A Locally Inertial Glimm Scheme for General Relativity

Jeff Groah, Blake Temple*

Reference:

Shock Wave Solutions of the Einstein Equations with Perfect Fluid Sources: Existence and Consistency by a Locally Inertial Glimm Scheme

Jeff Groah & B Temple

(Memoirs of AMS - Nov. 2004
& Springer Lecture Notes -)

Introduction

- 1915 Einstein Introduced his Field Equations for G.R.

- 1965 Glimm gave his theory of wave-interactions

Our Project: To put these two theories together

Simplest Setting: Spherically Symmetric Spacetimes

▪ The Central Issue:

"The gravitational metric appears to be singular at shock-waves in the coordinates where the analysis appears feasible"

▪ Assume: Spherical Symmetry + Standard Schwarzschild Coordinates

$$ds^2 = -A(r,t)\,dt^2 + B(r,t)\,dr^2 + r^2 d\Omega^2 \quad (*)$$

$$B = \left(1 - \frac{2M}{r}\right)^{-1}$$

• Einstein Eqn's:

$$\frac{A}{r^2 B}\left\{r\frac{B'}{B} + B - 1\right\} = \kappa A^2\, T^{00}(\rho, P, v) \quad (1)$$

$$-\frac{B_t}{rB} = \kappa AB\, T^{01}(\rho, P, v) \quad (2)$$

$$\frac{1}{r^2}\left\{r\frac{A'}{A} - (B-1)\right\} = \kappa B^2\, T^{11}(\rho, P, v) \quad (3)$$

$$-\frac{1}{rAB^2}\left\{B_{tt} - A'' + \Phi\right\} = \frac{2\kappa r}{B}\, T^{22}(\rho, P, v) \quad (4)$$

$$\Downarrow$$

$$\Phi = -\frac{BA_tB_t}{2AB} - \frac{B}{2}\left(\frac{B_t}{B}\right)^2 - \frac{A'}{r} + \frac{AB'}{rB} + \frac{A}{2}\left(\frac{A'}{A}\right)^2 + \frac{AA'B'}{2AB}$$

Note:

- No derivatives on fluid sources
- (1), (2), (3) are 1st order in A, B
- (4) is 2nd order in A, B

Conclude:

Shocks $\Rightarrow$ T discont.

- (1), (2), (3) 1st order $\Rightarrow$ $A, B \in C^{0,1}$
- (4) 2nd order $\Rightarrow$ (4) can only be satisfied weakly when $A, B \in C^{0,1}$

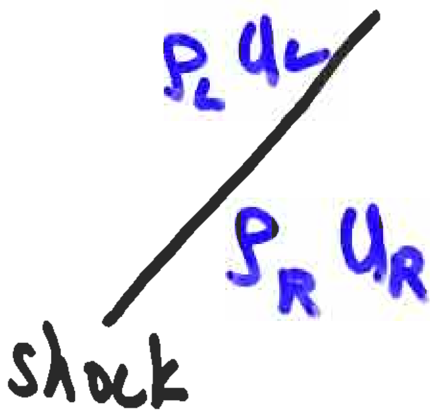
We Prove: $\exists$ weak soln of (1)-(4) with $A, B \in C^{0,1}$

Q: How smooth should shock wave soln's of $G = \kappa T$ be?

$$G = \kappa T$$

$$\partial^2 g = \kappa T(\rho, p, u)$$

Shock $\Rightarrow$ jump discont. in fluid



$G = \kappa T \Rightarrow$ jump in 2nd deriv of g

$\overset{?}{\Rightarrow}$ $g \in C^{1,1}$?

$C^{1,1} \equiv$ 1-derivative Lipschitz Cont.

