Periodic Solutions of Compressible Euler

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We have been working on a long term program to construct the first time-periodic solutions of the compressible Euler equations. Our program from the start has been to construct the simplest periodic wave structure, and then tailor the analysis in a rigorous existence proof to the explicit structure. In [26] have found the periodic structure, in [27, 28] we showed that linearized solutions with this structure are isolated it in the kernel of its linearized operator, expressing all of this terms of a new dimensionless formulation of the problem which converts linearized evolution into rotation of the representations of the Fourier modes. But the linearized operator has resonances on a measure zero set of periods, and small divisors at non-resonant periods. In [29] we have shown the consistency of a Liapunov-Schmidt decomposition and proven convergence subject to a finite Fourier mode cutoff. We are now trying to close the final proof by implementing a framework for period extraction into a Nash-Moser argument set out in [30]. Here is an accounting of our progress so far.

BACKGROUND: The compressible Euler equations are a system of five nonlinear first order partial differential equations that describe the time evolution of the density ρ, velocity $\mathbf{u} = (u^1, u^2, u^3)$ and temperature T of a gas under the assumption that there is no dissipation. In modern notation they are expressed in the form

$$\rho_t + div[\rho \mathbf{u}] = 0, \tag{1}$$

$$(\rho u^i)_t + div[\rho u^i \mathbf{u}] = -\nabla p, \tag{2}$$

$$E_t + div[(E + p)\mathbf{u}] = 0, \tag{3}$$

where the energy-density $E = \frac{1}{2}\rho \mathbf{u}^2 + \rho e$ is the sum of the kinetic energy of the center of mass motion $\frac{1}{2}\rho \mathbf{u}^2$, and e is the specific internal energy. To close the equations an equation of state relating p, ρ and e must be given. The equation of state for a gas composed of r-molecules can be derived from first principles using only the equipartition of energy principle and the second law of thermodynamics, and leads to the fundamental thermodynamic relations for a polytropic, or gamma law gas:

$$e = c_v T = \frac{c_v}{v^{\gamma-1}} exp\left\{\frac{s}{c_v}\right\}, \tag{4}$$

$$p = -\frac{\partial e}{\partial v}(s, v). \tag{5}$$

Here $v = 1/\rho$ is the specific volume, s is the specific entropy determined by the second law of thermodynamics $de = Tds - pdv$, and the adiabatic gas constant $\gamma = 1 + 2/3r$ can be shown to be equal to the (measurable) ratio c_p/c_v of specific heats c_p and c_v, [23]. It is well known that for shock-free solutions, the energy equation (3) can be replaced by the so called adiabatic constraint

$$(\rho s)_t + div(\rho s \mathbf{u}) = 0, \tag{6}$$

which expresses that specific entropy is constant along particle paths. For sound wave propagation in one direction, (say x), two degrees of freedom in the velocity drop out, and the equations reduce to a 3×3 system, three equations in three unknown functions of (x,t) : ρ, ρu and $(T, E, \text{or } s)$.

The compressible Euler equations (1)-(3) with polytropic equation of state (4), (5) is fundamental. In particular, it is the setting for the nonlinear theory of sound waves, and can be regarded as the essential extension of Newton's law $F = Ma$ to a continuous medium, neglecting dissipation. The question as to whether the compressible Euler equations admit time periodic solutions is as old as Euler who first derived the correct equations, linearized them to get the wave equation in the density, and thereby established the linear theory of sound by making the connection with the same equation for a vibrating string derived several years earlier by his colleague D'Alembert. Faced with the nonlinear equations, the very first question one would ask is *do the nonlinear equations admit time periodic solutions that propagate like the linear sound waves of the wave equation?* This is a question of great mathematical and historical interest, and we believe the existence of dissipation free transmission of waves will prove interesting physically as well. In particular, periodic solutions and their perturbations are a route to the study of large amplitude solutions, (for which a mathematical theory like Glimm's is pretty much non-existent), and doorway to new nonlinear phenomenon like quasi-periodicity and chaos.

