The Regularity Transformation Equations:

An Elliptic Mechanism for Smoothing Gravitational Metrics in General Relativity

(Dedicated to Joel Smoller)

Blake Temple
University of Michigan, UC-Davis
July 9, 2019

All Joint Work With: Moritz Reintjes

Starting Point:

My work with

Joel Smoller

CHAPTER 11

Solving the Einstein Equations by Lipschitz Continuous Metrics: Shock Waves in General Relativity

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[1] Supported in part by NSF Applied Mathematics Grant Number DMS-89-05205 and by the Institute of Theoret-

HANDBOOK OF MATHEMATICAL FLUID DYNAMICS, VOLUME II
Edited by S.J. Friedlander and D. Serre

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The RT-equations determine coordinate transformations which smooth a gravitational metric to optimal regularity…

The RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

$$Curl(J) \equiv \partial_j J_i^\mu - \partial_i J_j^\mu = 0 \ \ \text{on } \partial\Omega, \tag{5}$$

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\end{aligned}
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The 4-d coordinate Laplacian

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- We prove: existence of solutions to the RT-equations is equivalent to the existence of a coordinate transformation sufficient to smooth a crinkled map of spacetime to optimal metric regularity in General Relativity.

  The existence of such coordinate transformations rules out ``regularity singularities" in General Relativity.

Statement of Results

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(Moritz Rientjes will discuss details in his poster session afterward!)

- Our first existence theorem for the RT-equations is based on elliptic regularity in L^p–spaces…the result:

THEOREM: If a connection and its curvature tensor are both in, $W^{m,p}, m \geq 1, p > n$ then there always exists a coordinate transformation with Jacobian in $W^{m+1,p}$, such that in the new coordinate system, the connection is in $W^{m+1,p}$

- Our new result: (work in progress):

THEOREM: If $\Gamma, Riem(\Gamma) \in L^{\infty}$ in coordinate system x, then there always exists a coordinate transformation $x \to y$ such that in the new coordinate system,

$$\Gamma \in W^{1,p},\ \ p > n\,.$$

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Conclude: GR shock wave solutions are one derivative more regular than they appear to be in the coordinate system in which they are constructed…

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Conclude: This holds for vacuum, non-vacuum and GR shock wave solutions, without any symmetry or technical assumptions whatsoever…

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Corollary: GR shock wave solutions are nonsingular solutions…

- This tells us that we can solve the Einstein equations in coordinate systems in which the equations are simpler and solution metrics are one order below optimal, and still guarantee the existence of other coordinate systems in which the metric exhibits optimal regularity, i.e., two derivatives above its curvature tensor.

This tells us that if the metric loses a derivative relative to the curvature in a numerical simulation, it is always a failure of the coordinate system, and not a geometrical property of the spacetime…

- Comparison with other results:

Prior results regarding the regularity of solutions of Einstein's equations were derived from estimates for nonlinear wave equations in a 3+1 formulation of General Relativity.

For us the boost in regularity is derived from Jacobians which solve 4-d elliptic equations tailored to the cancelation of derivatives at leading order…

- Comparison with other results:

Cf Harmonic Coordinates:

Euclidean Case: $\Delta g_{ij} = R_{ij}$

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Cf Harmonic Coordinates:

Lorentzian Case: $\Box g_{ij} = R_{ij}$

(Regularity of **g** is derived from the boundary)

Optimal regularity of **g** is difficult problem…

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In the 3+1 setting, spacetime regularity is derived from the induced metric and induced second fundamental form on Cauchy hyper-surfaces.

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In the 3+1 setting, spacetime regularity is derived from the induced metric and induced second fundamental form on Cauchy hyper-surfaces.

But the second fundamental form is no more regular than the curvature for non-optimal connections.

- Comparison with other results:

Conclude: To deduce our theorem for non-optimal solutions from the 3+1 framework, the metric and induced second fundamental form needs to be regularized on the surface.

We gave up on the 3+1 approach early on because we didn't know how to do this.

- Comparison with other results:

Most work in the 3+1 framework applies to vacuum spacetimes $R_{ij} = 0$

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Our results based on the Regularity Transformation equations apply to vacuum and non-vacuum spacetimes alike, with no symmetry assumptions.

- Comparison with other results:

For example, let's compare our results with what can be derived from the celebrated resolution of the bounded L2 conjecture:

Overview of the proof of the Bounded L2 Curvature Conjecture

Sergiu Klainerman, Igor Rodnianski, Jeremie Szeftel

arXiv:1204.1772v2 [math.AP] 17 Jan 2013

Pages 1-149

THEOREM 1.6 (Main theorem). Let (M,g) an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that the Ricci curvature $Ric \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$, and Σ_0 has a strictly positive volume radius on scales ≤ 1, i.e. $r_{vol}(\Sigma_0, 1) > 0$. Then,

(1) L^2 regularity. There exists a time

$$T = T(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

and a constant

$$C = C(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

such that the following control holds on $0 \leq t \leq T$:

$$\|R\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C, \|\nabla k\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C, \text{ and } \inf_{0\leq t\leq T} r_{vol}(\Sigma_t, 1) \geq 1.$$

Shorter version…

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It's not an existence theory…

It assumes a solution of the Einstein equations…

—The RT-equations apply to any connection…

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$$\|R\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C, \|\nabla k\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C$$

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Applies to vacuum Einstein equations $G = 0$

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—Our theorem applies to any soln of $G = \kappa T$

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The assumption of the L2 Theorem is:

$$Ric \in L^2(\Sigma_0), \boxed{\nabla k \in L^2(\Sigma_0)}$$

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Thus: Their assumption requires the second fundamental form be one degree smoother than curvature…

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The RT-equations apply to any connection satisfying…

$$\Gamma, Riem(\Gamma) \in L^\infty$$

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holds …

But $\Gamma \in L^\infty$ means $k \in L^\infty$

on Cauchy hyper-surfaces…

so … $\boxed{\nabla k \notin L^2}$

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Conclude: The L2 theory does not apply when

$$\Gamma, Riem(\Gamma) \in L^\infty$$

Even for vacuum solutions…

(2) Higher regularity. Within the same time interval as in part (1) we also have the higher derivative estimates,

$$\sum_{|\alpha|\leq m} \|D^{(\alpha)}R\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C_m \sum_{i\leq m} \left[\|\nabla^{(i)}Ric\|_{L^2(\Sigma_0)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma_0)}\right],$$

where C_m depends only on the previous C and m.

Remark 1.7. Since the core of the main theorem is local in nature we do not need to be very precise here with our asymptotic flatness assumption...

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Remark 1.7. Since the core of the main theorem is local in nature we do not need to be very precise here with our asymptotic flatness assumption...

(Note their problem is local like ours...)

Conclude: As it stands…without constructing surfaces on which k is one order smoother…

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The Regularity Transformation equations apply at regularities too low to apply the L2 Existence Theory…

Conclude: As it stands...without constructing surfaces on which k is one order smoother...

The Regularity Transformation equations apply at regularities too low to apply the L2 Existence Theory...

I.e., in case $\Gamma, Riem(\Gamma), k \in L^{\infty}$

the L2 Existence Theory does not apply

Even in the case $\Gamma, k, Riem(\Gamma) \in W^{1,p}, p > n$

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$\ldots$L2 Regularity Theorem only gives $\Gamma \in H^1$

Even in the case $\Gamma, k, Riem(\Gamma) \in W^{1,p}, p > n$

...L2 Regularity Theorem only gives $\Gamma \in H^1$

...because you need to estimate

$$Ric \in L^2(\Sigma) \qquad k \in H^1(\Sigma)$$

Even in the case $\Gamma, k, Riem(\Gamma) \in W^{1,p}, p > n$

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…because you need to estimate

$$Ric \in L^2(\Sigma) \qquad k \in H^1(\Sigma)$$

which gives $Riem(\Gamma) \in L^2$ and $\Gamma \in H^1$

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The RT-equations give $\boxed{\Gamma \in W^{2,p}}$

Conclude: In the case $\Gamma, Riem(\Gamma) \in L^\infty$ the Regularity Transformation equations apply at regularities too low to apply the L2 Existence Theory...

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In the case $\Gamma, Riem(\Gamma) \in W^{1,p}$

…L2 Regularity Theory only gives $\Gamma \in H^1$

Conclude: In the case $\Gamma, Riem(\Gamma) \in L^\infty$ the Regularity Transformation equations apply at regularities too low to apply the L2 Existence Theory…

In the case $\Gamma, Riem(\Gamma) \in W^{1,p}$

…L2 Regularity Theory only gives $\Gamma \in H^1$

…below the regularity assumed at the start.

