=o
g=constant
s
$g = g_{max}$
1

- The Residual in general…

$$D(r, a, \phi) = \int_0^\infty \int_{-\infty}^\infty (\phi_t u_{ra} + \phi_x F(u_{ra}))\, dx\, dt + \int_{-\infty}^\infty \phi(x, 0)\psi_r(x)\, dx,$$

Initial Data

- Integrating by parts…

$$D(r, a, \phi) = \sum_{n=1}^\infty \int_{-\infty}^\infty \phi(x, n\sigma)(u_{ra}(x, n\sigma) - u_{ra}(x, n\sigma - 0))\, dx.$$

- The Residual for Glimm approximate soln…

$$D(r, a, \phi) = \sum_{m, n} D_{mn}(r, a, \phi).$$

$$D_{mn}(r, a, \phi) = \int_{(m-1)r}^{(m+1)r} \phi(x, n\sigma)(u_{mn}(a_{mn}) - u_{mn}(x))\, dx$$

- Glimm Orthoganality with respect to sampling…

Proposition 6.1 For and fixed ϕ and any fixed mesh length r, if $(m,n) \neq (m',n')$, then

$$< D_{mn}(r,\cdot,\phi_r), D_{m'n'}(r,\cdot,\phi_r) >_A = 0$$

…reduces the Residual to a sum of squares…

$$\|D(r,\cdot,\phi_r)\|_2^2 = \sum_{m,n} \|D_{mn}(r,\cdot,\phi_r)\|_2^2$$

- Estimate by value at fixed $\bar{a} \in A$ …

$$
\begin{aligned}
\|D(r,\cdot,\phi_r)\|_2^2 &= \sum_{m,n} \|D_{mn}(\cdot)\|_2^2 = \sum_{m,n} \int_A |D_{mn}(a)|^2 \, da \\
&= \int_A \sum_{m,n} |D_{mn}(a)|^2 \, da \\
&\leqslant \sup_{a \in A} \sum_{m,n} |D_{mn}(a)|^2 \\
&\leqslant \sum_{m,n} |D_{mn}(\bar{a})|^2 + \varepsilon^2
\end{aligned}
$$

- Decompose the sum into those corresponding to mesh rectangles R entirely in S_ϵ and those not…

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant \sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 + \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 + \varepsilon^2.$$

- Decompose the sum into those corresponding to mesh rectangles R entirely in S_ϵ and those not...

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant \sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 + \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 + \varepsilon^2.$$

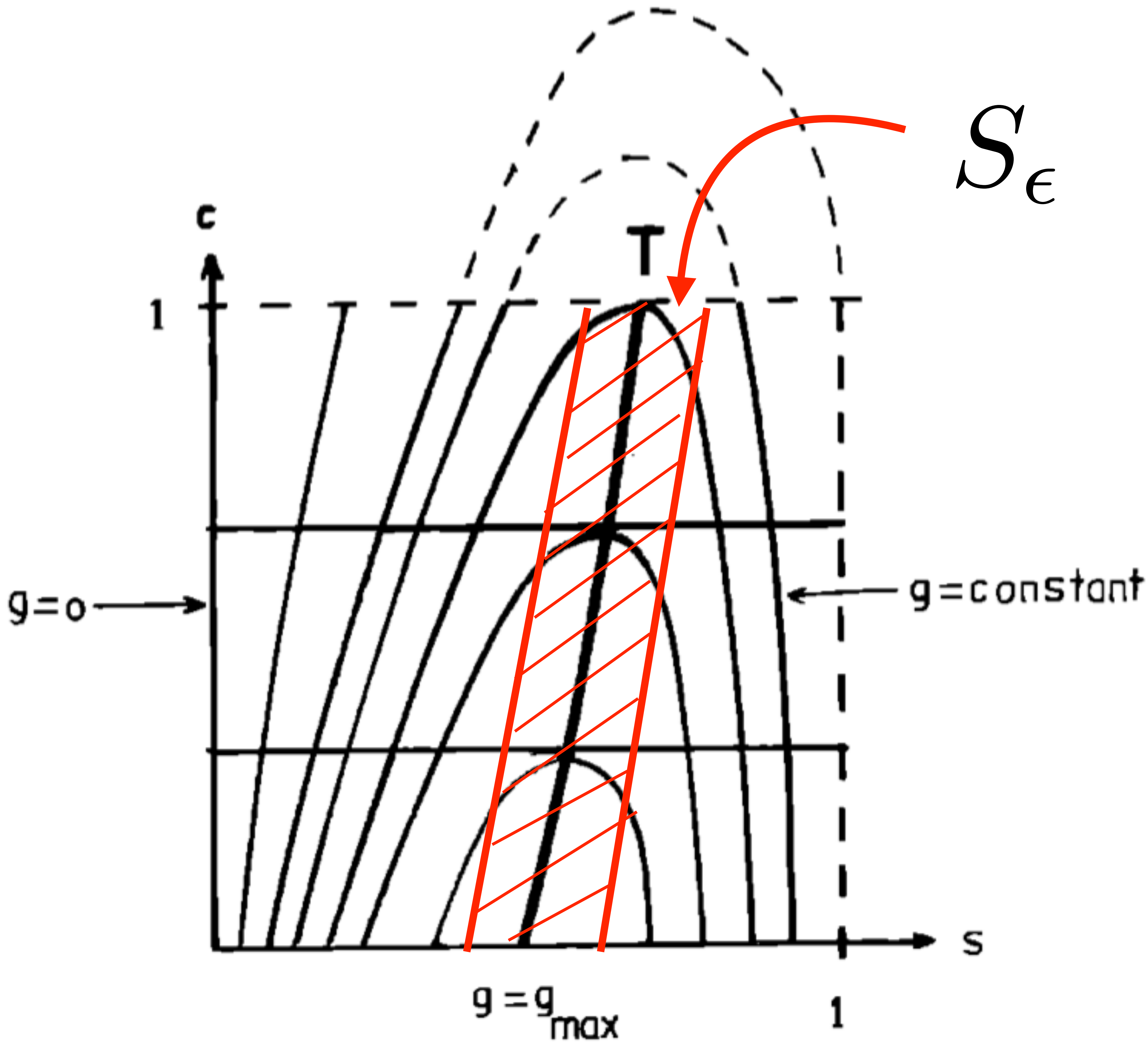
Mesh Rectangles
in S_ϵ

- Decompose the sum into those corresponding to mesh rectangles R entirely in S_ϵ and those not…

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant \sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 + \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 + \varepsilon^2.$$

Mesh Rectangles in S_ϵ

Mesh Rectangles not in S_ϵ


S_ϵ
c
1
T
g=o
g=constant
s
g = g_{max}
1

- For rectangles in S_ϵ estimate using the sup norm…

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant \sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 + \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 + \varepsilon^2$$

$$\boxed{\sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 \leq Const \sum_{m,n} r^2\epsilon^2}$$

$$D_{mn}(r,a,\phi) = \int_{(m-1)r}^{(m+1)r} \phi(x,n\sigma)\big(u_{mn}(a_{mn}) - u_{mn}(x)\big)\,dx$$

- For rectangles not in S_ϵ estimate by VAR in zc…

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant \sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 + \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 + \varepsilon^2$$

$$|D_{mn}(r,a,\phi_r)| \leq Const(\epsilon)\|\phi\|_\infty\, r\, Var_{zc} u_{mn}$$

$$\sum_m Var_{zc} u_{mn} \leq Const\, Var_{zc}\psi$$

$$\boxed{\sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 \leq Const(\epsilon^2) r}$$

- Therefore: The residual admits the estimate…

$$\|D(r,\cdot,\phi_r\|_2^2 \leq C\,\epsilon^2 + Const(\epsilon)\,r$$

Since $\epsilon << 1$ is arbitrary, we conclude

$$\boxed{\lim_{r\to 0}\|D(r,\cdot,\phi_r)\|_2^2 = 0}$$

Theorem: Off a set of zero measure $N \subset A$, the residual tends to zero.

Conclude: For almost every choice of sampling, the convergent Glimm Approximate Solutions converge to a weak solution of the equations…

Consider now the LAST paper Eli and I wrote:

Our LAST joint paper…

SIAM J. APPL. MATH.
Vol. 55, No. 3, pp. 625–640, June 1995

CONVERGENCE OF THE 2×2 GODUNOV METHOD FOR A GENERAL RESONANT NONLINEAR BALANCE LAW*

ELI ISAACSON† AND BLAKE TEMPLE‡

Abstract. We introduce a generalized solution of the Riemann problem for a general resonant nonlinear balance law, and we prove the convergence of the 2×2 Godunov numerical method based on these solutions. In particular, we obtain generic conditions that guarantee a canonical structure for the elementary waves in the solution of the Riemann problem, and an interesting multiplicity of time-asymptotic wave patterns is observed and characterized.

Key words. Godunov method, resonance, shock waves, balance law, Reimann problem

We study
generic resonance
between a
nonlinear wave family
and a
stationary source…

The framework...

An inhomogeneous scalar equation treated as system

$$u_t + f(a, u)_x = a' g(a, u)$$

$$a_t = 0$$

(so $a = a(x)$)

As a 2x2 system it takes the form…

$$\begin{pmatrix} a \\ u \end{pmatrix}_t + \begin{pmatrix} 0 \\ f(a,u) \end{pmatrix}_x = \begin{pmatrix} 0 \\ a' g(a,u) \end{pmatrix}$$

$$\boxed{U_t + F(U)_x = a' G(a,u)}$$

$$U = \begin{pmatrix} a \\ u \end{pmatrix} \qquad F(U) = \begin{pmatrix} 0 \\ f(a.u) \end{pmatrix} \quad G(U) = \begin{pmatrix} 0 \\ g(a.u) \end{pmatrix}$$

Motivation: Transonic flow in a variable area duct…

$$\rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u,$$

$$(\rho u)_t + (\rho u^2 + p)_x = -\frac{a'(x)}{a(x)}\rho u^2,$$

$$(\rho E)_t + (\rho E u + p u)_x = -\frac{a'(x)}{a(x)}(\rho E u + p u),$$

$$\boxed{U_t + F(U)_x = a' G(a, u)}$$

As a 2x2 system it has a line field and a contact field…

$$\begin{pmatrix} a \\ u \end{pmatrix}_t + \begin{pmatrix} 0 \\ f(a,u) \end{pmatrix}_x = \begin{pmatrix} 0 \\ a'g(a,u) \end{pmatrix}$$

Eigenvalues…

$\lambda = f_u(a,u)$ and $\lambda_0 = 0$

Ask: Under what generic conditions do you get resonance?

$$\boxed{\lambda = f_u(a, u) = 0}$$

That is: Solve the RP and initial value problem in a neighborhood of a state $U_* = (a_*, u_*)$

where generically

$$\boxed{\lambda_* = f_u(a_*, u_*) = 0}$$

Theorem: The RP has a canonical structure in a neighborhood of a point U_* where the following generic conditions hold:

Generic Conditions

(1) $f(U_*) = 0$ (resonance)

(2) $f_a(U_*) \neq 0$ (wlog <0)

(3) $g(U_*) - f_a(U_*) \neq 0$ (wlog <0)

(4) $f_{uu}(U_*) \neq 0$ (wlog <0)

(5) $g_u(U_*) \neq 0$

Theorem 1: The RP has a canonical structure in a neighborhood of a point where (1)-(5) hold

Theorem 1: The RP has a canonical structure in a neighborhood of a point where (1)-(5) hold

Theorem 2: Convergence can be demonstrated for both Glimm and Godunov methods

When $g = 0$, the RP looks exactly like the Polymer Equations...

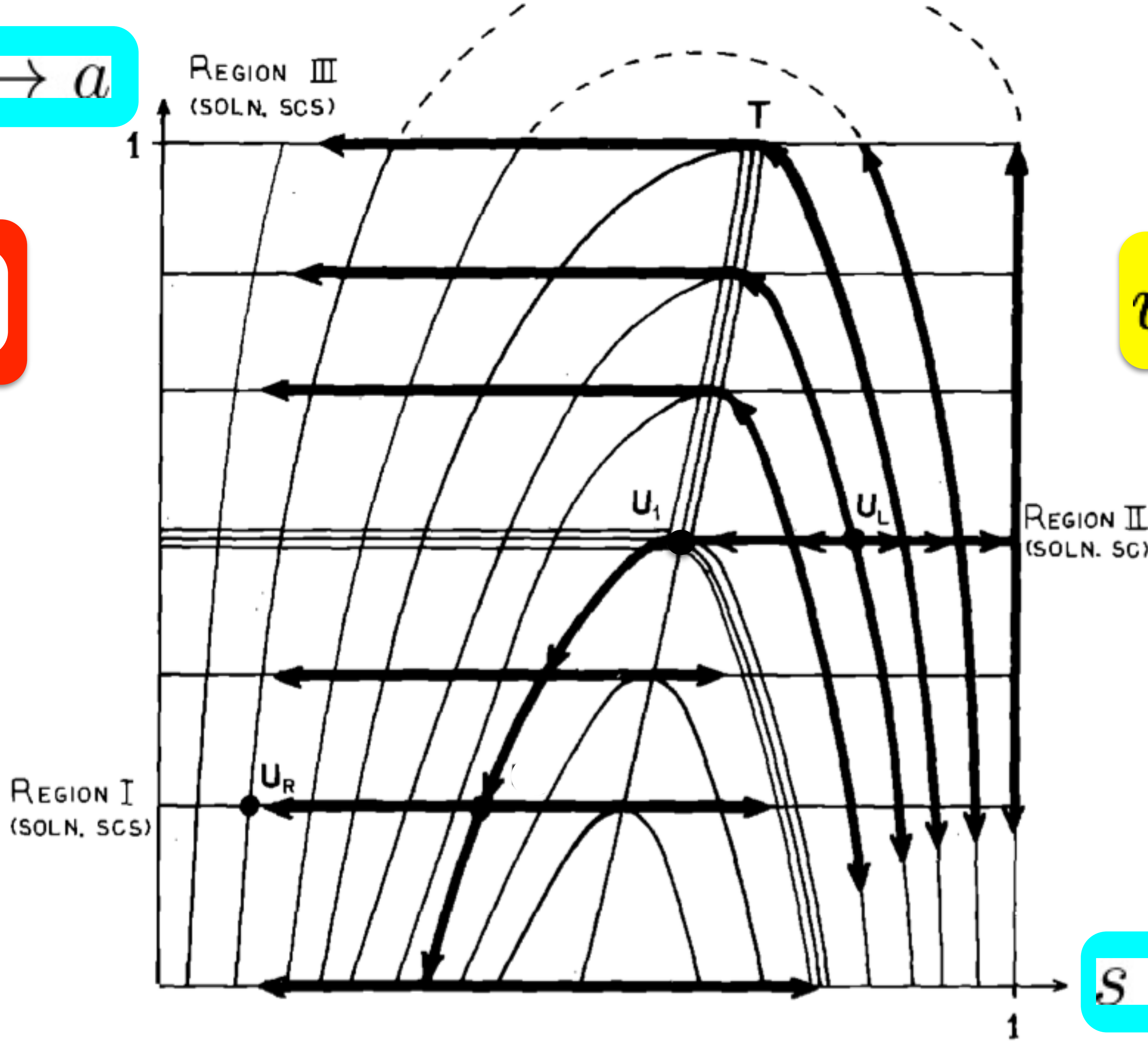
$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = \cancel{\begin{pmatrix} a' g(a,u) \\ 0 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = 0}$$

