Outline

I. Introduction to GR & Differential Geometry

II The Einstein Equations

III Matching gravitational metrics across shock-wave interfaces

IV FRW-TOV shock matching

V Some Exact Soln's

I. Introduction to General Relativity & Diff Geom

- GR is the modern theory of the gravitational field.
- In 1915, Albert Einstein introduced Einstein Gravitational Field Equations

$$\underbrace{G}_{\text{Einstein Curvature Tensor}} = K \underbrace{T}_{\text{Stress Energy Tensor}} \qquad \text{(EQ)}$$

- Energy & Momentum & their fluxes create spacetime curvature according to (EQ)

Goal 1: "Understand" (EQ)

Basic Principle of GR

"All properties of the gravitational field are determined by a signature $(-1,1,1,1)$ metric g defined on a 4-dimensional spacetime manifold"

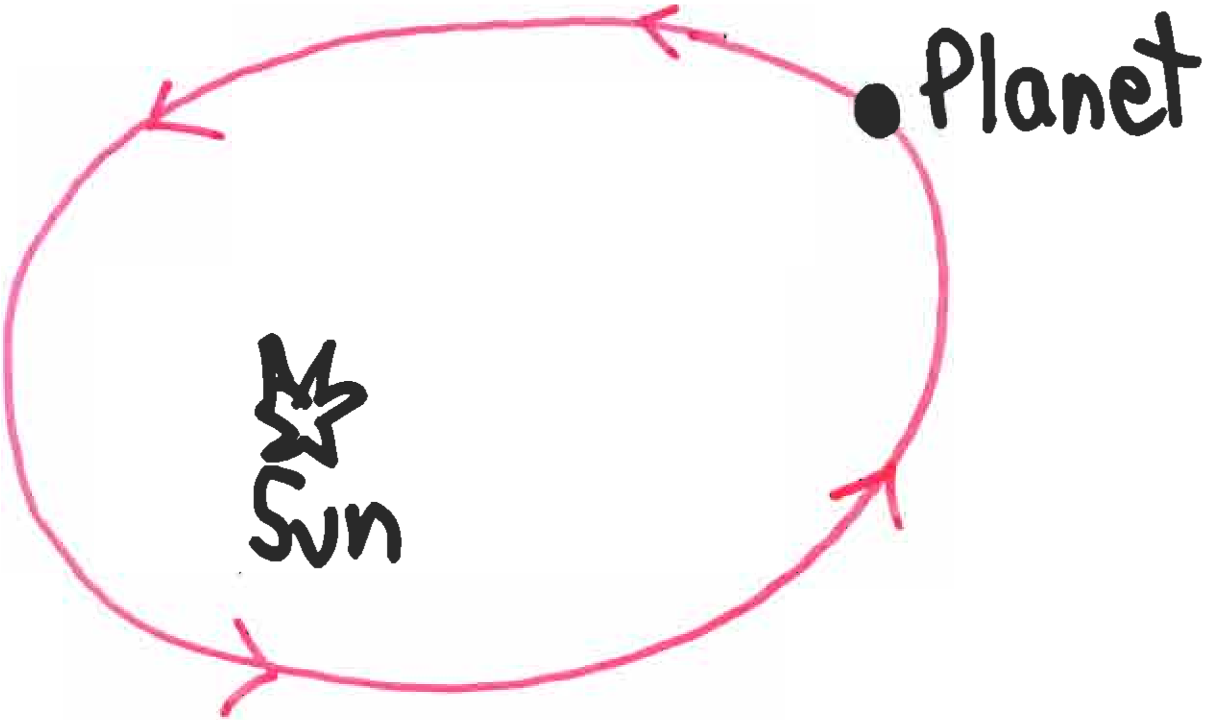
$$\mathcal{M} \equiv \text{Spacetime}$$

(Q1) What can you "measure" from g?

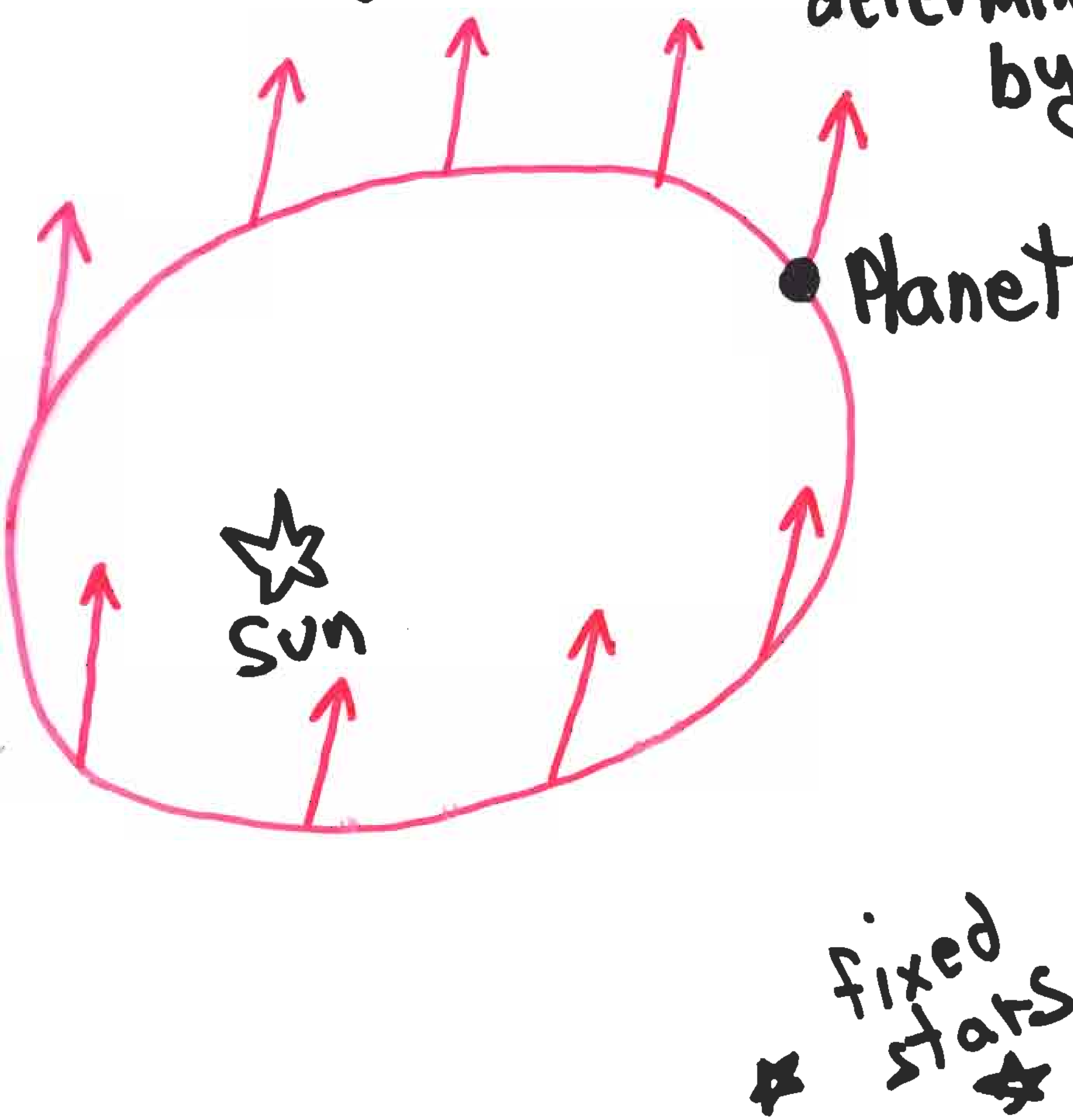
(Q2) What are the "constraints" on g that determine its time evolution?

■ What you can measure from g

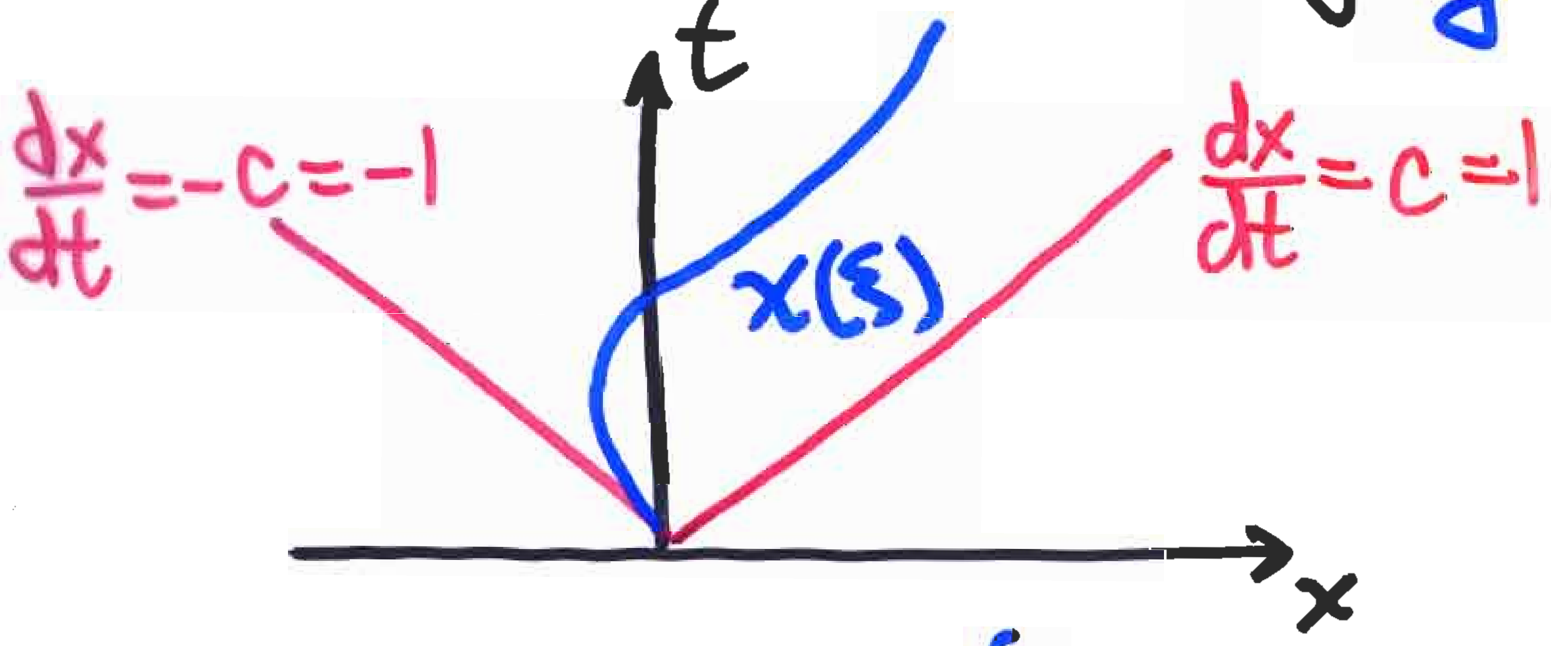
(1) Freefall paths through a gravitational field are geodesics of the spacetime metric g



(2) Non-rotating vectors carried by observers in freefall are parallel-transported by the unique symmetric connection determined by g

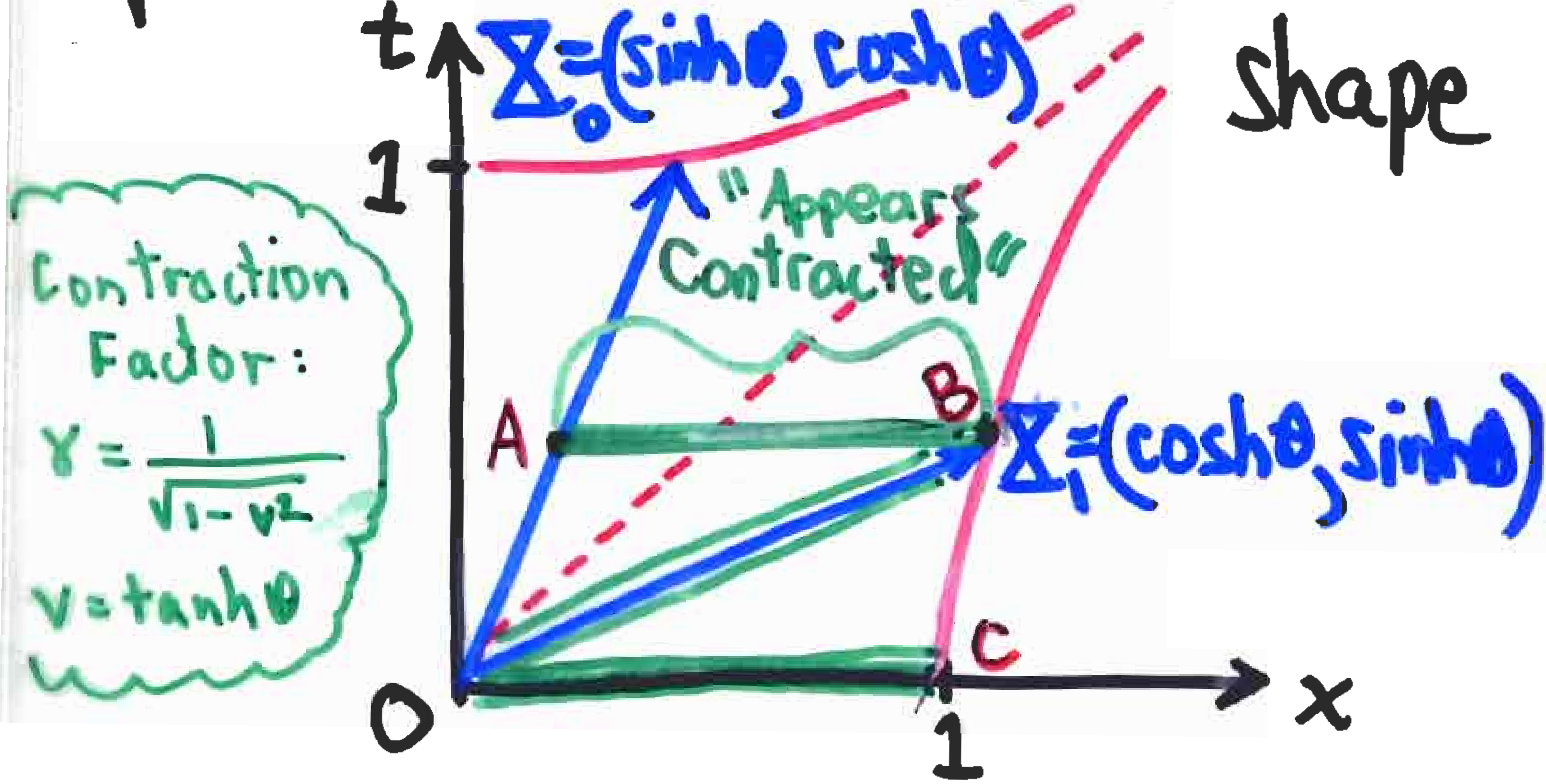


(3) Proper time change, or "**aging time**" as measured by an observer traversing a timelike curve through spacetime will be equal to the **arclength** Δs of the curve as measured by g



