ON THE REGULARITY IMPLIED BY THE ASSUMPTIONS OF GEOMETRY

MORITZ REINTJES AND BLAKE TEMPLE

NOVEMBER 6, 2020

ABSTRACT. We resolve the problem of optimal regularity and Uhlenbeck compactness for affine connections in General Relativity and Mathematical Physics. First, we prove that any connection Γ defined on the tangent bundle of an *arbitrary* differentiable manifold, satisfying the condition that the components of Γ together with the components of its Riemann curvature tensor are bounded in L^{∞} in a given coordinate system, can be smoothed by coordinate transformation to optimal regularity, $\Gamma \in W^{1,p}$ (one derivative smoother than the curvature), any $p < \infty$. For Lorentzian metrics in General Relativity this implies that shock wave solutions of the Einstein-Euler equations are non-singulargeodesic curves, locally inertial coordinates and the Newtonian limit, all exist in a classical sense. The proof is based on extending authors' existence theory for the *RT*-equations by one order, to the level of L^{∞} connections, and to accomplish this we introduce the reduced RT-equations, a system of *elliptic* partial differential equations for the Jacobians of the regularizing coordinate transformations. Secondly, we prove that this existence theory suffices to extend Uhlenbeck compactness from the case of connections on vector bundles over Riemannian manifolds, to the case of (affine) connections on the tangent bundle of arbitrary manifolds, including Lorentzian manifolds of relativistic Physics. By this, Uhlenbeck compactness and optimal regularity are pure logical consequences of the rule which defines how connections transform from one coordinate system to another-the starting assumption of geometry.

Contents

1.	Introduction	- 2
2.	Statement of our Uhlenbeck compactness and Optimal Regularity	
	Theorems for L^{∞} connections	11
3.	Preliminaries - The Euclidean Cartan algebra	15
4.	The reduced RT-equations and optimal regularity	17
5.	Proof of Theorems 2.1 and 2.3	23
6.	Applications of the main results	27
7.	How to recover the original RT-equations from the reduced RT-equations	
	by gauge transformation	31

M. Reintjes is currently supported by the German Research Foundation, DFG grant FR822/10-1, and was supported by FCT/Portugal through (GPSEinstein) PTDC/MAT-ANA/1275/2014 and UID/MAT/04459/2013 from January 2017 until December 2018.

8. Weak formulation of the RT-equations	35	
9. The gauge freedom - Proof of Theorem 4.1	43	
10. Existence theory for the reduced RT-equations - Proof of Theorem 4.3	56	
11. Proof of Lemmas 10.6, 10.7, 10.9 and 10.10	69	
Appendix A. Sobolev norms and inequalities	79	
Appendix B. Elliptic PDE theory	81	
Appendix C. Cauchy Riemann type equations at low regularities	84	
Appendix D. A lemma used in the proof of Theorem 4.1	91	
Acknowledgements		
References		

1. INTRODUCTION

In this paper we resolve two problems in Mathematical Physics by extending the multi-dimensional existence theory for the RT-equations¹ introduced in [21] to affine connections at the low level of L^{∞} regularity, with L^{∞} curvature tensor. First, this existence theory establishes that such connections can always be smoothed to optimal regularity $W^{1,p}$ by coordinate transformation, any $p \in (n, \infty)$, settling in the affirmative that spacetime singularities associated with GR shock waves are always removable.² In particular, this establishes for the first time that (weak) shock wave solutions of the Einstein-Euler equations constructed by the Glimm scheme are one order more regular than previously known [13], and that multi-dimensional shock wave solutions are always non-singular in the sense that the associated gravitational metric always solves the Einstein equations $G = \kappa T$ in the strong L^p -sense; and geodesic curves,³ locally inertial coordinates and the Newtonian limit all exist in a classical sense. Secondly, in the case of affine connections, our L^{∞} existence theory for the RT-equations suffices to extend, Uhlenbeck compactness⁴ from Riemannian to Lorentzian geometry. That is, we extend Uhlenbeck compactness from the case of $W^{1,p}$ connections with uniform L^p curvature bounds for connections defined on vector bundles over Riemannian manifolds (with positive definite metrics) [29], to the case of affine connections uniformly bounded in L^{∞} with L^{∞} curvature bounds, defined on the tangent bundle of arbitrary differentiable manifolds, including Lorentzian manifolds of relativistic Physics. Karen Uhlenbeck's compactness theorem in [29] for Riemannian geometry was a topic of the

¹The Regularity Transformation equations or the Reintjes-Temple equations.

²Because we work on bounded domains, establishing optimal regularity in L^p , p > n, directly implies it for $L^{p'}$, 1 < p' < p.

³For L^{∞} connections the basic existence theorem for ODE's at low regularities (Peano's Theorem) does not apply to construct geodesic curves or particle trajectories.

⁴By *Uhlenbeck compactness* we mean compactness of a sequence of connections Γ_i derived from a uniform bound on the un-differentiated connection and curvature components alone.

2019 Abel Prize and 2007 Steele Prize and was central to prove fundamental results in geometry, notably Donaldson's work in [9], see also [28, 32]. The importance of Uhlenbeck compactness is to provide a convergent subsequence of connections from uniform bounds on the curvature alone, without the need to bound all connection derivatives, with a convergence strong enough to pass limits through non-linear products.

The RT-equations are a system of nonlinear *elliptic* partial differential equations in matrix valued differential forms ($\tilde{\Gamma}, J, A$). These equations determine the Jacobians J of coordinate transformations which transforms a given connection Γ to optimal regularity. The unknown $\tilde{\Gamma}$ represents the regularized connection components, A is an auxiliary variable introduced to impose the integrability condition dJ = 0, and Γ , the connection components of an arbitrary given connection, appears as a source term on the right hand side of the RT-equations, along with a vector valued 0-form v free to be chosen. Our theory starts with no more than the component functions $\Gamma \equiv (\Gamma_{ij}^k)$ defined on some open set $\Omega \subset \mathbb{R}^n$, and we view Γ as the components of a connection in some given but arbitrary coordinate system x on Ω . The RT-equations, derived in [21] from the connection transformation law and first analyzed in [22], are given by

$$\Delta \tilde{\Gamma} = \delta d \left(\Gamma - J^{-1} dJ \right) + d (J^{-1} A), \qquad (1.1)$$

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \qquad (1.2)$$

$$d\vec{A} = \overrightarrow{\operatorname{div}}(dJ \wedge \Gamma) + \overrightarrow{\operatorname{div}}(J \, d\Gamma) - d(\overline{\langle dJ; \tilde{\Gamma} \rangle}), \qquad (1.3)$$

$$\delta \vec{A} = v, \tag{1.4}$$

with boundary data

$$dJ = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

The unknowns $(\tilde{\Gamma}, J, A)$ in the RT-equations, together with the given connection components Γ , are defined by their components in *x*-coordinates on Ω .⁵ The operations on the right hand side are defined in Section 3 in terms of the Cartan Algebra of differential forms in *x*-coordinates, and the RT-equations are reintroduced in Section 4.

The RT-equations are elliptic regardless of metric signature, because Δ is the Laplacian of the Euclidean metric in x-coordinates, and determine the Jacobians of coordinate transformations to optimal regularity. Thus the problem of optimal regularity and Uhlenbeck compactness is reduced to an existence theory for the RT-equations, c.f. [21, 23]. The RT-equations, and hence both Uhlenbeck compactness and optimal regularity, are mathematical consequences of only the rule which defines how connections transform

⁵Here $\Gamma \equiv \Gamma^{\mu}_{\nu i} dx^i$ and $\tilde{\Gamma} \equiv \tilde{\Gamma}^{\mu}_{\nu i} dx^i$ are matrix valued 1-forms, $J \equiv J^{\mu}_{\nu}$ and $A \equiv A^{\mu}_{\nu}$ are matrix valued 0-forms, and $\vec{A} \equiv \vec{A}^{\mu}_i dx^i$ is a vector valued 1-form, the vectorization of A, c.f. Sections 3 and 4 for details.

from one coordinate system to another, logically independent of any additional structure on the geometry, like positive definiteness of the metric or the Einstein equations.

In order to extend the existence theory for the RT-equations to the level of L^{∞} connections, we introduce the *reduced* RT-equations, (system (1.6) -(1.8) below), an elliptic system of equations equivalent to the original RTequations (1.1) - (1.4) as a consequence of the gauge freedom v inherent in those equations. The reduced RT-equations simplify the nonlinearities in the problem of optimal regularity to a degree sufficient to extend our analysis of the RT-equations by one order, from the level of connection components $\Gamma \in W^{1,p}$ achieved in [22], to $\Gamma \in L^{\infty}$, the level of GR shock waves. Our first main result, which follows from this existence theory, establishes that if the components of Γ and $\operatorname{Riem}(\Gamma)$ are in L^{∞} in a given coordinate system, then in a neighborhood of every point there exists a $W^{2,2p}$ coordinate transformation, such that in the transformed coordinates the components of Γ exhibit optimal regularity, $\Gamma \in W^{1,p}$, any $p \in (n, \infty)$, (i.e., the components of Γ are one derivative more regular than the components of its curvature tensor Riem(Γ)), c.f. Theorem 2.1 below.⁶ This new existence theory for the reduced RT-equations provides uniform $W^{1,p}$ estimates for the connection in the transformed coordinates, and this directly implies the new Uhlenbeck compactness theorem, stated in Theorem 2.3 below, which does not rely on any underlying Riemannian or Lorentzian metric. Although the derivation of the reduced RT-equations begins with the original RT-equations, the reduced RT-equations introduced in this paper represent a new starting point, and the subsequent proofs are self-contained and stand logically independent of the original RT-system.

1.1. The problem of optimal regularity. The existence of coordinates in which connections are non-optimal is a fundamental feature of Riemann's curvature tensor, following directly from the fact that the Riemann curvature transforms as a tensor by contraction with undifferentiated Jacobians, while the transformation of a connection involves derivatives of the Jacobian. So any transformation by a Jacobian which has the *same regularity* as a given connection, will lower the regularity of a connection of optimal regularity (one derivative more regular than its curvature) by one order due to the terms containing derivatives of the Jacobian in the transformation law

⁶With slight abuse of terminology we call this optimal regularity, because it is the extra derivative we are most concerned with, not the level of p, as the extra derivative alone suffices to imply Uhlenbeck compactness and the existence of locally inertial frames and geodesic curves. The highest possible regularity for Γ would be one full derivative above the curvature, c.f. [23]. One full derivative above L^{∞} is $C^{0,1} = W^{1,\infty}$, and one full derivative above L^p is $W^{1,p}$, and note that on bounded domains $W^{1,p}$ contains $W^{1,\infty}$ as a subspace [11]. Filling in the gap between them is a topic for future research, but currently of secondary importance.

for connections.⁷ The result is a transformed connection with components one order less regular, and in the same regularity class as the curvature, because the Riemann curvature tensor would preserve its regularity under tensor transformation. This holds for classical spaces of regularity like C^k , as well as weak regularity measured by Sobolev spaces $W^{m,p}$, and Hölder regularity $C^{m,\alpha}$. To prove the reverse direction, that non-optimal connections can always be smoothed to optimal regularity by coordinate transformation, one needs to undo the above process, and this requires constructing a singular transformation given only the information about the non-optimal connection and its curvature. For example, at the level of L^{∞} connections, such a coordinate transformation must be singular in the sense that jumps in derivatives of the Jacobian must be tuned to precisely cancel out the discontinuities in the given non-optimal connection in the transformation law for connections.

The existence of coordinate transformations which smooth connections to optimal regularity, one derivative more regular than the curvature, is surprising in light of the fact that the curvature, being a "curl" plus a "commutator", does not directly control every derivative of Γ , only $d\Gamma \equiv$ $\operatorname{Curl}(\Gamma)$. That is, the complementary derivatives, $\delta\Gamma \equiv \operatorname{div}(\Gamma)$, are not controlled directly by assuming a given regularity of the curvature. Since the basic compactness theorems for Sobolev spaces are based on controlling every derivative, it follows that optimal regularity is intimately connected to compactness. This principle, as expressed through the exterior derivative dand the co-derivative δ of the Cartan algebra of differential forms associated with an assumed positive definite metric, underlies Uhlenbeck's celebrated compactness result.

We note that the regularity of metric, connection and curvature is not altered by sufficiently smooth coordinate transformations, so in this sense regularity is a geometric property of the manifold when one takes the smooth atlas. Thus one can view the RT-equations as providing a low regularity transformation which lifts regularity, but that regularity then becomes a geometric property of the resulting manifold when again the atlas of smooth coordinate transformations is taken, c.f. the discussion in [23].

1.2. Uhlenbeck compactness. Uhlenbeck's compactness theorem, Theorem 1.5 of [29], applies to Riemannian metrics, and is based on establishing a uniform bound on the components of a connection in Coulomb gauge, the Coulomb gauge providing a coordinate system arranged to satisfy $\delta\Gamma = 0$ to bound the derivatives uncontrolled by the curvature through $d\Gamma$. Compactness in Coulomb gauge then follows from a uniform bound on the curvature. To illustrate the heart of the issue in [29], taking δ of Riem(Γ) = $d\Gamma + \Gamma \wedge \Gamma$, when $\delta\Gamma = 0$, results in an equation of (essentially) the form $\Delta\Gamma = \delta$ Riem(Γ), where $\Delta = d\delta + \delta d$ is the Laplacian of

⁷Note, this principle directly carries over to metric tensors which, by Christoffel's formula, are always exactly one derivative more regular than their connections.

the underlying Riemannian metric; so by elliptic regularity, a sequence of connections $\Gamma_i \in W^{1,p}$ with $\operatorname{Riem}(\Gamma_i)$ uniformly bounded in L^p , will be uniformly bounded in $W^{1,p}$ in Coulomb gauge, for $p < \infty$. Sobolev compactness then implies a subsequence converges weakly in $W^{1,p}$ and strongly in L^p in Coulomb gauge.

In the case of Lorentzian metrics, $d\delta + \delta d$ is the hyperbolic D'Alembert (wave) operator, and since hyperbolic operators propagate irregularities from initial data surfaces along characteristics, deducing optimal regularity for Lorentzian metrics in Coulomb gauge is at best problematic, c.f. [23, Ch. 9]. Our incoming point of view is that the Coulomb gauge condition $\delta \Gamma = 0$ is too restrictive for general connections, and instead of trying to eliminate the uncontrolled δ derivatives of Γ altogether, our idea is to bound them in the right space by the RT-equations, elliptic equations in $\tilde{\Gamma}$ and J. Different from Uhlenbeck's argument, the RT-equations are formulated in terms of the Cartan algebra of differential forms associated with the Euclidean metric of an arbitrary coordinate system x in which the components Γ_{ii}^k of Γ are given, not the invariant Cartan algebra of any underlying metric. Because they are based on the auxiliary Riemannian structure provided by the coordinate Euclidean metric, the RT-equations are *elliptic* regardless of any invariant metric structure for Γ . This allows us to obtain optimal regularity and Uhlenbeck compactness for arbitrary connections on the tangent bundle of arbitrary manifolds, without recourse to metric signature or even any underlying metric structure. Note that the RT-equations are not invariant in a tensorial sense, but a different version of them is given in each coordinate system. As a consequence the RT-equations have the same simple elliptic structure in every coordinate system, and this makes them inherently useful for analysis.⁸

We now compare our compactness theorem to Uhlenbeck's result in [29]. Theorem 1.5 of [29] assumes a sequence of connections $\Gamma_i \in W^{1,p}$, with curvature Riem(Γ_i) uniformly bounded in L^p , and from this concludes that in Coulomb gauge, the connection coefficients are uniformly bounded in $W^{1,p}$, with uniform bound provided by the original bound on the curvature in L^p . The uniform bound on the extra derivative in $W^{1,p}$ then implies Uhlenbeck compactness, i.e., the convergence of a subsequence of Γ_i weakly in $W^{1,p}$, and hence strongly in L^p , in Coulomb gauge. In contrast, our Theorem 2.3 stated below, assumes a sequence of connections $\Gamma_i \in L^\infty$, which need not lie in $W^{1,p}$ at the start, but assumes uniform bounds on both Γ_i and Riem(Γ_i) in L^∞ , (or equivalently on $\Gamma_i, d\Gamma_i$ in L^∞). From this, Theorem 2.3 concludes the existence of coordinate transformations $x \to y_i(x)$, (which play the role of Coulomb gauge), with Jacobians uniformly bounded in $W^{1,2p}$, such that in the new coordinates, the sequence of connection components Γ_{y_i} are uniformly bounded in $W^{1,p}$, with bound given by our original L^∞

⁸In contrast, the Einstein equations only take simpler forms in canonical coordinates like Standard Schwarzschild Coordinates [13] or wave coordinates [5].

bounds on Γ_i and $d\Gamma_i$. From this we again conclude Uhlenbeck compactness, i.e., strong convergence of a subsequence of Γ_{y_i} in L^p , with y_i converging in $W^{1,2p}$. Thus our theorem replaces the assumption $\Gamma_i \in W^{1,p}$, (which for us is the assumption of optimal regularity at the start), with the assumption that the components of Γ_i are uniformly bounded in L^∞ . In any event, our assumption of a uniform bound on both Γ_i and $d\Gamma_i$ is a small concession considering that it just assumes the two terms Γ_i and $d\Gamma_i$ which appear at different orders in the Riemann curvature tensor, are bounded separately, uniformly, in the same space as the curvature. Our assumption of a local L^∞ bound instead of a local L^p bound on the curvatures of a sequence of connections Γ_i is a bit stronger, but at this stage we are interested in L^∞ regularity as it includes the case of GR shock waves.

1.3. **GR shock waves.** The authors' present multi-dimensional theory of optimal regularity began with the special case of shock wave solutions of the Einstein-Euler equations constructed by Glimm's random choice method in [13], (see also [3]). The Lorentzian metrics associated with these shock wave solutions are only Lipschitz continuous $(C^{0,1})$, a regularity too low to construct geodesic curves and locally inertial coordinates directly by classical ODE methods. This motivated the question as to whether one can raise the metric regularity by coordinate transformation, to recover these basic objects of geometry, or whether the Lorentzian metrics of GR shock waves are exhibiting essential non-removable spacetime singularities. A coordinate transformation to optimal regularity would remove these singularities.⁹ Thus, since shock waves form generically in the compressible Euler equations and correctly model gas dynamics, resolving the question whether these singularities can be removed, directly addresses the basic consistency of the Einstein-Euler system.¹⁰

In his classic 1966 paper [15] Israel introduced the multi-dimensional theory of junction conditions and used it to prove that a metric $C^{0,1}$ across a single smooth shock surface can be locally smoothed to optimal regularity $C^{1,1}$ by coordinate transformation to Gaussian normal coordinates. But the optimal regularity results in [15] do not apply to shock wave interactions, and thus not to the $C^{0,1}$ metrics in [13], because the underlying Gaussian normal coordinate construction cannot be associated to intersecting shock surfaces. The only extension of Israel's result to shock wave interactions (before this paper) was accomplished for the special case of spherically symmetric shock wave interactions in [17, 18]. But it remained out of reach how to address these apparent singularities in shock wave solutions constructed in

⁹This is a perspicatious warm-up problem for the multi-dimensional theory of GR shock waves because the role played by non-optimal coordinates in spherically symmetric spacetimes is no different than the role they play in general multi-dimensional spacetimes: They exist simply because the Riemann curvature tensor involves second derivatives of the metric, but transforms as a tensor by first derivative Jacobians.

¹⁰Interestingly, metrics of a similar low regularity arise in the recent study of "wild" solutions of the non-relativistic Euler equations in [2].

[13], (or constructed in multi-dimensions by the junction conditions), when they contain more complex shock wave interactions.

Authors' paper [20] was a first step for the general problem of smoothing metrics and connections. In [20], we introduced a necessary and sufficient condition for the general problem of smoothing metrics with connection and Riemann curvature tensor in L^{∞} , the so-called *Riemann-flat condition*, which is the condition that there should exist a tensor $\tilde{\Gamma} \in C^{0,1}$ such that $\operatorname{Riem}(\Gamma - \tilde{\Gamma}) = 0$. Our main theorem in [20] then states that there exists a $C^{1,1}$ coordinate transformation which smooths an L^{∞} connection Γ by one order to Lipschitz continuous $C^{0,1} = W^{1,\infty}$ (hence optimal) if and only if the Riemann-flat condition holds.¹¹ The tensor $\tilde{\Gamma}$ gives rise to a coordinate system u in which Γ exhibits optimal regularity, and the components of $\tilde{\Gamma}$ and Γ agree in *y*-coordinates. However, even though the Riemann-flat condition gives a new geometric point of view on the problem of optimal regularity, it was entirely unclear how to construct such a tensor Γ , or whether this is always possible. The breakthrough in our research program came about in [21], when we derived, from two equivalent forms of the Riemann-flat condition, the RT-equations (1.1) - (1.4), a system of solvable elliptic equations in the sought after tensor Γ and Jacobian J.

In this paper we extend our current existence theory for the RT-equations in [22] by one order of regularity to Γ , Riem(Γ) $\in L^{\infty}$, (or equivalently $\Gamma, d\Gamma \in L^{\infty}$), and prove that any such connection can be locally smoothed to optimal regularity $\Gamma \in W^{1,p}$. This resolves the problem of optimal regularity at GR shock waves by establishing that for any weak solution of the Einstein equations satisfying $\Gamma, d\Gamma \in L^{\infty}$ in x-coordinates, there always exist local coordinate transformations $x \to y$ with Jacobian $J \in W^{1,2p}$, such that $\Gamma \in W^{1,p}$ in y coordinates. Here p > n can be taken to be arbitrarily large, but not yet $p = \infty$. So we do not obtain $\Gamma \in C^{0,1}$, $g \in C^{1,1}$ as Israel did for smooth shock surfaces, but we are arbitrarily close in the sense that pcan be arbitrarily large. For p > n, any $\Gamma \in W^{1,p}$ is Hölder continuous by Morrey's inequality, a regularity sufficient for geodesic curves to exist (by Peano's Theorem), for spacetime to admit locally inertial frames, and for the correct Newtonian limit to exist at each point in spacetime. An explicit construction of locally inertial frames is given in Corollary 6.3 below.

1.4. The reduced RT-equations. The key idea and innovation in the present paper is to introduce what we call the *reduced RT-equations*, a simplified elliptic system required to extend our previous existence theory for the RT-equations in [22] to L^{∞} connections. The derivation of the reduced RT-equations is based on using the gauge freedom v in the original RT-equations (1.1) -(1.4) to uncouple the system of PDE's for the unknown Jacobian J of the smoothing transformation from the PDE (1.1) for the sought after connection components $\tilde{\Gamma}$ of optimal regularity. This isolates

¹¹This equivalence extends easily to $\Gamma \in L^{\infty}$ and $\tilde{\Gamma} \in W^{1,p}$, the case address in this paper.

the nonlinearities in the $\tilde{\Gamma}$ system, and thereby eliminates them from the iteration scheme for J which is the basis of our existence theory. The gauge freedom v in the RT-equations reflects the multiplicity of coordinate maps which can smooth a connection to optimal regularity. In summary, the unknowns J and $\tilde{\Gamma}$ are coupled together in the original RT-equations, and the reduced RT-equations consist of the resulting uncoupled system of equations (1.6) - (1.8) in J alone. To finish the proof we use the $\tilde{\Gamma}$ equation to show directly that the coordinate system determined by a solution J of the reduced RT-equations, does indeed lift the original connection Γ to optimal regularity $\Gamma \in W^{1,p}$. To complete the picture, Theorem 4.1 below establishes that a solution J of the reduced RT-equation determines a $\tilde{\Gamma}$ which meets the Riemann-flat condition, and J together with this $\tilde{\Gamma}$ solve the original RT-equations. This again demonstrates that J is indeed a Jacobian which takes Γ to optimal regularity.

The reduced RT-equations are given by the following system,

$$\Delta J = \delta(J \cdot \Gamma) - B, \tag{1.6}$$

$$d\vec{B} = \vec{\operatorname{div}}(dJ \wedge \Gamma) + \vec{\operatorname{div}}(J \, d\Gamma), \qquad (1.7)$$

$$\delta \vec{B} = w, \tag{1.8}$$

where J is the Jacobian of the transformation to optimal regularity, B is an auxiliary matrix valued differential form introduced to impose the integrability of J to coordinates, and the new gauge freedom is the freedom to choose the vector valued function w. The operations on the right hand side of (1.6) - (1.8) are defined in Section 3 in terms of the Euclidean Cartan algebra of matrix valued differential forms. Equation (1.6) requires J to be viewed as a matrix valued differential form, but the integrability condition is expressed in terms the vector valued differential form $\vec{J} = J_i^{\mu} dx^i$ through $d\vec{J} = 0$, and one challenge is to incorporate both matrix valued differential forms and their vectorization within a single framework. The reduced RTequations are derived from the original RT-equations in Section 4. Although the reduced RT-equations are independent of $\tilde{\Gamma}$, we do not have a derivation of them independent of $\tilde{\Gamma}$ and the original (coupled) RT-system.

1.5. **Prior results.** It was shown by DeTurck and Kazdan in [8] that for (positive definite) Riemannian metrics, optimal regularity can always be achieved in harmonic coordinates. The first optimal regularity result in Lorentzian geometry is due to Anderson [1]. Anderson's results are based on using harmonic coordinates on the Riemannian hypersurfaces of a given foliation of spacetime, and establish curvature bounds for vacuum spacetimes and certain matter fields when the Riemann curvature is in L^{∞} , under some technical assumptions. A similar result for vacuum spacetimes was proven in [4]. As far as we can tell, these results do not apply to GR shock waves, and our result cannot be obtained from these prior methods, free of additional assumptions, even in the special case of vacuum spacetimes. (Keep in mind

that the setting of vacuum excludes fluid dynamical shock waves, and so is a warm-up problem from the point of view of shock wave theory. Historically shock waves are one of the main motivations for the study of low regularity solutions.) The results in [1, 4] require applying sophisticated analytical and geometric machinery on top of the classical harmonic coordinate construction in [8], and suggest strongly that metric signature is a central issue. Our results show that optimal regularity is entirely independent of metric and metric signature.¹²

1.6. Outline of the paper. In Section 2 we state our main results, Theorems 2.1 and 2.3. In Section 3 we introduce the Cartan calculus for matrix and vector valued differential forms required in this paper. In Section 4 we prove that solutions of the reduced RT-equations determine solutions of the full RT-equations, c.f. Theorem 4.1, and our main existence theorem for the reduced RT-equations is given in Theorem 4.3. The proofs of our main Theorems 2.1 and 2.3 are given in Section 5, assuming Theorems 4.1 and 4.3 are true. In Section 6 we give three applications of Theorems 2.1 and 2.3: A new compactness result for the vacuum Einstein equations in GR at low regularities as an application of Uhlenbeck compactness, a proof of optimal regularity for GR shock waves constructed by Glimm's theorem, and a construction of locally inertial frames for general L^{∞} connections. In particular, the existence of locally inertial frames rules out regularity singularities at GR shock waves by establishing that shock wave solutions of the Einstein-Euler equations are locally inertial.

Sections 7 -11 are devoted to the proofs of Theorems 4.1 and 4.3. In these sections we have given a careful development of the weak formulation of the RT-equations, and we present the existence theory for weak solutions of the reduced RT-equations starting with two basic theorems from linear elliptic PDE theory, Theorems B.1 and C.2 recorded in the appendix. In order to develop a rigorous existence theory for $\Gamma, d\Gamma \in L^{\infty}$ we introduce a calculus of L^2 adjoints for the differential operators on vector and matrix valued differential forms appearing in the RT-equations, required to make sense of the equations at the level of weak (distributional) solutions. The theory in Sections 7 - 11 is self-contained, and written to be accessible to mathematicians and physicists who need not be experts in PDE theory. An exposition of the development of the RT-equations in [23] summarizes the results and methods in [21, 22], and provides an outlook on the results established in this paper, stating without proof Theorems 2.1 and 2.3, as well as Corollaries 6.1 and 6.2.

¹²We note that the recent resolution of the bounded L^2 curvature conjecture for vacuum spacetimes, (c.f. Theorem 1.6 of [16]), does not address the issue of optimal regularity, essentially because initial data taken from non-optimal connections is one order less regular than the data assumed in [16], see the discussion in [23].

2. Statement of our Uhlenbeck compactness and Optimal Regularity Theorems for L^{∞} connections

Let Γ denote a connection on the tangent bundle $T\mathcal{M}$ of an arbitrary *n*dimensional differentiable manifold $\mathcal{M}, n \geq 2$. Since the problem of optimal regularity is local, we assume at the start a given coordinate system x defined on an open set $\Omega \subset \mathcal{M}$, such that $\Omega_x \equiv x(\Omega) \subset \mathbb{R}^n$ is bounded. That is, we work in a fixed chart (x, Ω) on \mathcal{M} . Without loss of generality we assume Ω_x has a smooth boundary. We use the notation Γ_x to denote the components of Γ in x-coordinates, $\Gamma_x \equiv \Gamma_{ij}^k(x)$. We say $\Gamma_x, d\Gamma_x \in L^\infty(\Omega_x)$ if all component functions are in $L^\infty(\Omega_x)$ in x-coordinates. Here $d\Gamma_x$ denotes the exterior derivative of Γ_x viewed as a matrix valued 1-form in x-coordinates, a noninvariant object in the sense that it transforms neither as a tensor nor as a connection. Since the Riemann curvature tensor can be expressed as

$$\operatorname{Riem}(\Gamma_x) = d\Gamma_x + \Gamma_x \wedge \Gamma_x, \qquad (2.1)$$

(c.f. (3.5) below), assuming $\Gamma_x, d\Gamma_x \in L^{\infty}(\Omega_x)$ is equivalent to assuming Γ_x , Riem $(\Gamma_x) \in L^{\infty}(\Omega_x)$. Given a coordinate transformation $x \to y$, we let $\Gamma_y \equiv \Gamma_{\alpha\beta}^{\gamma}(y)$ denote the connection components in *y*-coordinates defined on $\Omega_y \equiv y(\Omega)$. For coordinate transformations with Jacobians $J \in W^{1,p}(\Omega_x)$, p > n, (always assumed here), the assumption $\Gamma_x, d\Gamma_x \in L^{\infty}(\Omega_x)$ is an invariant statement,¹³ so we write $\Gamma, d\Gamma \in L^{\infty}(\Omega)$. Note, the statement that Γ_x is in $W^{1,p}$, (i.e., Γ_x has optimal regularity), is not an invariant statement for Jacobians at the low regularity $J \in W^{1,p}(\Omega_x)$. We introduce the coordinate dependent norm

$$\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x)} \equiv \|\Gamma_x\|_{L^{\infty}(\Omega_x)} + \|d\Gamma_x\|_{L^{\infty}(\Omega_x)},$$
(2.2)

which is central to our estimates. The norms we use in this paper are recorded carefully in Appendix A. Note that, for the purpose of this paper, we replace $||d\Gamma||_{L^{\infty}}$ by $||\operatorname{Riem}(\Gamma)||_{L^{\infty}}$ on the right hand side of (2.2), because (2.1) implies

$$\|d\Gamma\|_{L^{\infty}} \leq \|\operatorname{Riem}(\Gamma)\|_{L^{\infty}} + 2\|\Gamma\|_{L^{\infty}}^{2} \leq \|d\Gamma\|_{L^{\infty}} + 4\|\Gamma\|_{L^{\infty}}^{2}.$$
 (2.3)

We use the notation that subscript x on Γ_x can be dropped when it is clear from the context that we address only the x-components of Γ . Since all our analysis is done in fixed x-coordinates there should be no confusion.

Our main theorem regarding optimal regularity is the following:

Theorem 2.1' Assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in x-coordinates and let M > 0 be a constant such that

$$\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x)} \equiv \|\Gamma_x\|_{L^{\infty}(\Omega_x)} + \|d\Gamma_x\|_{L^{\infty}(\Omega_x)} \leq M.$$
(2.4)

¹³This holds since for such coordinate transformations $\operatorname{Riem}(\Gamma)$ transforms as a tensor, and contraction by Hölder continuous Jacobians does not lower the L^{∞} regularity. The L^{∞} regularity of $d\Gamma_y$ then follows from the regularity of $\operatorname{Riem}(\Gamma_y)$.

Then for any $n and any point <math>q \in \Omega$ there exists a neighborhood $\Omega' \subset \Omega$ of q and a coordinate transformation $x \to y$ with Jacobian $J = \frac{\partial y}{\partial x} \in W^{1,2p}(\Omega'_x)$, such that

$$\|\Gamma_y\|_{W^{1,p}(\Omega'_u)} \le C \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x)},\tag{2.5}$$

for some constant C > 0 depending only on Ω_x, p, n, q and M. That is, the connection components Γ_y in y-coordinates have optimal regularity, $\Gamma_y \in W^{1,p}(\Omega'_y)$.

Theorem 2.1' tells us that we can raise the connection regularity by essentially one derivative to $\Gamma_y \in W^{1,p}$, arbitrarily close to $W^{1,\infty}$ as $p \to \infty$.¹⁴ Note, since Γ and $d\Gamma$ are assumed in $L^{\infty}(\Omega)$, the statement of the theorem is sharper the larger p is (and extends trivially to $1 \leq p < \infty$ as Ω is bounded), and we can choose any $p < \infty$ but not yet $p = \infty$, a singular case in elliptic regularity theory, c.f. the discussion in [21]. By Morrey's inequality, Γ_y is Hölder continuous when p > n, and this is sufficient regularity to construct classical geodesic curves and locally inertial coordinates, as we prove in Corollary 6.3 below. Taken together, this resolves the open problem as to whether the spacetime singularities at GR shock waves are removable in the positive, establishing that every Lipschitz continuous metric of GR shock wave theory is regular enough to meet the physical requirements of spacetime.

Theorem 2.1' follows directly from the following more refined theorem, containing an improved version of estimate (2.5), which follows from interior elliptic estimates applied to the RT-equations in the proofs below.

Theorem 2.1. Assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in x-coordinates, satisfying the bound (2.4) in terms of a constant M > 0. Then for any n $and any point <math>q \in \Omega$ there exists a neighborhood $\Omega' \subset \Omega$ of q (depending only on Ω_x, p, n and M) and a coordinate transformation $x \to y$ with Jacobian $J = \frac{\partial y}{\partial x} \in W^{1,2p}(\Omega'_x)$, such that the connection components Γ_y in y-coordinates have optimal regularity $\Gamma_y \in W^{1,p}(\Omega''_y)$ on every open set Ω'' compactly contained in Ω' , where $\Omega''_y \equiv y(\Omega'')$. Moreover, for each Ω'' compactly contained in Ω', Γ_y satisfies the uniform bound

$$\|\Gamma_y\|_{W^{1,p}(\Omega''_{x})} \le C_1(M) \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega'_x)}, \tag{2.6}$$

for some constant $C_1(M) > 0$ depending only on $\Omega''_x, \Omega'_x, p, n, q$ and M.

The refinement of estimate (2.6) over (2.5) is that the regions in (2.6) are sets of roughly the same size. The proof of Theorem 2.1 is given in Section 5.

As a corollary, in the spirit of Uhlenbeck's earlier paper [30] for positive definite metrics, we immediately obtain that $d\Gamma \in L^{\infty}$ implies that

¹⁴Recall that the Sobolov space $W^{1,\infty}$ can be identified with the space of Lipschitz continuous functions, and $W^{1,p}$ can be identified with the space of Hölder continuous functions with Hölder coefficient $\alpha = \frac{n}{p}$ as long that p > n, [11].

singularities on sets of measure zero in non-optimal connections are always removable. The condition $d\Gamma \in L^{\infty}$ plays the role of a generalized Rankine Hugoniot jump condition [26], or "Junction Condition" [15], and it gives general expression to the condition that the curvature be free of "delta function sources", necessary and sufficient conditions introduced by Israel for smoothing discontinuous connections across *single* shock surfaces [15].

Corollary 2.2. Assume Γ , $d\Gamma$ are bounded and continuous off a set of measure zero in some open set Ω in x-coordinates. Then the additional condition that the L^{∞} extension of Γ to Ω satisfies $d\Gamma \in L^{\infty}(\Omega)$, is sufficient to imply that for any point $q \in \Omega$, there exists a neighborhood $\Omega' \subset \Omega$ of q, and a coordinate transformation $x \to y$ on Ω' , such that the connection components Γ_y in y-coordinates can be extended as Hölder continuous functions to Ω'_y with $\Gamma_y \in W^{1,p}(\Omega'_y)$.

This is a direct consequence of Theorem 2.1, keeping in mind that $W^{1,p}$ is embedded in the space of Hölder continuous functions for p > n by Morrey's inequality, c.f. (A.4).

To introduce our compactness theorem, let us briefly recall the relation between weak and strong convergence in Banach spaces $W^{m,p}$. Recall that whenever we have a uniform bound on a sequence of functions in $W^{m,p}$, there always exists a weakly convergent subsequence whose limit satisfies the same uniform bound $W^{m,p}$ as the original sequence. (By the Banach-Alaoglu Theorem, the closed unit ball is weakly compact in $W^{m,p}$.) But compactness is the statement that this weak limit is actually a strong limit in $W^{m,p}$. For this it suffices to have a uniform bound on the sequence of functions in $W^{m+1,p}$, in which case weak convergence in $W^{m+1,p}$ implies strong convergence in $W^{m,p}$. Weak convergence is not enough in non-linear problems because products are generally not continuous under weak limits, but are always continuous under strong limits, and weak limits cannot be estimated as close to the weakly convergent subsequence in the norms in which the global bounds are obtained.

We now develop some notation required to state our extension of Uhlenbeck's compactness result, Theorem 1.5 in [29]. Let $\{\Gamma_i\}_{i\in\mathbb{N}}$ be a sequence of connections Γ_i defined on the tangent bundle $T\mathcal{M}$, and let $(\Gamma_i)_x$ denote their components in fixed x-coordinates defined on $\Omega_x \subset \mathbb{R}^n$, bounded and open. Assume n . Our compactness theorem states the existence ofa*strongly* $convergent subsequence of <math>\{\Gamma_i\}_{i\in\mathbb{N}}$ in L^p under coordinate transformation, assuming only the bound

$$\|(\Gamma_i, d\Gamma_i)\|_{L^{\infty}(\Omega_x)} \le M,\tag{2.7}$$

for some constant M > 0 independent of Γ_i . More precisely, assuming (2.7), Theorem 2.1 implies that for each $i \in \mathbb{N}$, there exists a coordinate transformation $x \to y_i(x)$, such that the connection components $(\Gamma_i)_{y_i} \equiv \Gamma_{y_i}$ in y_i -coordinates are one order more regular and satisfy the uniform bound $\|\Gamma_{y_i}\|_{W^{1,p}(\Omega'_{y_i})} \leq C(M)$ for some constant C(M) > 0 and some open set Ω' ,

both depending only on M, independent of i. To conclude with a convergent subsequence of $\{\Gamma_{y_i}\}_{i\in\mathbb{N}}$, we express the components of each Γ_{y_i} as functions of the original x-coordinates $\Gamma_{y_i}(x)$, so that one does not loose regularity, (i.e., we transform the y-components back to x-components as scalars, in contrast to the connection transformation from Γ_{y_i} to $(\Gamma_i)_x$ which looses one derivative of regularity). The resulting components $\Gamma_{y_i}(x) \equiv (\Gamma_i)_{y_i}(y_i(x))$, where $\Gamma_{y_i}(x)$ are the y_i components of Γ_i viewed as functions of x, will again meet the uniform $W^{1,p}$ -bound $\|\Gamma_{y_i}\|_{W^{1,p}(\Omega'_x)} \leq C(M)$ but over the fixed region Ω'_x in x-coordinates and for a different constant C(M) > 0 which accounts for the Jacobian of the transformation from x to y_i . By the Banach-Alaoglu Theorem, one can now conclude with existence of a subsequence of $\Gamma_{y_i}(x)$ which converges weakly in $W^{1,p}(\Omega'_x)$ and hence strongly in $L^p(\Omega'_x)$. This is our compactness theorem. The proof is given in Section 5.

Theorem 2.3. Assume $\{(\Gamma_i)_x\}_{i\in\mathbb{N}}$ are the x-components of a sequence of connections Γ_i on the tangent bundle $T\mathcal{M}$ of an n-dimensional manifold \mathcal{M} in a fixed coordinate system x on Ω , and let $n . Assume <math>\Gamma_i, d\Gamma_i \in L^{\infty}(\Omega)$ in x-coordinates, such that the uniform bound

$$\|(\Gamma_i, d\Gamma_i)\|_{L^{\infty}(\Omega_x)} \equiv \|(\Gamma_i)_x\|_{L^{\infty}} + \|(d\Gamma_i)_x\|_{L^{\infty}} \le M$$

$$(2.8)$$

holds for some constant M > 0 independent of $i \in \mathbb{N}$. Then for any $q \in \Omega$ there exists a neighborhood $\Omega' \subset \Omega$ of q, and a subsequence of Γ_i , (also denoted by Γ_i), for which the following holds:

(i) There exists for each $(\Gamma_i)_x$ a coordinate transformation $x \to y_i(x)$ taking Ω'_x to Ω'_{y_i} , such that the components $(\Gamma_i)_{y_i} \equiv \Gamma_{y_i}$ of Γ_i in y_i -coordinates exhibit optimal regularity $\Gamma_{y_i} \in W^{1,p}(\Omega'_{y_i})$, with uniform bound (2.5) in $W^{1,p}(\Omega'_{y_i})$.

(ii) The y_i -components Γ_{y_i} , taken as functions of x, also exhibit optimal regularity $\Gamma_{y_i}(x) \equiv (\Gamma_i)_{y_i}(y_i(x)) \in W^{1,p}(\Omega'_x)$, with uniform bound (2.5) in $W^{1,p}(\Omega'_x)$.

(iii) The transformations $x \to y_i(x)$ are uniformly bounded in $W^{2,2p}(\Omega'_x)$, and converge to a transformation $x \to y(x)$, weakly in $W^{2,2p}(\Omega'_x)$, strongly in $W^{1,2p}(\Omega'_x)$.

(iv) Main Conclusion: There is a subsequence on which the y_i -components $\Gamma_{y_i}(x)$ converge to some $\Gamma_y(x)$, weakly in $W^{1,p}(\Omega'_x)$, strongly in $L^p(\Omega'_x)$, and Γ_y are the connection coefficients of Γ_x in y-coordinates, where Γ_x is the weak limit of $(\Gamma_i)_x$ in $L^p(\Omega'_x)$.