Conclude: The 3+1 theory is incomplete w/o accounting for non-optimal solutions in the sense that there are solutions in each Sobolev space for which...

(1) The curvature is estimated as one derivative below its regularity exhibited in that coordinate system...

(2) The connection is estimated as one derivative below its optimal regularity...

We do not know how to regularize the second fundamental form on Cauchy hyper surfaces. The point of view of the RT-equations is that the problem of regularizing a metric is a space-time problem…

The starting point for the derivation of the RT-equations is the Riemann-flat condition, a geometric condition for metric smoothing introduced previously by the authors.

Introduction to the RT-equations

Thesis Statement:

The Einstein equations of General Relativity are tensorial—Given independent of coordinates…

$$G = \kappa T$$

To solve them, one has to specify a coordinate system in which the equations take a form simple enough to solve by known methods of PDE's…

The spacetime metric may not exhibit its optimal level of regularity in the coordinates in which the equations can be solved.

In Einstein's theory of General Relativity:

The Einstein equations $G = \kappa T$

are equns for the gravitational metric $g = g_{ij}$

coupled to the fluid sources ρ, p, u

$$\boxed{\begin{gathered} G_{ij}[g_{ij}] = \kappa T_{ij}(\rho, p, u) \\ DivT = T^{\sigma}_{j;\sigma} = 0 \end{gathered}}$$

In Einstein's theory of General Relativity:

The gravitational metric tensor g determines the properties of spacetime…

geodesics, parallel translation, time dilation, arc length…
…as well as the connection Γ and curvature R

Connection: $$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma} \left\{ -g_{jk,i} + g_{i,jk} + g_{ki,j} \right\}$$

Riemann Curvature: $$R^l_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{j\sigma}\Gamma^\sigma_{ik} - \Gamma^l_{k\sigma}\Gamma^\sigma_{ij}$$

Our Question: Given a non-optimal solution of the Einstein equations, does there always exist a coordinate transformation which smooths the spacetime to its Optimal Regularity??

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By Optimal Regularity, we mean that the gravitational metric tensor is two full derivatives smoother than its curvature tensor.

The equations may be too complicated to solve in coordinates in which the metric exhibits its optimal regularity…

Motivation: Non-optimal solutions of the Einstein equations first came up in shock matching [Israel,Sm/Te] and when we constructed shock wave solutions of the Einstein equations by the Glimm Scheme in Standard Schwarzschild Coordinates (SSC)

Groah and I observed: ``The gravitational metric appears to be singular at the shocks in coordinates where the analysis is feasible (SSC)".

Although our problem began by considering solutions of the Glimm scheme in spherically symmetric spacetimes, the problem turns out to be much more general!

For example…assume g is two derivatives smoother than its curvature…

Given: $$g_{ij} \in W^{m+2,p},\ R^k_{lij} \in W^{m,p}$$

For example…assume **g** is two derivatives smoother than its curvature…

Given: $g_{ij} \in W^{m+2,p},\ R^k_{lij} \in W^{m,p}$

Do a ``singular" coordinate transformation with $J \in W^{m,p}$ …

For example…assume **g** is two derivatives smoother than its curvature…

Given: $g_{ij} \in W^{m+2,p},\ R^k_{lij} \in W^{m,p}$

$$g_{\mu,\nu} = g_{ij}\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu}$$

$W^{m+1,p}$ $W^{m+2,p}$ $W^{m+1,p}$

$$R^\alpha_{\beta\mu,\nu} = R^k_{lij}\frac{\partial y^\alpha}{\partial x^k}\frac{\partial x^l}{\partial y^\beta}\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu}$$

$W^{m,p}$ $W^{m+1,p}$

For example…assume **g** is two derivatives smoother than its curvature…

Given: $$g_{ij} \in W^{m+2,p}, \;\; R^k_{lij} \in W^{m,p}$$

$$g_{\mu,\nu} = g_{ij} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$$

$W^{m+1,p}$ $\quad W^{m+2,p}$ $\quad W^{m+1,p}$

$$R^\alpha_{\beta\mu,\nu} = R^k_{lij} \frac{\partial y^\alpha}{\partial x^k} \frac{\partial x^l}{\partial y^\beta} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$$

$W^{m,p}$ $\quad W^{m+1,p}$

Get: $$g_{\mu\nu} \in W^{m+1,p}, \;\; R^\alpha_{\beta\mu\nu} \in W^{m,p}$$

For example…assume **g** is two derivatives smoother than its curvature…

Given: $g_{ij} \in W^{m+2,p}, \;\; R^k_{lij} \in W^{m,p}$

$$g_{\mu,\nu} = g_{ij} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu} \qquad R^\alpha_{\beta\mu,\nu} = R^k_{lij} \frac{\partial y^\alpha}{\partial x^k} \frac{\partial x^l}{\partial y^\beta} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$$

$W^{m+1,p}$ $\quad W^{m+2,p}$ $\quad W^{m+1,p}$ $\quad W^{m,p}$ $\quad W^{m+1,p}$

(The connection Γ is always one derivative below the metric in each coordinate system)

Our problem…can you always reverse this?

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Given: $$g_{\mu\nu} \in W^{m+1,p},\ R^{\alpha}_{\beta\mu\nu} \in W^{m,p}$$

Our problem…can you always reverse this?

Given: $g_{\mu\nu} \in W^{m+1,p}, \; R^{\alpha}_{\beta\mu\nu} \in W^{m,p}$

Can you always find $x \to y$ such that

You get: $g_{ij} \in W^{m+2,p}, \; R^{k}_{lij} \in W^{m,p}$

The RT-equations resolve the problem of optimal regularity in full generality—

Answer: Yes

The Einstein Equations In SSC

Assume a gravitational metric ansatz of the SSC form:

$$ds^2 = -B(t,r)dt^2 + \frac{dr^2}{A(t,r)} + r^2 dr^2$$

Plug into the Einstein equations :

$$G = \kappa T$$

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}$$

Standard Schwarzschild Coordinates

Four PDE's

$$\left\{-r\frac{A_r}{A}+\frac{1-A}{A}\right\} = \frac{\kappa B}{A}r^2T^{00} \quad (1)$$

$$\frac{A_t}{A} = \frac{\kappa B}{A}rT^{01} \quad (2)$$

$$\left\{r\frac{B_r}{B}-\frac{1-A}{A}\right\} = \frac{\kappa}{A^2}r^2T^{11} \quad (3)$$

$$-\left\{\left(\frac{1}{A}\right)_{tt}-B_{rr}+\Phi\right\} = 2\frac{\kappa B}{A}r^2T^{22}, \quad (4)$$

where

$$\Phi = \frac{B_tA_t}{2A^2B}-\frac{1}{2A}\left(\frac{A_t}{A}\right)^2-\frac{B_r}{r}-\frac{BA_r}{rA} + \frac{B}{2}\left(\frac{B_r}{B}\right)^2-\frac{B}{2}\frac{B_r}{B}\frac{A_r}{A}.$$

$$(1)+(2)+(3)+(4) \Longleftrightarrow (1)+(3)+\mathrm{div}\, T=0$$

(weakly)

Theorem: (Te-Gr) The equations close in a "locally inertial" formulation of (1), (2) & Div T=0:

$$\left\{T_M^{00}\right\}_{,0} + \left\{\sqrt{AB}T_M^{01}\right\}_{,1} = -\frac{2}{r}\sqrt{AB}T_M^{01}, \quad (1)$$

$$\left\{T_M^{01}\right\}_{,0} + \left\{\sqrt{AB}T_M^{11}\right\}_{,1} = -\frac{1}{2}\sqrt{AB}\left\{\frac{4}{r}T_M^{11} + \frac{(1-A)}{Ar}(T_M^{00} - T_M^{11}) + \frac{2\kappa r}{A}(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4rT^{22}\right\}, \quad (2)$$

$$rA_r = (1-A) - \kappa r^2 T_M^{00}, \quad (3)$$

$$rB_r = \frac{B(1-A)}{A} + \frac{B}{A}\kappa r^2 T_M^{11}. \quad (4)$$

$$T_M^{00} = \frac{\rho c^2 + p}{1 - \left(\frac{v}{c}\right)^2} \qquad T_M^{01} = \frac{\rho c^2 + p}{1 - \left(\frac{v}{c}\right)^2}\frac{v}{c}$$

$$T_M^{11} = \frac{p + \left(\frac{v}{c}\right)^2}{1 - \left(\frac{v}{c}\right)^2}\rho c^2 \qquad T^{22} = \frac{p}{r^2} \qquad v = \frac{1}{\sqrt{AB}}\frac{u^1}{u^0}$$

$$
\begin{aligned}
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The metric components A,B…

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…are only one derivative

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The metric components A,B…

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Since $G = \kappa T$ $\longrightarrow$ The metric is only one order derivative smoother than the curvature tensor…

Conclude: For shock wave solutions of the Einstein equations $G=\kappa T$ generated by the Glimm Scheme:

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Conclude: Second derivatives of the metric contain delta function sources, but these cancel out in the curvature tensor…

This is a most natural setting for shock waves in GR because the Einstein equations $G = \kappa T$ place $G \in L^{\infty}$ when the bounded discontinuous fluid sources are $T \in L^{\infty}$.

