Nash-Moser for Euler Newton

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1 Nash Moser

We implement the Nash-Moser iteration, a Newton method that employs graded smoothing following the development in [1].[1] To start, let

$$U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \epsilon m_0) + \epsilon Z, \tag{1}$$

and

$$\mathcal{F}[U(\cdot)] = U(\theta, \cdot) - \mathcal{J}[U(\cdot)],$$

where $U(y,t)$ solves

$$U_y + \sigma(u) H U_t = 0. \tag{2}$$

We assume the following two estimates. The first is Taylor's Theorem,

$$\mathcal{F}[U - v] - \mathcal{F}[U] + D_U\mathcal{F}[v] \equiv Q_U[v], \tag{3}$$

where

$$\|\mathcal{Q}_U[v]\|_s \le K_2 \|v\|^2_{s+p_2}. \tag{4}$$

The second one is our estimate for the inverse of the linearized operator,

$$\|D\mathcal{F}^{-1}(y)\|_s \le \frac{K_1}{m_0 \epsilon} \|y\|_{s+p_1}. \tag{5}$$

The main deficiency of the standard Newton method is that the estimates (4), (5) entail a loss of derivatives at every stage, so to compensate for this, we smooth the errors by the smoothing operators S_λ of Alinhac [1]. The smoothing operators S_λ then take the union of all C^k to C^∞, and satisfy the following basic estimates which we use here: (Here e represents an error arising in the Newton method.)

$$\|e - S_\lambda e\|_{s-p} \le K_0 \lambda^{-p} \|e\|_s \quad p \ge 0, \tag{6}$$

[1] See [2] for an exposition of Nash-Moser Newton methods in analytic graded spaces.

$$\|S_\lambda e\|_s \le K_0 \lambda^p \|e\|_{s-p} \quad p \ge 0, \tag{7}$$

and

$$\|\frac{d}{d\lambda} S_\lambda e\|_s \le K_0 \lambda^p \|e\|_{s-p}, \quad p \le s. \tag{8}$$

The estimates (7) (8) imply that operating with S_λ compensates a gain of derivatives with a power of λ, and (6) implies that we recover the true errors e in the limit $\lambda \to \infty$. The use of (4)-(5) in the Newton method is then to replace a loss of derivatives with a power of λ, then use the quadratic convergence of the Newton method to overcome some choice of $\lambda \to \infty$, implying convergence of the modified Newton method.

We now define the modified Newton iteration. Assume for induction that $U_0, ..., U_n$, Y_n and E_n have been defined, with

$$Y_n = \sum_{k=1}^{n} y_k, \tag{9}$$

$$E_n = \sum_{k=1}^{n} e_k. \tag{10}$$

To define U_{n+1} by induction, set

$$U_{n+1} = U_n - v_{n+1}, \tag{11}$$

where v_{n+1} is defined by

$$D_{U_n}\mathcal{F}(v_{n+1}) = y_{n+1}, \tag{12}$$

and y_{n+1} is to be chosen. Before choosing y_{n+1}, define

$$e_{n+1} = \mathcal{F}[U_{n+1}] - \mathcal{F}[U_n] + D_{U_n}\mathcal{F}[v_{n+1}] = Q_{U_n}[v_{n+1}], \tag{13}$$

so by (4),

$$\|e_{n+1}\|_s \le K_2 \|v_{n+1}\|^2_{s+p_2} \tag{14}$$

holds for every s. Summing from $k = 0$ to n thus gives

$$E_n + e_{n+1} = \mathcal{F}[U_{n+1}] - \mathcal{F}[U_0] + Y_n + y_{n+1}, \tag{15}$$

which is an exact expression, expressing e_{n+1} in terms of y_{n+1}. For a regular Newton method, we would set e_{n+1} exactly equal to $\mathcal{F}[U_{n+1}]$, which would determine y_{n+1} through (15). To incorporate graded smoothing in such a way as to control the loss of derivatives and keep all errors higher than first order, we define,