Theorem 2.3 extends Uhlenbeck's compactness result, Theorem 1.5 of [29], as follows: Theorem 1.5 of [29] applies to connections on vector bundles (including tangent bundles) over Riemannian manifolds, and so far our Theorem 2.3 applies only to connections on tangent bundles, but for arbitrary (differentiable) manifolds, including Lorentizan manifolds of General Relativity. As we mentioned in Section 1, this extension requires a small modification of assumptions. Namely, Theorem 1.5 of [29] assumes a sequence of connections $(\Gamma_x)_i \in W^{1,p}$, with curvature Riem (Γ_i) uniformly

bounded in L^p , and from this concludes with a uniform $W^{1,p}$ bound on connection coefficients in Coulomb gauge (where $\delta \Gamma = 0$) and with resulting compactness in L^p . In contrast, our Theorem 2.3 assumes a sequence of connections $(\Gamma_x)_i$ which need not lie in $W^{1,p}$ at the start, but assumes uniform bounds on both Γ_i and $\operatorname{Riem}(\Gamma_i)$ in L^{∞} . We view our assumptions as being essentially equivalent to Uhlenbeck's assumptions, and natural for the setting of the RT-equations, as we now discuss. First note that a uniform bound on $\|\operatorname{Riem}(\Gamma_i)\|_{L^{\infty}}$ is implied by, but does not imply a uniform bound on $\|(\Gamma_i, d\Gamma_i)\|_{L^{\infty}} = \|\Gamma_i\|_{L^{\infty}} + \|d\Gamma\|_{L^{\infty}}$, since uncontrolled terms in Γ could cancel in the wedge-product in Riem $(\Gamma) = d\Gamma + \Gamma \wedge \Gamma$. In light of (2.3), for the bound on $\|(\Gamma_i, d\Gamma_i)\|_{L^{\infty}}$ to imply a bound on the curvature tensor would require starting in a coordinate system x in which $\|\Gamma_i\|_{L^{\infty}}$ is bounded by $||d\Gamma_i||_{L^{\infty}}$, or alternatively by $||\operatorname{Riem}(\Gamma_i)||_{L^{\infty}}$. For this one could take the locally inertial coordinate frames proven in Corollary 6.3 to exist for $W^{1,p}$ connections. This shows that our assumption of a uniform bound on $\|\Gamma_i\|_{L^{\infty}}$ is implied by an L^{∞} bound on the curvature alone, in natural coordinates, but not necessarily in all coordinate systems, which could involve transformations with arbitrarily large Jacobians.

Theorems 2.1 and 2.3 are based on authors' earlier discovery of the *RT*-equations ("Regularity Transformation equations" or "Reintjes-Temple equations"), a system of elliptic partial differential equations which determine whether coordinate systems exist in which the connection exhibits optimal regularity [21]. The RT-equations are elliptic independent of any underlying metric structure on the tangent bundle, hence our methods do not require the ellipticity of the Laplace-Beltrami operator of a metric, and by this we can extend Uhlenbeck's results to tangent bundles of arbitrary manifolds. (Again, using the Coulomb gauge method in the case of Lorentzian metrics would entail hyperbolic estimates, which are problematic, c.f. [23].) To formulate the RT-equations, we require an Euclidean Cartan algebra for matrix valued differential forms. This is the topic of the next section.

3. Preliminaries - The Euclidean Cartan Algebra

We now summarize the Cartan Calculus which we require to formulate the RT-equations and refer the reader to Section 2 in [21] for further detail and proofs. We work again in fixed x-coordinates defined on a open set $\Omega \equiv \Omega_x \subset \mathbb{R}^n$. By a matrix valued differential k-form ω we mean an $(n \times n)$ matrix whose components are k-forms, and we write

$$\omega = \omega_{[i_1\dots i_k]} dx^{i_1} \wedge \dots \wedge dx^{i_k} \equiv \sum_{i_1 < \dots < i_k} \omega_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \qquad (3.1)$$

for $(n \times n)$ -matrices $\omega_{i_1...i_k}$ such that total anti-symmetry holds in the indices $i_1, ..., i_k \in \{1, ..., n\}$. (We always sum over repeated indices, following Einstein's convention, but we never "raise" or "lower" indices.) We define the wedge product of a matrix valued k-form ω with a matrix valued l-form

 $u = u_{j_1 \dots j_l} dx^{j_1} \wedge \dots \wedge dx^{j_l}$ as

$$\omega \wedge u \equiv \frac{1}{l!k!} \omega_{i_1 \dots i_k} \cdot u_{j_1 \dots j_l} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}, \quad (3.2)$$

where "·" denotes standard matrix multiplication. In contrast to scalar valued differential forms, $\omega \wedge \omega$ can be non-zero, because matrices do in general not commute. The exterior derivative d is defined component wise on matrix-components,

$$d\omega \equiv \partial_l \omega_{[i_1...i_k]} dx^l \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}, \qquad (3.3)$$

and we define the co-derivative δ on a matrix valued k-form ω as

$$\delta\omega \equiv (-1)^{(k+1)(n-k)} * d * \omega,$$

where * is the Hodge star introduced in terms of the Euclidean metric in x-coordinates. That is, * satisfies the orthogonality condition

$$dx^{[i_1} \wedge \dots \wedge dx^{i_k}] \wedge * (dx^{[j_1} \wedge \dots \wedge dx^{j_k}]) = \begin{cases} dx^1 \wedge \dots \wedge dx^n, & \text{if } i_1 = j_1, \dots, i_k = j_k, \\ 0 & \text{otherwise,} \end{cases}$$
(3.4)

where indices are taken to be increasing. So δ is defined via the Euclidean metric in *x*-coordinates, while *d* requires no metric. Both *d* and δ act component wise on matrix components, so all properties of *d* and δ for scalar valued differential forms carry over to matrix valued forms. The Laplacian $\Delta \equiv d\delta + \delta d$ acts component wise on matrix-components and also on differential form components. By (3.4), one can show that Δ is in fact identical to the Laplacian of the Euclidean metric in *x*-coordinates,

$$\Delta = \partial_{x^1}^2 + \dots + \partial_{x^n}^2,$$

c.f. [6, 21] for more detail.

By (3.2) and (3.3), the Riemann curvature tensor can be written as

$$\operatorname{Riem}(\Gamma_x) = d\Gamma_x + \Gamma_x \wedge \Gamma_x, \qquad (3.5)$$

in x-coordinates. The exterior derivative satisfies the product rule

$$d(\omega \wedge u) = d\omega \wedge u + (-1)^k \omega \wedge du, \qquad (3.6)$$

where $\omega \in W^{1,p}(\Omega)$ is a matrix valued k-form and $u \in W^{1,p}(\Omega)$ is a matrix valued j-form, (c.f. Lemma 3.3 of [21]). Since the wedge product (3.2) for matrix valued 0-forms J is identical to matrix multiplication, and since $dJ^{-1} = -J^{-1} \cdot dJ \cdot J^{-1}$, the Leibnitz rule (3.6) implies that

$$d(J^{-1} \cdot dJ) = d(J^{-1}) \wedge dJ = -J^{-1}dJ \wedge J^{-1}dJ, \qquad (3.7)$$

c.f. Lemma 4.3 in [21]. Regarding the co-derivative δ , we require the following product rule

$$\delta(J \cdot w) = J \cdot \delta w + \langle dJ; w \rangle \tag{3.8}$$

16

where $J \in W^{1,p}(\Omega)$ is a matrix valued 0-form, $w \in W^{1,p}(\Omega)$ a matrix valued 1-form, and where $\langle \cdot ; \cdot \rangle$ is the matrix valued inner product defined on matrix valued k-forms ω and u by

$$\langle \omega ; u \rangle^{\mu}_{\nu} \equiv \sum_{\sigma=1}^{n} \sum_{i_1 < \dots < i_k} \omega^{\mu}_{\sigma \, i_1 \dots i_k} u^{\sigma}_{\nu \, i_1 \dots i_k}.$$
 (3.9)

So $\langle \omega ; u \rangle$ converts two matrix valued k-forms into a matrix valued 0-form. For multiplication by a matrix valued 0-form J we have the following multiplication property

$$J \cdot \langle \omega ; u \rangle = \langle J \cdot \omega ; u \rangle, \quad \langle \omega \cdot J ; u \rangle = \langle \omega ; J \cdot u \rangle, \quad \langle \omega ; u \cdot J \rangle = \langle \omega ; u \rangle \cdot J.$$
(3.10)

We also need to interpret matrix valued forms as vector valued differential forms. The two operations which convert matrix valued differential forms to vector valued forms on the right hand side of the RT-equations are $\vec{\cdot}$ and $\overrightarrow{\operatorname{div}}(\cdot)$. First, $\vec{\cdot}$ converts matrix valued k-forms ω into vector valued (k+1)-forms $\vec{\omega}$ by

$$\vec{\omega}^{\mu} \equiv \omega^{\mu}_{\nu \, i_1 \dots i_k} dx^{\nu} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}, \qquad (3.11)$$

with ω taken as in (3.1), c.f. (2.20) in [21] for the case k = 0, most relevant to us. Secondly, the operation $\overrightarrow{\operatorname{div}}(\cdot)$ converts matrix valued k-forms ω into vector valued k-forms $\overrightarrow{\operatorname{div}}(\omega)$ by the operation

$$\overrightarrow{\operatorname{div}}(\omega)^{\alpha} \equiv \sum_{l=1}^{n} \partial_{l} \left((\omega_{l}^{\alpha})_{i_{1}\ldots i_{k}} \right) dx^{i_{1}} \wedge \ldots \wedge dx^{i_{k}}.$$

Finally, for a matrix valued 1-form w and a matrix valued 0-form J, Lemma 2.4 of [21] gives the important identity

$$d(\overrightarrow{\delta(J \cdot w)}) = \overrightarrow{\operatorname{div}}(d(J \cdot w)) = \overrightarrow{\operatorname{div}}(dJ \wedge w) + \overrightarrow{\operatorname{div}}(J \cdot dw), \qquad (3.12)$$

which is crucial for the regularity to close in the RT-equations.

4. The reduced RT-equations and optimal regularity

In this section we derive the reduced RT-equations from the RT-equations, the system of elliptic PDE's introduced in [21] which determines whether a connection Γ can be mapped to optimal regularity, and prove their equivalence. We then state the main theorems concerning the existence of solutions of the reduced RT-equations and resulting optimal regularity for L^{∞} connections, Theorems 4.3 and 4.1 respectively, which are proven in Sections 8 - 11. In the end of this section we apply Theorems 4.1 and 4.3 to give the proof of our main results, Theorems 2.1 and 2.3. We begin by reviewing the RT-equations derived in [21]. The RT-equations consist of the following nonlinear elliptic system of PDE's

$$\Delta \tilde{\Gamma} = \delta d\Gamma - \delta (dJ^{-1} \wedge dJ) + d(J^{-1}A), \qquad (4.1)$$

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \qquad (4.2)$$

$$d\vec{A} = \overrightarrow{\operatorname{div}}(dJ \wedge \Gamma) + \overrightarrow{\operatorname{div}}(J \, d\Gamma) - d(\overline{\langle dJ; \tilde{\Gamma} \rangle}), \tag{4.3}$$

$$\delta \vec{A} = v, \tag{4.4}$$

together with boundary data

$$d\vec{J} = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

The connection $\Gamma \equiv \Gamma^{\mu}_{\nu k} dx^k$ is given and interpreted as a matrix valued 1-form, and Γ on the right hand side of (4.1) - (4.4) always denotes the components Γ_x in x-coordinates. The unknowns in the RT-equations are $(\tilde{\Gamma}, J, A)$ which are matrix valued differential forms as follows: $J \equiv J^{\mu}_{\nu}$ is the Jacobian of the sought after coordinate transformation which smooths the connection, viewed as a matrix-valued 0-form; $\tilde{\Gamma} \equiv \tilde{\Gamma}^{\mu}_{\nu k} dx^k$ represents the unknown tensor one order smoother than Γ such that $\operatorname{Riem}(\Gamma - \tilde{\Gamma}) =$ 0, viewed as a matrix-valued 1-form; and $A \equiv A^{\mu}_{\nu}$ is an auxiliary matrix valued 0-form introduced together with boundary data (4.5) to impose $d\vec{J} \equiv$ $\operatorname{Curl}(J) = 0$, the condition for the Jacobian J that guarantees it is integrable to a coordinate system, c.f. Theorem C.2 and [21]. See Section (3) for definitions of the remaining operations in (4.1) - (4.4).

The RT-equations (4.1) and (4.2) were derived by constructing Laplacians out of two equivalent forms of the Riemann-flat condition, a condition introduced in [20] equivalent to the existence of coordinates in which the connection has optimal regularity. These two starting conditions were that $\operatorname{Riem}(\Gamma - \tilde{\Gamma}) = 0$, or alternatively, that $\tilde{\Gamma} = \Gamma - J^{-1}dJ$ for some tensor $\tilde{\Gamma}$ one order smoother than Γ . If Γ_x can be smoothed to optimal regularity by the transformation $x \to y$ with Jacobian J, then, defining

$$\tilde{\Gamma}_J \equiv \Gamma - J^{-1} dJ, \tag{4.6}$$

the connection components Γ_y of optimal regularity are given by the tensor transformation rule

$$(\tilde{\Gamma}_J)_{ij}^k = (J^{-1})^k_{\gamma} J^{\alpha}_i J^{\beta}_j (\Gamma_y)^{\gamma}_{\alpha\beta}, \qquad (4.7)$$

and $\tilde{\Gamma}_J$ will solve the RT-equations (4.1) - (4.4) as well as the Riemann-flat condition Riem $(\Gamma - \tilde{\Gamma}_J) = 0$. That is, $\tilde{\Gamma}_J$, the tensor transformation by J of the components of the connection Γ_y in y-coordinates, will solve the RT-equations for some (A, v), when J is paired with $\tilde{\Gamma}_J$. Conversely, one can recover the connection of optimal regularity Γ_y via equation (4.7) from a general solution $\tilde{\Gamma}, J$ of the RT-equations, but this requires an existence theory which establishes the relationship between the solution $\tilde{\Gamma}$ and $\tilde{\Gamma}_J$, as we now explain.

18

In [21] we prove that if $\Gamma_x, d\Gamma_x \in W^{m,p}$, for $m \ge 1, p > n$, then there exists a coordinate transformation $x \to y$ which raises the regularity by one order to $\Gamma_y \in W^{m+1,p}$ if and only if there exists a solution $(J, \tilde{\Gamma}, A)$ of the RT-equations (4.1) - (4.4), (taking $\Gamma = \Gamma_x$ on the right hand side), with boundary data (4.5), and regularity $J, \tilde{\Gamma} \in W^{m+1,p}, A \in W^{m,p}$. In [22] we proved that such a solution $(J, \tilde{\Gamma}, A)$ exists for any such connection $\Gamma \in W^{m,p}$ with $d\Gamma \in W^{m,p}$, when $m \ge 1, p > n$. Extending this theory to the case of $\Gamma, d\Gamma \in L^{\infty}$, when the RT-equations only have meaning in a weak sense, is accomplished in the present paper, but was not possible with the methods used in our previous paper [22].

Given a suitable weak formulation of equations (4.1)-(4.4) and boundary condition (4.5), we could apply the Riemann-flat condition to conclude optimal regularity from existence of solutions. Thus the main obstacle is the problem of proving an existence theory for the RT-equations at the low level of regularity $\Gamma, d\Gamma \in L^{\infty}$ with $J \in W^{1,p}$. The problem is that the iteration scheme in [22] does not close because the gradient product $dJ^{-1} \wedge dJ$ on the right hand side of equation (4.1) fails to stay in a fixed L^p space under iteration. Alternatively, trying to construct solutions $J \in W^{1,\infty}$ is problematic as well, because $p = \infty$ is a singular case in elliptic regularity theory, and our iteration scheme in [22] would not close in L^{∞} for this different reason. We here extend the existence theory and consequent optimal regularity theory to the case $\Gamma, d\Gamma \in L^{\infty}$ by a serendipitous modification of the RT-equations.

In this paper we employ the gauge freedom of the RT-equations to circumvent the problem of incorporating the nonlinear product $dJ^{-1} \wedge dJ$ in (4.1) into an iteration scheme which closes in L^p spaces. The idea is to separate this term from the iteration scheme by using the gauge freedom v in the A equation (4.4) to consolidate $\tilde{\Gamma}$ and A into a single variable B, and thereby uncouple equations (4.2) - (4.4) for J from equation (4.1) for $\tilde{\Gamma}$. Defining

$$B \equiv A + \langle dJ; \tilde{\Gamma} \rangle, \tag{4.8}$$

$$w \equiv v + \delta \langle dJ; \tilde{\Gamma} \rangle, \tag{4.9}$$

observe now that we can write (4.2) - (4.4) as

$$\Delta J = \delta(J \cdot \Gamma) - B, \qquad (4.10)$$

$$d\vec{B} = \overrightarrow{\operatorname{div}}(dJ \wedge \Gamma) + \overrightarrow{\operatorname{div}}(J \, d\Gamma), \qquad (4.11)$$

$$\delta \vec{B} = w. \tag{4.12}$$

Equations (4.10)-(4.12) are the reduced RT-equations, c.f. (1.6)-(1.8). Since the transformation from v to w can be viewed as a gauge transformation, the gauge freedom of the RT-equations implies that we can ignore the dependence of $\tilde{\Gamma}$ on w, and view w as the independent gauge freedom in the RT-equations. Therefore equations (4.10) - (4.12) decouple the equations for J and B from $\tilde{\Gamma}$, and hence from the first RT-equation (4.1), which in terms of B becomes

$$\Delta \tilde{\Gamma} = \delta d\Gamma - \delta \left(dJ^{-1} \wedge dJ \right) + d \left(J^{-1} (B - \langle dJ; \tilde{\Gamma} \rangle) \right).$$
(4.13)

By this decoupling the Jacobians J which map Γ to optimal regularity can be constructed independently of $\tilde{\Gamma}$, (which we can now discard), by solving the reduced RT-equations (4.10) - (4.12) alone.

So, discard the original $\tilde{\Gamma}$, and assume (J, B) is a solution of the reduced RT-equations (4.10) - (4.12) with $J \in W^{1,2p}(\Omega'_x)$ and $B \in L^{2p}(\Omega'_x)$. The goal now is to use the solutions J, B of the reduced RT-equations to establish optimal regularity without reference to the original $\tilde{\Gamma}$. To show such a Jacobian J indeed maps Γ to optimal regularity, it suffices to prove that $\tilde{\Gamma} = \tilde{\Gamma}_J$ provides *another* solution of (4.13). In our next theorem we show that, incredibly enough, this is true: $\tilde{\Gamma} = \tilde{\Gamma}_J$ is an exact solution of the elliptic equation (4.13), an equation we could not solve by our previous methods at the low regularity $\Gamma, d\Gamma \in L^{\infty}$. The equation (4.13) establishes the requisite smoothness $\tilde{\Gamma}_J \in W^{1,p}$ when $\Gamma, d\Gamma \in L^{\infty}$.

To complete the circle, we now explain how to recover a solution of the full RT-equations form (4.10) - (4.12), when $\tilde{\Gamma}$ is replaced by $\tilde{\Gamma}_J$. For this, we need only show that $J, \tilde{\Gamma}_J$ solve the original RT-equations with a different choice of gauge A', v'. Reversing the above steps using $\tilde{\Gamma}_J$ in place of $\tilde{\Gamma}$, it follows that the back change of gauge

$$A' \equiv B - \langle dJ; \tilde{\Gamma}_J \rangle, \qquad (4.14)$$

$$v' \equiv w - \delta \langle dJ; \tilde{\Gamma}_J \rangle,$$
 (4.15)

takes a solution (J, B) of the reduced RT-equations back to a solution of the original RT-equations with the same J, but with $\tilde{\Gamma}$ replaced by $\tilde{\Gamma}_J$. These are recorded in parts (i) and (ii) of Theorem 4.1 below, which states that (J, Γ_J, A') defined in (4.6) - (4.14) indeed solves the full RT-equations (4.1) - (4.4), and, by this, $\tilde{\Gamma}_J$ has the requisite smoothness $\tilde{\Gamma}_J \in W^{1,p}(\Omega)$. Part (*iii*) of Theorem 4.1 establishes an estimate for $\tilde{\Gamma}_J$ from which we deduce the uniform $W^{1,p}$ bound (2.6) on Γ_{y} in Theorem 2.1, the bound that underlies Uhlenbeck compactness. The existence of solutions (J, B) of the reduced RT-equations, satisfying estimate (4.16), which are assumed in Theorem 4.1, are shown to exist in Theorem 4.3 below. For the low regularities considered in this paper, we need to establish the above equivalence and existence theory in a weak sense. Serendipitously, the RT-equations allow for a weak formulation because all lowest regularity terms on the right hand side have derivatives δ or d on them, making them amenable to integration by parts, (as in the theory of conservation laws [26]). This is accomplished in Sections 8 - 11.

Theorem 4.1. Assume Γ , $d\Gamma \in L^{\infty}(\Omega)$ in x-coordinates, and let n . Assume <math>(J, B) solves the reduced RT-equations (4.10) - (4.12) in a weak sense for some w on an open set $\Omega' \subset \Omega$, such that $J, J^{-1} \in W^{1,2p}(\Omega'_x)$ and $B \in L^{2p}(\Omega'_x)$. Then the following holds:

(i) $\tilde{\Gamma}_J$ solves (4.13), where $\tilde{\Gamma}_J$ is defined in (4.6). The tuple $(J, \tilde{\Gamma}_J, A')$ solves the full RT-equations (4.1) - (4.4) in Ω' for v = v', where A' and v' are defined in (4.14) and (4.15).

(ii) The regularity of $\tilde{\Gamma}_J$ is given by $\tilde{\Gamma}_J \in W^{1,p}(\Omega''_x)$ for any open set Ω'' compactly contained in Ω' , i.e., $\tilde{\Gamma}_J$ is one order more regular than the two terms are separately on the right hand side of its definition in (4.6).

(iii) Let M > 0 be a constant such that $\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x)} \leq M$, where $\|(\Gamma, d\Gamma)\|_{L^{\infty}}$ is defined in (2.2). Assume that (J, B) satisfies further the estimate¹⁵

$$\|I - J\|_{W^{1,2p}(\Omega'_x)} + \|I - J^{-1}\|_{W^{1,2p}(\Omega'_x)} + \|B\|_{L^{2p}(\Omega'_x)} \leq C_2(M) \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega'_x)},$$
(4.16)

for some constant $C_2(M) > 0$ depending only on Ω'_x , n, p and M. Then, on any open set Ω'' compactly contained in Ω' , $\tilde{\Gamma}$ satisfies the uniform bound

$$\|\overline{\Gamma}_J\|_{W^{1,p}(\Omega_x'')} \le C_3(M) \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x')}$$

$$(4.17)$$

where $C_3(M) > 0$ is some constant depending only on $\Omega''_x, \Omega'_x, n, p$ and M.

The key step in the proof of 4.1 is establish in Lemma 7.2 below, by proving that (4.6) is an exact formula for the solution $\tilde{\Gamma} = \tilde{\Gamma}_J$ of the first RTequation (4.1), from which the regularity gain of $\tilde{\Gamma}_J$ in (*ii*) can be deduced. To give the argument in its essence, we assume one more level of smoothness in Lemma 7.2. More care is required to extend the argument of Lemma 7.2 to the low regularities of Theorem 4.1 and prove the theorem rigorously, which is the subject of Section 9.¹⁶ Assuming only that (*ii*) of Theorem 4.1 holds, the equivalence of optimal regularity and the reduced RT-equations, in the spirit of our previous paper [21], can now be established as a corollary. This reduces the problem of optimal regularity to an existence theorem for the reduced RT-equations. ¹⁷

Corollary 4.2. Assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in x-coordinates, and let $n and <math>q \in \Omega$. Then there exists a neighborhood $\tilde{\Omega} \subset \Omega$ of q and a coordinate transformation $x \to y$ such that the connection components Γ_y in y-coordinates have optimal regularity $\Gamma_y \in W^{1,p}(\tilde{\Omega}_y)$ if and only if there exists a weak solution (J, B) of the reduced RT-equations (4.10) - (4.12), defined on some neighborhood Ω' of q, with $J, J^{-1} \in W^{1,2p}(\Omega'_x), B \in L^{2p}(\Omega'_x)$ and $d\vec{J} = 0$ in Ω'_x . The Jacobian of the coordinate transformation $x \to y$ is $dy = J \in W^{1,2p}(\Omega'_x)$.

¹⁵Estimate (4.16) bounds J and J^{-1} , but is expressed in terms of I - J and $I - J^{-1}$ to reflect the fact that J typically tends to the identity as M tends to zero.

¹⁶We actually prove Theorem 4.1 under the weaker assumption $\Gamma, d\Gamma \in L^{2p}(\Omega), p > n$.

¹⁷Although equation (4.1) can be bypassed for constructing solutions, equation (4.1) is required to prove optimal connection regularity in the coordinate system introduced by Jand is therefore a vital part of the RT-equations. Note also that one can use an underlying Cauchy-Riemann-type equation for $\tilde{\Gamma}$ instead of (4.1) and establish optimal regularity by applying Gaffney's inequality, but we prefer the Poisson type equation (4.1).

Proof. The forward implication of Corollary 4.2 is straightforward because the reduced RT-equations are derived from the full RT-equations which are in turn deduced from the Riemann-flat condition, a condition equivalent to optimal regularity, c.f. [21]. That is, given the Jacobian J and resulting connection Γ_y of optimal regularity, and defining

$$\tilde{\Gamma}_{ij}^k = (J^{-1})^k_{\gamma} J^{\alpha}_i J^{\beta}_j (\Gamma_y)^{\gamma}_{\alpha\beta},$$

then $\tilde{\Gamma}$ satisfies the Riemann-flat condition. It is now straightforward to check that $(\tilde{\Gamma}, J)$ solves the RT-equation for some A, and, defining B by (4.8), that (J, B) solves the reduced RT-equations (4.10) - (4.12). Recall that the condition $d\vec{J} = 0$ is equivalent to J being the Jacobian of a coordinate transformation, c.f. Theorem C.2.

To prove the reverse implication assume part (*ii*) of Theorem 4.1 holds. By Theorem (4.1), $\tilde{\Gamma}_J$ defined by (4.6) is in $W^{1,p}(\Omega'_x)$ and $\tilde{\Gamma}_J$ solves the first RT-equation (4.13) in terms of the solution (J, B) of the reduced RT-equations (4.10) - (4.12). Let $x \to y$ be a coordinate transformation with Jacobian dy = J. Now define the connection Γ_y in terms of $\tilde{\Gamma}_J$ in x-coordinates by (4.7), that is,

$$(\Gamma_y)^{\gamma}_{\alpha\beta} = J^{\gamma}_k (J^{-1})^i_{\alpha} (J^{-1})^j_{\beta} (\tilde{\Gamma}_J)^k_{ij}.$$
(4.18)

Since $\tilde{\Gamma}_J \in W^{1,p}(\Omega'_x)$ and $J, J^{-1} \in W^{1,2p}(\Omega'_x)$, p > n, Morrey's inequality implies that the components $\Gamma_y(x) \in W^{1,p}(\Omega'_x)$ in *x*-coordinates, and therefore when expressed in *y*-coordinates $\Gamma_y \in W^{1,p}(\Omega'_y)$. Substituting the definition of $\tilde{\Gamma}_J$ in (4.6) into (4.18) implies that

$$\begin{aligned} (\Gamma_y)_{\alpha\beta}^{\gamma} &= J_k^{\gamma} (J^{-1})_{\alpha}^i (J^{-1})_{\beta}^j \big(\Gamma_x - J^{-1} dJ \big)_{ij}^k \\ &= J_k^{\gamma} (J^{-1})_{\alpha}^i (J^{-1})_{\beta}^j (\Gamma_x)_{ij}^k - (J^{-1})_{\alpha}^i (J^{-1})_{\beta}^j \partial_i J_j^{\gamma}, \end{aligned}$$

from which we conclude that Γ_y are the connection components Γ_x transformed to a coordinate system y in which Γ exhibits optimal regularity, $\Gamma_y \in W^{1,p}(\Omega)$. This completes the proof.

Finally, to obtain the optimal regularity result stated in Theorem 2.1, together with the uniform estimate (2.6), we require the following theorem which establishes the existence of solutions to the reduced RT-equations satisfying the assumptions $J, J^{-1} \in W^{1,2p}(\Omega'_x), B \in L^{2p}(\Omega'_x)$ of Theorem 4.1, together with the additional estimate (4.16). (The existence theory is worked out in fixed x-coordinates, so we omit subscript x on Γ and Ω .)

Theorem 4.3. Assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in x-coordinates, and let M > 0 be a constant such that $\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x)} \leq M$, and let $n , <math>q \in \Omega$. Then there exists a neighborhood $\Omega' \subset \Omega$ of q, depending only on Ω, n, p, q, M , and there exists $J \in W^{1,2p}(\Omega')$ and $B \in L^{2p}(\Omega')$ such that (J, B) solves the reduced RT-equations (4.10) - (4.12) in a weak sense in Ω' . Moreover, J is invertible with $J^{-1} \in W^{1,2p}(\Omega')$, $d\vec{J} = 0$ in Ω' and (J, B) satisfies the uniform bound (4.16).

Theorem 4.1 in combination with the existence Theorem 4.3, imply our main result regarding optimal regularity, Theorem 2.1, as shown in Section 5 below. The proof of Theorem 4.3 is the main technical step in this paper and is presented in Sections 10 and 11. Note that the boundary data (4.5) is ill-defined at the low regularity $J \in W^{1,2p}$. So to make sense of this, we augmenting the reduced RT-equations with auxiliary elliptic PDE's for y, which allows us to replace (4.5) by Dirichlet data for J = dy, data one degree more regular than (4.5) and thus well-defined. By Theorem 4.1, Theorem 4.3 also establishes the first existence theory for the full RT-equations at the low regularities $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ associated with GR shock waves.

To summarize, the RT-equations reduce the nonlinear problem of regularizing connections to a linear existence problem for constructing the Jacobian J via the reduced RT-equations, followed by a regularity boost for $\tilde{\Gamma}_J$ provided by the first RT-equation (4.1). So for applications one only has to solve the linear reduced RT-equations for J to obtain the regularizing coordinate transformation, and our iteration scheme in Section 10 provides an algorithm for doing this. Although linear, the low regularity of the coefficients of the reduced RT-equations is still an issue in the existence theory below.

5. Proof of Theorems 2.1 and 2.3

In this section we give the proofs of our main results stated in Section 2, Theorem 2.1 on optimal regularity and Theorem 2.3 on Uhlenbeck compactness, assuming Theorems 4.1 and (4.3). The proof of Theorems 4.1 and (4.3) are the subject of Sections 7 - 11.

5.1. **Proof of Theorem 2.1.** Assume Theorems 4.1 and 4.3 hold, and assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in *x*-coordinates satisfy $\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega_x)} < M$ for some constant M > 0, and let $n , and <math>q \in \Omega$. To prove Theorem 2.1, it suffices to prove there exists a coordinate transformation $x \to y$ defined on a neighborhood Ω' of q such that $J = \frac{\partial y}{\partial x} \in W^{1,2p}(\Omega'_x)$ and

$$\|\Gamma_y\|_{W^{1,p}(\Omega''_u)} \le C_1(M) \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega'_x)},\tag{5.1}$$

for any Ω'' compactly contained in Ω' , and for some constant $C_1(M) > 0$ depending only on $\Omega''_x, \Omega'_x, p, n$ and M, while Ω'_x depends only on Ω_x, p, n, q and M.

By Theorem 4.3 there exists a solution (J, B) of the reduced RT-equations (4.10) - (4.12) defined in Ω'_x containing q, such that $J \in W^{1,2p}(\Omega'_x)$, $J^{-1} \in W^{1,2p}(\Omega'_x)$, $B \in L^{2p}(\Omega'_x)$, $d\vec{J} = 0$ in Ω'_x , and the uniform bound (4.16) holds. The condition $d\vec{J} \equiv \text{Curl}(J) = 0$ implies that J is integrable to a coordinate system, c.f. Theorem C.2 and [21, 23]. Theorem 4.1 states that J is the Jacobian of a coordinate transformation $x \to y$ such that the connection Γ_y has optimal regularity $\Gamma_y \in W^{1,p}(\Omega''_y)$, for any compactly contained subset Ω'' of Ω' . Moreover, part (*iii*) of Theorem 4.1 implies the uniform bound

(4.17) on Γ , and this gives the bound (5.1) on Γ_y by the relation between $\tilde{\Gamma}_J$ and Γ_y given in (4.7). That is, by

$$(\Gamma_y)^{\gamma}_{\alpha\beta} = J^{\gamma}_k (J^{-1})^i_{\alpha} (J^{-1})^j_{\beta} (\tilde{\Gamma}_J)^k_{ij}$$

which implies

$$\|\Gamma_y\|_{W^{1,p}(\Omega_u'')} \leq C \|J\|_{W^{1,p}} \|J^{-1}\|_{W^{1,p}}^2 \|\tilde{\Gamma}_J\|_{W^{1,p}},$$

where C > 0 is some constant resulting from applying Morrey's inequality several times. The sought after estimate (5.1) now follows by estimating $\tilde{\Gamma}$ by (4.17) and J, J^{-1} by (4.16). This completes the proof of Theorem 2.1, once we give the proofs of Theorems 4.1 and 4.3 in Sections 9 - 11 below.

5.2. **Proof of Theorem 2.3.** For the proof we use Morrey's inequality, the Banach-Alaoglu Theorem, Sobolev compactness and the following technical lemma which says products of strongly and weakly converging functions converge weakly. Recall that Morrey's inequality tells us that when p > n, functions uniformly bounded in $W^{1,p}$ are also uniformly bounded in L^{∞} , so $W^{1,p}$ is closed under products, and uniform bounds in $W^{1,p}$ norms extend to uniform bounds on products. Banach-Alaoglu tells us that the closed unit ball in L^p is weakly compact [24]. These together with Sobolev compactness and the boundedness of Ω tell us that sequences of functions uniformly bounded in $W^{1,p}$, strongly in L^p , with uniform bounds given by the original uniform $W^{1,p}$ and L^{∞} bounds.

Lemma 5.1. Let f_i, g_i be sequences of functions on a bounded set Ω such that $f_i \to f$ in $L^p(\Omega)$ with $||f_i||_{\infty}, ||f||_{\infty} \leq M$, and such that $g_i \to g$ weakly in L^p with $||g_i||_{L^p}, ||g||_{L^p} \leq M_p, p > n$. Then $f_ig_i \to fg$ weakly in $L^p(\Omega)$.

Proof. Since $||f_i||_{\infty}, ||f||_{\infty} \leq M$, and $f_i \to f$ in L^p on a bounded set, it follows that $f_i \to f$ in every L^p , $p \geq 1$. This follows by measure theory because the measure of the set on which $|f_i - f| > \epsilon$ tends to zero as $i \to \infty$ for every $\epsilon > 0$. Recall now that the dual space of L^p is L^{p^*} with $1/p + 1/p^* = 1$. Thus to prove that $f_i g_i - fg \to 0$ weakly in L^p , we must show that

$$\langle f_i g_i - f g, \phi \rangle_{L^2} \equiv \int_{\Omega} (f_i g_i - f g) \phi \to 0$$
 (5.2)

for every $\phi \in L^{p^*}$, where $\langle \cdot, \cdot \rangle_{L^2}$ is the L^2 inner product. But

$$\langle f_i g_i - f g, \phi \rangle_{L^2} = \langle f(g_i - g), \phi \rangle_{L^2} + \langle (f_i - f)g_i, \phi \rangle_{L^2}.$$
(5.3)

But $f \in L^{\infty}$ implies $f\phi \in L^{p^*}$, so the first term in (5.3) satisfies

$$\langle f(g-g_i), \phi \rangle_{L^2} = \langle (g-g_i), f\phi \rangle_{L^2} \to 0$$

because $g_i - g$ tends to zero weakly in L^p .

24

Consider now the second term in (5.3). Since $f_i - f \in L^{\infty}$, we have $(f_i - f)\phi \in L^{p^*}$, so we can apply Holder's inequality twice to obtain the estimate

$$\begin{aligned} |\langle (f_i - f)g_i, \phi \rangle_{L^2}| &= |\langle g_i, (f_i - f)\phi \rangle_{L^2}| \\ &\leq ||g_i||_{L^p} ||(f - f_i)\phi||_{L^{p^*}} \\ &\leq M_p ||(f - f_i)\phi||_{L^{p^*}}. \end{aligned}$$
(5.4)

Now let $E_N = \{x \in \Omega : |\phi|^{p^*} \ge N\}$. Then since $\phi^{p^*} \in L^1$, it follows that $\int_{E_N} |\phi|^{p^*} \to 0$ as $N \to \infty$. Thus

$$\begin{aligned} \|(f-f_{i})\phi\|_{L^{p^{*}}}^{p^{*}} &= \int_{\Omega} |f-f_{i}|^{p^{*}} |\phi|^{p^{*}} d\mu \qquad (5.5) \\ &= \int_{E_{N}} |f-f_{i}|^{p^{*}} |\phi|^{p^{*}} d\mu + \int_{E_{N}^{c}} |f-f_{i}|^{p^{*}} |\phi|^{p^{*}} d\mu \\ &\leq \|(f-f_{i})^{p^{*}}\|_{L^{\infty}} \int_{E_{N}} |\phi|^{p^{*}} d\mu + N \int_{E_{N}^{c}} |f-f_{i}|^{p^{*}} d\mu \\ &\leq (2M)^{p^{*}} \int_{E_{N}} |\phi|^{p^{*}} d\mu + N \int_{\Omega} |f-f_{i}|^{p^{*}} d\mu. \end{aligned}$$

Now we can make the first term arbitrarily small by choosing N sufficiently large, and the second term tends to zero with *i* because $f_i \to f$ in $L^{p^*}(\Omega)$. It follows that $\|(f - f_i)\phi\|_{L^{p^*}} \to 0$, and by this we conclude from (5.4) that the second term in (5.3) tends to zero as well. Thus $f_ig_i \to fg$ weakly in L^p as claimed. \Box

We can now give the proof of Theorems 2.3 assuming Theorem 4.1 and 4.3. So assume $\{(\Gamma_i)_x\}_{i\in\mathbb{N}}$ are the *x*-components of a sequence of connections Γ_i defined on the tangent bundle $T\mathcal{M}$ of an *n*-dimensional manifold \mathcal{M} in a fixed coordinate system *x*, let $n , and assume <math>(\Gamma_i)_x, d((\Gamma_i)_x) \in L^{\infty}(\Omega_x)$, such that

$$\|(\Gamma_i, d\Gamma_i)\|_{L^{\infty}(\Omega_x)} \le M,\tag{5.6}$$

for some constant M > 0 independent of $i \in \mathbb{N}$. We need to prove that for each $q \in \Omega$ there exists a fixed neighborhood $\Omega' \subset \Omega$ of q, and for each $(\Gamma_i)_x$ there exists a coordinate transformation $x \to y_i(x)$ taking Ω'_x to Ω'_{y_i} , such that the components $\Gamma_{y_i} \equiv (\Gamma_i)_{y_i}$ of Γ_i in y_i -coordinates exhibit optimal regularity $\Gamma_{y_i} \in W^{1,p}(\Omega'_{y_i})$, both in y_i -coordinates, and when expressed in x-coordinates $\Gamma_{y_i}(x) \equiv \Gamma_{y_i}(y_i(x)) \in W^{1,p}(\Omega'_x)$. We need to prove further that a subsequence of $y_i(x)$ converges to some y(x) weakly in $W^{2,2p}(\Omega'_x)$, strongly in $W^{1,2p}(\Omega'_x)$, and that a further subsequence $\Gamma_{y_i}(x)$ converges to $\Gamma_y(x)$ weakly in $W^{1,p}(\Omega'_x)$, strongly in $L^p(\Omega'_x)$, and that Γ_y is the connection Γ_x in y-coordinates, where Γ_x is the weak L^p -limit of $(\Gamma_i)_x$.

By Theorem 2.1, there exists a single neighborhood Ω' depending only on M, (assuming n, p, Ω fixed), on which a coordinate transformation $x \to y_i(x)$

exists for each $i \in \mathbb{N}$, taking Ω'_x to Ω'_{y_i} , which maps $(\Gamma_i)_x$ to optimal regularity on Ω' , so $\Gamma_{y_i} \in W^{1,p}(\Omega'_{y_i})$. Moreover, by estimate (2.6) of Theorem 2.1, $\|\Gamma_{y_i}\|_{W^{1,p}(\Omega'_{y_i})}$ are uniformly bounded. This proves (i) of Theorem 2.3.

For each $i \in \mathbb{N}$, the coordinate transformations $y_i(x)$ are obtained from the Jacobians constructed in Theorem 4.3. To obtain a uniform L^p -bound on $y_i(x)$, we choose $y_i(x(q)) = 0$ for each $i \in \mathbb{N}$. It follows from (4.16) of Theorem 4.3 that the Jacobians J_i of $x \to y_i(x)$ satisfy the uniform bound $\|J_i\|_{W^{1,2p}(\Omega'_x)} \leq C_2(M)$ and $\|J_i^{-1}\|_{W^{1,2p}(\Omega'_x)} \leq C_2(M)$. Since the Jacobians bound the derivatives of the coordinate maps $x \to y_i(x)$, and $y_i(x(q)) = 0$ bounds the L^p -norm (and supnorm), it follows that each map $y_i(x)$ as a function of x is uniformly bounded in $W^{2,2p}(\Omega'_x)$ by some constant $C_4(M)$, again depending only on M. It now follows from the basic compactness theorem for Sobolev spaces that there exists a subsequence, also denoted $y_i(x)$, on which $y_i(x)$ converges to y(x) weakly in $W^{2,2p}(\Omega'_x)$, strongly in $W^{1,2p}(\Omega'_x)$, such that $\|y\|_{W^{2,2p}(\Omega'_x)} < C_4(M)$. In particular, J_i converges to J weakly in $W^{1,2p}(\Omega'_x)$, strongly in $L^{2p}(\Omega'_x)$, and the uniform bound on J_i^{-1} implies invertibility of J. This proves (*iii*) of Theorem 2.3.

Now since $J_i \in W^{1,2p}(\Omega'_x)$, $\Gamma_{y_i}(x)$ are uniformly bounded in $W^{1,p}(\Omega'_x)$ by the chain rule. That is, by Morrey's inequality one can estimate products $\Gamma_{y_i}(x)$ times J_i to lie in $W^{1,p}(\Omega'_x)$, with norm bounded by some $C_5(M)$ depending only on M. This proves (*ii*) of Theorem 2.3.