For most of the last two hundred years, experts believed that time periodic solutions of the compressible Euler equations, that propagate like sound waves, did not exist. The caveat is "with sound wave propagation because there are trivial time periodic solutions that represent entropy variations which, in the absence of dissipative effects, are passively transported by (6). There is no nonlinear sound wave propagation in these solutions. When we speak of time periodic solutions, we always mean with nonlinear wave propagation. The basic intuition is that, since the wave is nonlinear, each period will decompose into a *rarefactive* region, (characteristics spreading out in forward time), and a *compressive* region, (characteristics impinging in forward time), and in the compressive part the "back will catch up to the front", causing it to break, something like a wave breaking on the beach, forming a shock-wave; then the wave amplitude will decay to zero by shock-wave dissipation. Entropy strictly increases in time like a Liapunov function when shock-waves are present, so the presence of shock-waves is inconsistent with time periodic evolution. Our work in [26] demonstrates that this picture is fundamentally incorrect.

Being a system of three coupled nonlinear first order equations, the rigorous mathematical theory of the compressible Euler equations in one space dimension began with warmup problems. The model scalar equation is the Burgers equation $u_t + uu_x = 0$, and for this, it is easy to prove directly that periodic solutions form shock-waves and decay in amplitude, (by the dissipation of the shock-waves), at a rate $1/t$, [23]. If one closes the first two Euler equations so that there is no dependence of the pressure p on temperature or entropy, the Euler system reduces to the 2×2 coupled nonlinear system of conservation laws (1), (2), in the unknown density and velocity of the fluid. In Lagrangian form, both the isothermal and isentropic equations are special cases of the p-system, (so named by Smoller because the pressure is assumed given in terms of the density by a function $p = p(\rho)$, [23]). In 1964, Lax proved an elegant blow-up result [12] that rules out the possibility of time periodic solutions in any 2×2 system for which the nonlinear fields are *genuinely nonlinear* like the p-system[3]. Lax's argument is sufficient to imply blow up in the derivative for any non-constant periodic solution of the p-system, thereby implying the formation of shock-waves, which are inconsistent with time periodic evolution. In 1970 this result was made definitive in the celebrated paper of Glimm and Lax, [7], in which they proved that solutions of a 2×2 genuinely nonlinear system like the Euler equations, evolving from periodic initial data, must form shock-waves and decay in the total variation norm like $1/t$. (See [8] for an interesting example of a time periodic solution of a 2×2 system that is not strictly hyperbolic and genuinely nonlinear like the p-system.) We believe that at the time it was generally thought that the nonlinear fields of the full three Euler equations would do the same. There was a significant effort to extend the blow up results of Lax to systems of three or more equations. The most famous example is the blow-up result of Fritz John, [10], (improved by Liu in [15], see also [13]), that partially extends Lax's result to more than two equations, but no one could prove a blow-up result sufficient to rule out the possibility of the existence of time periodic solutions of the compressible Euler equations that propagate like sound waves. The idea that time periodic solutions may exist was kindled by work of Majda, Rosales and Schonbeck [17, 18, 19], and in [20], Pego produced a periodic solution to an asymptotic model associated with the Euler system. In [25] the we got a deep understanding of how the Lie Bracket effects in the full 3×3 Euler system can fundamentally alter wave interactions, and warmup problems suggesting that periodic solutions may exist were investigated by Young in [33, 34, 35]. In [22, 31], Rosales and two students Shefter and Vaynblat produced detailed numerical simulations of the Euler equations starting from periodic initial data, and these numerical studies indicated that periodic solutions of the 3×3 compressible Euler equations do not decay like the 2×2 p-system, and they made observations about the possibility of periodic, or quasi-periodic attractor solutions.