$$ds^2 = -dt^2 + dx^2 \Rightarrow \Delta s = \int_{\xi_1}^{\xi_2} \sqrt{|-\dot{t}^2 + \dot{x}^2|}\, d\xi$$

④ Spatial lengths of objects correspond to g-lengths of space like curves that define their shape



$$ds^2 = -dt^2 + dx^2$$

$\overline{OC}$ = unit rod fixed in xt-coordinates

$$\Delta S = -0^2 + 1^2 = 1$$

$\overline{OB}$ = unit rod fixed in $\bar{x}\bar{t}$-coord's

$$\Delta S = -\sinh^2\theta + \cosh^2\theta = 1$$

"invariant length" = coordinate length as measured in inertial frame in which rod fixed

- **Arclength** is computed by integrating the element of arclength along the curve:

$$\boxed{ds^2 = g_{ij}\, dx^i dx^j}$$

Here: $g_{ij} \equiv g_{ij}(x)$

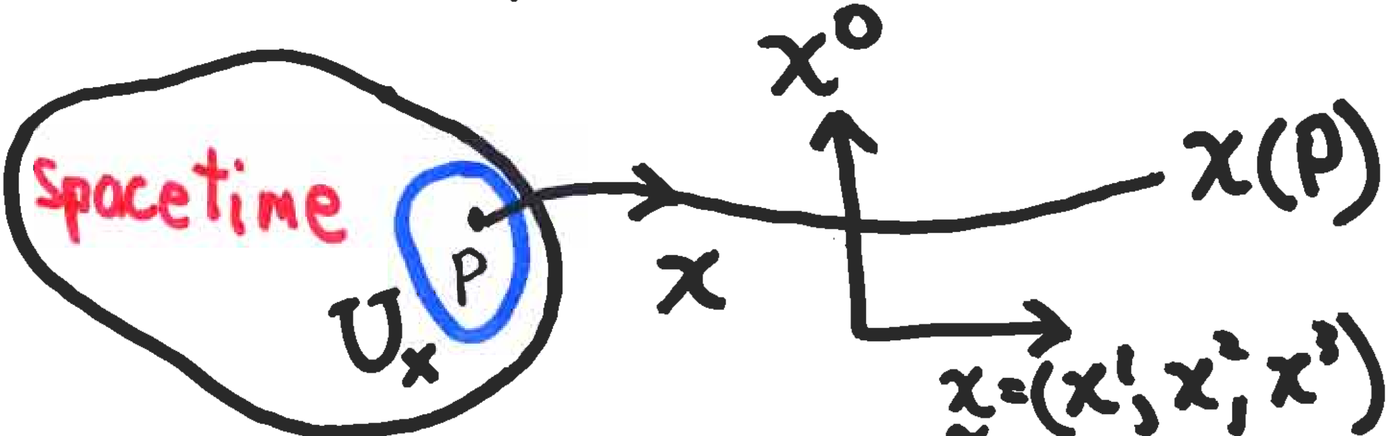
denote the components of g in coordinate system x

- **Einstein Summation Convention**
- Sum repeated up-down indices from 0 to 3

Notation:

- Coordinate system $x = (x^0, x^1, x^2, x^3)$

$$x : U_x \longrightarrow \mathbb{R}^4$$



- x denotes <u>both</u> coord map & pt in $\mathbb{R}^4$

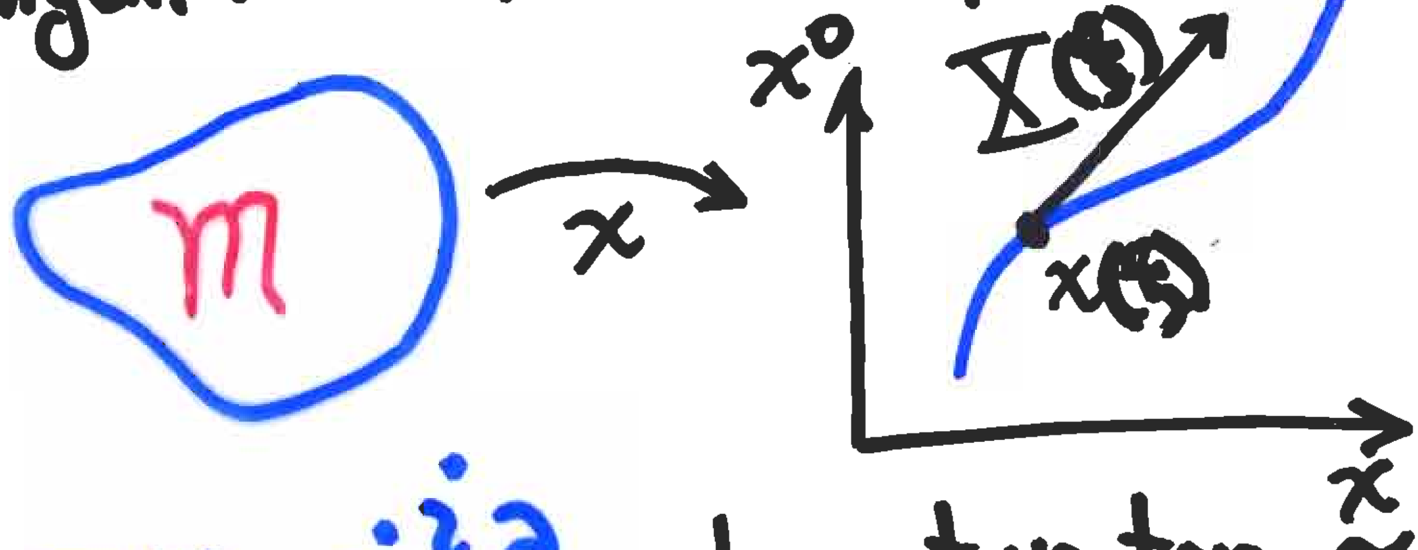
- $\left\{ \frac{\partial}{\partial x^i} \right\}_p$ x-basis for $T_p(M)$

- $\{dx^i\}_p$ x-basis for $T_p^* M$

☑ Metric

$$ds^2 = g_{ij}\, dx^i dx^j$$

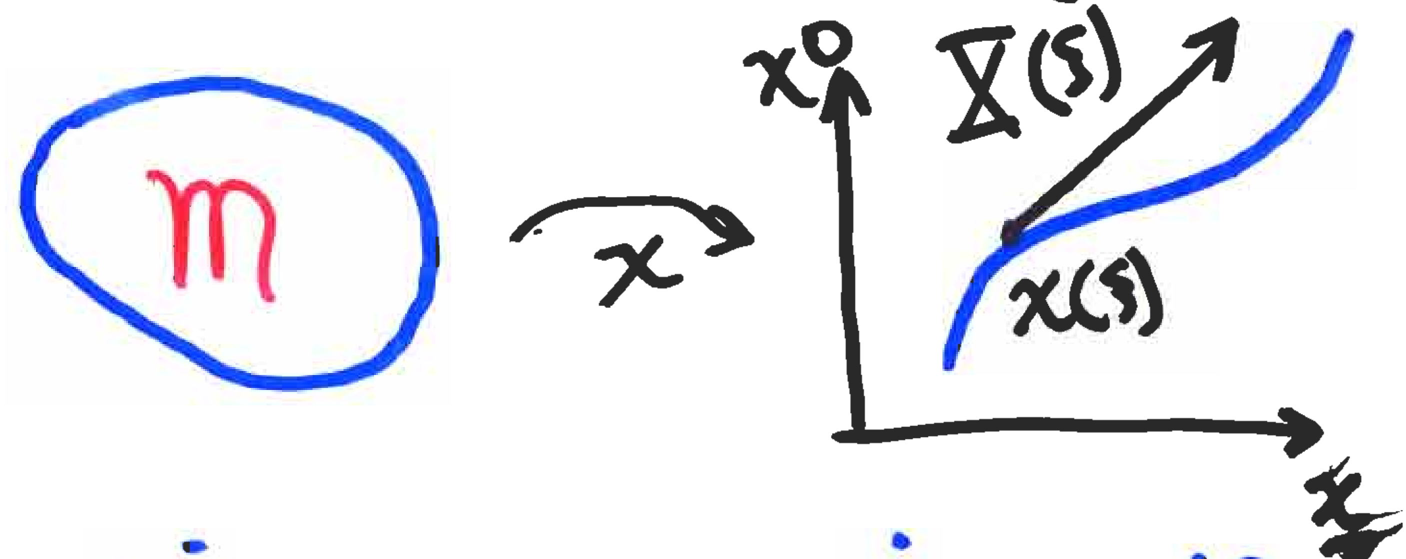
- $g_{ij}(x)$ = x-components of g
- $g_{ij}(x)$ computes the lengths of tangent vectors from x-components



$$\mathbf{X}(s) = \dot{x}^i \frac{\partial}{\partial x^i}$$ tangent vector

$$\|\mathbf{X}\|^2 = g_{ij}\, \dot{x}^i \dot{x}^j$$

View this another way:



dx^i = increment in x^i for $d\xi$

$$\Rightarrow \quad dx^i = \dot{x}^i \, d\xi$$

$$\Rightarrow \quad ds^2 = g_{ij}\, dx^i dx^j = g_{ij}\, \dot{x}^i \dot{x}^j \, d\xi^2$$

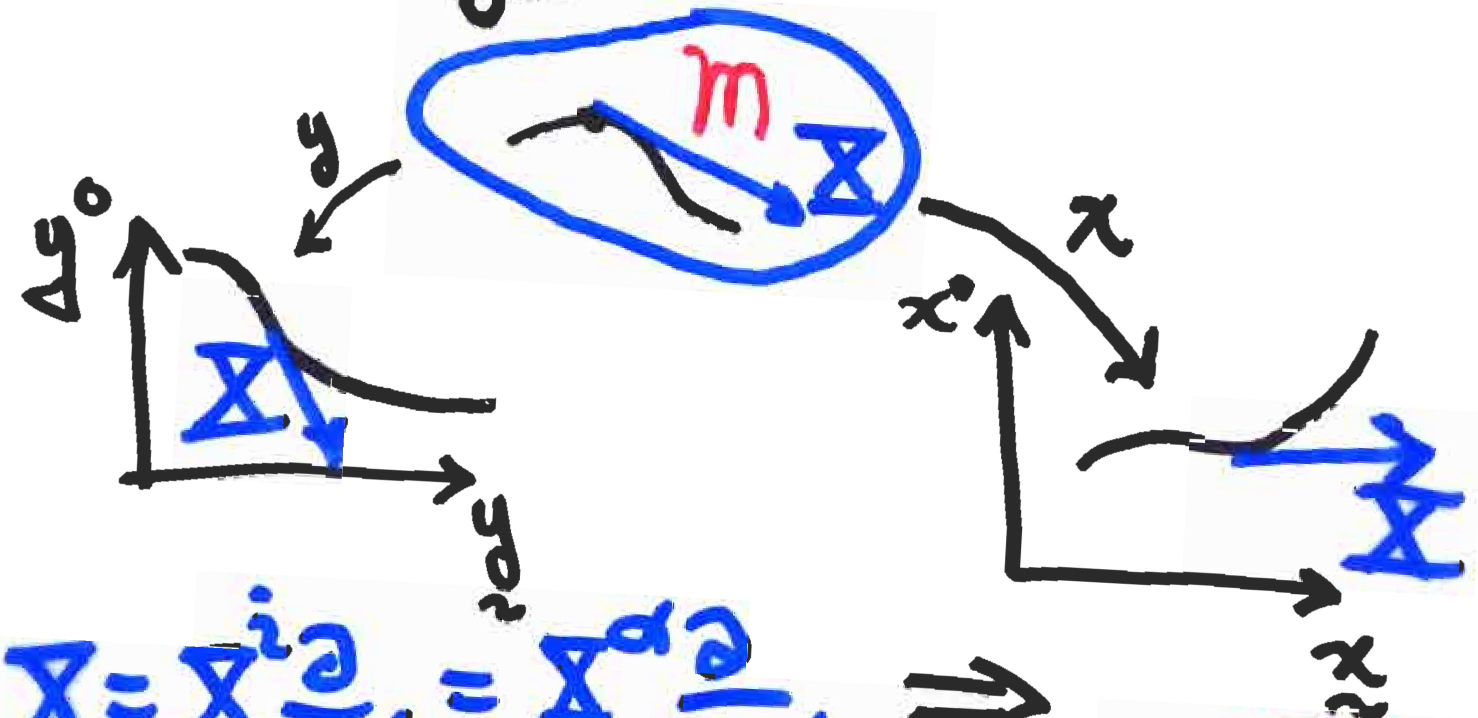
$$ds^2 = \|\mathbf{X}\|^2 d\xi^2$$

- Since $\mathbf{X} = \dot{x}^i \frac{\partial}{\partial x^i}$, $dx^i = \dot{x}^i d\xi$
- view dx^i as a linear operator on tangent vectors: $dx^i(\mathbf{X}) = \dot{x}^i$