By the uniform $W^{1,p}$ -bound on $\Gamma_{y_i}(x)$, it follows that a further subsequence of $\Gamma_{y_i}(x)$ converges weakly in $W^{1,p}(\Omega'_x)$ to a connection $\Gamma_y(x)$ which satisfies the same bound $C_5(M)$ in $W^{1,p}(\Omega'_x)$. Thus the coordinate map $x \to y$ is in $W^{2,2p}(\Omega'_x)$, and so Γ_y exhibits optimal regularity in ycoordinates.

Finally, by taking a further subsequence, $(\Gamma_i)_x$ converges to some Γ_x weakly in $L^p(\Omega_x)$ by the Banach Alaoglu Theorem, (i.e., the uniform L^{∞} bound (5.6) directly implies a uniform L^p -bound because Ω is bounded). To show that Γ_y is indeed the connection Γ_x in y-coordinates, we use that for each i, Γ_{y_i} is the connection $(\Gamma_i)_x$ in y_i -coordinates, so by the transformation law for connections (written in shorthand, suppressing indices) we have

$$J_i^{-1} \cdot J_i \cdot J_i \cdot \Gamma_{y_i} = (\Gamma_i)_x - J_i \, dJ_i.$$
(5.7)

Since J_i converges to J weakly in $W^{1,2p}(\Omega'_x)$, J_i converges to J strongly in L^p and dJ_i converges to dJ weakly in L^p . Similarly, J_i^{-1} converges to J^{-1} and $\Gamma_{y_i}(x)$ to $\Gamma_y(x)$ strongly in L^p and weakly in $W^{1,2p}(\Omega'_x)$, $W^{1,p}(\Omega'_x)$, respectively. Thus by Lemma 5.1, the left hand side of (5.7) converges to $J^{-1} \cdot J \cdot J \cdot \Gamma_y$ weakly in L^p , and the right hand side of (5.7) converges to $\Gamma_x - J \, dJ$ weakly in L^p . Taken on whole, (5.7) and Lemma 5.1 imply that the connection Γ_y is the connection Γ_x transformed to y-coordinates as L^p functions, which proves (*iv*) and completes the proof of Theorem 2.3. In this section we completed the proofs of our main results, Theorem 2.1 and 2.3, assuming Theorems 4.1 and (4.3). The remainder of the paper is devoted to a proof of these two theorems, in Sections 7 - (11), and applications of our main results are presented in Sections 6.3 and (6.1).

6. Applications of the main results

6.1. Application of Uhlenbeck compactness to the vacuum Einstein equations. As an application of Uhlenbeck compactness for Γ , Riem $(\Gamma) \in L^{\infty}$, we prove the following corollary of Theorem 2.3 which provides a new compactness theorem applicable to vacuum solutions of the Einstein equations. The main difficulty in a convergence proof for a PDE in an existence theory, is typically the problem of establishing a uniform bound on the highest order derivatives, suitable to apply Sobolev compactness. Uhlenbeck compactness tells us that it suffices to establish a bound on just the Riemann curvature, not all highest order derivatives of a connection, in order to imply subsequential convergence of connection and metric.

Corollary 6.1. Let g_i be a sequence of Lipschitz continuous metrics given on a manifold \mathcal{M} , and let Γ_i denote the Christoffel symbols of g_i for each $i \in \mathbb{N}$. Assume that $(g_i)_{i \in \mathbb{N}}$ is a sequence of approximate solutions of the vacuum Einstein equations such that, in a neighborhood of each point, there exists a coordinate system x in which $\operatorname{Ric}(g_i) \to 0$ weakly in L^p (some pwith $n), the sequence <math>\operatorname{Ric}(g_i)$ is uniformly bounded in L^∞ , and g_i satisfies the uniform bound

$$\|g_i\|_{L^{\infty}} + \|\Gamma_i\|_{L^{\infty}} + \|\operatorname{Weyl}(g_i)\|_{L^{\infty}} \le M \tag{6.1}$$

for some constant M > 0, together with the non-degeneracy condition that $|\det(g_i)|$ is uniformly bounded away from zero.¹⁸ Then, in each such coordinate system, there exists a subsequence of $(g_i)_{i\in\mathbb{N}}$ which converges componentwise and weakly in $W^{1,p}(\Omega)$ to some metric g which satisfies (6.1) and solves the vacuum Einstein equations $\operatorname{Ric}(g) = 0$. Furthermore, according to Theorem 2.1, for each $p \in (n, \infty)$ there exists locally, (i.e., in a neighborhood of each point), a $W^{2,2p}$ coordinate transformation $x \to y_i$ which lifts the components of g to $W^{2,p}$ and these are the $W^{2,p}$ -limits of g_{y_i} , the components of q_i in optimal coordinates y_i , as in (ii) of Theorem 2.3.

Note that if \mathcal{M} is a compact manifold, one can cover \mathcal{M} with a finite number of such optimal coordinate patches, and by a diagonal argument, extract a subsequence which converges to a solution of the vacuum Einstein equations in each coordinate system of the finite covering, and hence in all of \mathcal{M} .

Proof. To begin, note that the L^{∞} bound on Γ_i provides an L^{∞} bound on the derivatives of g_i , so the sequence $(g_i)_{i \in \mathbb{N}}$ is uniformly bounded in

¹⁸Here norms are taken in coordinate systems and $Weyl(g_i)$ denotes the Weyl curvature tensor of g_i .

 $W^{1,p}$, any $p \in (1,\infty)$. Thus the weak $W^{1,p}$ convergence of a subsequence of $(g_i)_{i\in\mathbb{N}}$ to some metric $g \in W^{1,p}$ in x-coordinates follows by the Banach Alaoglu Theorem for any $p \leq \infty$; (that g is a metric follows by our nondegeneracy assumption). But this is not enough to conclude that $\operatorname{Ric}(g) = 0$, because the convergence is not strong enough to pass weak limits through $\operatorname{Ric}(g_i)$. To prove that $\operatorname{Ric}(g) = 0$ we apply now Uhlenbeck compactness of Theorem 2.3. Since $\operatorname{Ric}(g) = 0$ is a point-wise condition, we assume without loss of generality that the coordinate transformations to optimal regularity $(x \to y_i)$, asserted to exist by Theorem 2.3, are defined on the entire coordinate patch in which each g_i is given.

Note first that assuming a uniform L^{∞} bound on $\operatorname{Ric}(g_i)$ and $\operatorname{Weyl}(g_i)$ implies that $\|\operatorname{Riem}(g_i)\|_{L^{\infty}}$ is uniformly bounded, since the Ricci tensor together with the Weyl tensor comprise the Riemann curvature tensor [5]. Thus, also taking into account the bound on the metric and connection in (6.1), Theorem 2.3 applies and yields the existence of a convergent subsequence of $(g_i)_{i \in \mathbb{N}}$, asserting weak $W^{2,p}$ and strong $W^{1,p}$ convergence, (any pwith $n). Namely, let <math>y_i$ be a coordinate system in which Γ_i and hence g_i has optimal regularity, and denote by g_{y_i} the metric $g_i(x)$ in y_i coordinates but with its components expressed as functions over x-coordinates, c.f. (ii) of Theorem 2.3. Then $g_{y_i} \in W^{2,p}$ in x-coordinates and

$$\|g_{y_i}\|_{W^{2,p}} \le \|g_{y_i}\|_{L^p} + \|\Gamma_{y_i}\|_{W^{1,p}}$$

is bounded uniformly by some constant C(M) > 0, since $\|\Gamma_{y_i}\|_{W^{1,p}}$ and the Jacobians $\|\frac{\partial y_i}{\partial x}\|_{W^{1,p}}$ are both uniformly bounded, c.f. (4.16) and (4.17). The asserted convergence of a subsequence now follows by the Banach Alaoglu Theorem. We denote this convergent subsequence by $(g_{y_i})_{i\in\mathbb{N}}$, where $g_{y_i} \equiv g_{y_i}(x)$ is to be understood as the metric in y_i -coordinates with components expressed in x-coordinates.

The main point then is that the curvature is linear in derivatives, and one can pass weak limits through such derivatives. That is, by assumption $\operatorname{Ric}(g_i) \to 0$ weakly in L^p for some $p \in (n, \infty)$ as $i \to \infty$, and to prove that the limit metric $g \equiv \lim_{i \to \infty} g_i$ solves the vacuum Einstein equations, we need only show that $\operatorname{Ric}(g_i)$ converges to $\operatorname{Ric}(g)$ weakly in L^p as $i \to \infty$. For this, observe that the weak $W^{1,p}$ convergence of Γ_i implies weak L^p convergence of $d\Gamma_i$ to $d\Gamma$, where Γ denotes the connection of g. Moreover, the strong L^p convergence of Γ_i implies strong convergence of $\Gamma_i \wedge \Gamma_i$ to $\Gamma \wedge \Gamma$ in $L^{\frac{p}{2}}$. This implies weak convergence of the Riemann curvature, namely for any matrix valued 2-form $\Psi \in W_0^{1,(p/2)^*} \subset W_0^{1,p^*}$ we have

$$\begin{array}{lll} \langle \operatorname{Riem}(\Gamma_i), \Psi \rangle_{L^2} &=& -\langle \Gamma_i, \delta \Psi \rangle_{L^2} + \langle \Gamma_i \wedge \Gamma_i, \Psi \rangle_{L^2} \\ & \stackrel{i \to \infty}{\longrightarrow} & -\langle \Gamma, \delta \Psi \rangle_{L^2} + \langle \Gamma \wedge \Gamma, \Psi \rangle_{L^2} = \langle \operatorname{Riem}(\Gamma), \Psi \rangle_{L^2}, \end{array}$$

which implies weak convergence in $L^{\frac{p}{2}}$ by denseness of $W_0^{1,(p/2)^*}$ in L^{p^*} . Since this applies to any p > n, we conclude that $\operatorname{Riem}(\Gamma_i)$ converges weakly to $\operatorname{Riem}(\Gamma)$ in L^p which implies the sought after convergence, $\operatorname{Ric}(q_i) \to \operatorname{Ric}(q)$ weakly in L^p . This proves g solves the vacuum Einstein equations $\operatorname{Ric}(g) = 0$ and this holds in any coordinate system by tensor transformation.

Note that without Uhlenbeck compactness Theorem 2.3, the uniform L^{∞} bound on a sequence of metric connections and their curvatures would not in general imply that the limit metric solves the vacuum Einstein equations. Indeed, weak L^p convergence of a subsequence of the metric connections is not in general sufficient to pass weak limits through nonlinear functions like products [7, Chapter 16]. As a result, even though the Ricci tensor would correctly converge to zero, the limit Ricci tensor would in general fail to be the Ricci tensor of the limit connection.

6.2. Optimal regularity in spherically symmetric spacetimes. The following corollary of Theorem 2.1 establishes for the first time that solutions of the Einstein equations constructed in Standard Schwarzschild Coordinates, including the Lipschitz continuous metrics associated with shock waves in [13], can always be smoothed to optimal regularity by coordinate transformation. Solutions of the Einstein equations in SSC have a long history in General Relativity going back to Schwarzschild and Birkhoff. The existence theory in [13] establishes (weak) shock wave solutions of the Einstein-Euler equations by Glimm's method, (see also [3]). The Einstein-Euler system couples the unknown metric g_{ij} to the unknown density ρ , pressure p and velocity u of a perfect fluid via $T^{ij} = (\rho + p)u^iu^j + pg^{ij}$ in $G = \kappa T$. The spacetime metrics of these solutions are non-optimal with curvature in L^{∞} , but optimal metric regularity would be required to introduce locally inertial frames and geodesic curves by standard methods.

For this consider a metric in Standard Schwarzschild Coordinates (SSC)

$$ds^{2} = -B(t,r)dt^{2} + \frac{dr^{2}}{A(t,r)} + r^{2}d\Omega^{2}.$$
(6.2)

This represents the coordinates in which the Einstein equations for a spherically symmetric spacetime metric (arguably) take their simplest form. Since the first three Einstein equations in SSC are

$$-rA_r + (1-A) = \kappa BT^{00}r^2 \tag{6.3}$$

$$A_t = \kappa B T^{01} r \tag{6.4}$$

$$r\frac{B_r}{B} - \frac{1-A}{A} = \frac{\kappa}{A^2}T^{11}r^2$$
 (6.5)

the metric can generically be only one level more regular than the curvature tensor, at every level of regularity, and is hence non-optimal. (See [13] for the full system of equations.) As an application of Theorem 2.1, we have the following result which establishes that shock wave solutions of the Einstein equations constructed by the Glimm scheme are one order more regular than previously known [13]. (The result here extends to every level of regularity, c.f. [23].)

Corollary 6.2. Let $T \in L^{\infty}$ and assume $g \equiv (A, B)$ is a (weak) solution of the Einstein equations in SSC satisfying $g \in C^{0,1}$ and hence $\Gamma \in L^{\infty}$ in an open set Ω . Then for any p > 4 and any $q \in \Omega$ there exists a coordinate transformation $x \to y$ defined in a neighborhood of q, such that, in y-coordinates, $g \in W^{2,p}$, $\Gamma \in W^{1,p}$.

Proof. In Standard Schwarzschild coordinates the Ricci and Riemann curvature tensor have the same regularity (as can be verified using Mathematica). So assuming T in L^{∞} implies $d\Gamma$ in L^{∞} , and Theorem 2.1 implies the corollary.

6.3. Construction of locally inertial coordinates. The standard method for constructing locally inertial frames does not apply to connections $\Gamma \in L^{\infty}(\Omega)$ because the classical Riemann normal coordinate construction requires Lipschitz continuity for a connection, and regularity $C^{1,1}$ for a metric, [18]. The following corollary of Theorem 2.1 establishes that locally inertial coordinates always exist in a Hölder sense, for any L^{∞} connection with $Riem(\Gamma) \in L^{\infty}(\Omega)$.

Corollary 6.3. Assume Γ , Riem $(\Gamma) \in L^{\infty}(\Omega)$ on a bounded spacetime domain $\Omega \subset \mathbb{R}^n$. Then for any $p \geq 1$ and any point $q \in \Omega$ there exists a neighborhood $\Omega' \subset \Omega$ of q and a coordinate transformation with Jacobian $J \in W^{1,2p}(\Omega')$ such that the connection in the resulting coordinates z has regularity $\Gamma \in W^{1,p}(\Omega')$ and satisfies

$$\Gamma^{\gamma}_{\alpha\beta}(q) = 0 \tag{6.6}$$

$$\left|\Gamma^{\gamma}_{\alpha\beta}(\hat{q})\right| \leq C \left|q - \hat{q}\right|^{\alpha},\tag{6.7}$$

where $\alpha \in (0,1)$ is the Hölder coefficient associated with 2p > n by Morrey's inequality and $|\cdot|$ is the Euclidean norm on \mathbb{R}^n applied to q - p in z-coordinates.

We call a coordinate system y in which the connection is in $W^{1,p}$ for p > nand satisfies (6.6) and (6.7), a locally inertial coordinate system with Hölder corrections to the gravitational field. The case $\alpha = 1$ in (6.7) would give the standard second order correction due to the gravitational field. For Lorentz metrics one can in addition arrange for the metric to be equal to the Minkowski metric at q by suitable multiplication with a constant Jacobian.¹⁹

Proof of Corollary 6.3. The assumptions of Corollary 6.3 are identical to those of Theorem 2.1. Applying Theorem 2.1 gives us a Jacobian $J \in W^{1,2p}(\Omega')$, as determined by the RT-equations, defined in some neighborhood $\Omega' \subset \Omega$ of q, such that the connection in the resulting coordinates y^{α}

30

¹⁹This Jacobian is the unique composition of the orthogonal matrix diagonalizing the metric at q multiplied with the diagonal matrix that has the inverse of the square root of each eigenvalue of the metric on its diagonal. See the construction in [20] for details. Since the Jacobian is constant, properties (6.6) - (6.7) are preserved.

has regularity $\Gamma \in W^{1,p}(\Omega')$. Without loss of generality, we assume that y(q) = 0.

To arrange for condition (6.6), following the development in Chapter 8 of [20], we introduce a smooth coordinate transformation $y \to z$, (hence preserving the regularity $W^{1,p}$ of Γ), such that Γ satisfied the sought after properties (6.6) - (6.7) in z-coordinates. Indeed, it is the Hölder continuity of Γ , implied by Morrey's inequality (A.4), which allows us to evaluate the connection in y-coordinates evaluated at the point q, $\Gamma^{\alpha}_{\beta\gamma}|_{q}$, and then introduce the coordinate transformation

$$z^{\mu}(y) \equiv \delta^{\mu}_{\alpha} \Gamma^{\alpha}_{\beta\gamma} \big|_{q} y^{\beta} y^{\gamma} + \delta^{\mu}_{\alpha} y^{\alpha}.$$
(6.8)

where δ^{μ}_{α} denotes the Kronecker symbol. Clearly (6.8) defines a smooth coordinate transformation and, by our incoming assumption y(q) = 0, it follows that

$$z(y(q)) = 0$$
 and $\frac{\partial z^{\mu}}{\partial y^{\alpha}}\Big|_{q} = \delta^{\mu}_{\alpha}.$ (6.9)

Moreover, and this is the main point of definition (6.8), we have

$$\frac{\partial^2 z^{\mu}}{\partial y^{\beta} \partial y^{\gamma}}\Big|_q = \delta^{\mu}_{\alpha} \Gamma^{\alpha}_{\beta\gamma}\Big|_q, \tag{6.10}$$

which implies that the connection $\Gamma^{\sigma}_{\mu\nu}$ in z-coordinates vanishes at q. Indeed, from the transformation law of connections we find that

$$\frac{\partial z^{\sigma}}{\partial y^{\alpha}}\Gamma^{\alpha}_{\beta\gamma} = \frac{\partial^2 z^{\sigma}}{\partial y^{\beta}\partial y^{\gamma}} + \Gamma^{\sigma}_{\mu\nu}\frac{\partial z^{\mu}}{\partial y^{\beta}}\frac{\partial z^{\nu}}{\partial y^{\gamma}}$$

so using (6.9) and (6.10) to evaluate $\Gamma^{\sigma}_{\mu\nu}$ at q gives

$$\delta^{\sigma}_{\alpha}\Gamma^{\alpha}_{\beta\gamma}\big|_{q} = \delta^{\sigma}_{\alpha}\Gamma^{\alpha}_{\beta\gamma}\big|_{q} + \Gamma^{\sigma}_{\mu\nu}\big|_{q}\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}$$

and this implies that in z-coordinates $\Gamma^{\sigma}_{\mu\nu}|_q = 0$ for all $\sigma, \mu, \nu \in \{1, ..., n\}$. This proves property (6.6) of Corollary 6.3.

Now property (6.7) follows directly from (6.6) together with the Hölder continuity of Γ in z-coordinates. Namely, since the coordinate transformation $y \to z$ is in $C^{\infty}(\Omega')$, we again have $\Gamma \in W^{1,p}(\Omega')$ in z-coordinates, so Morrey's inequality implies that $\Gamma \in C^{0,\alpha}$ for $\alpha = 1 - \frac{n}{p}$. This completes the proof of Corollary 6.3.

7. How to recover the original RT-equations from the reduced RT-equations by gauge transformation

7.1. Conceptual overview. We start by describing, more carefully, the logical connection between the full RT-equations (4.1) - (4.4) and the reduced RT-equations (4.10) - (4.12). Recall from Section 4 that the original RT-equations were derived by constructing the Laplacian $d\delta + \delta d$ starting from two equivalent formulations of the Riemann-flat condition, one involving dJ and one involving $d\tilde{\Gamma}$; and the first order A equation came by replacing $A = J\delta\tilde{\Gamma}$ in the J equation, setting d of the right hand side equal to zero,

and imposing dJ = 0 on the boundary in (4.5). Now in the existence theory set out in [22] for the case $\Gamma, d\Gamma \in W^{1,p}$, we saw that not every solution of the original RT-equations determines a solution in which J is paired with $\tilde{\Gamma} = \tilde{\Gamma}_J \equiv \Gamma - J^{-1}dJ$ given in (4.6). Here $\tilde{\Gamma}_J$ is the tensor transformation (4.7) of the connection coefficients obtained by transforming the original Γ by J. To complete the argument in [21], we proved that given a solution ($\tilde{\Gamma}, J, A, v$) of the full RT-equations, $\tilde{\Gamma}_J$ will solve a modified version of the first RT-equation (4.1). In [21], the role of this modified elliptic equation was to establish that $\tilde{\Gamma}_J$, (and hence also Γ_y by (4.7)), is of optimal regularity. This was established rigorously in [22] at the smoothness level $\Gamma, d\Gamma \in W^{1,p}$.

We now understand this more conceptually as follows. The variables for the original RT-equations are $(\tilde{\Gamma}, J, A, v)$. The transformation $(\tilde{\Gamma}, J, A, v) \rightarrow (\tilde{\Gamma}, J, B, w)$ effected by the change of variables (4.8) - (4.9) given by

$$B = A - \langle dJ; \hat{\Gamma} \rangle, w = v - \delta \langle dJ; \tilde{\Gamma}_J \rangle,$$

transforms the last three RT-equations (4.2)-(4.4) into the reduced RTequations (4.10)-(4.12), and transforms the first RT-equation (4.1) into equation (4.13), an elliptic equation for $\tilde{\Gamma}$ involving $(\tilde{\Gamma}, J, B)$ on the right hand side. Thus the original four RT-equations (4.1)-(4.4) are equivalent to the three reduced RT-equations (4.10)-(4.12) together with (4.13). Now the rather remarkable discovery, which is the basis for the present paper, is that $\tilde{\Gamma} = \tilde{\Gamma}_J$ turns out to exactly solve equation (4.13), but only on solutions (J,B) of the reduced RT-equations. That is, Γ_J does not in general solve the first RT-equation (4.1), but the transformation (4.8) - (4.9), which uncouples $\tilde{\Gamma}$ from the last three equations, also produces the elliptic equation (4.13) satisfied by Γ_J . The result then, is that we no longer need the original RT-equations, because optimal regularity is determined entirely from the reduced RT-equations (4.10) - (4.12) for (J, B), together with the elliptic equation (4.13) for the gain in regularity of Γ_J . At the end, the original Γ is out of the picture. To borrow words from Ludwig Wittgenstein (regarding his private language argument), the original RT-equations are a "ladder we climb" to obtain the reduced RT-equations within the gauge freedom of the original RT-equations, but that ladder can then be thrown away once we find them. But still, to complete the picture, it is interesting to understand the sense in which solutions (Γ_J, J, B, w) of the reduced RT-equations correspond to solutions $(\tilde{\Gamma}, J, A, v)$ of the original RT-equations.

To clarify this, recall that if we are given a coordinate transformation $x \to y$ and J is its Jacobian, applying the tensor transformation law (4.7) to $\tilde{\Gamma}_J$ produces the coefficients of the connection Γ_y in y coordinates. Now we know from [22] that the solution space of the original RT-equations (4.1)-(4.5) is larger than we want, because it contains solutions ($\tilde{\Gamma}, J, A, v$) for which $\tilde{\Gamma} \neq \tilde{\Gamma}_J$. That is, $\tilde{\Gamma}$ need not have anything to do with the transformation of our starting non-optimal connection Γ . In fact, we have discovered that

there exist solutions of the RT-equations with the same J, but different $(\tilde{\Gamma}, A, v)$. Define then the equivalence class $\mathcal{E}(J)$ of all solutions of the original RT-equations (4.1)-(4.5) which share the same Jacobian field J. (It could be empty). Recall that only the J-equation (4.2) comes with a boundary condition (4.5), so the "gauge freedom" in the RT-equations is the freedom to choose the free function v, and the freedom to impose boundary conditions for $\tilde{\Gamma}$ and A in equations (4.1) and (4.3), (4.4), respectively. Thus the equivalence class $\mathcal{E}(J)$ associated with a given Jacobian J is the set of all solutions ($\tilde{\Gamma}, J, A, v$) of the RT-equations (4.1)-(4.5), solutions determined by v and all the possible boundary conditions we might impose to determine $\tilde{\Gamma}$ and A from equations (4.1) and (4.3), (4.4). Now once we have established that $\tilde{\Gamma}_J$ solves (4.13), reversing the argument leading to (4.8) - (4.9) in Section 4, (which entails simply replacing $\tilde{\Gamma}$ by $\tilde{\Gamma}_J$ in the formulas for B and w), shows directly that ($\tilde{\Gamma}_J, J, A', v'$) will solve the original original RT-equations whenever ($\tilde{\Gamma}, J, A, v$) does, where

$$\begin{array}{rcl} A' &=& B - \langle dJ; \tilde{\Gamma}_J \rangle, \\ v' &=& w - \delta \langle dJ; \tilde{\Gamma}_J \rangle. \end{array}$$

Thus conceptually, starting with a solution (Γ, J, A, v) of the original RTequations, (J, B) will solve the reduced RT-equations, $\tilde{\Gamma}_J$ will solve the modified first RT-equation (4.13), and the back transformation (4.14), (4.15) determines a new solution of the original RT-equations within the equivalence class $\mathcal{E}(J)$, this new solution having as its first component $\tilde{\Gamma} = \tilde{\Gamma}_J$. That is, through a change of gauge, we can substitute $\tilde{\Gamma}$ for $\tilde{\Gamma}_J$ in any solution of the original RT-equations. This gives expression to the content of what is claimed in (i),(ii) of Theorem 4.1. In summary, we write this as a direct corollary of Theorem 4.1:

Corollary 7.1. If $(\tilde{\Gamma}, J, A, v)$ lies within the equivalence class $\mathcal{E}(J)$ of the RT-equations, then $(\tilde{\Gamma}_J, J, A', v') \in \mathcal{E}(J)$ as well.

That Γ_J given by (4.6) actually solves (4.13), the first RT-equation (4.1) modified by the substitution (4.8)), on solutions of the reduced RT-equations, is crucial because it is this equation which gives the requisite optimal regularity of $\tilde{\Gamma}_J$, the connection coefficients of Γ transformed by J. As laid out above, it really is quite remarkable that a change of gauge simultaneously eliminates $\tilde{\Gamma}$ from the last three RT-equations, and then on top of that, transforms the first RT-equations into a new elliptic equation satisfied by $\tilde{\Gamma}_J$. So as a preliminary to the proof of Theorem 4.1, we now go through the key idea in the proof of parts (i) and (ii) leading to Γ_J being a solution of (4.13), in the case when Γ is smooth, thereby displaying how it works without being distracted by the weak formulation of the equations.

7.2. Recovering solutions of the RT-equations from the reduced **RT-equations:** A complete proof of Theorem 4.1 is the subject of Section 9. As a preliminary, we explain in this section the key step in the equivalence

between the reduced and full RT-equations in its simplest setting, when Γ is a smooth connection. That is, we explain why $\tilde{\Gamma} = \tilde{\Gamma}_J \equiv \Gamma - J^{-1}dJ$ automatically solves the first RT-equation when defined in terms of a solution (J, B) of the reduced RT-equations. On the face of it, this is quite remarkable, because the *J*-equation appears to have lost all knowledge of $\tilde{\Gamma}$ once we gauge transform from *A* to *B*. The point, which we establish here, is that when we take the *B* gauge, the formula (4.6) for $\tilde{\Gamma}_J$ in terms of *J* and Γ alone gives an exact solution of the first RT-equation (4.1), provided *J* solves the reduced RT-equations. Thus the regularity of $\tilde{\Gamma}_J$ is determined by the first RT-equation, but is ultimately encoded in the reduced RT-equations.

Lemma 7.2. Let Γ be smooth and assume (J, B) is a smooth solution of the reduced RT-equations (4.10) - (4.12) for some given w, such that J is invertible. Then $\tilde{\Gamma} = \tilde{\Gamma}_J$ and A = A', defined in (4.6) and (4.14), satisfy the first RT-equation (4.1).

Proof. To prove Lemma 7.2, we first take the exterior derivative d of equation $\tilde{\Gamma}_J = \Gamma - J^{-1} dJ$, c.f. (4.6), to obtain

$$d\Gamma_J = d\Gamma - d(J^{-1}dJ)$$

= $d\Gamma - d(J^{-1}) \wedge dJ,$ (7.1)

where we use $d(J^{-1}dJ) = d(J^{-1}) \wedge dJ$ by the Leibniz rule (3.7). Taking now the co-derivative δ of (7.1) gives

$$\delta d\tilde{\Gamma}_J = \delta d\Gamma - \delta \big(dJ^{-1} \wedge dJ \big), \tag{7.2}$$

thus giving the first term of the Laplacian $\Delta \tilde{\Gamma}_J \equiv \delta d \tilde{\Gamma}_J + d \delta \tilde{\Gamma}_J$.

To determine the second term of $\Delta \tilde{\Gamma}_J$, we take δ of equation $\tilde{\Gamma}_J = \Gamma - J^{-1}dJ$, c.f. (4.6), to compute

$$\delta \tilde{\Gamma}_J = \delta \Gamma - \delta (J^{-1} dJ). \tag{7.3}$$

Using now the Leibniz rule for co-derivatives (3.8) we have

$$\delta \tilde{\Gamma}_J = \delta \Gamma - \langle d(J^{-1}); dJ \rangle - J^{-1} \delta dJ = \delta \Gamma - \langle d(J^{-1}); dJ \rangle - J^{-1} \Delta J, \qquad (7.4)$$

since $\Delta J = \delta dJ$ by $\delta J = 0$, because the co-derivative δ vanishes on 0-forms. Substituting now the reduced RT-equation (4.10) for ΔJ into (7.4) gives

$$\delta \tilde{\Gamma}_J = \delta \Gamma - \langle d(J^{-1}); dJ \rangle - J^{-1} \big(\delta(J \cdot \Gamma) - B \big), \tag{7.5}$$

and since $\delta(J \cdot \Gamma) = J\delta\Gamma + \langle dJ; \Gamma \rangle$ by (3.8), we obtain that

$$\delta \tilde{\Gamma}_J = J^{-1}B - \langle d(J^{-1}); dJ \rangle - J^{-1} \langle dJ; \Gamma \rangle.$$
(7.6)

The cancellation of the lowest regularity term $\delta\Gamma$ in the step from (7.5) to (7.6) is the essence of the gain of regularity implied by the RT-equations, c.f. Theorem 4.1. (To establish this cancellation for weak solutions takes some work, see Section 9.2.) Now the right hand side of (7.6) is equal to $J^{-1}A$ as a consequence of the definition of A in (4.14), as proven in Lemma 7.3

below. So assuming Lemma 7.3 to be true for the moment, we find equation (7.6) to be identical to

$$\delta \tilde{\Gamma} = J^{-1} A. \tag{7.7}$$

Taking now the exterior derivative d of (7.7) and adding the resulting equation to (7.2) gives us

$$\Delta \tilde{\Gamma}_J = \delta d\Gamma - \delta (dJ^{-1} \wedge dJ) + d (J^{-1}A)$$
(7.8)

which proves that $\tilde{\Gamma}_J$ solves the sought after equation (4.1).

To complete the proof of Lemma 7.2, it remains to prove Lemma 7.3.

Lemma 7.3. For A defined in (4.14), we have

$$A = B - J\langle d(J^{-1}); dJ \rangle - \langle dJ; \Gamma \rangle, \qquad (7.9)$$

so the right hand side of (7.6) is equal to $J^{-1}A$.

Proof. To verify (7.9), we substitute $\tilde{\Gamma}_J = \Gamma - J^{-1} dJ$, c.f. (4.6), into the definition $A = B - \langle dJ; \tilde{\Gamma}_J \rangle$, c.f. (4.14), to compute

$$A = B - \langle dJ; (\Gamma - J^{-1}dJ) \rangle$$

= $B + \langle dJ; J^{-1}dJ \rangle - \langle dJ; \Gamma \rangle.$ (7.10)

We now use the multiplication property (3.10) of the matrix valued inner product $\langle \cdot ; \cdot \rangle$ twice, to write the second term in (7.10) as

where the last equality holds since $d(J^{-1}) = -J^{-1} dJ \cdot J^{-1}$ by the Leibniz rule for gradients, (since J is a 0-form so d is the gradient). Substituting (7.11) into (7.10) gives the sought after identity (7.9) and proves Lemma 7.3.

This completes the proof of Lemma 7.2, the case of smooth solutions. Accomplishing this for weak solutions is subject of Section 9.2.

8. Weak formulation of the RT-equations

We now begin the existence theory for weak solutions of the reduced RTequations (4.10)-(4.12) on bounded domains $\Omega \subset \mathbb{R}^n$. This provides an existence theory for weak solutions of the full RT-equations (4.1)-(4.5) by using the change of gauge $(B, w) \to (A', v')$ given in (4.14) - (4.15).

The RT-equations are a nonlinear elliptic system of equations in unknowns J, A and $\tilde{\Gamma}$ determined by the assumed given connection Γ , and they allow for the freedom to choose the arbitrary function v in the second A-equation (4.4), together with boundary conditions for $\tilde{\Gamma}$ and A in (4.1) and (4.3), (4.4), within the appropriate regularity class. We refer to this as the gauge freedom in the full RT-equations, and loosely refer to v or A as the choice of

gauge. Note first that because the right hand sides involve the derivatives $D\Gamma$ of Γ , the RT-equations are consistent with the regularity J, Γ being one order more regular than $\Gamma, d\Gamma$ because this puts the right hand sides of the Γ and J equations (4.2) and (4.1) at the same regularity as $D\Gamma$, so long as A has the same regularity as Γ ; and this is consistent with the regularity of the right hand side of the A equations (4.3) - (4.4) being one order less regular than Γ . Thus, the RT-equations are consistent with elliptic PDE theory in the sense that Laplacians in (4.1) - (4.2) should raise the regularity of Γ and J two orders above the right hand side, which is one order above the regularity of Γ ; and the first order Cauchy Riemann system (4.3) -(4.4) should raise the regularity of A to the same regularity as Γ . In [22] authors proved that this consistency of the RT-equations is correct in the classical sense by giving an existence theory for the full RT-equations in the case $\Gamma, d\Gamma \in W^{1,p}$, and in Sections 9 - 11 we extent this classical theory to the case of weak solutions, when $\Gamma, d\Gamma \in L^{\infty}$. When $\Gamma, d\Gamma \in L^{\infty}$, we also have $\Gamma, d\Gamma \in L^p$ for every p, so our goal is to establish existence of weak solutions $J, \tilde{\Gamma} \in W^{1,p}, A \in L^p$, on sufficiently small domains. Note that in this case, the right hand sides of the Γ, J and A equations (4.1) - (4.4) are at the regularity $W^{-1,p}$, some $p < \infty$, a regularity too weak for classical solutions. So an existence theory requires a weak formulation of the RTequations. Note that the RT-equations do indeed admit a weak formulation because all of the lowest order terms on the right hand side of (4.1)-(4.4)are matrix valued differential forms with d or δ operating on them, so that integration by parts will raise the regularities one order. To accomplish the integration by parts and express a rigorous weak formulation, we need to construct an inner product and the adjoint operators associated with the matrix valued differential forms on the right hand side of the RT-equations. This is accomplished in this section.

The existence theory for weak solutions of the RT-equations is accomplished in Sections 9 - 11. A few preliminary comments are in order. First, the iteration scheme used in [22] only closes in L^p spaces for classical solutions, because of the bad nonlinear term $dJ^{-1} \wedge dJ$ on the right hand side of the $\tilde{\Gamma}$ equation (4.1). The problem is that products of functions in L^p are not in L^p , (or alternatively, in classical elliptic theory, the Laplacian does not lift L^{∞} to $C^{1,1}$), so the iteration scheme does not close in any L^p space. We overcome this problem with the reduced RT-equations. I.e., by showing that for solutions $(\tilde{\Gamma}, J, A, v)$ with $v \in W^{-1,p}$, the change of gauge $(A, v) \rightarrow (B, w)$, given in (4.8) - (4.9), uncouples the (J, B) equation from the Γ equation which contains the bad nonlinear term. We named the resulting linear system (4.10)-(4.12) in (J, B) the reduced RT-equations, and view this as the RT-equations written in a special gauge. Since the right hand sides of the reduced RT-equations are linear, we prove in Section 10 that our iteration scheme, modified to the weak formulation of the equations, does converge when $\Gamma, d\Gamma \in L^{\infty}$, for sufficiently small bounded sets
Ω . Even though the reduced RT-equations are linear, it is a system and the coefficients are at critical low regularity, so an iteration scheme is still required to handle the lower order terms. The smallness of the neighborhood Ω is used to rule out complications with the Fredholm alternative. Once we obtain a solution (J, B) of the reduced RT-equations, we then prove that (4.6) provides an exact solution $\tilde{\Gamma} = \tilde{\Gamma}_{I}$ of the first RT-equation (4.1) (with B substituted for A), and by this we obtain the desired regularity of the transformed connection coefficients $\tilde{\Gamma}_{J}$ from the now classical linear L^{p} theory of the Laplacian. In this way the bad nonlinear product in (4.1) can be handled by simply using solutions of the reduced RT-equations with larger $p, J \in W^{1,2p}$, so the bad nonlinear term $dJ^{-1} \wedge dJ$ on the right hand side of (4.1) is in $W^{-1,p}$, thereby placing the solution $\tilde{\Gamma} = \tilde{\Gamma}_J \in W^{1,p}$. Once we have a solution to the RT-equation in the (B, w) gauge, we no longer require a solution of the original RT-equations, but to complete the circle we show that the transformation back to (A', v') in (4.14), (4.15) provides a weak solution in the original gauge, thereby demonstrating the consistency of the whole theory for every gauge at the level $\Gamma, d\Gamma \in L^{\infty}$.

We finish this introduction of the existence theory to follow, by pointing out some of the obstacles our theory faces in the weak formulation required when $\Gamma, d\Gamma \in L^{\infty}$. One technical step in the argument is to prove that a weak solution of the B equation really does impose the integrability equation dJ = 0. That is, the boundary condition dJ = 0 (4.5), is not a classical Dirichlet boundary condition, and when $J \in W^{1,p}$, dJ = 0 is too weak to impose on a boundary. Fortunately, the way we handled this boundary condition in the iteration scheme introduced in [22], can be modified to the weak setting. The idea is to introduce an auxiliary elliptic equation for ysatisfying dy = J in the iteration. Then we can use Dirichlet boundary conditions for J which make sense at this low regularity, and thereby obtain the integrability from dy = J which implies dJ = 0. This provides a very clean way to handle the boundary condition since we can then apply classical linear L^p -elliptic theory for the Dirichlet problem at each stage of the iteration. However, for the low regularity considered here, this procedure is more technical because it involves two different version of the weak Laplacian combined with operations on the Cartan algebra of differential forms. This is accomplished in Sections 10 - 11. Finally, the proof that Γ_J solves the first RT-equation in a weak sense is more involved, because the weak reduced RT-equation for J cannot be used in a straightforward way to achieve the cancellation of the lowest regularity term $\delta\Gamma$ in equation (7.6). This is achieved in Section 9.2.

8.1. Integration by parts for matrix valued differential forms. To introduce the weak formulation of the RT-equations, we first define the following inner products over matrix and vector valued differential forms. On matrix valued k-forms A and B, we define the point-wise inner product

$$\langle A, B \rangle \equiv \operatorname{tr} \langle A; B^T \rangle$$

$$\stackrel{(3.9)}{=} \sum_{\nu,\sigma=1}^{n} \sum_{i_1 < \dots < i_k} A^{\nu}_{\sigma \ i_1 \dots i_k} B^{\nu}_{\sigma \ i_1 \dots i_k}, \tag{8.1}$$

where the matrix valued inner product $\langle A; B \rangle$ is defined in (3.9). We further introduce the L^2 -inner product

$$\langle A, B \rangle_{L^2} \equiv \int_{\Omega} \langle A, B \rangle dx$$
 (8.2)

where dx denotes the Lebesgue measure in \mathbb{R}^n . For vector valued k-forms v and w we define the point-wise inner product

$$\langle v, w \rangle \equiv \sum_{\alpha=1}^{n} \sum_{i_1 < \dots < i_k} v^{\alpha}_{i_1 \dots i_k} w^{\alpha}_{i_1 \dots i_k}, \qquad (8.3)$$

in terms of the Euclidean inner product, and we introduce the L^2 -inner product by

$$\langle v, w \rangle_{L^2} \equiv \int_{\Omega} \langle v, w \rangle dx.$$
 (8.4)

The inner products on matrix valued 0-forms and vector valued 1-forms are in fact identical,

$$\langle A, B \rangle = \langle \vec{A}, \vec{B} \rangle, \tag{8.5}$$

where A and B are matrix valued 0-forms. For $k \ge 1$, vectorization of matrix valued k-forms generally results in a loss of information due to cancellation of symmetries, and one can not expect (8.5) to be valid.

To introduce the weak formalism of the RT-equations below, we further require the following well-known partial integration formula for scalar valued differential forms,

(

$$d\alpha, \beta\rangle_{L^2} + \langle \alpha, \delta\beta \rangle_{L^2} = 0, \tag{8.6}$$

where α is a k-form and β a (k + 1)-form, such that either $\alpha|_{\partial\Omega} = 0$ or $\beta|_{\partial\Omega} = 0$, c.f. Theorem 1.11 in [6]. Equation (8.6) holds for regularity $\alpha \in W^{1,p}(\Omega)$ and $\beta \in W^{1,p^*}(\Omega)$, (where p, p^* are conjugate exponents, $\frac{1}{p} + \frac{1}{p^*} = 1$), and the condition $\alpha|_{\partial\Omega} = 0$ or $\beta|_{\partial\Omega} = 0$ is understood in the sense that $\alpha \in W_0^{1,p}(\Omega)$ or $\beta \in W_0^{1,p^*}(\Omega)$. Here $W_0^{1,p}(\Omega)$ is the closure of the space of smooth functions C_0^{∞} with respect to the $W^{1,p}$ -norm, so for p > n functions in $W_0^{1,p}(\Omega)$ vanish on $\partial\Omega$ in the sense of continuous functions. In our first lemma, we extend (8.6) to matrix and vector valued differential forms.

Lemma 8.1. Let u be a matrix valued k-form and ω be a matrix valued (k + 1)-form, $k \geq 0$, such that $u \in W_0^{1,p}(\Omega)$ and $\omega \in W_0^{1,p^*}(\Omega)$, where $\frac{1}{p} + \frac{1}{p^*} = 1$, then

$$\langle du, \omega \rangle_{L^2} + \langle u, \delta \omega \rangle_{L^2} = 0,$$
(8.7)

and (8.7) continues to hold if only one of the forms u and ω vanishes on the boundary, i.e., only $u \in W_0^{1,p}(\Omega)$ or $\omega \in W_0^{1,p^*}(\Omega)$. Moreover, (8.7) holds as

well when $u \in W_0^{1,p}(\Omega)$ is a vector valued k-form and $\omega \in W_0^{1,p^*}(\Omega)$ a vector valued (k+1)-form, (or if either $\omega|_{\partial\Omega} = 0$ or $u|_{\partial\Omega} = 0$ in the above sense).