OUR PROGRAM: Our long term program is to understand the mechanism, construct, and give the first proof of existence of time-periodic[1] solutions of the compressible Euler equations supporting sound wave propagation. I must confess that I have been certain several times in the past that we had a complete proof, only to find a gap in our argument. And I am disappointed that at this time I cannot claim that we have a complete proof. But I will again go out on a limb and say that I believe we have a correct proof strategy, and I believe we will have a proof once we build a mechanism for period extraction into our Nash-Moser Newton iteration method to beat the problem of small divisors that have been stubbornly difficult for us to overcome. We have been on this now a long time. But I won't apologize too much because we have found so much fascinating new mathematics along the journey, the problem is a 250 year old conjecture that has stumped many others as well, and I truly believe that a final proof is nearly at hand. We have published four papers [76-79] on

[3] Riemann was the first to address shock-wave formation in 1858, when he introduced the notion of *Riemann invariants* to prove that solutions of the Euler equations which arise from smooth initial conditions can develop infinite gradients in finite time. Riemann had the full Euler system available to analyze, so given his deep work in other areas of mathematics, one has to wonder whether Riemann ever made conjectures about the possible existence of (shock-free) time periodic solutions.

[1] By time-periodic we really mean any solution with a spacetime periodic tiling. Such solutions will not in general be exactly time-periodic for reasons of incommensurability.

this topic since the last NSF proposal was written in 2006. Although the publication dates are 2009-2011, we did have understanding of many of the basic ideas in these papers at the writing of my last NSF proposal.

I will first describe in words the history and current state of our program, and then explain in more detail how we propose to complete the first proof of existence of periodic solutions of compressible Euler. Our starting idea (in my opinion a great new idea) was to first discover the basic periodic wave structure, and then tailor the analysis to that wave structure. Then we would have an explicit comprehensible physical mechanism as well as a proof. So we set out at the start to find the simplest possible periodic wave structure consistent with time-periodic evolution for Euler. In [77] this we succeeded in our first step more successfully that we could have imagined—we discovered the simplest possible periodic wave structure consistent with time-periodic evolution for compressible Euler. This simplest but non-trivial structure is based on nonlinear waves propagating through two alternating constant entropy levels, connected by contact discontinuities. The key is that the periodic tiling of the solution in the xt-plane requires a half period shift in time after each spatial period—meaning that a sound wave (characteristic) must pass through eight different entropy levels before periodic return.

To us this was a thrilling discovery confirming we were on the right track. By carefully characterizing how nonlinear waves can change from Compressive (C) to Rarefactive (R) at entropy jumps, and then using combinatorial arguments, we discovered that there is a simplest formal wave pattern such that compression and rarefaction are are in balance along all forward and backward characteristics. The idea being that any imbalance would create shock waves in either forward or backward time, and since shock waves dissipate entropy, they are inconsistent with time-periodic wave propagation. More precisely, the requirement is that each characteristic pass through equal numbers of compressive and rarefactive regions, but it is a formal balance because in an exact periodic solution, the magnitudes of compression and rarefaction must balance as well. The pattern is diagrammed in [77]. In this simplest periodic pattern, each forward and backward characteristic traverses four regions of C and four regions of R before its periodic return.

In [78] and [76] we derived a non-trivial dimensionless formulation of the Lagrangian equations, casting the problem of proving existence of time-periodic solutions of compressible Euler into the simplest possible framework. Essentially, the new variables convert linear evolution of Fourier modes into rotation of their representations, and using this we identified a condition under which exact solutions exhibiting the periodic wave structure in [77] exist in the 1-mode kernel of the linearized operator—that is, the linearized operator obtained by linearizing the compressible Euler equations about the solution consisting of two alternating (normalized) constant states at two entropy levels separated by contact discontinuities. Under perturbation of this 1-mode linearized solution, nearby nonlinear solutions maintain the basic formal wave structure identified in [77] that balances R's and C's along characteristics. The idea then is that some tuning of the initial data would tune the formal balance into an exact balance in nearby nonlinear solutions.