$$y_{n+1} = \mathcal{F}[U_0] - Y_n + S_{\lambda_{n+1}} E_n, \tag{16}$$

where $S_{\lambda_{n+1}}$ are the smoothing operators of Alinhac. (The point is that the y_n are smooth, and this is compensated for in the e_n which may not be smooth.) The λ_n, chosen later to satisfy $\lambda_n \to \infty$, give the modulus of smoothing at the n'th iteration, and $S_{\lambda_{n+1}}E_n \to E_n$ as $n \to \infty$, c.f. [1]. This then completes the definition of the Newton iteration.

Subtracting, (16) gives

$$y_{n+1} - y_n = -Y_n + Y_{n-1} + S_{\lambda_{n+1}}E_n - S_{\lambda_n}E_{n-1}, \tag{17}$$

which simplifies to

$$y_{n+1} = \left(S_{\lambda_{n+1}} - S_{\lambda_n}\right)E_{n-1} + S_{\lambda_{n+1}}e_n, \tag{18}$$

because all the linear terms y_i telescope out. In particular, note that all remaining terms on the RHS are multiples of the quadratic errors e_i. That is, according to (14), (12) and (5), we have

$$\|e_k\|_s \leq K_2\|v_k\|^2_{s+p_2} \leq \frac{K_2K_1^2}{m_0^2\epsilon^2}\,\|y_k\|^2_{s+p}, \tag{19}$$

$k = 1, ..., n$, where we have set $p \equiv p_1 + p_2 > 0$.

To estimate (18), use the Mean Value Theorem for (18), (c.f. [1]), to get

$$\|y_{n+1}\|_s = (\lambda_{n+1} - \lambda_n)\,\|\frac{d}{d\lambda}S_\lambda E_{n-1}\|_s + \|S_{\lambda_{n+1}}e_n\|_s, \tag{20}$$

and estimate the two terms separately. For the second term, use (7) and (19) to estimate,

$$\|S_{\lambda_{n+1}}e_n\|_s \leq K_0\lambda^p_{n+1}\|e_n\|_{s-p} \leq \lambda^p_{n+1}K_*\|y_n\|^2_s, \tag{21}$$

where

$$K_* = K_0\frac{K_1^2K_2}{m_0^2\epsilon^2}. \tag{22}$$

For the first term in (20), use (7) and (19) to estimate

$$\begin{aligned}\|\left(S_{\lambda_{n+1}} - S_{\lambda_n}\right)e_k\|_s &\leq (\lambda_{n+1} - \lambda_n)\|\frac{d}{d\lambda}S_{\lambda_n}e_k\|_s \\ &\leq (\lambda_{n+1} - \lambda_n)K_0\lambda_n^{-q-1}\|e_k\|_{s+q} \\ &\leq (\lambda_{n+1} - \lambda_n)\lambda_n^{-q-1}K_*\|y_k\|^2_{s+q+p},\end{aligned} \tag{23}$$

and here we take $q \geq -s$. Combining (21) and (23) gives

$$\|y_{n+1}\|_s \leq (\lambda_{n+1} - \lambda_n)\lambda_n^{-q-1}K_*\sum_{k=1}^{n-1}\|y_k\|^2_{s+q+p} + \lambda^p_{n+1}K_*\|y_n\|^2_s, \tag{24}$$

which we use for $q \geq -p$. To remove the constant K_* from (24), set

$$z_n(s) = K_*\|y_n\|_s,$$

in which case (24) simplifies to our final form,

$$z_{n+1}(s) \leq (\lambda_{n+1} - \lambda_n)\,\lambda_n^{-q-1}\sum_{k=1}^{n-1} z_k^2(s+q+p) + \lambda_{n+1}^p\, z_n^2(s), \tag{25}$$

which holds for $q \geq -p$.

