Commun. Math. Phys. 306, 831–851 (2011)
Digital Object Identifier (DOI) 10.1007/s00220-011-1299-5

Communications in
**Mathematical
Physics**

Global Solutions to the Ultra-Relativistic Euler Equations

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Received: 4 November 2010 / Accepted: 25 February 2011
Published online: 12 July 2011 – © Springer-Verlag 2011

Abstract: We show that when entropy variations are included and special relativity is imposed, the thermodynamics of a perfect fluid leads to *two* distinct families of equations of state whose relativistic compressible Euler equations are of Nishida type. (In the non-relativistic case there is only one.) The first corresponds *exactly* to the Stefan-Boltzmann radiation law, and the other, emerges most naturally in the ultra-relativistic limit of a γ-law gas, the limit in which the temperature is very high or the rest mass very small. We clarify how these two relativistic equations of state emerge physically, and provide a unified analysis of entropy variations to prove global existence in one space dimension for the two distinct 3×3 relativistic Nishida-type systems. In particular, as far as we know, this provides the first large data global existence result for a relativistic perfect fluid constrained by the Stefan-Boltzmann radiation law.

It was shown in [6,10] that for non-relativistic perfect fluids a unique equation of state of the form $p = a^2\rho$ emerges from a γ-law gas in the (appropriately re-scaled) limit $\gamma \to 1$. A global existence theorem for the 3×3 non-relativistic compressible Euler equations was then proven for this model equation of state. This non-relativistic equation of state is unique, but has questionable physical interpretation as an isothermal gas, cf. [8]. Surprisingly, in contrast with the classical $\gamma \to 1$ limit, the equation of state $p = a^2\rho$ emerges in *two* fundamental limits, not one, when special relativity is imposed: it is *exact* in the case of the Stefan-Boltzmann radiation law, and also emerges in a most natural ultra-relativistic limit of a γ-law gas, the limit in which the temperature is very high or the rest mass very small [2], (not the awkward limit $\gamma \to 1$). Our results clarify how these two relativistic equations of state emerge physically, and provide a unified analysis of the entropy variations for the resulting two distinct relativistic Nishida systems that leads to a large data global existence theorem for both. In particular, as far as we know, this provides the first large data global existence result for a relativistic perfect fluid constrained by the Stefan-Boltzmann radiation law.

1. The Relativistic Euler Equations

The relativistic Euler equations in one spatial dimension form a system of conservation laws which can be written as [9],

$$U_t + F(U)_x = 0, \tag{1}$$

where,

$$U = \left(\frac{n}{\sqrt{1-v^2}}, (\rho + p)\frac{v}{1-v^2}, (\rho + p)\frac{v^2}{1-v^2} + \rho\right) \tag{2}$$

and

$$F(U) = \left(\frac{nv}{\sqrt{1-v^2}}, (\rho + p)\frac{v^2}{1-v^2} + p, (\rho + p)\frac{v}{1-v^2}\right). \tag{3}$$

We designate: v the gas velocity in a chosen Lorenz frame; ρ the proper energy density; p the pressure; ϵ the specific internal energy; n the particle number; S the specific entropy; and T the temperature. We choose units where the speed of light is one and note that the thermodynamic quantities are related by the second law of thermodynamics, $TdS = d\epsilon + pd\tau$, where $\tau = 1/n$ is the specific volume.

To close the system (1) we consider equations of state of the form

$$p = a^2\rho. \tag{4}$$

With (4) the system (1) contains special properties; in this limit one can prove global solutions exist and depend continuously on the initial data in the density, pressure and velocity variables, for initial data with arbitrarily large, but finite, variation [3,9]. We will use this existence result to prove a large data existence theorem for an ultra-relativistic gas that incorporates entropy and temperature variations via an equation of state of the form,

$$\epsilon(n, S) = A(S)n^{\gamma-1}, \tag{5}$$

with $1 < \gamma < 2$. Once the existence of bounded variation solutions is proven, its uniqueness and continuous dependence can be analyzed using techniques in [5]. We assume the function A satisfies the following conditions:

$$A \in C^1(\mathbb{R}^+, \mathbb{R}^+), \quad \text{and} \quad A'(S) > 0 \quad \text{for } S > 0. \tag{6}$$

This family includes the equations of state for a polytropic gas, $A(S) \approx e^{\frac{\gamma-1}{R}S}$ and one modeling a radiation dominated gas constrained by the Stefan-Boltzmann radiation law, $A(S) \approx S^\gamma$, [11]. (When $\gamma = \frac{4}{3}$ this gives the condition that $\rho \approx T^4$.)

The proper energy density ρ is the sum of the rest mass energy and internal energy. For a gas with particles of rest mass m and specific internal energy ϵ this gives $\rho = n(m+\epsilon)$. Based upon this relation the equations of state (5) do not reduce to (4); however, we observe now that they do in the ultra-relativistic limit where either the rest mass is very small (e.g. neutrinos or, in the limiting sense, massless thermal radiation) or the temperature very large, $\frac{m}{T} \ll 1$, [2]. In this limit it follows that

$$\rho = n(m + \epsilon) \approx n\epsilon. \tag{7}$$

Indeed using the second law of thermodynamics under assumption (7), (5) reduces to an equation of state of the form (4) with $a^2 = (\gamma - 1)$,

$$p = n^2 \frac{d\epsilon}{dn} = n^2(\gamma - 1)A(S)n^{\gamma-2} = (\gamma - 1)A(S)n^{\gamma} = (\gamma - 1)n\epsilon = (\gamma - 1)\rho.$$

We find it remarkable in this limit, when special relativity is assumed, the pressure is still a function of n and S, but reduces to (4) when viewed as a function of ρ alone. This model allows one to find the temperature evolution of the gas and still take advantage of the simplifying effects of an equation of state of the form (4). An analogous equation of state incorporating the entropy was obtained in [6,10] with a rescaled limit $\gamma \to 1$ in the non-relativistic case, and a corresponding global existence theorem for the classical Euler equations was given. In contrast to the classical $\gamma \to 1$ limit, the equation of state $p = a^2\rho$ emerges in *two* fundamental limits, not one, when special relativity is imposed. It is *exact* in the case of the Stefan-Boltzmann radiation law and also emerges in the ultra-relativistic limit of a perfect fluid.

With these two equations of state as motivation, the goal of this paper is now to prove the following:

Theorem 1. *Let $\rho_0(x)$, $v_0(x)$ and $S_0(x)$ be arbitrary initial data satisfying, $\rho_0(x) > 0$, $-1 < v_0(x) < 1$ and $S_0(x) > 0$. Let $\Sigma = \ln[A(S)]$ for $\epsilon(n, S) = A(S)n^{\gamma-1}$, $1 < \gamma < 2$, and A satisfying (6). Suppose further that*

$$Var\{\Sigma_0(\cdot)\} < \infty, \tag{8}$$
$$Var\{\ln(\rho_0(\cdot))\} < \infty, \tag{9}$$

and

$$Var\left\{\ln\left(\frac{1 + v_0(\cdot)}{1 - v_0(\cdot)}\right)\right\} < \infty. \tag{10}$$

Then there exists a bounded weak solution $(\rho(x,t), v(x,t), S(x,t))$ to (1) in the ultra-relativistic limit, satisfying

$$Var\{\Sigma(\cdot, t)\} < N, \tag{11}$$
$$Var\{\ln(\rho(\cdot, t))\} < N, \tag{12}$$

and

$$Var\left\{\ln\left(\frac{1 + v(\cdot, t)}{1 - v(\cdot, t)}\right)\right\} < N, \tag{13}$$

where N is a constant depending only on the initial variation bounds in (8), (9), and (10).

Theorem 1 is a generalization of the work by Smoller and Temple [9] that includes the entropy evolution. In other words, in this model we are able to prove global solutions exist including a physically relevant temperature profile. Smoller and Temple found that the relativistic Euler equations with equation of state (4) possessed the property that after each elementary wave interaction in a Glimm scheme, $Var\{\ln(\rho)\}$ is non-increasing. This functional, introduced by Liu, is used as a replacement for the quadratic potential in Glimm's original analysis, which can be used to show that (9) and (10) implies (12) and (13). Considering the ultra-relativistic limit, the solutions of Riemann problems are independent of S, enabling one to solve for the intermediate state in the projected state space and place a corresponding entropy wave between them.

In [9] it is shown that for an equation of state of the form (4), the shock curves are translationally invariant in the plane of Riemann invariants. In our case, this property continues to hold under certain coordinate changes in the three dimensional non-projected state space for an equation of state of the form (5). This can be viewed as the relativistic analogue of the large data existence result in [10] with a family of distinct temperature profiles.