Proof. The Lemma follows directly from (8.6) together with the fact that the exterior derivative and co-derivative act on matrix-components separately, c.f. [21]. Namely, assuming the case that u and v are matrix valued differential forms, we find from definition (8.2) that

$$\langle du, \omega \rangle_{L^2} = \sum_{\nu, \sigma=1}^n \int_{\Omega} \sum_{i_1 < \dots < i_{k+1}} (du)_{\sigma \ i_1 \dots i_{k+1}}^{\nu} \omega_{\sigma \ i_1 \dots i_{k+1}}^{\nu} dx.$$

Applying now the partial integration formula for scalar valued forms (8.6) to the above right hand side for fixed ν and σ , we obtain that

$$\begin{aligned} \langle du, \omega \rangle_{L^2} &= -\sum_{\nu, \sigma=1}^n \int_{\Omega} \sum_{i_1 < \dots < i_k} u^{\nu}_{\sigma \ i_1 \dots i_k} (\delta \omega)^{\nu}_{\sigma \ i_1 \dots i_k} dx \\ &= -\langle u, \delta \omega \rangle_{L^2}. \end{aligned}$$

This is the sought after identity (8.7) for matrix valued forms. The case for vector valued forms follows analogously.

Before we define the weak formulation of the RT-equations, we first introduce the weak formulations of Cauchy Riemann type and Poisson equations. So consider an equation of Cauchy Riemann type,

$$\begin{cases} du = f\\ \delta u = g, \end{cases}$$
(8.8)

where u, f and g are vector valued differential forms. In light of (8.7), we say that a vector valued k-form $u \in W^{1,p}(\Omega)$ solves (8.8) weakly, if

$$\begin{cases} \langle u, \delta \psi \rangle_{L^2} = -f(\psi) \\ \langle u, d\varphi \rangle_{L^2} = -g(\varphi), \end{cases}$$
(8.9)

for any vector valued (k + 1)-form $\psi \in W_0^{1,p^*}(\Omega)$ and any vector valued (k-1)-form $\varphi \in W_0^{1,p^*}(\Omega)$ both vanishing on $\partial\Omega$, where we assume that f is a scalar valued linear functional on the space of vector valued (k + 1)-forms in $W_0^{1,p^*}(\Omega)$ and g is a scalar valued linear functional on the space of vector valued (k - 1)-forms in $W_0^{1,p^*}(\Omega)$. (See Section C for a complete list of consistency conditions on f and g required for existence of solutions.) Weak solutions of (8.8) for matrix valued differential forms can be introduced in a similar fashion, but are not used in this paper since the third and fourth RT-equations 4.3 - (4.4) are vector valued 1-forms.

Now consider the Poisson equation

$$\Delta u = f. \tag{8.10}$$

We say that a matrix valued k-form $u \in W^{1,p}(\Omega)$ solves (8.10) weakly, if

$$\Delta u[\phi] = f(\phi), \tag{8.11}$$

for any matrix valued k-form $\phi \in W_0^{1,p^*}(\Omega)$, where

$$\Delta u[\phi] \equiv -\langle du, d\phi \rangle_{L^2} - \langle \delta u, \delta \phi \rangle_{L^2}$$
(8.12)

and f is a scalar valued linear functional on the space of matrix valued k-forms $\phi \in W_0^{1,p^*}(\Omega)$. To extend the notion of weak solutions of the Poisson equation (8.10) to vector valued differential forms, simply use vector valued test k-forms $\phi \in W_0^{1,p^*}(\Omega)$ and the corresponding inner product (8.4), which suffices since the exterior derivative d and the co-derivative δ act only on form-indices, but not on matrix or vector indices.

The following lemma clarifies that the weak formulation of the Laplacian in (8.12) for differential forms is identical to the standard weak form of the Laplacian (c.f., [11, 12]) taken component wise. By this we know we can use standard theorems of elliptic regularity theory for the analysis in this paper, c.f. Section B.

Lemma 8.2. Let $u \in W^{1,p}(\Omega)$ be a matrix (or vector) valued k-form, then

$$\Delta u[\phi] = -\langle \nabla u, \nabla \phi \rangle_{L^2}, \qquad (8.13)$$

for any matrix (or vector) k-form $\phi \in W_0^{1,p^*}(\Omega)$, where $\Delta u[\phi]$ is the weak Laplacian defined in (8.12), ∇ is the Euclidean gradient in x-coordinates taken on each component²⁰, and we set

$$\langle \nabla u, \nabla \phi \rangle_{L^2} \equiv \sum_{j=1}^n \int_{\Omega} \langle \partial_j u, \partial_j \phi \rangle dx.$$
 (8.14)

Moreover, (8.13) holds assuming only $u \in L^p(\Omega)$ with $du, \delta u \in L^p(\Omega)$, (the low regularity we encounter in the proof of Theorem 4.1).

Proof. By compactness of $C_0^{\infty}(\Omega)$ in $W_0^{1,p^*}(\Omega)$ it suffices to prove (8.13) for $\phi \in C_0^{\infty}(\Omega)$. So let $\phi \in C_0^{\infty}(\Omega)$ and use partial integration component wise to compute

$$\langle \nabla u, \nabla \phi \rangle_{L^2} = -\langle u, \nabla \cdot (\nabla \phi) \rangle_{L^2}.$$

Substituting now that $\nabla \cdot (\nabla \phi) = \Delta \phi = d\delta \phi + \delta d\phi$ and using partial integration for differential forms (8.7), we obtain that

$$\langle \nabla u, \nabla \phi \rangle_{L^2} = -\langle u, d\delta \phi \rangle_{L^2} - \langle u, \delta d\phi \rangle_{L^2} = \langle \delta u, \delta \phi \rangle_{L^2} + \langle du, d\phi \rangle_{L^2} = -\Delta u[\phi].$$

$$(8.15)$$

This proves (8.13) in the case $u \in W^{1,p}(\Omega)$.

To prove the supplement, assume that $u \in L^p(\Omega)$ with $du, \delta u \in L^p(\Omega)$. For this regularity the weak Laplacian (8.12) is well-defined, i.e.,

$$\Delta u[\phi] \equiv -\langle du, d\phi \rangle_{L^2} - \langle \delta u, \delta \phi \rangle_{L^2}, \qquad (8.16)$$

²⁰So ∇ is taken on each matrix, vector or differential form component separately, e.g., $\nabla u = (\nabla u)_{i_1...i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$ for $u = u_{i_1...i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$.

exists for any matrix valued k-form $\phi \in W_0^{1,p^*}(\Omega)$. Again, considering ϕ in the dense subspace $C_0^{\infty}(\Omega)$, $\Delta \phi = \delta d\phi + d\delta \phi$ is well-defined. We now apply partial integration (8.7) to write (8.16) as

$$\Delta u[\phi] = \langle u, \Delta \phi \rangle_{L^2} = \lim_{h \to 0} \langle u, \nabla_h \cdot \nabla \phi \rangle_{L^2}, \qquad (8.17)$$

where the last equality holds by convergence of the difference quotient $\nabla_h \phi$ to $\nabla \phi$, so that we have $\nabla_h \cdot \nabla \phi \to \Delta \phi$ as $h \to 0$. By partial integration for ∇_h , we find that

$$-\langle \nabla_h u, \nabla \phi \rangle_{L^2} = \langle u, \nabla_h \cdot \nabla \phi \rangle_{L^2}.$$
(8.18)

By (8.17) the right hand side in (8.18) converges, which implies that the left hand side in (8.18) converges as $h \to 0$ as well, and we have

$$\lim_{h \to 0} \langle \nabla_h u, \nabla \phi \rangle_{L^2} = \langle \nabla u, \nabla \phi \rangle_{L^2}.$$
(8.19)

Combing this with (8.17), we find that

$$\Delta u[\phi] = -\langle \nabla u, \nabla \phi \rangle_{L^2}$$

for any $\phi \in C_0^{\infty}(\Omega)$. By denseness of $C_0^{\infty}(\Omega)$ in $W_0^{1,p^*}(\Omega)$ this establishes (8.13) for the low regularity $u \in L^p(\Omega)$ with $du, \delta u \in L^p(\Omega)$. This completes the proof of Lemma 8.2.

8.2. The weak RT-equations and weak reduced RT-equations. We are now prepared to derive the weak formulation of the RT-equations (4.1) - (4.4), that is, of the system

$$\begin{split} \Delta \tilde{\Gamma} &= \delta d\Gamma - \delta \left(dJ^{-1} \wedge dJ \right) + d(J^{-1}A), \\ \Delta J &= \delta (J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \\ d\vec{A} &= \overrightarrow{\operatorname{div}} \left(dJ \wedge \Gamma \right) + \overrightarrow{\operatorname{div}} \left(J \, d\Gamma \right) - d \left(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle} \right), \\ \delta \vec{A} &= v. \end{split}$$

To begin, assume we are given a smooth solution $(\tilde{\Gamma}, J, A)$ of the RT-equations (4.1) - (4.4) with $\tilde{\Gamma} \in W^{1,p}(\Omega)$, $J \in W^{1,2p}(\Omega)$ and $A \in L^{2p}(\Omega)$, and assuming p > 1 suffices here. Now, in light of (8.12), for any matrix valued 1-form $\Phi \in W_0^{1,p^*}(\Omega)$, we write the left hand side of the first RT-equation (4.1) as

$$\Delta \tilde{\Gamma}[\Phi] \equiv -\langle \delta \tilde{\Gamma}, \delta \Phi \rangle_{L^2} - \langle d \tilde{\Gamma}, d \Phi \rangle_{L^2}, \qquad (8.20)$$

where we applied (8.7) to $\Delta = \delta d + d\delta$ in the last equality, where again $\frac{1}{p^*} + \frac{1}{p} = 1$. Applying now (8.7) to rewrite the right hand side of (4.1) in an analogous way, we find that the first RT-equation (4.1) can be written equivalently as

$$-\Delta \tilde{\Gamma}[\Phi] = \left\langle (d\Gamma - dJ^{-1} \wedge dJ), \ d\Phi \right\rangle_{L^2} + \left\langle J^{-1}A, \ \delta\Phi \right\rangle_{L^2}$$
(8.21)

for any matrix valued 1-form $\Phi \in W_0^{1,p^*}(\Omega)$. This is the weak formulation of (4.1). Similarly, we find the weak formulation of (4.2) to be

$$-\Delta J[\phi] = \left\langle J \cdot \Gamma, \ d\phi \right\rangle_{L^2} + \left\langle \left(\left\langle dJ; \tilde{\Gamma} \right\rangle + A \right), \ \phi \right\rangle_{L^2}, \tag{8.22}$$

for all matrix valued 0-form $\phi \in W_0^{1,(2p)^*}(\Omega)$, where $\Delta J[\phi] = -\langle dJ, d\phi \rangle_{L^2}$. Since we address the solution space $J \in W^{1,2p}(\Omega)$, we require the test space $W_0^{1,(2p)^*}(\Omega)$, where $(2p)^*$ denotes the conjugate exponent to 2p, i.e., $\frac{1}{(2p)^*} + \frac{1}{2p} = 1$, (and note that $(2p)^* \neq 2p^*$ in general). To derive the weak formulation of (4.3) - (4.4), we first clarify how to

To derive the weak formulation of (4.3) - (4.4), we first clarify how to shift \overrightarrow{div} over to test functions, i.e., we have to find the adjoint to the operation defined in (3.11). For this, we introduce \overrightarrow{div} as the mapping from vector valued 2-forms $\psi \in W_0^{1,(2p)^*}(\Omega)$ to matrix valued 2-forms in $L^{(2p)^*}(\Omega)$ defined by

$$\left(\underbrace{\operatorname{div}}_{\nu}(\psi)\right)_{\nu}^{\mu} = \partial_{\nu}\psi_{ij}^{\mu} \, dx^{i} \wedge dx^{j}, \qquad (8.23)$$

where the conjugate exponent $(2p)^*$ in used in light of the desired solution space $A \in L^{2p}(\Omega)$ below. Now, by applying partial integration component wise, it is easy to verify the following lemma.

Lemma 8.3. For any matrix valued 2-form ω , it is

$$\langle \vec{div}(\omega), \psi \rangle_{L^2} = -\langle \omega, \underline{div}(\psi) \rangle_{L^2}$$
(8.24)

for any vector valued 2-form $\psi \in W_0^{1,(2p)^*}(\Omega)$.

Proof. By the definition in (3.11) we have $\overrightarrow{\operatorname{div}}(\omega)^{\mu} \equiv \sum_{\nu=1}^{n} \partial_{\nu}(\omega_{\nu}^{\mu})_{ij} dx^{i} \wedge dx^{j}$, so components-wise partial integration gives us

$$\begin{split} \langle \overrightarrow{div}(\omega), \psi \rangle_{L^2} &= \sum_{\mu} \sum_{i < j} \int_{\Omega} \sum_{\nu} \partial_{\nu} (\omega^{\mu}_{\nu})_{ij} \psi^{\mu}_{ij} dx \\ &= \sum_{\mu,\nu} \sum_{i < j} \int_{\Omega} (\omega^{\mu}_{\nu})_{ij} \partial_{\nu} \psi^{\mu}_{ij} dx \\ &= -\langle \omega, \underline{div}(\psi) \rangle_{L^2}, \end{split}$$
(8.25)

which proves (8.24) and the lemma.

Applying now (8.24) together with (8.7) for vector valued differential forms, we find that (4.3) - (4.4) can be written equivalently as

$$\begin{cases} \langle \vec{A}, \delta\psi \rangle_{L^2} = \langle (dJ \wedge \Gamma + J \, d\Gamma), \underline{\operatorname{div}}(\psi) \rangle_{L^2} - \langle \langle \overline{dJ}; \tilde{\Gamma} \rangle, \delta\psi \rangle_{L^2} \\ \langle \vec{A}, d\varphi \rangle_{L^2} = -\langle v, \varphi \rangle_{L^2}, \end{cases}$$
(8.26)

for all vector valued 2-form $\psi \in W_0^{1,(2p)^*}(\Omega)$ and any vector valued function $\varphi \in W_0^{1,(2p)^*}(\Omega)$, (so $\psi|_{\partial\Omega} = 0$ and $\varphi|_{\partial\Omega} = 0$). This is the weak formulation of (4.3) - (4.4). For the purpose of this paper we only need the weak formulation when $\Gamma, d\Gamma \in L^{\infty}$, but we introduce it here more generally for L^p spaces.

Definition 8.4. Let Γ , $d\Gamma \in L^{2p}(\Omega)$, and assume $\tilde{\Gamma} \in W^{1,p}(\Omega)$, $J \in W^{1,2p}(\Omega)$ and $A \in L^{2p}(\Omega)$. We say $(\tilde{\Gamma}, J, A)$ is a weak solution of the RT-equations if (8.21), (8.22) and (8.26) hold for all test functions $\Phi \in W_0^{1,p^*}(\Omega)$ and

 $\phi, \psi, \varphi \in W_0^{1,(2p)^*}(\Omega)$ specified above, where p^* and $(2p)^*$ are the conjugate exponents defined by $\frac{1}{p^*} + \frac{1}{p} = 1$ and $\frac{1}{(2p)^*} + \frac{1}{2p} = 1$.

The weak formulation of the RT-equations in Definition 8.4 can be easily adapted to the reduced RT-equations, subject of the next definition. Recall first that the reduced RT-equations (4.10) - (4.12) are

$$\begin{aligned} \Delta J &= \delta(J \cdot \Gamma) - B, \\ d\vec{B} &= \overrightarrow{\operatorname{div}} (dJ \wedge \Gamma) + \overrightarrow{\operatorname{div}} (J \, d\Gamma), \\ \delta \vec{B} &= w, \end{aligned}$$

with unknowns J and B.

Definition 8.5. Let $\Gamma, d\Gamma \in L^{2p}(\Omega)$. For $J \in W^{1,2p}(\Omega)$ and $B \in L^{2p}(\Omega)$, we say that (J,B) is a weak solution of the reduced RT-equations (4.10) - (4.12), if

$$-\Delta J[\phi] = \langle J \cdot \Gamma, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2}$$
(8.27)

holds for any matrix valued 0-form $\phi \in W_0^{1,(2p)^*}(\Omega)$, (where $\Delta J[\phi] = -\langle dJ, d\phi \rangle_{L^2}$), and if

$$\begin{cases} \langle \vec{B}, \delta\psi \rangle_{L^2} = \langle (dJ \wedge \Gamma + J \, d\Gamma), \underline{div}(\psi) \rangle_{L^2} \\ \langle \vec{B}, d\varphi \rangle_{L^2} = w, \end{cases}$$
(8.28)

holds for any vector valued 2-form $\psi \in W_0^{1,(2p)^*}(\Omega)$ and any vector valued function $\varphi \in W_0^{1,(2p)^*}(\Omega)$.

9. The gauge freedom - Proof of Theorem 4.1

In this section we prove that, given a solution (J, B) of the reduced RTequations (4.10) - (4.12), then $(J, \tilde{\Gamma}, A)$ solves the full RT-equations (4.1) -(4.4) with $\tilde{\Gamma} = \tilde{\Gamma}_J$ and A = A', where $\tilde{\Gamma}_J$ and A' are defined in (4.6) and (4.14) as

$$\begin{split} \tilde{\Gamma}_J &= \Gamma - J^{-1} dJ, \\ A' &= B - \langle dJ; \tilde{\Gamma} \rangle. \end{split}$$

From this, using the first RT-equation (4.1) in terms of A = A' (which is equivalent to equation (4.13)), we then prove that $\tilde{\Gamma}_J$ gains one derivative over the regularity of the terms separately on the right hand side of its definition in (4.6). This is the subject of Theorem 4.1. Throughout the rest of this paper we only address the case $\tilde{\Gamma} = \tilde{\Gamma}_J$ and A = A', so for ease of notation from here on, we denote $\tilde{\Gamma}_J$ by $\tilde{\Gamma}$ and A' by A.

For completeness, and to connect our point of view here with [23], in Section 9.2, we prove Theorem 4.1 at the higher level of connection regularity, $\Gamma, d\Gamma \in W^{m,p}(\Omega), m \geq 1, p > n$, a regularity sufficient to take point-wise L^p estimates and apply Morrey's inequality (A.4). In Section 9.2 we prove the low regularity case $\Gamma, d\Gamma \in W^{m,p}(\Omega)$.

9.1. The smooth case of Theorem 4.1. We here prove the following proposition which establishes Theorem 4.1 in the case $\Gamma, d\Gamma \in W^{m,p}(\Omega)$, $m \geq 1, p > n$. This is proposition is not needed to prove the results in this paper. For ease of notation we let $\Omega \equiv \Omega_x$ denote the neighborhood Ω'_x of Theorem 4.1 in x-coordinates, and we denote the compactly contained subset Ω''_x by Ω' .

Proposition 9.1. Assume Γ , $d\Gamma \in W^{m+1,p}(\Omega)$ in x-coordinates, for $m \geq 1$, p > n and $p < \infty$. Assume (J, B) solves the reduced RT-equations (4.10) - (4.12) in a weak sense for some $w \in W^{m-1,p}(\Omega)$, such that $J, J^{-1} \in W^{m+1,p}(\Omega)$ and $B \in W^{m,p}(\Omega)$.²¹ Then the following holds:

(i) Defining $\tilde{\Gamma}_J$ by (4.6), $\tilde{\Gamma} \equiv \tilde{\Gamma}_J$ solves (4.13), and the tuple $(J, \tilde{\Gamma}, A)$ solves the full RT-equations (4.1) - (4.4) in Ω for $v \equiv v'$ and $A \equiv A'$, where A' and v' are defined in (4.14) and (4.15).

(ii) The regularity of $\tilde{\Gamma} \equiv \tilde{\Gamma}_J$ is given by $\tilde{\Gamma}_J \in W^{m+1,p}(\Omega')$ for any open set Ω' compactly contained in Ω , i.e., $\tilde{\Gamma}_J$ is one order more regular than the two terms are separately on the right hand side of its definition in (4.6). (iii) Let M > 0 be a constant such that

 $\|(\Gamma, d\Gamma)\|_{W^{m,p}(\Omega)} \equiv \|\Gamma\|_{W^{m,p}(\Omega)} + \|d\Gamma\|_{W^{m,p}(\Omega)} \le M.$

Assume that (J, B) satisfies further the estimate

 $\|I - J\|_{W^{m+1,p}(\Omega)} + \|I - J^{-1}\|_{W^{m+1,p}(\Omega)} + \|B\|_{W^{m,p}(\Omega)} \leq C_1(M) \|(\Gamma, d\Gamma)\|_{W^{m,p}(\Omega)},$ (9.1)

for some constant $C_1(M) > 0$ depending only on Ω, n, p and M. Then, on any open set Ω' compactly contained in Ω , $\tilde{\Gamma} \equiv \tilde{\Gamma}_J$ satisfies the uniform bound

$$\|\widetilde{\Gamma}\|_{W^{m+1,p}(\Omega'_x)} \le C_2(M) \|(\Gamma, d\Gamma)\|_{W^{m,p}(\Omega)}$$

$$(9.2)$$

where $C_2(M) > 0$ is some constant depending only on Ω', Ω, n, p and M.

Proof. Let $\Gamma, d\Gamma \in W^{m,p}(\Omega)$, for $m \geq 1$, p > n, and assume (J, B) is a solution of the reduced RT-equations (4.10) - (4.12) with $J, J^{-1} \in W^{m+1,p}(\Omega)$ and $B \in W^{m,p}(\Omega)$. For this regularity Lemma 7.2 applies and yields that $(J, \tilde{\Gamma}, A)$ solves the first RT-equation (4.1).

We now prove that (J, Γ, A) solves the second RT-equation (4.2). By assumption, (J, B) solves the reduced RT-equation (4.10), that is,

$$\Delta J = \delta(J \cdot \Gamma) - B. \tag{9.3}$$

By definition of A in (4.14), we have

$$B = A + \langle dJ; \tilde{\Gamma} \rangle, \tag{9.4}$$

so substitution of (9.4) into (9.3) gives

$$\Delta J = \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A,$$

which is the sought after RT-equation (4.2).

 $^{^{21}}$ The existence of such a solution follows from our existence theory in [22, 23].

We now prove that $(J, \tilde{\Gamma}, A)$ solves the last two RT-equations (4.3) - (4.4). By assumption, (J, B) solves (4.3) - (4.4) for some $w \in W^{m-1,p}(\Omega)$, that is,

$$d\vec{B} = \vec{\operatorname{div}}(dJ \wedge \Gamma) + \vec{\operatorname{div}}(J \, d\Gamma), \qquad (9.5)$$

$$\delta \vec{B} = w. \tag{9.6}$$

Substituting (9.4) into (9.5) and subtracting the resulting equation by $d\langle dJ; \tilde{\Gamma} \rangle$ gives us the equation

$$d\vec{A} = \overrightarrow{\operatorname{div}} (dJ \wedge \Gamma) + \overrightarrow{\operatorname{div}} (J \, d\Gamma) - d \overline{\langle dJ; \tilde{\Gamma} \rangle},$$

which is the sought after third RT-equation (4.3). Similarly, substituting (9.4) into (9.6) gives

$$\delta \vec{A} = w - \delta \langle dJ; \tilde{\Gamma} \rangle,$$

which is the sought after RT-equation (4.4) for v defined by (4.15), that is, for $v = w - \delta \langle dJ; \tilde{\Gamma} \rangle$. Taken together, we proved that $(J, \tilde{\Gamma}, A)$ solves the full RT-equations (4.1) - (4.4) for the gauge freedom in v fixed by the choice (4.15). This proves (i) of the proposition.

To prove (ii) of Proposition 9.1, we need to show the regularity $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ together with estimate (9.2). In a first step, we establish the lower regularity $\tilde{\Gamma} \in W^{m,p}(\Omega)$ and $A \in W^{m,p}(\Omega)$ from their definitions in (4.6) and (4.14). For this we use that by Morrey's inequality (A.4) the space $W^{m,p}(\Omega)$ is closed under multiplication when $m \geq 1$, p > n, c.f. [21]. Now $\tilde{\Gamma} \in W^{m,p}(\Omega)$, since $\Gamma \in W^{m,p}(\Omega)$, $J^{-1} \in W^{m+1,p}(\Omega)$ and $dJ \in W^{m,p}(\Omega)$ so that the closedness of $W^{m,p}(\Omega)$ yields $\tilde{\Gamma} = \Gamma - J^{-1}dJ \in W^{m,p}(\Omega)$ by (4.6). Moreover, the regularity $A \in W^{m,p}(\Omega)$ directly follows from (4.14), since $dJ, \tilde{\Gamma} \in W^{m,p}(\Omega)$ implies that $A = B - \langle dJ; \tilde{\Gamma} \rangle \in W^{m,p}(\Omega)$. This shows that $\tilde{\Gamma}$ and A are both in $W^{m,p}(\Omega)$.

We now prove that $\tilde{\Gamma} \in W^{m,p}(\Omega)$ is one derivative more regular, $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, by establishing estimate (9.2). For this, we use the first RT-equation (4.1),

$$\Delta \tilde{\Gamma} = \delta d\Gamma - \delta (dJ^{-1} \wedge dJ) + d(J^{-1}A).$$
(9.7)

But estimate (B.8) of elliptic regularity theory gives

$$\|\tilde{\Gamma}\|_{W^{m+1,p}(\Omega')} \le C_e \left(\|\Delta \tilde{\Gamma}\|_{W^{m-1,p}(\Omega)} + \|\tilde{\Gamma}\|_{W^{m,p}(\Omega)} \right)$$
(9.8)

for any open set Ω' compactly contained in Ω , where C > 0 is some constant depending only on Ω', Ω, p, n, m . The regularity gain of $\tilde{\Gamma}$ follows once we show $\Delta \tilde{\Gamma} \in W^{m-1,p}(\Omega)$, since $\tilde{\Gamma} \in W^{m,p}(\Omega)$. But to derive the sought after estimate (9.2) we need a more refined analysis. For this, we begin by substituting for $\Delta \tilde{\Gamma}$ the right hand side of (9.7) to obtain

$$\begin{split} &\|\tilde{\Gamma}\|_{W^{m+1,p}(\Omega')} \\ &\leq C_e \Big(\|\delta d\Gamma\|_{W^{m-1,p}(\Omega)} + \|\delta \big(dJ^{-1} \wedge dJ \big)\|_{W^{m-1,p}(\Omega)} + \|d \big(J^{-1}A \big)\|_{W^{m-1,p}(\Omega)} \Big) \\ &\leq C_e \Big(\|d\Gamma\|_{W^{m,p}(\Omega)} + \|\delta \big(dJ^{-1} \wedge dJ \big)\|_{W^{m-1,p}(\Omega)} + \|J^{-1}A\|_{W^{m,p}(\Omega)} \Big). \end{split}$$
(9.9)

(We subsequently often write $\|\cdot\|_{W^{m,p}}$ instead of $\|\cdot\|_{W^{m,p}(\Omega)}$.) The first term on the right hand side of (9.9) is bounded by assumption. Using first the product rule and then Morrey's inequality (A.4), the second term can be bounded by

$$\begin{split} \|\delta(dJ^{-1} \wedge dJ)\|_{W^{m-1,p}(\Omega)} \\ &\leq \||D(dJ^{-1})| \cdot |dJ|\|_{W^{m-1,p}} + \||dJ^{-1}| \cdot |D(dJ)|\|_{W^{m-1,p}} \\ &\leq \|D(dJ^{-1})\|_{W^{m-1,p}} \|dJ\|_{W^{m-1,\infty}} + \|dJ^{-1}\|_{W^{m-1,\infty}} \|D(dJ)\|_{W^{m-1,p}} \\ &\leq \|J^{-1}\|_{W^{m+1,p}} \|J\|_{W^{m,\infty}} + \|J^{-1}\|_{W^{m,\infty}} \|J\|_{W^{m+1,p}} \\ &\stackrel{(A.4)}{\leq} 2C_M \|J^{-1}\|_{W^{m+1,p}(\Omega)} \|J\|_{W^{m+1,p}(\Omega)}, \end{split}$$

which is bounded by our incoming assumptions $J^{-1}, J \in W^{m+1,p}(\Omega)$. Using that dJ = d(J - I), the previous estimate gives us

$$\|\delta(dJ^{-1} \wedge dJ)\|_{W^{m-1,p}(\Omega)} \le 2C_M \|J^{-1}\|_{W^{m+1,p}(\Omega)} \|I - J\|_{W^{m+1,p}(\Omega)}$$
(9.10)

To estimate the third term on the right hand side of (9.9), we use that by definition $\tilde{\Gamma} \equiv \tilde{\Gamma}_J = \Gamma - J^{-1} dJ$ and $A = B - \langle dJ; \tilde{\Gamma} \rangle$, to write

$$J^{-1}A = J^{-1}B - J^{-1}\langle dJ; \Gamma \rangle + \langle dJ^{-1}; dJ \rangle,$$

where we used the multiplication property (3.10) to get $J^{-1}\langle dJ; J^{-1}dJ \rangle = \langle dJ^{-1}; dJ \rangle$. We now estimate the third term on the right hand side of (9.9) as

$$\|J^{-1}A\|_{W^{m,p}(\Omega)} \le \|J^{-1}B\|_{W^{m,p}} + \|J^{-1}\langle dJ;\Gamma\rangle\|_{W^{m,p}} + \|\langle dJ^{-1};dJ\rangle\|_{W^{m,p}}.$$
(9.11)

We now use the closedness of $W^{m,p}(\Omega)$ under multiplication $(m \ge 1, n > p)$ by Morrey's inequality (A.4), to estimate the products in (9.11), for instance,

$$\begin{split} \|J^{-1}B\|_{W^{m,p}(\Omega)} &= \|D(J^{-1}B)\|_{W^{m-1,p}} + \|J^{-1}B\|_{W^{m-1,p}} \\ &\leq \|D(J^{-1})B\|_{W^{m-1,p}} + \|J^{-1}DB\|_{W^{m-1,p}} + \|J^{-1}B\|_{W^{m-1,p}} \\ &\leq \|D(J^{-1})\|_{W^{m-1,p}} \|B\|_{W^{m-1,\infty}} + \|J^{-1}\|_{W^{m-1,\infty}} \|DB\|_{W^{m-1,p}} \\ &+ \|J^{-1}\|_{W^{m-1,\infty}} \|B\|_{W^{m-1,p}} \\ &\leq C_M \|J^{-1}\|_{W^{m,p}} \|B\|_{W^{m,p}}. \end{split}$$

In this fashion, replacing dJ on the right hand side of (9.11) by d(I - J), we obtain

$$\|J^{-1}A\|_{W^{m,p}} \leq C\|J^{-1}\|_{W^{m+1,p}} \Big(\|B\|_{W^{m,p}} + \|J\|_{W^{m+1,p}}\|\Gamma\|_{W^{m,p}} + \|I - J\|_{W^{m+1,p}}\Big)$$

$$\leq C\|J^{-1}\|_{W^{m+1,p}} \Big(1 + \|J\|_{W^{m+1,p}}\Big) \Big(\|B\|_{W^{m,p}} + \|\Gamma\|_{W^{m,p}} + \|I - J\|_{W^{m+1,p}}\Big),$$

(9.12)

where C > 0 is some constant depending only on Ω, m, n, p . Substituting (9.10) and (9.12) back into the original estimate (9.9), we obtain

$$\|\tilde{\Gamma}\|_{W^{m+1,p}(\Omega')} \le \mathcal{P}(J) \Big(\|\Gamma\|_{W^{m,p}} + \|d\Gamma\|_{W^{m,p}} + \|I-J\|_{W^{m+1,p}} + \|B\|_{W^{m,p}} \Big) + C \|\tilde{\Gamma}\|_{W^{m,p}}$$
(9.13)

where $\mathcal{P}(J) \equiv C(1+\|J^{-1}\|_{W^{m+1,p}})(1+\|J\|_{W^{m+1,p}})$ for some constant C > 0depending only on Ω, m, n, p . By the definition of $\tilde{\Gamma}$ in (4.6), we bound $\|\tilde{\Gamma}\|_{W^{m,p}(\Omega)}$ using Morrey's inequality as

$$\|\tilde{\Gamma}\|_{W^{m,p}(\Omega)} \le C_M \big(\|\Gamma\|_{W^{m,p}(\Omega)} + \|J^{-1}\|_{W^{m+1,p}(\Omega)} \|J\|_{W^{m+1,p}(\Omega)} \big),$$

from which we obtain the simplified bound

$$\|\tilde{\Gamma}\|_{W^{m+1,p}(\Omega')} \le \mathcal{P}(J)\Big(\|\Gamma\|_{W^{m,p}} + \|d\Gamma\|_{W^{m,p}} + \|I - J\|_{W^{m+1,p}} + \|B\|_{W^{m,p}}\Big),$$
(9.14)

by changing the constant C > 0 in the definition on \mathcal{P} suitably. Finally, using our incoming assumption (9.1) to bound the above Sobolev norms on J, J^{-1} and B, we obtain the sought after uniform bound (9.2). Clearly, estimate (9.2) implies the sought after regularity $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$.²² This completes the proof of Proposition 9.1.

9.2. **Proof of Theorem 4.1.** We now prove Theorem 4.1, roughly following the steps in the proof of Proposition 9.1, but adapted to the weak formulation of the RT-equations to account for the low regularity addressed here. This is significantly more complicated because the substitution of the *J*-equation in the proof of Lemma 7.2, (required to get the cancellation of terms involving $\delta\Gamma$, on which our whole theories rests), is not a simple replacement when dealing with the weak form of the equations, due to the problem of multiplying distributions by low regularity functions, c.f. Lemma 9.2 below. Moreover, for the low regularity here, products must be estimated by Hölder inequality instead of Morrey, which we compensate for by putting *J* in the smaller Sobolev space $W^{1,2p}$ and estimating $\tilde{\Gamma}$ in $W^{1,p}$. Even though we begin here with a given solution of the reduced RT-equations, low regularity products have to be incorporated into the weak formulation of the $\tilde{\Gamma}$ equation, a reflection of the fact that the RT-equations are non-linear.

²²Note, the gain of one derivative to the required regularity $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$, is entirely based on the cancellation of $\delta\Gamma$ -terms in equation (7.6) of the proof of Lemma 7.2.

So assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in *x*-coordinates and $\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)} \leq M$ for some constant M > 0, and let n . Assume <math>(J, B) solves the reduced RT-equations (4.10) - (4.12) on Ω , such that

$$\|I - J\|_{W^{1,2p}(\Omega)} + \|I - J^{-1}\|_{W^{1,2p}(\Omega)} + \|B\|_{L^{2p}(\Omega)} \leq C_2(M)\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)},$$
(9.15)

where $C_2(M) > 0$ is some constant depending only on Ω, n, p and M. Define $\tilde{\Gamma}$ by (4.6) and A by (4.14), that is,

$$\tilde{\Gamma} = \tilde{\Gamma}_J \equiv \Gamma - J^{-1} dJ$$
 and $A = A' \equiv B - \langle dJ; \tilde{\Gamma} \rangle$.

Again we denote $\tilde{\Gamma}_J$ by $\tilde{\Gamma}$ and A' by A from here on, and we let $\Omega \equiv \Omega_x$ denote the neighborhood Ω'_x of Theorem 4.1 in *x*-coordinates, and we denote the compactly contained subset Ω''_x by Ω' . Theorem 4.1 (*i*) states that $(J, \tilde{\Gamma}, A)$ solves the full RT-equations (4.1) - (4.4) in Ω for v = v' defined in (4.15). Parts (*ii*) and (*iii*) of Theorem 4.1 states that $\tilde{\Gamma}$ is in $W^{1,p}(\Omega')$ and satisfies the uniform bound

$$\|\Gamma\|_{W^{1,p}(\Omega')} \le C_3(M) \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)},$$
(9.16)

on any open set Ω' compactly contained in Ω , and where $C_3(M) > 0$ is some constant depending only on Ω', Ω, n, p and M.

Anticipating future work, we give the proof of Theorem 4.1 under the weaker assumption that $\Gamma, d\Gamma \in L^{2p}(\Omega)$, assuming the uniform bound

$$\|(\Gamma, d\Gamma)\|_{L^{2p}(\Omega)} \equiv \|\Gamma_x\|_{L^{2p}(\Omega)} + \|d\Gamma_x\|_{L^{2p}(\Omega)} \le M,$$

in place of $\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)} \leq M$ and assuming the right hand side of (9.15) is bounded by $\|(\Gamma, d\Gamma)\|_{L^{2p}(\Omega)}$ in place of $\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)}$. Since

$$\|(\Gamma, d\Gamma)\|_{L^{2p}(\Omega)} \le \operatorname{vol}(\Omega)^{\frac{1}{2p}} \|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)}$$

for Ω bounded, Theorem 4.1 follows directly by substituting $\|(\Gamma, d\Gamma)\|_{L^{\infty}}$ for $\|(\Gamma, d\Gamma)\|_{L^{2p}}$.

Proof of Theorem 4.1. So assume $J \in W^{1,2p}(\Omega)$ and $B \in L^{2p}(\Omega)$ solve the reduced RT-equations (4.10) - (4.12). Since only the regularity of the gauge variable w in (4.12) is relevant for the proof, we assume here without loss of generality that w = 0, the case for which we prove existence of solutions in Section 10. So (J, B) is assumed to solve

$$\Delta J = \delta(J \cdot \Gamma) - B, \tag{9.17}$$

$$d\vec{B} = \vec{\operatorname{div}}(dJ \wedge \Gamma) + \vec{\operatorname{div}}(J \, d\Gamma), \qquad (9.18)$$

$$\delta \vec{B} = 0, \tag{9.19}$$

in the weak sense of Definition 8.5. That is, we assume

$$-\Delta J[\phi] = \langle J \cdot \Gamma, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2}$$
(9.20)

48

for any matrix valued 0-form $\phi \in W_0^{1,(2p)^*}(\Omega)$, where $\Delta J[\phi] = -\langle dJ, d\phi \rangle_{L^2}$, and we assume

$$\begin{cases} \langle \vec{B}, \delta \psi \rangle_{L^2} = \left\langle (dJ \wedge \Gamma + J \, d\Gamma), \underbrace{\operatorname{div}}(\psi) \right\rangle_{L^2} \\ \langle \vec{B}, d\varphi \rangle_{L^2} = 0, \end{cases}$$
(9.21)

for any vector valued 2-form $\psi \in W_0^{1,(2p)^*}(\Omega)$ and any vector valued function $\varphi \in W_0^{1,(2p)^*}(\Omega)$. Here $(2p)^*$ denotes the conjugate exponent of 2p, i.e., $\frac{1}{(2p)^*} + \frac{1}{2p} = 1$, so $(2p)^* = \frac{(2p)^*}{(2p)^*-1}$, (and $(2p)^* \neq 2p^*$ in general). In the proof of Theorem 4.1 we use C > 0 to denote a running constant depending only on Ω , n and p.

The proof of Theorem 4.1 requires several lemmas. Before we establish these lemmas, we derive the preliminary regularities $\tilde{\Gamma} \in L^{2p}(\Omega)$ and $A \in L^p(\Omega)$ from the definitions of $\tilde{\Gamma}$ in (4.6) and A in (4.14), regularities we require to bootstrap to the desired regularities $\tilde{\Gamma} \in W^{1,p}(\Omega)$ and $A \in L^{2p}(\Omega)$ below. For this, recall that J is assumed to be invertible with $J^{-1} \in W^{1,2p}(\Omega)$, so Morrey's inequality (A.4) implies that $J^{-1}dJ \in L^{2p}(\Omega)$. Thus, since $\Gamma \in L^{2p}(\Omega)$ by assumption, $\tilde{\Gamma} \in L^{2p}(\Omega)$ follows directly from its definition in (4.6). To show that $A \in L^p(\Omega)$, we first apply Hölder's inequality as in (A.7) to conclude that $\langle dJ; \tilde{\Gamma} \rangle \in L^p(\Omega)$, since dJ and $\tilde{\Gamma}$ are both in $L^{2p}(\Omega)$, which implies that $A \in L^p(\Omega)$ by our incoming assumption $B \in L^{2p}(\Omega)$. So we now have established the preliminary regularity $\tilde{\Gamma} \in L^{2p}(\Omega)$ and $A \in L^p(\Omega)$.

In Lemma 9.3 below we prove that $\tilde{\Gamma}$ defined in (4.6) solves the first RT-equation (4.1) for A defined by (4.14), in the weak sense (8.21). We then apply elliptic regularity theory to this equation to boost the regularity of $\tilde{\Gamma}$ one order to $\tilde{\Gamma} \in W^{1,p}(\Omega)$. By Morrey's inequality and (4.14), $\tilde{\Gamma} \in W^{1,p}(\Omega')$ then implies that $A \in L^{2p}(\Omega')$. These are the regularities stated in Theorem 4.1. The main step in the proof of Lemma 9.3 is accomplished in the following lemma by adapting the computation in the proof of Lemma 7.2 to the weak formulation required for the low regularity here.