In General: Given a Metric g_{ij} …

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Conclude: $\boxed{\Gamma \approx \partial g}$ $\boxed{R \approx \partial^2 g}$

Thus for Shock-Waves:

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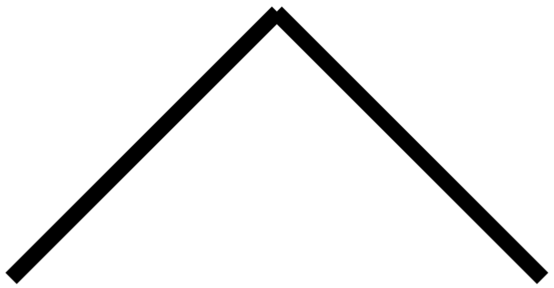
g is Lipschitz



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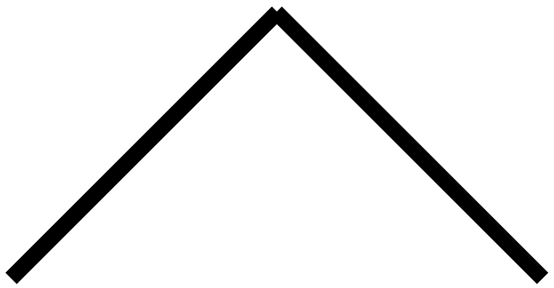
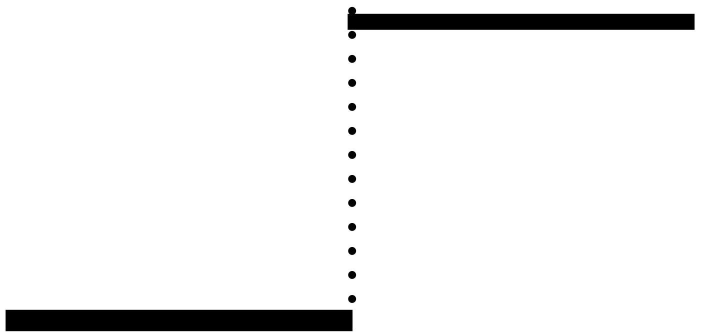


$\Gamma \approx \partial g$ is L^∞



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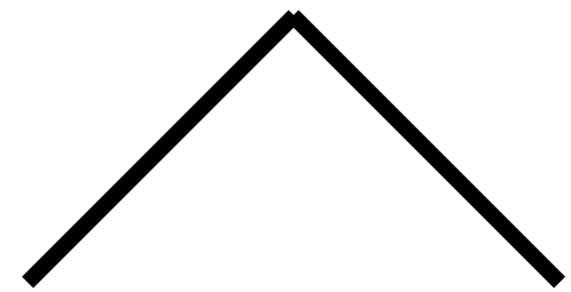

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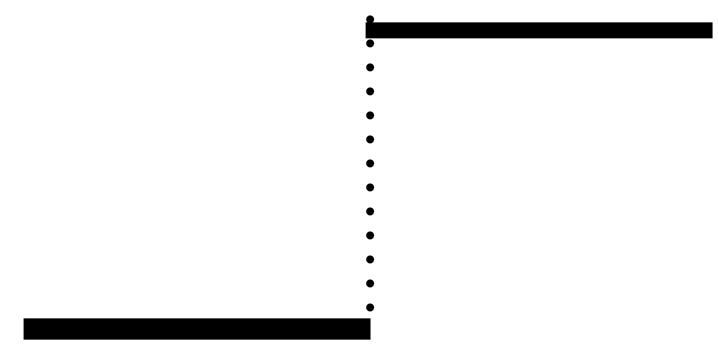

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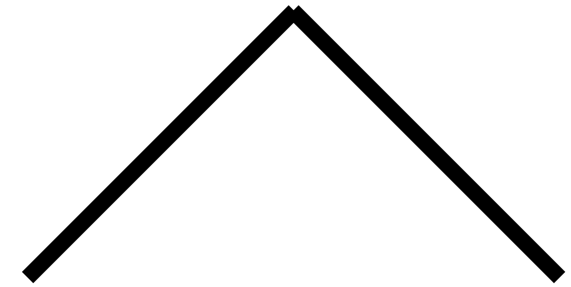

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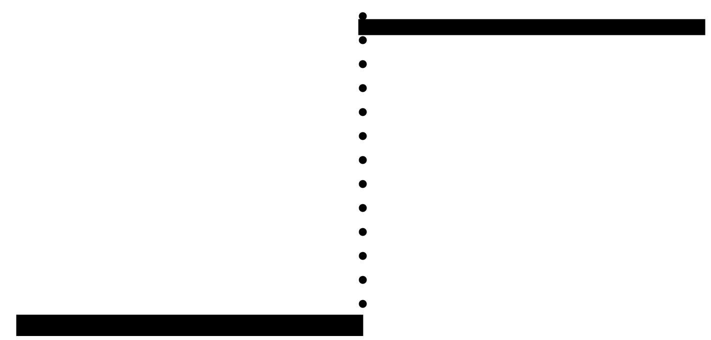

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In SSC…The curvature is only one derivative less smooth than g …

Thus for Shock-Waves something is special:

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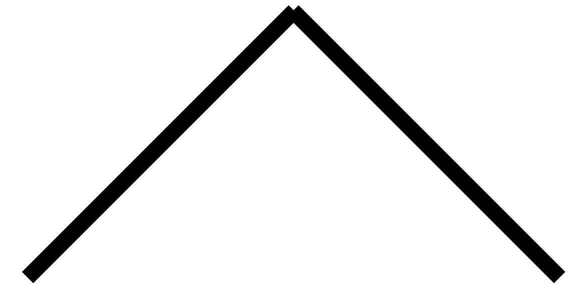

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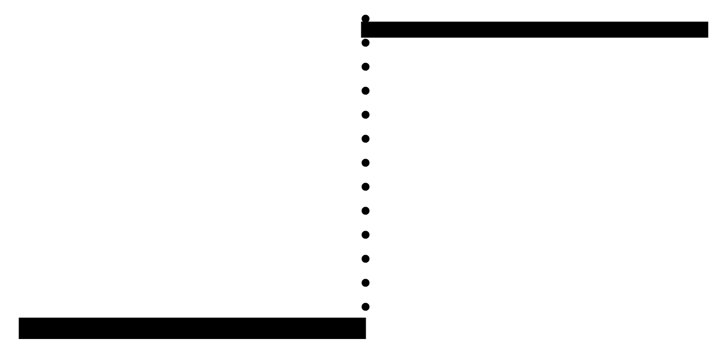

$R \approx \partial^2 g$ is L^∞ $\Longleftrightarrow$ $d\Gamma \in L^\infty$

So all delta functions cancel out in $d\Gamma$

Conclude: For shock waves we have....

$$\Gamma, d\Gamma \in L^\infty \quad \text{so} \quad g \in C^{0,1}$$

...in x-coordinates

We ask: "Does there exist a coordinate transformation $x \to y$ such that in y-coordinates, we have..."

$$\Gamma \in C^{0,1} \quad \text{so} \quad g \in C^{1,1} \quad ?$$

$$(d\Gamma, R \in L^\infty)$$

More Generally: Given....

$$\Gamma, d\Gamma \in W^{m,p} \quad \text{so} \quad g \in W^{m+1,p}$$

"Does there exist a coordinate transformation $x \to y$ such that in y-coordinates..."

$$\Gamma \in W^{m+1,p} \quad \text{so} \quad g \in W^{m+2,p} \quad ?$$

$$(d\Gamma, R \in W^{m,p})$$

In words: If the connection has the same regularity as the curvature in some coordinate system, does there always exist a coordinate transformation which smooths the connection, (and hence the metric), by one order ??