We then proved something really remarkable: The linearized operator is invertible on the complement of the 1-mode kernel for almost every choice of periods. That is, for almost every period, a spacetime periodic linear solution exhibiting the periodic wave pattern we derived as simplest possible, is *isolated* in the kernel of the linearized operator. If the linearized operator had a bounded inverse on the complement of the kernel, then an implicit function theorem based on a modified Newton method (after a straightforward Liapunov-Schmidt reduction) should directly prove that the linearized solutions perturb to nearby nonlinear periodic solutions of the compressible Euler equations. But we prove that the linearized operators are unbounded due to resonances at a rational set of periods, and small divisors are present on the complement of the resonant periods. This is pretty much the simplest possible resonance structure you could possibly imagine for a fully nonlinear problem. By comparison, the linearized operators associated with the periodic solutions of the semi-linear equation $\Box u = g(u)$ constructed in [craig/wayne] have a dense kernel. But our equations are fully nonlinear, making for other difficulties. In the paper [79] we prove essentially that the small divisors are the only obstacle to existence of spacetime periodic solutions of compressible Euler by proving that the Liapunov-Schmidt method succeeds with an arbitrarily large Fourier cutoff. For example, a numerical simulation presented in [77] shows that in an optimal case, the small divisors aren't all that small up to the $50'th$ Fourier mode.

But it remains for us to prove that these linearized solutions perturb to spacetime periodic solutions of the fully nonlinear compressible Euler equations. We are absolutely certain they do, but we still do not have a complete proof.

At the writing of my last NSF proposal, I was convinced that we had a correct argument proving existence of solutions by an iteration method based on expanding nearby solutions in a perturbation series with exponential decay in Fourier modes. Our problem had special simplifying features that other small divisor problems in nonlinear PDE's did not have, such as the one dimensional one mode kernel for almost every period. It probably is a correct method, but the coupling of modes due to the nonlinearities makes the analysis beyond our abilities to estimate, our argument had a gap in continuing back to real perturbations in the complex plane, and we couldn't fix it. We had to give up on that approach. Since that time I have come to learn that no one else has achieved a proof of convergence of such a perturbation series in any similar problem [3], and our hope here was probably a bit naive, but we learned a lot.

The method we have been developing for the last several years is based on Nash-Moser, a modified Newton method that uses the quadratic convergence of Newton to beat out the loss of a bounded inverse under iteration. Since we had little familiarity with Nash-Moser techniques at the start, this has been very much a learning experience. We first tried to prove convergence by the methods outlined in Deimling [], based on graded spaces of analytic functions. The idea here is that the small divisors can be interpreted as a loss of smoothness when you apply the inverse of the linearized operator under iteration by Newton's method, so that the inverse of the linearized operator is bounded from a space of analytic functions to a space with fewer derivatives (as measured by exponential decay rates on Fourier modes). The idea then is that a larger inverse, (less and less change of grade) is allowed at each subsequent Newton iterate, because the quadratic convergence of Newton requires for weaker and weaker estimates on the inverses as the iteration proceeds. The hope then is that you get convergence of the grading to a large space where the limit solution exists. But after a lot of trying, we could not get the required estimates in the graded analytic spaces. Results of Caflisch [et al] show that a continuous loss of exponential smoothness under evolution by even Burgers equation is optimal. So essentially, it appeared that a minimum loss of grade is required for any bound on the inverse at all. Giving up on that, more recently we have been trying to adapt the methods based on graded smoothing operators applicable to C^s and H^s spaces as outlined in [Alhinac]. The spaces H^s and C^s are more natural for fully nonlinear conservation laws because smooth solutions stay in the same H^s and C^s spaces under time evolution, and so don't lose their grading under evolution as for analytic spaces. The idea here is to use estimates for the inverse of linearized operators from say $H^s \to H^{s+2}$, then use graded smoothing operators S_λ to lift the solution back up to H^s at the cost of amplification by factors of λ which one controls by the quadratic convergence of Newton. This past year we worked out a nice proof by Nash Moser based on estimating the leading order correction to our linearized operator under perturbation, and we proved that the leading order correction killed the small divisors, and regularized the inverses in Newton's method. The estimates are remarkable. We declared victory in the spring of 2011. But this past summer we found a gap. The problem is that the errors after the first order correction involve taking derivatives, and the loss of derivatives in the errors cannot be controlled unless you can prove $\|U_n\|_{s+1} \leq C_n \|U_n\|_s\|$ for every s under Newton iteration, and let quadratic convergence of Newton beat the growing value of C_n. Without this estimate there are resonances in linearized operators near $\mathcal{L}_0$. We thought we could get this estimate out of the Nash-Moser Newton method with graded smoothing by induction, but this summer we realized that this estimate is not in the cards. I was pretty certain we had a complete proof this past spring, but this summer we see there is a gap. But I believe we now understand what is missing and where to go from here. Our main new insight is this: We thought we could exploit the simplified structure of our kernel to get a proof that the linearized solutions perturb for *every* period, avoiding the period extraction techniques that have been required in virtually every other proof of periodic solutions, [ref Wayne etc.] . We have come to believe that this is just not in the cards for the estimates we have available for Nash-Moser Newton. And we have already developed a comprehensive framework for period extraction. So I will now try to explain all of this in more detail.