$c \to a$

$g = 0$

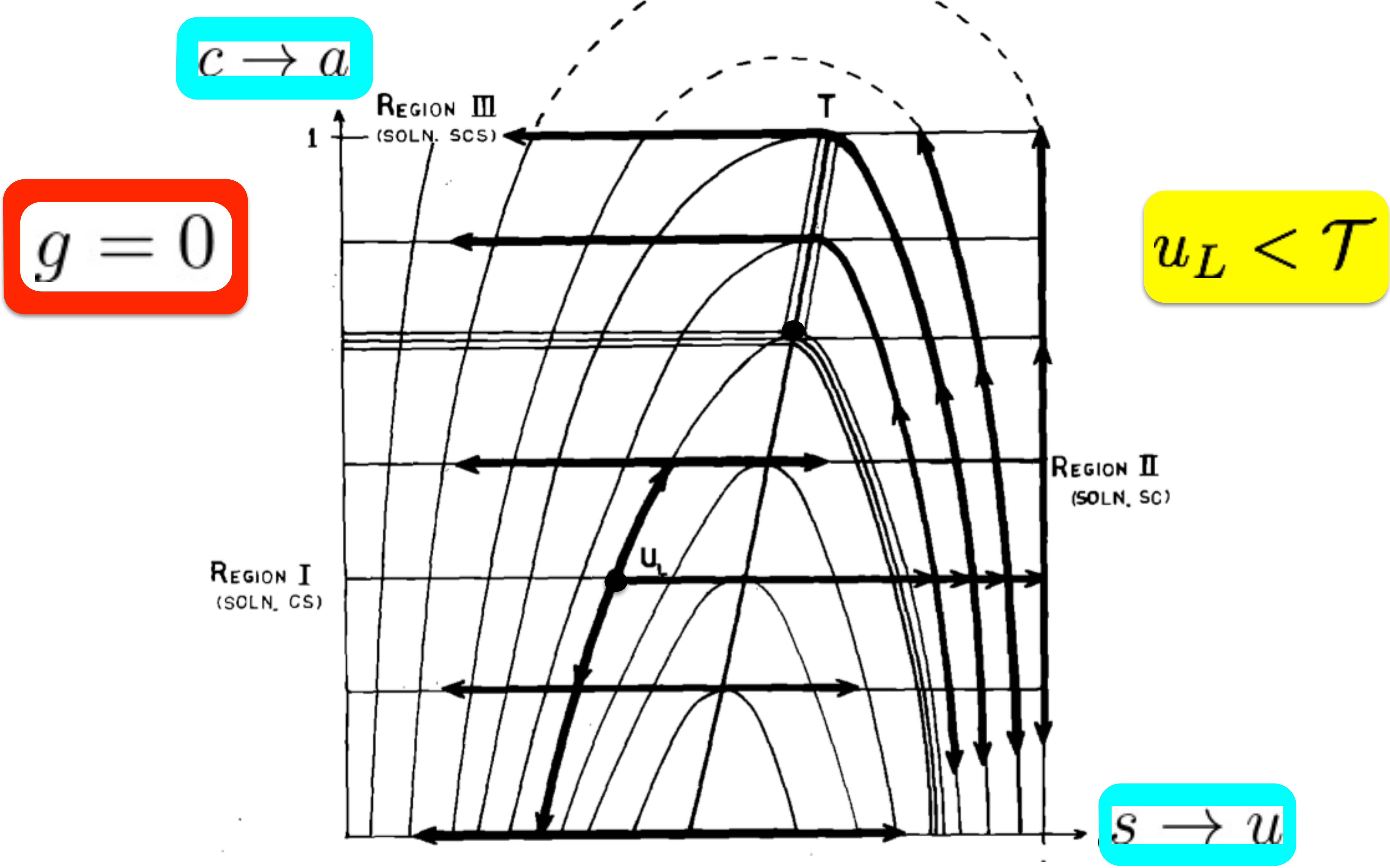
$u_L > \mathcal{T}$



$s \to u$

...RP the same as for the Polymer Eqns...

...except all c-waves have zero speed...



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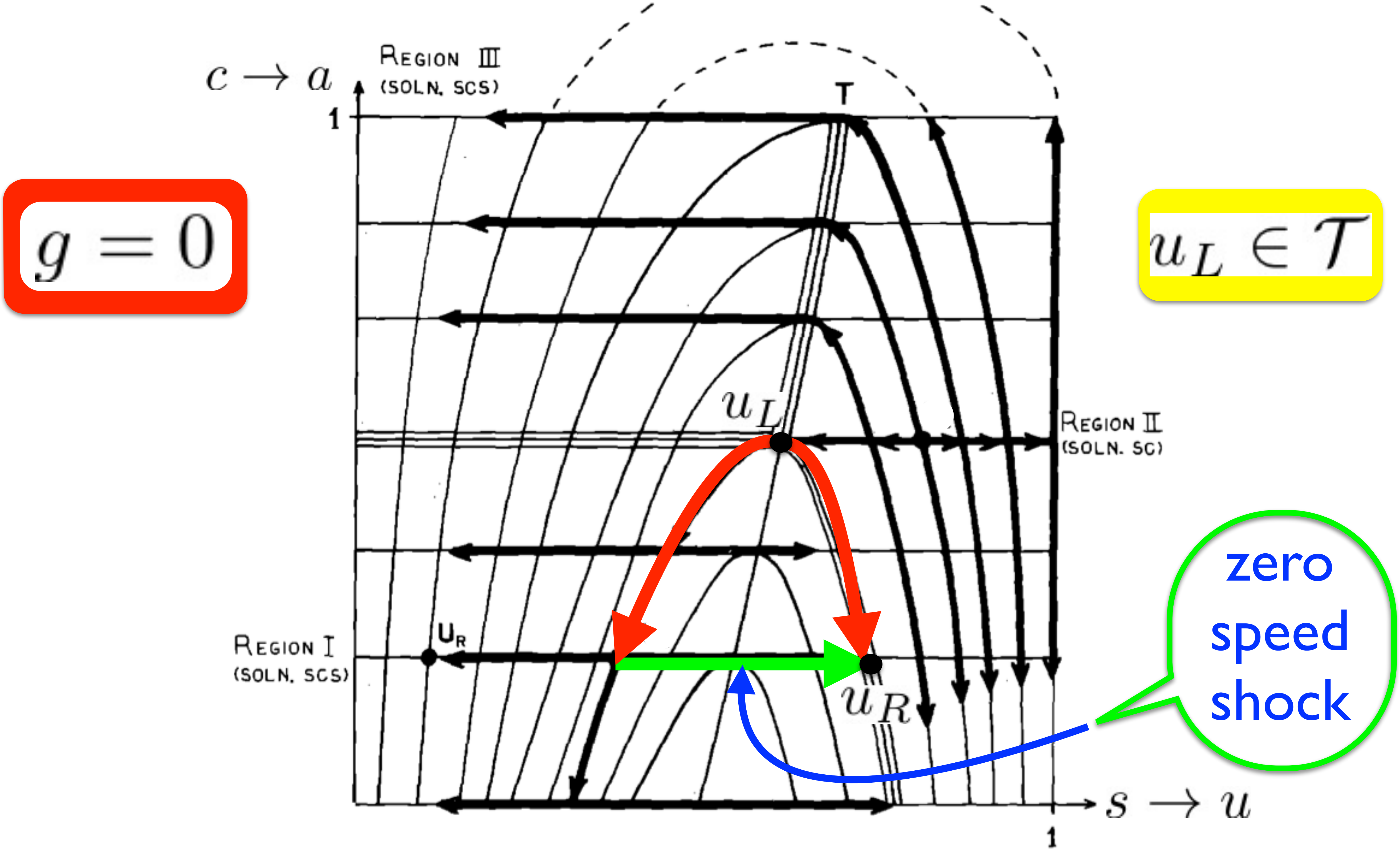
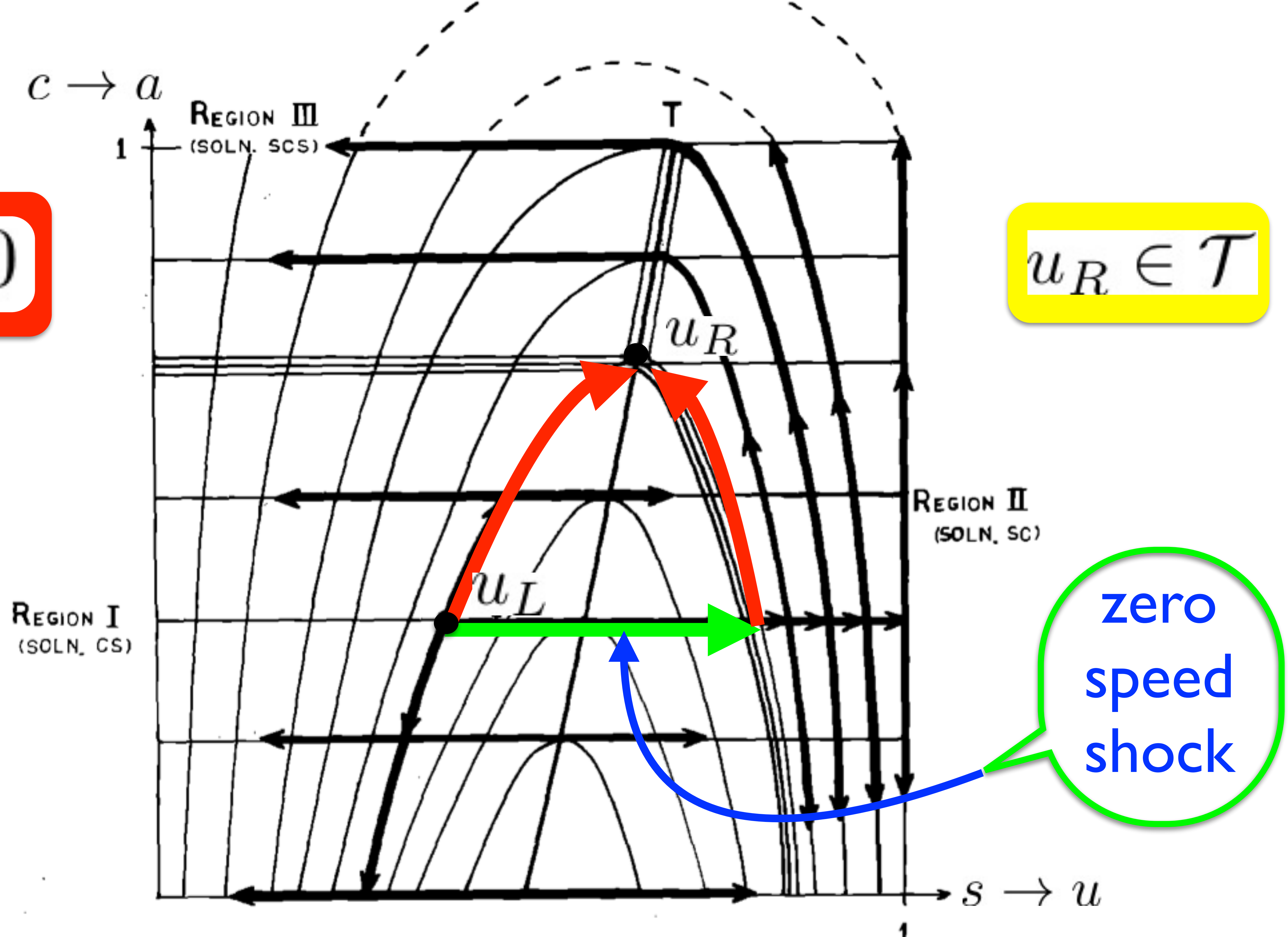


…and $g = const$ give BOTH the standing wave curves and the zero speed shock curves…

$g = 0$

$u_R \in \mathcal{T}$



...and $g = const$ give BOTH the standing wave curves and the zero speed shock curves...

The reason it looks like Polymer is because it is Polymer…

I.e., doing a Lagrangian change of variables using speed…

$$\frac{dx}{dt} = g(s, c) = \lambda_c$$

gives… $u_t + f(u, a)_x = 0$

with… $u = s \qquad a(x) = c(x)$

How does the source term work?

$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = \begin{pmatrix} a' g(a,u) \\ 0 \end{pmatrix}$$

Why the a' on the RHS??