Defn: We say that a connection has "optimal regularity" if it is two orders more regular than its curvature tensor.

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Our Question: Can "non-optimal" connections always be lifted to optimal regularity by coordinate transformation?

Defn: If Γ non-optimal and no such coordinate transformation exists at P, we call P a "Regularity Singularity"

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For example: If P is a regularity singularity on a shock wave, then spacetime is not regular enough to admit locally inertial coordinate frames:

$$g \in C^{0,1} \text{ (Lipschitz continuous)}$$

$$g_{ij}(P) = \eta \equiv diag(-1,1,1,1), \quad g_{ij,k}(P) = 0 \quad ???$$

Conclude: If shock-wave interaction can create a regularity singularity, then spacetime is not locally Minkowski…

I.e., the physics of GR does not reduce to Special Relativity…

New GR scattering effects would occur in a neighborhood of a shock-wave Regularity Singularity

. . . .

Regularity Singularities and the scattering of gravity waves in approximate locally inertial frames,

M. Reintjes and B.Temple, Meth Appl Anal, Vol. 23, No. 2, pp. 233-258, September 2016.

Our Question: Given

$$\Gamma, d\Gamma \in W^{m,p} \quad \text{so} \quad g \in W^{m+1,p}$$

"Does there exist a coordinate transformation $x \to y$ such that in y-coordinates..."

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$$(d\Gamma, R \in W^{m,p})$$

Question: How smooth should the Jacobian be?

Answer: The Jacobian should have the same regularity as the metric…

$$J \equiv \frac{\partial x^i}{\partial y^\mu} \in W^{m+1,p}$$

That is: …

$$\underbrace{\bar{g}_{\mu\nu}}_{W^{m+2,p}} = \underbrace{\frac{\partial x^i}{\partial y^\mu}}_{W^{m+1,p}} \underbrace{g_{ij}}_{W^{m+1,p}} \underbrace{\frac{\partial x^j}{\partial y^\nu}}_{W^{m+1,p}}$$

For shock-waves: $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

That is:

$$\bar{g}_{\mu\nu} = \frac{\partial x^i}{\partial y^\mu} g_{ij} \frac{\partial x^j}{\partial y^\nu}$$

$C^{1,1}$ $\qquad$ $C^{0,1}$ $\qquad$ $C^{0,1}$ $\qquad$ $C^{0,1}$

"We need discontinuities in derivatives to cancel out in the Leibniz products…"

For shock-waves: $g \in C^{0,1}, \ \Gamma \in L^{\infty}, \ J \in C^{0,1}$

That is: $$\bar{g}_{\mu\nu} = \frac{\partial x^i}{\partial y^\mu} g_{ij} \frac{\partial x^j}{\partial y^\nu}$$

$C^{1,1}$ $C^{0,1}$ $C^{0,1}$ $C^{0,1}$

"To smooth out a metric singularity requires a singular transformation…"

The connection at shock-waves:

$$g \in C^{0,1},\ \Gamma \in L^\infty,\ J \in C^{0,1}$$

That is:

$$\underbrace{\tilde{\Gamma}^{\alpha}_{\beta\gamma}}_{C^{0,1}} = \underbrace{\Gamma^{i}_{jk}}_{L^\infty} \underbrace{\frac{\partial y^\alpha}{\partial x^i} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma}}_{C^{0,1}} + \underbrace{\frac{\partial^2 y^\alpha}{\partial x^\sigma \partial x^\tau}}_{L^\infty} \underbrace{\frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial x^\tau}{\partial y^\gamma}}_{C^{0,1}}$$

"Discontinuities have to cancel out on the RHS to smooth the connection…"

The curvature at shock-waves:

$$g \in C^{0,1},\ \Gamma \in L^\infty,\ J \in C^{0,1}$$

That is:

$$\underbrace{\tilde{R}^\alpha_{\beta\gamma\delta}}_{L^\infty} = \underbrace{R^i_{jkl}}_{L^\infty} \underbrace{\frac{\partial y^\alpha}{\partial x^i} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} \frac{\partial x^l}{\partial y^\delta}}_{C^{0,1}}$$

"The curvature involves 2nd derivatives, but it is a tensor, so J maintains the regularity of the curvature R in L^∞ …"

Theorem (SmTe): $C^{1,1}$ coordinate transformations with $C^{0,1}$ Jacobians preserve the weak formulation of the Einstein equations $G = \kappa T$ at shocks

Our Question:

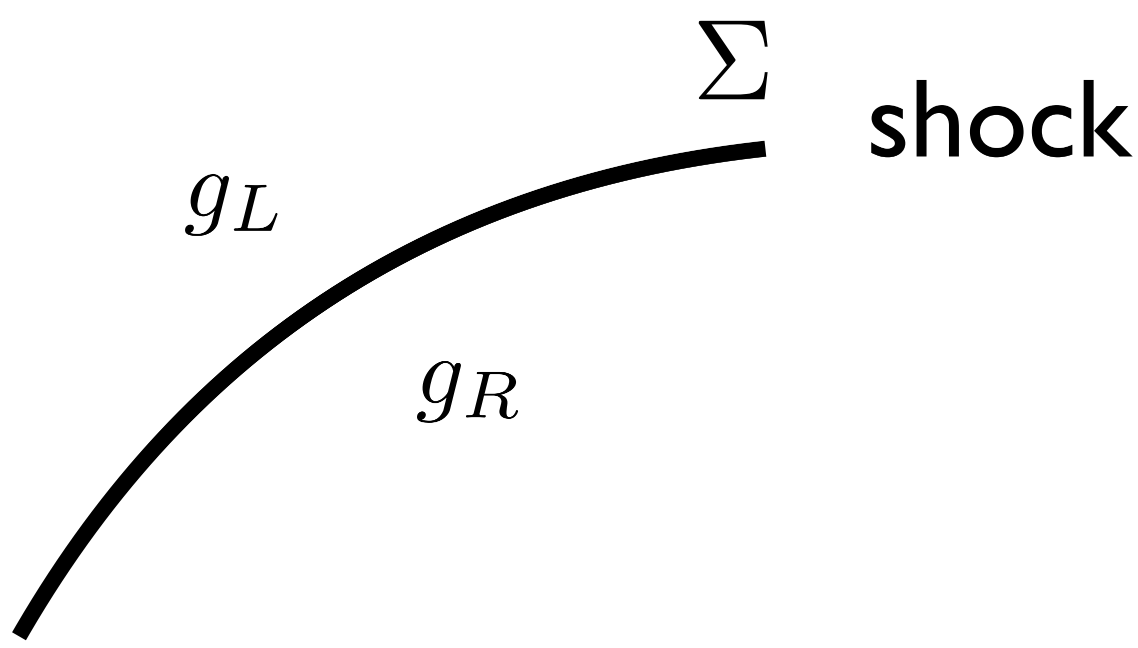
Given $\Gamma, d\Gamma \in W^{m,p}$, does there exist a $W^{m+2,p}$ coordinate transformation $x \to y$ with Jacobian $J \in W^{m+1,p}$, such that in y-coordinates, $\tilde{\Gamma} \in W^{m+1,p}$??

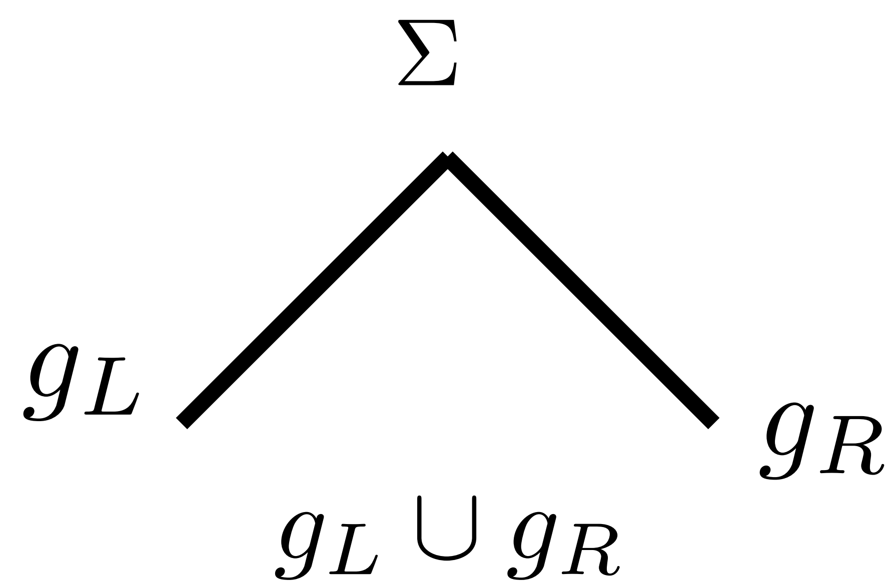
Or for shock-waves:

Given $\Gamma, d\Gamma \in L^{\infty}$, does there exist a $C^{1,1}$ coordinate transformation $x \to y$ with Jacobian $J \in C^{0,1}$, such that in y-coordinates, $\tilde{\Gamma} \in C^{0,1}$??