The main part of the analysis is showing that $Var\{S\}$ is bounded in our approximate solutions. We extend the analysis by Smoller and Temple for the ultra-relativistic regime with equation of state (5), by utilizing the geometry of the shock curves in the space of Riemann invariants. Considering only the change of S across shock waves, we find that $Var\{S\}$ is uniformly bounded by $Var\{\ln(\rho)\}$ for a polytropic equation of state; however, across the linearly degenerate entropy waves there is no change in pressure and hence no jump in proper energy density by (4). Thus, another method must be employed to estimate the strengths of these jumps. For a gas dominated by radiation or a general equation of state of the form (5), the change in entropy across a shock depends on the initial entropy value. It is not known *a priori* that this dependence does not lead to blow-up in the variation in S.

Furthermore, in certain elementary wave interactions, $Var\{S\}$ may actually increase while $Var\{\ln(\rho)\}$ remains invariant. Complicating matters, using $\Delta \ln(\rho)$ as the definition of wave strengths increases the technicality of the entropy wave estimates. For example, after the interaction of two shocks of the same family the new shock wave has strength strictly less than the sum of the two previous. In other words when two shock waves combine, the strengths are not simply additive, but the new wave strength is less than the simple sum of the incoming shock strengths.

To alleviate technicalities with the entropy estimates and decreasing shock strengths after interaction, we use the change of Riemann invariants as a measure of wave strength. Under this regime, wave strengths are now additive and the sum of all the strengths of shock waves is shown to be non-increasing in time. In conclusion, using $\Delta \ln(\rho)$ as a measure of wave strength dramatically simplifies the interaction estimates for the nonlinear waves, but complicates the problem dealing with the entropy waves.

The rest of this paper is outlined as follows:

In Sect. 2, we analyze the structure of simple wave solutions of (1) and derive the equations of state corresponding to both a γ-law gas and constrained by the Stefan-Boltzmann law. Using these properties, we prove global existence of solutions to Riemann problems. We then obtain *a priori* wave interaction estimates which are used to produce estimates on approximate solutions constructed using a Glimm scheme in Sect. 3. Section 4 contains the proof of our main theorem.

2. Relativistic Gas Dynamics

We consider a gas where the proper energy density ρ and pressure satisfy the relationship (4) where causality restricts the sound speed $c_s = \sqrt{dp/d\rho} = a$ to be less than one. Under assumption (4), the system (1) decouples so that we may solve for two variables first, then solve for the third afterward. In this section, we will show in the domain $\rho > 0, -1 < v < 1$, and $S > 0$, Riemann problems are globally solvable and their general structure consists of two waves separated by a jump in entropy traveling with the fluid.

2.1. Riemann invariants. The Riemann invariants for the system (1) with $p = a^2\rho$ are given by [9],

$$r = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) - \frac{a}{1+a^2}\ln(\rho) \quad \text{and} \quad s = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) + \frac{a}{1+a^2}\ln(\rho).$$

The function $r = r(\rho, v)$ is constant across 3-rarefaction waves and $s = s(\rho, v)$ is constant across 1-rarefaction waves. The entropy S is a third Riemann invariant constant across 1 and 3-rarefaction waves. In our analysis we will view state space in the coordinates of the Riemann invariants rather than the conserved variables. However, using S is not sufficient because the shock curves in (r, s, S) space are, in general, not translationally invariant. We will instead use $\Sigma = \ln[A(S)]$ as our third coordinate. It is shown in Sect. 2.3 that in (r, s, Σ) space the shock-rarefaction curves are independent of base point.

We now change our variables from the conserved quantities (U_1, U_2, U_3) to (ρ, v, S).

Proposition 1. *In the region, $\rho > 0, -1 < v < 1, S > 0$, the mapping $(\rho, v, S) \to (U_1, U_2, U_3)$ is one-to-one, and the Jacobian determinant of the map is both continuous and non-zero.*

Proof. The conserved quantities U_2 and U_3 depend only on v and ρ. It can be shown that that the mapping $(\rho, v) \to (U_2, U_3)$ is one-to-one for $\rho > 0$ and $-1 < v < 1$ [9].

Now we show that the mapping $(\rho, v, S) \to (U_1, U_2, U_3)$ is one-to-one. If (ρ_1, v_1, S_1) and (ρ_2, v_2, S_2) have the same image we must have $\rho_1 = \rho_2$ and $v_1 = v_2$, since U_2 and U_3 only depend on ρ and v, and the mapping $(\rho, v) \to (U_2, U_3)$ is one-to-one. Moreover, since $n = n(\rho, S)$ the equality $U_1(\rho_1, v_1, S_1) = U_1(\rho_2, v_2, S_2)$ reduces to $n(\rho, S_1) = n(\rho, S_2)$. Thus we are done if $\partial n/\partial S \neq 0$. Using $\rho = n\epsilon$ to rewrite the second law of thermodynamics as $nd\rho = n^2TdS + (a^2+1)\rho dn$ we conclude $\frac{\partial n}{\partial S} = -\frac{n^2T}{(a^2+1)\rho} \neq 0$, and the mapping $(\rho, v, S) \to (U_1, U_2U_3)$ is one-to-one.

Finally, the determinate of the Jacobian matrix of the map is $\det(\mathrm{J}) = \frac{n^2T(1-a^2v^2)}{(1-v^2)^2} > 0$, which is continuous for $\rho > 0, -1 < v < 1$ and $S > 0$. □

2.2. Jump conditions. For systems of conservation laws, the relations defining the dynamics of shock waves are the Rankine-Hugoniot jump conditions,

$$s[[U]] = [[F(U)]], \tag{14}$$

where s is the speed of the shock and $[[U]]$ and $[[F(U)]]$ the change of U and $F(U)$ respectively across the shock, [8].

Given a state U_L the Rankine-Hugoniot relations, for each $i = 1, \ldots, n$, define a 1−parameter family of states that can be connected on the right by a shock wave in the i^{th} characteristic family. Moreover, this curve has second order contact with the curve defining all the states that connect to U_L on the right by an i^{th} rarefaction wave given by the i^{th} integral curve. Only half of these curves are physically relevant. For the first and third genuinely non-linear characteristic field, with wave speeds $\lambda_1 = (v-a)/(1-va)$ and $\lambda_3 = (v+a)/(1+va)$, we take the portion of the integral curve extending from U that satisfies $\lambda_i(U) < \lambda_i(U')$. On the other hand, take the portion of the shock curve $\boldsymbol{S}_i$ that satisfies the Lax entropy condition, $\lambda_i(U') < s < \lambda_i(U)$ [8]. The second characteristic class is linearly degenerate with characteristic speed $\lambda_2 = v$.

We now give two lemmas that describe the structure of the shock curves.

Lemma 1 [7]. *Let $U = (\rho, v, n)$ and $U_L = (\rho_L, v_L, n_L)$ be two states separated by a shock wave. Then with* (4) *the following relation holds*:

$$\frac{n^2}{n_L^2} = \frac{\rho^2}{\rho_L^2} \frac{\left(1 + a^2 \frac{\rho_L}{\rho}\right)}{\left(1 + a^2 \frac{\rho}{\rho_L}\right)}. \tag{15}$$

The global structure of the solutions of the shock relations (14) for the relativistic Euler equations in the space of Riemann invariants was studied by Smoller and Temple [9] for an equation of state of the form (4). We summarize their results in the following lemma:

Lemma 2 [9]. *Let $p = a^2\rho$ with $0 < a < 1$. The projected shock curves $i = 1, 3$ onto the plane of Riemann invariants (r, s) at any entropy level satisfy the following*:

1. *The shock speed s is monotone along the shock curve $\mathbf{S}_i$ and for each state $(\rho_L, v_L) \neq (\rho_R, v_R)$ on $\mathbf{S}_i$ the Lax entropy condition holds.*
2. *The shock curves when parameterized by $\Delta \ln(\rho)$ are translationally invariant. Furthermore the 1 and 3−shock curves based at a common point $(\bar{r}, \bar{s})$ have mirror symmetry across the line $r = s$ through the point $(\bar{r}, \bar{s})$.*
3. *The i−shock curves are convex and*

$$0 \le \frac{ds}{dr} \le \frac{\sqrt{2K} - 1}{-\sqrt{2K} - 1} < 1$$

for $i = 1$ and

$$0 \le \frac{dr}{ds} \le \frac{\sqrt{2K} - 1}{-\sqrt{2K} - 1} < 1$$

for $i = 3$, where $K = 2a^2/(1 + a^2)^2$.

In light of Lemma 2 we can globally define the shock curves in the rs−plane and know that everywhere on this curve the Lax entropy conditions hold.

2.3. Equations of state. In this section we derive both the equations of state for an ultra-relativistic γ-law gas and one dominated by thermal radiation subject to the Stefan-Boltzmann law. We then show that the general class of equations of state (5) have the property that as a function of wave strength, the change in a certain function of entropy is independent of base point. Moreover, we will find that the change of this function of entropy and its derivative are monotonically increasing. These facts are used in our estimates on the entropy waves in Sect. 2.5.

