Journal of Mathematical Physics

Causal dissipation in the relativistic dynamics of barotropic fluids

Cite as: J. Math. Phys. **59**, 063101 (2018); https://doi.org/10.1063/1.5007831
Submitted: 03 October 2017 . Accepted: 02 June 2018 . Published Online: 21 June 2018

Heinrich Freistühler, and Blake Temple

JOURNAL OF MATHEMATICAL PHYSICS **59**, 063101 (2018)

Causal dissipation in the relativistic dynamics of barotropic fluids

Heinrich Freistühler[1] and Blake Temple[2]
[1]*Department of Mathematics, University of Konstanz, 78457 Konstanz, Germany*
[2]*Department of Mathematics, University of California, Davis, California 95616, USA*

(Received 3 October 2017; accepted 2 June 2018; published online 21 June 2018)

This paper develops two causal four-field theories of relativistic fluid dynamics, one for viscous fluids that are barotropic and the other for viscous heat-conductive fluids that are "thermo-barotropic," a property that is newly defined here. While similar to the deduction of a five-field theory in the authors' article *Causal dissipation for the relativistic dynamics of ideal gases* [Freistühler, H. and Temple, B., Proc. R. Soc. A **473**, 20160729 (2017)], the argumentation is consistently carried out in a way that stays closer to Eckart's flow frame (than to Landau's), and the result is directly compared with the four-field theories resulting from early proposals made by Eckart, Lichnerowicz, and Choquet-Bruhat. The key idea is to impose second-order symmetric hyperbolicity in the sense of Hughes, Kato, and Marsden [Arch. Rational Mech. Anal. **63**, 273–294 (1976)], in the natural Godunov variables that make the corresponding system of perfect-fluid conservation laws symmetric hyperbolic in the first order sense. The new theory for viscous heat-conductive thermo-barotropic fluids belongs to the Hughes-Kato-Marsden class; the new theory for viscous general barotropic fluids lies on the boundary of that class. As in the five-field theory, the coefficients of bulk viscosity ζ, shear viscosity η, and, in the case of thermo-barotropic fluids, that of heat conductivity χ are free parameters, and in terms of these, the relativistic dissipation tensor is uniquely determined under three conditions which the authors propose as definitive: *symmetric hyperbolicity*, *sharp causality*, and *first-order equivalence with Eckart*—the requirement that the resulting equations be equivalent to the Eckart equations to leading order in ζ, η, and χ. The new theory for general viscous barotropic fluids complements the one given in the abovementioned paper for massive ideal gases. The new theory for viscous heat-conductive thermo-barotropic fluids notably includes the case of pure radiation, providing as one application a quantitative correction to the authors' previous proposal in *Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation* [Freistühler, H. and Temple, B., Proc. R. Soc. A **470**, 20140055 (2014)]. *Published by AIP Publishing.* https://doi.org/10.1063/1.5007831

I. INTRODUCTION

In the special theory of relativity, the local state of a perfect barotropic fluid at a space-time point x^α in $\mathbb{R}^4$ is described by the values of its internal energy ρ and its 4-velocity U^α at that point, and the fluid as such is described by means of its pressure p as a function of its internal energy ρ,

$$p = \hat{p}(\rho). \tag{1.1}$$

The dynamics of such a fluid are governed by the Euler equations

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta}\right) = 0, \tag{1.2}$$

which, referring to the fluid's energy-momentum tensor

$$T^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + p\, g^{\alpha\beta}, \tag{1.3}$$

determine flows for positive times, $x^0 > 0$, from their initial data at $x^0 = 0$. As U^α is constrained by unitarity, $U_\alpha U^\alpha = -1$, one speaks of a four-field theory. Besides general barotropic fluids, this paper also specifically studies the subclass of "thermo-barotropic" fluids; by this, we mean fluids for which also the temperature depends only on the internal energy,

$$\theta = \hat{\theta}(\rho).$$

With internal energy given as a function $\rho = \rho(n, s)$ of particle number density n and specific entropy s, for example both massless ideal gases

$$\rho(n, s) = kn^\gamma \exp(s/c_v), \quad 1 < \gamma < 2, \tag{1.4}$$

and "double γ-law" fluids

$$\rho(n, s) = kn^\gamma s^\gamma, \quad 1 < \gamma < 2, \tag{1.5}$$

are barotropic, but of the two only the latter is thermo-barotropic. To see this, recall that the first law,

$$de = \theta ds - pd(1/n) \quad \text{with} \quad e = \rho/n,$$

implies

$$\theta = e_s(n, s).$$

Note that (1.5) with $\gamma = 4/3$ is pure radiation, and (1.4) is also used for not exactly massless gases in regimes of extremely high temperature, where the contribution of the particle mass to the internal energy is negligible (Ref. 15, p. 51). Throughout the paper, we will assume that the square of the sound speed,

$$c_s^2 = \frac{dp}{d\rho} = \hat{p}'(\rho),$$

lies strictly between 0 and 1; this makes (1.2) hyperbolic and causal.

The question that this paper is devoted to is whether small dissipative effects can be modeled appropriately through augmenting the Euler equations as

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}\right) = 0 \tag{1.6}$$

by an *energy-momentum dissipation tensor* $-\Delta T^{\alpha\beta}$ which is linear in the gradients of the state variables and causal. Or, to express the same differently, is there a causal four-field theory for the dynamics of dissipative barotropic fluids? We will answer this question in the affirmative.

In 1940, Eckart had proposed equations for dissipative relativistic fluid dynamics.[2] For viscous barotropic fluids, the Eckart theory reads

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}_{E0}\right) = 0, \tag{1.7}$$

with

$$-\Delta T^{\alpha\beta}_{E0} = \eta \Pi^{\alpha\gamma}\Pi^{\beta\delta}\left[\frac{\partial U_\gamma}{\partial x^\delta} + \frac{\partial U_\delta}{\partial x^\gamma} - \frac{2}{3} g_{\gamma\delta}\frac{\partial U^\epsilon}{\partial x^\epsilon}\right] + \zeta \Pi^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}, \tag{1.8}$$

where

$$\eta > 0 \quad \text{and} \quad \zeta \geq 0$$

denote the fluid's coefficients of shear and bulk viscosity. (Corresponding to the Minkowski metric $g^{\alpha\beta}$ with signature $- + + +$, we use the standard projection $\Pi^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta$.) The Eckart equations (1.7), together with their related formulation by Landau, are the historically first relativistic counterparts of the classical Navier-Stokes equations, representing the starting point for theories of relativistic dissipation. These equations are fundamental in the sense that they provide the simplest covariant extension of classical Navier-Stokes which, like the classical equations, is consistent with an equilibrium version of the second law of thermodynamics. However, like their classical analog, they do not have the property of finite speed of propagation. This violates causality, i.e., the principle that speeds of propagation must not be larger than that of light.

The causal viscosity tensor we propose in this paper for general barotropic fluids has the form

$$-\Delta T^{\alpha\beta}_{\square 0} = \eta \Pi^{\alpha\gamma}\Pi^{\beta\delta}\left[\frac{\partial U_\gamma}{\partial x^\delta} + \frac{\partial U_\delta}{\partial x^\gamma} - \frac{2}{3} g_{\gamma\delta}\frac{\partial U^\epsilon}{\partial x^\epsilon}\right] + \tilde{\zeta}\Pi^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}$$
$$+\sigma\left[U^\alpha U^\beta \frac{\partial U^\gamma}{\partial x^\gamma} - \left(\Pi^{\alpha\gamma}U^\beta + \Pi^{\beta\gamma}U^\alpha\right)U^\delta \frac{\partial U_\gamma}{\partial x^\delta}\right] \tag{1.9}$$

with certain coefficients $\tilde{\zeta} \neq \zeta$ and $\sigma \neq 0$. We obtain this new theory by considering the class of all dissipation tensors $-\Delta T_0^{\alpha\beta}$ that are linear in the gradients of the velocity and properly equivariant (in the sense of respecting the natural isotropy of the physics) as well as make

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T_0^{\alpha\beta}\right) = 0, \tag{1.10}$$

written in Godunov variables, second-order symmetric hyperbolic in the sense of Hughes-Kato-Marsden.[8] Our $-\Delta T^{\alpha\beta}_{\square 0}$ is the unique operator in the closure of this class which has its signal speeds bounded sharply by that of light and is *first-order equivalent*, in a sense that will be made precise below, to the Eckart viscosity tensor $-\Delta T^{\alpha\beta}_{E0}$. Concretely, $-\Delta T^{\alpha\beta}_{\square 0}$ is determined as (1.9) with

$$\sigma = ((4/3)\eta + \zeta)/(1 - c_s^2) \quad \text{and} \quad \tilde{\zeta} = \zeta + c_s^2\sigma. \tag{1.11}$$