We have reduced the problem of periodic solutions to finding an element of the kernel of a nonlinear operator of the form

$$\mathcal{F}[U] = \mathcal{S} \circ \mathcal{J}^{-1} \circ \overline{\mathcal{E}} \circ \mathcal{J} \circ \underline{\mathcal{E}}[U] - U = 0, \tag{7}$$

where $U = U(t)$ is 2π periodic initial data at $y = 0$, $\overline{\mathcal{E}}$ is nonlinear evolution in space by our dimensionless compressible Euler equations through spatial interval $y = \overline{\theta}$ at entropy level $\overline{s}$, (since entropy is constant, this is 2×2 evolution by a p-system), $\underline{\mathcal{E}}$ is corresponding nonlinear evolution at entropy level $\underline{s}$ through interval $y = \underline{\theta}$, $\mathcal{J}$ and its inverse $\mathcal{J}^{-1}$ are the contact jumps between the entropy levels, and $\mathcal{S}$ is a half period shift in time. That is, for our dimensionless version of the Langrangian formulation of compressible Euler, t is time, y is space, and the problem is to find periodic data $U(t)$ at $y = 0$ that evolves through two consecutive entropy levels $\overline{s}$ and $\underline{s}$ separated by contact discontinuities at $y = \overline{\theta}$ and $y = \overline{\theta} + \underline{\theta}$, such that the time periodic initial data $U(t)$ returns to $U(t)$ at $y = (\overline{y} + \underline{y})+$ after a half period shift. (In our simplest dimensionless formulation of the problem, $U = (u, v)$ is even in u (the dimensionless density), odd in v (the dimensionless velocity), the nonlinear evolution operators $\mathcal{E}$ are independent of the entropy, the jump operators $\mathcal{J}$ depend on a single real parameter J and act only on the velocity v variables (not u!), and in the simplest case we can assume $\underline{\theta} = \theta = \overline{\theta}$, c.f. [78].) The half period shift essentially mixes up the compressions and rarefactions and is responsible for creating the balance in R's and C's along characteristics. Linearizing about the solution with normalized constant states at each entropy level, we obtain the linearized operator

$$\mathcal{L}_0[U] = \mathcal{S} \circ \mathcal{J}^{-1} \circ L_0 \circ \mathcal{J} \circ L_0[U] - U = 0, \tag{8}$$

where L_0 is linear evolution by a non-dimensionalized wave equation that converts linear evolution of Fourier modes into rotation of their representations in R^2. Now the five operators in (8) are non-symmetric, non-commuting and diagonal on Fourier modes, so they are represented by 2×2 matrices in each Fourier mode. And evolution by L_0 is represented by 2×2 rotation through angle $n\theta$ in the n'th Fourier mode. Using this framework, we prove that if