We begin by assuming the ultra-relativistic limit (7) and (4) with $a^2 = \gamma - 1$ for $1 < \gamma < 2$. Finding the differential of $\epsilon = \rho\tau$ we get, $d\epsilon = \left(\frac{1}{\gamma-1}\right) p d\tau + \left(\frac{1}{\gamma-1}\right) \tau dp$. Plugging this into the second law, we get the two constraints on entropy function $S(p, \tau)$,

$$\frac{\partial S}{\partial \tau} = \left(\frac{\gamma}{\gamma - 1}\right) \frac{p}{T} \quad \text{and} \quad \frac{\partial S}{\partial p} = \left(\frac{1}{\gamma - 1}\right) \frac{\tau}{T}. \tag{16}$$

2.3.1. Ideal gas. We first consider a gas subject to the ideal gas law: $p\tau = RT$. Using the ideal gas law, rewrite (16) as,

$$\frac{\partial S}{\partial \tau} = \left(\frac{\gamma}{\gamma-1}\right)\frac{R}{\tau} \quad \text{and} \quad \frac{\partial S}{\partial p} = \left(\frac{1}{\gamma-1}\right)\frac{R}{p},$$

which has the solution,

$$S(p,\tau) = \left(\frac{\gamma}{\gamma-1}\right) R\ln(\tau) + \left(\frac{1}{\gamma-1}\right) R\ln(p) + C.$$

After dividing by $\frac{R}{\gamma-1}$, exponentiating and rearranging, we get,

$$\rho(n,S) = Ce^{\left(\frac{R}{\gamma-1}\right)S} n^{\gamma},$$

that returns the equation of state (5) with $A(S) = Ce^{\left(\frac{R}{\gamma-1}\right)S}$.

2.3.2. Stefan-Boltzmann. For the Stefan-Boltzmann equation of state, we now assume that the pressure, and hence energy density by (4), depends only on the temperature T. With this we equate the mixed partials of our constraint equations (16),

$$\frac{\partial}{\partial p}\left[\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{T}\right]\Bigg|_{\tau} = \frac{\partial}{\partial \tau}\left[\left(\frac{1}{\gamma-1}\right)\frac{\tau}{T}\right]\Bigg|_{p}.$$

After differentiation and simplifying we get the differential equation for T,

$$\frac{dT}{dp} = \left(\frac{\gamma-1}{\gamma}\right)\frac{T}{p}, \tag{17}$$

with solution, $p(T) = CT^{\frac{\gamma}{\gamma-1}}$. Equivalently by (4), $\rho(T) = bT^{\frac{\gamma}{\gamma-1}}$. (Note that when $\gamma = \frac{4}{3}$, this reduces to the fourth power law, $\rho = bT^4$.)

Now we put this equation of state in the form (5). To proceed we first find the entropy function $S(n,\rho)$, then solve for ρ. From the first equation of (16) we find,

$$S(p,\tau) = \left(\frac{\gamma}{\gamma-1}\right)\frac{p\tau}{T} + f(p).$$

The second constraint gives us,

$$\frac{df}{dp} = -\frac{\tau}{T}\left[1 - \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{T}\frac{dT}{dp}\right].$$

In light of (17) $f'(p) = 0$, this gives,

$$S(p,\tau) = \left(\frac{\gamma}{\gamma-1}\right)\frac{p\tau}{T}.$$

Now, replacing p, τ and T with their equivalent expressions in terms of ρ and n, we obtain,

$$S(n,\rho) = \gamma b^{\frac{\gamma-1}{\gamma}} \rho^{1/\gamma} n^{-1}.$$

Solving for ρ results in $\rho = b^{1-\gamma}\gamma^{-\gamma}S^{\gamma}n^{\gamma}$. Therefore, we have (5) with, $A(S) = b^{1-\gamma}\gamma^{-\gamma}S^{\gamma} \approx S^{\gamma}$.

2.3.3. Shockwave entropy change. For an equation of state of the form, $\epsilon(n, S) = A(S)n^{\gamma-1}$, with $A(S)$ satisfying (6), the second law of thermodynamics says, $p(n, S) = n^2 \frac{\partial\epsilon}{\partial n} = (\gamma-1)A(S)n^{\gamma} = (\gamma-1)\epsilon n$. In the ultra-relativistic limit this further reduces to $p(n, S) = (\gamma-1)\rho$, an equation of state of the form (4) with $a^2 = (\gamma-1)$.

Choose Σ by

$$\Sigma(S) = \ln[A(S)]. \tag{18}$$

We show that across a shock wave the difference $[\Sigma - \Sigma_L]$ is a function of the change of the corresponding Riemann invariant alone. Then the difference $[\Sigma - \Sigma_L]$ along the shock curve is independent of base point. Finally, we show that the difference $[\Sigma - \Sigma_L]$ and its derivative, as a function of the change of Riemann invariants, are monotonically increasing. Note that it is sufficient to show that the change $[\Sigma - \Sigma_L]$ and its derivative are monotonically increasing as viewed as a function of $\ln(\rho/\rho_L)$, because they satisfy the relationship as parameters, $\Delta r = \frac{2a}{a^2+1}\Delta\ln(\rho)$. (For 3−Shocks replace Δr with Δs.) Indeed,

$$\frac{d[S-S_L]}{d(r-r_L)} = \frac{d[S-S_L]}{d\ln(\rho/\rho_L)} \cdot \left|\frac{d\ln(\rho/\rho_L)}{d(r-r_L)}\right| = \frac{a^2+1}{2a} \cdot \frac{d[S-S_L]}{d\ln(\rho/\rho_L)}.$$

Using (15), we have for $\sigma = \ln(\rho/\rho_L)$,

$$[\Sigma - \Sigma_L](\sigma) = (1-\gamma)\sigma + \frac{\gamma}{2}\ln\left(\frac{1+(\gamma-1)e^{\sigma}}{1+(\gamma-1)e^{-\sigma}}\right).$$

After differentiating, we have

$$\frac{d[\Sigma-\Sigma_L]}{d\sigma} = \frac{(e^{\sigma}-1)^2(2-\gamma)(\gamma-1)}{2(1+e^{\sigma}(\gamma-1))(e^{\sigma}+(\gamma-1))},$$

which is non-negative in the domain $1 < \gamma < 2$ and $\sigma \geq 0$. Furthermore, the derivative is zero only when $\sigma = 0$. Thus, $[\Sigma - \Sigma_L](\sigma)$ is a monotonically increasing function. Differentiating a second time we find,

$$\frac{d^2[\Sigma-\Sigma_L]}{d\sigma^2} = \frac{\gamma^2(2-\gamma)(\gamma-1)(e^{3\sigma}-e^{\sigma})}{2(1+e^{\sigma}(\gamma-1))^2(e^{\sigma}+(\gamma-1))^2} > 0,$$

showing $d[\Sigma - \Sigma_L]/d\sigma$ is also monotonically increasing for $1 < \gamma < 2$ and $\sigma > 0$. Considering Lemma 2 we have proven:

Proposition 2. *Consider the ultra-relativistic Euler equations with the equation of state* (5), $1 < \gamma < 2$ *and A satisfying* (6). *Then the change in* $\Sigma = \ln[A(S)]$, *when regarded as a function of the change in the corresponding Riemann invariant, is independent of base state. Geometrically, the shock curves, as viewed in* (r, s, Σ)*−space, are translationally invariant.*

An interesting fact is that the change in Σ becomes nearly linear for strong shock waves. We state this as a corollary.

Corollary 1. *Under the assumptions of Proposition 2, the change in* Σ *becomes nearly linear for large* σ.

Proof.

$$\lim_{\sigma\to\infty}\frac{d[\Sigma-\Sigma_L]}{d\sigma} = \lim_{\sigma\to\infty}\frac{(e^{\sigma}-1)^2(2-\gamma)(\gamma-1)}{2(1+e^{\sigma}(\gamma-1))(e^{\sigma}+(\gamma-1))} = \frac{(2-\gamma)}{2}.$$

□

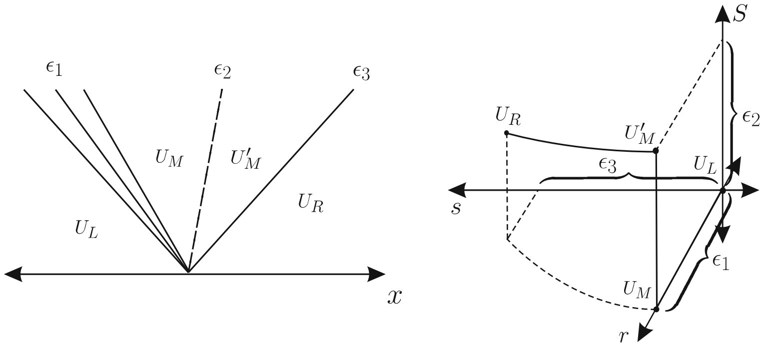


Fig. 1. A solution to the Riemann Problem $<U_L, U_R>$. The states U_M and U'_M differ only in S

2.4. The Riemann problem. The Riemann problem is a particular class of Cauchy problems with initial data of the form,

$$U_0(x) = \begin{cases} U_L & x < 0, \\ U_R & x > 0. \end{cases}$$

From the geometry of the shock-rarefaction curves in the coordinate system of Riemann invariants, we can globally solve Riemann problem for any two initial states in the region $\rho > 0$, $-1 < v < 1$ and $S > 0$.