Besides the viscosity tensor $-\Delta T^{\alpha\beta}_{E0}$, the energy-momentum dissipation tensor $-\Delta T^{\alpha\beta}_{E}$ that Eckart proposed in Ref. 2 actually contains another, thermal, part

$$-\Delta T^{\alpha\beta}_{E} = -\Delta T^{\alpha\beta}_{E0} + \Theta^{\alpha\beta}_{E}, \quad \Theta^{\alpha\beta}_{E} = \chi\left(\Pi^{\alpha\gamma}U^\beta + \Pi^{\beta\gamma}U^\alpha\right)\frac{\partial\theta}{\partial x^\gamma},$$

in which

$$\chi > 0$$

is a coefficient of heat conductivity. Obviously, the corresponding equations

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T_E^{\alpha\beta}\right) = 0 \tag{1.12}$$

are not closed for general barotropic fluids. But they do provide a closed four-field theory for viscous heat-conductive thermo-barotropic fluids. This theory, which is again not causal, can also be made causal. The viscosity-and-heat-conduction tensor we propose here for general thermo-barotropic fluids has the form

$$-\Delta T^{\alpha\beta}_{\square} = -\Delta T^{\alpha\beta}_{\square\chi} + \Theta^{\alpha\beta}_{\square} \tag{1.13}$$

with

$$\Theta^{\alpha\beta}_{\square} = \chi\left[\left(U^\alpha\frac{\partial\theta}{\partial x_\beta} + U^\beta\frac{\partial\theta}{\partial x_\alpha}\right) - g^{\alpha\beta}U^\gamma\frac{\partial\theta}{\partial x^\gamma}\right]. \tag{1.14}$$

This time, we consider the class of all dissipation tensors $-\Delta T^{\alpha\beta}$ that are linear in the gradients of the velocity *and the temperature* and properly equivariant as well as make (1.6), written in Godunov variables, second-order symmetric hyperbolic in the sense of Hughes-Kato-Marsden. Now, our $-\Delta T^{\alpha\beta}_{\square}$ is the unique operator *in* this class which has its signal speeds bounded sharply by that of light and is *first-order equivalent*, in a sense that will be made precise below, to the Eckart viscosity-and-heat-conduction tensor $-\Delta T^{\alpha\beta}_{E}$. It is determined through (1.13) and (1.14) with $-\Delta T^{\alpha\beta}_{\square\chi}$ equal to the right-hand side of (1.9) with now

$$\sigma = ((4/3)\eta + \zeta)/(1 - c_s^2) - c_s^2\chi\theta \quad \text{and} \quad \tilde{\zeta} = \zeta + c_s^2\sigma - c_s^2(1 - c_s^2)\chi\theta \tag{1.15}$$

in (1.9).

We thus construct two new theories,

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}_{\square 0}\right) = 0 \tag{1.16}$$

for general barotropic fluids with viscosity and

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}_{\square}\right) = 0 \tag{1.17}$$

for thermo-barotropic fluids with viscosity and heat conduction.

We will subsume the two theories, (1.16) with (1.11) and (1.17) with (1.15), under the form (1.17), (1.15) in the sense that specializing to $\chi > 0$ is good for thermo-barotropic fluids and specializing to $\chi = 0$ is appropriate for general barotropic fluids. This is consistent as (1.15) reduces to (1.11) for $\chi = 0$. Note that the rest-frame matrix representations of the tensors $-\Delta T_E^{\alpha\beta}, -\Delta T_\square^{\alpha\beta}$ are given by

$$-\Delta T_E|_0 = \begin{pmatrix} 0 & 0 \\ 0 & \eta\mathbf{Su} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix} + \begin{pmatrix} 0 & \chi(\nabla\theta + \theta\dot{\mathbf{u}}^\top) \\ \chi((\nabla\theta)^\top + \theta\dot{\mathbf{u}}) & 0 \end{pmatrix} \tag{1.18}$$

(cf. Ref. 15, p. 54) and

$$-\Delta T_\square|_0 = \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & -\sigma\dot{\mathbf{u}}^\top \\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{Su} + \tilde{\zeta}\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix} + \begin{pmatrix} -\chi\dot{\theta} & \chi\nabla\theta \\ \chi(\nabla\theta)^\top & -\chi\dot{\theta}\,\mathbf{I} \end{pmatrix}, \tag{1.19}$$

where $\mathbf{u}$ denotes 3-velocity. $\dot{h} = (\partial/\partial x^0)$, $\nabla = ((\partial/\partial x^1), (\partial/\partial x^2), (\partial/\partial x^3))$, and $\mathbf{Su}$ is twice the trace-free symmetric part of the velocity gradient,

$$\mathbf{Su} = D\mathbf{u} + (D\mathbf{u})^T - \frac{2}{3}\nabla\cdot\mathbf{u}\,\mathbf{I}.$$

Since Eckart's pioneering contribution,[2] the formulation of causal relativistic theories of dissipation has been the topic of many proposals and debates. From the rich history, here we point to two prominent pieces: (i) The problem of finite, notably subluminal, speed of propagation has motivated the development of rational thermodynamics from its beginning[11] and led to the creation of extended thermodynamics, in particular, relativistic extended thermodynamics.[12] (ii) The problem has just in recent years again become an area of lively efforts in the physics community; cf. Refs. 13 and 14 and the many references therein.

The authors of the present paper have [not aware of (ii) until very recently; cf. Ref. 4] made two previous contributions to this area. In Ref. 5, we have studied the causal-dissipation problem for the specific case of the pure-radiation fluid; in Ref. 4, we have proposed a new *five-field* theory of dissipation for *general* fluids. From this point of view, the questions of the present paper can be viewed as special cases of those considered in Ref. 4 and have thus partly been answered there. However the consistency of viscous barotropic fluids and viscous heat-conductive thermo-barotropic fluids at the simpler level of four-field theories makes them deserving of a self-consistent treatment in their own right. As well as simplifying the arguments in Ref. 4 and clarifying the connection with earlier proposals, the special four-field context also allows us to answer questions that remained unresolved in Ref. 4 at the five-field level of generality; this applies notably to the issue of subluminality and dissipativity of the Fourier-Laplace modes, which we can nicely resolve here completely. Finally, we record that the integrated treatment of pure radiation as a thermo-barotropic fluid reveals a natural slight correction to the theory given in Ref. 5. We refer to Refs. 4 and 5 for further general information and more references regarding the history of the topic.

The plan of this paper is as follows. In Sec. II, we characterize the general form of equations of state for barotropic and for thermo-barotropic fluids. The abovementioned assertions are proved in Secs. III and IV: In Sec. III, we construct the Hughes-Kato-Marsden class and in Sec. IV we show the connection with the theory of Eckart. Section V serves to demonstrate that our Eqs. (1.16) and (1.17), which combine first- and second-order terms, are causal and dissipative. In Sec. VI, we show compatibility of our theory for thermo-barotropic fluids with the second law of thermodynamics by proving that entropy production is non-negative to leading order in the small dissipation coefficients. Section VII is devoted to a comparison of our theory with four-field theories of dissipation proposed by Lichnerowicz[10] and Choquet-Bruhat.[1]

II. BAROTROPIC AND THERMO-BAROTROPIC FLUIDS

This section serves to characterize the possible equations of state of barotropic fluids and those of thermo-barotropic fluids. The result is as follows.

Theorem 1. *(i) A fluid is* barotropic *if and only if there exist functions* Σ *and* r *with* $\Sigma, r, r' > 0$ *such that*

$$\rho = r(n\Sigma(s)) \quad \text{and} \quad \rho + p = r'(n\Sigma(s))n\Sigma(s). \tag{2.1}$$

(ii) A fluid is thermo-barotropic if and only if there exists a function r *with* $r, r' > 0$ *such that*

$$\rho = r(ns), \quad \rho + p = r'(ns)ns, \quad \text{and} \quad \theta = r'(ns). \tag{2.2}$$

Proof. (i) A barotropic fluid $\rho = \rho(n, s)$ satisfies

$$\frac{\hat{p}(\rho)}{\rho} = \frac{n^2 \dfrac{\partial e(n,s)}{\partial n}}{ne(n,s)} = \frac{\dfrac{\partial (ne(n,s))}{\partial n}}{e(n,s)} - 1 = \frac{\dfrac{\partial \tilde{\rho}(\tilde{n},s)}{\partial \tilde{n}}}{\tilde{\rho}(\tilde{n},s)} - 1,$$

where we have used

$$\tilde{n} \equiv \log n \quad \text{and} \quad \tilde{\rho}(\tilde{n}, s) \equiv \rho(n, s).$$

This yields

$$\frac{\partial \tilde{\rho}}{\partial \tilde{n}} = \tilde{\rho} + \hat{p}(\tilde{\rho})$$

and thus

$$\tilde{F}(\rho) = \tilde{n} + \tilde{\Sigma}(s),$$

where

$$\tilde{F}(\tilde{\rho}) = \int_1 \frac{d\tilde{\rho}}{\tilde{\rho} + \hat{p}(\tilde{\rho})}$$

and $\tilde{\Sigma}$ is an arbitrary function. With $F = \exp \tilde{F}, \Sigma = \exp \tilde{\Sigma}$, we find

$$F(\rho) = n\Sigma(s),$$

which is $(2.1)_1$ with $r \equiv F^{-1}$. Now, $(2.1)_1$ implies

$$e(n, s) = \frac{r(n\Sigma(s))}{n}$$

and thus

$$p(n, s) = n^2 \frac{\partial e(n, s)}{\partial n} = -r(n\Sigma(s)) + nr'(n\Sigma(s))\Sigma(s),$$

which gives $(2.1)_2$.

