The Discovery and Significance of the RT-equations: Optimal Regularity and Uhlenbeck Compactness for General Relativity and Non-Riemannian Geometry

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April 1, 2021

"[Riemann...bound by Dirichlet...] would give acute, logical analyses of foundational questions, and would avoid long computations as much as possible."

—Felix Klein

Abstract

The RT-equations are a novel system of elliptic partial differential equations foundational for geometric analysis in General Relativity and Mathematical Physics: Solutions of the RT-equations furnish coordinate transformations which give an arbitrary non-optimal affine connection a gain of one derivative over its own Riemann curvature tensor, (i.e., to *optimal regularity*), and the equations are elliptic regardless of metric or metric signature. The *elliptic* RT-equations thus regularize singularities in the *hyperbolic* Einstein equations. As corollaries we establish that singularities at GR shock waves are always removable, implying geodesic curves, locally inertial coordinates and the Newtonian limit all exist; and we extend Uhlenbeck compactness from Riemannian to Lorentzian geometry. Uhlenbeck compactness is based on the curvature alone, making it intrinsic for nonlinear problems in General Relativity and Mathematical Physics. We note that removable Black Hole singularities, like the singularity at the Schwarzschild radius, are non-optimal solutions to which the RT-equations formally apply, but such singularities lie below the threshold L^∞ regularity associated with GR shock wave singularities, the threshold of our current existence theory.

Introduction: In his celebrated habilitation of 1864, "*On the hypotheses which lie at the foundations of geometry*", Bernhard Riemann explained how to solve the longstanding problem proposed to him by Gauss, the problem of defining curvature in spaces of dimension larger than two. His idea was to discover an essential mathematical construct formed from second derivatives of a metric, such that it transformed under coordinate change like a first derivative object–what we now call a *tensor*. Thus began one of the greatest efforts of modern mathematics, to set Riemann's ideas upon a solid mathematical framework, and extend them to Physics. It was Levi-Civita who explained curvature in terms of the *covariant derivative* which is defined in terms of a connection, and by this the *connection* replaced the *metric* as the starting point of Riemann's theory of curvature.

Riemann's theory of curvature applies to any affine connection defined on the tangent bundle of an arbitrary differentiable manifold. In 1915 Albert Einstein introduced General Relativity (GR), and based his equations on Riemann's theory of curvature in Lorentzian geometry. In Einstein's theory, the connection defines the "parallel translation" of non-rotating frames described physically by gyroscopic translation. But a fundamental consequence of the tensorial nature of Riemann's curvature is that it implies the existence of low regularity coordinate transformations which leave the regularity of the curvature unchanged, but lower the regularity of the connection to that of the curvature, thus making the connection "singular." This produces an *enormous redundancy* in the coordinate expressions for connection and curvature. When presented with a non-optimal connection, no regularizing coordinate transformation is given, and it has been long unknown whether all such singularities are removable by coordinate transformation, or whether classes of them exist which are "geometric". This is fundamental, for example, because one needs the regularity of the connection to be one order above its curvature to conclude the convergence of approximation sequences; and the extra derivative is required to evolve solutions within the framework of nonlinear wave equations. At the low regularity of GR shock waves, the singularities are so severe that they call into question the physical significance of the solutions themselves.