The Point: Writing the source as

$$\boxed{a' g(a, u)}$$

implies the standing waves are scale invariant, like a linear

Contact Family…

I.e., standing waves re-scale into jump discontinues.

Standing Waves Re-scale

Standing Waves Re-scale

CL: $$u_t + f(a,u)_x = a'g(a,u)$$

Standing Waves Re-scale

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SW: $$f(a,u)_x = a'g(a,u)$$

Standing Waves Re-scale

CL: $$u_t + f(a,u)_x = a' g(a,u)$$

SW: $$f(a,u)_x = a' g(a,u)$$

$$f_a \frac{da}{dx} + f_u \frac{du}{dx} = \frac{da}{dx} g(a,u)$$

Standing Waves Re-scale

CL: $$u_t + f(a,u)_x = a'g(a,u)$$

SW: $$f(a,u)_x = a'g(a,u)$$

$$f_a \frac{da}{\cancel{dx}} + f_u \frac{du}{\cancel{dx}} = \frac{da}{\cancel{dx}} g(a,u)$$

Standing Waves Re-scale

CL: $$u_t + f(a,u)_x = a' g(a,u)$$

SW: $$f(a,u)_x = a' g(a,u)$$

$$f_a \frac{da}{\cancel{dx}} + f_u \frac{du}{\cancel{dx}} = \frac{da}{\cancel{dx}} g(a,u)$$

$$\boxed{\frac{da}{du} = \frac{f_u}{g - f_a}}$$

Standing waves are determined by integral curves like a Contact Family:

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$$\boxed{\frac{da}{du} = \frac{f_u}{g - f_a}} \qquad g - f_a \neq 0 \quad (> 0)$$

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By assumption:

$$f_u(U_*) = 0$$

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By assumption:

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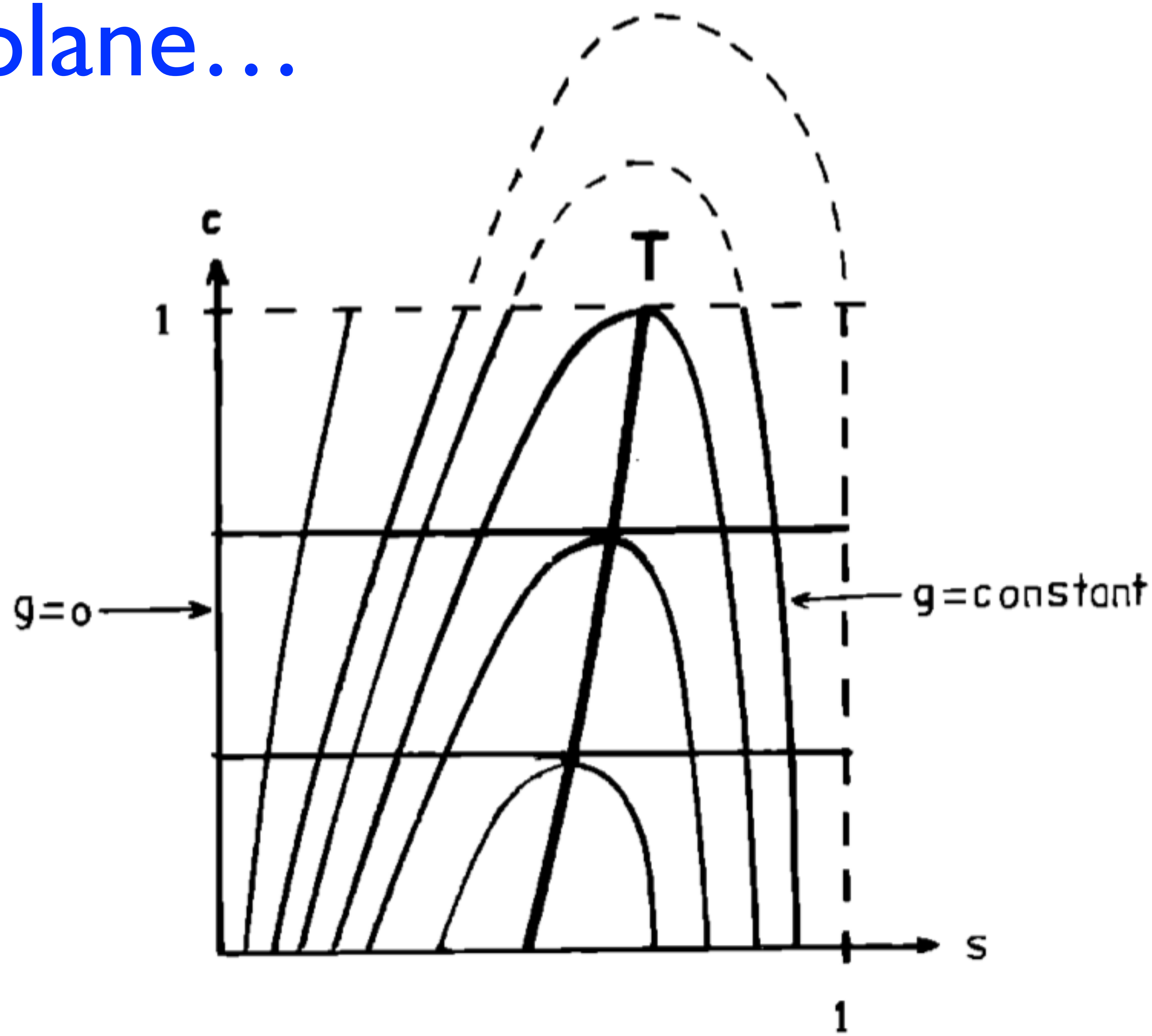
$$f_{uu}(U_*) \neq 0 \quad (< 0)$$

Thus:

$$\boxed{\frac{da}{du}(U_*) = 0 \qquad \frac{d^2a}{du^2}(U_*) = \frac{f_{uu}}{g - f_a} < 0}$$

Generically, standing waves look like c-waves of Polymer Equations…

Line Field (s,c)-plane…

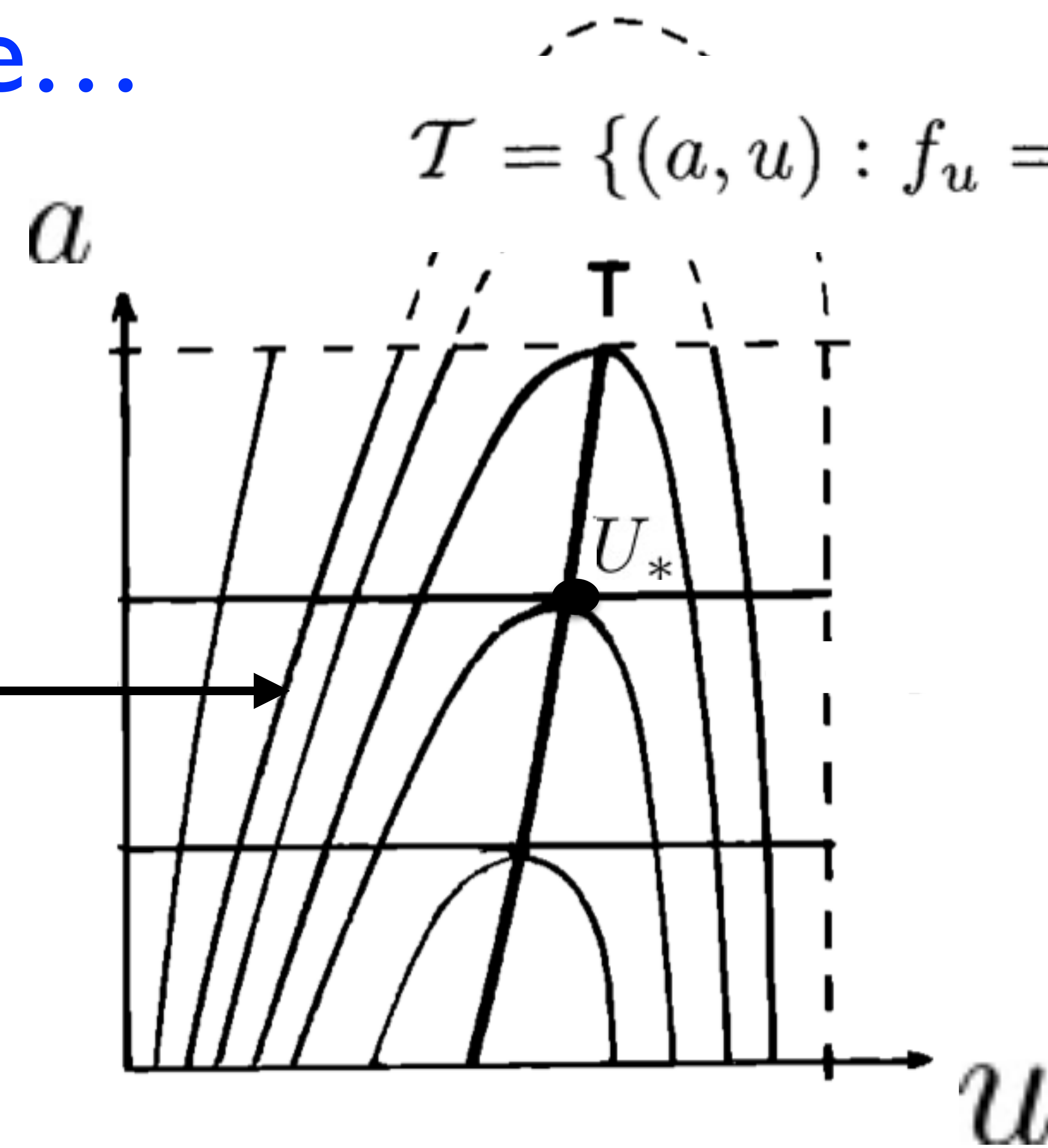


Generically, standing waves look like c-waves of Polymer Equations…

SW's in (a,u)-plane…

Integral curves of

$$\frac{da}{du} = \frac{f_u}{g - f_a}$$



The RP can be solved and the Glimm and Godunov methods converge, subject to an

Entropy Condition…

(E1) Standing waves do not cross T

(E2) The RP minimizes F

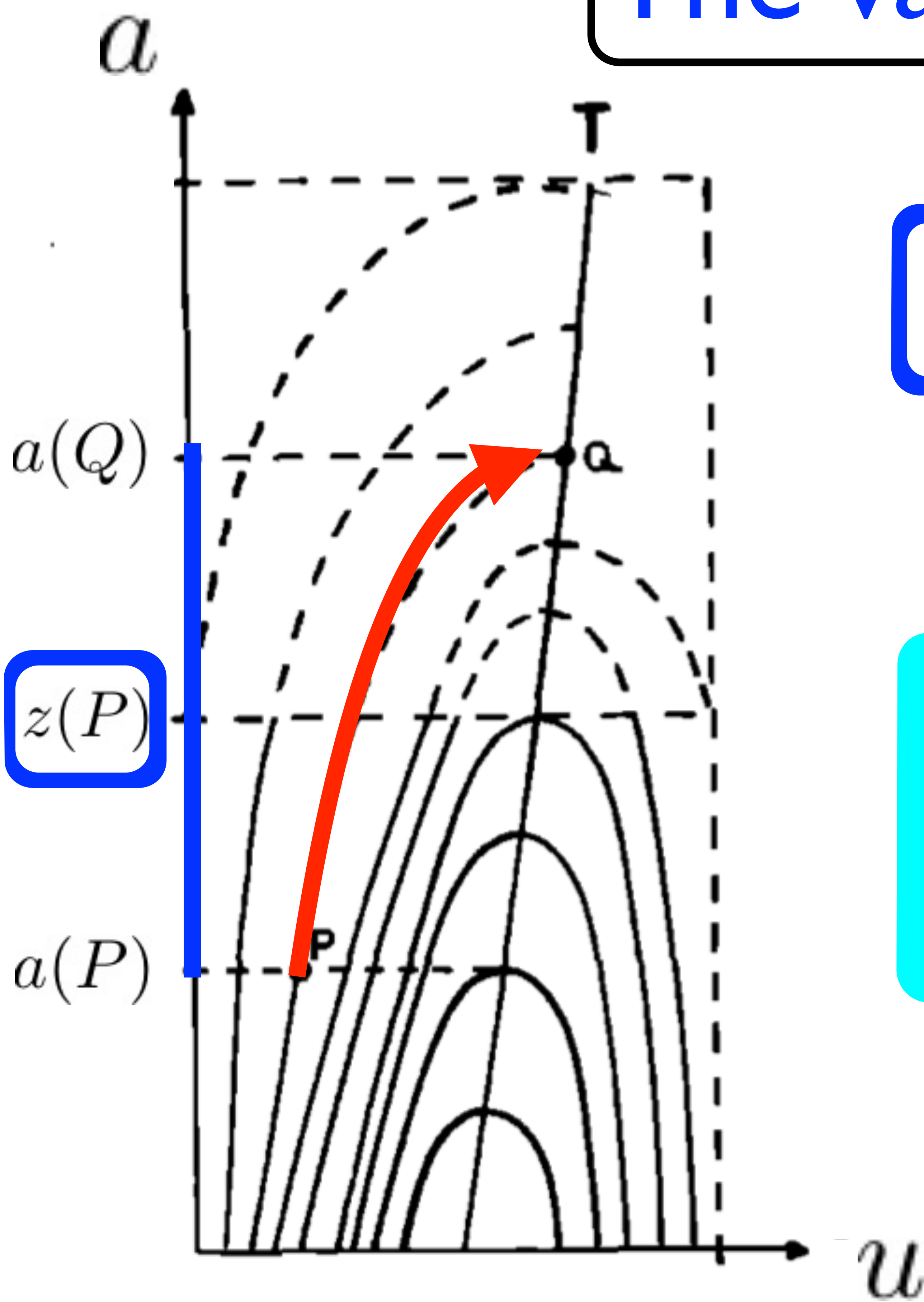
As for Polymer:

$$z(a,u) = \operatorname{sgn}(u - u_{\mathcal{T}})|a - a_{\mathcal{T}}|$$

$$|\gamma| = \begin{cases} |z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a nonlinear wave,} \\ 2|z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a standing wave with } u_R < u_L, \\ 4|z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a standing wave with } u_R > u_L \end{cases}$$

$$F[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|.$$

The variable **z**



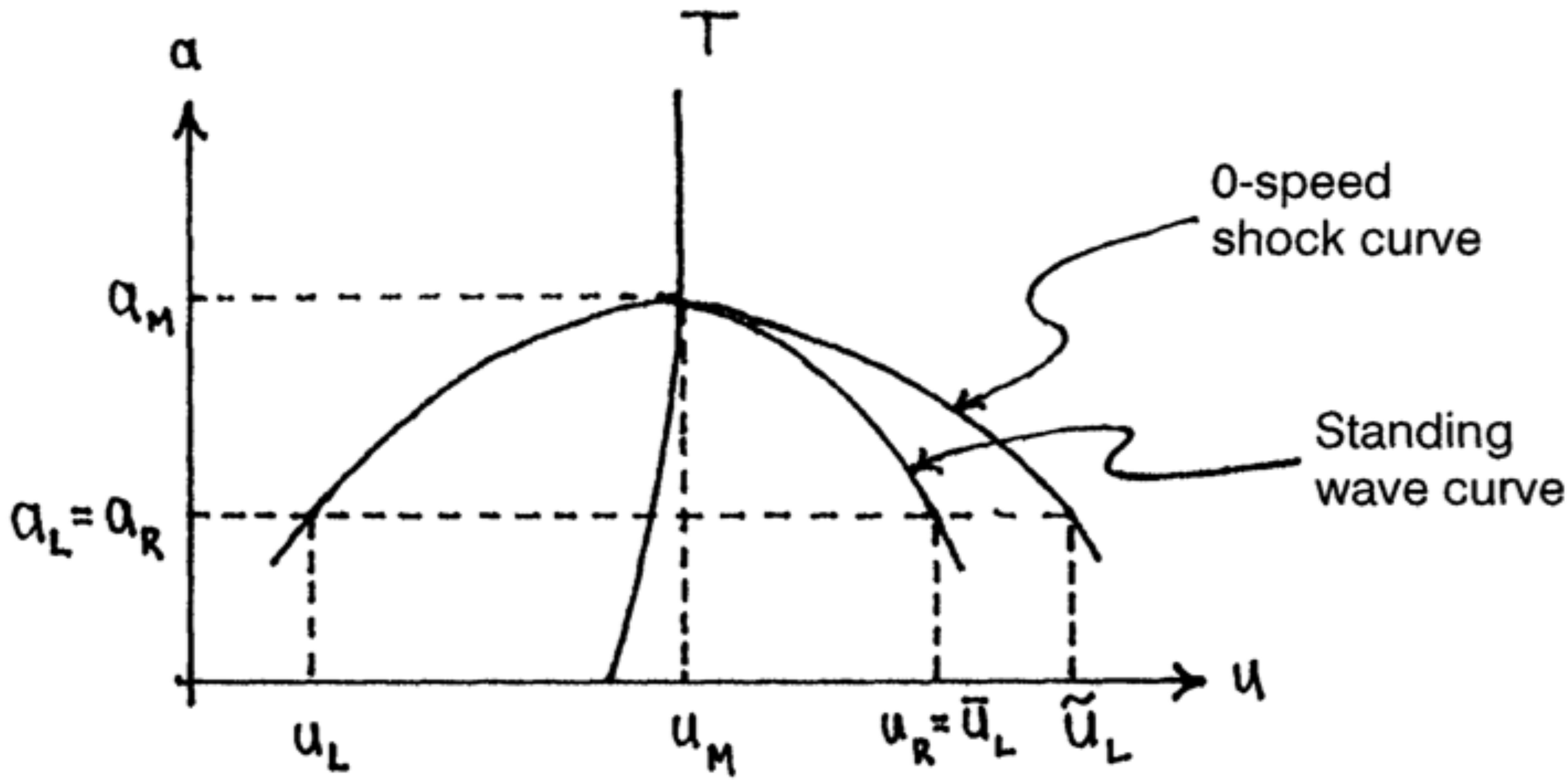
$$|z(P)| = |a(Q) - a(P)|$$

$$z < \mathcal{T} \quad \text{sign } z = +$$

$$z > \mathcal{T} \quad \text{sign } z = -$$

The admissible RP can be presented in four diagrams…

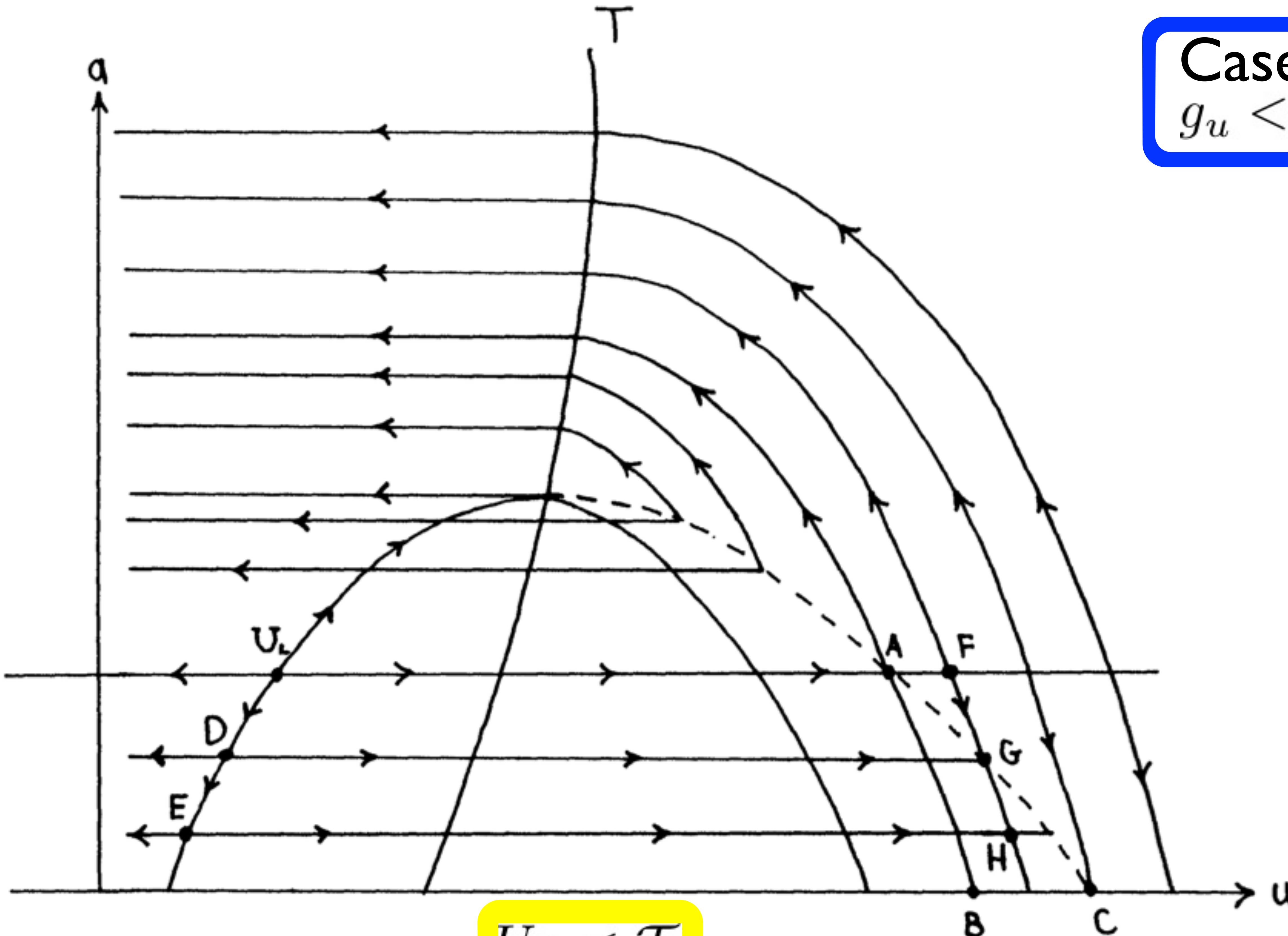
New phenomena because the zero speed shock curve diverges from the standing wave curve…


a
T
0-speed
shock curve
a_M
Standing
wave curve
$a_L = a_R$
u
u_L
u_M
$u_R = \bar{u}_L$
$\tilde{u}_L$