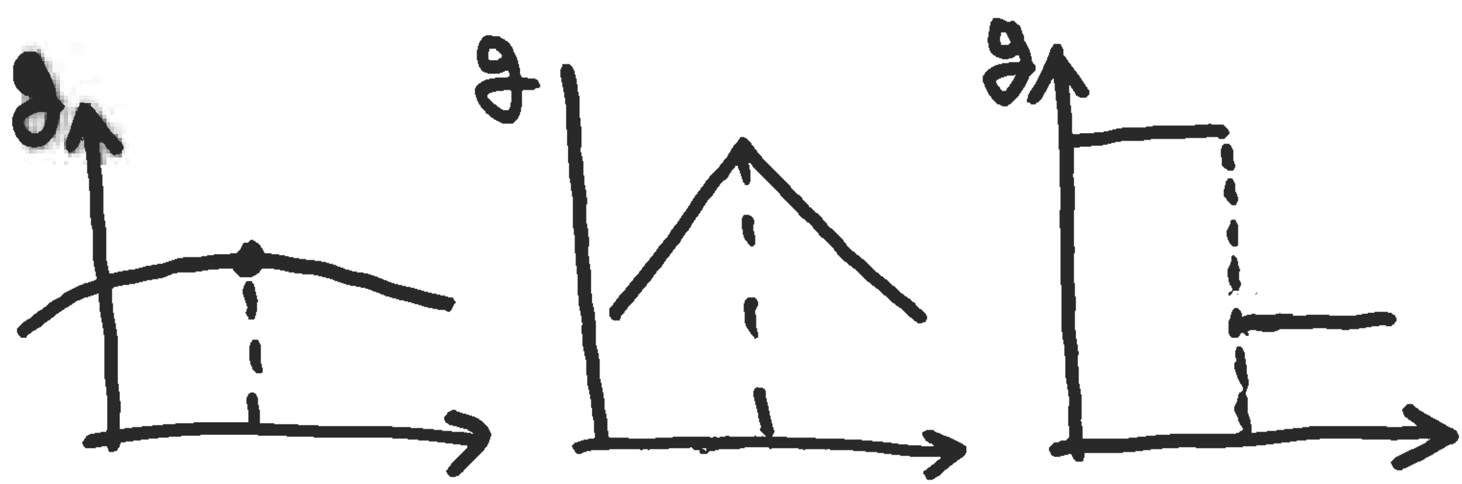
" $\partial^2 g_{ij} = \kappa T(\rho, p, u)$ "

RHS discontinuous

$$\Downarrow$$

LHS has one cont derivative

?? $g \in C^{1,1}$??



$g \in C^{1,1}$ $\quad$ $\partial g \in C^{0,1}$ $\quad$ $\partial^2 g$ jump disc.

- **Conclude**: Our solutions of the Einstein-Euler equations are only $C^{0,1}$ at shocks

$$\Downarrow$$

Soln's are one degree less smooth than general theory tells us they should be

- Open Question: Is there a coordinate transformation that smooths a $C^{0,1}$ weak solution to a $C^{1,1}$ strong solution of $G = \kappa T$?

- For single shock surfaces in $\mathbb{R}^4$ the answer is Yes

Ref: Israel / Smoller-Te

Open Problem: Given a $C^{0,1}$ shock-wave soln of $G = \kappa T$, does $\exists$ coord. trans.

$$x \longrightarrow y$$

such that

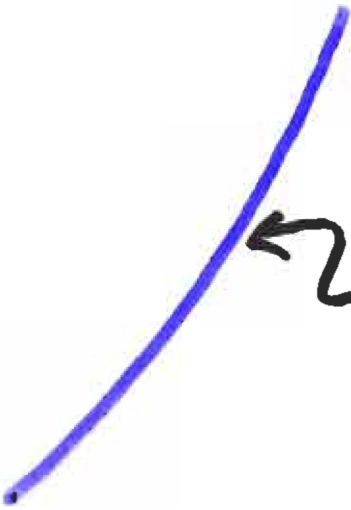
$$g_{ij}(x) \in C^{0,1}$$

but

$$g_{\alpha\beta}(y) = \frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta} \in C^{1,1} \; ?$$