We now estimate (25) by a bootstrap argument. To motivate this, observe first that (25) is a family of estimates for the same $z_n(s)$, the family depending on the choice of q. Note that the second term on the right hand side of (25) is bounded by the purely quadratic factor $z_n^2(s)$, and is dominated by the first term because the term $\sum_{k=1}^{n-1} z_k^2(s+q+p)$ should be bounded and small, but being a cumulative sum, does not tend to zero. Note also that because of the term $\sum_{k=1}^{n-1} z_k^2(s+q+p)$, the iteration (25) closes within the same $\|\cdot\|_s$ norm only in the case when $q = -p$. But we see presently that when $q = -p$, choosing $\lambda_n \to \infty$ as appropriate small powers of n, one can prove that $z_n(s) \to 0$, but the estimates are not sufficient to prove that $\sum_{n=1}^{\infty} z_n(s)$ is finite. The problem is that when $q \leq 0$, the power λ_n^{-q-1} in the first term is larger than the critical power λ_n^{-1}, and this is too slow for $z_n(s)$ to sum. In light of this, our strategy is to first show there exist optimal λ_n sufficient to prove the apriori estimate $\sum_{n=1}^{\infty} z_n^2(s) < \infty$, even though $\sum_{n=1}^{\infty} z_n(s) = \infty$ cannot be ruled out. We then use this apriori estimate in (25) with values of $q > 0$ to get an improved convergence rate, sufficient to conclude that $\sum_{n=1}^{\infty} z_n(s) \leq O(z_1(s+q+p))$, that is, so long as z_1 is measured $q+p$ derivatives higher than s.

Consider then the first case $q = -p$, in which case (25) becomes

$$z_{n+1}(s) \leq (\lambda_{n+1} - \lambda_n)\,\lambda_n^{p-1}\sum_{k=1}^{n-1} z_k^2(s) + \lambda_{n+1}^p\, z_n^2(s), \tag{26}$$

a closed iteration for $\underline{s}+p < s < \overline{s}$. (We needed p derivatives below s to estimate $\|e_k\|_{s-p}$ by $\|y_k\|_s^2$, c.f. (19).) We now find $\lambda_n \to \infty$ sufficient to prove that $z_n(s) \to 0$ and $\sum_{n=1}^{\infty} z_n^2(s) < \infty$ as $n \to \infty$. Since $\lambda_n \to \infty$ and $\sum_{n=1}^{\infty} z_n^2(s) \not\to \infty$ as $n \to \infty$, we must have $(\lambda_{n+1} - \lambda_n) \to 0$ in order for the first term on the right hand side of (26) to tend to zero. This then limits the growth rate on λ_n. For this, a serendipitous choice of λ_n is

$$\lambda_n = \lambda_1 n^b, \tag{27}$$

with

$$\lambda_1 = 1, \tag{28}$$

and b to be chosen, $0 < b < 1$. (One could in principle use λ_1 as an adjustable parameter, but for our purposes here, $\lambda_1 = 1$ suffices, c.f. [1]). [2]

[2] To motivate (27), note that the obvious choice for λ is a power of n, or r^n for some $r > 1$. Since the iteration indicates that z_n grows by some negative power $\lambda_n^{-\alpha}$ of λ, the choice $\lambda_n = r^n$ grows too fast, but the choice (27) works, and assuming this, we look to estimate z_n in (26)by n^{-a}, for a to be found, depending on b

Assuming (27), we can write

$$\lambda_{n+1} - \lambda_n = (n+1)^b - n^b = b\,(n+\theta)^{b-1} \le b n^{b-1}, \tag{29}$$

and using this in (26) gives

$$z_{n+1}(s) \le b n^{b-1} n^{b(p-1)} \sum_{k=1}^{n-1} z_k^2(s) + (n+1)^{bp} z_n^2(s). \tag{30}$$