The admissible solution of RP

The admissible solution of RP

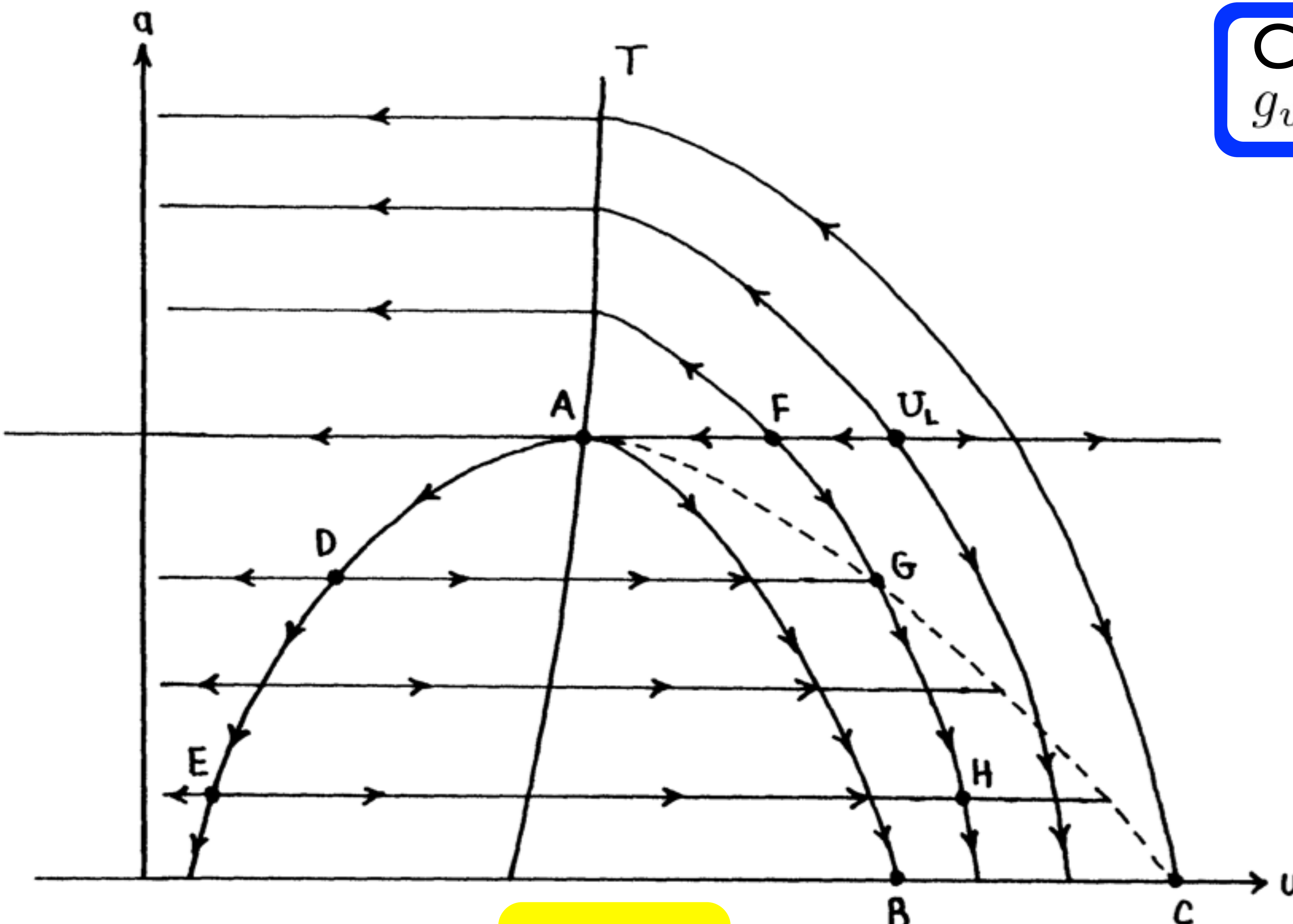
Case
$g_u < 0$



$U_L < T$

The admissible solution of RP

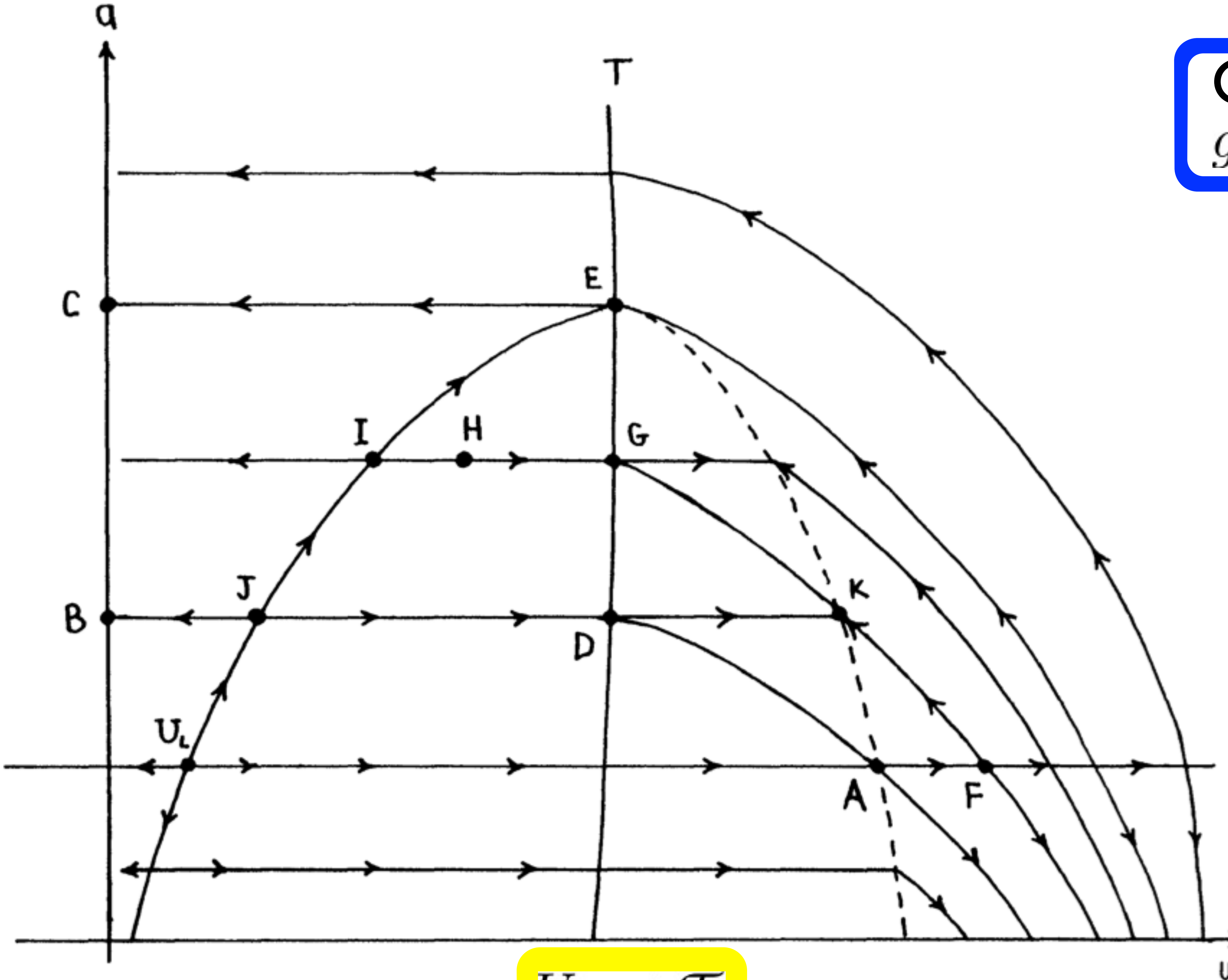
Case
$g_u < 0$



$U_L > T$

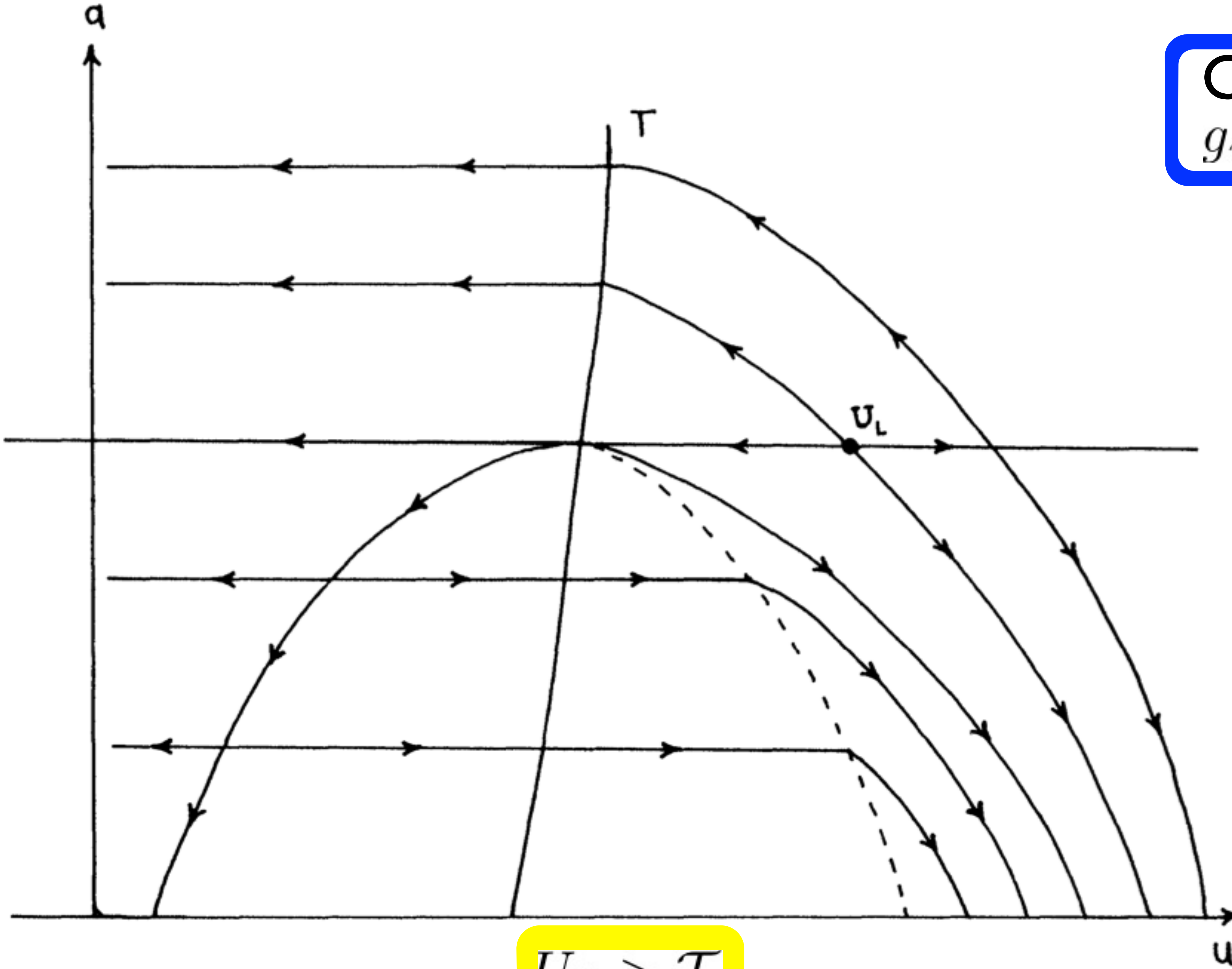
The admissible solution of RP

Case
$g_u > 0$



$U_L < T$

The admissible solution of RP

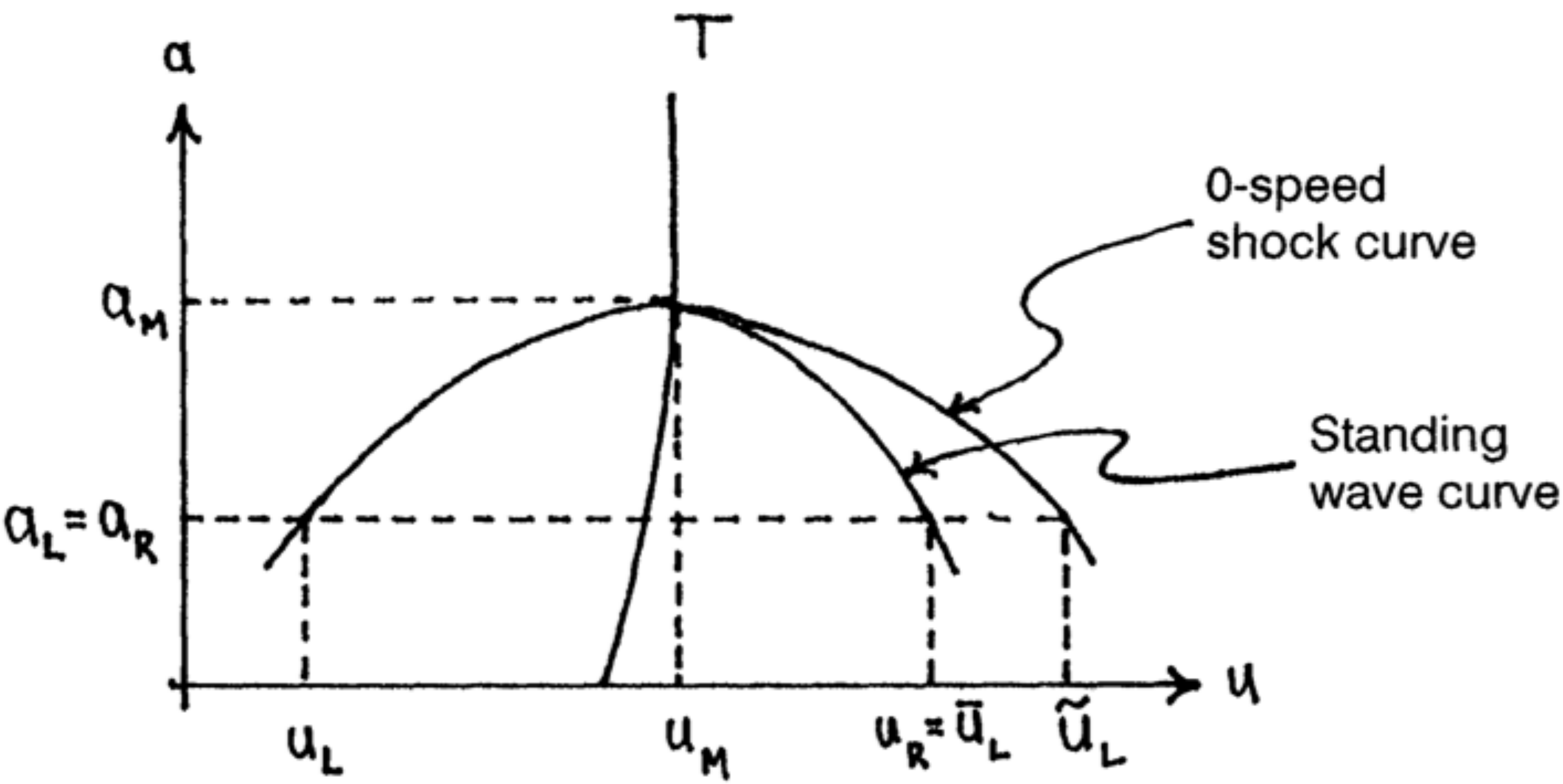


Case
$g_u > \hat{0}$

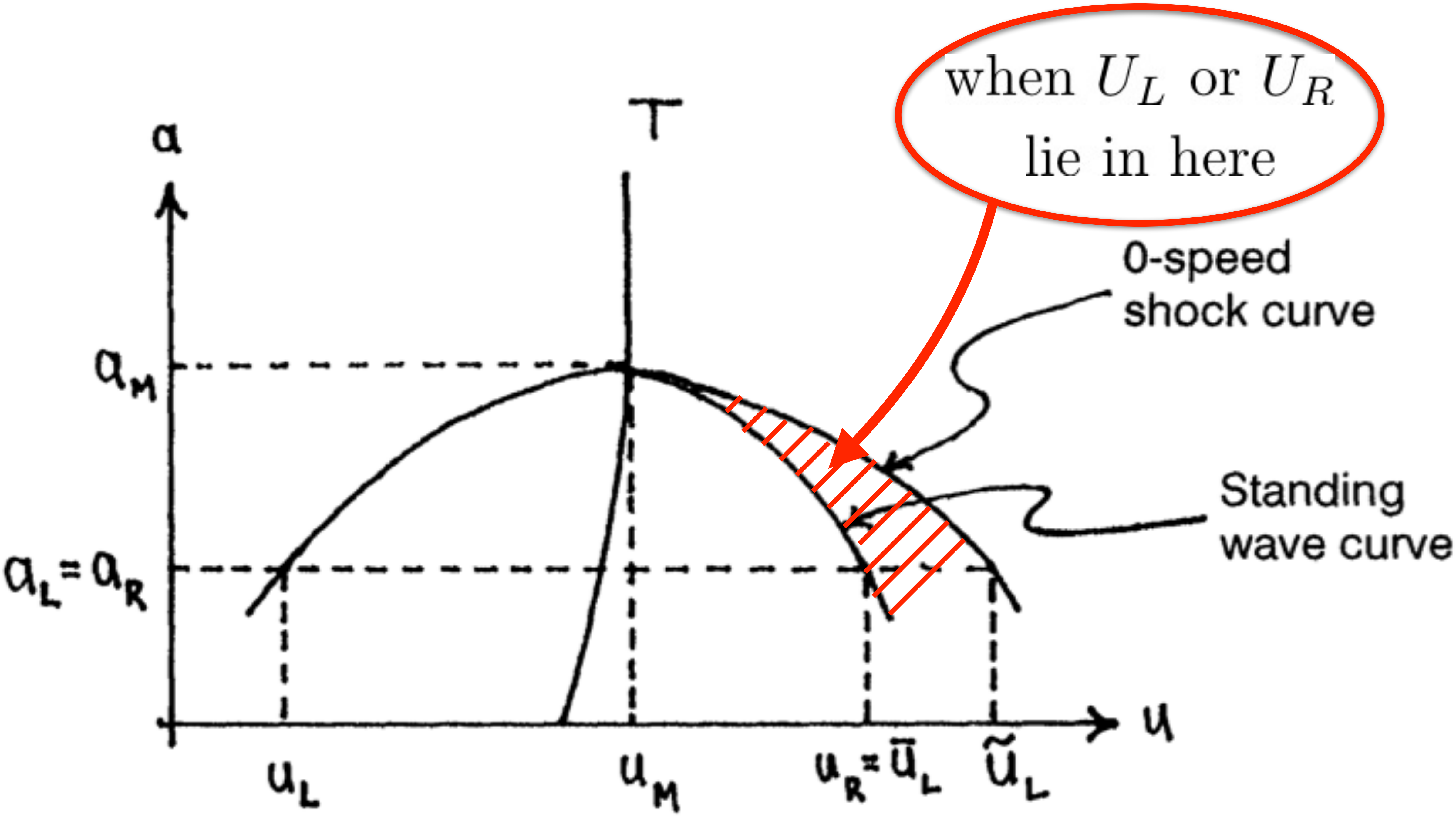
$U_L > \mathcal{T}$

Multiple solutions of RP ➡
Multiple time-asymptotic
wave patters...