Our work introduces a new set of partial differential equations for geometry, the RT-equations, which are elliptic regardless of metric or metric signature. We use the RT-equations to establish the regularity and compactness of general affine connections. (In GR, these *elliptic* equations determine the optimal regularity of solutions of the *hyperbolic* Einstein equations.) We establish that any affine connection, that is, any connection on the tangent bundle of a differentiable manifold, which satisfies the condition that its components together with the components of its Riemann curvature tensor are functions bounded in L^∞ in some coordinate system, can be smoothed by a (low regularity) coordinate transformation to lift the regularity of the connection by one order of differentiability. In seminal work, Kazdan and DeTurck [3] resolved the problem of optimal regularity for Riemannian (positive definite) metric geometries. Our theory of the RT-equations extends Kazdan and DeTurck to Lorentzian geometry and affine connections, it resolves the long-standing open problem of the essential regularity of GR shock waves, and it establishes Uhlenbeck compactness [19] in General Relativity, that any uniformly bounded sequence of L^∞ connections with L^∞ curvature has a convergent subsequence. Uhlenbeck compactness is a new starting point for analyzing the Einstein equations at low regularity, establishing that L^∞ bounds on the curvature alone are sufficient to imply weak-$W^{1,p}$ and strong L^p compactness for affine connections, including General Relativity.

As discussed further below, the existence of non-optimal metrics is a direct consequence of Riemann's original idea to construct a first derivative tensorial measure of curvature out of second derivatives of the metric, so the problem of optimal regularity has been an issue since Riemann introduced the Riemann curvature tensor in his famous lecture of 1864. In General Relativity, the RT-equations reinforce Einstein's viewpoint of spacetime as a four dimensional geometry, in contrast to the 3+1 approach required to apply classical methods of analysis in the study of the Cauchy problem. The theory of the *Regularity Transformation* (RT) equations reached a culmination in our paper "*On the regularity implied by the assumptions of geometry*", [17], a title which we chose to conjure up Riemann's original habilitation. An exposition of the theory of the RT-equations, including a summary of our results, is now published in the Proceedings of the Royal Society-A, [16].

Singularities of spacetime: It has been a central problem since Schwarzschild discovered his solution in 1915, whether a spacetime singularity in General Relativity is geometric, or whether it is removable by a regularizing coordinate transformation. It took several years until Eddington proved that the apparent singularity at the Schwarzschild radius in Schwarzschild's solution, what we now call the event horizon of a black hole, was only a coordinate singularity. At the time this question as to the nature of a spacetime singularity was essential in determining whether the equations of General Relativity, $G = 8\pi T$, were to be taken seriously as a physically meaningful theory. The celebrated Hawking-Penrose singularity theorems[1] address the opposite side of this issue, providing conditions under which a spacetime singularity is *non-removable*.[2] Our new theory of the RT-equations in [17], a theory unrestricted by dimension or symmetry assumptions, resolves in the affirmative the open problem as to whether the spacetime singularities at GR shock waves are removable.

Singularities at GR shock waves: In the year 1965 Glimm introduced the celebrated Glimm scheme of shock wave theory[3], and a year later Israel introduced the theory of Junction Conditions, the conditions under which two smooth gravitational metrics match continuously across an interface to form a general relativistic shock wave. To solve the Einstein equations $G = 8\pi T$ at a shock wave, the Einstein curvature tensor G will inherit the regularity of the fluid which comprises the energy momentum tensor T, so since the density, pressure and velocity are discontinuous at shock waves, this places a discontinuity in the curvature of spacetime at the shock, as well. The discontinuity in curvature is not a problem, but in Israel's theory, the regularity of the gravitational metric was only one order above the curvature at shock surfaces constructed by the Junction Conditions, and this left open the possibility of so called "delta function sources" *in the second derivatives of the metric* at the shock. Israel showed that ruling out the delta function sources in the Riemann curvature tensor at the shock was sufficient to imply the Rankine-Hugoniot jump conditions which enforce conservation at the shock, and under this assumption he proved that a low regularity transformation to Gaussian normal coordinates would regularize the apparent singularity in the metric and its connection at smooth shock surfaces. Israel's ideas were clarified and generalized in [18]. But the map to Gaussian normal coordinates is ill-defined as a continuous transformation when the shock surface is not smooth, i.e., when the normal vector is discontinuous, [10]. Israel's theory left open the problem of whether such singularities could be regularized at more complicated regions of "interacting" shock waves. If not, then shock waves could create a new kind of singularity in GR, and the authors named such potential singularities *Regularity Singularities*, c.f. [12].