Theorem 2. *Consider left and right states $U_L = (\rho_L, v_L, S_L)$ and $U_R = (\rho_R, v_R, S_R)$, such that $\rho_L, \rho_R > 0, -1 < v_L, v_R < 1$, and $S_L, S_R > 0$. With the equation of state* (5) *satisfying $1 < \gamma < 2$ and* (6)*, there exists a weak solution to the Riemann problem $<U_L, U_R>$ for system* (1) *in the ultra-relativistic limit. This solution is unique in the class of solutions with constant states separated by centered rarefaction, shock and contact waves.*

We parameterize the $1-$ (resp. 3)shock/rarefaction curve by the change in r (resp. s) and define the strength of a shock or rarefaction wave as the difference in the values of either r for a $1-$shock-rarefaction wave, or s for a $3-$shock-rarefaction wave. We choose the orientation on our parametrization so that we have a positive parameter along the rarefaction curve and negative parameter along the shock curve. Therefore, the solution of the Riemann problem can be given as a sequence of coordinates, $(\epsilon_1, \epsilon_2, \epsilon_3)$, where ϵ_1 denotes the change in the Riemann invariant r from U_L to U_M, ϵ_2 the change in S from U_M to U'_M and ϵ_3 the change in the Riemann invariant s from U'_M to U_R. In summary, for $i = 1, 3$ we have a shock wave of strength ϵ_i when $\epsilon_i < 0$ and a rarefaction wave of strength ϵ_i when $\epsilon_i > 0$ (Fig. 1).

We adopt the following notation: α, strength of $1-$shock wave; β, strength of $3-$shock wave; μ, strength of $1-$rarefaction wave; η, strength of $3-$rarefaction wave; and δ, strength of entropy wave. If $(\epsilon_1, \epsilon_2, \epsilon_3)$ is the solution to the Riemann problem with states U_L, U_R, we would have:

$$\alpha = \begin{cases} -\epsilon_1 & \epsilon_1 \le 0 \\ 0 & \text{otherwise} \end{cases}, \quad \beta = \begin{cases} -\epsilon_3 & \epsilon_3 \le 0 \\ 0 & \text{otherwise} \end{cases},$$
$$\mu = \begin{cases} \epsilon_1 & \epsilon_1 \ge 0 \\ 0 & \text{otherwise} \end{cases}, \quad \eta = \begin{cases} \epsilon_3 & \epsilon_3 \ge 0 \\ 0 & \text{otherwise} \end{cases}. \tag{19}$$

We define $\delta = \Sigma_R - \Sigma_L$ where $\Sigma = \ln[A(S)]$ and denote δ_ω as the absolute change of Σ across a shock wave of strength ω. More specifically, if two states are separated by a shock of strength ω the absolute change in Σ across the shock would be δ_ω for either a 1 or 3−shock. Since we have shown that the change in Σ is independent on the base state and dependent only on the strength of the wave, δ_ω is well defined.

2.5. Interaction estimates. Consider the following three states: $U_L = (\rho_L, v_L, \Sigma_L)$, $U_M = (\rho_M, v_M, \Sigma_M)$, and $U_R = (\rho_R, v_R, \Sigma_R)$. We wish to estimate the difference in the solutions of the three Riemann problems $<U_L, U_M>$, $<U_M, U_R>$, and $<U_L, U_R>$ with solutions denoted by a 1 subscript, 2 subscript and $'$ respectively.

Proposition 3. *Let Ω be a simply connected compact set in rs−space. Then there exists a constant C_0, $1/2 < C_0 < 1$, such that for any interaction $<U_L, U_M> + <U_M, U_R> \to$ $<U_L, U_R>$ in Ω at any value of Σ, one of the following holds:*

1. $A = -\xi \leq 0, \quad 0 \leq B \leq C_0\xi,$
   or
   $B = -\xi \leq 0, \quad 0 \leq A \leq C_0\xi.$
2. $A \leq 0$, *and* $B \leq 0$,

where $A = \alpha' - \alpha_1 - \alpha_2$ and $B = \beta' - \beta_1 - \beta_2$ are change in the strengths of the 1 *and* 3 *shock waves in the solutions.*

These estimates are proven by systematically looking at all possible wave interactions for which we show several representative examples in the Appendix. Because the interactions are independent of entropy level, we only consider interactions within the first and third characteristic classes. The main consequence is that *after an interaction, there cannot be an overall increase in the strengths of the shock waves*. This fact follows since as the solution progresses forward in time, cancelations and merging of shock and rarefaction waves of the same class lead to a decrease in shock strength. For example, when a shock wave is weakened by a rarefaction wave, a reflected shock wave is created in the opposite family. This interaction may increase the total strength of the shock waves in the opposite family, but the total gain in shock strength is uniformly bounded by the loss in the weakened or annihilated shock.

We choose the constant C_0 to be the maximum slope of the largest shock curve that lies within the compact set Ω or $1/2$ in order to bound the constant below. More specifically, let $\overline{\omega}$ be the strongest shock wave possible in Ω. Then we take C_0 to be

$$C_0 = \max\left\{\frac{1}{2}, \left.\frac{dr}{ds}\right|_{\overline{\omega}}, \left.\frac{ds}{dr}\right|_{\overline{\omega}}\right\}. \tag{20}$$

By Lemma 2, the slopes of the shock wave curves in a compact set in the rs−plane are strictly bounded away by 1. Therefore, we conclude $C_0 < 1$.

For interactions in a compact set, the variation in Σ across a shock wave is uniformly bounded by a constant times the strength of the shock, but the variation in Σ may increase after an interaction because of the creation of an entropy wave. Typically, across these waves the pressure is invariant and there is a jump in density; however, under the assumption (4), there must be no jump in energy density. Thus, we cannot use $\ln(\rho/\rho_L)$ or the change in the Riemann invariants r or s as a measure of wave strength. It should be noted that under certain interactions, such as an i−shock being weakened by an incoming i−rarefaction wave, an entropy wave is created with strength such that

$\Sigma_R - \Sigma_L$ is equal to the loss in entropy change across the shock, plus the change in the entropy across the new shock wave in the opposite family. We need a way to bound the variation in the entropy waves and it turns out that this increase is bounded by a corresponding decrease in the shock strengths.

Proposition 4. *For every simply connected compact set Ω in $rs-$space, there exists a constant $M > 0$ such that after every interaction in Ω, at any value Σ for the system (1) with (5) in the ultra-relativistic limit, the following holds:*

$$|\delta'| - |\delta_1| - |\delta_2| + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leq -M(A+B).$$

Proof. Choose C_0 so that Proposition 3 holds. Since Ω is a compact set, let

$$\overline{\omega} = \sup\{\|(r_1, s_1) - (r_2, s_2)\| : (r_1, s_1), (r_2, s_2) \in \Omega\}$$

so that the strength of the largest shock wave in Ω is bounded above by $\overline{\omega}$. Furthermore, let $M = (1 - C_0)^{-1}\,\overline{M}$, where

$$\overline{M} = 2\frac{d[\Sigma - \Sigma_L]}{d\omega}(\overline{\omega}), \tag{21}$$

which is twice the largest rate of change of Σ for all shocks contained in Ω. Also, since $[\Sigma - \Sigma_L](\omega)$ is positive and convex up, we have for strengths, $\omega' \geq \omega_1 + \omega_2$, $\delta_{\omega'} \geq \delta_{\omega_1} + \delta_{\omega_2}$.