Lemma 9.2. Under the assumption of Theorem 4.1, we have

$$\delta \tilde{\Gamma}[\phi] = \langle J^{-1}A, \phi \rangle_{L^2} \tag{9.22}$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, where $\delta \tilde{\Gamma}[\phi] \equiv -\langle \tilde{\Gamma}, d\phi \rangle_{L^2}$. That is, $\delta \tilde{\Gamma} = J^{-1}A$ in the sense of weak derivatives.²³

Proof. Assume J is a solution of the first reduced RT-equation in the weak sense (8.27) for some given $B \in L^{2p}(\Omega)$. The proof is based on adapting the computation (7.3) - (7.7) of the proof of Lemma 7.2 to regularities $J \in W^{1,2p}(\Omega)$ and $B \in L^{2p}(\Omega)$. This requires care because of the presence of

²³For the reader familiar with our previous work in [21, 23], note that establishing (9.22) reverses a basic identity in the derivation of the RT-equations from the Riemann-flat condition where we defined A in terms of $J\delta\tilde{\Gamma}$.

low regularity products. Instead of computing $\delta \tilde{\Gamma}$ directly as in (7.3), which would not yield the weak Laplacian on J (essential for the argument), we begin by taking δ of $J\tilde{\Gamma}$.²⁴ Using that $\tilde{\Gamma} \equiv \Gamma - J^{-1}dJ$ by its definition in (4.6), taking δ of $J\tilde{\Gamma}$ gives us

$$\delta(J\tilde{\Gamma})[\phi] = -\langle J\tilde{\Gamma}, d\phi \rangle_{L^2} = -\langle J\Gamma, d\phi \rangle_{L^2} + \langle dJ, d\phi \rangle_{L^2}, \qquad (9.23)$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$. Note that the expressions in (9.23) are finite by Hölder inequality (A.6), since $\Gamma \in L^{2p}(\Omega)$ and $J \in W^{1,2p}(\Omega)$, and since $L^{p^*}(\Omega) \subset L^{(2p)^*}(\Omega)$ because Ω is bounded and $p^* = \frac{p}{p-1}$ is larger than $(2p)^* = \frac{2p}{2p-1}$. For example,

$$\left| \langle dJ, d\phi \rangle_{L^2} \right| \le C \, \|dJ\|_{L^{2p}} \|d\phi\|_{L^{(2p)^*}} \le C \|dJ\|_{L^{2p}} \, \|d\phi\|_{L^{p^*}}^{\frac{p^*}{(2p)^*}} < \infty,$$

where C > 0 is some constant depending only on Ω , n and p, c.f. the use of Hölder's inequality in (A.8). Now, by the weak reduced RT-equation (9.20), we replace the the second term on the right hand side of (9.23) by $\langle dJ, d\phi \rangle_{L^2} = \langle J \cdot \Gamma, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2}$, which after the essential cancellation of the lowest regularity term $\langle J \cdot \Gamma, d\phi \rangle_{L^2}$ leads to

$$\delta(J\Gamma)[\phi] = \langle B, \phi \rangle_{L^2},$$

that is,

$$\langle J\tilde{\Gamma}, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2} = 0. \tag{9.24}$$

Our goal now is to move d to the other side of the first inner product in (9.24) as δ on the product $J\tilde{\Gamma}$ and isolate the weak derivative $\delta\tilde{\Gamma}$, from which then the sought after equation (9.22) follows. Note that at the start, $J\tilde{\Gamma}$ is not regular enough to apply the Leibnitz product rule. But by the consistency of the RT-equation we anticipate that $\tilde{\Gamma}$ is one order more regular and this explains why the following mollification argument works to establish that the product rule is valid for $\delta(J\tilde{\Gamma})$.

So now consider standard mollifiers $\Gamma_{\epsilon} \in C^{\infty}(\Omega)$ and $J_{\epsilon} \in C^{\infty}(\Omega)$ of Γ and J respectively, so that Γ_{ϵ} converges to Γ in $L^{2p}(\Omega)$ as $\epsilon \to 0$ and J_{ϵ} converges to J in $W^{1,2p}(\Omega)$ as $\epsilon \to 0$.²⁵ Define

$$\Gamma_{\epsilon} \equiv \Gamma_{\epsilon} - J^{-1} dJ_{\epsilon},$$

then $\tilde{\Gamma}_{\epsilon} \in W^{1,2p}(\Omega)$. To show that $\tilde{\Gamma}_{\epsilon} \longrightarrow \tilde{\Gamma}$ in $L^{2p}(\Omega)$ as $\epsilon \to 0$, recall $J^{-1} \in W^{1,2p}(\Omega)$ is bounded in $L^{\infty}(\Omega)$ by Morrey's inequality, so we have

$$\|J^{-1}(dJ_{\epsilon} - dJ)\|_{L^{2p}} \le \|J^{-1}\|_{L^{\infty}} \|dJ_{\epsilon} - dJ\|_{L^{2p}} \longrightarrow 0$$

 $^{^{24}}$ Note, applying the results of Lemma 7.2 in this setting at a mollified level, would entail the problem of controlling the zero mollification limit through the second reduced RT-equation (4.2), a system of inhomogeneous elliptic PDE's for which it would be difficult to avoid generalized eigenvalues.

²⁵Since the test functions have compact support, mollifying does not change the region of integration Ω .

as $\epsilon \to 0$, and this implies that $\tilde{\Gamma}_{\epsilon} \longrightarrow \tilde{\Gamma}$ in $L^{2p}(\Omega)$ as $\epsilon \to 0$. Now, since $\tilde{\Gamma}_{\epsilon} \longrightarrow \tilde{\Gamma}$ in $L^{2p}(\Omega)$ as $\epsilon \to 0$, equation (9.24) implies that

$$\lim_{\epsilon \to 0} \left(\langle J \tilde{\Gamma}_{\epsilon}, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2} \right) = 0.$$
(9.25)

The regularity $\tilde{\Gamma}_{\epsilon} \in W^{1,2p}(\Omega)$ allows us now to use partial integration (8.7), followed by the Leibniz rule (3.8), $\delta(J\tilde{\Gamma}_{\epsilon}) = J\delta\tilde{\Gamma}_{\epsilon} + \langle dJ;\tilde{\Gamma}\rangle$, to compute

$$\langle J\tilde{\Gamma}_{\epsilon}, d\phi \rangle_{L^{2}} + \langle B, \phi \rangle_{L^{2}} = -\langle \delta(J\tilde{\Gamma}_{\epsilon}), \phi \rangle_{L^{2}} + \langle B, \phi \rangle_{L^{2}} \stackrel{(3.8)}{=} -\langle J\delta\tilde{\Gamma}_{\epsilon}, \phi \rangle_{L^{2}} + \langle (B - \langle dJ; \tilde{\Gamma}_{\epsilon} \rangle), \phi \rangle_{L^{2}} = -\langle J\delta\tilde{\Gamma}_{\epsilon}, \phi \rangle_{L^{2}} + \langle A_{\epsilon}, \phi \rangle_{L^{2}},$$

$$(9.26)$$

where

$$A_{\epsilon} \equiv B - \langle dJ; \Gamma_{\epsilon} \rangle.$$

Note that $A_{\epsilon} \to A$ in $L^{p}(\Omega)$ as $\epsilon \to 0$ by the convergence properties of $\tilde{\Gamma}_{\epsilon}$ and J_{ϵ} . We now applying the multiplication property of $\langle \cdot; \cdot \rangle$ in (3.10) together with cyclic commutativity of matrix multiplication in the trace, to write (9.26) as

$$\langle J\tilde{\Gamma}_{\epsilon}, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2} = -\langle \delta\tilde{\Gamma}_{\epsilon}, \phi J \rangle_{L^2} + \langle J^{-1}A_{\epsilon}, \phi J \rangle_{L^2}.$$
(9.27)

To clarify this step, consider for example

$$\begin{split} \langle J\delta\tilde{\Gamma}_{\epsilon},\phi\rangle_{L^{2}} \stackrel{(8.2)}{=} & \int_{\Omega} \operatorname{tr}\langle J\delta\tilde{\Gamma}_{\epsilon};\phi^{T}\rangle dx \stackrel{(3.10)}{=} \int_{\Omega} \operatorname{tr}\left(J\cdot\langle\delta\tilde{\Gamma}_{\epsilon};\phi^{T}\rangle\right) dx \\ &= \int_{\Omega} \operatorname{tr}\left(\langle\delta\tilde{\Gamma}_{\epsilon};\phi^{T}\rangle\cdot J\right) dx \stackrel{(3.10)}{=} \int_{\Omega} \operatorname{tr}\left(\langle\delta\tilde{\Gamma}_{\epsilon};(J^{T}\phi)^{T}\rangle = \langle\delta\tilde{\Gamma}_{\epsilon},J^{T}\phi\rangle_{L^{2}} dx \right) dx \end{split}$$

Applying now the integration by parts formula (8.7) to the first term on the right hand side of (9.27) and defining $\psi \equiv J^T \phi$, we obtain

$$\langle J\tilde{\Gamma}_{\epsilon}, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2} = \langle \tilde{\Gamma}_{\epsilon}, d\psi \rangle_{L^2} + \langle J^{-1}A_{\epsilon}, \psi \rangle_{L^2}.$$
(9.28)

Since $J \in W^{1,2p}(\Omega)$, and $2p > n > p^*$, it follows that $\psi \equiv J^T \phi \in W_0^{1,p^*}(\Omega)$ is indeed a test function, (c.f. Lemma D.1 in the appendix). Thus, since $\tilde{\Gamma}_{\epsilon}$ converges to $\tilde{\Gamma}$ in $L^{2p}(\Omega)$ and since $A_{\epsilon} \to A$ in $L^p(\Omega)$ as $\epsilon \to 0$, we conclude that the right hand side in (9.28) converges as $\epsilon \to 0$. Moreover, by (9.25) the left hand side in (9.28) converges as well and the limit of (9.28) as $\epsilon \to 0$ vanishes, which gives

$$-\langle \tilde{\Gamma}, d\psi \rangle_{L^2} = \langle J^{-1}A, \psi \rangle_{L^2}, \qquad (9.29)$$

for $\psi = J^T \phi$. Since any test function $\psi \in W_0^{1,p^*}(\Omega)$ can be written as $\psi = J^T \phi$ for some $\phi \in W_0^{1,p^*}(\Omega)$, (c.f. Lemma D.1 in the appendix), we conclude that the sought after equation (9.29) holds for any $\psi \in W_0^{1,p^*}(\Omega)$. This proves Lemma 9.2.

We now apply Lemma 9.2 to establish that $\tilde{\Gamma}$ solves the first RT-equation for (4.1) for A = A' defined in (4.14), (which is equivalent to first RTequation (4.13) in terms of B), at the correct level of regularity. This lemma establishes the first statement in part (i) of Theorem 4.1 together with part (ii).

Lemma 9.3. Under the assumption of Theorem 4.1, $\tilde{\Gamma}$ solves the first RTequation (4.1) for A defined by (4.14), and $\tilde{\Gamma} \in W^{1,p}(\Omega), A \in L^{2p}(\Omega)$.

Proof. To begin recall that by (9.22) of Lemma 9.2, we have

$$\delta \tilde{\Gamma}[\phi] = \langle J^{-1}A, \phi \rangle_{L^2}$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, where $\delta \tilde{\Gamma}[\phi] \equiv -\langle \tilde{\Gamma}, d\phi \rangle_{L^2}$. Since $A \in L^p(\Omega)$ and $J^{-1} \in W^{1,2p}(\Omega)$, this directly implies that the weak co-derivative δ of $\tilde{\Gamma}$ is an L^p function, $\delta \tilde{\Gamma} \in L^p(\Omega)$, and it follows that

$$\delta \tilde{\Gamma} = J^{-1} A \in L^p(\Omega) \tag{9.30}$$

holds in the sense of L^p functions. We can now take the exterior derivative d of $\delta \tilde{\Gamma}$ in a weak sense, which gives in light of (9.30) that

$$\langle \delta \Gamma, \delta \Phi \rangle_{L^2} = \langle J^{-1} A, \delta \Phi \rangle_{L^2} \tag{9.31}$$

for any matrix valued 1-form $\Phi \in W_0^{1,p^*}(\Omega)$. This determines the second term of the weak Laplacian of $\tilde{\Gamma}$ in (8.20).

To determine the first term of the weak Laplacian $\Delta \tilde{\Gamma}[\Phi]$ in (8.20), we take the exterior derivative of $\tilde{\Gamma}_{\epsilon} \equiv \Gamma_{\epsilon} - J^{-1}dJ_{\epsilon}$, where Γ_{ϵ} and J_{ϵ} are the mollifications of Γ and J introduced in the proof of Lemma 9.2, (required to apply the Leibnitz rule to low regularity products). This gives us

$$d\tilde{\Gamma}_{\epsilon} = d\Gamma_{\epsilon} - d(J^{-1}dJ_{\epsilon}) = d\Gamma_{\epsilon} - dJ^{-1} \wedge dJ_{\epsilon}, \qquad (9.32)$$

where we applied the Leibniz rule (3.6) for the last equality. We now show that the right hand side of (9.32) converges in $L^p(\Omega)$ as $\epsilon \to 0$. For this recall first that $d\Gamma \in L^{2p}(\Omega)$, which implies that $d(\Gamma_{\epsilon}) = (d\Gamma)_{\epsilon} \longrightarrow d\Gamma$ in $L^{2p}(\Omega)$ as $\epsilon \to 0$. Moreover, by using Hölder's inequality (A.6) as in (A.9) we find that

$$\begin{aligned} \left\| dJ^{-1} \wedge \left(dJ_{\epsilon} - dJ \right) \right\|_{L^{p}}^{p} &\leq C \left\| |dJ^{-1}|^{p} \right\|_{L^{2}} \left\| |dJ_{\epsilon} - dJ|^{p} \right\|_{L^{2}} \\ &\leq C \left\| dJ^{-1} \right\|_{L^{2p}}^{p} \left\| dJ_{\epsilon} - dJ \right\|_{L^{2p}}^{p} \end{aligned}$$
(9.33)

where the point-wise norm $|\cdot|^p$ on matrix valued differential form is defined in (A.1). Estimate (9.33) implies by L^{2p} convergence of dJ_{ϵ} to dJ that $dJ^{-1} \wedge dJ_{\epsilon}$ converges to $dJ^{-1} \wedge dJ$ in $L^p(\Omega)$ as $\epsilon \to 0$. We conclude that the right hand side of (9.32) converges in $L^p(\Omega)$ as $\epsilon \to 0$, and this yields that

$$d\Gamma = d\Gamma - dJ^{-1} \wedge dJ \in L^p(\Omega).$$
(9.34)

Taking now the co-derivative δ of (9.34) gives

$$\delta d\tilde{\Gamma} = \delta d\Gamma - \delta (d(J^{-1}) \wedge dJ) \in W^{-1,p}(\Omega),$$

but of course in the weak sense

$$\langle d\tilde{\Gamma}, d\Phi \rangle_{L^2} = \langle \left(d\Gamma - dJ^{-1} \wedge dJ \right), d\Phi \rangle_{L^2}, \tag{9.35}$$

52

for any matrix valued 1-form $\Phi \in W_0^{1,p^*}(\Omega)$. Combing now (9.31) and (9.35), and using that by our definition in (8.20)

$$\Delta \tilde{\Gamma}[\Phi] = -\langle \delta \tilde{\Gamma}, \delta \Phi \rangle_{L^2} - \langle d \tilde{\Gamma}, d \Phi \rangle_{L^2},$$

we finally obtain that $\tilde{\Gamma}$ solves the sought after first RT-equation,

$$\Delta \tilde{\Gamma}[\Phi] = -\langle \left(d\Gamma - dJ^{-1} \wedge dJ \right), d\Phi \rangle_{L^2} - \langle J^{-1}A, \delta\Phi \rangle_{L^2}, \tag{9.36}$$

in the weak form (8.21).

To complete the proof of Lemma 9.3, it remains to show $\tilde{\Gamma}$ gains one derivative over Γ , i.e., $\tilde{\Gamma} \in W^{1,p}(\Omega)$. For this, note that we have already established that $\delta \tilde{\Gamma}$ and $d\tilde{\Gamma}$ are in $L^p(\Omega)$, c.f. (9.30) and (9.34). Thus, since the right hand side of (9.36) results from taking d of $\delta \tilde{\Gamma}$ and δ of $d\tilde{\Gamma}$, we conclude that the right hand side of (9.36) lies in $W^{-1,p}(\Omega)$, that is,

$$\Delta \tilde{\Gamma} \in W^{-1,p}(\Omega). \tag{9.37}$$

Applying now Lemma 8.2 for the case of regularity $\tilde{\Gamma} \in L^p(\Omega)$ with $d\tilde{\Gamma}, \delta\tilde{\Gamma} \in L^p(\Omega)$, we find that $\Delta \tilde{\Gamma}[\Phi] = -\langle \nabla \tilde{\Gamma}, \nabla \Phi \rangle_{L^2}$. Thus the standard weak Laplacian $-\langle \nabla \tilde{\Gamma}, \nabla \Phi \rangle_{L^2}$ lies in $W^{-1,p}(\Omega)$ by (9.37). Applying now basic elliptic regularity theory, c.f. Theorem B.2 in the appendix, we conclude with the sought after regularity $\tilde{\Gamma} \in W^{1,p}(\Omega')$ for any open set $\Omega' \subset \Omega$ compactly contained in Ω . Note finally, since $\tilde{\Gamma} \in W^{1,p}(\Omega)$, the regularity $A \in L^{2p}(\Omega)$ follows directly from its definition $A \equiv B - \langle dJ; \tilde{\Gamma} \rangle$ in (4.14). This completes the proof of Lemma 9.3.

In the next Lemma we prove the basic elliptic estimate (4.17) from which we later derive the curvature bound (2.6), using Hölder and Morrey inequality in combination with estimate (4.16) assumed on J, J^{-1} and B. Lemma 9.4 proves part (*iii*) of Theorem 4.1.

Lemma 9.4. Under the assumption of Theorem 4.1, in particular assuming $\|(\Gamma, d\Gamma)\|_{L^{2p}} \leq M$ and the bound (9.15) on (J, B), the weak solution $\tilde{\Gamma} \equiv \tilde{\Gamma}_J$ of the first RT-equation (4.1), where $\tilde{\Gamma}_J$ is defined in (4.6), satisfies

$$\|\Gamma\|_{W^{1,p}(\Omega')} \le C(M)\|(\Gamma, d\Gamma)\|_{L^{2p}(\Omega)},$$
(9.38)

for any open set Ω' compactly contained in Ω and some constant C(M) > 0depending only on Ω, Ω', p, n and M.

Proof. By Lemma 9.3, $\tilde{\Gamma} \equiv \tilde{\Gamma}_J$ solves the weak first RT-equation (9.36), that is,

$$\Delta \tilde{\Gamma}[\Phi] = -\langle (d\Gamma - dJ^{-1} \wedge dJ), d\Phi \rangle_{L^2} - \langle J^{-1}A, \delta\Phi \rangle_{L^2}, \qquad (9.39)$$

for any matrix valued 1-form $\Phi \in W_0^{1,p^*}(\Omega)$. Applying the basic elliptic estimates (B.8) to (9.39), we obtain

$$\|\Gamma\|_{W^{1,p}(\Omega')} \le C(\|\mathcal{F}\|_{W^{-1,p}(\Omega)} + \|\Gamma\|_{L^{p}(\Omega)})$$
(9.40)

for any open Ω' compactly contained in Ω , and where we define the functional

$$\mathcal{F}(\Phi) \equiv -\left\langle \left(d\Gamma - dJ^{-1} \wedge dJ \right), d\Phi \right\rangle_{L^2} - \langle J^{-1}A, \delta\Phi \rangle_{L^2} \tag{9.41}$$

for any matrix valued 1-form $\Phi \in W_0^{1,p^*}(\Omega)$. Recall that the operator norm $\|\cdot\|_{W^{-1,p}}$ is defined by

$$\|\mathcal{F}\|_{W^{-1,p}(\Omega)} \equiv \sup \left\{ |\mathcal{F}(\Phi)| \, \big| \, \Phi \in W^{1,p^*}_0(\Omega), \, \|\Phi\|_{W^{1,p^*}(\Omega)} = 1 \right\}.$$
(9.42)

From (9.41) we find

$$\left|\mathcal{F}(\Phi)\right| \leq \left|\left\langle \left(d\Gamma - dJ^{-1} \wedge dJ\right), d\Phi\right\rangle_{L^2}\right| + \left|\left\langle J^{-1}A, \delta\Phi\right\rangle_{L^2}\right|,\right.$$

which we further estimate using Hölder's inequality (A.6) as

$$\begin{aligned} \left| \mathcal{F}(\Phi) \right| &\leq C \left(\left\| \left(d\Gamma - dJ^{-1} \wedge dJ \right) \right\|_{L^{p}} \| d\Phi \|_{L^{p^{*}}} + \left\| J^{-1}A \right\|_{L^{p}} \| \delta\Phi \|_{L^{p^{*}}} \right) \\ &\leq C \left(\left\| d\Gamma \right\|_{L^{p}(\Omega)} + \left\| dJ^{-1} \wedge dJ \right\|_{L^{p}(\Omega)} + \left\| J^{-1}A \right\|_{L^{p}(\Omega)} \right), \quad (9.43) \end{aligned}$$

where we estimated $\|d\Phi\|_{L^{p^*}} \leq \|\Phi\|_{W^{1,p^*}} = 1$ and $\|\delta\Phi\|_{L^{p^*}} \leq 1$ for the last inequality. Throughout this proof C > 0 denotes a universal constant depending only on Ω, n, p We now use Hölder's inequality as in (A.9) to estimate the second term in (9.43) as

$$\begin{aligned} \|dJ^{-1} \wedge dJ\|_{L^{p}(\Omega)}^{p} &\leq C \langle |dJ^{-1}|^{p}, |dJ|^{p} \rangle_{L^{2}} \\ &\leq C \||dJ^{-1}|^{p}\|_{L^{2}(\Omega)} \||dJ|^{p}\|_{L^{2}(\Omega)} \\ &= C \|dJ^{-1}\|_{L^{2p}(\Omega)}^{p}\|dJ\|_{L^{2p}(\Omega)}^{p}, \end{aligned}$$
(9.44)

where we lost a little regularity from L^{2p} to L^p , as anticipated in our theory by starting with $J \in W^{1,2p}(\Omega)$. Now taking the *p*-th root of (9.44) and using that dJ = d(J - I), we obtain

$$\left\| dJ^{-1} \wedge dJ \right\|_{L^{p}(\Omega)} \leq \left\| J^{-1} \right\|_{W^{1,2p}(\Omega)} \left\| I - J \right\|_{W^{1,2p}(\Omega)}.$$
 (9.45)

To estimate the third term in (9.43), we substitute (4.6) and (4.14), that is, we substitute $\tilde{\Gamma} = \Gamma - J^{-1}dJ$ into $A = B - \langle dJ; \tilde{\Gamma} \rangle$. This leads to the identity

$$J^{-1}A = J^{-1}B - J^{-1}\langle dJ; \Gamma \rangle + \langle dJ^{-1}; dJ \rangle,$$

where we used the multiplication property (3.10) to write

$$J^{-1}\langle dJ; J^{-1}dJ \rangle = \langle dJ^{-1}; dJ \rangle.$$

We now obtain the bound

$$\begin{aligned} \left\| J^{-1}A \right\|_{L^{p}} &\leq \| J^{-1}B \|_{L^{p}} + \| J^{-1} \langle dJ; \Gamma \rangle \|_{L^{p}} + \| \langle dJ^{-1}; dJ \rangle \|_{L^{p}} \\ &\leq \| J^{-1}B \|_{L^{p}} + C_{M} \| J^{-1} \|_{W^{1,p}} \| \langle dJ; \Gamma \rangle \|_{L^{p}} + \| \langle dJ^{-1}; dJ \rangle \|_{L^{p}}, \quad (9.46) \end{aligned}$$

where we applied Morrey's inequality (A.4) in the last step to bound the L^{∞} -norm of J^{-1} . We now estimate the remaining product terms in (9.46) employing Hölder's inequality as in (9.44), to obtain

$$\left\|J^{-1}A\right\|_{L^{p}} \le C\|J^{-1}\|_{W^{1,p}}\left(\|B\|_{L^{2p}} + \|dJ\|_{L^{2p}}\|\Gamma\|_{L^{2p}} + \|d(I-J)\|_{L^{2p}}\right), \quad (9.47)$$

where C > 0 is some constant depending only on Ω, n, p . Applying dJ = d(J - I) we write bound (9.47) further as

$$\left\|J^{-1}A\right\|_{L^{p}} \le C\|J^{-1}\|_{W^{1,p}}\left(1+\|J\|_{W^{1,p}}\right)\left(\|B\|_{L^{2p}}+\|\Gamma\|_{L^{2p}}+\|I-J\|_{W^{1,2p}}\right).$$
(9.48)

Combining now (9.45) and (9.48) to bound the right hand side of (9.43), we conclude that

$$\left|\mathcal{F}(\Phi)\right| \le \mathcal{P}_0\left(\|d\Gamma\|_{L^p} + \|\Gamma\|_{L^{2p}} + \|B\|_{L^{2p}} + \|I - J\|_{W^{1,2p}}\right),\tag{9.49}$$

where $\mathcal{P}_0 = C(1+\|J^{-1}\|_{W^{1,2p}})(1+\|J\|_{W^{1,2p}})$. Substituting the bound (9.49) for the right hand side of (9.40), we find that

$$\|\tilde{\Gamma}\|_{W^{1,p}(\Omega')} \le \mathcal{P}_0\big(\|d\Gamma\|_{L^{2p}} + \|\Gamma\|_{L^{2p}} + \|B\|_{L^{2p}} + \|I - J\|_{W^{1,2p}}\big) + C\|\tilde{\Gamma}\|_{L^p(\Omega)}$$
(9.50)

From $\tilde{\Gamma} = \Gamma - J^{-1} dJ$ and Hölder inequality, we bound $\|\tilde{\Gamma}\|_{L^p(\Omega)}$ by

$$\|\tilde{\Gamma}\|_{L^{p}(\Omega)} \leq \operatorname{vol}(\Omega)^{\frac{1}{2p}} \|\Gamma\|_{L^{2p}(\Omega)} + C \, \|J^{-1}\|_{L^{2p}(\Omega)} \|dJ\|_{L^{2p}(\Omega)},$$

(where we estimated $\|\Gamma\|_{L^p} = \|\Gamma \cdot 1\|_{L^p} \leq \operatorname{vol}(\Omega)^{\frac{1}{2p}} \|\Gamma\|_{L^{2p}(\Omega)}$ using Hölder's inequality), from which we conclude with the bound

$$\|\tilde{\Gamma}\|_{W^{1,p}(\Omega')} \le \mathcal{P}_0\big(\|d\Gamma\|_{L^{2p}} + \|\Gamma\|_{L^{2p}} + \|B\|_{L^{2p}} + \|I - J\|_{W^{1,2p}}\big), \quad (9.51)$$

by modifying the constant C > 0 in the definition of \mathcal{P}_0 suitably.

To derive the sought after bound (9.38) we now use that (J, B) are assumed to meet the bound (9.15), that is,

$$||I - J||_{W^{1,2p}(\Omega)} + ||I - J^{-1}||_{W^{1,2p}(\Omega)} + ||B||_{L^{2p}(\Omega)} \leq C_2(M)||(\Gamma, d\Gamma)||_{L^{2p}(\Omega)}$$
(9.52)

So using (9.52) together with the definition $\|(\Gamma, d\Gamma)\|_{L^{2p}(\Omega)} = \|d\Gamma\|_{L^{2p}} + \|\Gamma\|_{L^{2p}}$ to bound the right hand side of (9.51), we conclude that there exists some constant $C_3(M) > 0$ depending only on Ω, n, p and M, such that

$$\|\Gamma\|_{W^{1,p}(\Omega')} \le C_3(M) \|(\Gamma, d\Gamma)\|_{L^{2p}(\Omega)}$$

which is the sought after bound (9.16). This completes the proof of Lemma 9.4. $\hfill \Box$

To establish part (i) of Theorem 4.1 it only remains to verify that $(J, \tilde{\Gamma}, A)$ solves the weak RT-equations (8.22) - (8.26).

Lemma 9.5. Under the assumption of Theorem 4.1, $(J, \tilde{\Gamma}, A)$ solves the second, third and fourth weak RT-equations (8.22) - (8.26) for v defined in (4.15) with w = 0, that is, $v \equiv \delta \langle dJ; \tilde{\Gamma} \rangle$.

Proof. By our assumption that $J \in W^{1,2p}(\Omega)$ and $B \in L^{2p}(\omega)$ solve the first weak reduced RT-equation (9.20), that is,

$$-\Delta J[\phi] = \langle J \cdot \Gamma, d\phi \rangle_{L^2} + \langle B, \phi \rangle_{L^2}$$
(9.53)

holds for any matrix valued 0-form $\phi \in W_0^{1,(2p)^*}(\Omega)$, where $\Delta J[\phi] = -\langle dJ, d\phi \rangle_{L^2}$. Substituting $B = \langle dJ; \tilde{\Gamma} \rangle + A$, (which follows from the definition of A in (4.14)), into (9.53), we directly obtain

$$-\Delta J[\phi] = \left\langle J \cdot \Gamma, \ d\phi \right\rangle_{L^2} + \left\langle \left(\left\langle dJ; \widetilde{\Gamma} \right\rangle + A \right), \ \phi \right\rangle_{L^2},$$

which is the sought after weak RT-equation (8.22).

We now show that the third and fourth weak RT-equations (8.26) hold. By assumption $J \in W^{1,2p}(\Omega)$ and $B \in L^{2p}(\Omega)$ solve the weak reduced RTequations (9.21), that is,

$$\begin{cases} \langle \vec{B}, \delta\psi \rangle_{L^2} = \left\langle (dJ \wedge \Gamma + J \, d\Gamma), \underbrace{\operatorname{div}}(\psi) \right\rangle_{L^2} \\ \langle \vec{B}, d\varphi \rangle_{L^2} = 0, \end{cases}$$

holds for any vector valued 2-form $\psi \in W_0^{1,(2p)^*}(\Omega)$ and any vector valued function $\varphi \in W_0^{1,(2p)^*}(\Omega)$. From the definition of A in (4.14), we find that $\vec{B} = \vec{A} + \overline{\langle dJ; \tilde{\Gamma} \rangle}$ and substituting this for \vec{A} in (9.54) gives us

$$\begin{cases} \langle \vec{A}, \delta\psi \rangle_{L^2} = \left\langle (dJ \wedge \Gamma + J \, d\Gamma), \underbrace{\operatorname{div}}(\psi) \right\rangle_{L^2} - \langle \overline{\langle dJ; \tilde{\Gamma} \rangle}, \delta\psi \rangle_{L^2} \\ \langle \vec{A}, d\varphi \rangle_{L^2} = -\langle v, \varphi \rangle_{L^2}, \end{cases}$$

for $v = \delta \langle dJ; \tilde{\Gamma} \rangle$. This proves that $(J, \tilde{\Gamma}, A)$ solves the weak RT-equations (8.22) - (8.26), completes the proof of Lemma 9.5.

Taken together, Lemmas 9.2 - 9.5 complete the proof of Theorem 4.1, under the weaker assumption $\Gamma, d\Gamma \in L^{2p}(\Omega)$. From here on we reimpose our original assumption $\Gamma, d\Gamma \in L^{\infty}(\Omega)$. To complete the proofs of Theorems 2.1 and 2.3 it remains only to prove Theorem 4.3, the subject of the next two sections.

10. Existence theory for the reduced RT-equations - Proof of Theorem 4.3

In this section we prove Theorem 4.3, regarding existence of solutions to the reduced RT-equations (4.10) - (4.12) which meet the assumptions of Theorem 4.1. This is the final step remaining to complete the proof of Theorems 2.1 and 2.3. The proof of Theorem 4.3 is based on an iteration scheme which reduces the problem to known estimates in elliptic PDE theory, recorded in Appendix B. To handle the first order system of equations for B (4.11) - (4.12), we extend the existence theory for Cauchy-Riemann type equations in [6] to the low regularity required here. This extension is presented in Appendix C. The proof of Theorem 4.3 is given in terms of several technical lemmas whose proofs are postponed to Section 11.

So assume $\Gamma, d\Gamma \in L^{\infty}(\Omega)$ in x-coordinates, let M > 0 be a constant such that

$$\|(\Gamma, d\Gamma)\|_{L^{\infty}(\Omega)} \equiv \|\Gamma\|_{L^{\infty}(\Omega)} + \|d\Gamma\|_{L^{\infty}(\Omega)} \leq M,$$

and assume $p > n, < \infty$ and let $q \in \Omega$. Again, we work in fixed x-coordinates and we write Ω for Ω_x and Γ for Γ_x throughout the remainder of this paper. Then, to prove Theorem 4.3, it suffices to prove that there exists a neighborhood $\Omega' \subset \Omega$ of x(q), depending only on Ω, n, p, M , and there exists $J \in W^{1,p}(\Omega')$ and $B \in L^p(\Omega')$ such that (J, B) solves the reduced RTequations (4.10) - (4.12) in Ω' in the weak sense of Definition 8.5, such that (J, B) satisfies the uniform bound (4.16), J is invertible with $J^{-1} \in W^{1,p}(\Omega')$ and

$$d\vec{J} \equiv \operatorname{Curl}(J) = 0 \tag{10.1}$$

in Ω' , so J is integrable to coordinates.²⁶ For simplicity we assume without loss of generality that w = 0 in (4.12), that is, we prove existence of a solution (J, B) of

$$\Delta J = \delta(J \cdot \Gamma) - B, \qquad (10.2)$$

$$d\vec{B} = \vec{\operatorname{div}}(dJ \wedge \Gamma) + \vec{\operatorname{div}}(J \, d\Gamma), \qquad (10.3)$$

$$\delta \vec{B} = 0, \tag{10.4}$$

in the weak sense specified in Definition 8.5. Without loss of generality we assume that Ω is the unit ball in \mathbb{R}^n centered at x(q) = 0, $\Omega = B_1(0)$. We show below that it suffices to take $\Omega' = B_{\epsilon}(0)$, the ball of radius $\epsilon > 0$ centered at x = 0, where $\epsilon > 0$ is taken sufficiently small for the iteration scheme to converge. We begin the proof of Theorem 4.3 by giving a formal introduction to the iteration scheme on which our existence proof is based.

10.1. The iteration scheme. Start with $J_0 = 1$. For induction, we show that (B_{k+1}, J_{k+1}) can be constructed from J_k for each $k \ge 0$. So assume J_k is given for some $k \ge 0$. Define B_{k+1} as a weak solution of

$$\begin{cases} d\vec{B}_{k+1} = \overrightarrow{\operatorname{div}} (dJ_k \wedge \Gamma) + \overrightarrow{\operatorname{div}} (J_k \, d\Gamma), \\ \delta\vec{B}_{k+1} = 0, \end{cases}$$
(10.5)

such that $B_{k+1} \in L^p(\Omega)$ satisfies a uniform bound in the L^p -norm. The regularity L^p is too low to impose boundary data in 10.5, and our theory does not require B_{k+1} meet any boundary conditions, B_{k+1} only needs to satisfy a uniform L^p bound. This is achieved by choosing B_{k+1} to be the zero mollification limit of a solution of the corresponding mollified equation with zero Dirichlet boundary data, c.f. Sections 10.3 and C.

Likewise, the regularity $J \in W^{1,2p}(\Omega)$ is too low to impose the boundary condition (4.5), $d\vec{J} = 0$ on $\partial\Omega$, a problem we circumvent by imposing $\vec{J} = dy$, for y solving an auxiliary elliptic equation. For this, define auxiliary variables Ψ_{k+1} and y_{k+1} in terms of J_k and B_{k+1} , but independent of the previous

²⁶For ease of notation we show $J \in W^{1,p}$ and $B \in L^p$, since P > n is arbitrary, the proof yields solutions with $J \in W^{1,2p}$ and $B \in L^{2p}$, as stated in Theorem 4.3.

iterates Ψ_k and y_k . That is, we define the vector valued 0-form $\Psi_{k+1} \in L^p(\Omega)$ as a weak solution of

$$d\Psi_{k+1} = \overrightarrow{\delta(J_k \cdot \Gamma)} - \overrightarrow{B_{k+1}}, \qquad (10.6)$$

with uniform L^p bounds, obtained again by mollification in the same manner as in the case for B_{k+1} . We then define the vector valued 0-form $y_{k+1} \in W^{2,p}(\Omega)$ as the solution of

$$\Delta y_{k+1} = \Psi_{k+1},\tag{10.7}$$

for zero Dirichlet data. Given B_{k+1}, Ψ_{k+1} and y_{k+1} , now define $J_{k+1} \in W^{1,p}(\Omega)$ as the weak solution of the following Dirichlet boundary value problem:

$$\Delta J_{k+1} = \delta(J_k \cdot \Gamma) - B_{k+1}, \qquad (10.8)$$

$$\overline{J_{k+1}} = dy_{k+1} \text{ on } \partial\Omega.$$
(10.9)

Equations (10.5) - (10.8) define our iteration scheme in a formal way. To prove convergence we need a small parameter $\epsilon > 0$. We incorporate ϵ into the iteration scheme in Sections 10.2 - 10.3, and prove convergence for $\epsilon > 0$ sufficiently small in Section 10.4.

Two clarifying remarks are in order. First note that (10.6) requires a solvability condition, namely that d of its right hand side must be zero. This condition is meet, because by (3.12) we have

$$d\left(\overrightarrow{\delta(J_k \cdot \Gamma)}\right) = \overrightarrow{\operatorname{div}} \left(dJ_k \wedge \Gamma \right) + \overrightarrow{\operatorname{div}} \left(J_k \cdot d\Gamma \right),$$

which implies in light of equation (10.5) for B_{k+1} that

$$d\left(\overrightarrow{\delta(J_k \cdot \Gamma)} - \overrightarrow{B_{k+1}}\right) = 0,$$

so the right hand side of (10.5) has a vanishing exterior derivative. That this consistency condition is necessary and sufficient for the low regularity here is shown in Appendix C.

Secondly, we remark on the role of auxiliary equations (10.6) - (10.7). The reason for introducing Ψ_k and y_{k+1} is that the $W^{1,p}$ -regularity of J_{k+1} is too low to impose the boundary data $d\vec{J}_{k+1} = 0$ which was required in [21] to arrange for the integrability condition of J (10.1). Now, augmenting the reduced RT-equations by equations (10.6) and (10.7), allows us to impose Dirichlet data for J_{k+1} which again gives rise to integrability of J_{k+1} to coordinates, as we show in the following lemma for smooth solutions. In Lemma 10.9 below we extend this result to the low regularities required by the above iteration scheme.

Lemma 10.1. Assume Γ is smooth and that $B_{k+1}, \Psi_{k+1}, y_{k+1}, J_{k+1}$ are defined by the iteration scheme (10.5) - (10.9) and are smooth. Then $dy_{k+1} = \vec{J}_{k+1}$, and hence J_{k+1} is integrable to coordinates y_{k+1} and $d\vec{J}_{k+1} = 0$.

Proof. By (10.5) - (10.9) it follows that

$$\Delta dy_{k+1} = d\Delta y_{k+1} = d\Psi_{k+1} = \overrightarrow{\delta(J_k \cdot \Gamma)} - \overrightarrow{B_{k+1}} = \Delta \vec{J}_{k+1},$$

where the last equality holds, since the operation vec commutes with the Laplacian Δ (which acts component wise). Thus,

$$\Delta(J_{k+1} - dy_{k+1}) = 0 \quad \text{in} \quad \Omega,$$

$$\vec{J} - dy = 0 \quad \text{on} \quad \partial\Omega,$$
 (10.10)

which implies by uniqueness of solutions of the Laplace equation that $\vec{J}_{k+1} = dy_{k+1}$ in Ω . Since second derivatives of y_{k+1} commute, we conclude that

$$d\vec{J}_{k+1} = Curl(\vec{J}_{k+1}) = 0 \quad \text{in } \Omega.$$

Moreover, J_{k+1} is the Jacobian of the coordinate system y_{k+1} .

Lemma 10.9 below generalizes the above result to the low regularity required in this paper. To prove convergence of the iteration scheme, we introduce a small parameter $\epsilon > 0$ by restricting to $\Omega' = B_{\epsilon}(0)$, and prove convergence for $\epsilon > 0$ sufficiently small.

10.2. The ϵ -rescaled reduced RT-equations. We first incorporate the small parameter $\epsilon > 0$ into the theory by deriving an ϵ -rescaled version of the reduced RT-equations, required to prove convergence of our iteration scheme. For this we use the fact that regularity is a local problem, so that we can suitably restrict and rescale Γ to isolate the small parameter ϵ , while maintaining the uniform bound (2.4) assumed in Theorems 2.1 and 2.3. This is accomplished in the following lemma.

Lemma 10.2. Assume $\Omega = B_1(0)$ and introduce the coordinate transformation $x \to \tilde{x}(x) = \frac{x}{\epsilon}$. Define Γ^* as the restriction of the components of Γ_x to $B_{\epsilon}(0)$, transformed to \tilde{x} -coordinates as scalars, $\Gamma^*(\tilde{x}) \equiv \Gamma_x(x(\tilde{x}))$. Then, $\Gamma_{\tilde{x}}$ satisfies in \tilde{x} -coordinates

$$\Gamma_{\tilde{x}}(\tilde{x}) = \epsilon \ \Gamma^*(\tilde{x}), \tag{10.11}$$

together with the bound

$$\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}(\Omega'_{\tilde{x}})} = \|\Gamma_x\|_{L^{\infty}(\Omega'_x)} + \epsilon \|d\Gamma_x\|_{L^{\infty}(\Omega'_x)},$$
(10.12)

where $\Omega'_{\tilde{x}} = B_1(0)$ and $\Omega'_x = B_{\epsilon}(0)$.

Proof. By the connection transformation law we have

$$(\Gamma_{\tilde{x}})^{\sigma}_{\mu\nu} = \frac{\partial \tilde{x}^{\sigma}}{\partial x^{k}} \left(\frac{\partial x^{i}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{j}}{\partial \tilde{x}^{\nu}} (\Gamma_{x})^{k}_{ij} + \frac{\partial^{2} x^{k}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \right) = \epsilon (\Gamma_{x})^{\sigma}_{\mu\nu}, \qquad (10.13)$$

since $\frac{\partial x^i}{\partial \tilde{x}^j} = \epsilon \, \delta^i_{\ j}$ under the transformation $\tilde{x}(x) = \frac{x}{\epsilon}$. It follows that for $\tilde{x} \in B_1(0)$, we have component wise

$$\Gamma_{\tilde{x}}(\tilde{x}) = \epsilon \ \Gamma_x(x(\tilde{x})) \equiv \epsilon \ \Gamma^*(\tilde{x}).$$

To prove (10.12), observe that by construction of Γ^* , as the scalar transformed components of the restriction of Γ_x to the ball of radius ϵ , we have

 $\|\Gamma^*\|_{L^{\infty}(\Omega'_{\tilde{x}})} = \|\Gamma_x\|_{L^{\infty}(\Omega'_x)}$, and by the chain rule we have $\|d\Gamma^*\|_{L^{\infty}(\Omega'_{\tilde{x}})} = \epsilon \|d\Gamma_x\|_{L^{\infty}(\Omega'_x)}$ since $\frac{\partial x}{\partial \tilde{x}} = \epsilon I$. In combination, this gives (10.12) and completes the proof.