But after Israel there were no examples of such interacting shock waves in GR until 2005 when Groah and Temple [6] proved that the Glimm scheme could be extended to General Relativity in spherically symmetric spacetimes. This introduced into GR the first rigorous existence theory for shock waves admitting interactions of arbitrary complexity. The validity of Glimm's method in GR re-introduced into General Relativity Israel's original problem with the Junction Conditions: The gravitational metric appeared to be singular at shock waves in the coordinate systems in which convergence of the Glimm scheme could be proven. That is, at the shock waves, the gravitational metric was singular

[1] Including the work for which Roger Penrose was awarded the 2020 Nobel prize in Physics, c.f. [9].

[2] Interestingly, Penrose's proof in [9] of the existence of a geodesic singularity requires the *apriori* assumption that the metric remain in $C^{1,1}$, which places the curvature in L^∞, c.f. [5]. Thus our theorems on optimal regularity would apply if the singularity formation actually involved a loss of optimal regularity, e.g., metric regularity falls to $C^{0,1}$ at the singularity, with curvature still in L^∞.

[3] Glimm's method was off-the-wall original and in stark contrast to the established method of analyzing hyperbolic PDE's at that time, *energy methods*, and even up until now, no one has succeeded in establishing Glimm's theorem by energy methods in one space dimension.

in the sense that the connection, defined in GR by the Christoffel symbols of the metric, degenerated to the level of regularity of its own Riemann curvature tensor, (which is discontinuous at shocks according to $G = 8\pi T$), making the gravitational metric only one derivative more regular than its curvature–a level of regularity so low at shock waves that the existence of locally inertial frames, geodesic curves and the Newtonian limit, were at issue. So as with Schwarzschild's original work, this raised the question as to whether these new shock wave solutions constructed by the Glimm scheme were physical. It is well known that shock waves form generically when a fluid is sufficiently compressive, so the physical interpretation of these solutions addressed the basic physical status of the Einstein-Euler system of equations. This singular structure at shock waves either required a new physical interpretation for the Einstein-Euler equations, or else there was a missing theory of regularizing coordinate transformations in General Relativity–some generalization of Israel's result to the case of shock wave interactions of arbitrary complexity.[4]

The fundamental question was then: Could these solutions constructed by the Glimm scheme be regularized by coordinate transformation at points of shock wave interaction? I.e., did there exist some unknown theory of low regularity coordinate transformations sufficient to regularize these singularities? To begin, Reintjes proved in [10] that metrics could always be smoothed to optimal regularity at points of regular shock interaction in spherically symmetric spacetimes. This was a complicated technical argument based on a non-local PDE, in which the Rankine-Hugoniot jump conditions came in again and again to make a system of seemingly over-determined PDE's, just barely solvable. But the argument was highly tuned to the structure of the particular interaction, and gave no hints as to what, if any, was the general principle working behind the scenes [10, 11].

Discovery of the RT-equations: After entertaining the possibility that there was some sort of new physics involved, (c.f. [12]), Reintjes and Temple became convinced that there must exist a fundamental, unknown theory of regularizing coordinate transformations, and they changed directions, and set out to find it. Thus we began a decade long journey to discover the theory of the missing regularizing coordinate transformations.

To keep the history straight, we record that it was difficult to pursue this program because panels of experts in GR who served as referees at NSF Applied Mathematics unanimously declared the program "not feasible", and denied NSF funding for this research several years straight. But fair enough. After all, we were proposing something *completely new* to General Relativity and Geometry: The possibility that there might exist an as yet undiscovered *elliptic* system of equations which regularized the singularities in solutions of the *hyperbolic* Einstein-Euler equations. We have now discovered these elliptic equations, and named them the "Regularity Transformation equations",