A result by Israel from the 60s resolves the issue for smooth shock surfaces in General Relativity:

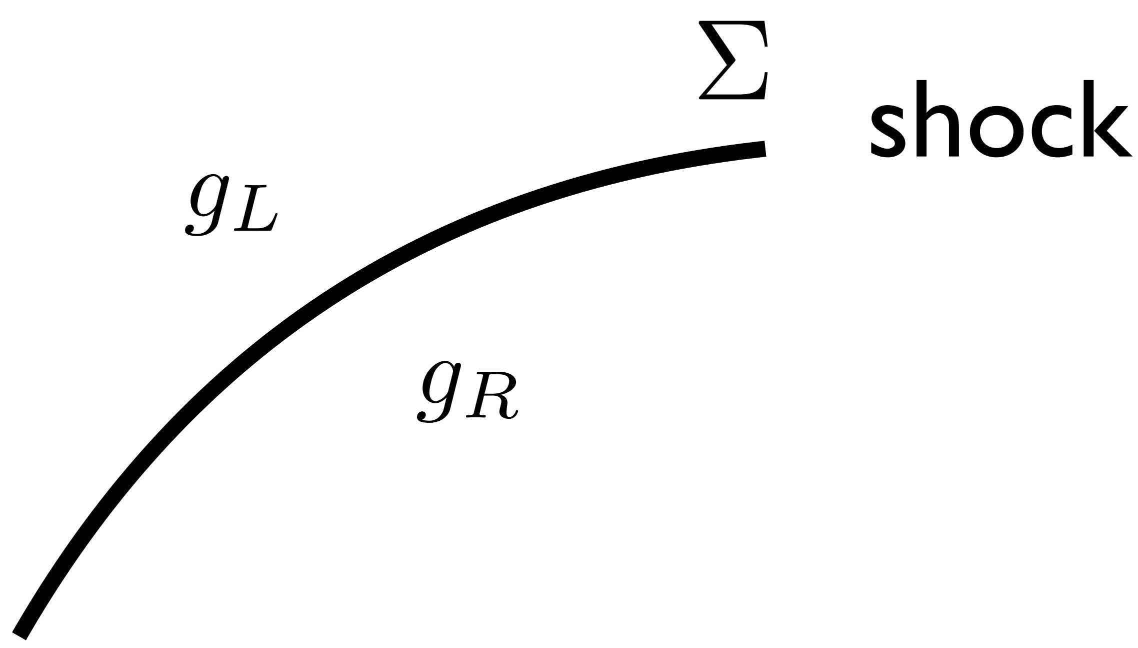
Assume g_L and g_R are smooth solutions of the Einstein equations which match Lipschitz continuously across a smooth shock surface Σ, and let... $g \equiv g_L \cup g_R$

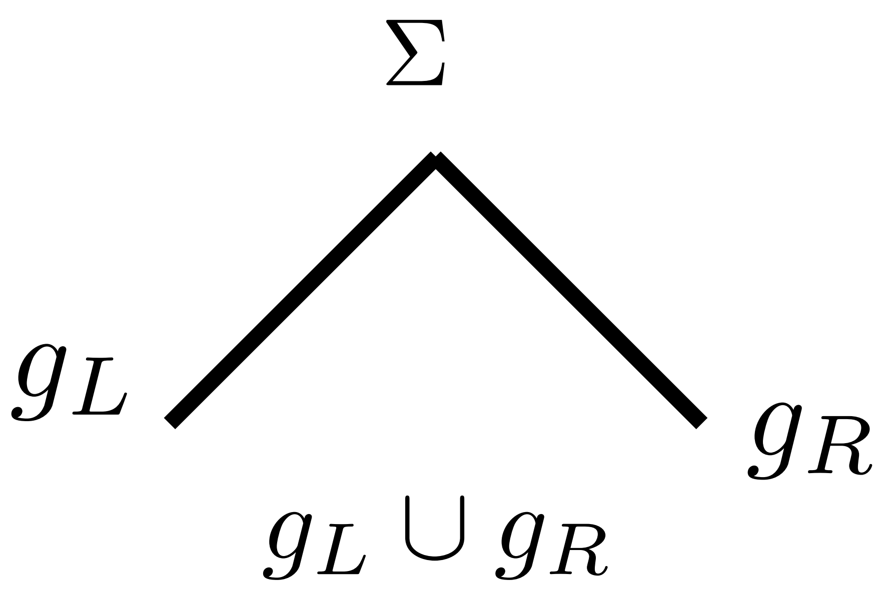




A result by Israel from the 60s resolves the issue for smooth shock surfaces in General Relativity:

Then $g \in C^{0,1}$, so delta functions exist in the second derivatives of the metric.   The following theorem gives conditions under which they all cancel out in the curvature tensor…





Theorem (Israel/SmTe): The follow are equivalent:

(1) $g = g_L \cup g_R$ is a weak solution of the Einstein equations with curvature in L^∞.

(2) All the delta functions cancel out in the Riemann curvature tensor.

(3) The Second Fundamental Forms from g_L and g_R agree on the surface Σ

(4) There exists a $C^{1,1}$ coordinate transformation in a neighborhood of Σ such that in the new coordinates,

$$g = g_L \cup g_R \in C^{1,1}$$

Theorem (Israel/SmTe):

Moreover, if any of the four equivalencies hold, then the Rankine-Hugoniot jump conditions (which express conservation of energy and momentum at the shock) also hold:

$$[G_{i\sigma}]n^{\sigma} = 0 = [T_{i\sigma}]n^{\sigma}$$

…on solutions of $G = \kappa T$

``Proof:'' In Gaussian Normal Coordinates the components of the second fundamental form are $g_{ij,n}$ so if the second fundamental form is continuous, then the metric and its derivatives match on Σ, which implies $g \in C^{1,1}$ in GNC…

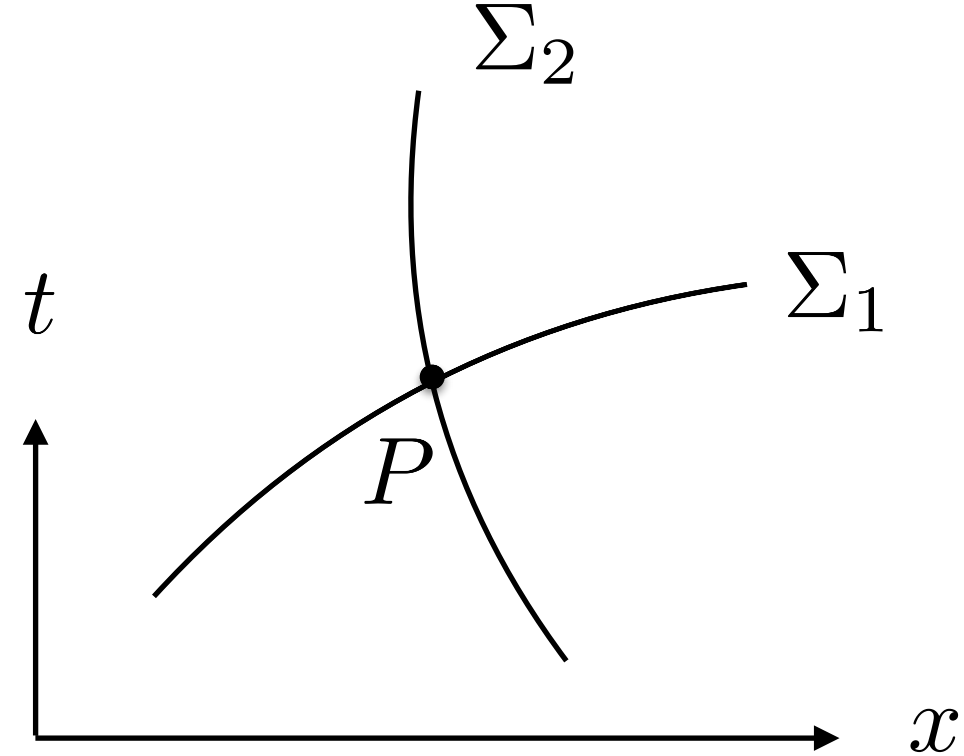
The map $x \to y$ to GNC is $C^{1,1}$ …

…with Jacobian in $J \in C^{1,1}$. $\blacksquare$

The first step forward from Israel's Theorem, was given by Moritz Reintjes:

Theorem (Reintjes): There always exists a $C^{1,1}$ transformation that smooths a $C^{0,1}$ metric to $C^{1,1}$ in a neighborhood of a point of regular shock-wave interaction in spherically symmetric spacetimes, between shock waves of different families…

Gaussian Normal Coordinates break down at P:



Reintjes' procedure for finding the local coordinate systems of optimal smoothness is orders of magnitude more complicated than the Riemann normal, or Gaussian normal construction process.