$\Rightarrow$ $\{dx^i\}$ is a basis for $T^*\mathcal{M}$

▪ Change of Coordinates – Tensors

- Let $y = (y^0, y^1, y^2, y^3)$ be another coordinate system:



- $X = X^i \dfrac{\partial}{\partial x^i} = X^\alpha \dfrac{\partial}{\partial y^\alpha} \Rightarrow$

$$X^i = \frac{\partial x^i}{\partial y^\alpha} X^\alpha$$

$$\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}$$

$i\, j\, k \ldots$ for x-coord

$\alpha\, \beta\, \gamma \ldots$ for y-coord

- Up indices transform <u>contravariantly</u>
Down indices transform <u>covariantly</u>

- This determines how g_{ij} transform

$$\|X\|^2 = g_{ij} X^i X^j = g_{\alpha\beta} X^\alpha X^\beta$$

$$\Downarrow$$

$$\boxed{g_{ij} = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}}$$

$$\Updownarrow$$

$$g = J^{tr} \bar{g} J \qquad J = \left(\frac{\partial y}{\partial x}\right)_{4\times 4}$$

$$ds^2 = g_{ij} dx^i dx^j = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$\Downarrow$$

$$\boxed{dx^i = \frac{\partial x^i}{\partial y^\alpha} dy^\alpha}$$

$$\omega = \omega_i dx^i = \omega_\alpha dy^\alpha \Rightarrow \omega^i = \frac{\partial x^i}{\partial y^\alpha} \omega^\alpha$$

■ Einstein Summation Convention – keeps track of how tensor components transform under change of coordinates –

- Coordinate indices up x^i
  Vector components up X^i
  Basis 1-forms up dx^i

- Vector basis down $\frac{\partial}{\partial x^i}$
  1-form components down ω_i
  metric components down g_{ij}

$\Rightarrow$ Match indices with coord system
$\{i\,j\,k\,l\cdots\} \longleftrightarrow x \qquad \{\alpha\,\beta\,\gamma\cdots\} \longleftrightarrow y$

$$\Rightarrow \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}, \quad g_{ij} = g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}$$

Defn: a tensor of type (k, l) has k indices up & l indices down

$$T^{i_1 \cdots i_k}_{j_1 \cdots j_l} \qquad x\text{-coord comp's}$$

$$T^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_l} \qquad y\text{-coord comp's}$$

$$T^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_l} = T^{i_1 \cdots i_k}_{j_1 \cdots j_l} \frac{\partial x^{j_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{j_l}}{\partial y^{\beta_l}} \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{i_k}}$$

- Can view $T^{i_1 \cdots i_k}_{\beta_1 \cdots \beta_l}$ as components of $\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_l} \right\}$

*basis for linear operators on

$$T^*M \times \cdots \times T^*M \times TM \times \cdots \times TM$$

- Lower an indice with metric

$$T_{ij} \equiv T_i^{\ \sigma} g_{\sigma j}$$

$$\Downarrow$$

T^{ij} is a $(0,2)$-tensor

- raise an indice with g^{-1}

$$T^{ij} = T^i_{\ \sigma} g^{\sigma j}$$

- Define $(g^{ij})_{4\times 4} = (g_{ij})^{-1}_{4\times 4}$

- Thm: contraction/lowering/raising preserves tensor transformation laws

☒ Freefall paths are geodesics of the spacetime metric

- **Examples:** $x^0 = ct$, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

① Schwarzschild Metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2 \qquad (S)$$

G = Newton's Constant

M = Mass of Sun at $r = 0$

$\Rightarrow$ Planets follow geodesics of (S)

(Schwarzschild radius $r = 2GM$)

[Birkhoff Thm: (S) is the only spherically symmetric gravitational field in empty space]

② Tolman-Oppenheimer-Volkoff metric

(TOV) $ds^2 = -B(r)dt^2 + \frac{1}{1-\frac{2GM}{r}}dr^2 + r^2 d\Omega^2$

$M(r) \equiv$ "total mass inside radius r"

Models the gravitational field inside a static fluid sphere

$\approx$ Gravitational Field inside a star

[Chandresekhar Stability limit
Buchdahl Stability limit]

■ FRIEDMANN-ROBERTSON-Walker Metric

$$(\text{FRW}) \quad ds^2 = -dt^2 + R(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right\}$$

$R(t) \equiv$ Cosmological Scale Factor

Big Bang → $0 \leq R(t) \leq 1$ ← Present Universe

$$H = \frac{\dot{R}}{R} = \text{Hubble Constant} \approx h_0 \frac{100\,\text{km}}{s\,\text{mpc}}$$

$\text{Mpc} = 3.26 \times 10^6$ ltyrs

- Galaxies follow geodesics of (FRW)

- $k<0 \iff \Omega_M = \frac{\rho_0}{\rho_{crit}} < 1$ (open)

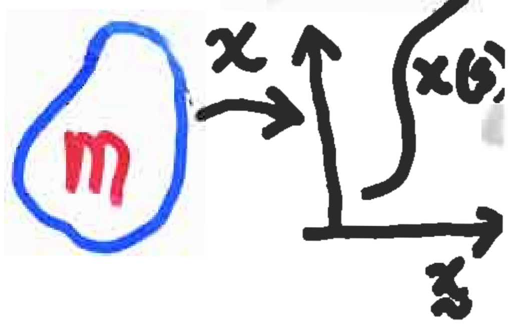
  $k=0 \iff \Omega_M = 1$ (critical)

  $k>0 \iff \Omega_M > 1$ (closed)

■ <u>The</u> <u>Geodesic</u> <u>Equations</u> - curve $x(s)$

Given a metric g, x-compt's $g_{ij}(x)$

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}$$



$$\Gamma^i_{jk} = \frac{1}{2}g^{i\sigma}\left\{-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\right\}$$

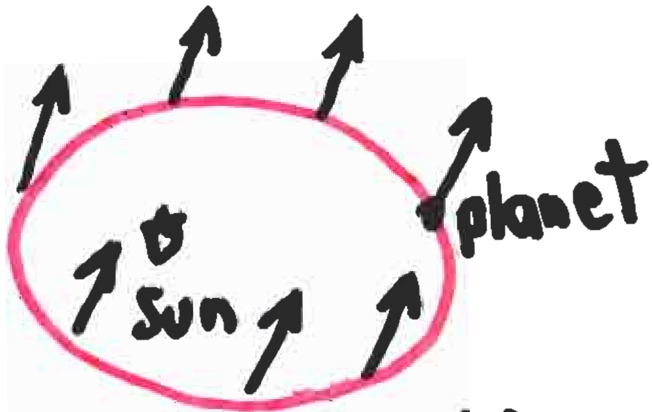
- Γ^i_{jk} ≡ Christoffel Symbols // Connection Coefficients

<u>Thm:</u> If $g_{ij,k} \equiv \frac{\partial}{\partial x^k} g_{ij}(x) = 0$,

then $\Gamma^i_{jk}(x) \equiv 0$

Γ^i_{jk} <u>not</u> tensorial	$\Gamma^i_{jk} = \Gamma^i_{\underline{jk}}$

■ Non-rotating vectors carried by an observer in freefall are parallel-transported by the unique symmetric connection determined by g —



• Y is parallel in direction X if

"$\nabla_X Y = 0$"

• $\nabla_X \equiv$ Covariant derivative defined by

$$\nabla_{\frac{\partial}{\partial x^i}} Y = \left\{ Y^j_{,i} - \Gamma^j_{i\sigma} Y^\sigma \right\} \frac{\partial}{\partial x^j} \equiv Y^j_{;i} \frac{\partial}{\partial x^j}$$

" Γ^i_{jk} converts differentiation of vector components to a "tensor operation"

- Given $\underline{X} = X^i \frac{\partial}{\partial x^i}$, it follows that

$$\nabla_{\underline{X}} \underline{Y} = \nabla_{X^i \frac{\partial}{\partial x^i}} \underline{Y} \equiv X^i \nabla_{\frac{\partial}{\partial x^i}} \underline{Y}$$

is a vector with x-components

$$X^i Y^j{}_{;i}$$

- <u>Theorem</u>: $Y^j{}_{;i}$ transforms as a $(1,1)$-tensor

- The Covariant Derivative Extends to ∇T for any tensor T:

$$T^{i}_{\ j;k} = T^{i}_{\ j,k} - \Gamma^{i}_{\sigma k} T^{\sigma}_{\ j} + \Gamma^{\sigma}_{jk} T^{i}_{\ \sigma}$$

A term like this $\forall$ index up

A term like this for every index down

- Note: Covariant Differentiation reduces to ordinary differentiation when $\Gamma^{i}_{jk} = 0$

Example:

$$\operatorname{div} T \underset{\substack{\uparrow \\ \text{components}}}{=} T^{i\sigma}_{\ \ ;\sigma} \Rightarrow T^{i\sigma}_{\ \ ,\sigma} \text{ if } \Gamma^{i}_{jk} = 0$$

▫ **Defn:** x is locally inertial (locally Lorentzian/Minkowskian)

at P in Spacetime M if

$$\begin{cases} g_{ij}(P) = \mathrm{diag}(-1, 1, 1, 1) \\ g_{ij,k}(P) = 0 \end{cases}$$

$$\Downarrow$$

$$\Gamma^i_{jk}(P) = 0$$

· **Conclude:** in a locally inertial coordinate system, covariant derivative agrees with classical derivative

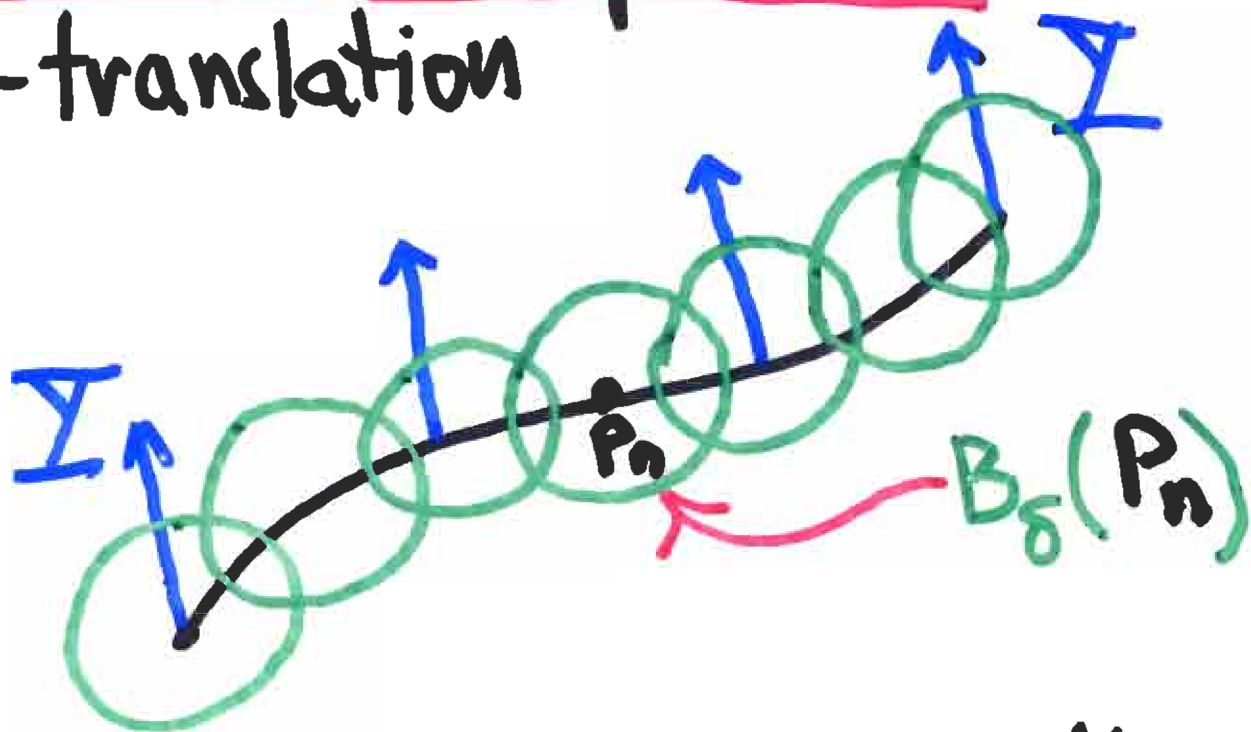
- **Corollary**: In a locally inertial coord. system

① Geodesics are (locally) straight lines

② Vectors are (locally) parallel translated by keeping components constant

(Just like flat Minkowski Space)

Geometric Interpretation of $\parallel$-translation



- Given path in spacetime
- Cover it with locally inertial coordinate frames $\{B_\delta(P_n)\}_{n=1}^{N}$

Translate the components as constant in each inertial frame

- Take limit $\delta \to 0$ to squeeze out errors

(Similar for Geodesics) (keep $g_{ij,k}$ & unif bounded!)