We now use (30) to prove by induction that for appropriately chosen a,

$$z_n(s) \le z_1(s) n^{-a}, \quad \forall\, n \ge 1. \tag{31}$$

Assuming (31) for induction, (30) gives

$$z_{n+1}(s) \le z_1^2(s) \left[b n^{bp-1} \sum_{k=1}^{n-1} k^{-2a} + (n+1)^{bp} n^{-2a} \right], \tag{32}$$

which implies the desired

$$z_{n+1}(s) \le z_1(s)(n+1)^{-a}, \tag{33}$$

under the sufficient condition

$$z_1(s) \left[b n^{bp-1} \sum_{k=1}^{n-1} k^{-2a} + (n+1)^{bp} n^{-2a} \right] \le (n+1)^{-a}. \tag{34}$$

By the integral test,

$$\sum_{k=1}^{\infty} k^{-2a} \le \frac{2a}{2a-1}, \tag{35}$$

provided $a > 1/2$, and using this in (34) gives

$$z_1(s) \le \min_{n \ge 1} \left\{ \frac{(n+1)^{-a}}{\left[b n^{bp-1} \sum_{k=1}^{n-1} d^{-2a} + (n+1)^{bp} n^{-2a} \right]} \right\}_*, \tag{36}$$

as a condition sufficient for (33). To meet this, set

$$a = 1 - bp, \quad \text{and} \quad 1/2 < a < 1, \tag{37}$$

forcing

$$0 < b < \frac{1}{2p}. \tag{38}$$

Putting (38) into (36) then leads to

$$\{\cdot\}_* = \left\{ \frac{(1+1/n)^{bp-1}}{\frac{2b(1-bp)}{1-2bp} + (1+1/n)^{bp} n^{2bp-1}} \right\}_*. \tag{39}$$

All terms in the bracket are now monotone in n, implying $\{\cdot\}_*$ is an increasing function of n, so replacing $n = 1$ gives

$$\begin{aligned}
\{\cdot\}_* &\geq \frac{2^{bp-1}}{\frac{2b(1-bp)}{1-2bp} + 2^{bp} n^{2bp-1}} = \frac{1-2bp}{2^{2-2bp}(1-bp)b + 2(1-2bp)} \\
&\geq \frac{1-2bp}{4b+2} \geq \left(\frac{1-2bp}{4}\right),
\end{aligned} \tag{40}$$

where we have used $0 < bp < 1/2$, and $p \geq 1$. We therefore conclude from (34) and (40) that $z_{n+1}(s) \leq z_1(s) n^{-a}$ under the sufficient conditions

$$z_1(s) \leq \frac{1-2bp}{4} = \frac{2a-1}{4}, \quad a = 1-bp, \quad 0 < bp < 1/2, \tag{41}$$

where $s \in (\underline{s}+p, \overline{s}]$ (to estimate $\|e_n\|_{s-p}$ by $z_n^2(s)$). We have thus proven the following lemma:

Lemma 1 *Assume $z_n(s)$ satisfy (26) for every $s \in [\underline{s}, \overline{s}]$, assume $p \geq 1$, $0 < b < 1/2p$, and set $a = 1 - bp$, so that*

$$1/2 < a < 1.$$

Assume that

$$z_1(s) \leq \frac{2a-1}{4}. \tag{42}$$

Then for every $s \in [\underline{s} + p, \overline{s}]$,

$$z_n(s) \leq z_1(s) n^{-a},$$

and

$$\sum_{k=1}^{\infty} z_n^2(s) \leq \frac{2a}{2a-1} z_1^2(s). \tag{43}$$

For the second part of the bootstrap, consider next the case $q \geq 0$, so that (25) gives,

$$z_{n+1}(s) \leq (\lambda_{n+1} - \lambda_n)\, \lambda_n^{-q-1} \sum_{k=1}^{n-1} z_k^2(s+q+p) + \lambda_{n+1}^p\, z_n^2(s).$$