By (10.12), and assuming without loss of generality that $\epsilon \leq 1$, we find that Γ^* satisfies the original uniform bound (2.4),

$$\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}(\Omega'_{\tilde{\pi}})} \le M.$$

We can thus construct solutions to the reduced RT-equations and apply Theorem 4.1 in \tilde{x} -coordinates, and obtain the uniform curvature bound (2.6) in \tilde{x} -coordinates, without ever scaling back to x-coordinates.²⁷

So now we can take the \tilde{x} -coordinates to be the original x-coordinates, and assume without loss of generality that the connection in x-coordinates has the form

$$\Gamma_x = \epsilon \ \Gamma^*, \tag{10.14}$$

for some Γ^* satisfying

$$\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}(\Omega_x)} < M, \tag{10.15}$$

and we assume without loss of generality that $\Omega_x = B_1(0)$. In light of (10.14), we introduce the scaling ansatz

$$J = I + \epsilon u, \qquad B = \epsilon a. \tag{10.16}$$

Since we only need to prove *existence* of a solution to establish optimal connection regularity via the RT-equations, assumption 10.16 is made without loss of generality. Note, the variables Ψ and y "inherit" their ϵ -scaling from B and J, c.f. Section 10.3 below. To derive the reduced RT-equations expressed in terms of the rescaled variables, we now substitute (10.14) and (10.16) into (10.2) - (10.4) and divide by ϵ . This yields the following equivalent set of equations:

Lemma 10.3. The reduced RT-equations (8.27) - (8.28) written in terms of the rescaled connection (10.14) and rescaled variables (10.16) are equivalent to

$$-\Delta u[\phi] = F_u(u,a)[\phi], \qquad (10.17)$$

$$\begin{cases} \langle \vec{a}, \delta\psi \rangle_{L^2} = F_a(u)[\psi] \\ \langle \vec{a}, d\varphi \rangle_{L^2} = 0, \end{cases}$$
(10.18)

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, any vector valued 2-form $\psi \in W_0^{1,p^*}(\Omega)$ and any vector valued function $\varphi \in W_0^{1,p^*}(\Omega)$, where we define the

²⁷A note might be in order here regarding our proof of Uhlenbeck compactness. In the proof we implicitly assumed that there is a uniform $\epsilon > 0$ which applies uniformly to each connection in the sequence. In the Proposition10.11 below we show that ϵ can be taken to be on the order of $\frac{1}{M}$, and therefore independent of connections in the sequence.

linear functionals

$$F_u(u,a)[\phi] \equiv \left\langle \Gamma^*, d\phi \right\rangle_{L^2} + \epsilon \left\langle u \cdot \Gamma^*, d\phi \right\rangle_{L^2} + \left\langle a, \phi \right\rangle_{L^2}, \tag{10.19}$$

$$F_a(u)[\psi] \equiv \left\langle d\Gamma^*, \underline{div}(\psi) \right\rangle_{L^2} + \epsilon \left\langle (u \cdot d\Gamma^* + du \wedge \Gamma^*), \underline{div}(\psi) \right\rangle_{L^2}.$$
(10.20)

Proof. Substituting $J = I + \epsilon u$ and $B = \epsilon a$ into (8.27) and dividing by ϵ , we obtain

$$\begin{aligned} -\Delta u[\phi] &\stackrel{(8.12)}{=} \langle du, d\phi \rangle_{L^2} \\ &= \langle (I + \epsilon u) \cdot \Gamma^*, d\phi \rangle_{L^2} + \langle a, \phi \rangle_{L^2} \\ &= \langle \Gamma^*, d\phi \rangle_{L^2} + \epsilon \langle u \cdot \Gamma^*, d\phi \rangle_{L^2} + \langle a, \phi \rangle_{L^2} \\ &= F_u(u, a)[\phi] \end{aligned}$$

which proves the equivalence between (10.2) and (10.17). Similarly, substituting our scaling ansatz $J = I + \epsilon u$ and $B = \epsilon a$ into (8.28), a division by ϵ gives

$$\begin{array}{lll} \langle \vec{a}, \delta \psi \rangle_{L^2} &= \left\langle (\epsilon du \wedge \Gamma^* + (I + \epsilon u) \cdot d\Gamma^*), \underline{\operatorname{div}}(\psi) \right\rangle_{L^2} \\ &= \left\langle d\Gamma^*, \underline{\operatorname{div}}(\psi) \right\rangle_{L^2} + \epsilon \left\langle (du \wedge \Gamma^* + u \cdot d\Gamma^*), \underline{\operatorname{div}}(\psi) \right\rangle_{L^2} \\ &= F_a(u) [\psi] \end{array}$$

as well as $\langle \vec{a}, d\varphi \rangle_{L^2} = 0$, which proves the equivalence between (8.28) and (10.18).

The existence result of Theorem 4.3 is a corollary of the following proposition, the proof of which is topic of Sections 10.3 - 11.

Proposition 10.4. Let $\Gamma^*, d\Gamma^* \in L^{\infty}(\Omega)$ satisfy the bound (10.15) and let $n . Then, for every <math>\epsilon > 0$ sufficiently small, there exists $u \in W^{1,p}(\Omega)$ and $a \in L^p(\Omega)$ which solve the ϵ -rescaled reduced RT-equation (10.17) - (10.18).

The proof of Proposition 10.4 is based on the iteration scheme introduced in Section 10.1, but adjusted to incorporate the small parameter ϵ . The resulting iteration scheme for the ϵ -rescaled reduced RT-equations is introduced in the next Section 10.3. The proof of our main existence result, Theorem 4.3, is completed in Section 10.5 by applying Proposition 10.4 together with additional arguments to establish the integrability and invertability of the Jacobian $J = I + \epsilon u$, claimed in the theorem, as well as the uniform bound (4.16).

10.3. The iteration scheme in the ϵ -rescaled variables. In this section we define the iterates $(u_k, a_k), k \geq 0$, for approximating solutions of (10.17)-(10.18), and set up the framework for proving convergence of the scheme in the appropriate Sobolev spaces for ϵ sufficiently small. The iteration scheme we introduce here differs from the iteration scheme in Section 10.1 in that it is adapted to the rescaled equations (10.17)-(10.18). Existence at each stage will be established in Lemma 10.7 below. From here on we often omit dependence of norms on Ω , e.g., writing $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(\Omega)}$. We define now the matrix valued 0-forms u_{k+1} and a_{k+1} by induction as follows.

To start the iteration, set

$$u_0 = a_0 = 0.$$

Given $u_k \in W^{1,p}(\Omega)$ and $a_k \in L^p(\Omega)$ for $k \ge 0$, we then construct a particular matrix valued 0-form $a_{k+1} \in L^p(\Omega)$ which solves

$$\begin{cases} \langle \overrightarrow{a_{k+1}}, \delta \psi \rangle_{L^2} = F_a(u_k)[\psi], \\ \langle \overrightarrow{a_{k+1}}, d\varphi \rangle_{L^2} = 0, \end{cases}$$
(10.21)

and satisfies the estimate

$$||a_{k+1}||_{L^p} \le C ||F_a(u_k)||_{W^{-1,p}}, \qquad (10.22)$$

for some constant C > 0 independent of k, where (10.21) are taken in the weak sense specified in Lemma 10.3. Existence is established Lemma 10.7 and relies on the algorithm developed in Appendix C for constructing particular solutions when regularity is too low to impose Dirichlet data in a classical sense.

We next introduce the vector valued 0-form $\Psi_{k+1} \in L^p(\Omega)$ as a weak solution of

$$d\Psi_{k+1} = \overrightarrow{\delta(J_k \cdot \Gamma^*)} - \overrightarrow{a_{k+1}}, \qquad (10.23)$$

where $J_k \equiv I + \epsilon u_k$, such that Ψ_{k+1} meets the bound

$$\|\Psi_{k+1}\|_{L^p} \le C \|F_a(u_k)\|_{W^{-1,p}}, \tag{10.24}$$

for some constant C > 0 independent of k. That is, $\Psi_{k+1} \in L^p(\Omega)$ meets the bound (10.24) and satisfies

$$\langle \Psi_{k+1}, \delta \vec{\phi} \rangle_{L^2} = F_{\Psi}(u_k, a_{k+1})[\phi], \qquad (10.25)$$

for any matrix-valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, where we set

$$F_{\Psi}(u_k, a_{k+1})[\phi] \equiv \langle J_k \Gamma^*, d\phi \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \overrightarrow{\phi} \rangle_{L^2}.$$
(10.26)

The definition in (10.26) is based on the product rule $\langle \vec{\delta w}, \vec{\phi} \rangle_{L^2} = -\langle w, d\phi \rangle_{L^2}$ for matrix valued 1-forms $w \in W^{1,p}(\Omega)$, established in the proof of Lemma 10.5 below. Because of this product rule it is convenient to interpret the test forms in (10.25) and (10.26) as matrix valued 0-forms instead of vector valued 1-forms. In Lemma 10.5 below we show that the weak formulation of (10.25) is equivalent to the strong formulation (10.23) in the case of smooth solutions. Existence of Ψ_{k+1} also follows from Lemma 10.7 by use of the algorithm in Appendix C.

We next define the vector valued 0-form $y_{k+1} \in W^{2,p}(\Omega)$ as the solution of

$$\begin{cases} \Delta y_{k+1} = \Psi_{k+1}, \\ y_{k+1} \Big|_{\partial \Omega} = 0. \end{cases}$$
(10.27)

Similar to Lemma 10.1, equations (10.21), (10.25) and (10.27) again arrange for the integrability of $J_{k+1} = I + \epsilon u_{k+1}$.

62

Finally, we define $u_{k+1} \in W^{1,p}(\Omega)$ as the unique weak solution satisfying

$$-\Delta u_{k+1}[\phi] = F_u(u_k, a_{k+1})[\phi], \qquad (10.28)$$

for every matrix valued 0-forms $\phi \in W_0^{1,p^*}(\Omega)$, with Dirichlet boundary data

$$u_{k+1}|_{\partial\Omega} = dy_{k+1}|_{\partial\Omega}. \tag{10.29}$$

Equations (10.21) - (10.29) define our iteration scheme. For completeness we show in the next lemma that smooth solutions Ψ_{k+1} of the weak equation (10.25) are indeed strong solutions of (10.23).

Lemma 10.5. Let $\Psi_{k+1} \in W^{1,p}(\Omega)$ be a vector valued 0-form, then Ψ_{k+1} solves (10.25) if and only if Ψ_{k+1} solves (10.23).

Proof. We first prove the following statement: Let $w \in W^{1,p}(\Omega)$ be a matrix valued 1-form and $\phi \in W_0^{1,p^*}(\Omega)$ a matrix valued 0-form, then

$$\langle \overrightarrow{\delta w}, \overrightarrow{\phi} \rangle_{L^2} = -\langle w, d\phi \rangle_{L^2}.$$
 (10.30)

To prove (10.30), let $w = w^{\mu}_{\nu i} dx^i$ and $\phi = \phi^{\mu}_{\nu}$, then

$$\delta w = \sum_{i=1,\dots,n} \partial_i w^{\mu}_{\nu i} \qquad \text{and} \qquad \overrightarrow{\delta w}^{\mu} = (\delta w)^{\mu}_{\nu} dx^{\nu} = \sum_{i=1}^n \partial_i w^{\mu}_{\nu i} dx^{\nu}.$$

Using partial integration component wise, we compute

$$\begin{split} \langle \overrightarrow{\delta w}, \vec{\phi} \rangle_{L^2} &= \sum_{\mu, \nu} \int_{\Omega} \delta w^{\mu}_{\nu} \phi^{\mu}_{\nu} \, dx = \sum_{\mu, \nu, i} \int_{\Omega} \partial_i w^{\mu}_{\nu i} \phi^{\mu}_{\nu} dx \\ &= -\sum_{\mu, \nu, i} \int_{\Omega} w^{\mu}_{\nu i} \partial_i \phi^{\mu}_{\nu} dx = -\langle w, d\phi \rangle_{L^2}, \end{split}$$

c.f. the definition of inner products on matrix and vector valued differential forms (8.2) and (8.4). This proves the sought after equation (10.30).

We now apply (10.30) to prove Lemma 10.5. So assume $\Psi_{k+1} \in W^{1,p}(\Omega)$ solves (10.25), that is,

$$\langle \Psi_{k+1}, \delta \vec{\phi} \rangle_{L^2} = \langle J_k \Gamma^*, d\phi \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \vec{\phi} \rangle_{L^2}, \qquad (10.31)$$

for any matrix-valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$. From the partial integration formula (8.7) for vector valued forms, we find that

$$\langle \Psi_{k+1}, \delta \vec{\phi} \rangle_{L^2} = -\langle d\Psi_{k+1}, \vec{\phi} \rangle_{L^2}, \qquad (10.32)$$

and by (10.30), we have

$$\langle J_k \Gamma^*, d\phi \rangle_{L^2} = -\langle \overline{\delta(J_k \Gamma^*)}, \vec{\phi} \rangle_{L^2}.$$
 (10.33)

Combining (10.32) and (10.33), we write (10.31) as

$$\langle d\Psi_{k+1}, \vec{\phi} \rangle_{L^2} = \left\langle \left(\overrightarrow{\delta(J_k \Gamma^*)} - \overrightarrow{a_{k+1}} \right), \vec{\phi} \right\rangle_{L^2},$$

and since this equation holds for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, we conclude by Riesz representation that the strong form (10.23) holds. The

opposite implication is straightforward. This completes the proof of Lemma 10.5. $\hfill \square$

The iteration scheme on which the existence theory for the reduced RTequations stated in Proposition 10.4 is based, is defined in (10.21) - (10.29). Our strategy for completing the proof of Proposition 10.4 is to first state the main technical lemmas regarding the iteration scheme being well-defined and convergent, including elliptic estimate for differences of iterates to establish convergence in suitable Sobolev spaces. The statement of these lemmas is the topic of the next section.

10.4. Well-posedness and convergence of the iteration scheme. In this section we state the main lemmas required for the proof of Proposition 10.4, and assuming these we give the proof of Proposition 10.4. Proofs of the supporting lemmas are postponed until Section 11 below. The first lemma provides an apriori estimate for the source terms.

Lemma 10.6. Let $\Gamma^*, d\Gamma^* \in L^{\infty}(\Omega)$ and assume $u \in W^{1,p}(\Omega)$ and $a \in L^p(\Omega)$, for $n , then there exists a constant <math>C_s > 0$ depending only on Ω, n, p , such that

$$||F_u(u,a)||_{W^{-1,p}} \le ||a||_{L^p} + C_s \left(\operatorname{vol}(\Omega) + \epsilon \, ||u||_{L^p} \right) ||(\Gamma^*, d\Gamma^*)||_{L^{\infty}} \quad (10.34)$$

$$\|F_{a}(u)\|_{W^{-1,p}} \leq C_{s} \left(\operatorname{vol}(\Omega) + \epsilon \|u\|_{W^{1,p}} \right) \|(\Gamma^{*}, d\Gamma^{*})\|_{L^{\infty}}.$$
(10.35)

Lemma 10.6 is proven in Section 11.1 below. Our second lemma gives the elliptic estimates required to establish that the iteration scheme is well-defined.

Lemma 10.7. Assume $u_k \in W^{1,p}(\Omega)$ is given, $n . Then there exists <math>a_{k+1} \in L^p(\Omega)$ which solves (10.21), there exists the auxiliary iterates $\Psi_{k+1} \in L^p(\Omega)$ and $y_{k+1} \in W^{2,p}(\Omega)$ which solve (10.25) - (10.27), and there exists $u_{k+1} \in W^{1,p}(\Omega)$ which solves (10.28) with boundary data (10.29). In addition, the iterates satisfy the following elliptic estimates:

- $\|a_{k+1}\|_{L^{p}(\Omega)} \leq C_{e} \|F_{a}(u_{k})\|_{W^{-1,p}(\Omega)}, \qquad (10.36)$
- $\|\Psi_{k+1}\|_{L^{p}(\Omega)} \leq C_{e} \|F_{u}(u_{k}, a_{k+1})\|_{W^{-1,p}(\Omega)},$ (10.37)
- $\|y_{k+1}\|_{W^{2,p}(\Omega)} \leq C_e \|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)},$ (10.38)

$$\|u_{k+1}\|_{W^{1,p}(\Omega)} \leq C_e \|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)},$$
(10.39)

for some constant $C_e > 0$ depending only on n, p and Ω .

The proof of Lemma 10.7, given in Section 11.2, is based on the L^p elliptic estimate (B.3) of Theorem B.1 and Gaffney's inequality (C.3) of Theorem C.2.²⁸ Lemma 10.7 directly implies the following corollary.

Corollary 10.8. The iteration scheme is well-defined.

64

²⁸Note that the boundary data (10.29) for J, i.e. J = dy on $\partial\Omega$, does not enter estimate (10.39), since it can be bounded by $||F_u(u_k, a_{k+1})||_{W^{-1,p}(\Omega)}$ using (10.38).

Before we establish convergence of the iteration scheme, we show that each Jacobian $J_k = I + \epsilon u_k$ is integrable to coordinates for each $k \in \mathbb{N}$. This is the subject of the next lemma, proven in Section 11.4 below, which extends Lemma 10.1 to the low regularities here.

Lemma 10.9. Let $u_{k+1} \in W^{1,p}(\Omega)$ be a solution of (10.28) with boundary data (10.29), and let $y_{k+1} \in W^{2,p}(\Omega)$ be a solution of (10.27). Then

$$d\overrightarrow{u_{k+1}} = 0 \tag{10.40}$$

in Ω and $J_{k+1} \equiv I + \epsilon u_{k+1}$ is the Jacobian of the coordinate transformation $x \to x + \epsilon y_{k+1}(x)$.

We now discuss convergence of the iteration scheme. Lemma 10.7 yields a sequence of iterates $(u_k, a_k)_{k \in \mathbb{N}}$. To establish convergence of this sequence in $W^{1,p} \times L^p$, we require estimates on the differences

$$\overline{a_k} \equiv a_k - a_{k-1}$$
 and $\overline{u_k} \equiv u_k - u_{k-1}$ (10.41)

in terms of the corresponding previous difference of iterates, $\overline{a_{k-1}}$ and $\overline{u_{k-1}}$. This is accomplished in the following lemma, proven in Section 11.3. The proof of the lemma combines the elliptic estimates (10.36) - (10.37) with suitable bounds on differences of source terms by previous differences of iterates in the fashion of the estimates of Lemma 10.6.

Lemma 10.10. Assume $\Gamma^*, d\Gamma^* \in L^{\infty}(\Omega)$, then

$$\|\overline{a_{k+1}}\|_{L^p} \leq \epsilon C_d \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} \|\overline{u_k}\|_{W^{1,p}}, \qquad (10.42)$$

$$\|\overline{u_{k+1}}\|_{W^{1,p}} \leq \epsilon C_d \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} \|\overline{u_k}\|_{W^{1,p}},$$
(10.43)

where $C_d \equiv C_s C_e(1 + C_e) > 0$ depends only on n, p, Ω , where $C_s > 0$ and $C_e > 0$ are the constants of Lemmas 10.6 and 10.7 respectively.

Convergence of the iteration scheme will follow from Lemma 10.10, because

$$\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}(\Omega)} \le M_{2}$$

by (10.15). This is proven in the following proposition, which completes the proof of Proposition 10.4, assuming Lemmas 10.6, 10.7 and 10.10 hold.

Proposition 10.11. Assume Lemmas 10.6, 10.7 and 10.10 hold. Let $\Gamma^*, d\Gamma^* \in L^{\infty}(\Omega)$ satisfy the initial bound (10.15), $\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} < M$ for some constant M > 0, and assume

$$0 < \epsilon < \frac{1}{C_d M}, \tag{10.44}$$

where $C_d > 0$ is the constant from Lemma 10.10. Then the sequence of iterates $(u_k, a_k)_{k \in \mathbb{N}}$ defined by (10.21) - (10.29) converges in $W^{1,p}(\Omega) \times L^p(\Omega)$, and the corresponding limits

$$u \equiv \lim_{k \to \infty} u_k \in W^{1,p}(\Omega)$$
$$a \equiv \lim_{k \to \infty} a_k \in L^p(\Omega),$$

solve the reduced RT-equations (10.17) - (10.18) and satisfy the bound

 $\|u\|_{W^{1,p}(\Omega)} + \|a\|_{L^{p}(\Omega)} \le C_{2}(M)\|(\Gamma^{*}, d\Gamma^{*})\|_{L^{\infty}(\Omega)},$ (10.45)

for some constant $C_2(M) > 0$ depending only on Ω, n, p and M.

Proof. We prove Proposition 10.11 under the assumption that Lemmas 10.6, 10.7 and 10.10 are valid, and postpone their proofs to Section 11. So by Lemma 10.7 there exist a sequence of iterates (u_k) . Given two such iterates $u_k, u_l \in W^{1,p}(\Omega), (k \ge l)$, estimate (10.43) of Lemma 10.10 in combination with our incoming bound $\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} \le M$, implies

$$\|u_k - u_l\|_{W^{1,p}} \leq \sum_{j=l+1}^k \|\overline{u_j}\|_{W^{1,p}} \leq \|\overline{u_{l+1}}\|_{W^{1,p}} \sum_{j=l+1}^k (\epsilon C_d M)^j.$$

By the bound (10.44) on ϵ , the above geometric series converges as $k \to \infty$. This implies that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $W^{1,p}(\Omega)$. Therefore, $(u_k)_{k\in\mathbb{N}}$ converges to some u in $W^{1,p}(\Omega)$. Similarly, (10.42) together with the bound (10.15) implies

$$||a_k - a_l||_{L^p} \leq \sum_{j=l+1}^k ||\overline{a_j}||_{L^p} \leq ||\overline{u_{l+1}}||_{L^p} \sum_{j=l+1}^k (\epsilon C_d M)^j,$$

which in light of (10.44) is a convergent geometric series, and we conclude with convergence of $(a_k)_{k\in\mathbb{N}}$ to some a in the Banach space $L^p(\Omega)$.

The limit (u, a) solves (10.17) and (10.18) because each term in the equations (10.21) and (10.28) converge to the corresponding terms in (10.17) and (10.18) with respect to the L^p -norm on Ω . For example, using Hölder inequality we find from (10.28) that

$$\Delta u[\phi] + F_u(u, a)[\Phi] = \lim_{k \to \infty} \left(\Delta u_{k+1}[\phi] + F_u(u_k, a_{k+1})[\Phi] \right) = 0,$$

which shows that $u = \lim_{k \to \infty} u_k$ is indeed a solution of (10.17).

To derive estimate (10.45), we use the bounds on source terms of Lemma 10.6 in combination with the above convergence $(u_k, a_k) \rightarrow (u, a)$. That is, using that we initiated the iteration with $a_0 = 0$ and $u_0 = 0$, we find

$$||a||_{L^p} = ||a - a_0||_{L^p} \le \sum_{k=1}^{\infty} ||a_{k+1} - a_k||_{L^p} + ||a_1||_{L^p}.$$

Using first (10.42) and then (10.43) successively, we estimate the above sum as

$$\|a\|_{L^{p}} \leq \sum_{k=1}^{\infty} \left(\epsilon C_{d} \| (\Gamma^{*}, d\Gamma^{*}) \|_{L^{\infty}} \right)^{k} + \|a_{1}\|_{L^{p}}.$$
 (10.46)

We now use the elliptic estimate (10.36) in combination with the bound (10.35) on $F_a(u_0)$ and $u_0 = 0$ to obtain

$$\|a_1\|_{L^p} \le C_e \operatorname{vol}(\Omega) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}}$$
(10.47)

Substituting this back into (10.46) and using (10.15) to estimate $\|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}}$ by M > 0, we obtain

$$\|a\|_{L^p} \le \left(C_e \operatorname{vol}(\Omega) + \epsilon C_d \sum_{k=1}^{\infty} \left(\epsilon C_d M\right)^{k-1}\right) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}},$$

and our ϵ -bound (10.44) implies the above infinite sum converges, so we conclude that

$$||a||_{L^p} \le C_2(M) ||(\Gamma^*, d\Gamma^*)||_{L^{\infty}}$$
(10.48)

for some constant $C_2(M) > 0$ depending only on Ω, n, p and M.

We now derive an estimate on u in a similar way. Using $u_0 = 0$, we begin by writing

$$\|u\|_{W^{1,p}} = \|u - u_0\|_{W^{1,p}} \le \sum_{k=1}^{\infty} \|u_{k+1} - u_k\|_{W^{1,p}} + \|u_1\|_{W^{1,p}},$$

and applying (10.43) together with our initial bound (10.15) yields

$$\|u\|_{W^{1,p}} \le \epsilon C_d \sum_{k=1}^{\infty} \left(\epsilon C_d M\right)^{k-1} \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} + \|u_1\|_{W^{1,p}}, \quad (10.49)$$

where the sum is finite by our ϵ -bound (10.44). Using the elliptic estimate (10.39) in combination with the bound (10.34) on $F_u(u_0, a_1)$, we obtain

$$\begin{aligned} \|u_1\|_{W^{1,p}} &\leq C_e \big(\|a_1\|_{L^p} + \operatorname{vol}(\Omega)\big) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} \\ &\leq C_e (C_e + 1) \operatorname{vol}(\Omega) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}}. \end{aligned}$$

Substituting this estimate into (10.49), we obtain the estimate

$$||u||_{W^{1,p}} \le C_2(M) ||(\Gamma^*, d\Gamma^*)||_{L^{\infty}}, \qquad (10.50)$$

for some constant $C_2(M) > 0$ depending only on Ω, n, p and M. Adding (10.48) and (10.50) yields the sought after estimate (10.45).

Theorem 10.11 is a refined restatement of Proposition 10.4, and thereby completes the proof of Proposition 10.4, once we give the proofs of Lemmas 10.6, 10.7 and 10.10. This is accomplished in Section 11 below.

10.5. **Proof of Theorem 4.3.** We now give the proof of our main existence result, Theorem 4.3, assuming Proposition 10.11 and Lemma 10.9 hold. So given a weak solution $u \in W^{1,p}(\Omega)$ and $a \in L^p(\Omega)$ of the rescaled reduced RT-equations (10.17) - (10.18) constructed in Proposition 10.11, we obtain a solution $J \in W^{1,p}(\Omega)$ and $B \in L^p(\Omega)$ of the reduced RT-equations (10.2) - (10.4) by setting

$$J = I + \epsilon u$$
 and $B = \epsilon a$, (10.51)

as can be verified by inspection, c.f. our scaling ansatz (10.16) and Lemma 10.3. It remains to prove that (J, B) satisfy estimate 4.16, and that J is integrable to coordinates as well as invertible for any $\epsilon > 0$ subject to some upper bound depending only on Ω, n, p and M.

We first prove that J is invertible assuming $\epsilon > 0$ meets (10.44) together with the upper bound

$$\epsilon < \frac{1}{2C_M C_2(M)M},\tag{10.52}$$

where $C_M > 0$ is the constant from Morrey's inequality (A.4), $C_2(M)$ is the constant from estimate (10.45) of Proposition 10.11, and M > 0 is our incoming bound on $\|(\Gamma, d\Gamma)\|_{L^{\infty}}$ in (4.16). For this we use the following lemma, which was proven in [22, Lemma 6.1].

Lemma 10.12. Let $J = I + \epsilon u$ for some matrix valued 0-form $u \in W^{1,p}(\Omega)$, p > n, and assume

$$0 < \epsilon < \frac{1}{2C_M \|u\|_{W^{1,p}}},\tag{10.53}$$

where $C_M > 0$ is the constant from Morrey's inequality (A.4). Then J is invertible and there exists a matrix valued 0-form $u^- \in W^{1,p}(\Omega)$ such that

$$J^{-1} = I + \epsilon \, u^- \tag{10.54}$$

and there exists a constant $C_{-} > 0$ depending only on Ω, n, p such that

$$\|u^{-}\|_{W^{1,p}(\Omega)} \le C_{-} \|u\|_{W^{1,p}(\Omega)}.$$
(10.55)

To apply Lemma 10.12 to the matrix valued 0-form $u \in W^{1,p}(\Omega)$ constructed in Proposition 10.11, it suffices to show that the ϵ -bound (10.52) implies the ϵ -bound (10.53) of Lemma 10.12. By estimate (10.45) of Proposition 10.11 and our initial bound (10.16) on Γ^* , we find

$$||u||_{W^{1,p}(\Omega)} \leq C_2(M)||(\Gamma^*, d\Gamma^*)||_{L^{\infty}(\Omega)} \leq C_2(M) M,$$

which implies

$$\frac{1}{2C_M C_2(M)M} \le \frac{1}{2C_M \|u\|_{W^{1,p}}},$$

and this shows that our ϵ -bound (10.52) implies (10.53). We conclude that the Jacobian $J = I + \epsilon u$ is invertible with $J^{-1} \in W^{1,p}(\Omega)$, c.f. (10.54). Moreover, by (10.54) and (10.55) it follows that

$$\|I - J^{-1}\|_{W^{1,p}(\Omega)} \le \epsilon C_{-} \|u\|_{W^{1,p}(\Omega)},$$
(10.56)

where $C_{-} > 0$ is the constant from Lemma 10.12, which depends only on Ω, n, p .

We now prove estimate (4.16). For this, observe first that by (10.51), we have $||I - J||_{W^{1,p}} = \epsilon ||u||_{W^{1,p}}$ and $||B||_{L^p} = \epsilon ||a||_{L^p}$. Applying now estimate (10.45) of Proposition 10.11, we obtain

$$\|I - J^{-1}\|_{W^{1,p}} + \|I - J\|_{W^{1,p}} + \|B\|_{W^{1,p}} \le \epsilon (1 + C_{-}) (\|u\|_{W^{1,p}} + \|a\|_{L^{p}})$$

$$\le C_{2}(M)\epsilon \|(\Gamma^{*}, d\Gamma^{*})\|_{L^{\infty}(\Omega)},$$

absorbing $(1 + C_{-}) > 0$ into the constant $C_2(M) > 0$. Now our scaling assumption $\Gamma_x = \epsilon \Gamma^*$ in (10.14) directly gives

$$||I - J^{-1}||_{W^{1,p}(\Omega)} + ||I - J||_{W^{1,p}(\Omega)} + ||B||_{W^{1,p}(\Omega)} \le C(M)||(\Gamma_x, d\Gamma_x)||_{L^{\infty}(\Omega)},$$

68

which is the sought after estimate 4.16 of Theorem 4.3.

We finally show that $J \equiv I + \epsilon u$ is indeed a Jacobian which is integrable to coordinates. For this recall that by Lemma 10.9, for each $k \geq 1$, the Jacobian $J_k \equiv I + \epsilon u_k$ is integrable to coordinates, that is,

$$d\overline{J}'_k = 0 \tag{10.57}$$

holds, c.f. (10.40), where $u_{k+1} \in W^{1,p}(\Omega)$ is defined by (10.28) - (10.29) of the iteration scheme. By the convergence $u_k \to u$ in $W^{1,p}$ as $k \to \infty$, it follows that J_k converges to J in $W^{1,p}$ as $k \to \infty$ as well. Thus dJ_k converges to dJ in L^p , and this implies dJ = 0 by (10.57). That is, J is integrable to some coordinate system y. It then follows directly that y_k defined by our iteration scheme converges to some y in $W^{2,p}(\Omega)$, and that J is the Jacobian of the coordinate transformation $x \to x + \epsilon y(x)$. This completes the proof of Theorem 4.3.

It remains only to prove Lemmas 10.6, 10.7 and 10.10, used to prove Proposition 10.11, and to prove Lemma 10.9, which together with Proposition 10.11 was used to prove Theorem 4.3.

11. Proof of Lemmas 10.6, 10.7, 10.9 and 10.10

The proof of Theorem 4.3 in Section 10.5 above followed from Lemma 10.9 and Proposition 10.11, which assumed Lemmas 10.6, 10.7 and 10.10 to be valid. In this section we prove these lemmas and thereby complete the proof of Theorem 4.3.

11.1. Proof of Lemma 10.6 (Estimate for the source terms). Recall Lemma 10.6 provides the basic estimates for the terms on the right hand side of equations (10.17) - (10.18). Lemma 10.6 is required in the proofs of Lemmas 10.7 and 10.10. So let $\Gamma^*, d\Gamma^* \in L^{\infty}(\Omega)$, and assume $u \in W^{1,p}(\Omega)$ and $a \in L^p(\Omega)$, for n . Then Lemma 10.6 states that (10.34) and(10.35) hold, namely

$$\begin{aligned} \|F_u(u,a)\|_{W^{-1,p}} &\leq \|a\|_{L^p} + C_s \left(\operatorname{vol}(\Omega) + \epsilon \, \|u\|_{L^p} \right) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}(\Omega)} \\ \|F_a(u)\|_{W^{-1,p}} &\leq C_s \left(\operatorname{vol}(\Omega) + \epsilon \, \|u\|_{W^{1,p}} \right) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}(\Omega)}, \end{aligned}$$

where $C_s > 0$ is some constant depending only on Ω , n, p, and where $F_u(u, a)$ and $F_a(u)$ are defined in (10.19) and (10.20).

Proof. Recall that the operator norm on a linear functional $F \in W^{-1,p}(\Omega)$, $F: W_0^{1,p^*}(\Omega) \longrightarrow \mathbb{R}$, is defined as

$$\|F\|_{W^{-1,p}} \equiv \sup_{\phi \in \mathcal{T}} \left|F[\phi]\right|,\tag{11.1}$$

where

$$\mathcal{T} \equiv \left\{ \phi \in W_0^{1,p^*}(\Omega) \, \middle| \, \|\phi\|_{W^{1,p^*}} = 1 \right\},\$$

and p^* is the conjugate of p, $\frac{1}{p} + \frac{1}{p^*} = 1$. We first derive (10.34). For this, recall the definition of $F_u(u, a)$ in (10.19),

$$F_u(u,a)[\phi] \equiv \left\langle \Gamma^*, d\phi \right\rangle_{L^2} + \epsilon \left\langle u \cdot \Gamma^*, d\phi \right\rangle_{L^2} + \left\langle a, \phi \right\rangle_{L^2},$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$. From this together with (11.1), we directly obtain that

$$\|F_{u}(u,a)\|_{W^{-1,p}} \leq \sup_{\phi \in \mathcal{T}} \left(\left| \left\langle \Gamma^{*}, d\phi \right\rangle_{L^{2}} \right| + \epsilon \left| \left\langle u \cdot \Gamma^{*}, d\phi \right\rangle_{L^{2}} \right| + \left| \left\langle a, \phi \right\rangle_{L^{2}} \right| \right).$$
(11.2)

We now estimate the right hand side of (11.2) term by term. In this proof we use C > 0 to denote a running constant depending only on Ω, n, p . For the first term we apply Hölder's inequality (A.6) to obtain

$$\begin{aligned} \left| \left\langle \Gamma^*, d\phi \right\rangle_{L^2} \right| &\leq C \left\| \Gamma^* \right\|_{L^p} \| d\phi \|_{L^{p^*}} \\ &\leq C \left| \Omega \right| \left\| \Gamma^* \right\|_{L^{\infty}} \| \phi \|_{W^{1,p^*}} \\ &\leq C \left| \Omega \right| \left\| (\Gamma^*, d\Gamma^*) \right\|_{L^{\infty}}, \end{aligned}$$
(11.3)

for $|\Omega| \equiv \text{vol}(\Omega)$, and where the last estimate follows from $\|\phi\|_{W^{1,p^*}} = 1$ for $\phi \in \mathcal{T}$. Using again Hölder's inequality and (10.15), we estimate the second term in (11.2) by

$$\begin{aligned} \left| \left\langle u \cdot \Gamma^*, d\phi \right\rangle_{L^2} \right| &\leq C \left\| u \cdot \Gamma^* \right\|_{L^p} \left\| d\phi \right\|_{L^{p^*}} \\ &\leq C \left\| \Gamma^* \right\|_{L^{\infty}} \left\| u \right\|_{L^p} \left\| \phi \right\|_{W^{1,p^*}} \\ &\leq C \left\| u \right\|_{L^p} \left\| (\Gamma^*, d\Gamma^*) \right\|_{L^{\infty}} \end{aligned}$$
(11.4)

and the third term by

$$\left| \left\langle a, \phi \right\rangle_{L^2} \right| \leq C \, \|a\|_{L^p} \|\phi\|_{L^{p^*}} \leq C \, \|a\|_{L^p}. \tag{11.5}$$

Substituting (11.3) - (11.5) into (11.2), we obtain

$$||F_u(u,a)||_{W^{-1,p}} \le C \left(\operatorname{vol}(\Omega) + \epsilon \, ||u||_{L^p} \right) ||(\Gamma^*, d\Gamma^*)||_{L^{\infty}} + ||a||_{L^p}$$

which implies the sought after estimate (10.34).

We next prove (10.35). The functional F_a is defined in (10.20) as

$$F_a(u)[\psi] \equiv \left\langle d\Gamma^*, \underbrace{\operatorname{div}}(\psi) \right\rangle_{L^2} + \epsilon \left\langle (u \cdot d\Gamma^* + du \wedge \Gamma^*), \underbrace{\operatorname{div}}(\psi) \right\rangle_{L^2}$$

for any vector valued 2-form $\psi \in W_0^{1,p^*}(\Omega)$, where

$$\left(\underline{\operatorname{div}}(\psi)\right)_{\nu}^{\mu} = \partial_{\nu}\psi_{ij}^{\mu} \, dx^{i} dx^{j}$$

by (8.23). Thus we have

$$\|F_{a}(u)\|_{W^{-1,p}} \leq \sup_{\psi \in \mathcal{T}} \left(\left| \left\langle d\Gamma^{*}, \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}} \right| + \epsilon \left| \left\langle (u \cdot d\Gamma^{*} + du \wedge \Gamma^{*}), \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}} \right| \right),$$
(11.6)

where \mathcal{T} is now taken as the space of vector valued 2-forms in $W_0^{1,p^*}(\Omega)$ having unit length with respect to the W^{1,p^*} -norm. Applying again Hölder's inequality, we estimate the first term in (11.6) by

$$\left|\left\langle d\Gamma^*, \underline{\operatorname{div}}(\psi)\right\rangle_{L^2}\right| \quad \leq \quad C \, \|d\Gamma^*\|_{L^p} \, \|\underline{\operatorname{div}}(\psi)\|_{L^{p^*}}$$

$$\leq C |\Omega| ||d\Gamma^*||_{L^{\infty}} ||\psi||_{W^{1,p^*}}$$

$$\leq C |\Omega| ||(\Gamma^*, d\Gamma^*)||_{L^{\infty}},$$
(11.7)

since $\|\psi\|_{W^{1,p^*}} = 1$ for all $\psi \in \mathcal{T}$. Likewise, we estimate the second term in (11.6) using Hölder's inequality by

$$\begin{aligned} \left| \left\langle (u \cdot d\Gamma^* + du \wedge \Gamma^*), \underbrace{\operatorname{div}}_{L^2} \right| &\leq C \left\| u \cdot d\Gamma^* + du \wedge \Gamma^* \right\|_{L^p} \left\| \underbrace{\operatorname{div}}_{L^{p^*}}(\psi) \right\|_{L^{p^*}} \\ &\leq C \left(\|u\|_{L^p} \| d\Gamma^*\|_{L^{\infty}} + \| du\|_{L^p} \|\Gamma^*\|_{L^{\infty}} \right) \|\psi\|_{W^{1,p^*}} \\ &\leq C \|u\|_{W^{1,p}} \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}}, \end{aligned}$$

$$(11.8)$$

again using $\|\psi\|_{W^{1,p^*}} = 1$. Substituting (11.7) and (11.8) into (11.6), we finally obtain

$$||F_a(u)||_{W^{-1,p}} \le C \left(\operatorname{vol}(\Omega) + \epsilon \, ||u||_{W^{1,p}} \right) ||(\Gamma^*, d\Gamma^*)||_{L^{\infty}},$$
(11.9)

which is the sought after estimate (10.35). This completes the proof. \Box

11.2. Proof of Lemma 10.7 (Well-posedness of the iteration scheme). We now prove Lemma 10.7 regarding well-posedness of the iteration scheme. For this, assume $u_k \in W^{1,p}(\Omega)$ is given and let $n , <math>n \ge 2$. Lemma 10.7 then states that there exists $a_{k+1} \in L^p(\Omega)$ which solves (10.21), there exists $\Psi_{k+1} \in L^p(\Omega)$ and $y_{k+1} \in W^{2,p}(\Omega)$ which solve (10.25) - (10.27), and there exists $u_{k+1} \in W^{1,p}(\Omega)$ which solves (10.28) with boundary data (10.29), and these solutions satisfy the elliptic estimates (10.36) - (10.39),

$$\|a_{k+1}\|_{L^{p}(\Omega)} \leq C_{e} \|F_{a}(u_{k})\|_{W^{-1,p}(\Omega)}, \qquad (11.10)$$

$$\|\Psi_{k+1}\|_{L^{p}(\Omega)} \leq C_{e} \|F_{u}(u_{k}, a_{k+1})\|_{W^{-1,p}(\Omega)}, \qquad (11.11)$$

$$\|y_{k+1}\|_{W^{2,p}(\Omega)} \leq C_e \|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)}, \qquad (11.12)$$

$$\|u_{k+1}\|_{W^{1,p}(\Omega)} \leq C_e \|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)}, \qquad (11.13)$$

for some constant $C_e > 0$ depending only on n, p and Ω .

Proof. We begin by proving existence of a weak solution a_{k+1} to the first order system (10.21), namely

$$\begin{cases} \langle \overrightarrow{a_{k+1}}, \delta \psi \rangle_{L^2} = F_a(u_k)[\psi], \\ \langle \overrightarrow{a_{k+1}}, d\varphi \rangle_{L^2} = 0, \end{cases}$$

subject to the bound (10.36), by applying Proposition C.4 of Appendix C. Proposition C.4 gives the existence of solutions to Cauchy Riemann type systems in a scalar variable at low level of regularity $a_{k+1} \in L^p(\Omega)$. We obtain such solutions by solving mollified equations with classical Dirichlet data, and then taking the zero mollification limit to obtain solutions $a_{k+1} \in L^p(\Omega)$. (Note that $a_{k+1} \in L^p$ is too weak to impose Dirichlet data directly.) To start, note that the incoming assumption $u_k \in W^{1,p}(\Omega)$ together with the source estimates of Lemma 10.6 show that $F_a(u_k) \in W^{-1,p}(\Omega)$, which is the regularity assumed in Proposition C.4. We now show that each vector component of (10.21) is a Cauchy Riemann type system in scalar variables, each component satisfying the assumptions of Proposition C.4. The right hand side of the second equation in (10.21) is zero, and hence of the form assumed in Proposition C.4. To apply Proposition C.4 to the first equation in (10.21), it suffices to show that there exists a vector valued 1-form $w \in W^{-1,p}(\Omega)$ such that $F_a(u) = dw$ in a weak sense, which then also implies the standard consistency conditions $dF_a(u) = 0$ a weak sense. This is accomplished in the next lemma.

Lemma 11.1. Assume $u \in W^{1,p}(\Omega)$ is given, then there exists a vector valued 1-form $w \in W^{-1,p}(\Omega)$ such that $F_a(u) = dw$ in the weak sense $F_a(u)[\varphi] = -w[\delta\varphi]$ for any vector valued 2-form $\varphi \in W_0^{2,p^*}(\Omega)$. Moreover, $dF_a(u) = 0$ holds in the weak sense that $F_a(u)[\delta\varphi] = 0$ for any vector valued 3-form $\varphi \in W_0^{2,p^*}(\Omega)$.