- **Conclude**: Parallel Translation must agree with $\nabla_{\mathbf{X}} \mathbf{Y} = 0$ in order that spacetime have (locally) the same inertial properties as flat Minkowski space

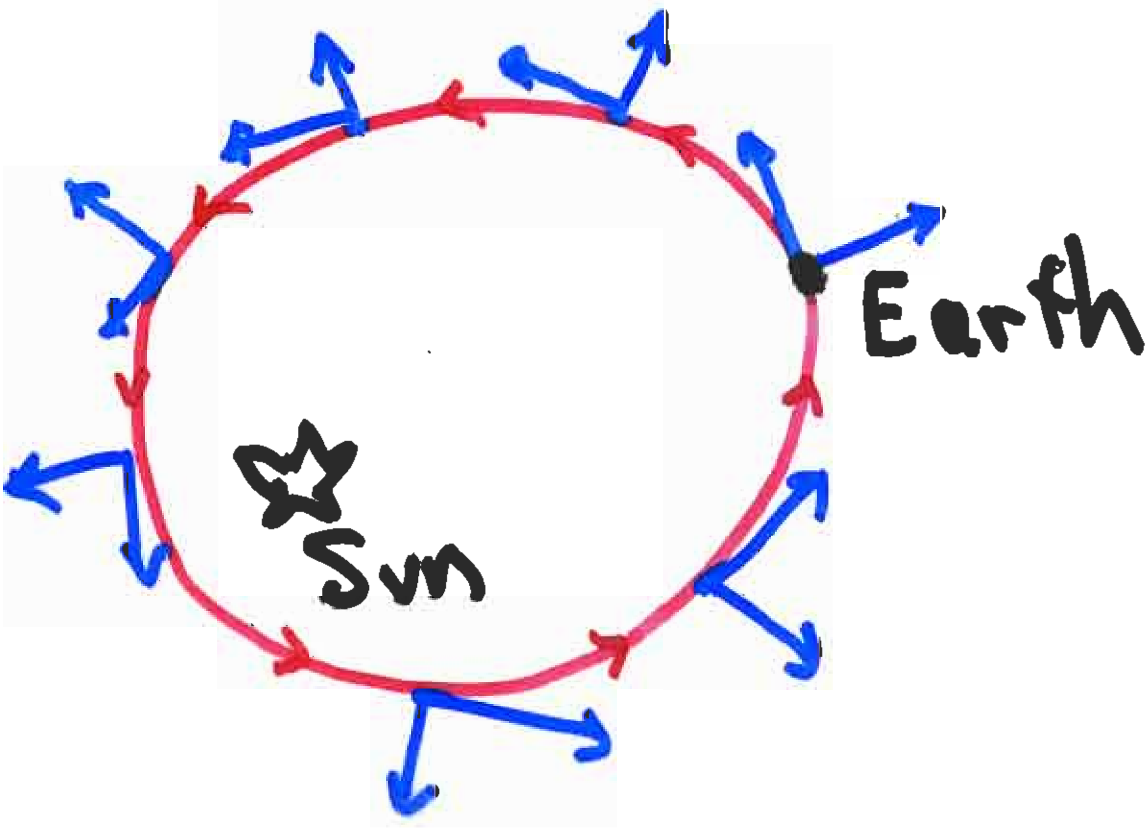
- **Reverse It**: "$\nabla_{\mathbf{X}} \mathbf{Y} = 0$ gives a coordinate independent (covariant) description of parallel translation by locally "inertial" frames"

▫ Fundamental Tenet of General Relativity

"When gravitational fields are present, there is no global inertial coordinate system on spacetime"

"In general relativity, inertial coordinate systems are local properties of space time that change from point to point"

■ A Picture: The earth moves "unaccelerated" thru each local inertial frame, but these frames change from point to point, thus producing apparent accelerations in a global coordinate system in which metric component $\neq$ diag(-1,1,1,1)



■ A Point of View: One can view the gravitational metric g as a "bookkeeping device" for keeping track of the local inertial coordinate systems as they change from point to point in spacetime

- The fact that the earth moves in a periodic orbit around the sun is proof that $\nexists$ a coord system that is globally inertial

- This is an expression of the fact that gravitational fields produce non-zero spacetime

* Curvature *

- **Thm**: You cannot in general remove the 2nd derivatives $g_{ij,kl}(P)$ at the center of a locally inertial coordinate system, & these measure

Spacetime Curvature

- **Riemann** 1854

- Introduced Riemann Curvature Tensor

"A tensorial measure of the 2nd derivatives $g_{ij,kl}(P)$ that cannot be removed by coordinate transformation"

$$R^i_{jkl} = \underbrace{\Gamma^i_{jk,l} - \Gamma^i_{jl,k}}_{\text{Curl}} + \underbrace{\left\{ \Gamma^\sigma_{jl}\Gamma^i_{\sigma k} - \Gamma^\sigma_{jk}\Gamma^i_{\sigma l} \right\}}_{\text{Commutator}}$$

Γ not tensor but R is

II. Introduction to the Einstein Equation

Mach's Principle

- Once one makes the leap to the idea that inertial frames change from point to point, it becomes remarkable that our non-rotating frames here on earth are also non-rotating relative to the fixed stars —
- Mach's Principle is that the stars must have had something to do with the determination of the inertial coordinate frames here on Earth

• Indeed: not every Lorentzian metric g can describe a gravitational field —

– The gravitational metrics must satisfy a constraint that describes how inertial frames at different points interact and evolve

• In Einstein's theory of gravity this constraint is given by the Einstein Equations — (1915)

$$G = \kappa T$$

$$G = \kappa T$$

Einstein Curvature Tensor

Fundamental Constant

Stress-Energy Tensor

- In x-coordinates:

$$G_{ij}(x) = \kappa T_{ij}(x)$$

$$G_{ij} \equiv R^{\sigma}_{i\sigma j} - \frac{1}{2} R^{\sigma\tau}_{\sigma\tau} g_{ij}$$

- For a perfect fluid:

$$T^{ij} = (\rho + p) u^i u^j + p g^{ij}$$

u = 4-velocity vetor $\quad u = \frac{dx}{ds}$

ρ = energy density $= \rho c^2$

p = pressure

Einstein Equations for Perfect Fluid

$$G_{ij}[g_{ij}] = \kappa T_{ij}[\rho, p, u]$$

2nd order diff. operator on g_{ij}

0-order source term (no derivatives)

- Equation Count: $G_{ij} = G_{ji}$ symmetric 4×4

$\Rightarrow$ 10 independent equations

- # of unknowns: $g_{ij} = g_{ji}$ symmetric 4×4

$\Rightarrow$ 10 independent metric components

$+$ 4 fluid variables ρ plus 3 of 4 components of u

$p = p(\rho)$

14 Unknowns

- 4 free coordinate transformations

$\Rightarrow$ Equations can be closed

- Where did $G = \kappa T$ come from?

- The constant κ is determined so that the theory corresponds with Newton's Law of gravity in that limit of low velocities & weak gravitational fields —

$$\kappa = \frac{8\pi G}{c^4}$$

- G = Newton's Constant enters through the inverse square force law

$$\text{Force} = M\vec{a} = -G\frac{MM_0}{r^3}\vec{r}$$

■ Force $= \cancel{M}\vec{a} = -G\dfrac{\cancel{M}M_0\vec{r}}{r^3}$ (N)

Inertial Mass (I-M)

Gravitational mass (G-M)

I-M = G-M

$\Downarrow$

- (N) is independent of any properties of the earth!!
- The earth & a feather will traverse the same path thru gravitational field (subject to same initial condition)
- Newton's "Gravitational Force" is different on different objects but it adjusts itself perfectly so that every object traverses same Path

- From this, Einstein was led to suspect that Newton's Gravitational Force was some sort of "artificial devise", and that the fundamental objects in a gravitational field were the "freefall paths", not the forces —

- Conclude: In this sense, $G = kT$ is more reasonable than $\vec{F} = -G\frac{Mm\hat{r}}{r^2}$ because at the start, $G = kT$ is an equation for the gravitational metric g, which describes free-fall paths via the geodesic equation of motion

■ **Comment:** In Newton's theory of gravity, the non-rotating frame on Earth are aligned with stars because there is a global inertial coordinate system that connects them.

• In Einstein's Theory this happens (according to Cosmology) because they are aligned in the FRW metric.

$$ds^2 = -dt^2 + R(t)\{dr^2 + r^2 d\Omega^2\}$$

and the FRW metric solves the Einstein Equations (for appropriate $R(t)$)

The Continuum Version of Newton's Law of Gravity

- In limit pt masses $\{m_i\} \to$ density ρ

$$\vec{a} = -G \sum_i \frac{m_i}{|r-r_i|^3} (\overrightarrow{r-r_i})$$

$$\Downarrow$$

$$\vec{a} = -\nabla\phi = \int_{\mathbb{R}^3} \frac{G}{|x-y|^3}(x-y)\rho(y)\,d^3y$$

$$\Downarrow$$

$$\phi(x) = \int_{\mathbb{R}^3} \frac{G}{|x-y|} \rho(y)\,dy$$

$\rho \in C_0^\infty$

$$\Downarrow$$

$$\boxed{-\Delta\Phi = 4\pi G\rho}$$ (Poisson)

▪ Conclude:

Newton's Theory: $-\Delta\Phi = 4\pi G\rho$

Einstein's Theory: $G[g] = \frac{8\pi G}{c^4} T[\rho, u, p]$

- Important difference: In Einstein's theory, the evolution of ρ, u, p are determined along with g
- For Newton's theory, we must augment $-\Delta\Phi = 4\pi G\rho$ with conservation laws (Euler Eqn's) that give the evolution of ρ

Euler-Poisson Equations:

$$\rho_t + \mathrm{div}_x(\rho \underset{\sim}{v}) = 0$$

$$(\rho v^i)_t + \mathrm{div}_x(\rho v^i v + p e^i) = -\rho \nabla \phi$$

$$-\Delta \phi = 4\pi G \rho$$

- Rewrite with sources on left:

$$-\rho \nabla \begin{bmatrix} 0 \\ \phi \end{bmatrix} = \mathrm{div}_{t,x} \begin{bmatrix} \rho & \rho \underset{\sim}{v}^t \\ \rho \underset{\sim}{v} & \rho v^i v^j + p \delta^{ij} \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} -\rho \underset{\sim}{\nabla} \phi &= \mathrm{div}_{t,x}(\rho v^i v^j + p \delta^{ij}) \\ -\Delta \phi &= 4\pi G \rho \end{aligned}}$$

$v = (1, \underset{\sim}{v}) \qquad \delta = (0, \underset{\sim}{\delta})$

Compare

(Euler-Poisson) $-\rho \nabla \phi = \mathrm{div}(\rho v^i v^j + p\delta^{ij})$

$$-\Delta \phi = 4\pi G \rho$$

(Einstein) $G[g] = \frac{8\pi G}{c^4}\left((\rho+p)u^i u^j + p g^{ij}\right)$

- $T^{ij} = (\rho+p)u^i u^j + p g^{ij}$

is just the relativistic version of

$\hat{T}^{ij} = \rho v^i v^j + p\delta^{ij}$

$\Rightarrow$ In Einstein's theory, conservation of energy & momentum should be given by

$$\mathrm{div}\, T = 0$$

where we take <u>covariant divergence</u> so it agrees with regular div in inertial frames

- **Problem:** We did the equation count for

$$G = kT \qquad \text{(Ein)}$$

and got 14 equations in 14 unknowns

$\Rightarrow$ no more freedom to impose

$$\operatorname{div} T = 0$$

$\Rightarrow$ G must be chosen so that

$$\operatorname{div} G = 0 \qquad \text{(Evl)}$$

is an identity (Ein $\Rightarrow$ Evl!!)