[4] In fact, solutions of the Einstein equations constructed in Standard Schwarzschild Coordinates (SSC), the spherically symmetric coordinate system used to construct the Glimm scheme solutions in [6], are non-optimal at every order of regularity. Non-optimality is built into SSC at the start by the choice of radial coordinate, i.e., chosen to be proportional to the area of the sphere of symmetry. Interestingly, the non-optimality of the SSC coordinate system has the effect of converting second order Einstein equations into first order equations. So on the one hand SSC is a degenerate non-optimal coordinate system, but intriguingly, it is precisely this degeneracy that makes it feasible to implement the first order Glimm scheme in SSC. The SSC coordinate system has played an important role in the history of GR, starting with Schwarzschild and Birkhoff. Our theorem clarifies the logical status of the SSC coordinate system within GR by demonstrating that non-optimal metrics in SSC can be converted to optimal regularity by coordinate transformation at all levels of regularity, L^∞ and above. Interestingly, no one knows how to implement Glimm's method, or any other method for constructing shock wave solutions of the Einstein equations, in a coordinate system in which the shock waves exhibit optimal regularity.

or RT-equations.[5] The challenge was to obtain existence of solutions to the RT-equations for the low regularity of shock waves, and the important space was the space of connections in L^∞ with L^∞ Riemann curvature tensor. This space automatically extends Israel's condition that there be "no delta functions sources" at shocks to the case of general interacting shock waves. Indeed, since Einstein built his equations to satisfy $Div\, T = 0$ by the Bianchi identities of geometry, the condition that the curvature be locally bounded in L^∞ naturally replaces the Rankine-Hugoniot jump conditions and the extension of this by the weak formulation of solutions in terms of test functions, as the condition on weak solutions which imposes conservation at shock waves in General Relativity, (see [18]).

Our program to address optimal regularity in our current general setting began with the insight that the existence of the apparent singularities at GR shock waves might be due simply to the fact that the Riemann curvature tensor is formed from second derivatives of the metric tensor, but being a tensor, transforms like a first derivative object. In fact, the problem lies more fundamentally at the level of an affine *connection*, the fundamental object more general than a metric, to which Riemann's theory of curvature applies. The problem is this: because the transformation law for connections involves second derivatives, a transformation whose Jacobian has the same regularity as the connection, will in general transform a connection of *optimal* regularity, (one which has a level of regularity one derivative above its curvature tensor), to a connection at the same regularity as its curvature, because the curvature maintains its regularity under tensor transformation.[6] In fact, every singularity that arises from the application of a low regularity coordinate transformation to a metric of optimal regularity, will (generically) be of this non-optimal character.[7] Our conjecture, then, was that this was the *only* way non-optimal connections came into existence in geometry.[8] That is, we conjectured that this process could always be reversed. But to prove the reverse direction, that non-optimal connections could always be smoothed to optimal regularity by coordinate transformation, one needs to undo the above process, and this requires constructing a suitable low regularity, (*singular* if you will), coordinate transformation, given only the information about the non-optimal connection and its curvature. For example, at the level of L^∞ connections associated with shock waves, such a coordinate transformation must be singular in the sense that jumps in derivatives of the Jacobian must be tuned to precisely cancel out the discontinuities in the given non-optimal connection in the transformation law for connections. But what sort of equation would the Jacobians of such regularizing transformations satisfy? And how would one find such an equation? And what theory of mathematics would be available to solve such an equation?

[5] In choosing this name, we were fully aware of both the originality, and the fundamental nature of these new equations.

[6] Note that the essential regularity of a connection, including whether it is optimal or non-optimal, is a geometric, coordinate independent notion when the manifold is restricted to the atlas of smooth coordinate transformations, but optimal regularity becomes coordinate dependent when the smooth atlas is extended to include low regularity coordinate transformations, c.f. [16].

[7] The black hole singularities at the Schwarzschild radius are examples of non-optimal metrics by this principle, but the regularity of the connection at such singularities is below the L^∞ level of our current existence theory, [17].