Proof. By definition (10.20), we have

$$F_{a}(u)[\psi] \equiv \left\langle d\Gamma^{*}, \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}} + \epsilon \left\langle (u \cdot d\Gamma^{*} + du \wedge \Gamma^{*}), \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}}$$
$$= \left\langle \left((I + \epsilon u) \cdot d\Gamma^{*} + d(I + \epsilon u) \wedge \Gamma^{*} \right), \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}}$$

for any vector valued 2-form $\psi \in W_0^{1,p^*}(\Omega)$. Let Γ_{ρ}^* denote a standard mollifier of $\Gamma^* \in L^{\infty}(\Omega)$ and u_{ρ} a mollifier of $u \in W^{1,p}(\Omega)$, then $\Gamma_{\rho}^* \to \Gamma^*$ in L^{∞} and $d\Gamma_{\rho}^* \to d\Gamma^*$ in L^{∞} as $\rho \to 0$, while $u_{\rho} \to u$ in $W^{1,p}(\Omega)$ as $\rho \to 0$. As a consequence, setting

$$\mathcal{F}_{\rho}[\psi] \equiv \lim_{\rho \to 0} \left\langle \left((I + \epsilon u_{\rho}) \cdot d\Gamma_{\rho}^{*} + d(I + \epsilon u_{\rho}) \wedge \Gamma_{\rho}^{*} \right), \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}},$$

Hölder inequality (A.6) implies convergence

$$F_a(u)[\psi] = \lim_{\rho \to 0} \mathcal{F}_{\rho}[\psi]. \tag{11.14}$$

We now show that $\mathcal{F}_{\rho}[\delta\varphi] = 0$. For this we begin by using the Leibnitz rule for differential forms (3.6) to compute

$$\mathcal{F}_{\rho}[\psi] = \left\langle d\left((I + \epsilon u_{\rho}) \cdot \Gamma_{\rho}^{*} \right), \underline{\operatorname{div}}(\psi) \right\rangle_{L^{2}},$$

Application of the adjoint property (8.24) gives

$$\mathcal{F}_{\rho}[\psi] = \left\langle \overrightarrow{\operatorname{div}} \left(d \left((I + \epsilon u_{\rho}) \cdot \Gamma_{\rho}^{*} \right) \right), \psi \right\rangle_{L^{2}}.$$

We now apply (3.12) to commute d and \overrightarrow{div} , from which we obtain

$$\mathcal{F}_{\rho}[\psi] = \left\langle d \, \overline{\delta((I + \epsilon u_{\rho}) \cdot \Gamma_{\rho}^{*})}, \psi \right\rangle_{L^{2}} \\ = \left\langle \overline{\delta((I + \epsilon u_{\rho}) \cdot \Gamma_{\rho}^{*})}, \delta \psi \right\rangle_{L^{2}}, \qquad (11.15)$$

where the last equality follows from partial integration for differential forms (8.7). Now, since $\Gamma_{\rho}^* \to \Gamma^*$ in L^{∞} and $u_{\rho} \to u$ in $W^{1,p}(\Omega)$ as $\rho \to 0$, it
follows that the expression on the right hand side converges in $W^{-1,p}(\Omega)$ and defines the vector valued 1-form $w \in W^{-1,p}(\Omega)$ as

$$w[\psi] \equiv \lim_{\rho \to 0} \left\langle \overline{\delta \left((I + \epsilon u_{\rho}) \cdot \Gamma_{\rho}^* \right)}, \delta \psi \right\rangle_{L^2}$$

Combining (11.14) with (11.15) imply that $F_a(u)[\psi] = w[\delta \psi]$ for any vector valued 2-form $\varphi \in W_0^{2,p^*}(\Omega)$, which is the sought after equation.

To prove the supplement, substitute $\psi = \delta \varphi$ into (11.15), the identity $\delta^2 = 0$ then gives us

$$\mathcal{F}_{\rho}[\delta\varphi] = \left\langle \overline{\delta\big((I + \epsilon u_{\rho}) \cdot \Gamma_{\rho}^{*}\big)}, \delta\delta\varphi \right\rangle_{L^{2}} = 0,$$

which implies by (11.14) that

$$F_a(u)[\delta\varphi] = \lim_{\rho \to 0} \mathcal{F}_{\rho}[\delta\varphi] = 0$$

for any vector valued 3-form $\varphi \in W_0^{2,p^*}(\Omega)$. This proves Lemma 11.1.

By Lemma 11.1, the desired condition $dF_a(u) = 0$ holds for each vector component in the weak sense, since there exists of a vector valued 1-form $w \in W^{-1,p}(\Omega)$ such that $F_a(u) = dw$. We conclude that Proposition C.4 applies component wise and yields the existence of a solution $a_{k+1} \in L^p(\Omega)$ to (10.21). Moreover, the solution constructed in Proposition C.4 meets the L^p -bound (C.14), which by application to each vector component directly implies the sought after L^p -bound (10.36) on a_{k+1} .

Next, we prove existence of a weak solution $\Psi_{k+1} \in L^p(\Omega)$ of (10.25), namely of

$$\langle \Psi_{k+1}, \delta \phi \rangle_{L^2} = F_{\Psi}(u_k, a_{k+1})[\phi]$$

for any matrix-valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, subject to the L^p bound (10.24) by applying Proposition C.5, which is a version of Proposition C.4 applying to the simpler case of 0-forms. For this, we need to verify that each vector component of (10.25) meets the consistency condition df = 0, in the weak sense $f(\delta \psi) = 0$, of Proposition C.5, which is achieved in the next Lemma.

Lemma 11.2. Assume $a_{k+1} \in L^p(\Omega)$ solves (10.21) for some $u_k \in W^{1,p}(\Omega)$, then F_{Ψ} , defined in (10.25), satisfies the weak consistency condition

$$F_{\Psi}(u_k, a_{k+1})[\delta \psi] = 0 \tag{11.16}$$

for any vector valued 2-form $\psi \in W_0^{2,p^*}(\Omega)$, (so $\psi|_{\partial\Omega} = 0$ and $\delta \psi|_{\partial\Omega} = 0$).²⁹ Proof. By (10.26), F_{Ψ} is defined as

$$F_{\Psi}(u_k, a_{k+1})[\phi] = \langle J_k \Gamma^*, d\phi \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \overrightarrow{\phi} \rangle_{L^2},$$

²⁹Note, $\delta\psi$ is a vector valued 1-form, and any such form can always be interpreted as a matrix valued 0-form. So $\delta\psi$ is an admissible argument for F_{Ψ} .

for matrix valued 0-forms $\phi \in W_0^{1,p^*}(\Omega)$, where $J_k \equiv I + \epsilon u_k$. In order for F_{Ψ} to act on the vector valued 1-form $\delta(\psi_{ij}^{\mu}dx^i \wedge dx^j) = (\delta\psi)_{\nu}^{\mu}dx^{\nu}$, we express $\delta\psi$ as the associated matrix valued 0-form $(\delta\psi)_{\nu}^{\mu}$, then

$$F_{\Psi}(u_k, a_{k+1})[\delta \psi] = \langle J_k \Gamma^*, d(\delta \psi) \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \overline{\delta \psi} \rangle_{L^2}, \qquad (11.17)$$

where $d(\delta\psi) = \partial_i(\delta\psi)^{\mu}_{\nu}dx^i$ is the exterior derivative of the matrix valued 0-form $\delta\psi$, and where $\overrightarrow{\delta\psi}$ denotes the original vector valued 1-form $\delta\psi^{\mu} = (\delta\psi)^{\mu}_{\nu}dx^{\nu}$. The main technical step of this proof is to show by a mollification argument that the first term on the right hand side of (11.17) equals

$$\langle J_k \Gamma^*, d(\delta \psi) \rangle_{L^2} = - \left\langle \left(J_k \cdot d\Gamma^* + dJ_k \wedge \Gamma^* \right), \underline{\operatorname{div}}(\psi) \right\rangle_{L^2}.$$
(11.18)

Assuming for the moment (11.18) holds, we substitute $J_k = I + \epsilon u_k$ to write (11.18) as

$$\langle J_k \Gamma^*, d(\delta \psi) \rangle_{L^2} = - \langle d\Gamma^*, \operatorname{div}(\psi) \rangle_{L^2} - \epsilon \left\langle (u_k \cdot d\Gamma^* + du_k \wedge \Gamma^*), \operatorname{div}(\psi) \right\rangle_{L^2} = - \langle \overrightarrow{a_{k+1}}, \overrightarrow{\delta \psi} \rangle_{L^2},$$

$$(11.19)$$

where the last equality follows from (10.21), the equation for a_{k+1} . Substituting (11.19) into (11.17) gives the sought after consistency condition, $F_{\Psi}(u_k, a_{k+1})[\delta \psi] = 0$ for any vector valued 2-form $\psi \in W_0^{1,p^*}(\Omega)$, which completes the proof of Lemma 11.2 once we prove equation (11.18) holds.

To verify (11.18), we consider a standard mollifier Γ_{ρ}^{*} of Γ^{*} together with a mollifier $(u_{k})_{\rho}$ of u_{k} , as in the proof of Lemma 11.1. For ease of notation we omit writing out the mollifier $(u_{k})_{\rho}$ in the subsequent argument, that is, whenever Γ_{ρ}^{*} appears we assume J_{k} denotes the mollification $(J_{k})_{\rho} =$ $I + \epsilon(u_{k})_{\rho}$. Now, since $\Gamma_{\rho}^{*} \to \Gamma^{*}$ in $L^{\infty}(\Omega)$ and $(u_{k})_{\rho} \to u_{k}$ in $W^{1,p}(\Omega)$, it follows that

$$\langle J_k \Gamma^*, d(\delta \psi) \rangle_{L^2} = \lim_{\rho \to 0} \langle J_k \Gamma^*_\rho, d(\delta \psi) \rangle_{L^2}.$$

Using the partial integration formula (8.7), we obtain

$$\langle J_k \Gamma_{\rho}^*, d(\delta \psi) \rangle_{L^2} = -\langle \delta(J_k \Gamma_{\rho}^*), (\delta \psi) \rangle_{L^2} = -\langle \overline{\delta(J_k \Gamma_{\rho}^*)}, \overline{\delta \psi} \rangle_{L^2},$$

where for the last equality we used the inner product identity (8.5) for matrix and valued forms, using again the notation $\vec{\delta\psi} = (\delta\psi)^{\mu}_{\nu} dx^{\nu}$. Applying now partial integration (8.7) for vector valued 1-forms, we get

$$\langle J_k \Gamma_{\rho}^*, d(\delta \psi) \rangle_{L^2} = \langle d\overline{\delta(J_k \Gamma_{\rho}^*)}, \psi \rangle_{L^2},$$

and using (3.12) to commute d and $\overrightarrow{\text{div}}$ as

$$d\overline{\delta(J_k\Gamma_{\rho}^*)} = \overrightarrow{\operatorname{div}} (d(J_k\Gamma_{\rho}^*)).$$

we find that

$$\langle J_k \Gamma_{\rho}^*, d(\delta \psi) \rangle_{L^2} = \langle \overrightarrow{\operatorname{div}} \left(d(J_k \Gamma_{\rho}^*) \right), \psi \rangle_{L^2}$$

$$\stackrel{(8.24)}{=} -\langle d(J_k \Gamma_{\rho}^*), \underline{\operatorname{div}}(\psi) \rangle_{L^2},$$
(11.20)

using the adjoint property (8.24) for $\overrightarrow{\operatorname{div}}$ in the last step. Now, by the Leibnitz rule, the L^{∞} convergence of Γ_{ρ}^{*} and $d\Gamma_{\rho}^{*}$ and the $W^{1,p}$ convergence of $(J_k)_{\rho}$, it follows that

$$d((J_k)_{\rho}\Gamma_{\rho}^*) = d(J_k)_{\rho} \wedge \Gamma_{\rho}^* + (J_k)_{\rho} \cdot d\Gamma_{\rho}^* \xrightarrow{\rho \to 0} dJ_k \wedge \Gamma^* + J_k \cdot d\Gamma^*$$

converges in $L^p(\Omega)$. Thus the left and right hand sides in (11.20) both converge (as can be shown using Hölder inequality) and yield

$$\langle J_k \Gamma^*, d(\delta \psi) \rangle_{L^2} = - \left\langle \left(dJ_k \wedge \Gamma^* + J_k \cdot d\Gamma^* \right), \underline{\operatorname{div}}(\psi) \right\rangle_{L^2},$$

which is the sought after identity (11.18). This completes the proof of Lemma 11.2. $\hfill \Box$

Lemma 11.2 establishes the consistency condition required by Proposition C.5 for existence of a solution to the first order Cauchy Riemann type system. To apply Proposition C.5 and conclude with the sought after existence of a vector valued 0-form $\Psi_{k+1} \in L^p(\Omega)$ which solves (10.25), it remains only to show that $F_{\Psi}(u_k, a_{k+1}) \in W^{-1,p}(\Omega)$. For this, recall that by (10.26), F_{Ψ} is defined as

$$F_{\Psi}(u_k, a_{k+1})[\phi] = \langle J_k \Gamma^*, d\phi \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \overrightarrow{\phi} \rangle_{L^2},$$

for any matrix valued 0-form ϕ , where $J_k \equiv I + \epsilon u_k$. Comparing this F_u in (10.19),

$$F_u(u,a)[\phi] \equiv \left\langle (I+\epsilon \, u) \cdot \Gamma^*, d\phi \right\rangle_{L^2} + \left\langle a, \phi \right\rangle_{L^2},$$

where $\phi \in W^{1,p}(\Omega)$ can be any matrix valued 0-form, we conclude that

$$||F_{\Psi}(u_k, a_{k+1})||_{W^{-1,p}(\Omega)} = ||F_u(u_k, a_{k+1})||_{W^{-1,p}(\Omega)}.$$

which is finite by the source estimate (10.34) of Lemma 10.6. We can now apply Proposition (C.5) and conclude with existence of a vector valued 0-form $\Psi_{k+1} \in L^p(\Omega)$ which solves (10.25) and satisfies the sought after estimate (11.11).

We now prove the existence of a solution $y_{k+1} \in W^{2,p}(\Omega)$ to the Dirichlet problem (10.27),

$$\begin{cases} \Delta y_{k+1} = \Psi_{k+1}, \\ y_{k+1} \big|_{\partial \Omega} = 0, \end{cases}$$

together with the elliptic estimate (11.12). By Lemma 10.6, $F_u(u_k, a_{k+1})$ is in $W^{-1,p}(\Omega)$ and we can apply the basic existence result for the Poisson equation with L^p sources, Theorem B.1. This yields the existence of a solution $y_{k+1} \in W^{2,p}(\Omega)$. To prove estimate (10.39), we now apply the elliptic estimate (B.4) of Theorem B.1 component wise to (10.27), which gives us

 $\|y_{k+1}\|_{W^{2,p}(\Omega)} \leq C \|\Psi_{k+1}\|_{L^{p}(\Omega)}.$

Using now estimate (10.38) on $\|\Psi_{k+1}\|_{L^p(\Omega)}$, we obtain

$$||y_{k+1}||_{W^{2,p}(\Omega)} \leq C ||F_u(u_k, a_{k+1})||_{W^{-1,p}(\Omega)},$$

where we have absorbed the constant from the estimate on $\|\Psi_{k+1}\|_{L^p}$ into the universal constant C > 0. This is the sought after estimate (11.12).

We now prove existence of a solution $u_{k+1} \in W^{1,p}(\Omega)$ of (10.28) with boundary data (10.29), that is,

$$-\Delta u_{k+1}[\phi] = F_u(u_k, a_{k+1})[\phi], \qquad (11.21)$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, and with Dirichlet boundary data $u_{k+1}|_{\partial\Omega} = dy_{k+1}|_{\partial\Omega}$. By Lemma 10.6 we have $F_u(u_k, a_{k+1}) \in W^{-1,p}(\Omega)$, so existence of a solution $u_{k+1} \in W^{1,p}(\Omega)$ of (11.21) follows directly from Theorem B.1. To prove estimate (10.37), we apply estimate (B.3) of Theorem B.1 component wise to equation (11.21) and obtain

$$\|u_{k+1}\|_{W^{1,p}(\Omega)} \le C\Big(\|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)} + \|dy_{k+1}\|_{W^{1,p}(\Omega)}\Big), \quad (11.22)$$

where the second terms on the right hand side results from the boundary data, $u_{k+1} = dy_{k+1}$ on $\partial\Omega$. Applying now estimate (11.12) to bound the boundary term by

 $\|dy_{k+1}\|_{W^{1,p}(\Omega)} \le \|y_{k+1}\|_{W^{2,p}(\Omega)} \le C \|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)},$

we obtain

 $\|u_{k+1}\|_{W^{1,p}(\Omega)} \le C \|F_u(u_k, a_{k+1})\|_{W^{-1,p}(\Omega)},$

which is the sought after estimate (10.37). We now choose the maximum over all constants in the above estimates as the constant $C_e > 0$ stated in Lemma 10.7. This completes the proof of Lemma 10.7.

11.3. Proof of Lemma 10.10 (Bounds on differences of iterates). We prove the closeness of subsequent iterates required to conclude with convergence of the iteration scheme in the proof of Proposition 10.11. So assume $\Gamma^*, d\Gamma^* \in L^{\infty}(\Omega)$, and let $C_e > 0$ denote the constant from the elliptic estimates of Lemma 10.7, which depends only on n, p, Ω . Then, to prove Lemma 10.10, it suffices to show that differences of iterates satisfy

$$\|\overline{a_{k+1}}\|_{L^p} \leq \epsilon C_e \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} \|\overline{u_k}\|_{W^{1,p}}, \qquad (11.23)$$

$$\|\overline{u_{k+1}}\|_{W^{1,p}} \leq \epsilon C_e(1+C_e) \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}} \|\overline{u_k}\|_{W^{1,p}}, \quad (11.24)$$

for any $k \in \mathbb{N}$. To prove Lemma 10.10, we require the following lemma which gives bounds on differences of source terms,

$$\overline{F_a(u_k)} \equiv F_a(u_k) - F_a(u_{k-1}),$$

$$\overline{F_u(u_k, a_{k+1})} \equiv F_u(u_k, a_{k+1}) - F_u(u_{k-1}, a_k),$$
(11.25)

which by linearity of F_a and F_u is a straightforward modification of the proof of Lemma 10.6.

Lemma 11.3. Assume (u_k, a_k) are defined by the iteration scheme (10.21) - (10.28), then the differences of source terms defined in (11.25) satisfy

$$\left\|\overline{F_{u}(u_{k}, a_{k+1})}\right\|_{W^{-1,p}} \le \|\overline{a_{k+1}}\|_{L^{p}} + \epsilon \|\overline{u_{k}}\|_{L^{p}} \|(\Gamma^{*}, d\Gamma^{*})\|_{L^{\infty}}, \qquad (11.26)$$

$$\|F_a(u_k)\|_{W^{-1,p}} \le \epsilon \|\overline{u_k}\|_{W^{1,p}} \|(\Gamma^*, d\Gamma^*)\|_{L^{\infty}}.$$
(11.27)

Proof. We prove the lemma by using linearity of the source terms F_u and F_a and following the steps in the proof of Lemma 10.6. In more detail, we find from the definition of $F_u(u, a)$ in (10.19), that

$$\overline{F_u(u_k, a_{k+1})[\phi]} = \epsilon \left\langle \overline{u_k} \cdot \Gamma^*, d\phi \right\rangle_{L^2} + \left\langle \overline{a_{k+1}}, \phi \right\rangle_{L^2}$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$. Following the steps in the proof of Lemma 10.6, then yields (11.26). Similarly, from the definition of F_a in (10.20) that

$$\overline{F_a(u_k)[\psi]} \equiv \epsilon \left\langle (\overline{u_k} \cdot d\Gamma^* + d\overline{u_k} \wedge \Gamma^*), \underbrace{\operatorname{div}}(\psi) \right\rangle_{L^2}$$

for any vector valued 2-form $\psi \in W_0^{1,p^*}(\Omega)$, and following the steps in the proof of Lemma 10.6 gives us (11.27). This completes the proof of Lemma 11.3.

Lemma 10.10 now follows from the elliptic estimates (10.36) and (10.39) together with the bounds on differences of sources in Lemma 11.3. That is, by linearity of (10.36), we have

$$\begin{aligned} \|\overline{a_{k+1}}\|_{L^{p}(\Omega)} &\leq C_{e} \|\overline{F_{a}(u_{k})}\|_{W^{-1,p}(\Omega)} \\ &\leq \epsilon C_{e} \|\overline{u_{k}}\|_{W^{1,p}} \|(\Gamma^{*}, d\Gamma^{*})\|_{L^{\infty}}. \end{aligned}$$
(11.28)

Likewise, by linearity of (10.39), we find

$$\begin{split} \|\overline{u_{k+1}}\|_{W^{1,p}(\Omega)} &\leq C_{e} \|\overline{F_{u}(u_{k},a_{k+1})}\|_{W^{-1,p}(\Omega)} \\ &\leq C_{e} \left(\|\overline{a_{k+1}}\|_{L^{p}} + \epsilon \|\overline{u_{k}}\|_{L^{p}} \|(\Gamma^{*},d\Gamma^{*})\|_{L^{\infty}}\right) \\ &\leq \epsilon C_{e}(1+C_{e}) \|\overline{u_{k}}\|_{W^{1,p}} \|(\Gamma^{*},d\Gamma^{*})\|_{L^{\infty}}. \quad (11.29) \end{split}$$

This completes the proof of Lemma 10.10.

11.4. Proof of Lemma 10.9 (Integrability of J). On smooth k-forms the Laplacian acts component wise, (i.e., on components of matrix-, vectorand differential forms separately), and the relation between vector and matrix valued solutions of the Poisson equations in a classical sense is straightforward. That is, we have $\overrightarrow{\Delta u} = \Delta \vec{u}$ in a classical sense. This is used in Lemma 10.1 to prove that the Jacobian J produced by the iteration scheme is integrable to coordinates. The next lemma establishes the relation $\overrightarrow{\Delta u} = \Delta \vec{u}$ for the weak Laplacian.

Lemma 11.4. Let $u \in W^{1,p}(\Omega)$ be a matrix valued 0-form, then

$$\Delta(u)[\phi] = \Delta \vec{u}[\vec{\phi}] \tag{11.30}$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$.

Proof. From the weak form of the Laplacian in (8.12), using that $\delta u = 0$ and $du = \nabla u$ for matrix valued 0-forms, (where again ∇u denotes the gradient acting on each component of u), we find

$$\begin{aligned} -\Delta(u)[\phi] &= \langle du, d\phi \rangle_{L^2} + \langle \delta u, \delta\phi \rangle_{L^2} &= \langle du, d\phi \rangle_{L^2} \\ &= \langle \nabla u, \nabla\phi \rangle_{L^2} &= \sum_j \langle \partial_j u, \partial_j \phi \rangle_{L^2}, \end{aligned}$$

where the last equality follows from the definition in (8.14), c.f. Lemma 8.2. Using now that for fixed j the inner product is invariant under vectorization for matrix valued 0-forms, c.f. (8.5), we obtain

$$-\Delta(u)[\phi] = \langle \nabla \vec{u}, \nabla \vec{\phi} \rangle_{L^2}.$$

Now, let u^{ϵ} be a standard mollification of u. Then $u^{\epsilon} \to u$ in $W^{1,p}(\Omega)$ as $\epsilon \to 0$, which allows us to compute

$$\begin{aligned} -\Delta(u)[\phi] &= \lim_{\epsilon \to 0} \langle \nabla \vec{u}^{\epsilon}, \nabla \phi \rangle_{L^{2}} \\ &= -\lim_{\epsilon \to 0} \langle \Delta \vec{u}^{\epsilon}, \vec{\phi} \rangle_{L^{2}} \\ &= \lim_{\epsilon \to 0} \left(\langle d\vec{u}^{\epsilon}, d\vec{\phi} \rangle_{L^{2}} + \langle \delta \vec{u}^{\epsilon}, \delta \vec{\phi} \rangle_{L^{2}} \right) \\ &= \langle d\vec{u}, d\vec{\phi} \rangle_{L^{2}} + \langle \delta \vec{u}, \delta \vec{\phi} \rangle_{L^{2}} \\ &= -\Delta \vec{u}[\vec{\phi}]. \end{aligned}$$

This completes the proof.

We now prove Lemma 10.9, which states that

$$d\overrightarrow{u_{k+1}} = 0, \tag{11.31}$$

where $u_{k+1} \in W^{1,p}(\Omega)$ is a solution of (10.28) with boundary data (10.29). Equation (11.31) implies directly that $J_{k+1} \equiv I + \epsilon u_{k+1}$ is integrable to coordinates. Moreover, Lemma 10.9 states that J_{k+1} is the Jacobian of the coordinate transformation $x \to x + \epsilon y_{k+1}(x)$, where $y_{k+1} \in W^{2,p}(\Omega)$ is the solution of (10.27).

Proof of Lemma 10.9. The idea of proof is similar to that of Lemma 10.1, but adapted to the weak formulation of (10.23), to take into account the regularity of $\Psi_{k+1} \in L^p(\Omega)$ and $u_{k+1} \in L^p(\Omega)$. That is, we need to show that

$$\Delta\left(\overrightarrow{u_{k+1}} - dy_{k+1}\right)\left[\vec{\phi}\right] = 0, \qquad (11.32)$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$. Assume for the moment (11.32) is true. Then, since $\overrightarrow{u_{k+1}} - dy_{k+1}$ vanishes on $\partial\Omega$ by the boundary condition (10.29), equation (11.32) implies that

$$\overrightarrow{u_{k+1}} - dy_{k+1} = 0 \tag{11.33}$$

in Ω , which is the sought after equation (10.40). Moreover, (11.33) directly implies that

$$d(x + \epsilon y_{k+1}) = \overrightarrow{J_{k+1}},$$

where $J_{k+1} \equiv I + \epsilon u_{k+1}$. Thus J_{k+1} is in fact the Jacobian of the coordinate transformation $x \to x + \epsilon y_{k+1}(x)$. This proves Lemma 10.9 once we establish (11.32).

To prove (11.32), recall that Ψ_{k+1} satisfies (10.25),

$$\langle \Psi_{k+1}, \delta \vec{\phi} \rangle_{L^2} = \langle J_k \Gamma^*, d\phi \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \vec{\phi} \rangle_{L^2}, \qquad (11.34)$$

for any matrix-valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$, where $J_k \equiv I + \epsilon u_k$. Moreover, $y_{k+1} \in W^{2,p}(\Omega)$ solves

$$\Delta y_{k+1} = \Psi_{k+1}, \tag{11.35}$$

with boundary data $y_{k+1}|_{\partial\Omega} = 0$, c.f. (10.27).

Combining (11.34) and (11.35), and using $d^2y_{k+1} = 0$, we obtain from the definition of the weak Laplacian in (8.12) that

$$-\Delta(dy_{k+1})[\vec{\phi}] = \langle \delta dy_{k+1}, \delta \vec{\phi} \rangle_{L^2} + \langle d^2 y_{k+1}, d \vec{\phi} \rangle_{L^2} = \langle \delta dy_{k+1}, \delta \vec{\phi} \rangle_{L^2} = \langle \Delta y_{k+1}, \delta \vec{\phi} \rangle_{L^2}, \qquad (11.36)$$

since y_{k+1} is a vector valued 0-form, so that $\delta y_{k+1} = 0$ and $\Delta y_{k+1} = (\delta d + d\delta)y_{k+1} = \delta dy_{k+1}$. Substituting now (11.35) for Δy_{k+1} , we write (11.36) as

$$-\Delta(dy_{k+1})[\vec{\phi}] \stackrel{(11.35)}{=} \langle \Psi_{k+1}, \delta\vec{\phi} \rangle_{L^2}$$
$$\stackrel{(11.34)}{=} \langle J_k \Gamma^*, d\phi \rangle_{L^2} + \langle \overrightarrow{a_{k+1}}, \vec{\phi} \rangle_{L^2}.$$
(11.37)

Now, recall that u_{k+1} solves (10.28), that is,

$$-\Delta u_{k+1}[\phi] = \left\langle J_k \, \Gamma^*, d\phi \right\rangle_{L^2} + \left\langle a_{k+1}, \phi \right\rangle_{L^2}. \tag{11.38}$$

By definition of the inner products we have $\langle a_{k+1}, \phi \rangle_{L^2} = \langle \overrightarrow{a_{k+1}}, \overrightarrow{\phi} \rangle_{L^2}$, c.f. (8.5). Thus, (11.37) in combination with (11.38) gives us

$$\Delta(dy_{k+1})\left[\vec{\phi}\right] = \Delta u_{k+1}\left[\phi\right] \tag{11.39}$$

for any matrix valued 0-form $\phi \in W_0^{1,p^*}(\Omega)$. Finally, applying Lemma 11.4 to the right hand side of (11.39), we obtain

$$\Delta(dy_{k+1})\left[\vec{\phi}\right] = \Delta \overrightarrow{u_{k+1}}\left[\vec{\phi}\right],$$

which directly gives the sought after equation (11.32). This completes the proof of Lemma 10.9. $\hfill \Box$

This finishes the proof of Theorem 2.1 and 2.3, thereby establishing optimal regularity and Uhlenbeck compactness for L^{∞} connections.

APPENDIX A. SOBOLEV NORMS AND INEQUALITIES

We first give an overview of the norms used in this paper. These norms are coordinate dependent, so we assume at the start a given coordinate system x defined on an open set $\Omega \subset \mathcal{M}$ such that $\Omega_x \equiv x(\Omega) \subset \mathbb{R}^n$ is bounded. In this paper we always write Ω instead of Ω_x when there is no confusion, c.f. Section 2. In this section Ω always refers to Ω_x . We denote by $\|\cdot\|_{W^{m,p}(\Omega)}$ the standard $W^{m,p}$ -norm, defined as the sum of the L^p -norms of derivatives from order zero up to m [11]. When applied to matrix or vector valued differential forms ω , $\|\omega\|_{W^{m,p}(\Omega)}$ denotes the p-th root of the sum of the p-th powers of the $W^{m,p}$ -norms of all components, summed over all matrix and differential form components for matrix valued differential forms, and over all vector and differential form components for vector valued differential forms. Specifically, for matrix valued k-form $\omega = \omega_{\nu i_1...i_k}^{\mu} dx^{i_1} \wedge ... \wedge dx^{i_k}$ we define

$$\begin{split} |\omega|^{p} &\equiv \sum_{\mu,\nu,i_{1},\dots,i_{k}} |\omega^{\mu}_{\nu i_{1}\dots i_{k}}|^{p} \\ \|\omega\|^{p}_{L^{p}(\Omega)} &\equiv \sum_{\mu,\nu,i_{1},\dots,i_{k}} \|\omega^{\mu}_{\nu i_{1}\dots i_{k}}\|^{p}_{L^{p}} &= \int_{\Omega} |\omega|^{p} dx \\ \|\omega\|_{W^{m,p}(\Omega)} &\equiv \sum_{|l| \leq m} \|\partial^{l}\omega\|_{L^{p}(\Omega)} \end{split}$$
(A.1)

where $1 \leq p < \infty$, l is a multi-index, so $l = (l_1, ..., l_n)$, $|l| = l_1 + ... + l_n$ and $\partial^l \omega \equiv (\partial_{l_1} \omega, ... \partial_{l_n} \omega)$ taken component wise. Likewise, define the L^{∞} -norm

$$\|\omega\|_{L^{\infty}(\Omega)} \equiv \sum_{\mu,\nu,i_{1},\dots,i_{k}} \|\omega^{\mu}_{\nu \,i_{1}\dots i_{k}}\|_{L^{\infty}(\Omega)},\tag{A.2}$$

and the L^2 -inner product on matrix valued forms ω and u,

$$\langle \omega, u \rangle_{L^2} \equiv \int_{\Omega} \operatorname{tr} \left(\langle \omega ; u^T \rangle \right) = \int_{\Omega} \sum_{\nu, \sigma=1}^n \sum_{i_1 < \dots < i_k} \omega^{\nu}_{\sigma \, i_1 \dots i_k} u^{\nu}_{\sigma \, i_1 \dots i_k}, \qquad (A.3)$$

c.f. (8.2), where tr(·) is the matrix trace and $\langle \cdot ; \cdot \rangle$ the matrix valued inner product (3.9), and where we integrate with respect to Lebesgue measure in a fixed coordinate system x. By this we introduce the Hilbert-Schmidt inner product on the matrix components of matrix valued differential forms. When convenient we drop the dependence of the region Ω of norms, for example, writing $\|\cdot\|_{W^{m,p}}$ instead of $\|\cdot\|_{W^{m,p}(\Omega)}$.

We now summarize the basic integral inequalities we apply in this paper, see [11] for details. The space $W^{1,p}$ for p > n, is embedded in the space of Hölder continuous functions $C^{0,\alpha}(\overline{\Omega})$. Namely, for p > n Morrey's inequality gives

$$||f||_{C^{0,\alpha}(\overline{\Omega})} \le C_M ||f||_{W^{1,p}(\Omega)},$$
 (A.4)

where $\alpha \equiv 1 - \frac{n}{p}$ and $C_M > 0$ is a constant depending only on n, p and Ω [11].³⁰ Morrey's inequality (A.4) extends unchanged to components of matrix valued differential forms. By Morrey's inequality we can estimate products of $W^{1,p}$ functions f and g on bounded domains Ω as

$$\|fg\|_{W^{1,p}(\Omega)} \le C_M \|f\|_{W^{1,p}(\Omega)} \|g\|_{W^{1,p}(\Omega)}, \tag{A.5}$$

³⁰When applying (A.4) for higher derivative estimates in Section 9.1, we often encounter C_M multiplied by some positive combinatorial factor depending only on m which we again denote by C_M .

by pulling L^{∞} norms of undifferentiated functions out of L^p norms and applying (A.4) to bound the resulting L^{∞} norms. This shows that $W^{1,p}(\Omega)$ is closed under multiplication on bounded domains.

To handle products in the RT-equations at the lowest order of regularity we employ Hölder's inequality, which states

$$\|fg\|_{L^{1}(\Omega)} \leq \|f\|_{L^{p}(\Omega)} \|g\|_{L^{p^{*}}(\Omega)}, \tag{A.6}$$

where p and p^* are conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. Now, assuming $f, g \in L^{2p}(\Omega)$, (A.6) implies the estimate

$$\|fg\|_{L^{p}(\Omega)} = \||fg|^{p}\|_{L^{1}(\Omega)}^{\frac{1}{p}}$$

$$\stackrel{(A.6)}{\leq} \||f|^{p}\|_{L^{2}(\Omega)}^{\frac{1}{p}} \||g|^{p}\|_{L^{2}(\Omega)}^{\frac{1}{p}}$$

$$= \|f\|_{L^{2p}(\Omega)} \|g\|_{L^{2p}(\Omega)},$$
(A.7)

which shows in particular that $fg \in L^p(\Omega)$. Estimate (A.7) allows us to control the gradient product $dJ^{-1} \wedge dJ$ in (4.1) in the proof of Theorem 4.1, which is a key step in our analysis. Hölder's inequality (A.6) and estimate (A.7) extend to matrix valued differential forms, only a little care must be taken to handle index summation, for example,

$$\begin{split} \|A \cdot B\|_{L^{1}(\Omega)} &\equiv \sum_{\mu,\nu} \|(A \cdot B)^{\mu}_{\nu}\|_{L^{1}(\Omega)} \\ &\leq \sum_{\mu,\nu,\sigma} \|A^{\mu}_{\sigma}B^{\sigma}_{\nu}\|_{L^{1}(\Omega)} \\ &\stackrel{(A.6)}{\leq} \left(\sum_{\mu,\nu} \|A^{\mu}_{\nu}\|_{L^{p}(\Omega)}\right) \left(\sum_{\mu,\nu} \|B^{\mu}_{\nu}\|_{L^{p^{*}}(\Omega)}\right) \\ &\leq C \left(\sum_{\mu,\nu} \|A^{\mu}_{\nu}\|_{L^{p}(\Omega)}^{p}\right)^{\frac{1}{p}} \left(\sum_{\mu,\nu} \|B^{\mu}_{\nu}\|_{L^{p^{*}}(\Omega)}^{p^{*}}\right)^{\frac{1}{p^{*}}} \\ &\equiv C \|A\|_{L^{p}(\Omega)} \|B\|_{L^{p^{*}}(\Omega)} \end{split}$$
(A.8)

for matrix valued 0-form A and matrix valued k-form B, where C > 0 is a constant depending only on n (just to compensate for the order of taking roots and summation) and we omitted summation over form indices $i_1, ..., i_k$. Similarly one obtains that

$$\|A \wedge B\|_{L^{1}(\Omega)} \leq C \|A\|_{L^{p}(\Omega)} \|B\|_{L^{p^{*}}(\Omega)}, \|A \wedge B\|_{L^{p}(\Omega)} \leq C \|A\|_{L^{2p}(\Omega)} \|B\|_{L^{2p}(\Omega)},$$
 (A.9)

for A and B being general matrix valued differential forms.

APPENDIX B. ELLIPTIC PDE THEORY

We now summarize the estimates we use from elliptic PDE theory. We assume throughout that $n , <math>n \ge 2$ and that $\Omega \subset \mathbb{R}^n$ is a bounded open domain, simply connected and with smooth boundary. Our estimates are based on the following two theorems, which directly extend to matrix

valued and vector valued differential forms since the Laplacian acts component wise, c.f. Lemma 8.2. That is, we take the weak Laplacian here as $\Delta u[\phi] = -\langle \nabla u, \nabla \phi \rangle_{L^2}$ for scalar functions $u \in W^{1,p}(\Omega)$ and for test functions $\phi \in W_0^{1,p^*}(\Omega)$, where $W_0^{1,p^*}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ with respect to the W^{1,p^*} -norm (so $\phi|_{\partial\Omega} = 0$). Our first theorem is based on Theorem 7.2 in [25], but adapted to the case of solutions to the Poisson equation with non-zero Dirichlet data.

Theorem B.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$, assume $f \in W^{-1,p}(\Omega)$ and $u_0 \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ for n . Then the Dirichlet boundary value problem³¹

$$\Delta u[\phi] = f[\phi], \quad in \ \Omega \tag{B.1}$$

$$u = u_0 \quad on \ \partial\Omega, \tag{B.2}$$

for any $\phi \in W_0^{1,p^*}(\Omega)$, has a unique weak solution $u \in W^{1,p}(\Omega)$ with boundary data $u - u_0 \in W_0^{1,p}(\Omega)$. Moreover, any weak solution³² u of (B.1) -(B.2) satisfies

$$\|u\|_{W^{1,p}(\Omega)} \le C\left(\|f\|_{W^{-1,p}(\Omega)} + \|u_0\|_{W^{1,p}(\Omega)}\right),\tag{B.3}$$

for some constant C depending only on Ω, n, p , and if $f \in L^p(\Omega)$ and $u_0 \in W^{2,p}(\Omega)$, then the solution u satisfies

$$\|u\|_{W^{2,p}(\Omega)} \le C \left(\|f\|_{L^{p}(\Omega)} + \|u_{0}\|_{W^{2,p}(\Omega)} \right).$$
(B.4)

Proof. Theorem 7.2 in [25] yields existence of a unique solution $u \in W^{1,p}(\Omega)$ to (B.1) - (B.2) satisfying estimate (B.3) in the case of zero Dirichlet data, i.e. when $u_0 = 0$ in Ω . Note, Theorem 7.2 in [25] applies since the weak Laplacian is a strongly uniformly elliptic operator in the sense of equation (1.8) of [25, Def 1.3].³³ To extend this result to non-zero Dirichlet data, let $\tilde{u} \in W^{1,p}(\Omega)$ be the solution of the Laplace equation $\Delta \tilde{u} = 0$ with boundary data $\tilde{u} = u_0$ on $\partial \Omega$ in the sense that $u_0 - \tilde{u} \in W_0^{1,p}(\Omega)$; note that u can be constructed via Green's representation formula [12, Eqn. (2.21)] for $W^{1,p}$ -data. Assume now $w \in W^{1,p}$ is the solution of (B.1) with zero Dirichlet data satisfying (B.3), $\Delta w = f$ in a weak sense and $w \in W_0^{1,p}(\Omega)$, which exists by Theorem 7.2 in [25]. Then $u \equiv w + \tilde{u}$ solves (B.1) - (B.2), since $\Delta u = \Delta w = f$ (in a weak sense) and $u - u_0 \in W_0^{1,p}(\Omega)$. To show that estimate (B.3) holds, we begin by using the triangle inequality twice to get

$$||u||_{W^{1,p}} \leq ||w||_{W^{1,p}} + ||\tilde{u} - u_0||_{W^{1,p}} + ||u_0||_{W^{1,p}}.$$
(B.5)

³¹Note that $u \in W^{1,p}(\Omega)$ is Hölder continuous by Morrey's inequality (A.4), since we assume here p > n. So boundary data can be assigned in the sense of continuous functions.

 $^{^{32}}$ It suffices to assume that u is regular enough to make sense of the weak formulation of the Laplacian, for example, $du, \delta u \in L^p(\Omega)$ for a differential form u, as in Section 9.2.

³³That in fact any such solution satisfies estimate (B.3) follows from Theorem 6.1 in [25], equation (6.2), where we can take $C_2 = 0$ since Δ is strongly uniformly elliptic.

We can now apply the established case of estimate (B.3), for the case of zero Dirichlet data, to the first two terms, since $w, \tilde{u} - u_0 \in W^{1,p}(\Omega)$. This yields

$$||w||_{W^{1,p}} \leq C||f||_{W^{-1,p}(\Omega)}, \tag{B.6}$$

while the second term, using in addition that $\Delta(\tilde{u} - u_0) = \Delta u_0$ and that $\|\Delta u_0\|_{W^{m-1,p}(\Omega)} \leq \|u_0\|_{W^{m+1,p}(\Omega)}$, is bounded by

$$\begin{aligned} \|\tilde{u} - u_0\|_{W^{1,p}(\Omega)} &\leq C \|\Delta(\tilde{u} - u_0)\|_{W^{-1,p}(\Omega)} \\ &\leq C \|u_0\|_{W^{1,p}(\Omega)}. \end{aligned} \tag{B.7}$$

Substitution of estimates (B.6) and (B.7) into (B.5) yields the sought after estimate (B.3) in the general case of non-zero Dirichlet data. We proved that there exists a solution to (B.1) - (B.2) which satisfies estimate (B.3).