The coordinate systems of optimal $C^{1,1}$ regularity are constructed by solving a complicated non-local PDE highly tuned to the structure of the interaction…

Trying to guess the coordinate system of optimal smoothness apriori, eg harmonic or Gaussian normal coordinates, didn't work.

Several apparent miracles happen in which the Rankine-Hugoniot jump conditions come in to make seemingly over-determined equations consistent…

… but…the principle behind what PDE's must be solved to smooth the metric in general, or when this is possible, appears entirely mysterious.

M. Reintjes, *Spacetime is Locally Inertial at Points of General Relativistic Shock Wave Interaction between Shocks from Different Characteristic Families*, Adv. Theor. Math. Phys., arXiv:1409.5060.

M. Reintjes and B. Temple, *No Regularity Singularities Exist at Points of General Relativistic Shock Wave Interaction between Shocks from Different Characteristic Families*,
Proc. R. Soc. A **471**:20140834.
http://dx.doi.org/10.1098/rspa.2014.0834

The first breakthrough in discovering the general principles at play in metric smoothing came with our discovery of the Riemann-flat condition…

The Riemann-flat Condition

The Riemann-flat Condition

“Shock Wave Interactions in General Relativity: The Geometry behind Metric Smoothing and the Existence of Locally Inertial Frames”

Moritz Reintjes, Blake Temple

https://arxiv.org/abs/1610.02390

The Riemann-flat condition:

Assume $\Gamma, R \in L^{\infty}$.

Then: There exists a $C^{1,1}$ coordinate transformation that smooths an L^{∞} connection Γ by one order to $C^{0,1}$ if and only if there exists a tensor $\tilde{\Gamma} \in C^{0,1}$ such that $Riem(\Gamma + \tilde{\Gamma}) = 0$.

In words: A smoothing transformation exists at shock-waves if and only if there exists a tensor, one order smoother than the original connection, such that when added to the original connection, the new connection is Riemann-flat.

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$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

$$\Gamma \in L^\infty \qquad \tilde{\Gamma} \in C^{0,1}$$

The same proof works at other orders of smoothness, for example: $\Gamma, \tilde{\Gamma} \in W^{m,p}$

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The same proof works at other orders of smoothness, for example: $\Gamma, \tilde{\Gamma} \in W^{m,p}$

A smoothing transformation $J \in W^{m+2,p}$ exists if and only if $\exists \ \tilde{\Gamma} \in W^{m+1,p}$ st

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

$\Gamma \in W^{m,p}$

$\Gamma \in W^{m+1,p}$

This shows that whether or not you can smooth the connection is a geometrical problem for connections, independent of the signature of the metric...

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

"Proof": Assume $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

and

$$\Gamma^{\alpha}_{\beta\gamma} = \underbrace{\hat{\Gamma}^{i}_{jk}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial x^{j}}{\partial y^{\beta}}\frac{\partial x^{k}}{\partial y^{\gamma}}}_{-\tilde{\Gamma}^{\alpha}_{\beta\gamma}} + \underbrace{\frac{\partial^{2} y^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\frac{\partial x^{\sigma}}{\partial y^{\beta}}\frac{\partial x^{\tau}}{\partial y^{\gamma}}}_{\hat{\hat{\Gamma}}^{\alpha}_{\beta\gamma}}$$

L^{∞} $\quad C^{0,1}$ $\quad C^{0,1}$ $\quad Flat$

so $$Riem(\Gamma + \tilde{\Gamma}) = 0$$

The "**hard**" part is to show that if

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

…then a coordinate transformation exists which takes the connection $\Gamma + \tilde{\Gamma}$ to zero, and this is the coordinate transformation which smooths the original connection… $\blacksquare$

The Riemann-flat condition reduces the problem of smoothing Lorentzian spacetime metrics to an equation on a tensor $\tilde{\Gamma}$ that has all the remarkable properties of the zero curvature condition of Riemann.

Since $\tilde{\Gamma}$ continuous implies $\Gamma + \tilde{\Gamma}$ has the same jump discontinuities (shock set) as Γ at first we looked for a Nash-type embedding theorem for extending the shock set to a neighborhood as a flat connection in order to prove the metric can be smoothed.

However, our point of view changed again with another new idea in our next paper…

"The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity"

Moritz Reintjes, Blake Temple
https://arxiv.org/abs/1805.01004

We set out to use the Riemann-flat condition for metric smoothing to derive a system of elliptic equations in unknowns $\tilde{\Gamma}$ and J ...

The Regularity Transformation Equations:

(RT-equations)

The RT-equations

"The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity"

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The RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

$$Curl(J) \equiv \partial_j J_i^\mu - \partial_i J_j^\mu = 0 \ \text{ on } \partial\Omega, \tag{5}$$

Here: $\tilde{\Gamma}$ is a matrix valued 1-form, J and A are matrix valued 0-forms, and $\vec{J}$, $\vec{A}$ are vector valued 1-forms as follows:

$$\tilde{\Gamma} \equiv \tilde{\Gamma}^{\mu}_{\nu i} dx^i$$

$$J \equiv J^{\mu}_{\nu} \qquad \vec{J} \equiv J^{\mu}_{i} dx^i \qquad dJ = Curl(J)$$

$$A \equiv A^{\mu}_{\nu} \qquad \vec{A} \equiv A^{\mu}_{i} dx^i \qquad dA = Curl(A)$$

The integrability condition for J is: $Curl(J) = 0$

We introduce two new operations on matrix valued differential forms:

$$\overrightarrow{\mathrm{div}}(\omega)^{\alpha} \equiv \sum_{l=1}^{n} \partial_l \Big((\omega_l^{\alpha})_{i_1,,,i_k} \Big) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

("take divergence in each component to create a vector valued form out of matrix valued form")

$$\langle A\,;B\rangle_{\nu}^{\mu} \equiv \sum_{i_1 < \ldots < i_k} A^{\mu}_{\sigma\; i_1 \ldots i_k} B^{\sigma}_{\nu\; i_1 \ldots i_k}$$

("creates a matrix valued 0-form out of the ``inner product" of two matrix valued k-forms")

The RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

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$$\delta\vec{A} = v, \tag{4}$$

free to be chosen

$$Curl(J) \equiv \partial_j J_i^{\mu} - \partial_i J_j^{\mu} = 0 \;\; \text{on } \partial\Omega, \tag{5}$$

The gauge freedom in the RT-equations is the freedom to choose v, together with the freedom to choose the boundary conditions in the $\tilde{\Gamma}$ and A equations…

Theorem (RT): Assume Γ is defined in a fixed coordinate system x on $\Omega \subset \mathbb{R}^n$, and $\Gamma, d\Gamma \in W^{m,p}(\Omega),\ m \geq 1,\ p > n.$

If there exists $J \in W^{m+1,p}(\Omega)$ invertible, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, $A \in W^{m,p}(\Omega)$ which solve the RT-equations,

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If there exists $J \in W^{m+1,p}(\Omega)$ invertible, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, $A \in W^{m,p}(\Omega)$ which solve the RT-equations,

Then

$$\tilde{\Gamma}' = \Gamma + J^{-1} dJ$$

solves the Riemann-flat condition.

Theorem (RT): Assume Γ is defined in a fixed coordinate system x on $\Omega \subset \mathbb{R}^n$, and $\Gamma, d\Gamma \in W^{m,p}(\Omega)$, $m \geq 1$, $p > n$.

If there exists $J \in W^{m+1,p}(\Omega)$ invertible, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, $A \in W^{m,p}(\Omega)$ which solve the RT-equations,

then there exists a neighborhood $\Omega' \subset \Omega$ of p such that J is the Jacobian of a coordinate transformation $x \mapsto y$ on Ω', and the components of Γ in y-coordinates are in $W^{m+1,p}(\Omega')$.