- This is the key to the discovery of the Einstein equations —

▪ The road to $G = \kappa T$:

(1) Look for equation of form

$$G = \kappa T \qquad (E)$$

G measures the curvature

T measures energy & momentum densities & their fluxes

"(E) is the simplest tensor eqn that couples sources to spacetime curvature"

(2) Equation count ⇒ look for G that satisfies 1st order differential identity

$$\operatorname{div} G = 0$$

(3) The Riemann Curvature Tensor satisfies 1st order differential identities called Bianchi Identities

$$R^{i}{}_{j[kl;m]} = 0$$

cyclic sum

(4) **Theorem** : The simplest (0,2)-tensor G_{ij} constructed from $R^{i}_{jk\ell}$ s.t.

$$\operatorname{div} G \equiv 0$$

is

$$G_{ij} = R^{\sigma}_{i\sigma j} - \frac{1}{2} R g_{ij}$$

$R = R^{\sigma\tau}_{\sigma\tau} \equiv$ Ricci Scalar Curvature

The next simplest is

$$G_{ij} = R^{\sigma}_{i\sigma j} - \frac{1}{2} R g_{ij} + \Lambda g_{ij}$$

Cosmological Constant

(*) There are no others that can be constructed via "simple operations" on $R^{i}_{jk\ell}$ & g_{ij}

(5) Calculation $\Rightarrow$ Newton's Equations emerge to leading order in the limit of low velocities & weak grav. fields, $\left|\frac{v}{c}\right| \ll 1$

$$G_{ij} = \kappa T_{ij} \underset{\left|\frac{v}{c}\right| \ll 1}{\Longrightarrow} G_{00} = \kappa T_{00}$$

$$\Downarrow$$

$$\Delta g_{00} = \frac{8\pi G}{c^4} \rho c^2$$

$$-g_{00} = 1 + \frac{2\phi(x)}{c^2} + O\left(\frac{1}{c^3}\right)$$

$$\Downarrow$$

$$\boxed{-\Delta\phi = 4\pi G \rho}$$

- Calculation $\Longrightarrow$

$$\begin{cases} \Gamma^i_{00} \approx -\frac{1}{2}\frac{\partial g_{00}}{\partial x^i} + O\left(\frac{1}{c^3}\right) \\ \Gamma^i_{jk} \ll 1 \quad \text{o.w.} \end{cases}$$

$\Downarrow$ geodesic equation

$$\ddot{x}^i = \Gamma^i_{jk}\dot{x}^i\dot{x}^j \approx \Gamma^i_{00}\dot{x}^0\dot{x}^0 = \Gamma^i_{00}$$

$\frac{dx^0}{ds} \approx 1$

$\Downarrow$

$$\ddot{x} = -\nabla\varphi$$

Since $\quad g_{00} \approx 1 + 2\varphi$

- **Conclude**: Einstein Equations for a Perfect Fluid

$$G = \frac{8\pi G}{c^4} T$$

$$\operatorname{div} G = 0 \Rightarrow \operatorname{div} T = 0$$

$$\operatorname{div} T = 0$$

$$\Updownarrow$$

$$\operatorname{div}\left[(\rho + p)u^i u^j + p\, g^{ij}\right] = 0$$

$$\Updownarrow$$

Relativistic Version of Compressible Euler Equations in locally inertial coordinate frames $\Rightarrow$ Shock Waves

II Shock-wave Solutions of $G=\kappa T$

- Since $\mathrm{div}T=0$ is a subsystem of $G=\kappa T$, we expect shock-waves are as fundamental to Einstein Equations with perfect fluid sources as they are for Compressible Euler

- Heuristically: Shock-waves must form because, in principle, we could drive fluid into a shock-wave while keeping everything in a local inertial frame where equations are a small perturbation of Euler

A. General Theory of matching gravitational metrics across 3-d Shock surface
[Israel]

B. Construct 1st example of exact shock wave soln of
$G = \kappa T$ [Sm/Te]

- Spherical Blast wave
- Generalizes Oppenheimer Snyder to $p \neq 0$ (1939)

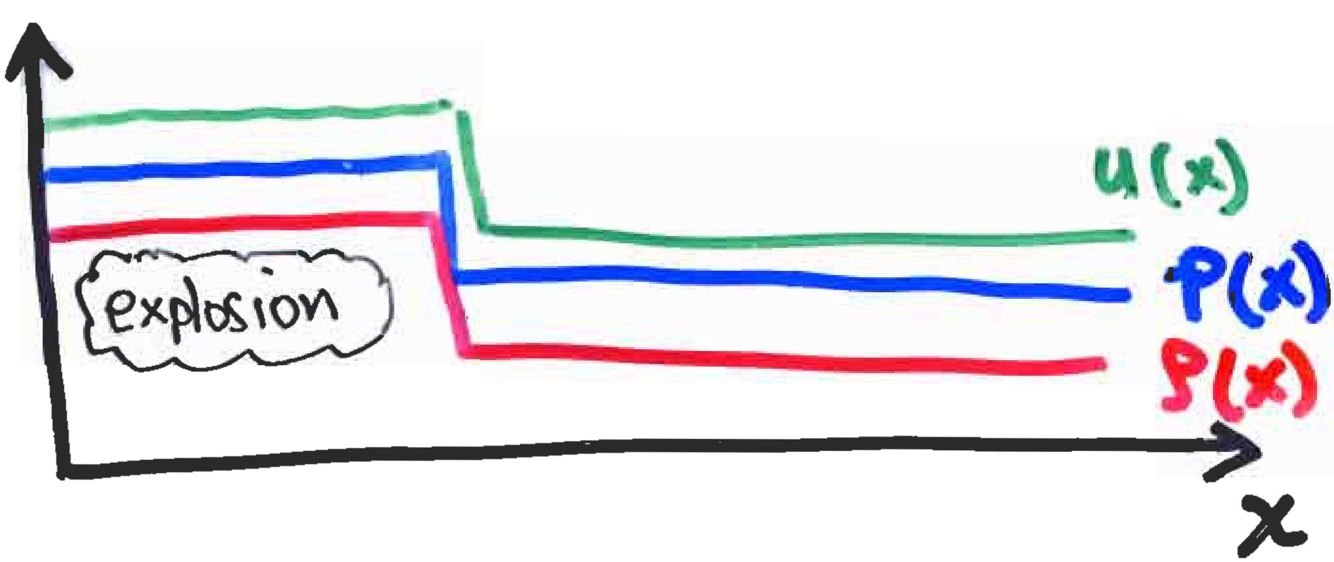
■ Shock-Waves:

Einstein Equations:

$$G[g_{ij}(x)] = \frac{8\pi G}{c^4} T_{ij}[\rho, P, u]$$

$$\text{“} \partial^2 g = \kappa T(\rho, P, u) \text{”}$$

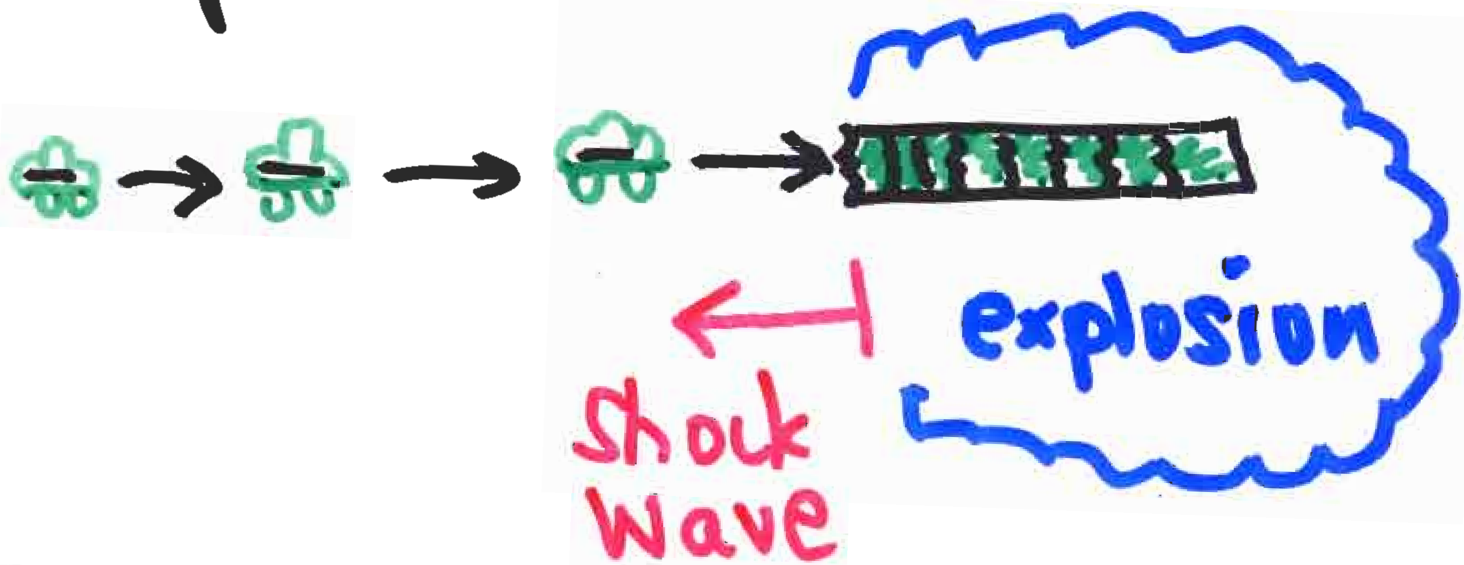
- At a shock-wave, ρ, P, u are discontinuous



■ "$\partial^2 g_{ij} = K T_{ij}(\rho, P, u)$" 3.4

At shock, ρ, P, u discontinuous

Example: Traffic Picture

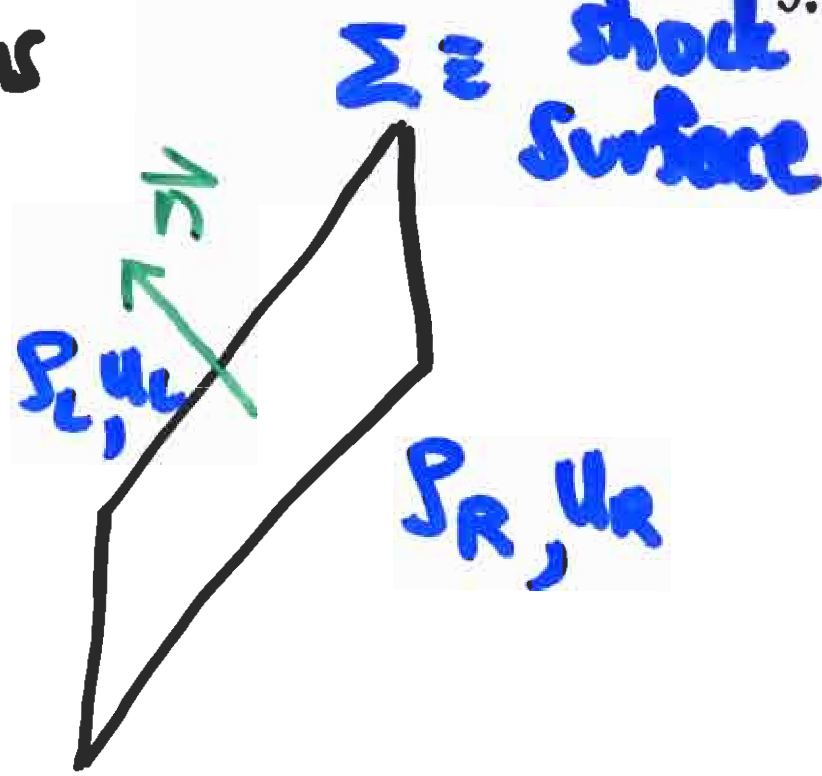


"Shock waves always propagate into side with lower density"

$\Updownarrow$

entropy Condition

■ Two Conditions @ Shock –



(1) Conservation

$\Updownarrow$

$$[T^{i\sigma}]n_\sigma = 0$$

$$[f] = f_L - f_R$$

$[T^{i\sigma}]n_\sigma$ <u>tensorial</u> $\Rightarrow$ extends from $\mathrm{div}\,T = 0$ to $G = \kappa T$ <u>unchanged</u>

(2) Entropy Condition

- breaks time symmetry
- $\Rightarrow$ time-irreversibility
- $\Rightarrow$ dissipation

☒ Entropy Condition:

- Lax Characteristic condt:
  ⇔ shock is compressive
- Density & pressure larger on side that recieves the mass flux

- ⇔ Shock wave is dissipative

"∃ dissipation in zero-dissipation limit"

Smoothness of Metric at Shock

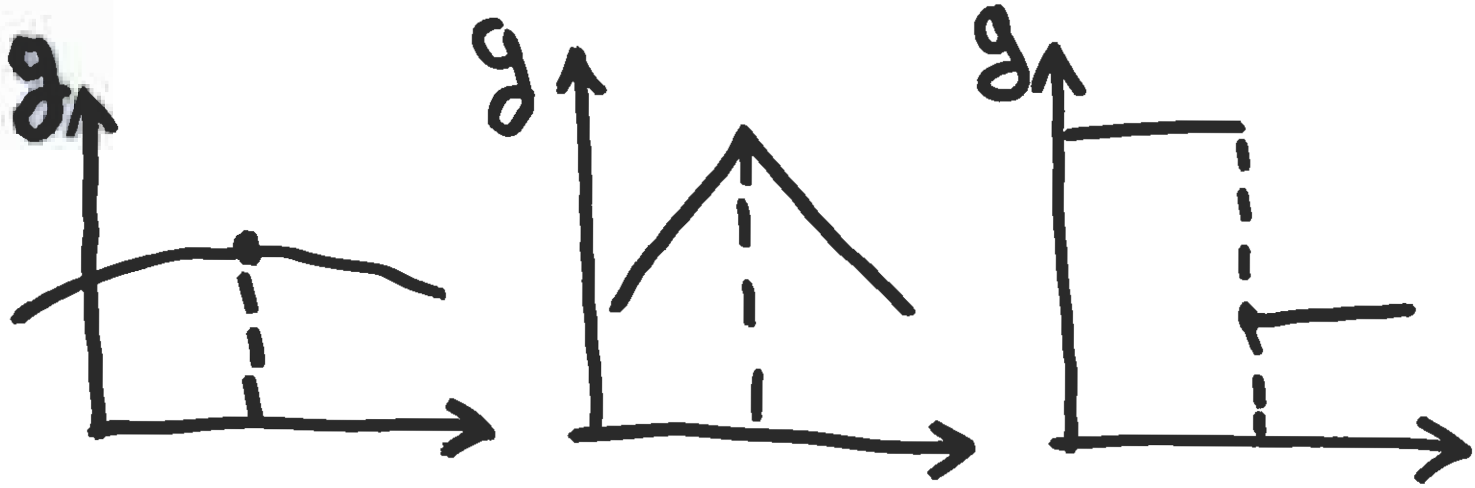
$$\text{"}\partial^2 g_{ij} = K\, T_{ij}(\rho, p, u)\text{"}$$

RHS discontinuous

$\Downarrow$

LHS has one cont derivative

?? $g \in C^{1,1}$??