Conversely, if (2.1) holds with a strictly monotone function r, then

$$p = r'(r^{-1}(\rho))r^{-1}(\rho) - \rho,$$

a function of ρ.

(ii) A barotropic fluid (1.4) has the temperature

$$\theta = \frac{\partial e(n,s)}{\partial s} = \frac{1}{n}\frac{\partial \rho(n,s)}{\partial s} = r'(r^{-1}(\rho))\Sigma'(s), \tag{2.3}$$

which is a function of ρ alone if and only if $\Sigma'(s)$ is constant. By a trivial redefinition of r, one achieves $\Sigma(s) = s$. [Here we assume that $r'(0+) = 0$ and, as an abstraction of Nernst's 3rd law, $s \to 0$ in the limit $\theta \to 0$.] Then, (2.3) implies $(2.2)_3$. □

Considerations on physical dimensions restrict r further. While a purely mathematical point of view would notably allow sums of powers, pure powers

$$r(y) = ky^\gamma$$

are the most natural candidates, as indeed realized in cases (1.4) and (1.5). In particular, the "double γ-law" in (1.5) appears almost as a mathematical necessity for a fluid to be thermo-barotropic. Note also the side-detail that in view of (2.1), $\gamma > 1$ is needed to have $p > 0$.

We now equivalently re-express (1.1) as

$$\rho = \hat{\rho}(p) \tag{2.4}$$

and note the following property for later use.

Lemma 1. For a thermo-barotropic fluid (2.4), *the Lichnerowicz index*[10]

$$\exp\left(\int_1 \frac{dp}{\hat\rho(p)+p}\right) \tag{2.5}$$

can be chosen as identical with the temperature.

Proof.

$$\exp\left(\int_1 \frac{dp}{\hat\rho(p)+p}\right)=\exp\left(\int_1 \frac{r''(ns)d(ns)}{r'(ns)}\right)=\exp\left(\int_1 (\log r')'(ns)d(ns)\right)=r'(ns).$$

□

We will also need the following.

Lemma 2. When expressed in the rest frame of their pointwise reference state, gradients of solutions to the Euler equations (1.2) *for thermo-barotropic fluids satisfy*

$$\dot{\mathbf{u}}=-\theta^{-1}\nabla\theta, \qquad \dot\theta=-c_s^2\theta\nabla\cdot\mathbf{u}. \tag{2.6}$$

Proof. Such gradients satisfy

$$\dot\rho+(\rho+p)\nabla\cdot\mathbf{u}=0,$$
$$(\rho+p)\dot{\mathbf{u}}+\nabla p=0.$$

Using (2.2), this yields

$$(ns)^{\dot{}}+ns\nabla\cdot\mathbf{u}=0,$$
$$r'(ns)\dot{\mathbf{u}}+r''(ns)\nabla(ns)=0.$$

Relations (2.6) now follow using relations (2.2) as well as their consequences

$$d\theta=r''(ns)d(ns)$$

and

$$c_s^2=\frac{dp}{d\rho}=\frac{r''(ns)ns}{r'(ns)}.$$

□

III. SYMMETRIC HYPERBOLICITY

Beginning with Friedrichs' paper[6] on first-order systems of partial differential equations, *symmetric hyperbolicity* has been identified as a basic requirement of fundamental equations in finite-speed-of-propagation continuum mechanics. Godunov showed that the symmetry of coefficient matrices in first-order systems arises naturally when one uses a particular choice of variables that is deeply motivated from physical considerations.[7] Later, Hughes, Kato, and Marsden introduced a notion of symmetric hyperbolicity for second-order equations, providing at the same time important applications in physics.[8] This background situation is what motivated us to think that a four-field theory (1.6) of dissipative relativistic fluid dynamics should be a second, or mixed, order symmetric hyperbolic system in the same Godunov variables that symmetrize the first-order Euler equations (1.2). In this section, we establish a mathematical class of four-field theories which have this property.

In analogy to our approach in Refs. 4 and 5, we start from the general equivariant form of a tensor $-\Delta T^{\alpha\beta}$ that is linear in the gradients of the state variables. This form is

$$-\Delta T^{\alpha\beta}\equiv U^\alpha U^\beta P+(\Pi^{\alpha\gamma}U^\beta+\Pi^{\beta\gamma}U^\alpha)Q_\gamma+\Pi^{\alpha\beta}R+\Pi^{\alpha\gamma}\Pi^{\beta\delta}S_{\gamma\delta} \tag{3.1}$$

with

$$P=\tau U^\gamma\frac{\partial f}{\partial x^\gamma}+\sigma\frac{\partial U^\epsilon}{\partial x^\epsilon},\quad Q_\gamma\equiv\nu\frac{\partial f}{\partial x^\gamma}+\mu U^\delta\frac{\partial U_\gamma}{\partial x^\delta},\quad R=\omega U^\gamma\frac{\partial f}{\partial x^\gamma}+\hat\zeta\frac{\partial U^\epsilon}{\partial x^\epsilon}, \tag{3.2}$$

$$S_{\gamma\delta} \equiv \eta\left(\frac{\partial U_\gamma}{\partial x^\delta} + \frac{\partial U_\delta}{\partial x^\gamma} - \frac{2}{3} g_{\gamma\delta} \frac{\partial U^\epsilon}{\partial x^\epsilon}\right), \tag{3.3}$$

where the scalar f could be taken to be any strictly monotone function of p. Given this, we ask for which choices of the parameters $\tau, \sigma, \nu, \mu, \omega, \hat{\zeta}, \eta$ the resulting tensor has the property that, when written in the Godunov variables associated with (1.2), the operator

$$-\frac{\partial}{\partial x^\beta}\left(\Delta T^{\alpha\beta}\right) \tag{3.4}$$

is symmetric hyperbolic in the sense of Ref. 8. For ease of notation, we choose f as the Lichnerowicz index (2.5),

$$f = \exp\left(\int_1 \frac{dp}{\hat{\rho}(p) + p}\right),$$

so that the Godunov variables read (see Ref. 3)

$$\Upsilon^\alpha = \frac{U^\alpha}{f}.$$

The Hughes-Kato-Marsden class is characterized by properties that the coefficient fields of the second-order derivatives must satisfy. Certain matrices composed from these fields must be symmetric and positive, respectively, negative and definite.[8] Correspondingly, we write the second-order part of (3.4) as

$$B^{\alpha\beta\gamma\delta} \frac{\partial^2 \Upsilon_\gamma}{\partial x^\beta \partial x^\delta}$$

and study properties of the $\beta\delta$-symmetrized coefficients

$$\tilde{B}^{\alpha\beta\gamma\delta} \equiv \frac{1}{2}(B^{\alpha\beta\gamma\delta} + B^{\alpha\delta\gamma\beta}).$$

The central result of this section is the following theorem.