[8] We record here that earlier investigations starting with Anderson [1], and more recently [2], made inroads into the problem of optimal regularity in GR from a $3+1$ point of view, based on using Kazdan and DeTurck [3] to regularize initial data on spacelike hypersurfaces where the gravitational metric restricts to positive definite, and then obtaining conditions (like the "geodesic ball condition") which control the regularity under time evolution. Our work does not build on this. Based on our stated conjecture here regarding the source of optimal regularity, our view from the start was that to get a theorem general enough to prove the optimal regularity of GR shock waves, we would have to build a new framework in which the connection and Riemann curvature tensor were treated fundamentally as 4-dimensional geometrical objects.

Our first breakthrough for the general problem came in [13], where we obtained a condition on a non-optimal connection equivalent to the existence of a regularizing coordinate transformation. We named this the *Riemann-flat condition.* The Riemann-flat condition states that a regularizing coordinate transformation exists for a given non-optimal connection if and only if there exists a (1,2)-tensor $\tilde{\Gamma}$, one order more regular than the given non-optimal connection Γ, such that the connection $\Gamma - \tilde{\Gamma}$ is Riemann-flat, i.e., $Riem(\Gamma - \tilde{\Gamma}) = 0$. Since this new Riemann flat connection had the same shock structure as the original connection, we wondered for a while whether we might obtain optimal regularity by some sort of Nash embedding theorem. The breakthrough came after we decided to try to construct an elliptic system of equations by coupling equations for the unknown Jacobians J, to equations for the above mentioned unknown (1,2)-tensor $\tilde{\Gamma}$, via the Riemann flat condition and the coordinate Laplacian. This opened the door to the discovery of the *Regularity Transformation* equations.

The new idea required to derive the RT-equations was to use the coordinate Laplacian instead of an invariant metric[9] to define a co-derivative, the idea being that the leading order part $d\Gamma$ of the Riemann curvature tensor in terms of a connection Γ, can be converted into an elliptic operator via $\Delta = d\delta + \delta d$. We then employ two equivalent forms of the Riemann-flat condition, one involving $d\tilde{\Gamma}$ and one involving dJ, where $\tilde{\Gamma}$ is the unknown (1,2)-tensor which completes a non-optimal connection to Riemann flat, and J is the unknown Jacobian of the sought after regularizing coordinate transformation. That is, by manipulating the Riemann-flat condition, we are able to show that equations for both $\tilde{\Gamma}$ and J can be completed to form a pair of coupled nonlinear Poisson equations with left hand sides given by $\Delta\tilde{\Gamma}$ and ΔJ, where Δ is the coordinate Laplacian constructed using the coordinate co-derivative δ. The RT-equations consist of these coupled Poisson equations, together with Cauchy-Riemann type equations which guarantee the integrability of J.

Our existence theory for the RT-equations is a new application of the celebrated L^p theory of elliptic regularity, a theory set out by the great analysts of the 1950's and 60's, including Agmon, Nirenberg, Lax, Milgram and others, [4]. The extra derivative on the connection and metric implied by our proof of optimal regularity directly implies Uhlenbeck compactness for Lorentzian metrics. The result applies to solutions of the Einstein equation at every order of regularity, for it tells us that bounding a connection and its curvature in the same Sobolev space in a sequence of approximate solutions, implies the existence of a convergent subsequence with convergence *regular enough* for its limit to be a solution.