Then assuming $s \in [\underline{s} + p, \overline{s} - q - p]$, we can use (43) to estimate

$$z_{n+1}(s) \leq (\lambda_{n+1} - \lambda_n)\, \lambda_n^{-q-1}\, z_1^2(s+p+q)\, \frac{2a}{2a-1} + \lambda_{n+1}^p\, z_n^2(s),$$

and now using (27) and (29) in this we get

$$\begin{aligned}
z_{n+1}(s) &\leq n^{-1-bq}\, z_1^2(s+p+q)\, \frac{2ab}{2a-1} + (n+1)^{bp}\, z_n^2(s) \\
&= M n^{-1-bq} + (n+1)^{bp}\, z_n^2(s),
\end{aligned} \tag{44}$$

where (using $bp = 1 - a$)

$$M = \frac{2a(1-a)}{p(2a-1)} z_1^2(s+p+q). \tag{45}$$

We now prove by induction that

$$z_n(s) \le z_1(s) n^{-1-r}, \tag{46}$$

true by definition at $n = 1$. So assume (46) at n. We find conditions on z_1 sufficient for (46) to hold at $n+1$. Multiplying (44) by $(n+1)^{1+r}$ gives

$$(n+1)^{1+r} z_{n+1}(s) \;\; \le \;\; M\left(1+\frac{1}{n}\right)^{1+r} n^{r-bq} + \left(1+\frac{1}{n}\right)^{1+r+bp} n^{-1-r+bp} z_1^2(s). \tag{47}$$

Now for this to be bounded, the powers of n must be bounded, and the condition for this is

$$-1 + bp \le r \le bq. \tag{48}$$

Since $bp < 1/2$, the first term in (47) dominates, so we now fix r at the optimal choice for $q > 0$, namely, choose

$$r = bq.$$

Using this in (47) gives

$$(n+1)^{1+r} z_{n+1}(s) \;\; \le \;\; M 2^{1+bq} + 2^{1+bq+bp} z_1^2(s),$$

which gives

$$(n+1)^{1+r} z_{n+1}(s) \;\; \le \;\; z_1(s), \tag{49}$$

provided (c.f. (45),

$$2^{2+bq} \frac{a(1-a)}{p(2a-1)} z_1^2(s+p+q) + 2^{1+bq+bp} z_1^2(s) \le z_1(s).$$

Using $z_1(s) \le z_1(s+p+q)$ and $bp < 1$, we obtain following condition on $z_1(s)$ sufficient to conclude (46) by induction:

$$2^{2+bq}\left\{\frac{a(1-a)}{p(2a-1)} + 1\right\} z_1^2(s+p+q) \le z_1(s).$$

A convenient value is $a = 1 - bp = 3/4 \in (1/2, 1)$, so $bp = 1/4 \in (0, 1/2)$, $q = 4pr$, $p \ge 1$ and

$$2^{2+bq}\left\{\frac{a(1-a)}{p(2a-1)} + 1\right\} \le 2^{3+r},$$

give the condition

$$2^{3+r} z_1^2(s+p+q) \le z_1(s) \tag{50}$$

as sufficient for $z_n(s) \le z_1(s) n^{-1-r}$. Putting the two inductions together, we have the following Theorem:

Theorem 2 *Assume $p \geq 1$, $q = 4pr$. and assume $s \in (\underline{s} + p, \overline{s} - p - q)$. Assume further that $z_1(s)$ satisfies*

$$z_1(s + p + q) \leq \frac{1}{8}, \tag{51}$$

and

$$z_1^2(s + p + q) \leq 2^{-3-r} z_1(s), \tag{52}$$

(so that (52) is sufficient for (52) when $z_1(s) < 1$.) Then the Newton iterates $z_n(s)$ converge and satisfy

$$z_n(s) \leq z_1(s) n^{-1-r}. \tag{53}$$

Proof: Condition (51) is condition (42) of Lemma 1 of the first induction in the boostrap argument, and condition (53) is (50) of the second induction.