Theorem 2. *(i) Under the assumption*

$$(\sigma + \mu) = (\omega + \nu) f, \tag{3.5}$$

the coefficients $\tilde{B}^{\alpha\beta\gamma\delta}$ are symmetric in α, γ and for V^β with $V^\beta U_\beta = 0$, $V^\beta V_\beta = 1$, the temporal, and spatial, coefficient tensors

$$B^{\alpha\beta\gamma\delta} U_\beta U_\delta, \qquad B^{\alpha\beta\gamma\delta} V_\beta V_\delta$$

have the rest-frame matrix representations

$$\begin{pmatrix} \tau f^2 & 0 \\ 0 & \mu f \delta^{ij} \end{pmatrix}, \qquad \begin{pmatrix} \nu f^2 & 0 \\ 0 & \eta f \delta^{ij} + (\frac{1}{3}\eta + \hat{\zeta}) f\, V^i V^j \end{pmatrix}. \tag{3.6}$$

(ii) If moreover

$$\mu, \tau < 0 \quad \textit{and} \quad \eta, \nu > 0, \quad \hat{\zeta} \geq -\frac{4}{3}\eta, \tag{3.7}$$

then system (1.6) with (3.1)–(3.3) is symmetric hyperbolic in the sense of Hughes, Kato, Marsden,[8] pointwise with respect to the fluid's rest frame.
(iii) Choosing, in particular,

$$\nu = -\tau = -\omega = \chi$$

and

$$\mu = -\sigma, \quad \hat{\zeta} = \tilde{\zeta},$$

the dissipation tensor assumes the form

$$
\begin{aligned}
-\Delta T^{\alpha\beta} &= U^{\alpha\beta} + F^{\alpha\beta} \\
&= \Big\{ \eta \Pi^{\alpha\gamma}\Pi^{\beta\delta}\left[\frac{\partial U_\gamma}{\partial x^\delta} + \frac{\partial U_\delta}{\partial x^\gamma} - \frac{2}{3} g_{\gamma\delta}\frac{\partial U^\epsilon}{\partial x^\epsilon}\right] + \hat\zeta \Pi^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma} \\
&\qquad + \sigma\left[U^\alpha U^\beta \frac{\partial U^\gamma}{\partial x^\gamma} - \left(\Pi^{\alpha\gamma}U^\beta + \Pi^{\beta\gamma}U^\alpha\right)U^\delta \frac{\partial U_\gamma}{\partial x^\delta}\right]\Big\} \\
&\quad + \chi\left[\left(U^\alpha \frac{\partial f}{\partial x_\beta} + U^\beta \frac{\partial f}{\partial x_\alpha}\right) - g^{\alpha\beta}U^\gamma \frac{\partial f}{\partial x^\gamma}\right]
\end{aligned}
\tag{3.8}
$$

with $\hat\zeta = \tilde\zeta$.

Proof. (i) This assertion and its proof are analogous to those of Lemmas 1 and 2 in Ref. 4, the only difference being that f assumes the role that the temperature θ played in Ref. 4. (ii) The assertion follows as these conditions make the first matrix in (3.6) negative definite and the second one positive definite. (iii) Follows by direct insertion. □

Note that the property in assertion (ii) does not imply that the expressions of the same system in all other Lorentz frames were also HKM symmetric-hyperbolic (cf. second paragraph in Sec. IV of Ref. 4). But, being an open property, it does imply HKM symmetric hyperbolicity in nearby frames, and this is sufficient to obtain wellposedness through localization.

Note also that the last summand, $F^{\alpha\beta}$, can be adapted to the case of thermo-barotropic fluids on the one hand or to that of other barotropic fluids on the other.

For thermo-barotropic fluids, we see from Lemma 1 that

$$F^{\alpha\beta} = \Theta^{\alpha\beta}_{\square}$$

from (1.14).

For other barotropic fluids, we set $\chi = 0$. Note that in the latter case, we can alternatively still interpret $F^{\alpha\beta}$ with $\chi > 0$ as turning (3.8) into an artificial regularization. While in general $f \neq \theta$ for non-thermo-barotropic fluids, this might be called "artificial heat conduction" in analogy to "artificial viscosity." Also, the limit $\chi \searrow 0$ confirms that $\Delta T^{\alpha\beta}_{\square 0}$ lies on the boundary of the Hughes-Kato-Marsden class.

IV. CONNECTION WITH THE ECKART THEORY

The purpose of this section is to prove the following.

Theorem 3. *The Navier-Stokes system* (1.6) *with* $\Delta T^{\alpha\beta} = \Delta T^{\alpha\beta}_{\square}$ *and* (1.15) *is first-order equivalent to the Eckart system* (1.12).

Before showing this, we first have to define what we mean by first-order equivalence. The following discussion parallels, but is simpler than the five-field argument in Ref. 4. We study transformations between different four-field theories (1.6) of dissipative barotropic fluid dynamics. Consider the space $\mathcal{F}_4$ of all linear gradient forms

$$\Delta T^{\alpha\beta} = T_U^{\alpha\beta\gamma\delta}\frac{\partial U_\gamma}{\partial x^\delta} + T_n^{\alpha\delta}\frac{\partial n}{\partial x^\delta} \tag{4.1}$$

and express the smallness of dissipation by giving them a common small factor $\epsilon > 0$; i.e., view $\Delta T^{\alpha\beta} \in \mathcal{F}_4$ as representing the four-field theory

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \epsilon \Delta T^{\alpha\beta}\right) = 0. \tag{4.2}$$

We characterize a group of transformations that establishes formal equivalences between different elements of $\mathcal{F}_4$ up to $O(\epsilon^2)$.

The idea of the transformations is that instead of working with the quantities U^α, n which appear in the evolution equations (4.2), one might alternatively base one's considerations on certain local spatiotemporally anisotropic averages $\tilde U^\alpha$, $\tilde n$. This is expressed through an ansatz

$$U^\alpha = \tilde U^\alpha + \epsilon\Delta\tilde U^\alpha + O(\epsilon^2),$$
$$n = \tilde n + \epsilon\Delta\tilde n + O(\epsilon^2), \tag{4.3}$$

in which

$$\Delta\tilde U^\alpha = \tilde U_U^{\alpha\gamma\delta}\frac{\partial\tilde U_\gamma}{\partial x^\delta} + \tilde U_n^{\alpha\delta}\frac{\partial\tilde n}{\partial x^\delta},$$
$$\Delta\tilde n = \tilde n_U^{\gamma\delta}\frac{\partial\tilde U_\gamma}{\partial x^\delta} + \tilde n_n^\delta\frac{\partial\tilde n}{\partial x^\delta} \tag{4.4}$$

are further linear gradient forms whose coefficients $\tilde U_U^{\alpha\gamma\delta},\ldots,\tilde n_n^\delta$ depend on $\tilde U^\alpha$, $\tilde n$.

For any fixed element (4.1) of $\mathcal{F}_4$, substituting (4.3) and (4.4) in (4.2) and writing

$$\tilde T^{\alpha\beta} = (\tilde\rho+\tilde p)\tilde U^\alpha\tilde U^\beta + \tilde p g^{\alpha\beta} \quad \text{with} \quad \tilde\rho = \rho(\tilde n),\ \tilde p = p(\tilde n)$$

and analogously $\Delta\tilde T^{\alpha\beta}$ will result in a modification $\delta\Delta\tilde T^{\alpha\beta}$ in the equations of motion

$$\frac{\partial}{\partial x^\beta}\left(\tilde T^{\alpha\beta} + \epsilon\left(\Delta\tilde T^{\alpha\beta} + \delta\Delta\tilde T^{\alpha\beta}\right)\right) = 0 \tag{4.5}$$

for $\tilde U^\alpha$, $\tilde n$. The new ingredients will be of the form

$$\delta\Delta\tilde T^{\alpha\beta} = \tilde\Delta\tilde T^{\alpha\beta} + O(\epsilon)$$

with a unique element $\tilde\Delta\tilde T^{\alpha\beta}$ of $\mathcal{F}_4$.

Definition 1. We call the assignment

$$\mathcal{F}_4 \to \mathcal{F}_4, \quad \Delta T^{\alpha\beta} \mapsto \Delta\tilde T^{\alpha\beta} + \tilde\Delta\tilde T^{\alpha\beta}$$

the first-order equivalence *generated by the gradient transformation* (4.3) *and* (4.4).

We consider first-order equivalences of three kinds, the first of which corresponds to changes of what is often referred to as "flow frames."

1. *Velocity shifts.* Only the velocity transforms, while $\Delta\tilde n = 0$. One finds that

$$\delta\Delta\tilde T^{\alpha\beta} = (\tilde\rho+\tilde p)(\tilde U^\alpha(\Delta\tilde U^\beta) + (\Delta\tilde U^\alpha)\tilde U^\beta)) + O(\epsilon) \tag{4.6}$$

with

$$\tilde U_\alpha(\Delta\tilde U^\alpha) = 0. \tag{4.7}$$

2. *Thermodynamic shifts*. Only n transforms, while $\Delta U^\alpha = 0$. One finds that

$$\delta\Delta\tilde T^{\alpha\beta} = (\Delta\tilde\rho)\tilde U^\alpha\tilde U^\beta + (\Delta\tilde p)\tilde\Pi^{\alpha\beta} + O(\epsilon) \tag{4.8}$$

with

$$\Delta\tilde p = c_s^2(\tilde p)\Delta\tilde\rho. \tag{4.9}$$

The constraints (4.7) and (4.9) reflect the fact that the transformations respect, to leading order, the unitarity of the 4-velocity and the equations of state.