To complete the proof, note that estimate (B.4) in the case of zero Dirichlet data $(u_0 = 0)$ is already proven in [25, Thm 7.2], (c.f. Lemma 9.17 in [12]). The case of estimate (B.4) for non-zero Dirichlet data follows by an argument analogous to (B.5) and (B.7). Namely, let \tilde{u} be the solution of $\Delta \tilde{u} = 0$ with boundary data $u_0 - \tilde{u} \in W_0^{1,p}(\Omega)$, and let $w \in W^{2,p}(\Omega)$ be the solution of $\Delta w = f$ with $w \in W_0^{1,p}(\Omega)$ established in [25, Thm 7.2]. Then setting again $u \equiv w + \tilde{u}$ and applying estimate (B.4) in the case of vanishing Dirichlet data (y = 0) to w and $\tilde{u} - u_0$ yields the sought after estimate (B.4):

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\leq \|w\|_{W^{2,p}} + \|\tilde{u} - u_0\|_{W^{2,p}} + \|u_0\|_{W^{2,p}} \\ &\leq C(\|f\|_{L^p(\Omega)} + \|\Delta(\tilde{u} - u_0)\|_{L^p(\Omega)}) + \|u_0\|_{W^{2,p}(\Omega)} \\ &\leq C(\|f\|_{L^p(\Omega)} + \|u_0\|_{W^{2,p}(\Omega)}), \end{aligned}$$

where C > 0 was again used as a running constant. This completes the proof of Theorem B.1.

We require the following interior elliptic estimates in the proof of Theorem 4.1 in Section 9 in the case m = 0, and for higher regularities $m \ge 1$ to prove Proposition 9.1. Note that interior elliptic estimates usually are established earlier in the development of elliptic PDE theory, but for completeness we derive the interior estimate from (B.3).

Theorem B.2. Let $f \in W^{m-1,p}(\Omega)$, for $m \ge 0$ and n . Assume <math>u is a weak solution of (B.1). Then $u \in W^{m+1,p}(\Omega')$ for any open set Ω' compactly contained in Ω and there exists a constant C depending only on Ω, Ω', m, n, p such that

$$\|u\|_{W^{m+1,p}(\Omega')} \le C(\|f\|_{W^{m-1,p}(\Omega)} + \|u\|_{W^{m,p}(\Omega)}).$$
(B.8)

Proof. We only need to prove the case m = 0. The case for $m \ge 1$ can easily be obtained by differentiating and applying the estimate for the case m = 0; (c.f. Appendix A in [22].)

We apply estimate (B.3) to ϕu where ϕ is a standard smooth cutoff function, $\phi = 1$ in Ω' , $\phi = 0$ on $\partial\Omega$. Then

$$\Delta(\phi u) = \phi \Delta u + 2\nabla \phi \cdot \nabla u + u \Delta \phi \equiv \hat{f}.$$
 (B.9)

Then applying (B.3) together with the assumption that we have a solution of the Poisson equation (B.1), we have

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega')} &= \|\phi u\|_{W^{1,p}(\Omega')} \le \|\phi u\|_{W^{1,p}(\Omega)} \le C \|\hat{f}\|_{W^{1,p}(\Omega)} \quad (B.10) \\ &= C \|\phi\|_{C^2} \left(\|f\|_{W^{-1,p}(\Omega)} + \|\nabla u\|_{W^{-1,p}(\Omega)} + \|u\|_{W^{-1,p}(\Omega)} \right), \end{aligned}$$

from which (B.8) follows, since by definition of Sobolev norms we have $\|\nabla u\|_{W^{-1,p}(\Omega)} \leq \|u\|_{L^p}$ and $\|u\|_{W^{-1,p}(\Omega)} \leq \|u\|_{L^p}$. This completes the proof.

Appendix C. Cauchy Riemann type equations at low regularities

In this appendix we prove Propositions C.4 and C.5 which give existence of weak solutions to Cauchy Riemann type equation for scalar valued differential forms, required in the proof of Lemma 10.7 for well-posedness of the iteration scheme. For this, in Theorems C.1, C.2 and C.3 below, we first collect the theorems from [6] regarding classical $W^{1,p}$ solutions of first order Cauchy-Riemann type equations

$$du = f$$
 and $\delta u = 0$, in Ω , (C.1)

where the Cartan algebra of differential forms is determined by the Euclidean metric in \mathbb{R}^n . We extend these theorems in Proposition C.4 and C.5 below to prove existence of weak L^p solutions, the lower regularity required for well-posedness of our iteration scheme, in the special case when boundary data is free to assign.

To begin, we state the following partial integration formula for non-zero boundary data,

$$\int_{\Omega} \langle du, w \rangle dx + \int_{\Omega} \langle u, \delta w \rangle dx = \int_{\partial \Omega} \langle N \wedge u, w \rangle = \int_{\partial \Omega} \langle u, N \cdot w \rangle, \quad (C.2)$$

where u is a k-form and w a (k + 1)-form, N denotes the outward-pointing unit normal of $\partial\Omega$ and $N \cdot w$ denotes the contraction of N and w, c.f. Theorem 3.28 in [6], (and (8.6) for the case of vanishing boundary data).

We now state the basic elliptic estimate for (C.1), which mirrors estimate (B.3) for the Poisson equation, the so-called Gaffney inequality, (c.f. Theorem 5.21 in [6]). The Gaffney inequality shows that d and δ control all derivatives of u.

Theorem C.1. (Gaffney Inequality): Let $u \in W^{m+1,p}(\Omega)$ be a k-form for $m \ge 0$, $1 \le k \le n-1$ and $n \ge 2$. Then there exists a constant C > 0depending only on Ω , m, n, p, such that

$$\|u\|_{W^{m+1,p}(\Omega)} \le C\Big(\|du\|_{W^{m,p}(\Omega)} + \|\delta u\|_{W^{m,p}(\Omega)} + \|u\|_{W^{m+\frac{p-1}{p},p}(\partial\Omega)}\Big).$$
(C.3)

The following special case of Theorem 7.4 in [6], provides the existence theorem sufficient for our purposes, and contains a refinement of Gaffney's

inequality (C.3) for 1-forms and 0-forms.

Theorem C.2. (i) Let $f \in W^{m,p}(\Omega)$ be a 2-form with df = 0, where $m \ge 0$, $n \ge 2$, 1 . Assume further that <math>f = dv for some 1-form $v \in W^{m,p}(\Omega)$.³⁴ Then there exists a 1-form $u = u_i dx^i \in W^{m+1,p}(\Omega)$ which solves

$$du = f \quad and \quad \delta u = 0 \qquad in \quad \Omega, \tag{C.4}$$

together with the boundary condition

$$u \cdot N = 0 \qquad on \quad \partial\Omega, \tag{C.5}$$

where N is the unit normal on $\partial\Omega$ and $u \cdot N \equiv u_i N^i$. Moreover, there exists a constant C > 0 depending only on Ω , m, n, p, such that

$$\|u\|_{W^{m+1,p}(\Omega)} \le C \|f\|_{W^{m,p}(\Omega)}.$$
(C.6)

(ii) Let $f \in W^{m,p}(\Omega)$ be a 1-form with df = 0. Then there exists a 0-form $u \in W^{m+1,p}(\Omega)$ such that u solves du = f, has zero average $\int_{\Omega} u dx = 0$ and satisfies estimate (C.6).

Proof. Theorem C.2 and its proof are taken from [22], and the proof is included for completeness, c.f. Theorem 2.4 in [22]. Part (i) is a special case of Theorem 7.4 in [6] for 1-forms with zero boundary conditions. Namely, our assumption df = 0 together with zero boundary data, ($\omega_0 = 0$, following notation in [6]), directly gives condition (C1) of [6, Thm 7.4]. The first equation of condition (C2) of [6, Thm 7.4] follows trivially from our assumptions; g = 0 and $\omega_0 = 0$ in the notation of [6]. The second equation in (C2), that $\int_{\Omega} \langle f; \Psi \rangle = 0$ for any harmonic form Ψ (i.e. $\delta \Psi = 0$) with vanishing normal components (i.e. $N \cdot \Psi = 0$) on the boundary ($\Psi \in \mathcal{H}_N$ in the notation of [6]), follows by application of the integration by parts formula (C.2) for differential forms to f = dv,

$$\langle f, \Psi \rangle_{L^2} = -\langle v, \delta \Psi \rangle_{L^2} + \langle v, N \cdot \Psi \rangle_{L^2} = 0.$$

Theorem 7.4 in [6] now yields the existence of a solution $u \in W^{m+1,p}(\Omega)$ to (C.4) - (C.5) satisfying estimate (C.6).

Part (ii) of Theorem C.2, can be thought of as a version of Theorem [6, Thm 7.4], in the special case of 0-forms, which does not require condition (C2) by abandoning boundary data. That is, we seek a 0-form u solving the gradient equation du = f such that estimate (C.6) holds. (No boundary data is required for our purposes). To begin the proof, observe that a solution $u \in W^{m+1,p}(\Omega)$ of du = f, in the case $m \ge 1$, is given by the path integral

$$u(x) = \int_{x_0}^x f \cdot d\vec{r} + u_0$$
 (C.7)

along any differentiable curve connecting x_0 and x, where $x_0 \in \Omega$ is some point we fix, and the constant u_0 is the value of u at x_0 , which is free to be

³⁴Since $d^2 = 0$, the assumption f = dv implies df = 0, and is a slightly stronger assumption than df = 0, convenient for our purposes.

chosen. Note, since df = 0, the integral (C.7) is path independent, as can be shown by applying Stokes Theorem to integration of df over the region enclosed by two curves connecting x_0 and x. We now choose u_0 such that the average of u is zero, $\int_{\Omega} u dx = 0$. Then Poincaré's inequality [12, Eqn. (7.45)] implies that $||u||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)}$ for a suitable constant C > 0. Thus, since $||du||_{L^p(\Omega)} = ||f||_{L^p(\Omega)}$ follows directly from du = f, we have

$$||u||_{W^{1,p}(\Omega)} \le C ||f||_{L^p(\Omega)}.$$
 (C.8)

Estimate (C.6) follows by suitable differentiation of du = f and application of estimate (C.8). Existence of a solution u to du = f in the case m = 0follows again from (C.7) by mollifying f, and using that this mollification is controlled by estimate (C.8). This completes the proof of Theorem C.2. \Box

We finally require the so-called Hodge-Morrey decomposition, taken from Theorem 6.12 in [6]:

Theorem C.3. (Hodge-Morrey decomposition): (i) Let $\Phi \in L^p(\Omega)$ be a 1-form for $1 . Then there exists 1-forms <math>w_1, w_2 \in W^{2,p}(\Omega)$ such that

$$\Phi = d\alpha + \delta\beta + h, \tag{C.9}$$

where $\alpha = \delta w_1$ and $\beta = dw_2$ such that $N \wedge \alpha |_{\partial\Omega} = 0$ and $N \cdot \beta |_{\partial\Omega} = 0$, where N is interpreted as either a 1-form or a vector normal to $\partial\Omega$, and where h is a harmonic 1-form in the sense that $dh = 0 = \delta h$. Moreover, there exists a constant C > 0 depending only on Ω , n, p such that

$$\|w_1\|_{W^{2,p}(\Omega)} + \|w_2\|_{W^{2,p}(\Omega)} + \|h\|_{L^p(\Omega)} \le C \|\Phi\|_{L^p(\Omega)}.$$
 (C.10)

(ii) Let $\Phi \in L^p(\Omega)$ be a 0-form, $1 , then there exist 0-forms <math>w \in W^{2,p}(\Omega)$ and a constant h_0 such that

$$\Phi = \delta\beta + h_0, \tag{C.11}$$

where $\beta = dw$ and $N \cdot \beta |_{\partial \Omega} = 0$, and exists a constant C > 0 depending only on Ω, n, p such that

$$\|w\|_{W^{2,p}} \le C \|\Phi\|_{L^p}. \tag{C.12}$$

Proof. Part (i) of Theorem C.3 is the case of Theorem 6.12 (iii) in [6] for 1-forms Φ . Part (ii) follows from (iii) of [6, Thm 6.12] for 0-forms Φ , by observing that any harmonic 0-form h is constant, (since dh = 0 is the vanishing gradient condition for h), so $h = h_0.^{35}$

³⁵One can understand Theorem C.3 (ii) quite easily from the point of view of the Poisson equation. Namely the sought after function w is the solution to the Poisson equation $\Delta w = \Phi - h_0$ with Neumann data $N \cdot dw = 0$ on $\partial\Omega$, where h_0 is a constant chosen such that $\Phi - h_0$ satisfies the consistency condition $\int_{\Omega} (\Phi - h_0) dx = 0$ existence of w, (required by the divergence theorem applies to the equation). The solution w is unique up to addition by a constant, and we choose this constant for w to have zero average $\int_{\Omega} w dx = 0$. Now, the Poincaré inequality implies the L^p -norm of w to be bounded by the L^p norm of dw, and from this estimate (C.12) follows from standard elliptic estimates. (Compare also with Theorem 9.2 in [6] and its proof.)

We are now prepared to establish the existence theorems for 1-forms and 0-forms required in our iteration scheme in Section 11.2, Proposition C.4 and C.5 below. We begin with the case of 1-forms. That is, given $f \in W^{-1,p}(\Omega)$ for $1 , we prove existence of a 1-form <math>a \in L^p(\Omega)$ which is a weak solution

$$\begin{cases} da = f\\ \delta a = 0, \end{cases}$$
(C.13)

such that

$$\|a\|_{L^p} \le C \|f\|_{W^{-1,p}} \tag{C.14}$$

for some constant C > 0 depending only on n, p, Ω . No boundary data is imposed. Here *a* is a scalar valued 1-form and *f* a linear functional over the space of 2-forms with components in $W_0^{1,p^*}(\Omega)$, $f : W_0^{1,p^*}(\Omega) \longrightarrow \mathbb{R}$, where $W_0^{1,p^*}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ with respect to the W^{1,p^*} -norm, and where $\frac{1}{p} + \frac{1}{p^*} = 1$. We refer to such a linear functional again as a 2-form in $W^{-1,p}(\Omega)$. Equations (C.13) are interpreted in the following weak sense,

$$\begin{cases} \langle a, \delta \phi \rangle_{L^2} = -f(\phi) \\ \langle a, d\psi \rangle_{L^2} = 0, \end{cases}$$
(C.15)

for all 2-forms ϕ with components in $W_0^{1,p^*}(\Omega)$ and all 0-forms $\psi \in W_0^{1,p^*}(\Omega)$, (so $\phi|_{\partial\Omega} = 0$ and $\psi|_{\partial\Omega} = 0$), where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the standard L^2 -inner product on differential forms.

Proposition C.4. Let $f \in W^{-1,p}(\Omega)$ be a 2-form satisfying df = 0 in the weak sense that $f(\delta \psi) = 0$ for all 3-forms ψ with components in $W_0^{2,p^*}(\Omega)$; assume further that f = dv for some 1-form $v \in W^{-1,p}(\Omega)$ in the sense that $f[\phi] = -v(\delta \phi)$ for any 2-forms $\phi \in W_0^{1,p^*}(\Omega)$.³⁶ Then there exists a solution $a \in L^p(\Omega)$ of (C.15) satisfying (C.14).

Proof. The proof consists of the following three steps: (1) Construct approximate solutions a^{ϵ} . (2) Derive an ϵ -independent bound on the approximate solutions which implies existence of a convergent subsequence. (3) Prove that the limit of this convergent subsequence is a solution of (C.14) which satisfies estimate (C.15).

To implement step (1), we mollify the functional f, that is, we introduce $f^{\epsilon}(x) \equiv f(\varphi^{\epsilon}(\cdot - x))$, where $\varphi^{\epsilon} \equiv \varphi^{\epsilon}_{ij} dx^i \wedge dx^j$ is a 2-form with components $\varphi^{\epsilon}_{ij} \in C_0^{\infty}(\Omega)$ that are a standard mollifier function. So $f^{\epsilon} \in C^{\infty}(\Omega)$, and f^{ϵ} converges to f in $W^{-1,p}$ component-wise. For each $\epsilon > 0$, we now introduce a^{ϵ} as the solution of

$$\begin{cases} da^{\epsilon} = f^{\epsilon} \\ \delta a^{\epsilon} = 0, \end{cases}$$
(C.16)

³⁶As in Theorem 8.8, assuming f = dv is a slightly stronger assumption than df = 0, convenient in our proof of well-posedness of the iteration scheme, c.f. Lemma 10.7.

with boundary data $N^j a_j^{\epsilon} = 0$ on $\partial \Omega$, where N is the outward pointing unit normal of $\partial \Omega$. The solution a^{ϵ} does indeed exist by Theorem C.2 (i), since $f^{\epsilon} = dv^{\epsilon}$ for the 1-form $v^{\epsilon} \equiv v[\varphi^{\epsilon}] \in C^{\infty}(\Omega)$. Namely, our assumption $f[\phi] = -v(\delta\phi)$ for any 2-forms $\phi \in W_0^{1,p^*}(\Omega)$ implies that

$$f^{\epsilon} = f(\varphi^{\epsilon}) = -v[\delta\varphi^{\epsilon}] = dv[\varphi^{\epsilon}] = dv^{\epsilon},$$

by definition of the distributional derivative dv. Clearly, $df^{\epsilon} = d^2 v^{\epsilon} = 0$. Thus Theorem C.2 (i) applies and yields a solution $a^{\epsilon} \in W^{1,p}(\Omega)$ for each $\epsilon > 0$, establishing step (1).

To establish step (2), we now derive a uniform bound on $||a^{\epsilon}||_{L^{p}}$ in order to conclude convergence of a subsequence to the sought after solution a. The uniform bound we derive can be thought of as a version of Gaffney's inequality at the lower level of L^{p} regularity, when boundary data cannot be imposed strongly. To begin, since the operator norm is equivalent to the L^{p} norm, we find that

$$\|a^{\epsilon}\|_{L^{p}} = \sup_{\Phi \in \mathcal{F}} |\langle a^{\epsilon}, \Phi \rangle_{L^{2}}|, \qquad (C.17)$$

where

$$\mathcal{F} \equiv \left\{ \Phi \in L^{p^*}(\Omega) \text{ a 1-form with } \|\Phi\|_{L^{p^*}} = 1 \right\}$$

is the space of test functions. Now, fix $\Phi \in \mathcal{F}$ and apply the Hodge-Morrey decomposition of Theorem C.3 to write

$$\Phi = d\alpha + \delta\beta + h, \tag{C.18}$$

where $\alpha = \delta w_1$ and $\beta = dw_2$ for 1-forms $w_1, w_2 \in W^{2,p^*}(\Omega)$, such that $N \wedge \alpha |_{\partial\Omega} = 0$ and $N \cdot \beta |_{\partial\Omega} = 0$, and where *h* is a harmonic 1-form. Next, applying the existence theory of Theorem C.2 (*ii*), we define the 0-form $\Psi \in W_0^{1,p^*}$ as a solution of

$$d\Psi = h \tag{C.19}$$

which exists, since dh = 0 for h harmonic; no boundary data imposed. We now substitute the decomposition (C.18) for Φ to write $\langle a^{\epsilon}, \Phi \rangle_{L^2}$ in (C.17) equivalently as

$$\begin{aligned} \langle a^{\epsilon}, \Phi \rangle_{L^2} &= \langle a^{\epsilon}, (d\alpha + \delta\beta + h) \rangle_{L^2} \\ &= \langle a^{\epsilon}, d\alpha \rangle_{L^2} + \langle a^{\epsilon}, \delta\beta \rangle_{L^2} + \langle a^{\epsilon}, h \rangle_{L^2}. \end{aligned}$$
 (C.20)

Applying now the partial integration formula (C.2) to each term, we obtain

$$\langle a^{\epsilon}, d\alpha \rangle_{L^2} = -\langle \delta a^{\epsilon}, \alpha \rangle_{L^2} + \langle a^{\epsilon}, N \wedge \alpha \rangle_{L^2(\partial\Omega)} = -\langle \delta a^{\epsilon}, \alpha \rangle_{L^2}, \quad (C.21)$$

where the last equality follows from $N \wedge \alpha |_{\partial\Omega} = 0$, c.f. Theorem C.3. Similarly, partial integration together with $N \cdot \beta |_{\partial\Omega} = 0$ gives

$$\langle a^{\epsilon}, \delta\beta \rangle_{L^{2}} = -\langle da^{\epsilon}, \beta \rangle_{L^{2}} + \langle a^{\epsilon}, N \cdot \beta \rangle_{L^{2}(\partial\Omega)} = -\langle da^{\epsilon}, \beta \rangle_{L^{2}}, \qquad (C.22)$$

and by (C.19),

$$\begin{aligned} \langle a^{\epsilon}, h \rangle_{L^2} &= \langle a^{\epsilon}, d\Psi \rangle_{L^2} \\ &= -\langle \delta a^{\epsilon}, \Psi \rangle_{L^2} + \langle N \cdot a^{\epsilon}, \Psi \rangle_{L^2(\partial\Omega)} \end{aligned}$$

$$= -\langle \delta a^{\epsilon}, \Psi \rangle_{L^2}, \tag{C.23}$$

since $N \cdot a^{\epsilon} = 0$ on $\partial \Omega$ by assumption. Now, substituting (C.21) - (C.23) into (C.20), and using that a^{ϵ} solves (C.16), we obtain

$$\langle a^{\epsilon}, \Phi \rangle_{L^{2}} = -\langle \delta a^{\epsilon}, \alpha \rangle_{L^{2}} - \langle da^{\epsilon}, \beta \rangle_{L^{2}} - \langle \delta a^{\epsilon}, \Psi \rangle_{L^{2}} = -\langle f^{\epsilon}, \beta \rangle_{L^{2}}.$$
 (C.24)

Now (C.24) and the definition of the operator norm $\|\cdot\|_{W^{-1,p}}$ imply

$$|\langle a^{\epsilon}, \Phi \rangle_{L^2}| \leq ||f^{\epsilon}||_{W^{-1,p}} ||\beta||_{W^{1,p^*}}.$$
 (C.25)

By Theorem C.3, we have $\beta = dw_2$ so estimate (C.10) gives us

$$\|\beta\|_{W^{1,p^*}} = \|dw_2\|_{W^{1,p^*}} \le \|w_2\|_{W^{2,p^*}} \stackrel{(C.10)}{\le} C\|\Phi\|_{L^{p^*}}, \qquad (C.26)$$

where C > 0 is some constant only depending on p, n, Ω . Substituting (C.26) into (C.25) and using that $\|\Phi\|_{L^{p^*}} = 1$ for any $\Phi \in \mathcal{F}$, we obtain the estimate

$$|\langle a^{\epsilon}, \Phi \rangle_{L^2}| \leq C ||f^{\epsilon}||_{W^{-1,p}} \leq 2C ||f||_{W^{-1,p}}, \qquad (C.27)$$

for all $\epsilon > 0$ sufficiently small, since f^{ϵ} converges to f in $W^{-1,p}$ by standard mollification. Finally, substituting (C.27) into (C.17), we obtain the sought after uniform bound

$$||a^{\epsilon}||_{L^{p}} \le C||f||_{W^{-1,p}}, \tag{C.28}$$

where C > 0 is some constant only depending on p, n, Ω .

We now complete step (3). By (C.28), $||a^{\epsilon}||_{L^p}$ is bounded independent of ϵ , so the Banach Alaoglu Theorem implies convergence of a subsequence to some differential form $a \in L^p$ weakly in L^p . We now show that this limit a solves (C.13). For this, let $\epsilon_k > 0$ such that $\epsilon_k \to 0$ as $k \to \infty$ and assume $a_k = a^{\epsilon_k}$ is the convergent subsequence, so $a_k \to a$ weakly in L^p as $k \to \infty$. By (C.2), we have for any ϕ and $\psi \in W_0^{1,p^*}(\Omega)$ that

$$\langle da^{\epsilon}, \phi \rangle_{L^{2}} = -\langle a^{\epsilon}, \delta \phi \rangle_{L^{2}}, \langle \delta a^{\epsilon}, \psi \rangle_{L^{2}} = -\langle a^{\epsilon}, d\psi \rangle_{L^{2}}.$$
 (C.29)

So using that a_k solves (C.16), we write (C.29) as

$$\begin{cases} \langle a_k, \delta \phi \rangle_{L^2} = -\langle f^{\epsilon_k}, \phi \rangle_{L^2} \\ \langle a_k, d\psi \rangle_{L^2} = 0, \end{cases}$$
(C.30)

which converges to the sought after equation (C.15). We conclude that a is the sought after weak solution of (C.15). Moreover, the sought after estimate (C.14) follows from the uniform bound (C.28), since

$$|a||_{L^{p}} = \sup_{\psi \in L^{p^{*}}} |\langle a, \psi \rangle| = \lim_{k \to \infty} \sup_{\psi \in L^{p^{*}}} |\langle a_{k}, \psi \rangle|$$
$$= \lim_{k \to \infty} ||a_{k}||_{L^{p}} \stackrel{(C.28)}{\leq} C ||f||_{W^{-1,p}}.$$
(C.31)

This completes the proof of Proposition C.4.

89

Our final existence result for 0-forms u is required to extend Theorem C.2 (*ii*) to solutions in the space $L^p(\Omega)$. It is an extension of Poincaré's Lemma to linear functionals. We seek weak solutions $u \in L^p(\Omega)$ of the first order equation

$$du = f, (C.32)$$

satisfying

$$||u||_{L^p} \le C ||f||_{W^{-1,p}},\tag{C.33}$$

for any 1-form f with components in $W^{-1,p}(\Omega)$, where C > 0 is a constant depending only on n, p, Ω . That is, we prove existence of weak solutions $u \in L^p(\Omega)$ of (C.32) in the sense that

$$\langle u, \delta \phi \rangle = -f(\phi),$$
 (C.34)

for any 1-form ϕ with components in $W_0^{1,p^*}(\Omega)$ subject to estimate (C.33).

Proposition C.5. Assume the 1-form $f \in W^{-1,p}(\Omega)$ satisfies df = 0 in the sense that $f(\delta \phi) = 0$ for any 2-form ϕ with components in $W_0^{2,p^*}(\Omega)$. Then there exists a solution $u \in L^p(\Omega)$ of (C.34) satisfying (C.33).

Proof. The proof is similar to that of Proposition C.4, consisting of the same three steps. To begin with the first step, we mollify the functional f, setting again $f^{\epsilon}(x) \equiv f(\varphi^{\epsilon}(\cdot - x))$, where $\varphi^{\epsilon} \in C_0^{\infty}(\Omega)$ is a 1-form whose components are standard mollifier functions. So $f^{\epsilon} \in C^{\infty}(\Omega)$, and f^{ϵ} converges to f in $W^{-1,p}$ component wise. For each $\epsilon > 0$, we have $df^{\epsilon} = -f(\delta\varphi^{\epsilon}) = 0$ by assumption. Thus Theorem C.2 (*ii*) applies, and yields the existence of a 0-form $u^{\epsilon} \in W^{1,p}(\Omega)$ solving

$$du^{\epsilon} = f^{\epsilon} \tag{C.35}$$

such that u^{ϵ} has zero average, $\int_{\Omega} u^{\epsilon} dx = 0$.

In the next step we derive a uniform bound on $||u^{\epsilon}||_{L^{p}}$. That is, we express the L^{p} -norm in terms of the operator norm,

$$\|u^{\epsilon}\|_{L^{p}} = \sup_{\Phi \in \mathcal{F}} |\langle u^{\epsilon}, \Phi \rangle_{L^{2}}|, \qquad (C.36)$$

where

$$\mathcal{F} \equiv \left\{ \Phi \in L^{p^*}(\Omega) \text{ a function with } \|\Phi\|_{L^{p^*}} = 1 \right\}$$

is the space of test functions and $\langle \cdot, \cdot \rangle_{L^2}$ denotes the standard L^2 inner product. We fix some $\Phi \in \mathcal{F}$ and apply the Hodge-Morrey decomposition in Theorem C.3 (ii) to write

$$\Phi = \delta\beta + h_0, \tag{C.37}$$

where $\beta = dw$ for some 0-form $w \in W^{2,p^*}(\Omega)$ and $N \cdot \beta|_{\partial\Omega} = 0$. From (C.37) we find that

$$\langle u^{\epsilon}, \Phi \rangle_{L^2} = \langle u^{\epsilon}, \delta \beta \rangle_{L^2} + \langle u^{\epsilon}, h_0 \rangle_{L^2} = \langle u^{\epsilon}, \delta \beta \rangle_{L^2},$$

since $\langle u^{\epsilon}, h_0 \rangle_{L^2} = h_0 \int_{\omega} u^{\epsilon} dx = 0$ by our zero average assumption on u^{ϵ} . Integration by parts (C.2) gives further

$$\langle u^{\epsilon}, \Phi \rangle_{L^2} = -\langle du^{\epsilon}, \beta \rangle_{L^2} + \langle u^{\epsilon}, N \cdot \beta \rangle_{L^2(\partial\Omega)} = \langle f^{\epsilon}, \beta \rangle_{L^2}, \quad (C.38)$$

90

where the last equality follows by substituting (C.32) for u^{ϵ} and $N \cdot \beta|_{\partial\Omega} = 0$. From (C.38) and the definition of the operator norm $\|\cdot\|_{W^{-1,p}}$, we obtain

$$|\langle u^{\epsilon}, \Phi \rangle_{L^2}| \leq ||f^{\epsilon}||_{W^{-1,p}} ||\beta||_{W^{1,p^*}}.$$
 (C.39)

Using now that $\beta = dw$ in combination with estimate (C.12) of the Hodge-Morrey decomposition, we obtain

$$\|\beta\|_{W^{1,p^*}} \le \|w\|_{W^{2,p^*}} \le C \|\Phi\|_{L^{p^*}}, \tag{C.40}$$

for some constant C > 0 only depending on p, n, Ω . Substituting now (C.40) into (C.39) and using that $\|\Phi\|_{L^{p^*}} = 1$ for any $\Phi \in \mathcal{F}$, we obtain the uniform bound

$$|\langle u^{\epsilon}, \Phi \rangle_{L^2}| \le C \|f^{\epsilon}\|_{W^{-1,p}} \le 2C \|f\|_{W^{-1,p}}, \tag{C.41}$$

for all $\epsilon > 0$ sufficiently small, because f^{ϵ} converges to f in $W^{-1,p}$ by standard mollification. Finally, substituting (C.41) into (C.36), we obtain the sought after uniform bound

$$\|u^{\epsilon}\|_{L^{p}} \le C \|f\|_{W^{-1,p}},\tag{C.42}$$

where C > 0 is some constant only depending on p, n, Ω .

The uniform bound (C.42) implies the existence of a subsequence which converges to some function $u \in L^p(\Omega)$ subject to the L^p -bound (C.33), and an argument similar to that of step (3) in the proof of Proposition C.4 shows that u solves (C.35). This completes the proof of Proposition C.5.

Appendix D. A lemma used in the proof of Theorem 4.1

For completeness we prove the following technical lemma which was used in the proof of Lemma 9.2 above. The point was that $J^T \phi$ can be taken as test function in the same space as the test function ϕ used in the argument. It suffices to prove the lemma for products ϕJ instead of $J^T \phi$, since only regularity of components is at issue here.

Lemma D.1. Let $J \in W^{1,2p}(\Omega)$ for $p > n \ge 2$, and assume J is invertible with inverse $J^{-1} \in W^{1,2p}(\Omega)$. Then $\phi J \in W_0^{1,p^*}(\Omega)$ for any $\phi \in W_0^{1,p^*}(\Omega)$, and for every $\psi \in W_0^{1,p^*}(\Omega)$ there exists some $\phi \in W_0^{1,p^*}(\Omega)$ such that $\psi = \phi J$. That is, $J \cdot W_0^{1,p^*}(\Omega) = W_0^{1,p^*}(\Omega)$.

Proof. So let $J \in W^{1,2p}(\Omega)$ for p > n. We first show that $\phi J \in W^{1,p^*}_0(\Omega)$ for any $\phi \in W^{1,p^*}_0(\Omega)$. To begin, observe that by the Leibniz rule, we obtain

$$\begin{aligned} \|\phi J\|_{W^{1,p^*}} &\equiv \|\phi J\|_{L^{p^*}} + \|d(\phi J)\|_{L^{p^*}} \\ &\leq \|\phi J\|_{L^{p^*}} + \|d\phi \cdot J\|_{L^{p^*}} + \|\phi \, dJ\|_{L^{p^*}}. \end{aligned} \tag{D.1}$$

(Note that d is simply the gradient for the matrix valued 0-forms J and ϕ .) By Morrey's inequality the first term in (D.1) can be bounded as

$$\|\phi J\|_{L^{p^*}} \le C \|\phi\|_{L^{p^*}} \|J\|_{L^{\infty}} \le C \|\phi\|_{L^{p^*}} \|J\|_{W^{1,p}}$$

and similarly

$$\|d\phi \cdot J\|_{L^{p^*}} \le \|d\phi\|_{L^{p^*}} \|J\|_{W^{1,p}}.$$

So we only have to show that the third term in (D.1) is bounded to prove that $\phi J \in W_0^{1,p^*}(\Omega)$. For this, we apply Hölder's inequality as in (A.8) to estimate

$$\|\phi \, dJ\|_{L^{p^*}}^{p^*} \le \int_{\Omega} |\phi|^{p^*} |dJ|^{p^*} dx \le C \, \||\phi|^{p^*}\|_{L^q} \, \||dJ|^{p^*}\|_{L^{\frac{2p}{p^*}}}, \tag{D.2}$$

where q is the conjugate exponent of $\frac{2p}{p^*}$ resulting from Hölder's inequality, $\frac{1}{q} + \frac{p^*}{2p} = 1$, and C > 0 denotes a running constant depending only on Ω, n, p . Observe now that by (A.1) we have $\||\phi|^{p^*}\|_{L^q} \leq \|\phi\|_{L^{qp^*}}^{p^*}$ and $\||dJ|^{p^*}\|_{L^{\frac{2p}{p^*}}} \leq \|dJ\|_{L^{2p}}^{p^*}$, so substitution into (D.2) gives

$$\|\phi \, dJ\|_{L^{p^*}} \le C \, \|\phi\|_{L^{qp^*}} \, \|dJ\|_{L^{2p}}. \tag{D.3}$$

We conclude that to prove $\phi J \in W_0^{1,p^*}(\Omega)$, it suffices to show that $\phi \in W_0^{1,p^*}(\Omega)$ implies $\|\phi\|_{L^{qp^*}}$ to be finite. For this, we apply the Sobolev embedding Theorem for bounded domains, which states that $W^{1,p^*}(\Omega) \subset L^{\frac{np^*}{n-p^*}}(\Omega)$. Thus the boundedness of $\|\phi\|_{L^{qp^*}}$ follows from the Sobolev embedding Theorem, as long as

$$qp^* \le \frac{np^*}{n-p*},\tag{D.4}$$

c.f. [11]. To verify (D.4), we first compute that

$$q = \frac{2}{3 - p^*},$$
 (D.5)

as follows: Inserting into the defining identity $\frac{1}{q} + \frac{p^*}{2p} = 1$ that $p^* = \frac{p}{p-1}$ and solving for q, we find that $q = \frac{2(p-1)}{2(p-1)-1} = \frac{2}{2-\frac{1}{p-1}} = \frac{2}{3-p^*}$, where the last equality follows from the identity $p^* = \frac{p}{p-1} = 1 + \frac{1}{p-1}$. To continue, we substitute (D.5) into the left hand side of (D.4), and show that (D.4) holds if and only if 2p > n. For this, note first that $p^* = \frac{p}{p-1} \in (1,2)$ and recall that $n \ge 2$. So $n - p^* \ge 0$ and $3 - p^* \ge 0$, which allows us to write (D.4) equivalently as $\frac{2p^*}{p^*-1} \ge n$, and using that $p^* = \frac{p}{p-1}$ we find that this is equivalent to $2p \ge n$. Since we assumed that p > n, we verified (D.4) and thereby proved that the Sobolev embedding Theorem implies that $\phi J \in W_0^{1,p^*}(\Omega)$ for any $\phi \in W_0^{1,p^*}(\Omega)$. This proves the forward implication of Lemma D.1.

To prove the backward implication, that for every $\psi \in W_0^{1,p^*}(\Omega)$ there exists a $\phi \in W_0^{1,p^*}(\Omega)$ such that $\psi = \phi J$, we make the ansatz $\phi \equiv \psi J^{-1}$. Since $J^{-1} \in W^{1,2p}(\Omega)$ by assumption, we use the forward implication of Lemma D.1 to conclude that $\phi = \psi J^{-1} \in W_0^{1,p^*}(\Omega)$, while $\phi J = \psi$ holds

92

trivially. This proves the backward implication and completes the proof of Lemma D.1. $\hfill \Box$

Acknowledgements

We thank Craig Evans for directing us to references [6] and [25] which contain the basic elliptic estimates which are the starting point for our analysis of convergence of the iteration scheme in the L^{∞} case addressed here.

References

- M.T. Anderson, "Regularity for Lorentz metrics under curvature bounds", Jour. Math. Phys. 44 (2003), 2994-3012.
- [2] A. Acharya, G.-Q. G. Chen, S. Li, M. Slemrod and D. Wang, "Fluids, elasticity, geometry and the existence of wrinkled solutions", Arch. Rat. Mech. Anal., 226 (2017), 1009-1060.
- [3] A.P. Barnes, P.G. LeFloch, B.G. Schmidt and J.M. Stewart, "The Glimm scheme for perfect fluids on plane-symmetric Gowdy spacetimes", Class. Quantum Grav., 21 (2004), 5043-5074.
- [4] B.-L. Chen and P.G. LeFloch, "Local foliations and optimal regularity of Einstein spacetimes", J. Geom. Phys. 59 (2009), 913-941.
- [5] Y. Choquet-Bruhat, General Relativity and the Einstein Equations, Oxford University Press, 2009.
- [6] G. Csató, B. Dacorogna and O. Kneuss, *The Pullback Equation for Differential Forms*, Birkhäuser, Progress in Nonlinear Differential Equations and Their Applications, Vol. 83, ISBN: 978-0-8176-8312-2.
- [7] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer Verlag (1999), second edition.
- [8] D. M. DeTurck and J. L. Kazdan, "Some Regularity Theorems in Riemannian Geometry", Ann. scient. Éc. Norm. Sup., 4^e série, t. 14, 1981, p. 249 á 260.
- S. K. Donaldson, "An application of gauge theory to four-dimensional topology", J. Differ. Geom. 18 (1983), 279-315. (doi:10.4310/jdg/1214437665)
- [10] A. Einstein, "Die Feldgleichungen der Gravitation", Preuss. Akad. Wiss., Berlin, Sitzber. 1915b, 844-847.
- [11] L. C. Evans, *Partial Differential Equations*, Berkeley Mathematics Lecture Notes, 3A, 1994.
- [12] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 3rd edition, (1997), Springer Verlag, ISBN 3-540-41160-7.
- [13] J. Groah and B. Temple, Shock-Wave Solutions of the Einstein Equations with Perfect Fluid Sources: Existence and Consistency by a Locally Inertial Glimm Scheme, Memoirs AMS, Vol. 172, Number 813, (2004), ISSN 0065-9266.
- [14] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Spacetime, Cambridge University Press, 1973.
- [15] W. Israel, "Singular hypersurfaces and thin shells in general relativity", Il Nuovo Cimento, Vol. XLIV B, N. 1, (1966), 1-14.
- [16] S. Klainermann, I. Rodnianski and J. Szeftel, "The Bounded L2 Curvature Conjecture", Invent. Math. 202 (1), (2015), 91-216.
- [17] M. Reintjes, "Spacetime is Locally Inertial at Points of General Relativistic Shock Wave Interaction between Shocks from Different Characteristic Families", Adv. Theor. Math. Phys. 21.6 (2017), 1525-1611. arXiv:1409.5060

- [18] M. Reintjes and B. Temple, "No Regularity Singularities Exist at Points of General Relativistic Shock Wave Interaction between Shocks from Different Characteristic Families", Proc. R. Soc. A 471:20140834. arXiv:1506.0408
- [19] M. Reintjes and B. Temple, Regularity Singularities and the Scattering of Gravity Waves in Approximate Locally Inertial Frames, Meth. Appl. Anal. 23.3 (2016), 233-258. arXiv:1506.04074.
- [20] M. Reintjes and B. Temple, "Shock Wave Interactions and the Riemann-flat Condition: The Geometry behind Metric Smoothing and the Existence of Locally Inertial Frames in General Relativity", Arch. Rat. Mech. Anal. 235 (2020), 1873-1904. arXiv:1610.02390
- [21] M. Reintjes and B. Temple, "The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity", (2018), Adv. Theor. Math. Phys. (to appear). arXiv:1805.01004
- [22] M. Reintjes and B. Temple, "Optimal metric regularity in General Relativity follows from the RT-equations by elliptic regularity theory in L^p-spaces", (2018), Meth. Appl. Anal. (to appear). arXiv:1808.06455
- [23] M. Reintjes and B. Temple, "How to smooth a crinkled map of spacetime: Uhlenbeck compactness for L^{∞} connections and optimal regularity for general relativistic shock waves by the Reintjes-Temple-equations", Proc. R. Soc. A 476: 20200177. arXiv:1812.06795.
- [24] W. Rudin, Functional Analysis, McGraw-Hill, 1973, ISBN-13: 978-0070542259.
- [25] C. Simander, On Dirichlet's Boundary Value Problem, Springer Lecture Notes 268, (1972), ISBN 3-540-05903-2.
- [26] J. Smoller, Shock Waves and Reaction Diffusion Equations, Springer-Verlag, 1983.
- [27] J. Smoller and B. Temple, "Shock wave solutions of the Einstein equations: The Oppenheimer-Snyder model of gravitational collapse extended to the case of non-zero pressure", Arch. Rat. Mech. Anal. 128 (1994), 249-297.
- [28] C. H. Taubes, "Karen Uhlenbeck's contributions to gauge theoretic analysis", Celebratio Mathematica (2019).
- [29] K. Uhlenbeck, "Connections with L^p Bounds on Curvature", Commun. Math. Phys. 83, 31-42, (1982).
- [30] K. Uhlenbeck, "Removable Singularities in Yang-Mills Fields", Commun. Math. Phys. 83, 11-29, (1982).
- [31] S. Weinberg, Gravitation and Cosmology, John Wiley & Sons, New York, 1972.
- [32] K. Wehrheim, Uhlenbeck Compactness, EMS Lectures in Mathematics, (2004), ISBN 3-03719-004-3.

FACHBEREICH FÜR MATHEMATIK UND STATISTIK, UNIVERSITÄT KONSTANZ, D-78467, GERMANY

Email address: moritzreintjes@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA 95616, USA

Email address: temple@math.ucdavis.edu