$g \in C^{1,1}$ $\quad$ $\nabla g \in C^{0,1}$ $\quad$ $\nabla^2 g$ jump disc.

- **Remarkable Fact:**

"Two metrics can match only Lipschitz continuously across a shock so long as an additional conservation constraint is met"

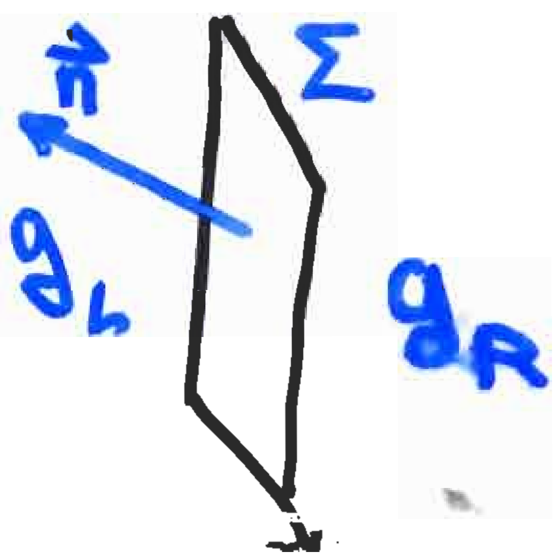
EG.

$$[T_{ij}n^i n^j] = 0$$

Main Theorem:

Let Σ be a 3-d shock surface with spacelike normal vector $\vec{n}$, such that $g = g_L \cup g_R$ match Lipschitz continuously across Σ:

Then the following are equivalent:



■ **Main Thm**: The following are **equivalent**: 3.10

① $[K] = 0 \quad \forall\, p \in \Sigma$

$K \equiv$ 2nd fundamental form

② R^i_{jkl}, G_{ij} viewed as 2nd order operators on g_{ij}, produce no δ-fn sources

③ $\exists$ a $C^{1,1}$ coord. trans. that improves g to $C^{1,1}$

④ $\exists$ locally Lorentzian coord. frame at each $p \in \Sigma$

$$\Downarrow$$

$$[G^{i\sigma}]\, n_\sigma = 0$$

Q: Why is the 2nd Fundamental Form K on Σ relevant

- Defn: 2nd Fundamental Form

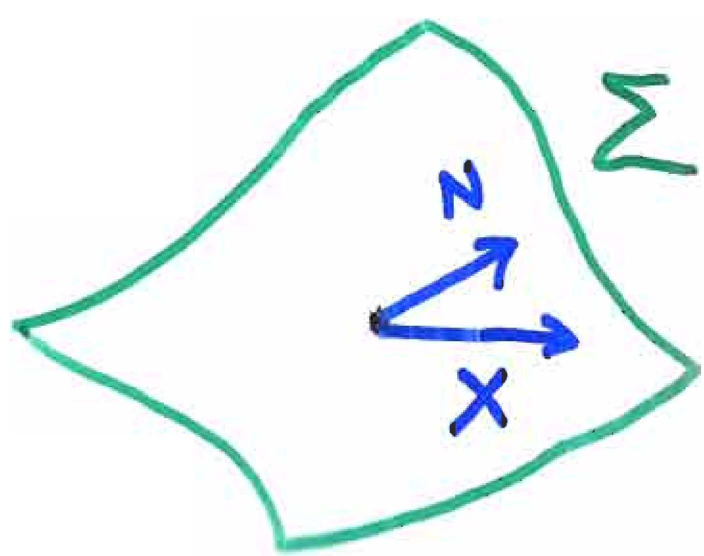
$$K : T_p\Sigma \to T_p\Sigma$$

$$K(X) = -\nabla_X N$$

where N is unit normal to Σ

- Since N is unit normal

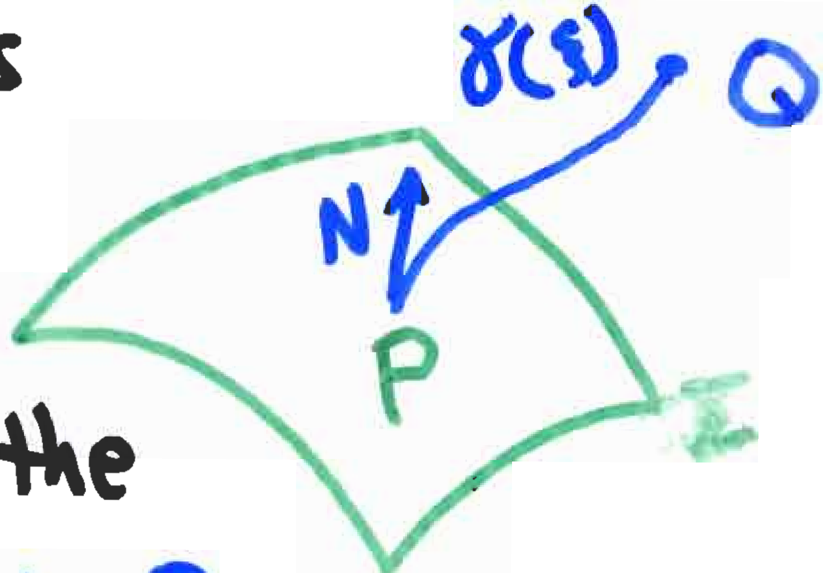
$\nabla_X N \in T_p\Sigma$



- <u>Claim:</u> K gives an invariant measure of how g changes in direction normal to Σ

To See This:

Introduce — Gaussian Normal Coords



- Choose y-coords so $\Sigma \leftrightarrow y^n = 0$
- For $P \in \Sigma$, let $\gamma(s)$ denote the geodesic

$$\gamma(0) = P$$
$$\dot{\gamma}(0) = N$$

- <u>Define</u>

$$w^n(Q) = s, \quad w^i(Q) = y^i \quad i = 1, \ldots, n-1$$

• In (GNC), the metric g_{ij} takes the form

$$ds^2 = d(w^n)^2 + \underset{\sim}{g}_{ij}\, dw^i dw^j$$

$i = 1, \dots, n-1$

• The components of K^i_σ in (GNC)

$$-X^\sigma K^i_\sigma = (\nabla_X N)^i = X^\sigma \underbrace{\cancel{N^i_{,\sigma}}}_{0} + \Gamma^i_{\sigma\tau} X^\sigma \underbrace{N^\tau}_{(0,\dots,0,1)} = \Gamma^i_{\sigma n} X^\sigma$$

But

$$\Gamma^i_{\sigma n} = \frac{1}{2} g^{i\tau}\{-g_{\sigma n,\tau} + g_{\tau\sigma,n} + g_{n\tau,\sigma}\} = \frac{1}{2} g^{i\tau} g_{\tau\sigma,n}$$

• Thus $$\boxed{K_{i\sigma} = -\frac{1}{2} g_{i\sigma,n}}$$

Theorem: Assume Σ is a 3-d shock surface across which $g = g^L \cup g^R$ matches Lipschitz continuously in x-coordinates

Assume $[K] = K^L - K^R = 0$

Then

$$G(g_{ij}(x))$$

Einstein tensor 2nd order diff operator

Produces no δ-fn sources

"**Proof**": go to (GNC) w for Σ

$$[K]=0 \Rightarrow [\underset{\sim}{g}_{ij,n}]=0$$

$$g \text{ Lipschitz} \Rightarrow [\underset{\sim}{g}_{ij,k}]=0$$

$$(GNC) \Rightarrow ds^2 = -d(w^n)^2 + \underset{\sim}{g}_{ij}\, dw^i dw^j$$

$$\Downarrow$$

$$g^L \cup g^R \in C^{1,1}$$

$$\Downarrow$$

$G[g(w)]$ has no δ-fn's

The fact that G is 2nd order tensor operator $\Rightarrow$ this can be mapped back thru singular transformation ✓ $w \leftrightarrow x$

☐ **Theorem** (Israel) the jump condts

$$[G^\sigma_j]\, n_\sigma = 0 \quad j = 0, \ldots, 3$$

hold at $p \in \Sigma$ iff both

$$[(\operatorname{tr} K)^2 - \operatorname{tr}(K^2)] = 0$$

$$[\operatorname{div} K - d(\operatorname{tr} K)] = 0$$

(where div & d are computed in the surface Σ)

Conclude: If $[K] = 0$ on Σ, then conservation holds —

$$[G^\sigma_j]\, n_\sigma = 0$$

- <u>Conclude</u> —

- **Compressible Euler:** $\mathrm{div}\, T = 0$

At a <u>shock wave</u> you must take <u>weak formulation</u>

$$\Updownarrow$$

$$[T^{i\sigma}]n_\sigma = 0 \qquad \text{(R-H)}$$

- **Einstein:** $G = \kappa T$

If $g \in C^{1,1}$ across shock surface

$$\Downarrow$$

$$0 = [G^{i\sigma}]n_\sigma = [T^{i\sigma}]n_\sigma \Leftrightarrow \text{(R-H)}$$

"The <u>weak</u> formulation of $\mathrm{div}\, T = 0$ is implied by <u>strong formulation</u> of $G = \kappa T$"

General Principle:

"The Einstein equations convert directions of $C^{1,1}$ smoothness into directions of conservation of the sources"

■ Corollaries of the Main Theorem

① If $g \in gR$ solves

$$G = \kappa T = 0 \quad (T \equiv 0)$$

then g Lipschitz cont at $\Sigma \Rightarrow$ $\exists$ coord system in which g is arbitrarily smooth at Σ

($\Rightarrow$ No shocks in empty space)

② Ricci Scalar Curvature R never has δ-fn sources at shocks

IV <u>Exact</u> Shock-Wave <u>Soln's</u> of

$G = \kappa T$

- The simplest setting in which mass can be localized —

Spherical Symmetry

- <u>Note</u>: The Mathematical Theory of shock-waves was developed first in 1-dimension

But: $\rho = \rho(x) \Rightarrow M = M(x) = \infty$

$\leadsto$ not so physically meaningful given that GR is all about the <u>mass</u>

General Spherically Symm. metric

$$ds^2 = -B(r,t)dt^2 + A(r,t)dr^2 + E(r,t)\,dr\,dt + C(r,t)\,d\Omega^2$$

$\Downarrow$ coord trans.

$$ds^2 = -B\,dt^2 + A\,dr^2 + C\,d\Omega^2$$

$\Downarrow$ $\frac{\partial C}{\partial r} \neq 0$

$$\boxed{ds^2 = -B\,dt^2 + A\,dr^2 + r^2\,d\Omega^2}$$

Standard Schwarzschild Coords-

- **Main Thm**: (Spherical Symmetry)
- **Assume**: $g = g^L \cup g^R$ match Lipschitz continuously across a radial shock surface Σ
- **Assume**: smooth matching of the spheres of symmetry ($C(r,t)\,d\Omega^2$ term)

Then:

All the equivalencies of the Main Thm are implied by the single condition:

$$[G^{ij}]\,n_i n_j = 0$$

"The Conservation Constraint"

TO CONSTRUCT EXACT SOLN'S

① Given 2 soln's of $G = \kappa T$:

$$ds^2 = -B(r,t)\,dt^2 + A(r,t)\,dr^2 + C(r,t)\,d\Omega^2$$

$$d\bar{s}^2 = -\bar{B}(\bar{r},\bar{t})\,d\bar{t}^2 + \bar{A}(\bar{r},\bar{t})\,d\bar{r}^2 + \bar{C}(\bar{r},\bar{t})\,d\Omega^2$$

② Map both to standard coords —

$$ds^2 = -B(r,t)\,dt^2 + A(r,t)\,dr^2 + r^2\,d\Omega^2$$

$$d\bar{s}^2 = -\bar{B}(\bar{r},\bar{t})\,d\bar{t}^2 + \bar{A}(\bar{r},\bar{t})\,d\bar{r}^2 + \bar{r}^2\,d\Omega^2$$

③ Set $\bar{r} = r$ $\Rightarrow$ smooth matching of spheres of symm.

④ Find a shock surface & a coord mapping $\bar{t} = \bar{t}(r,t)$ st metrics match Lipshitz across surface.