The formulation of the RT-equations, expressed within the language of matrix valued differential forms, was directed toward making them amenable to the classical methods of elliptic regularity theory at the low regularity of L^p spaces, a regularity low enough to include the L^∞ GR shock waves constructed by Glimm's method [6]. Our publication in [14] is devoted to a careful derivation of the RT-equations, and in [15] we established their viability by proving the first existence theory for the RT-equations applicable to non-optimal connections in $W^{m,p}$, $m \geq 1$, $p > n$, a level of regularity one order above the regularity of GR shock waves. We then identify the role played by a new type of gauge transformation in the equations, and by employing gauge freedom, we were able to de-couple the equations from a subsystem which we call the *reduced* RT-equations, a system of elliptic PDE's amenable to an existence theory at a level of regularity one order below what

[9] The method of Kazden and DeTurck [3] is based applying elliptic regularity to the operator which appears in the leading order part of the Ricci tensor as a consequence of the existence of an underlying invariant positive definite metric. For non-Riemannian metrics this method leads to a wave type (hyperbolic) operator, and for general affine connections, there would be no associated invariant metric or elliptic operator.

we could achieve with the original RT-equations, and by this we were able to extend the theory to L^∞ connections with L^∞ curvature. This level of regularity is low enough to include both GR shock waves constructed by the Glimm scheme, as well as shock solutions constructed in multi-d by Israel's theory of junction conditions, i.e., all of the known examples of GR shock waves [16].

Optimal Regularity in the context of the Cauchy Problem in GR: Current methods of the Cauchy problem are inadequate regarding how non-optimal solutions fit into the general picture. For example, referees have brought up the *celebrated* reference Klainerman-Rodnianski-Szeftel, [8]. Reference [8] *only* applies to vacuum solutions of the Einstein equations, (a setting which excludes shock waves, the setting of our papers), but what is interesting in light of our new result, is that the KRS theorem does not apply unless the second fundamental form, and hence the connection, is one derivative more regular than the curvature on a Cauchy Surface, i.e, the KRS theory only applies to solutions in vacuum which can be shown to exhibit *optimal regularity* at the start. The referees of our papers from the field of GR seem to think that non-optimal solutions can eventually be ruled out by *well-posedness* considerations, or that the initial value problem will automatically regularize all non-optimal solutions, or that non-optimal solutions are just an anomaly of spherical symmetry. Our opinion is that these statements are *not correct*, (c.f. our discussions in [15, 16]).

Non-optimal solutions exist in spherically symmetric spacetimes for precisely the same reason they exist in general spacetimes–the 4-dimensional Riemann curvature tensor is a second derivative construct which transforms like a tensor.

To clarify this point, start with the fact that non-optimal initial data and non-optimal solutions exist in every coordinate system due to the tensorial nature of Riemann's curvature tensor. Then it is easy to conjecture that all optimal solutions do not "fit" within any "one" coordinate system, in the sense that there is no single coordinate ansatz which simultaneously lifts all non-optimal solutions to optimal regularity at once. That is, you have to solve the RT-equations, an essentially 4-dimensional, not $3+1$ dimensional system. Without the RT-equations, the Cauchy problem in GR is thus faced with the problem as to what to do with non-optimal solutions: I.e., they aren't regular enough to evolve in time, but if they are "physical", you can't throw them out either. Now at higher levels of regularity, say connection and curvature of regularity above $W^{1,p}$, one can simply accept estimating non-optimal solutions as one order less regular than they really are, but at the lowest level of regularity, there is no room left, in the sense that L^p connections with L^p curvature are not regular enough to evolve in any regularity class.

Now if one did not know, as our new theorem demonstrates for $p=\infty$, that all of these non-optimal solutions are perfectly "good" optimal solutions written in a "bad" coordinate system, then it might make sense to rule out the bad one's by "well-posedness" considerations. After all, if the connection essentially lacks that extra derivative of regularity, it won't be stable in derivative norms at that level. The hope, then, would be the idea that some one gauge for the initial value problem would somehow make "good" solutions optimal, and rule out "bad" ones by "well-posedness".