3. *Eulerian gradient re-expressions*. These are modifications of $\Delta T^{\alpha\beta}$ which do not change the fields, but simply replace gradients by other gradients which differ only by $O(\varepsilon)$. We call any covariant linear gradient expression

$$W^{\alpha\beta} = W^{\alpha\beta\gamma\delta}\frac{\partial\Upsilon^\gamma}{\partial x^\delta}$$

for which

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta} = 0 \quad \Longrightarrow \quad W^{\alpha\beta} = 0$$

holds as an implication for arbitrary Υ^α, an *Eulerian constraint*. For any Eulerian constraint $W^{\alpha\beta}$, arbitrary solutions of (4.2) satisfy (4.5) with $\tilde T^{\alpha\beta} = T^{\alpha\beta}$, $\Delta\tilde T^{\alpha\beta} = \Delta T^{\alpha\beta}$, and

$$\delta\Delta\tilde T^{\alpha\beta} = W^{\alpha\beta} + O(\varepsilon).$$

It is obvious that *velocity shifts, thermodynamic shifts, and Eulerian gradient re-expressions form a group of first-order equivalences on* $\mathcal{F}_4$.

While we have intentionally carried out the above considerations in the covariant form, the practical use of such transformations is more nicely handled in a rest-frame notation. For this purpose, we represent dissipation tensors $-\Delta T^{\alpha\beta}$ in the form

$$\begin{pmatrix} -\Delta T^{00}|_0, & -\Delta T^{0j}|_0 \\ -\Delta T^{i0}|_0, & -\Delta T^{ij}|_0 \end{pmatrix} \in \begin{pmatrix} \Lambda & \Lambda^{1\times 3} \\ \Lambda^{3\times 1} & \Lambda^{3\times 3} \end{pmatrix},$$

where Λ denotes any real-valued linear form in the gradients of the state variables $\mathbf{u}$ and n and $\Lambda^{l\times m}$ is an $l\times m$-matrix of such objects.

In this notation, a velocity shift is any assignment

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} * & * + \Delta\mathbf{u}^\top \\ * + \Delta\mathbf{u} & * \end{pmatrix}$$

with some $\Delta\mathbf{u} \in \Lambda^{3\times 1}$, and a thermodynamic shift is any assignment

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} * + \Delta\rho & * \\ * & * + \Delta p\,\mathbf{I} \end{pmatrix}$$

with some pair $(\Delta\rho, \Delta p) \in \Lambda^2$ that satisfies (4.9), and an Eulerian gradient re-expression is any assignment

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} \tilde{*} & \tilde{*} \\ \tilde{*} & \tilde{*} \end{pmatrix}$$

for which each respective entrywise transition

$$\tilde{}: * \mapsto \tilde{*}$$

replaces, if anything, one gradient form Λ by another gradient form $\tilde{\Lambda}$ with the property that the values of Λ and $\tilde{\Lambda}$ agree on arbitrary Eulerian gradients, i.e., gradients that can be realized as those of solutions to the Euler equations (1.2).

Proof of Theorem 3. Starting from the Eckart tensor (1.18), we first perform a velocity shift

$$\begin{pmatrix} 0 & \chi(\nabla\theta + \theta\dot{\mathbf{u}}^\top) \\ \chi((\nabla\theta)^\top + \theta\dot{\mathbf{u}}) & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix} \to \begin{pmatrix} 0 & \chi\nabla\theta - \sigma\dot{\mathbf{u}}^\top \\ \chi(\nabla\theta)^\top - \sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix}$$

corresponding to a shift vector

$$\Delta\mathbf{u} = -(\chi\theta + \sigma)\dot{\mathbf{u}},$$

tentative inasmuch as σ is to be determined later. Then we carry out a first thermodynamic shift

$$\begin{pmatrix} 0 & \chi\nabla\theta - \sigma\dot{\mathbf{u}}^\top \\ \chi(\nabla\theta)^\top - \sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix} \to \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & \chi\nabla\theta - \sigma\dot{\mathbf{u}}^\top \\ \chi(\nabla\theta)^\top - \sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + (\zeta + c_s^2\sigma)\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix}$$

with

$$\Delta\rho = \sigma\nabla\cdot\mathbf{u}, \quad \Delta p = c_s^2\sigma\nabla\cdot\mathbf{u}$$

and a second thermodynamic shift

$$\begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & \chi\nabla\theta - \sigma\dot{\mathbf{u}}^\top \\ \chi(\nabla\theta)^\top - \sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + (\zeta + c_s^2\sigma)\nabla\cdot\mathbf{u}\,\mathbf{I} \end{pmatrix} \to \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} - \chi\dot{\theta} & \chi\nabla\theta - \sigma\dot{\mathbf{u}}^\top \\ \chi(\nabla\theta)^\top - \sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + ((\zeta + c_s^2\sigma)\nabla\cdot\mathbf{u} - c_s^2\chi\dot{\theta})\,\mathbf{I} \end{pmatrix}$$

with

$$\Delta\rho = -\chi\dot{\theta}, \quad \Delta p = -c_s^2\chi\dot{\theta}.$$

Lemma 2 shows that

$$-c_s^2\chi\dot{\theta} = -\chi\dot{\theta} - c_s^2(1-c_s^2)\chi\theta\nabla\cdot\mathbf{u}$$

is an Eulerian constraint. Using it and $(1.15)_2$ takes us to

$$\begin{pmatrix} \sigma\nabla\cdot\mathbf{u} - \chi\dot{\theta} & \chi\nabla\theta - \sigma\dot{\mathbf{u}}^\top \\ \chi(\nabla\theta)^\top - \sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + (\tilde{\zeta}\nabla\cdot\mathbf{u} - \chi\dot{\theta})\,\mathbf{I} \end{pmatrix},$$

which is (1.19). □

So far, we have not used $(1.15)_1$. We will explain and use it in Sec. V.

V. CAUSALITY AND DISSIPATIVITY

Theorem 4. *Whenever*

$$c_s^2(1-c_s^2)\chi\theta \le \frac{1+c_s^2}{3}\eta + \zeta, \tag{5.1}$$

our theory (1.17), (1.15) *is causal in the sense that its rest-frame Fourier-Laplace modes are subluminal or luminal and dissipative in the sense that they are decaying or neutral.*

We show the following more general fact.

Lemma 3. For any values

$$\eta > 0,\ \hat{\zeta} \ge -\frac{1}{3}\eta,\ \sigma = \frac{4}{3}\eta + \hat{\zeta},\ \chi \ge 0, \tag{5.2}$$

the Navier-Stokes system (1.6) *with* $\Delta T^{\alpha\beta}$ *as in* (3.8) *is causal in the sense that its rest-frame Fourier-Laplace modes are subluminal or luminal and dissipative in the sense that they are decaying or neutral.*

Proof. As in Ref. 5, we consider the linearized equations of motion in the rest frame, here

$$\frac{\rho+p}{f}\left\{c_s^{-2}\frac{\partial f}{\partial t} + f\nabla\cdot\mathbf{u}\right\} + \left\{\chi\left(\frac{\partial^2 f}{\partial t^2} - \Delta f\right)\right\} = 0,$$

$$\frac{\rho+p}{f}\left\{f\frac{\partial\mathbf{u}}{\partial t} + \nabla f\right\} + \left\{\sigma\frac{\partial^2\mathbf{u}}{\partial t^2} - (\eta\,\nabla\cdot\mathcal{S}\mathbf{u} + \hat{\zeta}\,\nabla(\nabla\cdot\mathbf{u}))\right\} = 0.$$

The existence of Fourier-Laplace modes

$$\begin{pmatrix}\hat{f}\\ \hat{\mathbf{u}}\end{pmatrix} e^{\lambda t + i\boldsymbol{\xi}\cdot\boldsymbol{x}} = \begin{pmatrix}\hat{f}\\ \hat{\mathbf{u}}\end{pmatrix} e^{Re\{\lambda\}t} e^{i(Im\{\lambda\}t + \boldsymbol{\xi}\cdot\boldsymbol{x})}, \quad (\lambda, \boldsymbol{\xi}) \in \mathbb{C}\times\mathbb{R}^3 \tag{5.3}$$

is controlled by the associated dispersion relation

$$\det\tilde{M}(\lambda,\boldsymbol{\xi}) = 0 \tag{5.4}$$

with

$$\tilde{M}(\lambda,\boldsymbol{\xi}) = \frac{\rho+p}{f}\begin{pmatrix} c_s^{-2}\lambda & if\boldsymbol{\xi}^\top \\ i\boldsymbol{\xi} & f\lambda I\end{pmatrix} + \begin{pmatrix}\chi(\lambda^2+|\boldsymbol{\xi}|^2) & 0 \\ 0 & N(\lambda,\boldsymbol{\xi})\end{pmatrix},$$

where

$$N(\lambda,\boldsymbol{\xi}) = \left(\sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2\right)I + \left(\hat{\zeta} + \frac{1}{3}\eta\right)\boldsymbol{\xi}\boldsymbol{\xi}^\top.$$

We call a mode (5.3) corresponding to any solution $(\lambda,\boldsymbol{\xi}) \in \mathbb{C}\times\mathbb{R}^3$ of (5.4) (a) decaying, neutral, or growing if $\mathrm{Re}\{\lambda\} < 0, = 0,$ or >0, respectively, and (b) subluminal, luminal, or superluminal if $|\mathrm{Im}\{\lambda\}|/|\boldsymbol{\xi}| < 1, = 1,$ or >1, respectively.