Theorem (RT): The converse also holds:

If there exists a coordinate transformation which smooths the connection,

Then the Jacobian together with the smoothed out connection solve the RT-equations for some A.

Conclude: The existence of a coordinate transformation which smooths a non-optimal connection by one order reduces to proving an existence theorem for the RT-equations, with

$\tilde{\Gamma}$ one order smoother than Γ

Existence for the RT-equations

"Optimal metric regularity in General Relativity follows from the RT-equations by elliptic regularity theory in Lp-spaces"

Moritz Reintjes, Blake Temple
https://arxiv.org/abs/1808.06455

Our First Existence Theorem:

Theorem (RT): Assume $\Gamma, d\Gamma \in W^{m,p}(\Omega)$ for $m \geq 1$, $p > n \geq 2$ in some coordinate system x. Then for each $q \in \Omega$ there exists a solution $(\tilde{\Gamma}, J, A)$ of the RT-equations defined in a neighborhood Ω_q of q such that

$$\tilde{\Gamma} \in W^{m+1,p}(\Omega_q),\; J \in W^{m+1,p}(\Omega_q),\; A \in W^{m,p}(\Omega_q).$$

Main Steps in the Derivation of the RT-equations

- Start with the Riemann-flat condition:

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

This can be viewed as an equation for $d\tilde{\Gamma}$

- Start with the Riemann-flat condition:

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This can be viewed as an equation for $d\tilde{\Gamma}$

We now use the Cartan exterior algebra in the Euclidean coordinate system in which the connection is given...

- Augment to a first order Cauchy-Riemann system…

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

$$\delta\tilde{\Gamma} = h$$

But…this is not a solvable system

- To obtain a solvable system, we look to to couple this Cauchy-Riemann system to an equation for the unknown Jacobian J.

Recall: $g \in C^{0,1},\ \Gamma \in L^\infty,\ J \in C^{0,1}$

and

$$\Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^i_{jk}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma} + \underbrace{\frac{\partial^2 y^\alpha}{\partial x^\sigma \partial x^\tau}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial x^\tau}{\partial y^\gamma}}_{\substack{\hat{\hat{\Gamma}}^\alpha_{\beta\gamma} \\ Flat}}$$

L^∞ $\quad$ $C^{0,1}$ $\quad$ $C^{0,1}$

$$\underbrace{C^{0,1}\qquad C^{0,1}}_{-\tilde{\Gamma}^\alpha_{\beta\gamma}}$$

so $\quad Riem(\Gamma + \tilde{\Gamma}) = 0$

Recall: $g \in C^{0,1},\ \Gamma \in L^\infty,\ J \in C^{0,1}$

and

$$\underset{\substack{\uparrow \\ L^\infty}}{\Gamma^\alpha_{\beta\gamma}} = \underbrace{\underset{\substack{\uparrow \\ C^{0,1}}}{\hat{\Gamma}^i_{jk}}\ \underset{\substack{\uparrow \\ C^{0,1}}}{\frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}}}_{-\tilde{\Gamma}^\alpha_{\beta\gamma}} + \underbrace{\frac{\partial^2 y^\alpha}{\partial x^\sigma \partial x^\tau}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial x^\tau}{\partial y^\gamma}}_{\substack{\hat{\hat{\Gamma}}^\alpha_{\beta\gamma} \\ Flat}}$$

so $Riem(\Gamma + \tilde{\Gamma}) = 0$ **or** $\boxed{\Gamma - \tilde{\Gamma} = J^{-1}dJ}$

- To obtain a solvable system, we look to to couple this Cauchy-Riemann system to an equation for the unknown Jacobian J.

Theorem: The Riemann-flat condition is equivalent to...

$$J^{-1}dJ = \Gamma - \tilde{\Gamma}$$

“Proof”…

Lemma: The following identity holds:

$$d(J^{-1}dJ) = J^{-1}dJ \wedge J^{-1}dJ$$

So assume $(J^{-1}dJ) = \Gamma - \tilde{\Gamma}.$

Then $d(J^{-1}dJ) = d\Gamma - d\tilde{\Gamma}$

which implies the Riemann-flat condition

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

- Thus we try to construct a closed system in $(\tilde{\Gamma}, J)$ out of two equivalent forms of the Riemann-flat condition…

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

$$\delta\tilde{\Gamma} = h$$

$$dJ = J(\Gamma - \tilde{\Gamma})$$

(They start out as equivalent!!)

- Thus we try to construct a closed system in $(\tilde{\Gamma}, J)$ out of two equivalent forms of the Riemann-flat condition…

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

$$\delta\tilde{\Gamma} = h$$

$$dJ = J(\Gamma - \tilde{\Gamma}) \qquad \delta J = 0$$

(for 0-forms)

(They start out as equivalent!!)

- We next employ the identity

$$\Delta = d\delta + \delta d$$

to derive two semi-linear elliptic Poisson equations, one for $\Delta\tilde{\Gamma}$ and one for ΔJ

- Apply $\Delta = d\delta + \delta d$ to

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

$$\delta\tilde{\Gamma} = h$$

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$\delta J = 0$ (delta of a 0-form is 0!)

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…to obtain

…to obtain

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1} dJ\big) + d(J^{-1}A),$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A,$$

where $\quad A = Jh \quad$ is free…

- To impose the integrability condition

$$Curl(\vec{J}) = 0$$

...we require that d of the vectorized right hand side of the $J-$equation vanish

$$\Delta J = \delta(J\,\Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A,$$

which gives the A- equation

$$d\vec{A} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(J\, d\Gamma) - d(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle})$$

This leads to the RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \quad (1)$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \quad (2)$$

$$d\vec{A} = \overrightarrow{\text{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\text{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \quad (3)$$

$$\delta\vec{A} = v, \quad (4)$$

$$Curl(J) \equiv \partial_j J_i^{\mu} - \partial_i J_j^{\mu} = 0 \ \text{ on } \partial\Omega, \quad (5)$$

- What I haven't shown you is how terms involving $\delta\Gamma$ which initially appear to be one derivative too low on the RHS, can be replaced by terms involving $d\Gamma$

  To make the RHS smooth enough so that Δ formally lifts $\tilde{\Gamma}$ to one derivative above Γ

This property comes about by a rather miraculous identity…

The RT-equations:

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$$\delta\vec{A} = v, \tag{4}$$

$$Curl(J) \equiv \partial_j J_i^\mu - \partial_i J_j^\mu = 0 \ \text{ on } \partial\Omega, \tag{5}$$

Consider the A-equation:

$$\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), && (1)\\
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d\vec{A} &= \underbrace{\overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big)}_{d\left(\overrightarrow{\delta(J\,\Gamma)}\right)} - d\Big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\Big), && (3)
\end{aligned}
$$

Lemma:

Let $\Gamma \in W^{m,p}(\Omega)$ and $J \in W^{m+1,p}(\Omega)$ for $p > n$ and $m \geq 1$, then

$$d\big(\overrightarrow{\delta(J\,\Gamma)}\big) = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big)$$

$d\delta\,\Gamma \in W^{-1,p}$

$\nabla(d\,\Gamma) \in W^{0,p}$

$d\Gamma$ one derivative smoother than $\delta\Gamma$!

- Finally, the $\tilde{\Gamma}$ which solve the RT-equations may not solve the Riemann-flat condition…

- Finally, the $\tilde{\Gamma}$ which solve the RT-equations may not solve the Riemann-flat condition…

Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations, then

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

- Finally, the $\tilde{\Gamma}$ which solve the RT-equations may not solve the Riemann-flat condition…

Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations, then

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is Riemann-flat…

$$Riem(\Gamma - \tilde{\Gamma}') = 0$$

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And… $\tilde{\Gamma}'$ has the same regularity as Γ

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Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations, then

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

$\tilde{\Gamma}' \in W^{m+1,p}$ $\qquad$ $\Gamma \in W^{m,p}$ $\qquad$ $dJ \in W^{m,p}$

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$\tilde{\Gamma}' \in W^{m+1,p}$ $\qquad \Gamma \in W^{m,p}$ $\qquad dJ \in W^{m,p}$

$\tilde{\Gamma}'$ has the same regularity as $\tilde{\Gamma}$!

- Finally, the $\tilde{\Gamma}$ which solve the RT-equations may not solve the Riemann-flat condition…

Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations, then

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

$\tilde{\Gamma}' \in W^{m+1,p}$ $\Gamma \in W^{m,p}$ $dJ \in W^{m,p}$

The jumps in the (m+1)-derivatives of $\tilde{\Gamma}'$

cancel out on the RHS!