⑤ Impose Conservation: $[T^{ij}]n_i n_j = 0$

FRW-TOV Shock-Matching

- Given arbitrary FRW & TOV metrics

$$ds^2 = -dt^2 + R(t)^2\left\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right\} \quad \text{(FRW)}$$

$$d\bar{s}^2 = -B(\bar{r})\,d\bar{t}^2 + \frac{1}{A(\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \quad \text{(TOV)}$$

$$A(r) = 1 - \frac{2GM}{\bar{r}}$$

- We look for a coordinate mapping

$$(t, r) \longleftrightarrow (\bar{t}, \bar{r})$$

such that, under this identification, the metrics agree along a shock surface

$$r = r(t) \Longleftrightarrow \bar{r} = \bar{r}(\bar{t})$$

<u>Thm</u> (formally) this can always be done

▪ **Step I**: Require that spheres of symmetry have equal areas —

$$\text{(FRW)}\quad ds^2 = -dt^2 + R(t)^2\left\{\frac{dr^2}{1-kr^2}\right\} + \underbrace{R(t)^2 r^2}_{\updownarrow} d\Omega^2$$

$$\text{(TOV)}\quad d\bar{s}^2 = -B(\bar{r})\,dt^2 + \frac{1}{A(\bar{r})}d\bar{r}^2 + \overbrace{\bar{r}^2}\,d\Omega^2$$

Define:

$$\boxed{\bar{r} = R(t)\,r \equiv \bar{r}(t,r)}$$

$$\Downarrow$$

$$d\bar{r} = R\,dr + \dot{R}\,r\,dt$$

$$dr = \frac{1}{R}d\bar{r} - \frac{\dot{R}}{R}r\,dt$$

$$\Downarrow$$

$$dr^2 = \frac{1}{R^2}d\bar{r}^2 + \frac{\dot{R}^2}{R^2}r^2dt^2 - 2\frac{\dot{R}}{R^2}\bar{r}\,dt\,d\bar{r}$$

Step II : Transform FRW metric to $(t, \bar{r})$ - coordinates:

$$\Downarrow$$

$$ds^2 = \frac{1}{R^2 - k\bar{r}^2}\left\{-R^2\left(1 - \frac{8\pi G}{3}\rho R^2 r^2\right)dt^2 + R^2 d\bar{r}^2\right\} - \frac{2R\dot{R}}{R^2 - k\bar{r}^2}\, dt\, d\bar{r} + \bar{r}^2 d\Omega^2$$

mixed term

• Step III: Define

$$\bar{t} = f(t, \bar{r})$$

So that in $(\bar{t}, \bar{r})$ coords, the mixed term in FRW vanishes

$$\Downarrow$$

$$\bar{t} = \bar{t}(t, r) = f(t, R(t)\, r)$$

This completes the mapping

$$(t, r) \longleftrightarrow (\bar{t}, \bar{r})$$

Step III (details) - removing the mixed term

$$ds^2 = -C\,dt^2 + D\,d\bar{r}^2 + 2E\,dt\,d\bar{r}$$

• Define:

$$\begin{cases} \text{(A)} & d\bar{t} = \psi(t,\bar{r})\{C(t,\bar{r})dt - E(t,\bar{r})d\bar{r}\} \\ \text{(B)} & \frac{\partial}{\partial \bar{r}}(\psi C) = -\frac{\partial}{\partial t}(\psi E) \end{cases}$$

[(B) is an equation for integrating factor ψ

(B) $\Rightarrow$ (A) is an exact differential

$\Rightarrow$ (A) defines $\bar{t} = f(t,\bar{r})$

$\Downarrow$

$$ds^2 = -(\psi^{-2}C^{-1})d\bar{t}^2 + \left(D + \frac{E^2}{C}\right)d\bar{r}^2$$

Applying this to (FRW) :

$$(FRW)\quad ds^2 = \frac{1}{R^2 - k\bar{r}^2}\left\{ -F\,d\bar{t}^2 + G\,d\bar{r}^2 \right\} + \bar{r}^2 d\Omega^2$$

$$F = \frac{1}{\psi^2 R^2 \left\{ 1 - \frac{8\pi G}{3}\rho R^2 r^2 \right\}}$$

$$G = R^2 + \frac{\dot{R}^2 R^2 r^2}{1 - \frac{8\pi G}{3}\rho R^2 r^2}$$

$$(TOV)\quad d\bar{s}^2 = -B\,d\bar{t}^2 + \frac{1}{A}\,d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

- Next: match the components to determine shock surface

Step IV : Equate coeff of $d\bar{r}^2$

(FRW) $ds^2 = \frac{1}{R^2 - k\bar{r}^2}\left\{-F\,d\bar{t}^2 + G\,d\bar{r}^2\right\} + \bar{r}^2 d\Omega^2$

(TOV) $ds^2 = -B(\bar{r})\,d\bar{t}^2 + \frac{1}{A(\bar{r})}\,d\bar{r}^2 + \bar{r}^2 d\Omega^2$

$$\Downarrow$$

$$\frac{1}{A(\bar{r})} = \frac{G}{R^2 - k\bar{r}^2}$$

$\Downarrow$ (simplify)

$$\boxed{M(\bar{r}) = \frac{4\pi}{3}\rho(t)\,\bar{r}^3} \qquad (A)$$

(A) implicitly defines the shock surface $\bar{r} = \bar{r}(t)$

Step V: Equate coeff of $d\bar{t}^2$

$$(\text{FRW}) \quad d\bar{s}^2 = \frac{1}{R^2 - k\bar{r}^2}\left\{-F\,d\bar{t}^2 + G\,d\bar{r}^2\right\} + \bar{r}^2 d\Omega^2$$

$$(\text{TOV}) \quad d\bar{s}^2 = -B(\bar{r})\,d\bar{t}^2 + \frac{1}{A(\bar{r})}\,d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$$\Downarrow$$

$F = \frac{(\text{stuff})}{\psi^2}$ $\qquad$ $$\boxed{\frac{F}{R^2 - k\bar{r}^2} = -B(\bar{r})} \qquad (B)$$

- **But**: We only need that (B) hold **on the shock surface**

(B) is initial data for $\qquad$ $$\boxed{\frac{\partial}{\partial \bar{r}}(\psi C) = -\frac{\partial}{\partial t}(\psi E)}$$

- Solve for ψ in nbhd of shock

☒ <u>Conclude</u>: as long as the shock surface is non-char. for PDE

$$\frac{\partial}{\partial \bar{r}}(\psi C) = -\frac{\partial}{\partial t}(\psi E),$$

it follows that ψ and hence $\bar{t}$ is defined in a nbhd of shock surface

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3$$

and metrics match Lipschitz cont across shock

- Other than it's existence we never need any explicit information about ψ or $\bar{t}$

What is remarkable is that the shock surface equation uncouples from the equation for $\bar{t} = \bar{t}(t, r)$ when we match this way:

$$d\bar{t} = \psi(t, \bar{r})\{C(t, \bar{r})\,dt - E(t, \bar{r})\,d\bar{r}\}$$

which is exact when ψ satisfies

$$\frac{\partial}{\partial \bar{r}}(\psi C) = -\frac{\partial}{\partial t}(\psi E)$$

$$C = R^2\left\{1 - \frac{8\pi G}{3}\rho R^2 r^2\right\}$$

$$D = R^2$$

$$E = -R\dot{R}\bar{r}$$

with initial data on shock surface (which provides the match)

$$\psi = \frac{1}{B(R^2 - k\bar{r}^2)\,C}$$

Non-Characteristic Condt.

$$\frac{d\bar{r}}{dt} \neq \frac{C}{E} \qquad (*)$$

Thm (Te/Sm) If $A \geq 0$, then $(*)$ holds

"The shock surface is non-char. outside the Black Hole"

If $A < 0$, then $(*)$ holds so long as

$$\rho \geq \bar{\rho}, \quad p \geq \bar{p}$$

"The shock surface is non-char for explosions" inside the Black Hole – no mute point!

- The <u>Conservation</u> <u>Constraint</u>

- Given FRW & TOV metrics that solve Einsteins Equations & that match Lipschitz continuously across shock-surface

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\,\bar{r}^3$$

- <u>Main</u> <u>Lemma</u> If

$$[T^{ij}]n_i n_j = 0 \qquad (C)$$

then conservation holds.

- (C) leads to an equation that determines the TOV metric from a given FRW metric

FRW eqn of state ⇒ TOV eqn of state

Modified Strategy for Black Hole Shock 4.17

- Cons. Constraint: $[T_{ij}]n^i n^j = 0$

Thm if $\rho > \bar{\rho}$, $p > \bar{p}$ (Explosion) then shock surface is non-char for Ψ PDE

$\Rightarrow$ Lip cont matching

Thm Lip cont matching $+ [T_{ij}]n^i n^j = 0$

$\Rightarrow$ conservation holds

Problem $[T_{ij}]n^i n^j = 0$ is a complicated cubic polyn in $p, \bar{p}, \rho, \bar{\rho}, A$ with a degenerate soln that corresponds to a characteristic shock surface

- For our new black hole shock –

Instead of $[T^{ij}]n_i n_j = 0$, we use

$$\det[T^{ij}] = 0$$

Use this to solve for density

$$\bar{\rho} = \bar{\rho}(\bar{p}, p, \rho, N)$$

<u>Check</u>: $\rho > \bar{\rho}$, $p > \bar{p} \Rightarrow$ ✓ PDE

non-char $\Rightarrow$ Soln's match Lip Cont across shock

<u>Final check</u>: $n^i \in \ker[T^{ij}]$

$\Rightarrow$ Cons holds $(\Leftrightarrow [T^{ij}]n_i = 0)$

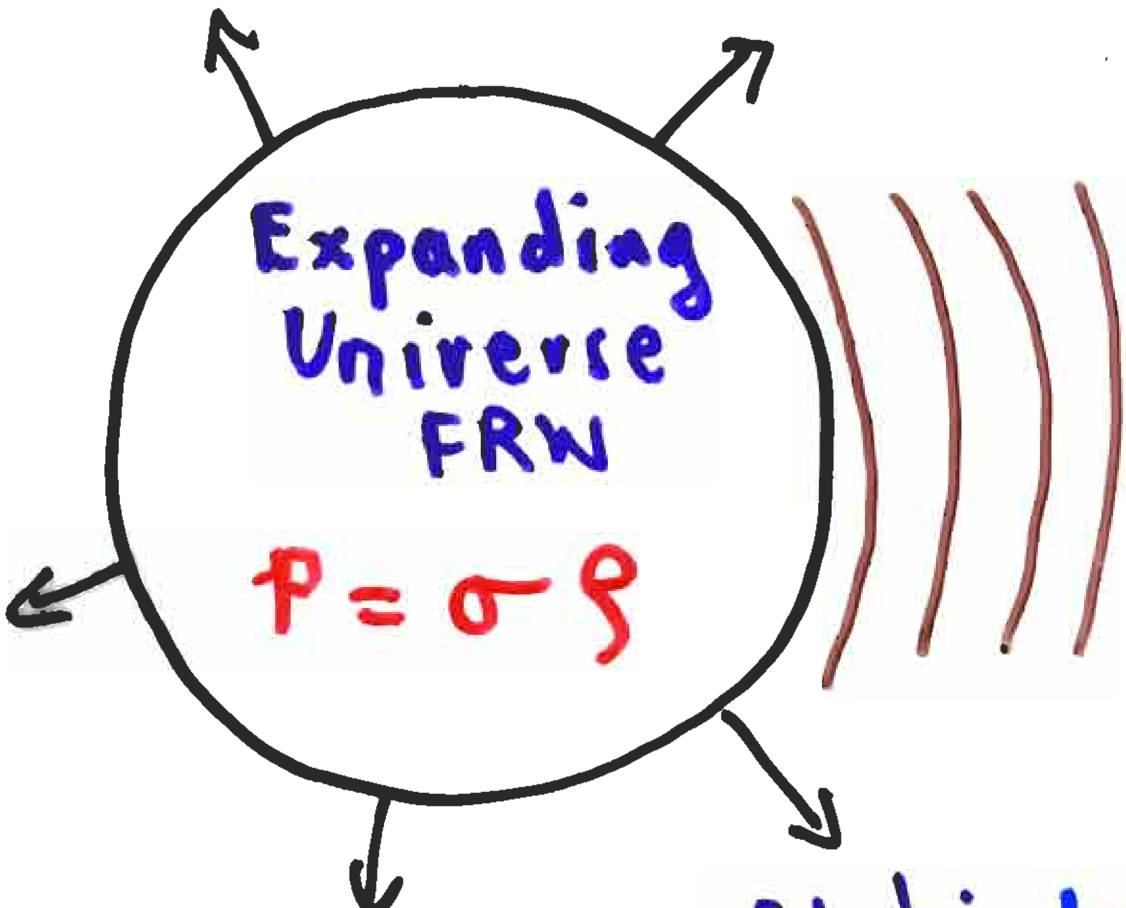
An Exact Shock-wave Soln of The Einstein Equations Modeling An Explosion

Sm/Te Phys Rev D '95

"Astrophysical Shock Wave Solutions of the Einstein Equations"

- Our Shock-wave model:

"when $p \neq 0$, interface should be a shock-wave"