We can now prove convergence of our Newton method as follows:

Recall that

$$y_1 = \mathcal{F}(U_0), \tag{54}$$

where U_0 is our ellipse

$$U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \epsilon m_0) + \epsilon Z. \tag{55}$$

Now we have that there exists a constants K_3 and K_4 such that

$$\|y_1\|_s = \|\mathcal{F}(U_0)\|_s \leq K_3 \epsilon^2, \tag{56}$$

and

$$\|\mathcal{F}(U_0)\|_s \geq K_4 \epsilon^2, \tag{57}$$

for every $s \in [\underline{s}, \overline{s}]$. Also, by definition,

$$z_n(s) = K_* \|y_n\|_s. \tag{58}$$

It follows that conditions (52) of Theorem 2 becomes

$$K_* \|\mathcal{F}(U_0)\|_{s+p+q}^2 \leq 2^{-3-r} \|\mathcal{F}(U_0)\|_s. \tag{59}$$

Using (56)with $s = s + p + q$, and the definition of K_*, we get

$$K_* \|y_1\|_{s+p+q}^2 \leq \frac{K_0 K_1^2 K_2 K_3^2}{m_0^2} \epsilon^2,$$

and from (57), we get

$$2{-}3 - r\|y_1\|_s \geq 2^{-3-r} K_4 \epsilon^2,$$

so (59) holds provided we choose m_0 large enough, namely

$$m_0^2 \geq K_0 K_1^2 K_2 K_3^2 2^{3+r} / K_4. \tag{60}$$

The reason we are getting conditions on m_0 is: although $\|y_i\| = O(\epsilon^2)$, the constant K_* has $\epsilon^2 m_0^2$ in the denominator, so the z_i are only $O(m_0^{-2})$. This means that we need m_0 large to make the z_1 small enough that the induction works.

Thus the conclusion of (53) Theorem holds, namely

$$z_n(s) \leq z_1(s) n^{-1-r}.$$

This then implies

$$\|y_n\|_s \leq \|y_1\|_s n^{-1-r} \leq \frac{r+1}{r} \|y_1\|_s.$$

But now we can estimate

$$\|U_N - U_0\|_s \leq \sum_{k=1}^{N} \|v_k\|_s \leq \frac{K_1}{m_0 \epsilon} \sum_{k=1}^{N} \|y_k\|_{s+p} \leq \frac{K_1}{m_0 \epsilon} \frac{r+1}{r} \|y_1\|_{s+p}.$$

which by (56) gives

$$\|U_N - U_0\|_s \leq K_1 \frac{r+1}{r} K_3 \frac{\epsilon}{m_0} \leq \frac{1}{2} \epsilon,$$

so long as

$$m_0 > 2 K_1 K_3 \frac{r+1}{r}.$$

This completes the proof.

Keep in mind that this all requires our estimate on $D\mathcal{F}^{-1}$ for $\epsilon/m_0 << 1$. That is, we need

$$U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \epsilon m_0) + \epsilon Z + \frac{\epsilon}{m_0} W,$$

with $\epsilon/m_0 << 1$ must give us

$$\|D\mathcal{F}^{-1}(y)\|_s \geq \frac{K_1}{m_0 \epsilon} \|y\|_{s+p}.$$

2 Leading Order Estimates for the Inverses:

We write down the formulas for the ϵ-order corrections to our linearized operator with small divisors and show that, on solutions with enough regularity, or under Fourier cutoff of high modes, the ϵ-order part kills the small divisors. We incorporate corrections that account for a zero mode drift term $\delta = m_0\epsilon$ to kill the element of the kernel that has zeros in all the v-entries. We use the notation in [5], Incorporating a zero mode drift term ϵm_0 as follows:

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon m^0 \\ 0 \end{pmatrix} + \epsilon Z^0 + \epsilon^2 W, \tag{61}$$

with the otherwise unchanged notation,

$$U = \begin{pmatrix} w^* \\ v^* \end{pmatrix}; \quad Z^0 = \begin{pmatrix} w^0 \\ v^0 \end{pmatrix}; \quad W = \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}, \tag{62}$$

so that

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon m^0 + w^0 \\ v^0 \end{pmatrix} + \begin{pmatrix} w_2 \\ v_2 \end{pmatrix},$$

where m^0 is the zero mode drift constant, $Z^0 = (w^0, v^0)$ is the sinusoidal function of t giving the elliptical $1-$ mode kernel, and $W = (w_2, v_2)$ is an arbitrary perturbation. Note that m_0 only changes the formula for $\mathcal{A}$ at the bottom of page 21, entering by just adding the constant $m^0\theta$ to the formulas for $I^{\pm}(t)$ given at the end. This then adds an order epsilon operator of the form "constant times derivative", which thus takes n-modes to n times n-modes.

Discussion: A motivation for putting the drift ϵm^0 into the ansatz is that really, the zero mode and 1-mode together form a two dimensional kernel, and even though we have set the constant average density to $(1, 0)$ in non-dimensionalizing the fully nonlinear problem, the epsilon order linearized equations do not respect conservation of mass, so it is reasonable to incorporate a parameter that sets the relative size of the zero and one mode kernels. In fact, even if it turns out that only the entropy drift $\mathcal{J} + \epsilon\mathcal{J}'$ and zero order drift $1 + \epsilon m^0$ are required to kill the kernel of the resonant operator, the main purpose of putting Z^0 remains to keep the nonlinear perturbation from giving the constant state solution under Newton iteration. That is, it could well be that the fact that our linearized operator is invertible off the kernel for almost every period, really tells us that we can solve for all of the Fourier modes, we just can't solve for them uniformly in ϵ, so genuine nonlinearity may not be the main issue in killing the small divisors. That is, our kernel is so fantastically isolated, that a small drift in the zero mode and entropy jump is all that is required to kill the small divisors and solve for the nonlinear modes of the periodic solution.

The formulas for the ϵ-order operators recorded that incorporate a zero mode drift term to kill the element of the kernel that has zeros in all the v-entries, is recorded in the following Theorem, (c.f. [5]):

Theorem 3 *Let our four component operator be denoted*

$$D_U\mathcal{E} = \begin{pmatrix} D_{U_1}\mathcal{E} \\ D_{U_2}\mathcal{E} \\ D_{U_3}\mathcal{E} \\ D_{U_4}\mathcal{E} \end{pmatrix}, \tag{63}$$

where

$$U_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon m^0 \\ 0 \end{pmatrix} + \epsilon Z_k^0 + \epsilon^2 W_k, \tag{64}$$

and the 2×2 *linearized operator* $D_{U_k}\mathcal{E}$ *is given by:*

$$D_{U_k}\mathcal{E} = \mathcal{L}_k[V] + \epsilon d\mathcal{A}_k[V] + \epsilon d\mathcal{B}_k[V] + O(\epsilon^2), \tag{65}$$

where

$$\mathcal{L}_k[V](t) = \begin{pmatrix} r(t+\theta_k) \\ s(t-\theta_k) \end{pmatrix}; \quad V = \begin{pmatrix} w \\ v \end{pmatrix}; \quad \begin{pmatrix} r = \frac{w-v}{2} \\ s = \frac{w+v}{2} \end{pmatrix}, \tag{66}$$

$$\mathcal{A}_k[V](t) = \mathcal{A}_k^0[V](t) + \mathcal{A}_k^1[V](t) \tag{67}$$

where

$$\mathcal{A}_k^0[V](t) = \begin{pmatrix} -m^0\theta_k r'(t+\theta_k) \\ +m^0\theta_k s'(t-\theta_k) \end{pmatrix} \tag{68}$$

$$\mathcal{A}_k^1[V](t) = \begin{pmatrix} -I_k^-(t) r'(t+\theta_k) \\ +I_k^+(t) s'(t-\theta_k) \end{pmatrix} \tag{69}$$