Multiple solutions of RP ➡ Multiple time-asymptotic wave patters…

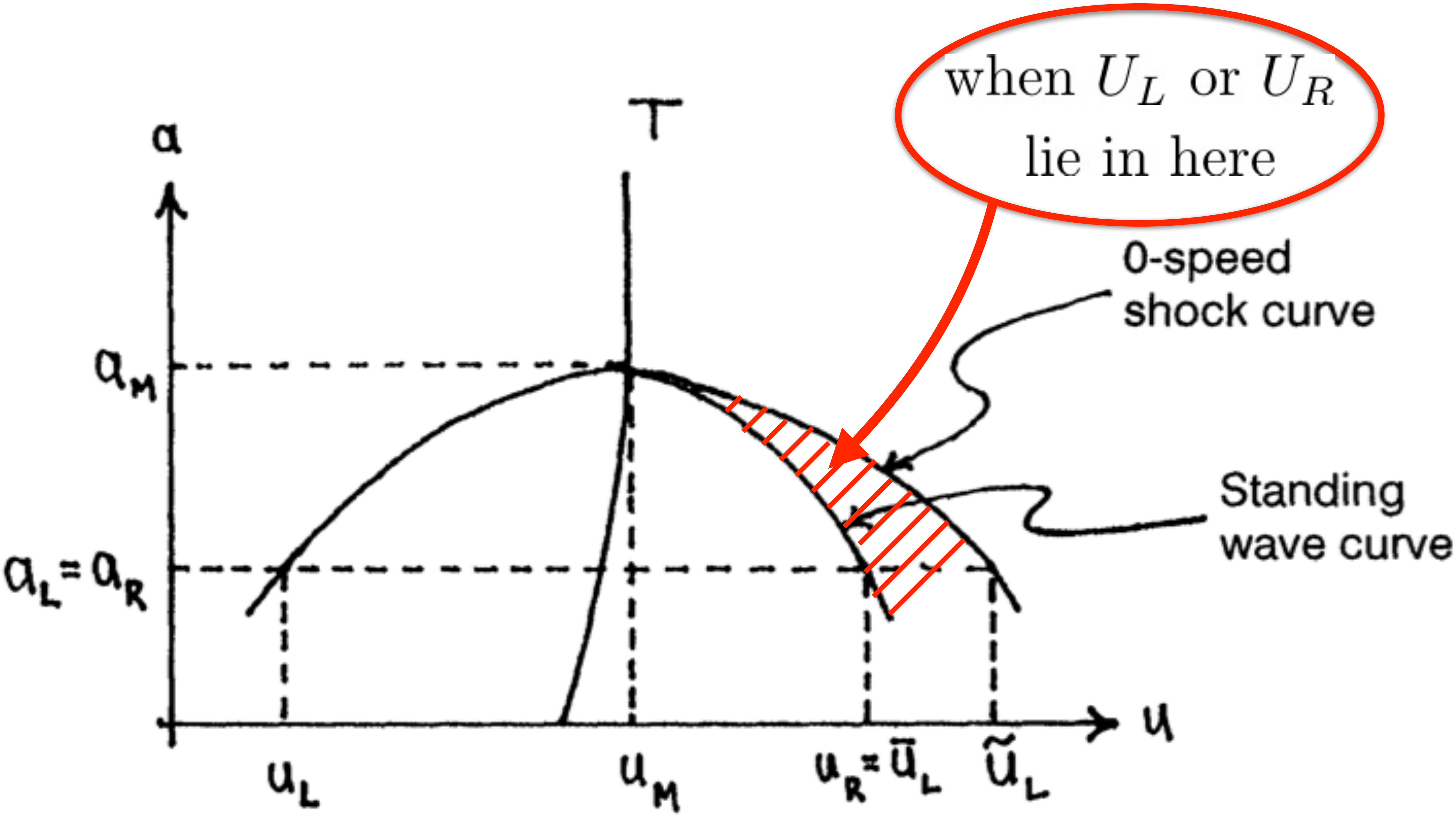


Multiple solutions of RP ➔ Multiple time-asymptotic wave patters...



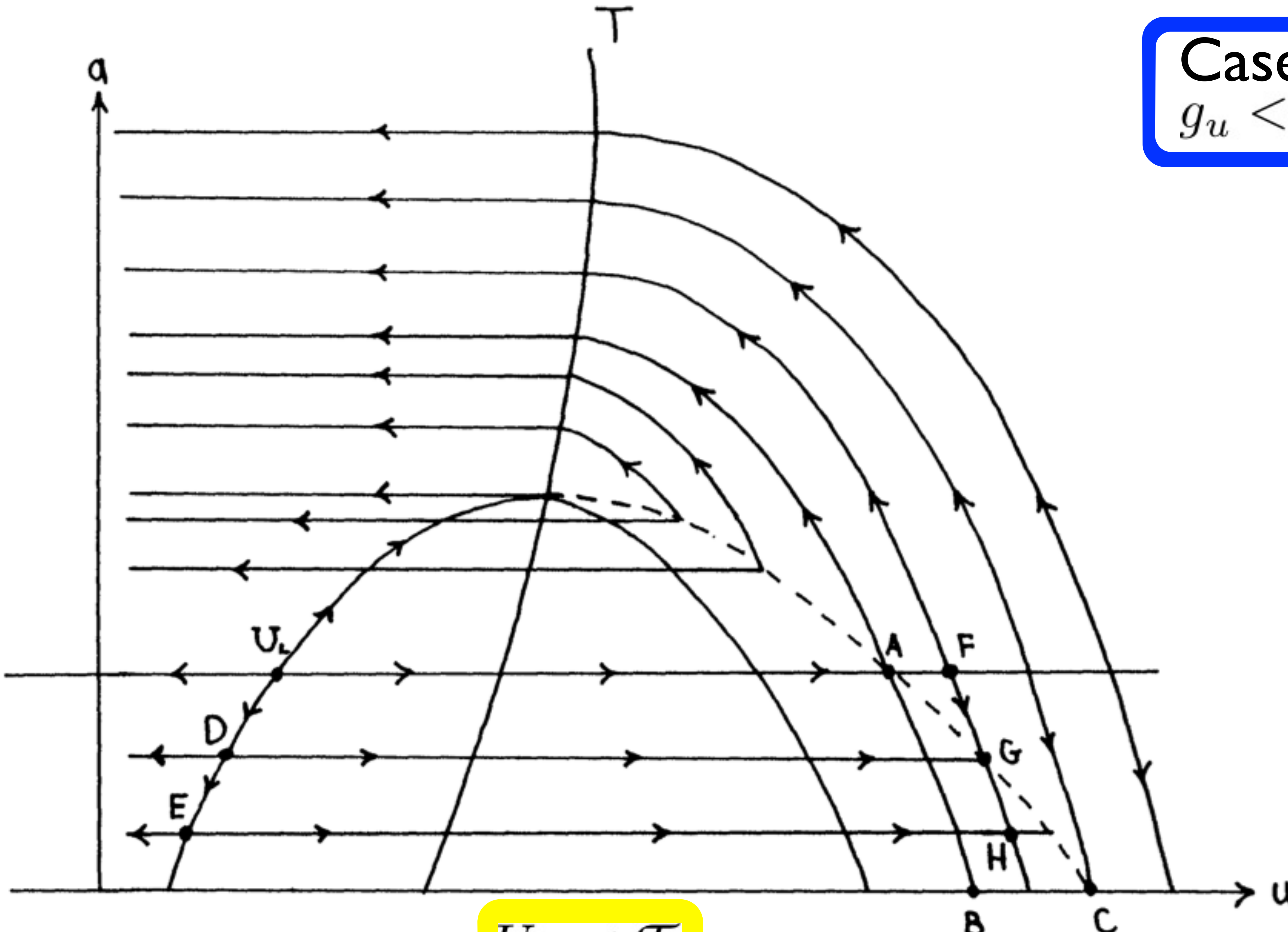
Multiple solutions of RP

For Example:



Multiple solutions of RP $U_R = H$

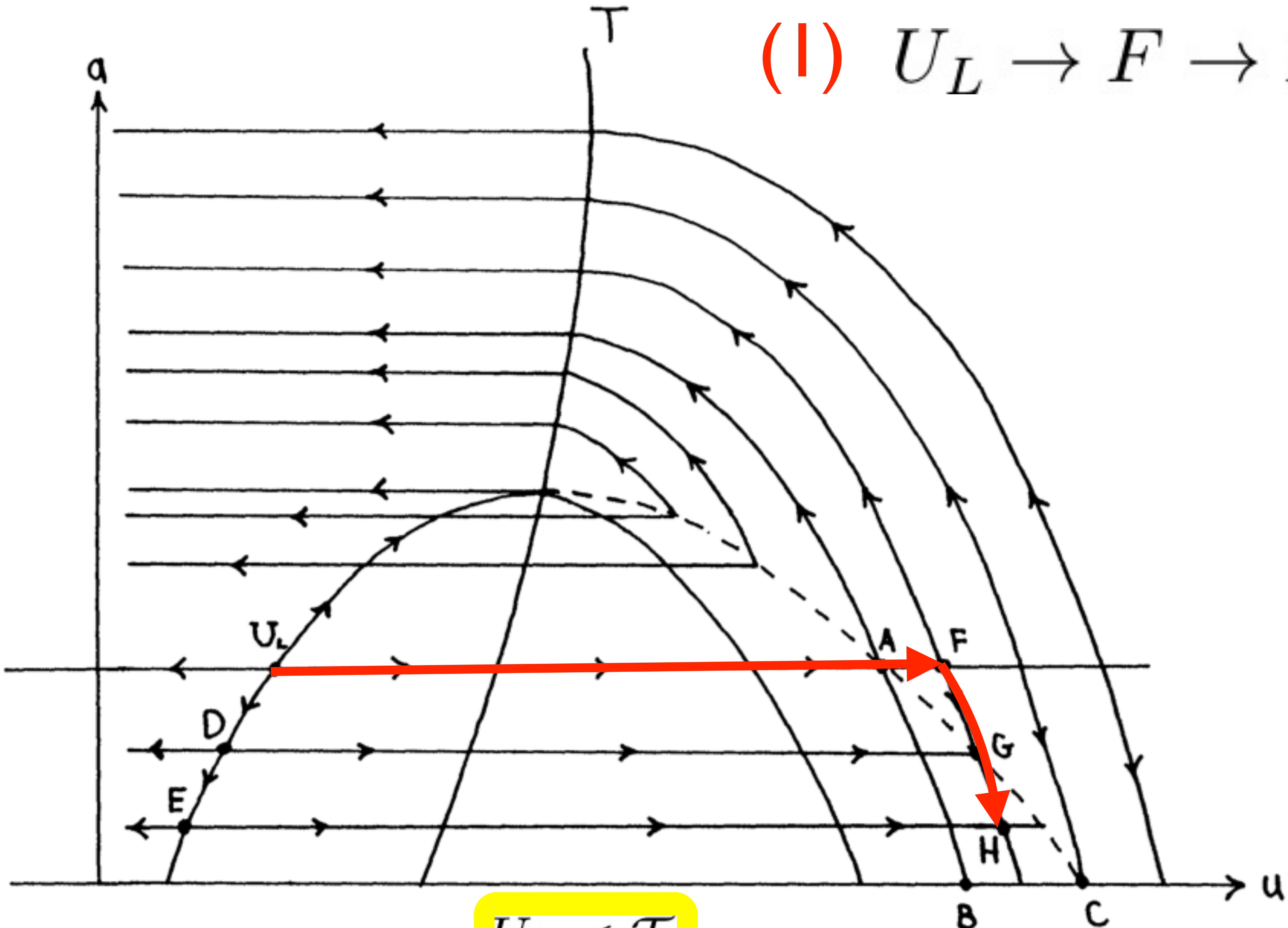
Case
$g_u < 0$



$U_L < T$

Multiple solutions of RP $U_R = H$

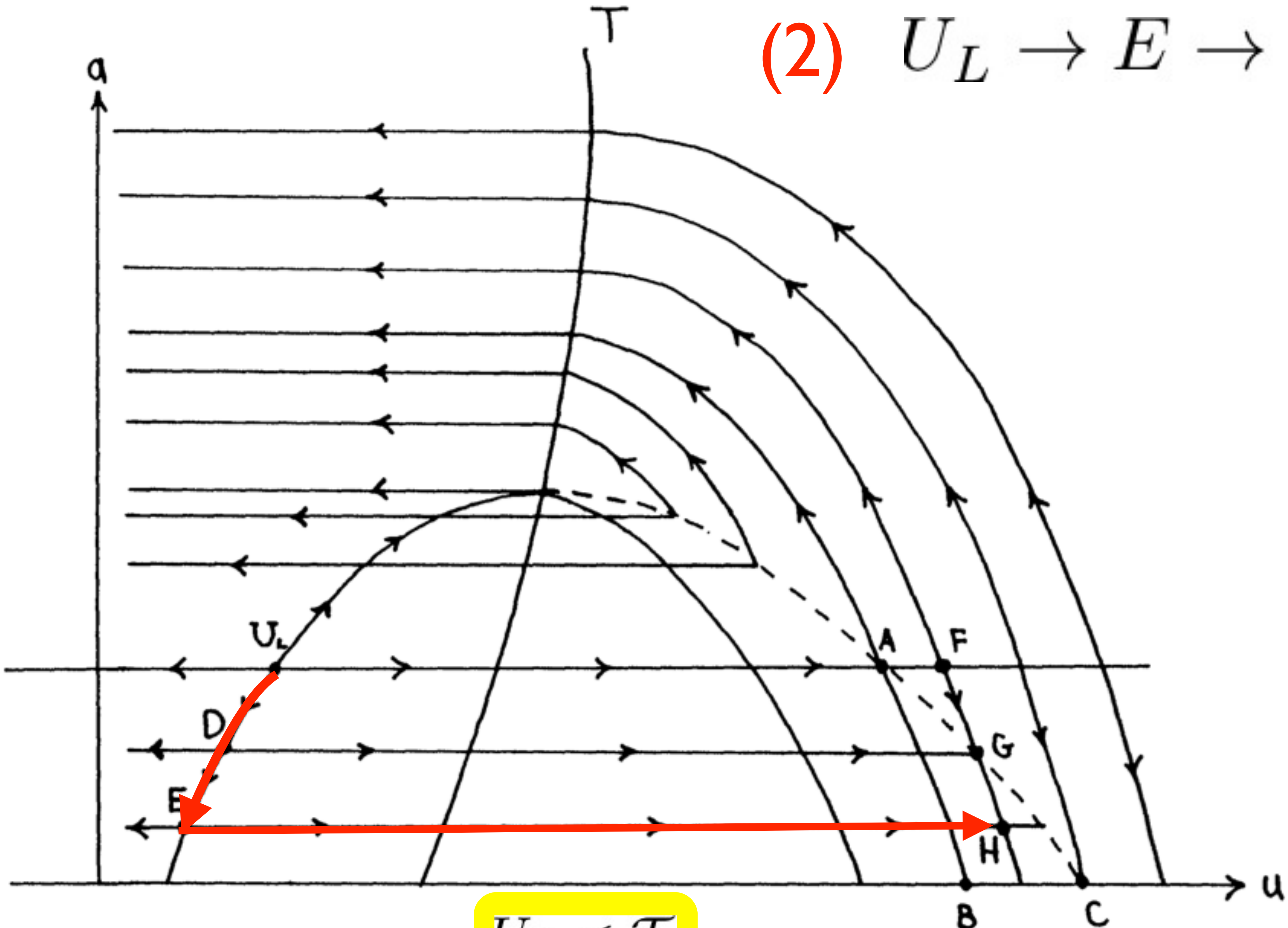
(I) $U_L \to F \to H$



$U_L < T$

Multiple solutions of RP $U_R = H$

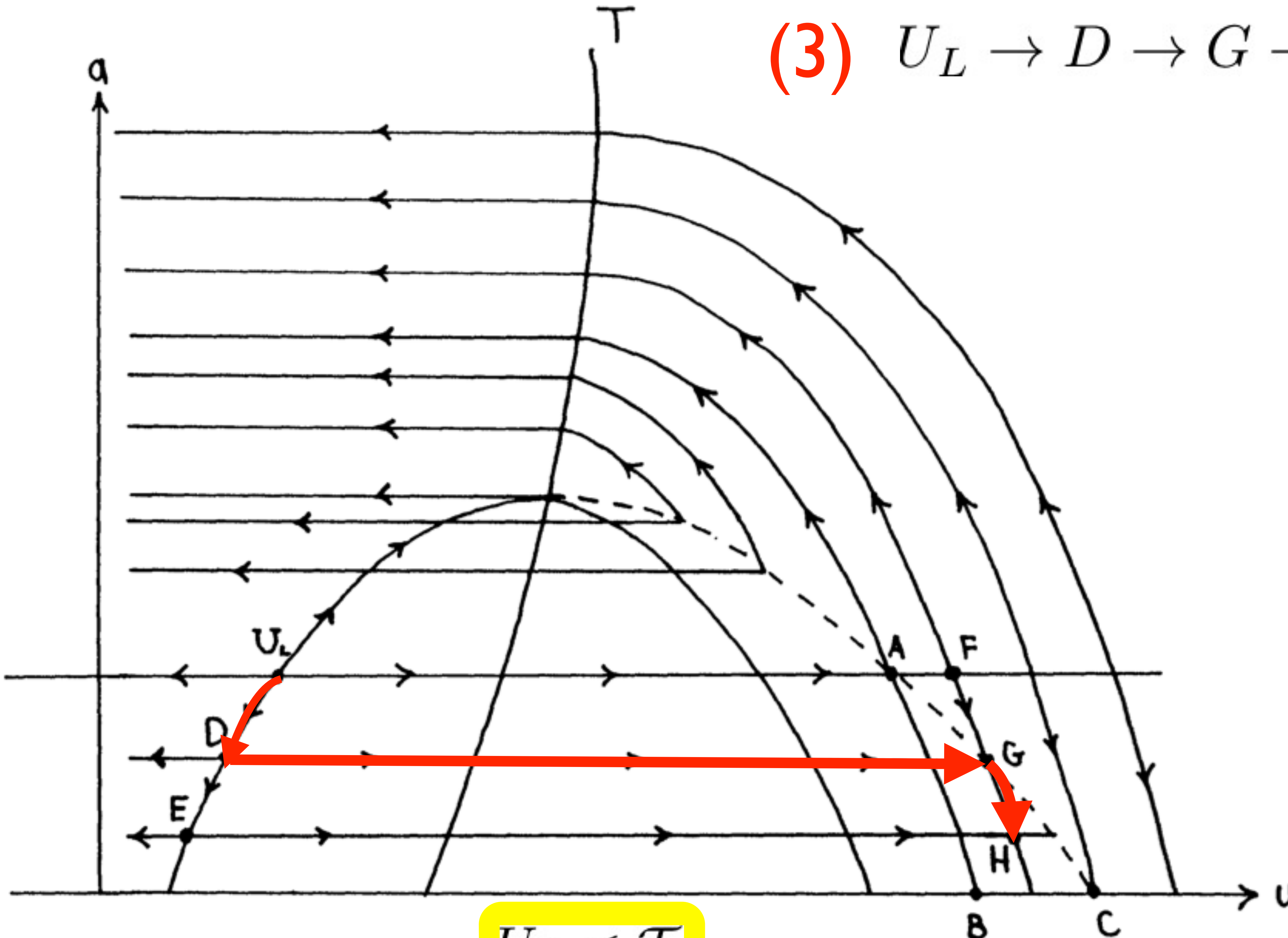
(2) $U_L \to E \to H$



$U_L < T$

Multiple solutions of RP $U_R = H$

(3) $U_L \to D \to G \to H$



$U_L < T$

The Multiple solutions of RP

The Multiple solutions of RP

- Always only 1 or 3 solutions of RP

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- Always only 1 or 3 solutions of RP

- All three solutions have the same F-value

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…so they don't affect Glimm Conv proof

The Multiple solutions of RP

- Always only 1 or 3 solutions of RP
- All three solutions have the same F-value

…so they don't affect Glimm Conv proof

- Multiple solutions do not depend continuously in (x,t)-plane…

Theorem: It doesn't matter which admissible solution you pick, there always is a convergent subsequence for both the Glimm and Godunov methods…

Main Lemma: $\boxed{F(J_2) \leq F(J_1)}$

Conclude: The three solutions of the RP represent distinct time-asymptotic wave patterns to which solutions with the same state left and right states at + and - infinity can converge!

Open Problem: For a smooth duct, find conditions on the initial data which tells which time-asymptotic wave pattern the solution will converge to!!!

…when the left and right states are the same…

My former student
John Hong
(Prof in Taiwan)
Found a Mistake in the proof of convergence of the residual…this led to his doctoral thesis…

He corrected the proof of convergence of the residual…

…and we answered the question: Can we find a bound for the Total Variation of the Conserved Quantities?

SIAM J. APPL. MATH.
Vol. 64, No. 3, pp. 819–857

A BOUND ON THE TOTAL VARIATION OF THE CONSERVED QUANTITIES FOR SOLUTIONS OF A GENERAL RESONANT NONLINEAR BALANCE LAW*

JOHN HONG† AND BLAKE TEMPLE‡

Abstract. We introduce a new potential interaction functional and use it to define a new Glimm-type functional that bounds the total variation of the conserved quantities at time $t > 0$ by the total variation at time $t = 0+$ in Glimm approximate solutions of a general resonant nonlinear balance law.

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Pages 819-857

Consider first the proof of convergence of the Residual…

The Residual:

$$R(a,u,\phi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{u\phi_t + f\phi_x + a'g\phi\}\, dxdt$$

$$+ \int_{-\infty}^{+\infty} u_0(x)\phi(x,0)dx$$

The Residual:

$$R(a,u,\phi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{u\phi_t + f\phi_x + a'g\phi\}\,dx dt$$

$$+ \int_{-\infty}^{+\infty} u_0(x)\phi(x,0)dx$$

Weakly imposes the initial data…

The Residual:

$$R(a,u,\phi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{u\phi_t + f\phi_x + a' g\phi\}\, dx dt$$

$$+ \underbrace{\int_{-\infty}^{+\infty} \cancel{u_0(x)\phi(x,0)}dx}$$

Ignore this...

The Residual:

$$R(a, u, \phi) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ u\phi_t + f\phi_x + a' g\phi \right\} dxdt$$

The Residual:

$$R(a, u, \phi) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{u\phi_t + f\phi_x + a' g\phi\}\, dx dt$$

Put in the Glimm approximates with discontinuous a and u...

$$(a, u) = (a_{\Delta x}, u_{\Delta x})$$

The Residual:

$$R(a_{\Delta x}, u_{\Delta x}, \phi) =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{u_{\Delta x}\phi_t + f(u_{\Delta x})\phi_x + a'_{\Delta x} g(a_{\Delta x}, u_{\Delta x})\}\, dx dt$$

The Residual:

$$R(a_{\Delta x}, u_{\Delta x}, \phi) =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ u_{\Delta x} \phi_t + f(u_{\Delta x}) \phi_x + \underbrace{a'_{\Delta x} g(a_{\Delta x}, u_{\Delta x})} \right\} dx dt$$

Delta-Function
times
Discontinuous

The Residual:

$$R(a_{\Delta x}, u_{\Delta x}, \phi) =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{u_{\Delta x} \phi_t + f(u_{\Delta x}) \phi_x + \underline{a'_{\Delta x} g(a_{\Delta x}, u_{\Delta x})}\} \, dx dt$$

Since you can't multiply distributions, these aren't approximations in the distributional sense!

The Residual:

To prove convergence of Residual…

$$R(a_{\Delta x}, u_{\Delta x}, \phi) \xrightarrow[\Delta x \to 0]{} 0$$

…requires 3 small parameters…

…and requires $a(x)$ be

Lipschitz Continuous

The three small parameters:

(1) A parameter to smooth out $g(a_{\Delta x}, u_{\Delta x})$

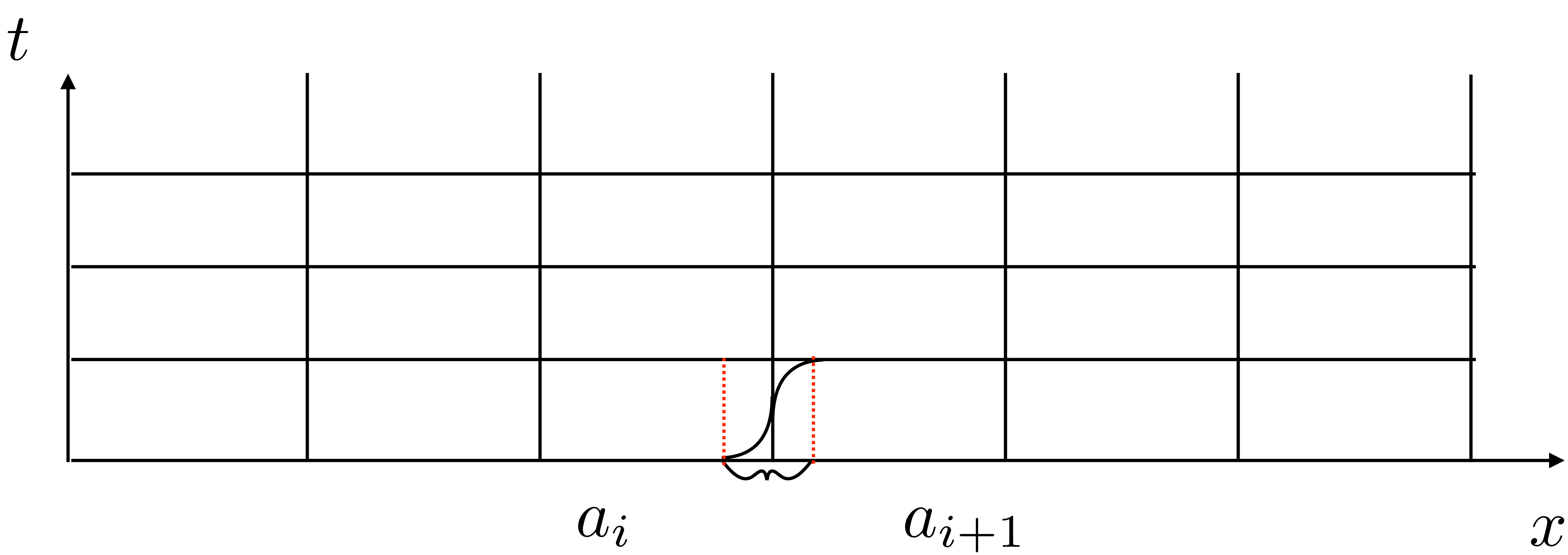
$$g(a_{\Delta x}, u_{\Delta x})_\delta = g(a_{\Delta x}, u_{\Delta x}) * \psi_\delta$$

$$\psi_\delta = \frac{1}{\delta^2}\psi\left(\frac{x}{\delta}, \frac{t}{\delta}\right)$$

...standard convolution kernel

The three small parameters:

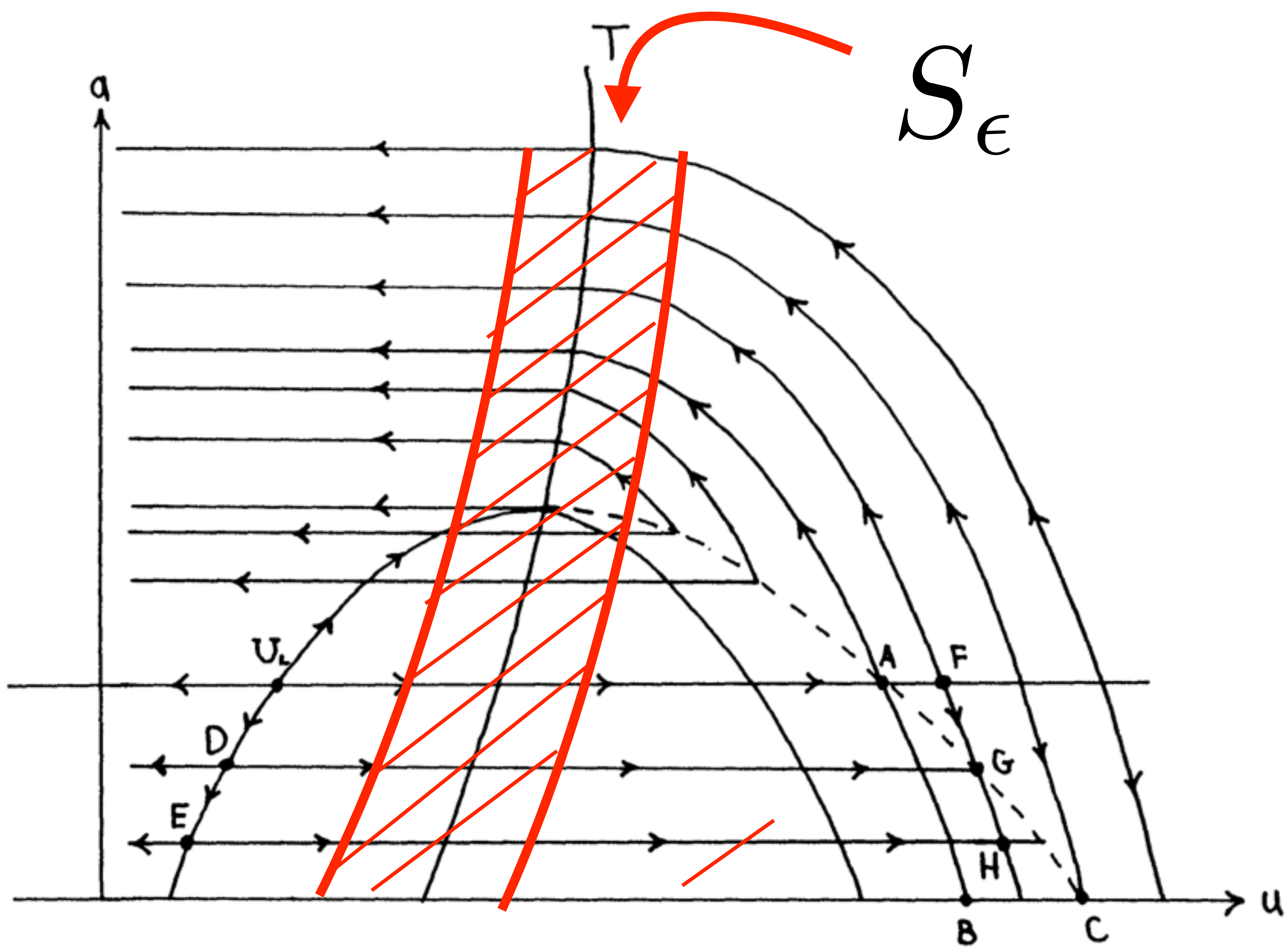
(2) A parameter to smooth out $a_{\Delta x}$



$\hat{\epsilon}\,\Delta x$

The three small parameters:

(3) Estimate the residual differently in S_ϵ



The Residual:

Modifying the argument you can get:

$$
\begin{aligned}
\int_{\mathcal{A}} R(a_{\Delta x}, u_{\Delta x}, \phi)^2 d\theta \\
&\leq O(1)\left\{\hat{\epsilon} + \left[\hat{K}(\hat{\epsilon})\Delta x\right]_1 + \left[\epsilon(\hat{\epsilon} + \hat{K}(\hat{\epsilon})\right]_2\right\} \\
&\quad + O(1)\left\{\left[\epsilon + \int\int_E \left|(g \cdot U^\epsilon)_\delta - g(U^\epsilon)\right| \, dx\, dt\right.\right. \\
&\qquad\qquad \left. + \int\int_E \left|(g \cdot U^\epsilon)_\delta - g(U)\right| \, dx\, dt\right]_3 \\
&\qquad\qquad\qquad \left. + \left[o(\Delta x)K(\epsilon) + \frac{\Delta x}{\delta}\right]_4\right\}^2
\end{aligned}
$$

The point is that you can make all of terms independent of Δx small by $\epsilon, \hat{\epsilon}, \delta << 1$, then choose $\Delta x << 1$ small to obtain...

$$\int_{\mathcal{A}} R(a_{\Delta x}, u_{\Delta x}, \phi)^2 d\theta < \tau$$

for any $\tau > 0$, implying

$$\int_{\mathcal{A}} R(a_{\Delta x}, u_{\Delta x}, \phi)^2 d\theta \underset{\Delta x \to 0}{\longrightarrow} 0$$

That is,

$$
\begin{aligned}
\int_{\mathcal{A}} R(a_{\Delta x}, u_{\Delta x}, \phi)^2 d\theta
&\le O(1)\left\{\hat{\epsilon} + \left[\hat{K}(\hat{\epsilon})\Delta x\right]_1 + \left[\epsilon(\hat{\epsilon} + \hat{K}(\hat{\epsilon})\right]_2\right\} \\
&\quad + O(1)\left\{\left[\epsilon + \int\int_E |(g\cdot U^{\epsilon})_\delta - g(U^{\epsilon})|\, dx\, dt \right.\right. \\
&\qquad \left. + \int\int_E |(g\cdot U^{\epsilon})_\delta - g(U)|\, dx\, dt\right]_3 \\
&\qquad \left. + \left[o(\Delta x)K(\epsilon) + \frac{\Delta x}{\delta}\right]_4\right\}^2
\end{aligned}
$$

Implies:

$$
\int_{\mathcal{A}} R(a_{\Delta x}, u_{\Delta x}, \phi)^2 d\theta \underset{\Delta x \to 0}{\longrightarrow} 0
$$

Key Step: Integration by Parts produces the $\dfrac{\Delta x}{\delta}$ term...

$$\left| \int \int_{t \geq 0} \frac{d}{dx}(a_\epsilon - a)\,(g \cdot U^{\epsilon}_{\Delta x})_\delta\, \phi\, dx\, dt \right|$$

$$\leq O(1) \int \int_{t \geq 0} |a_\epsilon - a| \left| \frac{d}{dx}(g \cdot U^{\epsilon})_\delta \right| \, dx\, dt \leq O(1) \frac{\Delta x}{\delta}.$$

Again we can conclude…

Theorem: Off a set of zero measure $N \subset A$, the residual tends to zero.

That is: For almost every choice of sampling, the convergent Glimm Approximate Solutions converge to a weak solution of the equations…

Using this correct convergence theorem, Hong was able to reduce Glimm's theorem for inhomogeneous systems to Glimm's original argument by treating the source term as a contact discontinuity field…

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An extension of Glimm's method to inhomogeneous strictly hyperbolic systems of conservation laws by "weaker than weak" solutions of the Riemann problem

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Available online 19 August 2005

THEOREM: Any $n \times n$ system of the form

$$u_t + f(a,u)_x = a' g(a,u)$$

satisfies Glimm's theorem as an $(n+1) \times (n+1)$ augmented system

$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = \begin{pmatrix} a' g(a,u) \\ 0 \end{pmatrix}$$

By treating the source term as a zero speed contact field!

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By treating the source term as a zero speed contact field…

(Proof for Non-resonant case only!)

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By treating the source term as a zero speed contact field...

(Resonant case is completely open!)

A local Glimm existence theory in a neighborhood of a point of resonance would solve a long open problem…

…transonic flow in a variable area duct…

Special Case: Transonic flow in a variable area duct...

$$\rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\,\rho u,$$

$$(\rho u)_t + (\rho u^2 + p)_x = -\frac{a'(x)}{a(x)}\,\rho u^2,$$

$$(\rho E)_t + (\rho E u + p u)_x = -\frac{a'(x)}{a(x)}\,(\rho E u + p u),$$

For systems: A Glimm type bound would require a bound on the total variation of the conserved quantities…it appears unlikely that for systems, wave interactions can be controlled by a singular transformation…

Such a bound would be required to extent the scalar results to systems like transonic flow in a duct…

To start, we set out to establish a bound on the total variation of the conserved variables for the scalar resonant case…

We accomplished this in…

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We discovered a Complicated Potential Interaction Functional that works…

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Pages 819-857

The Potential Interaction Functional
that bounds the
Total Variation of the Conserved Variables
for
Generic Resonance
between a
Nonlinear Wave Family
and a
Stationary Source…

That is: Find an functional F such that

$$\boxed{Var_{a,u}(J) \leq Const.\, F(J)}$$

$$\boxed{F(J_2) \leq F(J_1)}$$

…it is constructed as follows…

Start by defining
preliminary wave strengths…

- Replace the singular variable z by w which bounds the total variation in u…

$$w(a,u) = \begin{cases} u - u_T & \text{if } u < T, \\ u_T - \bar{u} & \text{if } u > T \end{cases}$$

- Impose the 1-2-4 weights on wave strengths

$$|\gamma|_w^* = \begin{cases} |w(U_R) - w(U_L)| & \text{if } \gamma \text{ is a nonlinear wave,} \\ 2|w(U_R) - w(U_L)| & \text{if } \gamma \text{ is a weak standing wave,} \\ 4|w(U_R) - w(U_L)| & \text{if } \gamma \text{ is a strong standing wave.} \end{cases}$$

- Use these to define the preliminary linear functional~TV in u…

$$L_w^*[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_w^*.$$

L_w^* is not continuous across multiple solutions of RP

- Adjust the wave strengths to make L continuous across multiple RP solns…

- Adjust the wave strengths to make L continuous across multiple RP solns…

$$|\gamma|_w = \begin{cases} |P(\gamma)|^*_w + \delta(\gamma) & \text{if } \gamma \text{ is a standing wave on the right of } \mathcal{T}, \\ |\gamma|^*_w & \text{otherwise,} \end{cases}$$

- Adjust the wave strengths to make L continuous across multiple RP solns…

$$|\gamma|_w = \begin{cases} |P(\gamma)|_w^* + \delta(\gamma) & \text{if } \gamma \text{ is a standing wave on the right of } \mathcal{T}, \\ |\gamma|_w^* & \text{otherwise,} \end{cases}$$

1. Correct to make triple RP's continuous

- Adjust the wave strengths to make L continuous across multiple RP solns…

$$|\gamma|_w = \begin{cases} \underbrace{|P(\gamma)|_w^*}_{} + \underbrace{\delta(\gamma)}_{} & \text{if } \gamma \text{ is a standing wave on the right of } \mathcal{T}, \\ |\gamma|_w^* & \text{otherwise,} \end{cases}$$

1. Correct to make triple RP's continuous

2. Correct new RP discontinuities Created by 1.

- Adjust the wave strengths to make L continuous across multiple RP solns…

$$|\gamma|_w = \begin{cases} |P(\gamma)|_w^* + \delta(\gamma) & \text{if } \gamma \text{ is a standing wave on the right of } \mathcal{T}, \\ |\gamma|_w^* & \text{otherwise,} \end{cases}$$

1. Correct to make triple RP's continuous

2. Correct new RP discontinuities Created by 1.

- Define: $L_w[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_w.$

- Adjust the wave strengths to make L continuous across multiple RP solns...

$$|\gamma|_w = \begin{cases} |P(\gamma)|^*_w + \delta(\gamma) & \text{if } \gamma \text{ is a standing wave on the right of } \mathcal{T}, \\ |\gamma|^*_w & \text{otherwise,} \end{cases}$$

1. Correct to make triple RP's continuous

2. Correct new RP discontinuities Created by 1.

$$L_w[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_w.$$

- Conclude: L_w is continuous wrt RP's

- Show that L_w only increases due to the interaction of rarefaction waves and standing waves…
- Define a potential for the interaction of rarefaction waves and standing waves…

$$P(J) = \sum_{(\alpha,\beta)\in App(J)} d(\gamma_0^\alpha, \gamma_r^\beta),$$

- Define the Nonlinear Functional...

$$F(J) = L_w(J) + P(J),$$

- In a case by case study, prove...

THEOREM: If J_2 is a successor of J_1, then

$$F(J_2) \leq F(J_1)$$

- Conclude: If the F-value of the waves at time t+ is finite, then

$$TV\,\{u(\cdot,t)\} \leq F(J_0)$$

- Corollary: If there are only shocks and standing waves initially, then the total variation of the solution at time t>0 is bounded by

  4x(TV of the waves at time zero).

Theorem: A sharp bound on the total variation of the conserved quantities in resonant scalar balance laws of the form:

$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = \begin{pmatrix} a'g(a,u) \\ 0 \end{pmatrix}$$

Theorem: A sharp bound on the total variation of the conserved quantities in resonant scalar balance laws of the form:

$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = \begin{pmatrix} a'g(a,u) \\ 0 \end{pmatrix}$$

(Only works for SCALAR functions f)

- Open Problem: Prove that a similar total variation bound holds in a neighborhood of a point where

$$\lambda_i = 0$$

in a resonant SYSTEM of form

$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} f(a,u) \\ 0 \end{pmatrix}_x = \begin{pmatrix} a' g(a,u) \\ 0 \end{pmatrix}$$

Difficult problem of transonic flow!!!

- References: Many people have addressed the transonic flow problem from varying perspectives..

Eg. TP Liu, Keyfitz & Kranzer, N Risebro, H Holden, P Lefloch…

Many more!