$\sigma, \bar{\sigma}$ constant,

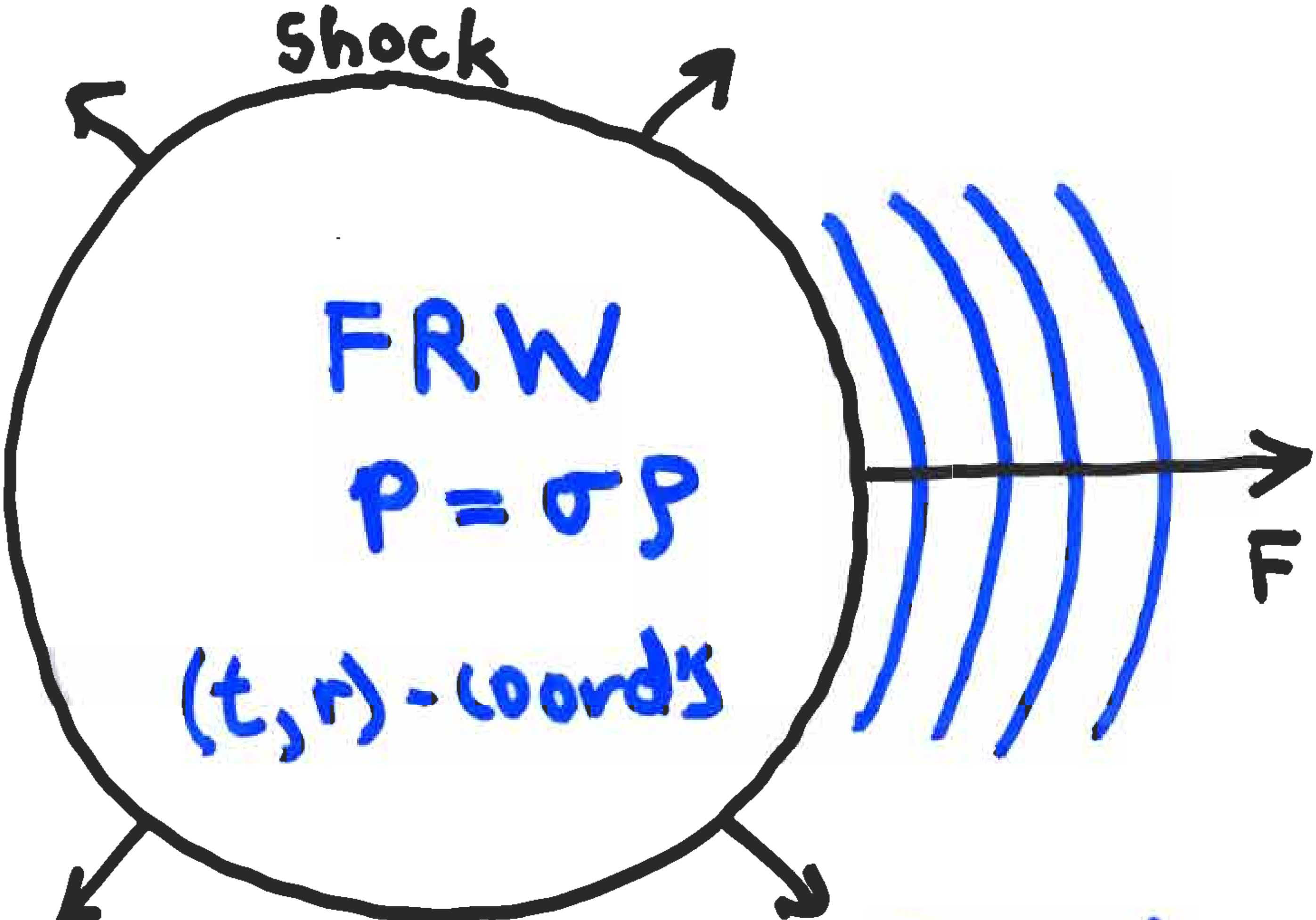
$\sigma = H(\bar{\sigma}) > \bar{\sigma}$

$\sqrt{\sigma} \equiv$ sound speed

Static Isothermal Fluid Sphere

$\bar{p} = \bar{\sigma} \bar{\rho}$

The Picture:



TOV

$\bar{p} = \bar{\sigma}\bar{\rho}$

$(\bar{t},\bar{r})$-coord's

Conservation

$\Downarrow$

$\bar{\sigma} = H(\sigma) < \sigma$

■ <u>Theorem</u>: $\exists$ exact solutions of FRW & TOV when $\sigma \equiv$ const.

- TOV soln is GR version of <u>static</u>, <u>singular</u>, <u>isothermal</u>, <u>sphere</u>

$$\Downarrow$$

Inverse Square density profile

EXACT SOLUTION OF TOV TYPE:

(TOV) $ds^2 = -B(\bar{r})\,d\bar{t}^2 + A(\bar{r})^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$

- $G = \kappa T$ + Co-moving Perfect Fluid

$$\Rightarrow \quad (*) \quad \boxed{\begin{aligned} \frac{dM}{d\bar{r}} &= 4\pi \bar{r}^2 \bar{\rho} \\ \frac{d\bar{p}}{d\bar{r}} &= -\frac{GM\bar{\rho}}{\bar{r}^2}\left(1+\frac{\bar{p}}{\bar{\rho}}\right)\left(1+\frac{4\pi\bar{r}^3\bar{p}}{M}\right)\left(1-\frac{2GM}{\bar{r}}\right)^{-1} \end{aligned}}$$

- Unknowns: $(M(\bar{r}), \bar{\rho}(\bar{r}))$

$$\Rightarrow \quad A = 1 - \frac{2GM}{\bar{r}}$$

$$\frac{B'}{B} = -2\,\frac{\bar{p}'}{\bar{\rho}+\bar{p}}$$

- Plug $\bar{p} = \bar{\sigma}\bar{\rho}$ into (*) $\Rightarrow$

EXACT SOLUTION!!

EXACT TOV SOLUTION:

- Let $\gamma = \frac{1}{2\pi G}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)$

- **Solution:**

$$\boxed{\begin{aligned} \bar{p} &= \bar{\sigma}\,\bar{\rho} \\ \bar{\rho}(\bar{r}) &= \frac{\gamma}{\bar{r}^2} \\ M(\bar{r}) &= 4\pi\gamma\bar{r} \\ A(\bar{r}) &= 1-8\pi G\gamma \equiv \text{Const.} \\ B(\bar{r}) &= \bar{r}^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}} \end{aligned}}$$

$\bar{\sigma} = \frac{1}{3} \Rightarrow \gamma = \frac{3}{56\pi G}$

- **Note:** $\bar{\sigma} \to 0 \Rightarrow A \to 1,\ B \to 1$

- **Note:** $\bar{\rho}(0) = \infty,\ \bar{p}(0) = \infty \Rightarrow$

"∞ pressure at $\bar{r}=0$ required to hold the configuration up"

The Shock-Wave Solution:

$$\bar{r}(t) = \pm\sqrt{18\pi G \gamma}\,(1+\sigma)(t-t_0) + \bar{r}_0$$

$$\rho(t) = \frac{3\gamma}{\bar{r}(t)^2}$$

Physical Review-D 1995

$$R(t) = R_0 \left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{\frac{2}{3+3\sigma}}$$

$$r(t) = \bar{r}_0 R_0^{-1} \left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{\frac{1+3\sigma}{3+3\sigma}}$$

- Note: Choose + for outgoing shock
- All are functions of $\bar{r}(t) \Rightarrow$

Singularity in backward time:

$\bar{r}(t_*) = 0$, $R(t_*) = 0$, $\rho(t_*) = \infty$ for

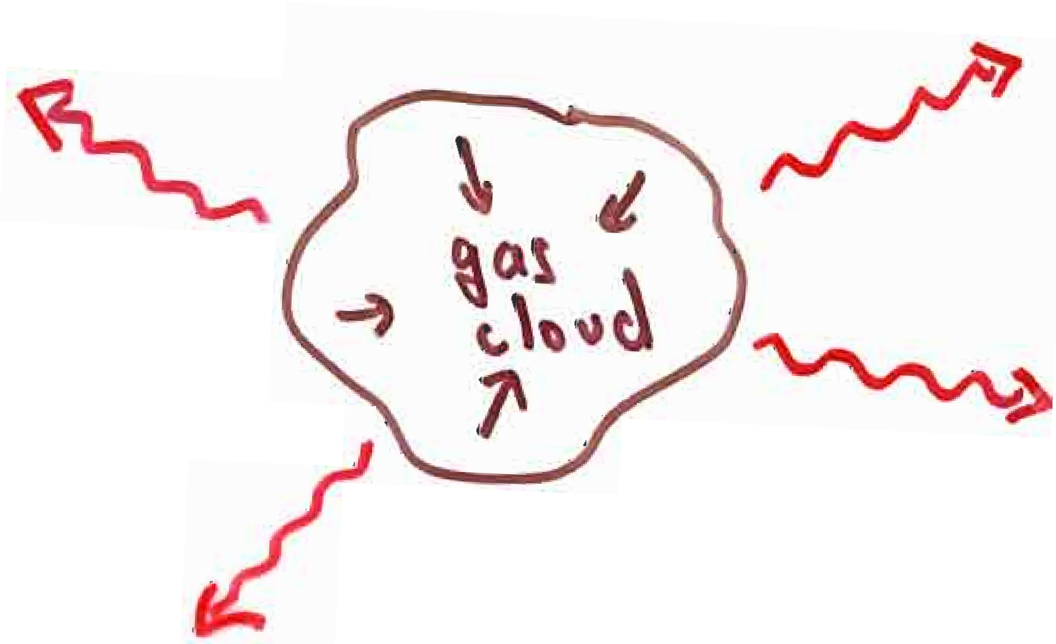
$$t_* = t_0 - \frac{\bar{r}_0}{\sqrt{18\pi G\gamma}\,(1+\sigma)}$$

"The Big Bang with Shock-Wave"

■ **Application**: Scenario for star formation (4)

(Ref: Christodoulou)

- A Star begins as a diffuse gas
- Contraction proceeds by converting grav. potential energy into kinetic energy & radiating energy out thru cloud



- Contraction in the gas cloud "stops" when the mean free path is small enough that light is <u>scattered</u> instead <u>transmitted</u>

⇒ gas cloud drifts toward the static solution that balances the pressure when equation of state is <u>isothermal</u>

⇓ <u>isothermal</u>

- The spherical static soln that balances the pressure when the equation of state is isothermal is a static, singular, isothermal sphere. (The outer solution in our model)
- This has an inverse square density profile $\Rightarrow \bar{\rho} = \frac{c}{r^2}$

  $\rho, p \to \infty$ as $r \to 0$
- Thus $\rho, p \to \infty$ at center of cloud - this ignites thermonuclear reactions $\Rightarrow$ shock wave explosion

Problem: In FRW-TOV shock matching, the shock wave cannot get out beyond

One Hubble Length

$\Leftrightarrow$ Critical distance beyond which the mass behind the shock wave lies inside a Black Hole:

$$\frac{2GM_{(F)}}{\bar{r}} > 1$$

▪ <u>Thm</u> (Sm/Te) Soln's of TOV cannot be continued into a Black Hole unless

$$\rho = p = 0$$

$$\Downarrow$$

To get FRW shock-wave models with shock-wave beyond $\frac{c}{H}$, match to new metric

"TOV metric inside Black Hole"

- To obtain a shock wave beyond one Hubble Length, match FRW to the

TOV metric inside the black hole:

$$ds^2 = B(\bar{t})\,d\bar{t}^2 + \frac{d\bar{r}^2}{1 - \frac{2M(\bar{r})}{\bar{r}}} + \bar{r}^2 d\Omega^2$$

- Assume $\frac{2M}{\bar{r}} > 1$, fluid co-moving

$\Rightarrow$ $\bar{r}$ $\equiv$ timelike variable

$\bar{t}$ $\equiv$ spacelike variable

- Most natural metric to cut off total mass

$M(\bar{r}) \equiv$ constant at each time $\bar{r}$

$\Rightarrow$ Finite mass @ each fixed time!

Some Interesting Aspects of Shock-Matching

- $\bar{r} \equiv$ arclength distance at FRW $t = \text{const}$

  $\Rightarrow \frac{\partial}{\partial \bar{r}}$ spacelike in FRW $(t, \bar{r})$-coords

  $\Updownarrow$ identified via Shock-Matching

- $\bar{r} \equiv$ timelike coord in TOV metric

  $\Rightarrow \frac{\partial}{\partial \bar{r}}$ timelike in TOV $(\bar{r}, \bar{t})$-coords

No $\not\Leftrightarrow$ as $\frac{\partial}{\partial \bar{r}}$ depends on complementary coordinate

Some Interesting Aspects of Shock-matching

- $M_{FRW}(t,\bar{r}) \equiv$ total mass inside radius $\bar{r}$ at FRW $t=const$

  $\Updownarrow$ identified via shock-matching

- $M_{TOV}(\bar{r}) \equiv$ from timelike component of TOV metric

  $\Rightarrow$ no interpretation as total mass in TOV metric

"No cons. of mass principle inside the Black Hole"

Some Interesting Aspects of Shock-Matching

- Procedure requires shock surface to be non-characteristic rel. to PDE that defines the integrating factor Ψ

Problem: Inside Black Hole $\frac{2M}{r} > 1$

$\exists$ degenerate shock surface that solves the conservation constraint but is everywhere characteristic

$\Rightarrow$ Work with new formulation of conservation constraint