and

$$\mathcal{B}_k[V](t) = \begin{pmatrix} -r_k^{0\prime}(t+\theta_k)\left(r(t+\theta_k)\theta_k + \int_0^{\theta_k} s(t+\theta_k-2y)dy\right) \\ +s_k^{0\prime}(t-\theta_k)\left(s(t-\theta_k)\theta_k + \int_0^{\theta_k} r(t-\theta_k+2y)dy\right) \end{pmatrix}. \tag{70}$$

Then $I^{\pm}(t) = \int_0^{\theta} w^0(y, t \pm y \mp \theta)dy$ *leads to:*

$$\begin{matrix} k=1 \\ \theta_1 = \overline{\theta} \end{matrix} : \begin{matrix} I_1^-(t) &=& c(t+\frac{\overline{\theta}}{2})\frac{\overline{\theta}+s(\overline{\theta})}{2} \\ I_1^+(t) &=& c(t-\frac{\overline{\theta}}{2})\frac{\overline{\theta}+s(\overline{\theta})}{2} \end{matrix} : \begin{matrix} r_1^{0\prime}(t+\underline{\theta}) = -\frac{1}{2}s(t+\frac{\overline{\theta}}{2}) \\ s_1^{0\prime}(t-\underline{\theta}) = -\frac{1}{2}s(t-\frac{\overline{\theta}}{2}) \end{matrix} \tag{71}$$

$$\begin{matrix} k=2 \\ \theta_2 = \underline{\theta} \end{matrix} : \begin{matrix} I_2^-(t) = +\rho s(t+\frac{\underline{\theta}}{2})\frac{\underline{\theta}-s(\underline{\theta})}{2} \\ I_2^+(t) = -\rho s(t-\frac{\underline{\theta}}{2})\frac{\underline{\theta}-s(\underline{\theta})}{2} \end{matrix} : \begin{matrix} r_2^{0\prime}(t+\underline{\theta}) = -\frac{1}{2}\rho c(t+\frac{\underline{\theta}}{2}) \\ s_2^{0\prime}(t-\underline{\theta}) = +\frac{1}{2}\rho c(t-\frac{\underline{\theta}}{2}) \end{matrix} \tag{72}$$

$$\begin{matrix} k=3 \\ \theta_3 = \overline{\theta} \end{matrix} : \begin{matrix} I_3^-(t) &=& -c(t+\frac{\overline{\theta}}{2})\frac{\overline{\theta}+s(\overline{\theta})}{2} \\ I_3^+(t) &=& -c(t-\frac{\overline{\theta}}{2})\frac{\overline{\theta}+s(\overline{\theta})}{2} \end{matrix} : \begin{matrix} r_3^{0\prime}(t+\underline{\theta}) = +\frac{1}{2}s(t+\frac{\overline{\theta}}{2}) \\ s_3^{0\prime}(t-\underline{\theta}) = +\frac{1}{2}s(t-\frac{\overline{\theta}}{2}) \end{matrix} \tag{73}$$

$$\begin{matrix} k=4 \\ \theta_4 = \underline{\theta} \end{matrix} : \begin{matrix} I_4^-(t) = -\rho s(t+\frac{\underline{\theta}}{2})\frac{\underline{\theta}-s(\underline{\theta})}{2} \\ I_4^+(t) = +\rho s(t-\frac{\underline{\theta}}{2})\frac{\underline{\theta}-s(\underline{\theta})}{2} \end{matrix} : \begin{matrix} r_4^{0\prime}(t+\underline{\theta}) = +\frac{1}{2}\rho c(t+\frac{\underline{\theta}}{2}) \\ s_4^{0\prime}(t-\underline{\theta}) = -\frac{1}{2}\rho c(t-\frac{\underline{\theta}}{2}) \end{matrix} \tag{74}$$

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