Unfortunately for this idea our new theorem tells us that *all* of the non-optimal solutions are actually "good" in some other coordinate system, but the coordinate transformation which regularizes a given non-optimal solution, is highly tuned to that particular solution, (you have to solve the RT-equations), so there won't be any one ansatz that regularizes them "all at once". On the other hand, by properties of the regularizing coordinate transformations obtained by solving the elliptic

RT-equations, one should expect two non-optimal L^∞ solutions with connection and curvature "close to each other" in L^p, will be mapped to two solutions close in $W^{1,p}$. So if one didn't know about the RT-equations, one might well be puzzling for a long time over ruling out solutions by "well-posedness" considerations, when the issue is basically a problem of optimal regularity instead.

Our conclusion, based on our new point of view, is that what the theory of the Cauchy problem lacks, besides theorems which extend to low regularity beyond the vacuum, is exactly what we have provided–a proof that non-optimal solutions can always be regularized by low regularity coordinate transformations.

Numerical Relativity: The Reintjes-Temple result gives an L^∞ based answer to the question as to whether loss of optimal regularity under time evolution in numerical or theoretical approximations, (one example being the formation of shock waves), represents the formation of real spacetime singularities, or simply a loss of regularity of the coordinate system employed. By this new result, solutions can always be regularized to optimal regularity by coordinate transformation in neighborhoods where the components of the connection and curvature are locally L^∞ functions. The iteration scheme in our existence theory, based on solving linearized Poisson equations, provides an explicit numerical algorithm for constructing the Jacobians J of the regularizing transformation, and the optimal connection is then given by an exact formula in terms of $J, \tilde{\Gamma}, \Gamma$. We would be excited to see applications of this algorithm in Numerical Relativity.

Uhlenbeck compactness: As a serendipitous corollary, our result on optimal regularity establishes the first extension of Karen Uhlenbeck's compactness theorem to General Relativity and affine connections in general.[10] The Reintjes-Temple work here extends Uhlenbeck compactness from the setting of connections on vector bundles over Riemannian manifolds, to (affine) connections on the tangent bundle of Lorentzian manifolds, the setting of Relativistic Physics. As a first example for how one might apply Uhlenbeck compactness in General Relativity, we give in [17] a new compactness theorem for approximate solutions of the Einstein equations in vacuum spacetimes. The new result establishes compactness of solutions under the simple assumption that connection and curvature be bounded in L^∞. Essentially, the weak $W^{1,p}$ convergence together with strong L^p convergence of the connection, i.e. what we get from our Uhlenbeck compactness in general, is sufficient to pass limits through nonlinear functions in Einstein equations, and in [17] we give a proof of this in the setting of vacuum spacetimes. We expect that this principle can be extended to non-vacuum solutions when combined with further analysis of the matter field equations. For example, we expect Uhlenbeck compactness to be applicable to zero viscosity limits of the coupled Einstein-Euler equations for self-gravitating relativistic fluids when combined with the method of compensated compactness.

Uhlenbeck's results in the Riemannian case are much more general regarding the bundle structure, as her main interest was gauge theory and the Yang-Mills equations, while our main concern is General Relativity. Our current methods, based on the elliptic RT-equations, apply to affine connections, but still hold the promise of further extending Uhlenbeck compactness to general vector bundles and general L^p connections over Lorentzian manifolds, $p \geq 2$. Uhlenbeck compactness is ideal for application to geometric PDE's, including the relativistic Einstein equations and Yang-Mills equations, because compactness follows from estimates based on the curvature alone, the

[10] Uhlenbeck's work on compactness for L^∞ and L^p connections in Riemannian (positive definite metric) geometry [19], was highly celebrated, including the 2019 Abel Prize and 2007 Steele Prize.

observable in the theory, without having to deal with the other potentially uncontrolled derivatives of the connection; and the regularity of convergence obtained is precisely what is needed to pass limits through nonlinear products. Compactness is the starting point of analysis, and there are scant few fundamental compactness theorems applicable to hyperbolic PDE's.

The new Uhlenbeck compactness result of Reintjes and Temple provides a new starting point for geometrical analysis in Mathematics and Physics.

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