$$J = \cot^2(\theta/2),$$

then $\mathcal{L}_0$ has a solution $U(t) = Z(t)$ in the Fourier 1-mode, (given explicitly in terms of sines and cosines), and $\mathcal{L}_0$ is invertible with small divisors on the complement of the 0- and 1-mode kernels for almost every period. And $Z(t)$ has the wave structure that balances compression and rarefaction identified in [77]. Moreover, we prove in [78] that for special periods associated with Liouville numbers, we can estimate the small divisors (the eigenvalues in each 2×2 Fourier mode) from below by powers of n^{-k}, the best case being $k = 1$. We then look for solutions of (7) of the form

$$U(t) = \begin{pmatrix} 1+\delta \\ 0 \end{pmatrix} + \epsilon Z(t) + \epsilon^2 W(t). \tag{9}$$

Here $U(0) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ produces the normalized constant solution whose linearization is L_0, the parameter δ parameterizes the zero mode kernel, and $Z(t)$ is the one mode kernel. To solve $\mathcal{F}[U] = 0$ by a Newton method then, we need bounds on the inverses of "nearby linearized operators"; that is, for the linear operators near $\mathcal{L}_0$ obtained by linearizing the Euler evolutions in (7) around solutions $U(y,t)$ obtained by evolving (9) from $y = 0$ to $y = 2\theta$. The problem then is that linear operators arbitrarily close to $\mathcal{L}_0$ can have resonances and hence not be invertible on the complement of the kernel of $\mathcal{L}_0$.

So here is our latest proposal. First, we vectorize the problem by making the two evolutions independent. For this, redefine $U(t) \equiv (\overline{U}(t), \underline{U}(t))$, a 4-component vector. This has the effect of separating the jump and shift operators from the evolutions in the re-expression of (7), which can now be written in the form

$$\mathcal{F}[U] = (\mathcal{E} - \mathcal{J})\,[U],$$

where now $\mathcal{E}$ is two 2×2 Euler evolutions in non-dimensionalized variables operating on the 4-vector $U(t)$, one p-system at each entropy level; and $\mathcal{J}$ is the linear operator on the four component function $U(t)$ consisting of the jumps and shifts that reproduce the effect of the jumps and shifts on the upper and lower entropy levels in (7). In this framework the linearized operator associated with initial data $U(t)$ takes the simplified form

$$D_U\mathcal{F} = D_U\mathcal{E} - \mathcal{J}, \tag{10}$$

which operators on $V(t)$ in some space, say C^s or H^s. Now the linearization of the evolution $\mathcal{E}$ around $U(y,t)$ consists of a first order operator that transports as a linear conservation law along the frozen characteristics of $U(y,t)$, with lower order corrections due to a zero order term on the right hand side of this linear conservation law. Ignoring the lower order term (which can be incorporated into the framework later), we can express the linearized evolution along the frozen characteristics of $U(y,t)$ as a shift operator which transports Riemann invariants as constant along characteristics, and write

$$D_U\mathcal{E} = L_\Phi, \tag{11}$$

where L_Φ shifts the inputs of the Riemann invariants by the shift in the constant coefficient operator L_0, (which has constant speed one characteristics in non-dimensionalized variables), plus a correction based on the non-constant characteristics of the nearby linearized operator. E.g., with some abuse of notation, we write $L_\Phi[V] = V(t + \Phi(t))$. (This is really a four component shift after transformation to Riemann invariants, but we can think of it like this as a pure shift.) Now for perturbations of the form (9),

$$\Phi(t) = \delta + \epsilon\phi(t),$$

where δ represents a constant shift from L_0 due to the change of constant state, an important free parameter in the problem. It is not difficult to show that a change δ in the 0-mode is equivalent to a change in period $\theta \to \theta + \delta$. The shift function $\phi = \phi_0 + \epsilon\phi_1$, where ϕ_0 is the leading order correction due to the non-constant characteristics of $Z(t)$, and $\epsilon\phi_1$ is the correction for an arbitrary nearby solution $U(y,t)$ of form (9). With this notation we recover L_0 when $\delta = \epsilon = 0$. (Again, these are representing four component shifts in the Riemann invariants for pure linearized Euler evolutions.)