- Finally, the $\tilde{\Gamma}$ which solve the RT-equations may not solve the Riemann-flat condition…

Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations, then

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

For fixed $J \in W^{m+1,p}$:

The transformation: $\tilde{\Gamma} \to \tilde{\Gamma}' \in W^{m+1,p}$

Represents a change of gauge: $A \to A' \in W^{m,p}$

Theorem: $(\tilde{\Gamma}, J, A)$ is a solution of the RT-equations

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\delta\vec{A} &= v, && (4)
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\delta\vec{A} &= v, && (4)
\end{aligned}
$$

$$Curl(J) \equiv \partial_j J_i^\mu - \partial_i J_j^\mu = 0 \ \text{ on } \partial\Omega,$$

if and only if $(\tilde{\Gamma}', J, A')$ is a solution.

- Summary: Starting with two equivalent forms of the RT-equations, we turn the first order equations into independent second order equations which allow for more general boundary conditions.

The second order equations don't imply the first order equations, but miraculously, a gauge transformation converts any solution into one which does satisfy the Riemann-flat condition.

Steps in the existence proof for the RT-equations

"Optimal metric regularity in General Relativity follows from the RT-equations by elliptic regularity theory in Lp-spaces"

Moritz Reintjes, Blake Temple
https://arxiv.org/abs/1808.06455

- The existence proof is based on an iteration scheme which applies the L^p theory of elliptic regularity at each stage

  The L^p-theory of derivatives is a linear theory, and the RT-equations are nonlinear, so an iteration scheme is required...

The proof that the iterates converge relies on only two theorems from classical elliptic PDE theory…

Theorem (Elliptic Regularity): *Let* $f \in W^{m-1,p}(\Omega)$*,* $m \geq 1$*, and* $u_0 \in W^{m+\frac{p-1}{p},p}(\partial\Omega)$ *both be scalar functions. Assume* $u \in W^{m+1,p}(\Omega)$ *solves the Poisson equation* $\Delta u = f$ *with Dirichlet data* $u|_{\partial\Omega} = u_0$*. Then there exists a constant* $C > 0$ *depending only on* Ω*,* m, n, p *such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|f\|_{W^{m-1,p}(\Omega)} + \|u_0\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$

Theorem (Elliptic Regularity): *Let $f \in W^{m-1,p}(\Omega)$, $m \geq 1$, and $u_0 \in W^{m+\frac{p-1}{p},p}(\partial\Omega)$ both be scalar functions. Assume $u \in W^{m+1,p}(\Omega)$ solves the Poisson equation $\Delta u = f$ with Dirichlet data $u|_{\partial\Omega} = u_0$. Then there exists a constant $C > 0$ depending only on Ω, m, n, p such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|f\|_{W^{m-1,p}(\Omega)} + \|u_0\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$

Theorem (Gaffney Inequality): *Let $u \in W^{m+1,p}(\Omega)$ be a k-form for $m \geq 0$, $1 \leq k \leq n-1$ and (for simplicity) $n \geq 2$. Then there exists a constant $C > 0$ depending only on Ω, m, n, p, such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\Big(\|du\|_{W^{m,p}(\Omega)} + \|\delta u\|_{W^{m,p}(\Omega)} + \|u\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$

(Actually, what we found recorded:)

Theorem (Elliptic Regularity): *Let $u \in W^{2,p}(\Omega)$ be a scalar, $1 < p < \infty$. Then there exists a constant $C > 0$ depending only on Ω, m, n, p, such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C\Big(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{2-\frac{1}{p},p}(\partial\Omega)}\Big).$$

(What we need:)

Theorem (Elliptic Regularity): *Let $f \in W^{m-1,p}(\Omega)$, $m \ge 1$, and $u_0 \in W^{m+\frac{p-1}{p},p}(\partial\Omega)$ both be scalar functions. Assume $u \in W^{m+1,p}(\Omega)$ solves the Poisson equation $\Delta u = f$ with Dirichlet data $u|_{\partial\Omega} = u_0$. Then there exists a constant $C > 0$ depending only on Ω, m, n, p such that*

$$\|u\|_{W^{m+1,p}(\Omega)} \le C\Big(\|f\|_{W^{m-1,p}(\Omega)} + \|u_0\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$

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S. Agmon, A.Douglas, L.Nirenberg, *Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions*, Comm. Pure Appl. Math, Vol **12**, 623-727 (1959)

- One of the main obstacles to overcome was how to reduce the existence theorem at each iterate to a problem with Dirchlet boundary conditions... so standard linear elliptic regularity applies...

For this we introduce ancillary variable y so that $dy = \vec{J}$, $d^2 y = d\vec{J} = Curl(J)$

- Details of Proof in Moritz's Poster!

Conclusion

- We show that proving non-optimal metrics (or connections) can be smoothed one order by coordinate transformation is equivalent to proving existence for the RT-equations…

- We show that proving non-optimal metrics (or connections) can be smoothed one order by coordinate transformation is equivalent to proving existence for the RT-equations…

- We prove existence for the RT-equations above the threshold smoothness of…

$$\Gamma, d\Gamma \in W^{m,p} \quad m \geq 1,\ m > n$$

- As a Corollary we have that solutions constructed in SSC can be smoothed one order by coordinate transformation…

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  Q: What do the SSC metrics look like in coordinates of optimal regularity?

- As a Corollary we have that solutions constructed in SSC can be smoothed one order by coordinate transformation…

Q: What do the SSC metrics look like in coordinates of optimal regularity?

Ans: We don't know!

- Q: Can this be done within the 3+1 framework of the initial value problem?

  Ans: We don't know how to do this!

  Ref: Anderson, Kleinerman, Rodnianski, Dafermos, LeFloch…

- Our theorem is geometric, applies independent of matter sources or metric signature, requires no symmetry assumptions, and makes no apriori assumptions on the spacetime other than its regularity…

- Q: Can the existence theory for the RT-equations extend to the case of GR shock-waves (topic of our current research)?

  ...the case $\Gamma, d\Gamma \in L^\infty$?

  Ans: Yes! (Work in progress...) Our new theorem:

Thm: If $\Gamma, d\Gamma \in L^\infty$ then there exists a coordinate transformation x to y such that in y-coordinates, $\Gamma \in W^{1,p}, p > n$

Cor: It suffices to solve the Einstein equations in coordinates where the metric is non-optimal and be assured that the solutions exhibit optimal regularity in other coordinate systems…

- Main Obstacles to overcome when

$$\Gamma, Riem(\Gamma) \in L^\infty$$

- Quadratic Nonlinearities on the RHS of RT-equations imply iterations may not stay in space…

  L^∞ is closed under nonlinear products

  L^p is NOT closed under nonlinear products

Elliptic regularity applies to L^p spaces, but nonlinear products are uncontrolled…

L^∞ is closed under products, but elliptic regularity does not apply…

(For the case $\Gamma, Riem(\Gamma) \in W^{m,p}, m \geq 1$ we control the nonlinear products by assumption $p > n$ so continuity controls the nonlinear products.)

To prove the L^∞ theorem, we exploit the gauge freedom in the RT-equations…

The RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

free to be chosen

$$Curl(J) \equiv \partial_j J^\mu_i - \partial_i J^\mu_j = 0 \;\; \text{on } \partial\Omega, \tag{5}$$

Theorem: Assume J solves the RT-equations for $\tilde{\Gamma}=0$. Then

$$\tilde{\Gamma}=J^{-1}dJ$$

solves the Riemann-flat condition

$$Riem(\Gamma-\tilde{\Gamma})=0$$

and $\tilde{\Gamma}$ is one order smoother than Γ

Since all the nonlinear products drop out when $\tilde{\Gamma} = 0$ the RHS of the RT-equations stays bounded in under iteration...

Moritz Reintjes will discuss details!

Open Problem…

Given $\Gamma, d\Gamma \in L^{\infty}$, does there exist a coordinate transformation which achieves $\Gamma_y \in C^{1,1}$?

Final thought (speculative open problem)...

If the regularity of the Ricci tensor and the connection control the regularity on the RHS of the RT-equations, then one can bootstrap the case $R_{ij} = 0$ all the way up to C^k, for any $k > 0$.

Then solutions of the vacuum Einstein equations are all smooth solutions written in non-optimal coordinates...

Moritz Reintjes will discuss further details!

References (Reintjes Temple):

arXiv:1610.0239

arXiv:1808.06455

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END

Thank You!