The proof will be split into two cases, one for each of the two cases from Proposition 3. First let us assume that $A \leq 0$ and $B \leq 0$, i.e., $\alpha' - \alpha_1 - \alpha_2 = -\xi_\alpha \leq 0$ and $\beta' - \beta_1 - \beta_2 = -\xi_\beta \leq 0$. We have $\alpha_1 + \alpha_2 - \xi_\alpha = \alpha'$ and hence, $\delta_{(\alpha_1+\alpha_2-\xi_\alpha)} = \delta_{\alpha'}$. It follows that

$$\delta_{\alpha_1} + \delta_{\alpha_2} - \frac{1}{2}\overline{M}\xi_\alpha \leq \delta_{\alpha_1+\alpha_2} - \frac{1}{2}\overline{M}\xi_\alpha \leq \delta_{\alpha'},$$

and

$$\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \leq \frac{1}{2}\overline{M}\xi_\alpha \leq -\frac{1}{2}MA. \tag{22}$$

Also, the change in entropy across the two Riemann problems before and the resulting one are equal:

$$\delta_{\alpha'} + \delta' - \delta_{\beta'} = \delta_{\alpha_1} + \delta_1 - \delta_{\beta_1} + \delta_{\alpha_2} + \delta_2 - \delta_{\beta_2}. \tag{23}$$

Rearranging (23) and using (22), we find

$$\left(\delta' - \delta_1 - \delta_2\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) = \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) \leq -\frac{1}{2}MA. \tag{24}$$

Adding the inequality (22) to (24),

$$\left(\delta' - \delta_1 - \delta_2\right) + \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) \leq -\frac{1}{2}MA - \frac{1}{2}MA = -MA.$$

By a similar argument we also have

$$-\left(\delta' - \delta_1 - \delta_2\right) + \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) \leq -MB.$$

Since $0 \le -MA$ and $0 \le -MB$ by assumption, and $|\delta' - \delta_1 - \delta_2| \ge |\delta'| - |\delta_1| - |\delta_2|$, we deduce

$$|\delta'| - |\delta_1| - |\delta_2| + \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) \le -M(A+B),$$

which concludes the proof of the first case.

Now, without loss of generality assume $A = -\xi \le 0$ and $0 \le B \le C_0\xi$. The other case when $0 \le A$ is similar. As before we find $\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \le \frac{1}{2}\overline{M}\xi$ and

$$\left(\delta' - \delta_1 - \delta_2\right) + \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) \le \overline{M}\xi. \tag{25}$$

Since $\beta' \ge \beta_1 + \beta_2$, we have $\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'} \le 0$ and so by adding this inequality twice to (23),

$$-\left(\delta' - \delta_1 - \delta_2\right) + \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) \le 0. \tag{26}$$

Therefore, from (25) and $|\delta' - \delta_1 - \delta_2| \ge |\delta'| - |\delta_1| - |\delta_2|$,

$$|\delta'| - |\delta_1| - |\delta_2| + \left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) + \left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) \le \overline{M}\xi.$$

But, $\overline{M}\xi = M(1 - C_0)\xi = M(\xi - C_0\xi) \le M(-A - B) = -M(A+B)$, where we used the fact $-C_0\xi \le -B$ following from the assumption that $0 \le B \le C_0\xi$ and $A = -\xi$. □

3. Glimm's Difference Scheme

In 1965 Glimm [4] proved existence of solutions to general systems of strictly hyperbolic conservation laws with genuinely non-linear or linearly degenerate characteristic fields. Glimm's method takes a piecewise constant approximate solution at one time step and uses Riemann problems, defined at each point of discontinuity, to evolve the solution to a later time. After the approximate solution is brought forward in time, the solution is randomly sampled and a new piecewise constant approximate solution is obtained. In this section we use a Glimm scheme to construct approximate solutions to (1).

3.1. Glimm difference scheme. Begin by partitioning space into intervals of length Δx and time into intervals of length Δt. In order to keep neighboring Riemann problems from colliding, we impose the following CFL condition: $\frac{\Delta x}{\Delta t} > 1 > |\lambda_i|$, $i = 1, 2, 3$. For $1 < \gamma < 2$ this condition is satisfied since the characteristic speeds are bounded above and below by 1 and -1.

We inductively define our approximate solution. To begin suppose that we have an approximate solution at time $t = n\Delta t$, $U(x, n\Delta t)$, which is constant on the intervals, $(k\Delta x, (k+2)\Delta x)$, where $k + n$ is odd. At each point $x = k\Delta x$ a Riemann problem is defined. Solve each Riemann problem for time $t = \Delta t$. This evolves our approximate solution forward in time from $t = n\Delta t$ to $t = (n+1)\Delta t$. To finish, we must construct a new piecewise constant function at time $t = (n+1)\Delta t$. Choose $a \in [-1, 1]$ and define, $U(x, (n+1)\Delta t) = U((k+1+a)\Delta x, (n+1)\Delta t-)$ for $x \in (k\Delta x, (k+2)\Delta x)$ and $k+n+1$ odd. Here $\Delta t-$ denotes the lower limit. To begin this process at $t = 0$, obtain a piecewise constant function from the initial data $U_0(x)$ by again choosing $a \in [-1, 1]$ and defining, $U(x, 0) = U_0((k+a)\Delta x)$ for k odd.

Consider, $\theta \in \prod_{i=0}^{\infty}[-1,1]$. We call $U_{\theta,\Delta x}(x,t)$ the approximate solution given by a mesh size of Δx with sampling points at the n^{th} time step given by θ_n. In order to estimate the change in the variation of our approximate solutions, we define piecewise linear, space-like curves, called I-curves, which connect sample points at different time levels. If an I-curve J passes through the sampling point $((k+\theta_n)\Delta x, n\Delta t)$, then J is only allowed to connect to $((k+1+\theta_{n\pm 1})\Delta x, (n\pm 1)\Delta t)$ on the right and to $((k-1+\theta_{n\pm 1})\Delta x, (n\pm 1)\Delta t)$ on the left.

We consider two functionals defined on $I-$curves. Define for an $I-$curve J:

$$F(J) = \sum_J \alpha_i + \sum_J \beta_i + V \tag{27}$$

and

$$L(J) = \sum_J \left(\alpha_i - M_0\delta_{\alpha_i}\right) + \sum_J \left(\beta_i - M_0\delta_{\beta_i}\right) - M_0 \sum_j |\delta| + V, \tag{28}$$

where the sums are taken over all waves that cross J. The constant M_0 will be chosen later and $V = Var\{U_0(\cdot)\}$ is the variation of the initial data.

The main problem in our analysis is to show that the variation in the entropy waves stays bounded for all time. To do this we need to bound the possible change in Σ across shock waves. This is accomplished by first showing that the variation in r and s stays finite for all time. This implies that all the interactions, as projected onto the $rs-$plane, occur in a compact set. Thus, there is a largest possible shock strength in this compact set. Using the fact that the derivative of the entropy change as a function of wave strength is monotonically increasing, there is a constant such that the entropy change is bounded by a constant times the wave strength. Moreover, we can then use Proposition 4 to estimate the increase in the variation in entropy in our approximate solutions.

3.2. Estimates on approximate solutions. For initial data $U_0(x)$ and the corresponding approximate solution $U_{\theta,\Delta x}(x,t)$, define $\overline{U}_0(x)$ and $\overline{U}_{\theta,\Delta x}(x,t)$ as the initial data and approximate solutions viewed as functions of r and s only. The first estimate will show that the variation in the Riemann invariants across an I-curve J is bounded above by the functional F on J.

Proposition 5. *Let $\overline{U}_0(\cdot)$ be of finite variation, J an I-curve and suppose that the approximate solution $\overline{U}_{\theta,\Delta x}(x,t)$ is defined on J. Then, $Var_{rs}(J) \le 4F(J)$.*

Proof. Let $Var_r^-(J)$ and $Var_r^-(J)$ denote the variation across J given by a decrease and increase in r respectively. The only waves that contribute to the decrease in r are 1 and 3$-$shocks and increase 1$-$rarefactions. Therefore,

$$Var_r^-(J) \le \sum_J \alpha_i + \sum_J \beta_i \quad \text{and} \quad Var_r^+(J) = \sum_J \mu_i, \tag{29}$$

where the sum is over all waves of the particular type crossing J. Following this line of reasoning for s, we also have

$$Var_s^-(J) \le \sum_J \alpha_i + \sum_J \beta_i \quad \text{and} \quad Var_s^+(J) = \sum_J \eta_i. \tag{30}$$

The initial data $\overline{U}_0$ may be written as a function of the Riemann invariants r and s, $\overline{U}_0(x) = (r_0(x), s_0(x))$. Since $\overline{U}_0(\cdot)$ is of finite variation, the limits $\lim_{x\to\pm\infty} r_0(x) = r^{\pm}$ and $\lim_{x\to\pm\infty} s_0(x) = s^{\pm}$ exist. For any I-curve J, the end states at $\pm\infty$ are given by $(r^{\pm}, s^{\pm})$. From this we obtain, $|Var_r^+(J) - Var_r^-(J)| = |r^+ - r^-| \le V$, and hence $Var_r^+(J) \le Var_r^-(J) + V$. Using (29) and similarly from (30), $\sum_J \mu_i \le \sum_J \alpha_i + \sum_J \beta_i + V$ and $\sum_J \eta_i \le \sum_J \alpha_i + \sum_J \beta_i + V$. Combining these together we have $\sum_J \mu_i + \sum_J \eta_i \le 2\left(\sum_J \alpha_i + \sum_J \beta_i + V\right)$. Thus,

$$\begin{aligned} Var_{rs}(J) &\le 2\left(\sum_J \alpha_i + \sum_J \beta_i\right) + \sum_J \mu_i + \sum_J \eta_i, \\ &\le 4\left(\sum_J \alpha_i + \sum_J \beta_i\right) + 2V \le 4F(J). \end{aligned}$$

□

We now show that the functional F on the I-curves is non-increasing. We define a partial ordering on the I-curves by saying that $J \prec J'$ if the curve J' never lies below the curve J. Furthermore, we say that J' is an immediate successor to J if $J \prec J'$ and J and J' share all the same sample points except for one. It is clear that for any pair of I-curves such that $J \prec J'$, there is a sequence of immediate successors that begins at J and ends at J'. The next proposition shows that if our approximate solution is defined on an I-curve, it can be defined for all following I-curves.