One could now summarize the main problem in applying Nash-Moser Newton to contruct periodic solutions of a fully nonlinear problem. Namely, the linearized operators coming from the conservation law are essentially shift operators, difficult to express in Fourier modes, while the baseline operator $\mathcal{L}_0$ that we try to estimate from is constant coefficient, and gets it meaning in the Fourier decomposition that diagonalizes it. The two are highly incompatible.

So now we can describe the main issues and how we propose to address them. To apply a Newton method to solve $\mathcal{F}[U] = 0$ for $0 < \epsilon << 1$, we need to apply the inverse $D_U^{-1}[U]$ on the complement of the kernel of $\mathcal{L}_0$. (Assume we have Liapunov-Schmidted away the kernel.) That is, we need "bounds on the inverses of the nearby linearized operators"; operators near $\mathcal{L}_0$ based on linearizing solutions of form (9). With this notation, we can write

$$D_U\mathcal{F} = L_0 - \mathcal{J} + (L_{\delta+\epsilon\phi} - L_0) = \mathcal{L}_0 - (L_{\delta+\epsilon\phi} - L_0)\,. \tag{12}$$

That is, each nearby linearized operator is our baseline linearized operator $\mathcal{L}_0$ (constant coefficient with small divisors) plus a linear operator that shifts the inputs relative to $\mathcal{L}_0$, based on the divergence of the characteristics of $U(y,t)$ from the constant characteristics of $\mathcal{L}_0$. Now it is easy to see that $(L_{\delta+\epsilon\phi} - L_0)$ is a bounded operator from $H^s \to H^s$ because these shift operators derive from evolution of a linear PDE that takes H^s boundedly into H^s. The problem is that this is not an $O(\delta+\epsilon)$ perturbation of $\mathcal{L}_0$.

Lemma 1 $(L_{\delta+\epsilon\phi} - L_0)$ *is a bounded linear operator from* $H^s \to H^s$*, but its norm is not* $O(\delta+\epsilon)$*.*

Proof: *(Scalar case.) By the mean value theorem,*

$$\begin{aligned}(L_{\delta+\epsilon\phi} - L_0)[V] &= V(t+\delta+\epsilon\phi(t)) - V(t) \\ &= V'(t+\sigma(t))(\delta+\epsilon\phi(t)),\end{aligned} \tag{13}$$

and operating on $V(t) = sin(nt)$*,*

$$\|V'(t+\sigma(t))\|_s \geq O(n)\|V(t)\|_s.$$

The problem, of course, is that $(L_{\delta+\epsilon\phi} - L_0)$ is only $O(\delta+\epsilon)$ in the low modes $n \leq O(\delta+\epsilon)^{-1}$.

The next lemma shows that this is an $O(\epsilon+\delta)$ perturbation from $H^{s+1} \to H^s$.

Lemma 2

$$\|\,(L_{\delta+\epsilon\phi} - L_0)\,[V(t)]\|_s \leq O(\delta+\epsilon)\|V\|\|_{s+1}.$$

Proof: *(Scalar case.) By the mean value theorem,*

$$\|\,(L_{\delta+\epsilon\phi} - L_0)\,[V]\|_s = \|\int_0^{\delta+\epsilon\phi} V'(t+\nu)d\nu\|_s \leq \|\delta+\epsilon\phi(t)\|_s\|V\|_{s+1}.$$