By rescaling λ and $\boldsymbol{\xi}$ in obvious ways, we equivalently consider

$$0 = \Pi(\lambda,\boldsymbol{\xi}) \equiv \det M(\lambda,\boldsymbol{\xi})$$

instead of (5.4), with

$$M(\lambda,\boldsymbol{\xi}) = \begin{pmatrix} c_s^{-2}\lambda & i\boldsymbol{\xi}^\top \\ i\boldsymbol{\xi} & \lambda I \end{pmatrix} + \begin{pmatrix} \chi f(\lambda^2+|\boldsymbol{\xi}|^2) & 0 \\ 0 & N(\lambda,\boldsymbol{\xi}) \end{pmatrix}.$$

For any given wave vector $\boldsymbol{\xi} \neq 0$, $M(\lambda,\boldsymbol{\xi})$ has the two invariant subspaces

$$\mathcal{V}^L_{\boldsymbol{\xi}} = \mathbb{C}\times\mathbb{C}\boldsymbol{\xi} \quad \text{and} \quad \mathcal{V}^T_{\boldsymbol{\xi}} = \{0\}\times\{\boldsymbol{\xi}\}^\perp,$$

and the dispersion relation correspondingly factors as

$$0 = \Pi^L(\lambda,\xi)\Pi^T(\lambda,\xi), \quad \text{with} \quad \xi = |\boldsymbol{\xi}|.$$

Consider now first briefly the infinite wave number/infinite frequency limit, where

$$0 = \Pi^L_\infty(\lambda,\xi) = \chi f(\lambda^2+\xi^2)\left(\sigma\lambda^2 + \left(\frac{4}{3}\eta + \hat{\zeta}\right)\xi^2\right)$$

and

$$0 = \Pi^T_\infty(\lambda,\xi) = \sigma\lambda^2 + \eta\xi^2.$$

In this limit, the squared wave speeds are

$$1, \left(\frac{4}{3}\eta + \hat{\zeta}\right)/\sigma \text{ (longitudinal)} \ \text{ and } \ \eta/\sigma \text{ transversal,}$$

and here is where we motivate our choices

$$\sigma = \frac{4}{3}\eta + \hat{\zeta} \quad \text{and} \quad \hat{\zeta} \geq -\frac{1}{3}\eta. \tag{5.5}$$

Choice $(5.5)_1$ makes the longitudinal wave speeds bounded sharply by the speed of light, and then condition $(5.5)_2$ is the requirement for the transverse wave speeds to be subluminal.

We turn to finite wave numbers/frequencies. As

$$M(\lambda,\boldsymbol{\xi})\begin{pmatrix} 0 \\ \boldsymbol{\xi}^\perp \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda + \left(\sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2\right))\boldsymbol{\xi}^\perp \end{pmatrix} \quad \text{for any} \quad \boldsymbol{\xi}^\perp \perp \boldsymbol{\xi},$$

the transverse part is

$$0 = \Pi^T(\lambda,\xi) = \lambda + \sigma\lambda^2 + \eta\xi^2,$$

which means

$$\lambda = \frac{1}{2\sigma}(-1 \pm \sqrt{1-4\sigma\eta\xi^2}).$$

Since this implies

$$\mathrm{Re}\{\lambda\} < 0 \quad \text{and} \quad |\mathrm{Im}\{\lambda\}|/\xi < \eta/\sigma \leq 1,$$

the transverse modes are indeed all decaying and subluminal.

As for the longitudinal modes, $M(\lambda,\boldsymbol{\xi})$ is represented on $\mathcal{V}^L_{\boldsymbol{\xi}}$ by

$$M^L(\lambda,\xi) = \begin{pmatrix} c_s^{-2}\lambda & i\xi \\ i\xi & \lambda \end{pmatrix} + \begin{pmatrix} \chi f(\lambda^2+\xi^2) & 0 \\ 0 & \sigma\left(\lambda^2+\xi^2\right) \end{pmatrix}.$$

By simple scaling and introducing $\hat{\chi} \equiv c_s^2\chi f/\sigma \geq 0$, we may write

$$\Pi^L(\lambda,\xi) = \{\lambda + \hat{\chi}(\lambda^2+\xi^2)\}\{\lambda + \left(\lambda^2+\xi^2\right)\} + c_s^2\xi^2.$$

We first show that any neutral mode would be superluminal. To see this, consider a pair $(\beta,\xi) \in \mathbb{R}\times(0,\infty)$. As

$$\begin{aligned}\Pi^L(i\beta,\xi) &= \{i\beta + \hat{\chi}z\}\{i\beta + z\} + c_s^2(z+\beta^2) \\ &= (\hat{\chi}z^2 + c_s^2(z+\beta^2) - \beta^2) + i\beta z(\hat{\chi}+1)\end{aligned}$$

with $z = \xi^2 - \beta^2$, the assumption $\Pi^L(i\beta,\xi) = 0$ readily implies $\beta = 0$ and $z < 0$, i. e., superluminality.

Next we show that no mode can be luminal. Assume to the contrary that there existed some $(\alpha, \xi) \in \mathbb{R} \times (0, \infty)$ with

$$0 = \Pi^L(\alpha \pm i\xi, \xi) = \{\alpha \pm i\xi + \hat{\chi}(\alpha^2 \pm 2i\alpha\xi)\}\{\alpha \pm i\xi + (\alpha^2 \pm 2i\alpha\xi)\} + c_s^2\xi^2. \tag{5.6}$$

As

$$\mathrm{Re}\, \Pi^L(\alpha \pm i\xi, \xi) = (\alpha/2)^2\pi_1(\alpha) - \pi_2(\alpha)\xi^2$$

and

$$\mathrm{Im}\, \Pi^L(\alpha \pm i\xi, \xi) = \pm\alpha\xi\pi_3(\alpha)$$

with

$$\begin{aligned}\pi_1(\alpha) &= 4\hat{\chi}\alpha^2 + 4(1 + \hat{\chi})\alpha + 4,\\ \pi_2(\alpha) &= 4\hat{\chi}\alpha^2 + 2(1 + \hat{\chi})\alpha + (1 - c_s^2),\\ \pi_3(\alpha) &= 4\hat{\chi}\alpha^2 + 3(1 + \hat{\chi})\alpha + 2,\end{aligned}$$

assumption (5.6) gives

$$\pi_3(\alpha) = 0 \quad \text{and} \quad \pi_1(\alpha)\pi_2(\alpha) > 0. \tag{5.7}$$

But $\pi_3(\alpha) = 0$ implies

$$\pi_1(\alpha) = (1 + \hat{\chi})\alpha + 2 \quad \text{and} \quad \pi_2(\alpha) = -(1 + \hat{\chi})\alpha - (1 + c_s^2),$$

and then $\pi_1(\alpha)\pi_2(\alpha) > 0$ entails

$$\alpha \in I := \left(-\frac{2}{1 + \hat{\chi}}, -\frac{1 + c_s^2}{1 + \hat{\chi}}\right).$$

However, as

$$(1 + \hat{\chi})^2\pi_3\left(\frac{-2}{1 + \hat{\chi}}\right) = 16\hat{\chi} - 4(1 + \hat{\chi})^2 = -4(1 - \hat{\chi})^2$$

and

$$(1 + \hat{\chi})^2\pi_3\left(\frac{-1}{1 + \hat{\chi}}\right) = 4\hat{\chi} - (1 + \hat{\chi})^2 = -(1 - \hat{\chi})^2,$$

we find

$$\pi_3(\alpha) < 0, \qquad \text{for all } \alpha \in I.$$

This shows that (5.7) cannot hold.

