Comments on thesis work of Vogler and Reintjes

Blake Temple, October, 2011

I discuss the doctoral thesis work of my last two students, Zeke Vogler, who finished in the spring of 2010, and Moritz Rientjes who finished September 2011. The work is summarized in preprints [84] and [86] on my webpage under publications.

The context for these results is my work with Groah in [64] in which we gave the first existence theory for shock wave solutions of the Einstein equations that admitted initial data general enough to allow for shock wave interactions. To quote myself at the time, *the gravitational metric appears to be singular at shock waves in the only coordinate system [Standard Schwarzschild Coordinates (SSC)] where the analysis appears to be feasible.* Our proof essentially put the spacetime metric in $C^{0,1}$ (metric components only Lipshitz continuous) but no smoother, at shock waves, in SSC. But we knew by Israel's theorem and my earlier work with Smoller that Gaussian normal coordinates will smooth the components of the gravitational metric to $C^{1,1}$ (derivatives of the metric components Lipshitz continuous) at points on a single shock surface, so this begged the question as to whether the Groah-Temple solutions could be smoothed to $C^{1,1}$ by coordinate transformation like single shock surfaces. In particular, the regularity $C^{1,1}$ is the threshold smoothness required for the existence of locally inertial coordinate frames at a given point.

Within this context, the two theses of Vogler and Rientjes dovetail in a nice and serendipitous way. To say it all at once, Vogler gave the first definitive numerical simulation of a point of GR shock wave interaction, and Moritz gave a complete mathematical proof that such points of shock wave interaction represent a new kind of singularity in general relativity, which he calls a *regularity singularity*, where the spacetime metric cannot be smoothed beyond $C^{0,1}$ by coordinate transformation, and so does not admit locally inertial coordinate frames. This resolves the issue we raised as to whether the Groah-Temple solutions can be smoothed to $C^{1,1}$ by coordinate transformation. The answer: they *cannot* once they are complicated enough to admit points of shock wave interaction.

This is a big surprise because it was generally believed that the gravitational metric is at least $C^{1,1}$ at shock waves. For example, the assumption of $C^{1,1}$ smoothness is taken in singularity theorems of Hawking and Penrose.

The work opens the door to a number of fascinating questions, including: What plays the role of locally inertial coordinate systems at points of shock wave interaction? Since special relativity does not tell the local structure of the spacetime metric at such points, are there observable general relativistic effects when shock waves collide? For example, even if dissipation is introduced, the gradients in derivatives of the spacetime metric cannot be bounded uniformly as the dissipation parameters tend to zero at points of shock wave interaction. And dissipation is a delicate issue in GR because viscosity has to be hyperbolic in order to preserve the fundamental light speed bound.

The work of Rientjes also provides a rigorous foundation for the principle that apparent singularities in the second derivatives of the metric in GR really can be points of a perfectly good weak solution of the equations amenable to numerical simulation, in part justifying computing right through the apparent singularities generated by numerical codes that have arisen, for example, in simulations of black hole collapse, something that is routinely done now.

Since no one has actually explicitly constructed an exact solution of the Einstein equations around a point of shock wave interaction, the simulation of Vogler backs up and justifies the assumptions Rientjes takes concerning the structure of the spacetime metric at points of shock wave interaction in Standard Schwarzschild Coordinates (SSC), these starting assumptions being the starting point of Rientjes' proof. There is nothing illusive in these assumptions—they just codify what it means for a metric to be no better than Lipschitz continuous across two crossing shock curves in the rt-plane. But it would be interesting to get a complete mathematical proof of the existence of an isolated point of GR shock wave interaction that meets these structural assumptions. Vogler's numerical solutions provide a natural starting point for such a proof.

In his thesis Vogler develops a new numerical method, and applies it to new initial data that meets the constraints of the Einstein equations weakly. To simulate a point of shock wave interaction requires finding initial data that meets the Einstein constraint equations and generates such a point. His new data is obtained by matching a Friedmann-Robertson-Walker (FRW) spacetime written in self-similar SSC coordinates (the starting point of [80]) to a Tolmann-Oppenheimer-Volkov (TOV) spacetime at a point of Lipschitz continuity. My work with Groah in [64] showed that it was consistent to match two solutions of the constraint equations Lipschitz continuously and insure a weak shock wave solution in the class of spacetime metrics of class $C^{0,1}$. Our theory in [64] confirms that such metrics satisfy the Einstein equations weakly, and the delta function sources in the second derivatives of the metric cancel out in the curvature tensors yielding a perfectly good weak solution of the Einstein equations with gravitational metrics only Lipschitz continuous at shock waves.

Vogler's numerical method is a fractional step method that approximates solutions in locally inertial grid cells by Riemann problems, and then accounts for effects of spacetime curvature by dilating the clocks appropriately in each cell. Applying his method to the new initial data, he produced what is, as far as we know, the first *detailed* simulation of two shock waves emanating from a point of interaction at the interface between two metrics.

Moritz's proof is very clever, and he struggled with this proof for a long time. Keep in mind that to prove a metric *can* be smoothed it suffices to construct a coordinate mapping that does it directly. But to prove that no such mapping exists at all is much more subtle and problematic. Add to this that, based on Israel's theorem for single shock surfaces, almost everyone expected that gravitational metrics *could* always be smoothed by coordinate transformation at shock waves, so it took courage to dive in and build a framework sufficient to prove they could not. It turns out, the problem is very delicate, and it wasn't until the very last step that we knew the answer was negative, the metric *cannot* be smoothed by coordinate transformation in a neighborhood of a point of shock wave interaction.

Moritz's idea is to construct a canonical form for functions that are Lipschitz continuous across two interacting shock surfaces in spherical symmetry, and meeting the structural conditions consistent with the regularity of the Groah-Temple weak solutions and verified in detail in Vogler's simulations. He then uses the Rankine-Hugoniot jump conditions for shocks to isolate conditions on the Jacobian derivatives J^μ_α of a coordinate transformation that would smooth the metric were the Jacobians to meet the integrability condition for coordinate functions. It turns out that Moritz's canonical form has enough free functions to formally meet the Jacobian smoothing conditions *and* the integrability conditions on the crossing shock surfaces, and so this lead us to believe up until the very end that one *could* construct coordinates in which components $g_{\alpha\beta}$ of the gravitational metric were $C^{1,1}$.

But at the very last step, taking the limit to the point p of shock wave interaction, Moritz proved that the Rankine-Hugoniot jump conditions together with the $C^{1,1}$ condition $[g_{\alpha\beta,\gamma}] = 0$ on both shock curves, has the effect of freezing out all the freedom in the free functions, thereby forcing the condition that the determinant of the Jacobian must vanish at p. The answer was not apparent *until the very last step*, making this a subtle, surprising and remarkable new result. I believe it is also an important and fundamental result for general relativity.