Proposition 6. *Let J and J' be $I-$curves, $J \prec J'$, and suppose that J is in the domain of definition of $\overline{U}_{\theta,\Delta x}$. If $F(J) < \infty$, then J' is in the domain of definition of $\overline{U}_{\Delta x,\theta}$, and $F(J') \le F(J)$. Moreover, if $Var_{rs}\{U_0(\cdot)\} < \infty$ then $\overline{U}_{\theta,\Delta x}$ can be defined for $t \ge 0$.*

Proof. We proceed by induction. Suppose first that J' is an immediate successor to J. Then the difference $F(J') - F(J)$ is given by the change in shock wave strengths across the diamond enclosed by J' and J. This is a consequence of the fact that the waves that head into the diamond from the left and right solve the same Riemann problem as the outgoing waves in the new single Riemann problem. If we denote J_0' and J_0 as the diamond portion of J' and J, we have by Proposition 3,

$$\begin{aligned} F(J') - F(J) &= \sum_{J'} \alpha_i + \sum_{J'} \beta_i + V - \left(\sum_J \alpha_i + \sum_J \beta_i + V\right), \\ &= \sum_{J_0'} \alpha_i + \sum_{J_0'} \beta_i - \sum_{J_0} \alpha_i - \sum_{J_0} \beta_i, \\ &= \left(\alpha' - \alpha_1 - \alpha_2\right) + \left(\beta' - \beta_1 - \beta_2\right) \le A + B \le 0. \end{aligned}$$

Thus, $F(J') \le F(J)$ for immediate successors. For any a general J and J' such that $J \prec J'$, we produce a sequence of immediate successors that take J to J'. At each step the functional F is non-increasing, thus $F(J') \le F(J)$ continues to hold.

By Proposition 5, $Var_{rs}(J') \le 4F(J') \le 4F(J)$, so, J' is in the domain of definition of $\overline{U}_{\theta,\Delta x}$. Moreover, if $Var_{rs}\left\{\overline{U}_0(\cdot)\right\} < \infty$, then $Var_{rs}(\mathbf{0}) < \infty$ for the unique $I-$curve $\mathbf{0}$ that lies along the line $t = 0$. In order to show that $\overline{U}_{\Delta x,\theta}$ can be defined for $t \ge 0$, we must show that $Var_{rs}\left\{\overline{U}_{\theta,\Delta x}(\cdot, t)\right\} < \infty$ for all time. But, this condition is equivalent to showing the variation across any $I-$curve J is always finite. Since for any $I-$curve J, $Var_{rs}(J) \le 4F(J) \le 4F(\mathbf{0}) \le 8Var_{rs}\{U_0(\cdot)\}$, the result follows. □

Again, Proposition 6 shows that the variation of our approximate solution in the variables r and s is finite. Thus, there exists a compact set in the $rs-$plane that contains all the interactions in our approximate solution.

Corollary 2. *Suppose that $Var_{rs}\{U_0(\cdot)\} < \infty$. Then there exists a simply connected compact set Ω in the $rs-$plane such that all possible interactions are contained in Ω.*

Proof. From Proposition 5 and Proposition 6 we know that for any I-curve J,

$$Var_{rs}(J) < 4F(J) < 4F(\mathbf{0}) < 8Var_{rs}\{U_0(\cdot)\} = N < \infty.$$

Thus, the distance between any two states occurring anywhere in our approximate solution is bounded by N. Consider the left limit state of $\overline{U}_0(\cdot)$, (r^-, s^-). Therefore, all states must be contained within the ball of radius $2N$ centered around (r^-, s^-). □

Now, we show that the variation of our approximate solution, including the variation in Σ, is bounded above by the functional $L(\cdot)$.

Proposition 7. *Suppose $Var\{U_0(\cdot)\} < \infty$ and J is an $I-$curve that is in the domain of definition of $U_{\theta,\Delta x}$. Then there exists constants $M_0 > 0$ and $K > 0$, independent of Δx and θ, such that $Var(J) \le K \cdot L(J)$.*

Proof. The variation across the $I-$curve J is bounded by

$$\begin{aligned} Var(J) \le\ & Var(\text{Shock Waves}) + Var(\text{Rarefaction Waves}) \\ & + Var(\Sigma\text{-Waves}) + Var(\Sigma \text{across Shocks}). \end{aligned}$$

Since $Var_{rs}\left\{\overline{U}_0(\cdot)\right\} \le Var\{U_0(\cdot)\} < \infty$, we have from Corollary 2 that all the interactions projected into the $rs-$plane occur in a compact set Ω. Therefore there exists a constant $\overline{M} > 0$ such that for a shock wave of strength ω, $\delta_\omega \le \overline{M}\omega$. Let $M = (1-C_0)^{-1}\overline{M}$ as in Proposition 4. Since, $\overline{M} < M$ we have for a shock wave of strength ω, $\delta_\omega < M\omega$.

From the proof of Proposition 5, we can bound the variation from the shock waves and rarefaction waves by the shock waves crossing J and the initial variation V. Thus,

$$\begin{aligned} Var(J) &\le 2\left(\sum_J \alpha_i + \sum_J \beta_i\right) + \sum_J \mu_i + \sum_J \eta_i + \sum_J |\delta| + \sum_J \delta_{\alpha_i} + \sum_J \delta_{\beta_i}, \\ &\le 4\left(\sum_J \alpha_i + \sum_J \beta_i + V\right) + \sum_J |\delta| + M\left(\sum_J \alpha_i + \sum_J \beta_i\right), \\ &\le (4+M)\left(\sum_J \alpha_i + \sum_J \beta_i + V\right) + \sum_J |\delta|. \end{aligned}$$

Let $M_0 \le 1/2M$. Then, $M_0\delta_\omega \le \frac{1}{2M}\delta_\omega \le \frac{1}{2M}(M\omega) \le \frac{1}{2}\omega$. Thus, for a shock wave of strength ω, $\omega \le 2(\omega - M_0\delta_\omega)$. Using this we find,

$$Var(J) \le 2(4+M)\left(\sum_J \left(\alpha_i - M_0\delta_{\beta_i}\right) + \sum_J \left(\beta_i - M_0\delta_{\beta_i}\right) + V\right) + \sum_J |\delta|.$$

Finally, since $M_0 \cdot 2(4+M) \geq 2MM_0 \geq 1$, we move the sum of the strengths of the entropy waves inside,

$$Var(J) \leq 2(4+M)\left(\sum_J \left(\alpha_i - M_0\delta_{\beta_i}\right) + \sum_J \left(\beta_i - M_0\delta_{\beta_i}\right) + M_0 \sum_J |\delta| + V\right).$$

Therefore, $Var(J) \leq K \cdot L(J)$, with $K = 2(4+M)$. □

Proposition 8. *Suppose that* $Var\{U_0(\cdot)\} < \infty$ *and* J, J' *are I-curves such that* $J \prec J'$ *and* $L(J) < \infty$*. Then* J' *is in the domain of definition of* $U_{\theta,\Delta x}(x,t)$, $L(J') \leq L(J)$ *and* $U_{\Delta x,\theta}(x,t)$ *is defined for* $t \geq 0$.

Proof. Since $Var_{rs}\left\{\overline{U}_0(\cdot)\right\} < Var\{U_0(\cdot)\} < \infty$ there exists a compact set Ω that contains all possible interactions. Define M as in Proposition 4 and take $M_0 \leq 1/2M$. As Proposition 6, we prove the result by induction on the I curves. First let J' be an immediate successor to J. Let J_0' and J_0 be the parts of J' and J that bound the diamond formed by J and J'. Using this and the definition of $L(J)$,

$$\begin{aligned} L(J') - L(J) \leq & \left[\sum_{J_0'} \left(\alpha_i - M_0\delta_{\alpha_i}\right) + \sum_{J_0'} \left(\beta_i - M_0\delta_{\beta_i}\right) + M_0 \sum_{J_0'} |\delta|\right] \\ & - \left[\sum_{J_0} \left(\alpha_i - M_0\delta_{\alpha_i}\right) + \sum_{J_0} \left(\beta_i - M_0\delta_{\beta_i}\right) + M_0 \sum_{J_0} |\delta|\right], \\ = & \left(\alpha' - \alpha_1 - \alpha_2\right) + \left(\beta' - \beta_1 - \beta_2\right) + M_0\left(\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}\right) \\ & + M_0\left(\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\right) + M_0\left(|\delta'| - |\delta_1| - |\delta_2|\right). \end{aligned}$$

Now we refer to Proposition 3 and 4. We see that the first two terms are equal to $(A+B)$ and the others are bounded above by $-M(A+B)$. Putting this together,

$$L(J') - L(J) \leq (A+B) - MM_0(A+B) \leq \frac{1}{2}(A+B) \leq 0.$$

For immediate successors, we have $L(J') \leq L(J)$. Moreover, by Proposition 7 we have that the variation along J' is bounded by $L(J')$ and hence $L(J)$. Thus, J' is in the domain of definition of $U_{\Delta x,\theta}$.