By continuity, the assertion of Lemma 3 will be proved once we know that *for some* $\hat{\chi} > 0$ *there exist* $\xi > 0$ *such that all associated modes are subluminal and decaying*. We show this for $\hat{\chi} = 1$. In this case, we have

$$\Pi^L(\lambda, \xi) = \{\lambda + (\lambda^2 + \xi^2)\}^2 + c_s^2\xi^2$$

so that $\Pi^L(\lambda, \xi) = 0$ implies

$$\lambda = \lambda_\delta^\pm(\xi) = \frac{1}{2}\left(-1 \pm \sqrt{1 - 4(\xi^2 + i\delta c_s\xi)}\,\right),\ \delta \in \{-1, 1\}.$$

As

$$\lambda_\delta^\pm(\xi) = -\frac{1}{2} \pm \left(\frac{1}{2} - i\delta c_s\xi + (c_s^2 - 1)\xi^2\right) + O(\xi^3),$$

we have indeed $\mathrm{Re}\{\lambda\} < 0$ and $|\mathrm{Im}\{\lambda\}|/\xi < 1$ for small $\xi > 0$. □

Proof of Theorem 4. While we chose $(1.15)_2$ to enable Theorem 3, our motivation for $(1.15)_1$ was that together with $(1.15)_2$, it implies

$$\sigma = \frac{4}{3}\eta + \tilde{\zeta}. \tag{5.8}$$

In view of (5.8), the assertion follows from Lemma 3 once we show that inequality (5.1) implies

$$\tilde{\zeta} \geq -\frac{1}{3}\eta. \tag{5.9}$$

However, as (1.15) yields

$$\frac{1}{3}\eta + \tilde{\zeta} = \frac{1+c_s^2}{3}\eta + \zeta - c_s^2(1-c_s^2)\chi\theta,$$

condition (5.1) is equivalent to (5.9). □

Remark 1. The question of subluminality and decay of Fourier-Laplace modes with respect to Lorentz frames that differ from the fluid's rest frame will be covered in a different paper.

VI. ENTROPY PRODUCTION IN THERMO-BAROTROPIC FLUIDS

For general barotropic fluids, the entropy production is, just as for general fluids at all, not determined by the conservation laws (1.6) of energy-momentum alone, but depends also on a particle number balance law, i.e., $\partial(nU^\beta)/\partial x^\beta = 0$ or a variant thereof. As we have chosen (1.6) as the natural frame of this paper, we do not address this interesting topic here, but refer to Refs. 4 and 3. However, remarkably, in the case of thermo-barotropic fluids, it *is* possible to determine the entropy production exclusively from (1.6). This is what we study in the rest of this section.

Theorem 5. *For thermo-barotropic fluids* (2.2), *the conservation laws* (1.6) *of energy-momentum imply that*

$$\frac{\partial}{\partial x^\beta}\left(nsU^\beta\right) = \frac{U_\alpha}{\theta}\frac{\partial}{\partial x^\beta}\Delta T^{\alpha\beta}. \tag{6.1}$$

Proof. Using (2.2), we compute

$$\begin{aligned}
\frac{U_\alpha}{\theta}\frac{\partial}{\partial x^\beta}\Delta T^{\alpha\beta} &= \frac{U_\alpha}{\theta}\frac{\partial}{\partial x^\beta}\left(r'(ns)nsU^\alpha U^\beta + (r'(ns)ns - r(ns))g^{\alpha\beta}\right)\\
&= -\frac{1}{\theta}\frac{\partial}{\partial x^\beta}\left(r'(ns)nsU^\beta\right) + \frac{U^\beta}{\theta}\frac{\partial}{\partial x^\beta}(r'(ns)ns - r(ns))\\
&= -\frac{r'(ns)}{\theta}\left(ns\frac{\partial U^\beta}{\partial x^\beta} + U^\beta\frac{\partial(ns)}{\partial x^\beta}\right)\\
&= -\frac{\partial}{\partial x^\beta}\left(nsU^\beta\right),
\end{aligned}$$

and (1.6) implies (6.1). □

As in Refs. 5 and 4, we define the entropy current for (1.17) as

$$S^\beta \equiv nsU^\beta - \frac{U_\alpha}{\theta}\Delta T^{\alpha\beta}_\square$$

and consider the net entropy production

$$\mathcal{Q} \equiv \frac{\partial S^\beta}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\left(\frac{U_\alpha}{\theta}\right)\left(-\Delta T^{\alpha\beta}_\square\right).$$

Theorem 6. *The net entropy production on Eulerian gradients is*

$$\mathcal{Q} = \frac{\eta}{2\theta}||\mathbf{S}\mathbf{u}||^2 + \frac{\zeta}{\theta}(\nabla\cdot\mathbf{u})^2.$$

Proof. From (1.19), we compute

$$\begin{aligned}\mathcal{Q} &= -\frac{1}{\theta^2}\frac{\partial\theta}{\partial t}\Delta T_\square^{00}|_0 - \frac{1}{\theta^2}\left(\frac{\partial\theta}{\partial x^i} + \theta\frac{\partial u_i}{\partial t}\right)\Delta T_\square^{i0}|_0 - \frac{1}{\theta}\frac{\partial u_i}{\partial x^j}\Delta T_\square^{ij}|_0 \\ &= \frac{1}{\theta^2}\frac{\partial\theta}{\partial t}\left(-\chi\frac{\partial\theta}{\partial t} + \sigma\frac{\partial u^l}{\partial x^l}\right) + \frac{1}{\theta^2}\left(\frac{\partial\theta}{\partial x^i} + \theta\frac{\partial u_i}{\partial t}\right)\left(\chi\frac{\partial n}{\partial x_i} - \sigma\frac{\partial u^i}{\partial t}\right) \\ &\quad + \frac{1}{\theta}\frac{\partial u_i}{\partial x^j}\eta\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} - \frac{2}{3}\frac{\partial u^l}{\partial x^l}\delta^{ij}\right) + \frac{1}{\theta}\frac{\partial u_i}{\partial x^j}\left(\tilde\zeta\frac{\partial u^l}{\partial x^l} - \chi\frac{\partial\theta}{\partial t}\right)\delta^{ij} \\ &\equiv \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4.\end{aligned}$$

While obviously

$$Q_2 = 0,$$

$$Q_3 = \frac{\eta}{2\theta}||\mathbf{S}\mathbf{u}||^2,$$

we use (1.15) and Lemma 2 to conclude that

$$\begin{aligned}Q_1 + Q_4 &= \frac{1}{\theta^2}(-\chi\dot\theta^2 + \sigma\dot\theta\nabla\cdot\mathbf{u}) + \frac{1}{\theta}(\tilde\zeta(\nabla\cdot\mathbf{u})^2 - \chi\dot\theta\nabla\cdot\mathbf{u}) \\ &= (\nabla\cdot\mathbf{u})^2\left\{\frac{\zeta}{\theta} + \frac{-\chi c_s^4\theta^2 - \sigma c_s^2\theta}{\theta^2} + \frac{(\tilde\zeta-\zeta) + \chi c_s^2\theta}{\theta}\right\} \\ &= (\nabla\cdot\mathbf{u})^2\left\{\frac{\zeta}{\theta} + \frac{-\chi c_s^4\theta - \sigma c_s^2}{\theta} + \frac{c_s^2\sigma - c_s^2(1-c_s^2)\chi\theta + \chi c_s^2\theta}{\theta}\right\} \\ &= \frac{\zeta}{\theta}(\nabla\cdot\mathbf{u})^2.\end{aligned}$$

□

To appropriately express the meaning of Theorem 6 for general solutions of (1.17), we also consider, as in Sec. IV and Ref. 4,

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \epsilon\Delta T_\square^{\alpha\beta}\right) = 0 \tag{6.2}$$

and state:

Corollary 1. For general solutions to (6.2), *the entropy production is*

$$\mathcal{Q} = \epsilon\left(\frac{\eta}{2\theta}||\mathcal{S}\mathbf{u}||^2 + \frac{\zeta}{\theta}(\nabla\cdot\mathbf{u})^2\right) + O(\epsilon^2).$$

As in the non-barotropic case,[4] $\mathcal{Q}$ is not necessarily non-negative on arbitrary solutions of (6.2). Note, however, that we are using the equilibrium definition of entropy, and there should exist choices of entropy and entropy current which are more appropriate in this non-equilibrium situation. As it stands, Corollary 1 does say that the entropy production on arbitrary solutions is non-negative to leading order in the small dissipation coefficients.

For further interpretation and the connection with the vanishing viscosity limit, cf. a remark at the end of Sec. VI in Ref. 4.