Lemma 3 $L_\delta - L_0$ *has resonances.*

Proof: *(*2×2 *case:) Using our representation of evolution by* L_0 *as rotation by* $n\theta$ *on the* 2*-dimensional representation the* n*'th mode gives:*

$$\begin{aligned}(L_\delta - L_0)\,\tau_n q &= \tau_n\left[(R(n\theta\sigma(1+\delta) - R(n\theta))\,q\right] \\ &= \tau_n\left[R(n\theta\sigma(1+\delta)-1)R(n\theta)q\right] \\ &\geq |q|Kdist\left(\frac{n\theta\,(\sigma(1+\delta)-1)}{2\pi}, \mathcal{Z}\right) \geq K\frac{1}{n^r}.\end{aligned} \tag{14}$$

(Here, $q \in R^2$ is the representation of the Fourier n-mode $\tau_n q$, c.f. [76].) That is we can obtain the latter upper bound for special periods associated with Liouville numbers, but the argument shows that you cannot rule out small divisors or resonances for any δ. The main point: A change in supnorm corresponds to a change in period, and we have a complete understanding of the resonant periods and estimates for the small divisors under change of period through our representation of L_0 evolution by rotations of the representations in R^2. From this we conclude that all nearby linearized operators have resonances as well. This leads to two possible ways to go for a Nash-Moser Newton method based on Fourier analysis. We can (1) control the inputs of the inverses of the linearized operators so as to apply the inverses on functions whose higher derivatives are bounded by lower derivatives, like analytic functions. This is what we thought we could do in the spring of 2011, but the required estimate for relative sizes of derivatives fall on the difference between successive Newton approximations, not the approximations themselves, and this estimate is not in the cards for the Newton method. Alternatively, (2) we must face the reality that we have resonances in nearby linearized operators, and cannot get the basic loss of derivative estimate (5) of [30] for a Newton method on the whole complement of the kernel of $\mathcal{L}_0$ at every iteration.

Thus, our proposal, essentially the only way left to go for a Nash-Moser Newton method based on Fourier modes, is to expunge periods in a framework that builds in a finite Fourier mode cutoff at each iteration. This looks most promising because we have the effect of L_δ completely characterized through the rotation of the representations of Fourier modes.

So here is the basic outline of the proposal. Consider the three steps of the Newton method:

$$(I)\ y_{n+1} = \mathcal{F}(U_n); \quad (II)\ V_{n+1} = D\mathcal{F}^{-1}_{U_n}[y_{n+1}] \quad (III)\ U_{n+1} = U_n - V_{n+1}.$$

Assume we have U_n and induct U_{n+1}. Cut off the y_{n+1} at the $n+1$-Fourier mode to make $\bar{y}_{n+1}$, and set

$$\bar{V}_{n+1} = D\mathcal{F}^{-1}_{U_n}[\bar{y}_{n+1}],$$

where by (12),

$$D\mathcal{F}[V] = \mathcal{L}[V] + \left(L_\delta \circ L_{\epsilon\phi(t)} - L_0\right)[V].$$

Now because we essentially know everything about L_δ, we expunge the small measure set of δ on which our inverse in (II) has resonances on the finite dimensional space of $\bar{y}_{n+1}$. Moreover, using the graded smoothing and Nash-Moser method in $Nash-Moser-Writeup$, we can allow the estimates on the inverse to get larger as $n \to 0$, allowing us to expunge smaller and smaller sets of δ as we iterate. The conclusion will be that a set of finite measure in δ will remain at the end on which Nash-Moser-Newton converges to a periodic solution of the nonlinear problem. Of course this is the basic proposal, and we must build the framework in the context of the full 4×4 problem.

In conclusion: new idea is to use our understanding of the supnorm shift=period shift to control the size of derivative on a finite Fourier mode cutoff on the complement of a small neighborhood of the resonant periods at each iteration. Then by exploiting the quadratic convergence of Newton we can allow for growing derivative estimates, and hence a smaller measure of the expunged periods at each iteration, implying that a positive measure set periods for convergence will remain at the end.

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