For general J and J' such that $J \prec J'$, the same conclusion holds by constructing a sequence of immediate successors to move from J to J'. Along each step, the results above continue to hold. Finally, if $Var\{U_0(\cdot)\} < \infty$, we have $L(\mathbf{0}) < \infty$ and for any $I-$curve J, $L(J) \leq L(\mathbf{0})$. Thus we can conclude that

$$Var(J) \leq 2(4+M)L(J) \leq 2(4+M)L(\mathbf{0}) < \infty,$$

so our approximate solution can be defined for $t \geq 0$. □

4. Existence of Weak Solutions

We use Glimm's Theorem [4] to prove existence of solutions to (1) in the ultra-relativistic limit with an equation of state of the form (5). For θ fixed and $x_n = 1/2^n$, the set of approximate solutions $\left\{U_{\theta,\Delta x_n}(x,t)\right\}_{n=1}^{\infty}$ has uniformly bounded variation by Proposition 7. Furthermore, since the variation is bounded and each approximate solution has the same limits at infinity, the sup norm is also uniformly bounded and are L^1 Lipschitz in time. At this point Helly's Theorem [1] provides a convergent subsequence, $U_{\theta,\Delta x_{n_i}}(x,t)$, that converges to a function $U(x,t)$ with finite variation for each fixed time. However, there is no justification that this limit is actually a weak solution. Glimm's Theorem guarantees that there exists a subsequence that converges to a weak solution.

Theorem 3 [4]. *Assume that the approximate solution $U_{\theta,\Delta x_i}$ satisfies,*

$$Var\left\{U_{\theta,\Delta x_i}(\cdot,t)\right\} < N < \infty \tag{31}$$

for $x_i = 1/2^i, \theta \in \Theta = \prod_{i=0}^{\infty}[-1,1]$, and all $t \geq 0$. Then there exists a subsequence of mesh lengths Δx_{i_k} such that $U_{\theta,\Delta x_{i_k}} \to U$ in $\mathbf{L^1_{Loc}}$, where $U(x,t)$ satisfies

$$Var\left\{U(\cdot,t)\right\} < N.$$

Furthermore, there exists a set of measure zero $\overline{\Theta} \subset \Theta$ such that if $\theta \in \Theta - \overline{\Theta}$ then $U(x,t)$ is a weak solution to (1).

We now prove Theorem 1 by showing that our approximate solutions meet the assumptions of Glimm's Theorem.

Proof. Assume the initial data satisfies, (8), (9), and (10). We show that for all Δx_i and sample points θ, $Var\left\{U_{\Delta x,\theta}(\cdot,t)\right\} < N < \infty$, where $U_{\theta,\Delta x}(\rho(x,t), v(x,t), S(x,t)) = (U_1, U_2, U_3)_{\theta,\Delta x}$. First we show that the variation in ρ, v, and S is bounded for all time in the approximate solutions.

From Proposition 5 and Proposition 6 we have that the variation of our approximate solution in r and s is uniformly bounded for all time. More specifically,

$$\begin{aligned} Var_{rs}\left\{\overline{U}_{\theta,\Delta x}(\cdot,t)\right\} < 4F(\mathbf{0}) &< 4\left[\sum_{\mathbf{0}}\alpha_i + \sum_{\mathbf{0}}\beta_i + Var_{rs}\{U_0\}\right] \\ &< 8\cdot Var_{rs}\{U_0(\cdot)\}. \end{aligned}$$

From this the variation of $\ln(\rho)$ and $\ln\left(\frac{1+v}{1-v}\right)$ are also bounded for all time. Using $\ln\left(\frac{1+v}{1-v}\right) = \frac{1}{2}(r+s)$ we have

$$\begin{aligned} Var\left\{\ln\left(\frac{1+v(\cdot,t)}{1-v(\cdot,t)}\right)\right\} &= \frac{1}{2}\sup_N \sum_{i=1}^{N} |(r(x_{i+1},t)+s(x_{i+1},t)) - (r(x_i,t)+s(x_i,t))|, \\ &\leq \frac{1}{2}Var_{rs}\left\{\overline{U}_{\Delta x,\theta}(\cdot,t)\right\} + \frac{1}{2}Var_{rs}\left\{\overline{U}_{\Delta x,\theta}(\cdot,t)\right\}, \\ &\leq 8\cdot Var\left\{U_0(\cdot)\right\}. \end{aligned}$$

Similarly, using $\ln(\rho) = \frac{1+a^2}{a}(s-r)$ we find, $Var\left\{\ln(\rho(\cdot,t))\right\} \leq 16\left(\frac{1+a^2}{a}\right) Var\left\{U_0(\cdot)\right\}$.

The variation in Σ is also bounded for all time in approximate solutions. This is clear from Proposition 7 and Proposition 8 because there exists a constant M so that $Var\left\{\Sigma_{\theta,\Delta x}(\cdot,t)\right\} \le 2(4+M)L(\mathbf{0})$.

We can now show that the variation in ρ, v and S is bounded for all time. Since $Var\{\ln(\rho(\cdot,t))\} < \infty$ for all $t > 0$ there exists a constant $b > 0$ such that $\rho(x,t) < b$. Let $c = max\{1,b\}$, then $Var\{\rho(\cdot,t)\} \le c \cdot Var\{\ln(\rho(\cdot,t))\}$.

For v we have,

$$
\begin{aligned}
Var\{v(\cdot,t)\} &= \sup_N \sum_{i=1}^{N} |v(x_{i+1},t) - v(x_i,t)|, \\
&\le \sup_N \sum_{i=1}^{N} \left| \ln\left(\frac{1+v(x_{i+1},t)}{1-v(x_{i+1},t)}\right) - \ln\left(\frac{1+v(x_i,t)}{1-v(x_i,t)}\right)\right|, \\
&\le Var\left\{\ln\left(\frac{1+v(\cdot,t)}{1-v(\cdot,t)}\right)\right\}.
\end{aligned}
$$

For S we need to find a constant C such that $|S(x,t)-S(y,t)| \le C\,|\Sigma(x,t)-\Sigma(y,t)|$. Since Σ is of finite variation for all time, there exists a largest and smallest value of S, say S_{max} and S_{min} with $0 < S_{min} \le S_{max}$. Define C by

$$
C = \max_{S\in[S_{min},S_{max}]} \left(\frac{d\Sigma}{dS}\right)^{-1} = \max_{S\in[S_{min},S_{max}]} \frac{A(S)}{A'(S)}.
$$

It follows that $Var\{S(\cdot,t)\} \le C \cdot Var\{\Sigma(\cdot,t)\}$.

Finally, from Proposition 1 the determinant of the Jacobian is bounded away from zero for all approximate solutions. Thus, the variation in conserved variables, (U_1, U_2, U_3), are bounded for all $t \ge 0$, θ and Δx_i.

Therefore, Theorem 3 provides existence of a set measure zero $\overline{\Theta} \subset \Theta$ such that if we choose $\theta \in \Theta - \overline{\Theta}$ there exists a subsequence of mesh refinements, $\Delta x_{i_k} \to 0$ such that $U_{\theta,\Delta x_{i_k}}$ converges pointwise almost everywhere in L^1_{loc} to a weak solution, $U(x,t)$ of (1). Moreover, this solution satisfies (11), (12) and (13) for some $N > 0$, all $t > 0$ and is L^1 Lipschitz in time. □

Acknowledgements. This paper is part of a doctoral thesis written under Professor J.B. Temple at the University of California, Davis.

5. Appendix: Interaction Estimates

In this section we discuss four cases of the interaction estimates needed to prove Proposition 3. In total there are sixteen possible incoming wave profiles, corresponding to whether each of the four incoming waves are a shock or rarefaction wave, and between one and four outgoing wave configurations. The main consequence of our estimates is that after an interaction there can be an increase in strengths of the shock waves in one class, but it is accompanied by a corresponding decrease in overall shock strength in the other class. We assume that all the interactions occur in a simply connected compact set $\Omega \subset \mathbb{R}^2$ and, as in (20), we define C_0 as the max of $1/2$ and the maximum slope of the largest shockwave contained in Ω.