We also wish to draw the reader's attention to the following.

Remark 2. In our previous paper[5] *on pure radiation, we had, instead of* $-\Delta T_\square^{\alpha\beta}$, *considered the tensor* (3.8) with

$$\sigma = (4/3)\eta + \zeta \quad \textit{and} \quad \hat\zeta = \zeta.$$

Our new different choice of (1.13) *with* (1.15) *corrects the results of* Ref. 5. *It not only reconciles our ansatz with the Eckart theory but also leads to a clearer picture of entropy production. In particular, Theorem* 6 *and Corollary* 1 *show that heat conduction does, at least to leading order, not produce entropy for thermo-barotropic fluids, including pure radiation. The only restriction on the dissipation coefficients besides* $\eta > 0$, $\zeta \geq 0$, $\chi \geq 0$ *is the natural causality requirement* (5.1).

VII. COMPARISON WITH THE PROPOSALS BY LICHNEROWICZ AND CHOQUET-BRUHAT

We conclude by comparing our energy-momentum dissipation tensor with those proposed by Lichnerowicz and Choquet-Bruhat.

Choquet-Bruhat has proposed dissipation tensors of the form

$$\Gamma^{\alpha\beta} \equiv \upsilon\left(\frac{\partial C^\alpha}{\partial x_\beta} + \frac{\partial C^\beta}{\partial x_\alpha}\right) + \varsigma g^{\alpha\beta}\frac{\partial C^\gamma}{\partial x_\gamma} \tag{7.1}$$

with

$$C^\gamma = f\, U^\gamma$$

being the "dynamic velocity";[1] Lichnerowicz[10] suggested

$$\Lambda^{\alpha\beta} = \Pi^{\alpha\gamma}\Gamma_{\gamma\delta}\Pi^{\delta\beta}.$$

We use the letters $\mathcal{C}$ and $\mathcal{L}$ to denote the class of all such tensors $\Gamma^{\alpha\beta}$ and $\Lambda^{\alpha\beta}$, respectively, and write $\mathcal{D}$ for the class of all tensors $-\Delta T^{\alpha\beta}$ of the form (3.8) with

$$\sigma = \frac{4}{3}\eta + \hat{\zeta},$$

which we have considered in Secs. III and V. We show the following.

Theorem 7. *(i) The intersection $\mathcal{C} \cap \mathcal{D}$ consists precisely of those elements of $\mathcal{D}$ for which*

$$\chi = \eta/f \quad \textit{and} \quad \hat{\zeta} = -\frac{1}{3}\eta. \tag{7.2}$$

(ii) The intersection $\mathcal{L} \cap \mathcal{D}$ is empty.

Proof. (i) On the one hand, any element of $\mathcal{C}$ reads

$$\Gamma^{\alpha\beta} = f\left[\upsilon\left(\frac{\partial U^\alpha}{\partial x_\beta} + \frac{\partial U^\beta}{\partial x_\alpha}\right) + \varsigma g^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}\right] + \left[\upsilon\left(U^\beta\frac{\partial f}{\partial x^\alpha} + U^\alpha\frac{\partial f}{\partial x^\beta}\right) + \varsigma g^{\alpha\beta}U^\gamma\frac{\partial f}{\partial x^\gamma}\right]. \tag{7.3}$$

Consider on the other hand for each element of $\mathcal{D}$ its decomposition (3.8). If

$$\hat{\zeta} = -\frac{1}{3}\eta$$

holds, the velocity-gradient part reads

$$U^{\alpha\beta} = \eta\left[\left(\frac{\partial U^\alpha}{\partial x_\beta} + \frac{\partial U^\beta}{\partial x_\alpha}\right) - g^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}\right].$$

It is only in this case and for

$$\upsilon = \eta/f \quad \text{and} \quad \varsigma = -\upsilon$$

that the velocity-gradient part agrees with the first summand in (7.3). Assuming these conditions, the gradient f part

$$F^{\alpha\beta} = \chi\left[\left(U^\alpha\frac{\partial f}{\partial x_\beta} + U^\beta\frac{\partial f}{\partial x_\alpha}\right) - g^{\alpha\beta}U^\gamma\frac{\partial f}{\partial x^\gamma}\right]$$

agrees with the second summand if and only if

$$\chi = \upsilon = \eta/f.$$

(ii) Assume $\Lambda^{\alpha\beta} = -\Delta T^{\alpha\beta}$ is an element of $\mathcal{D} \cap \mathcal{L}$. As an element of $\mathcal{L}$, this tensor obviously has the property $U_\alpha U_\beta \Lambda^{\alpha\beta} = 0$. As an element of $\mathcal{D}$, it satisfies

$$U_\alpha U_\beta\left(-\Delta T^{\alpha\beta}\right) = \sigma\frac{\partial U^\gamma}{\partial x^\gamma} - \chi U^\gamma\frac{\partial f}{\partial x^\gamma},$$

which will not generally vanish. □

We now comment briefly on the unique dissipation tensor

$$\Gamma_*^{\alpha\beta} \equiv \frac{\eta}{f}\left(\frac{\partial(f\,U^\alpha)}{\partial x_\beta} + \frac{\partial(f\,U^\beta)}{\partial x_\alpha} - g^{\alpha\beta}\frac{\partial(f\,U^\gamma)}{\partial x_\gamma}\right) \tag{7.4}$$

that lies in both $\mathcal{C}$ and $\mathcal{D}$. Due to $(7.2)_2$, an attempt to interpret $\hat{\zeta}$ as $\tilde{\zeta}$ and combine this with our choice (1.15) leads to the negative bulk viscosity

$$\zeta = -\frac{1}{3}((1 - c_s^2)^2 + 2c_s^4)\eta,$$

which shows that $\Gamma_*^{\alpha\beta}$ is *not* an instance of our proposed tensor $-\Delta T_\square^{\alpha\beta}$. But our considerations in Sec. III have a nice implication.

As Choquet-Bruhat has pointed out,[1] Eqs. (1.6) with $-\Delta T^{\alpha\beta} \in \mathcal{C}$ are causal and hyperbolic in the sense of Leray and Ohya.[9] By Theorem 2, Eqs. (1.6), in the special case of the Choquet-Bruhat dissipation tensor $-\Delta T^{\alpha\beta} = \Gamma_*^{\alpha\beta}$, also are a second-order symmetric hyperbolic system in the sense of Hughes, Kato, and Marsden[8] and thus well-posed in L^2-based Sobolev spaces.

[1] Choquet-Bruhat, Y., *General Relativity and the Einstein Equations* (Oxford University Press, Oxford, 2009).
[2] Eckart, C., "The thermodynamics of irreversible processes. 3: Relativistic theory of the simple fluid," Phys. Rev. **58**, 919–924 (1940).
[3] Freistühler, H., "Godunov variables in relativistic fluid dynamics," preprint arXiv:1706.06673.
[4] Freistühler, H. and Temple, B., "Causal dissipation for the relativistic dynamics of ideal gases," Proc. R. Soc. A **473**, 20160729 (2017).
[5] Freistühler, H. and Temple, B., "Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation," Proc. R. Soc. A **470**, 20140055 (2014).
[6] Friedrichs, K. O., "Symmetric hyperbolic linear differential equations," Commun. Pure Appl. Math. **7**, 345–392 (1954).
[7] Godunov, S. K., "An interesting class of quasilinear systems," Dokl. Akad. Nauk SSSR **139**, 525–523 (1961).
[8] Hughes, T., Kato, T., and Marsden, J., "Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity," Arch. Ration. Mech. Anal. **63**, 273–294 (1976).
[9] Leray, J. and Ohya, Y., "Équations et systèmes non-linéaires, hyperboliques nonstricts," Math. Ann. **170**, 167–205 (1967).
[10] Lichnerowicz, A., "Théories relativistes de la gravitation et de l'électromagnétisme," in *Relativité Générale et Théories Unitaires* (Masson et Cie., Paris, 1955).
[11] Müller, I., "Zur Ausbreitungsgeschwindigkeit von Störungen in kontinuierlichen Medien," Doctoral dissertation (TH Aachen, 1966)
[12] Müller, I. and Ruggeri, T., *Rational Extended Thermodynamics,* 2nd ed., Springer Tracts in Natural Philosophy Vol. 37 (Springer-Verlag, New York, 1998); Hydrodynamics (University Press, Oxford, 2013).
[13] Romatschke, P., "New developments in relativistic viscous hydrodynamics," Int. J. Mod. Phys. E **19**, 1–53 (2010).
[14] Van, P. and Biró, T., "First order and stable relativistic dissipative hydrodynamics," Phys. Lett. B **709**, 106–110 (2012).
[15] Weinberg, S., *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).