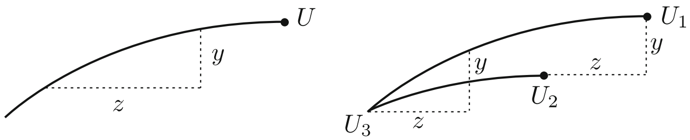


Fig. 2. The change in Riemann invariants along shock curves satisfy $y/z < C_0$

For these estimates we repeatedly utilize that the shock curves in the space of Riemann invariants are traslationally invariant, convex and whose derivatives are bounded above by C_0. Since our definition of wave strength is determined by the change in Riemann invariant r for 1-waves and s for 3-waves, we use the following two facts: one, the change in s, (r) along a 1, (3)-shock is uniformly bounded by the change in r, (s); and two, if two shock waves of the same family begin at two distinct states U_1 and U_2 and meet at a common third state U_3, then the ratio of the distances along the r and s axes from U_1 to U_2 are bounded above by C_0. These two facts are shown geometrically in Fig. 2. We also note that interaction estimates are often similar for cases where the shock and rarefaction waves are permuted in either the incoming or outgoing waves. For example, one can show the estimates hold in a similar manner for the four permutations of three incoming shock waves and one rarefaction wave.

We begin by noticing that after an interaction, the strengths of the shock waves in both families cannot increase. Suppose that $B > 0$. If the outgoing $1-$wave is a rarefaction wave, we are done since $A = -\alpha_1 - \alpha_2 \leq 0$. Suppose now that the outgoing $1-$wave is a shock. Since the starting and ending states, U_L and U_R, are fixed before and after an interaction, the total change in Riemann invariants is the same. Equating the change in r we find,

$$-\alpha' - \Delta r_{\beta'} = -\alpha_1 - \alpha_2 + \mu_1 + \mu_2 - \Delta r_{\beta_1} - \Delta r_{\beta_2},$$

where Δr_β is the change in r of the β shock. Rearranging terms we get,

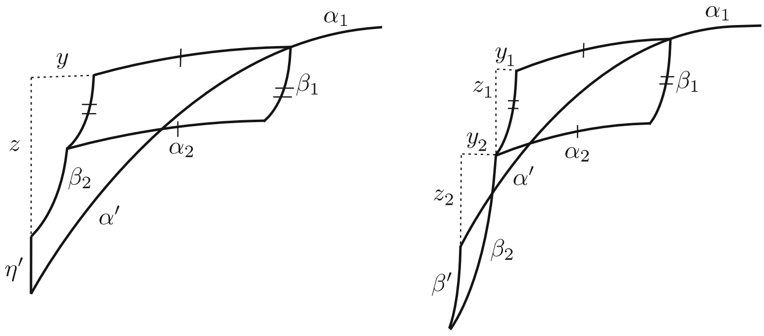
$$\alpha' - \alpha_1 - \alpha_2 = -\mu_1 - \mu_2 + \Delta r_{\beta_1} + \Delta r_{\beta_2} - \Delta r_{\beta'}.$$

By strict concavity of the shock curves and $\beta' > \beta_1+\beta_2$, we have $\Delta r_{\beta_1}+\Delta r_{\beta_2}-\Delta r_{\beta'} < 0$ and thus $A = \alpha' - \alpha_1 - \alpha_2 \leq 0$. Proposition 3 refines this result further. It states that the increase in B is strictly bounded above by the decrease in A.

For our first wave interaction estimate consider the case with four incoming shock waves, $(\alpha_1, \beta_1)+(\alpha_2, \beta_2)$ for which there are three possible outgoing wave profiles: two shock waves, (α', β'), or one rarefaction wave and one shock wave, (μ', β') or (α', η). Consider the interaction $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) \to (\alpha', \beta')$ and suppose that $A \geq 0$. (The case with $B \geq 0$ is similar.) See Fig. 3. We have $B = \beta' - \beta_1 - \beta_2 = -z_1 - z_2 = -\xi$ and $A = \alpha' - \alpha_1 - \alpha_2 = y_1 + y_2 < C_0(z_1 + z_2) = -C_0\xi$. Now consider the same interaction, but with outgoing waves, (α', η'). In this case, $B = -\beta_1 - \beta_2 = -z$ and $A = \alpha' - \alpha_1 - \alpha_2 = y < C_0 z$.

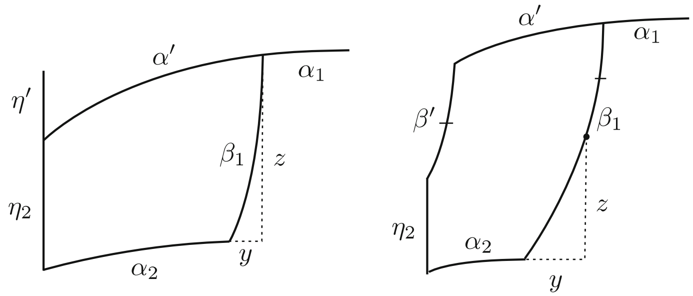
The interaction $(\alpha_1, \beta_1) + (\alpha_2, \eta_2)$ has two possible outgoing profiles, (α', β') or (α', η'). See Fig. 4. For the first case, $B = \beta' - \beta_1 = -z$ and $\alpha' = \alpha_1 + \alpha_2 + y$. Hence, $A = y < C_0 z$. For the outgoing waves, (α', η'), $B = -z$ and $A = y < C_0 z$.

Now consider the incoming waves, (α_1, η_1) and (μ_2, β_2). If two rarefaction waves are produced, $A, B \leq 0$ and we are done. For one or two outgoing shocks we look at outgoing cases (α', η') and (α', β'). See Fig. 5. With one shock, either $A, B \leq 0$



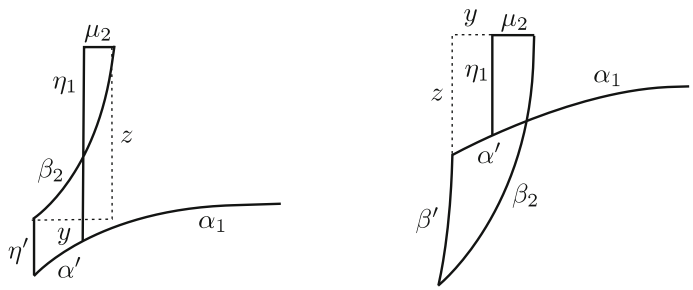
(a) **(b)**

Fig. 3. Interaction $(\alpha_1, \beta_1) + (\alpha_2, \beta_2)$, with result, (**a**), (α', η') and (**b**), (α', β')



(a) **(b)**

Fig. 4. Interaction $(\alpha_1, \beta_1) + (\alpha_2, \eta_2)$, with result, (**a**), (α', η') and (**b**), (α', β')



(a) **(b)**

Fig. 5. Interaction $(\alpha_1, \eta_1) + (\mu_2, \beta_2)$, with result, (**a**), (α', η') and (**b**), (α', β')

or $B = -\beta_2 = -z$ and $A = y - \mu_2 \leq y < C_0 z$. For two outgoing shocks with $A \geq 0$, $B = -z$ and $A = y < C_0 z$.

Lastly, we consider the interaction of three rarefaction waves and a shock wave, $(\alpha_1, \eta_1) + (\mu_2, \eta_2)$, Fig. 6. The cases with outgoing waves, (μ', η') and (α', η') have $A, B \leq 0$. For $(\alpha_1, \nu_1) + (\mu_2, \eta_2) \rightarrow (\mu', \beta')$, we have $A = -\alpha_1 = -z$ and $B = \beta' \leq y < C_0 z$ and for $(\alpha_1, \eta_1) + (\mu_2, \nu_2) \rightarrow (\alpha', \beta')$ we have $A = \alpha' - \alpha_1 = -z$ and $B = \beta' \leq y < C_0 z$.

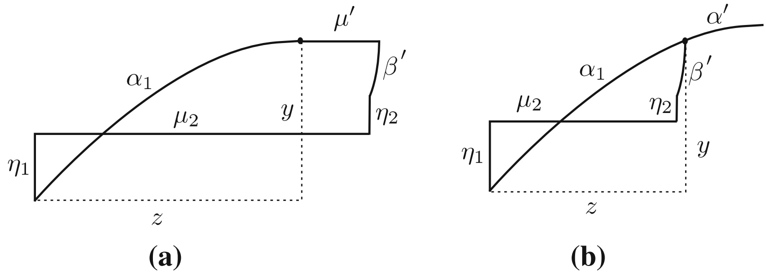


Fig. 6. Interaction $(\alpha_1, \eta_1) + (\mu_2, \eta_2)$, with result, (**a**), (μ', β') and (**b**), (α', β')

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Communicated by P. Constantin