- For a <u>single</u> <u>shock</u> <u>surface</u>

we know its <u>true</u>



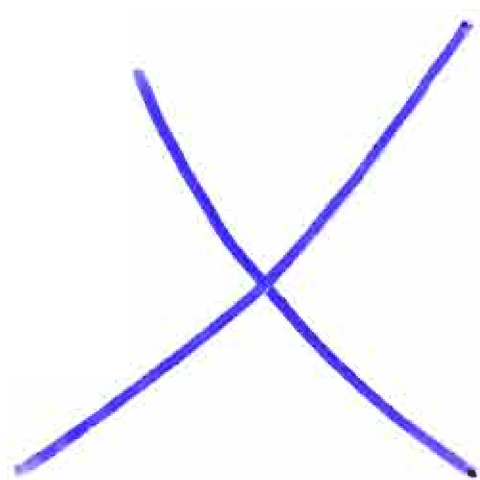
$\Sigma^3 \subseteq \mathbb{R}^4$

Gaussian normal coordinates $\Rightarrow$

$$g \in C^{0,1} \Rightarrow g \in C^{1,1}$$

Ref: Israel, Smoller-Te

- Q: what about when 2 shocks cross?



Open Problem

c.f. Hawking & Ellis

Theorem 1 *Let Σ denote a smooth, 3-dimensional shock surface in spacetime with spacelike normal vector* $\mathbf{n}$. *Assume that the components g_{ij} of the gravitational metric g are smooth on either side of Σ, (continuous up to the boundary on either side separately), and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are equivalent:*

(i) $[K] = 0$ at each point of Σ.

(ii) The curvature tensors R^i_{jkl} and G_{ij}, viewed as second order operators on the metric components g_{ij}, produce no delta function sources on Σ.

(iii) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of P, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates.

(iv) For each $P \in \Sigma$, there exists a coordinate frame that is locally Lorentzian at P, and can be reached within the class of $C^{1,1}$ coordinate transformations.

Any One of These $\Rightarrow$ Rankine-Hugoniot Jump Cndt's

$$[G]^\sigma_i \, n_\sigma = 0$$

[Sm/Te ~'94]

■ To Construct $C^{0,1}$ weak solutions of (1)-(4), we introduce a "locally inertial Glimm Scheme"

- This is satisfying because Einstein's theory of gravity is fundamentally a locally inertial theory

- Even tho physics is most naturally expressed in a locally inertial coordinate system, analysis can only be done in global coordinate systems that hide the locally inertial simplicity

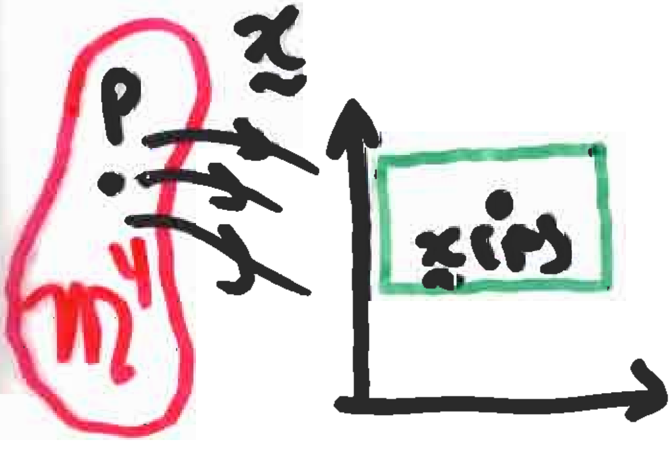
General Relativity as a locally inertial theory of gravity

(Towards a "Locally Inertial Glimm Scheme")

$$G = \kappa T$$

$$\partial^2 g_{ij} = \kappa T(\rho, p, u)$$

$\underset{\sim}{x} : \mathcal{M}^4 \to \mathbb{R}^4$ locally inertial @ P



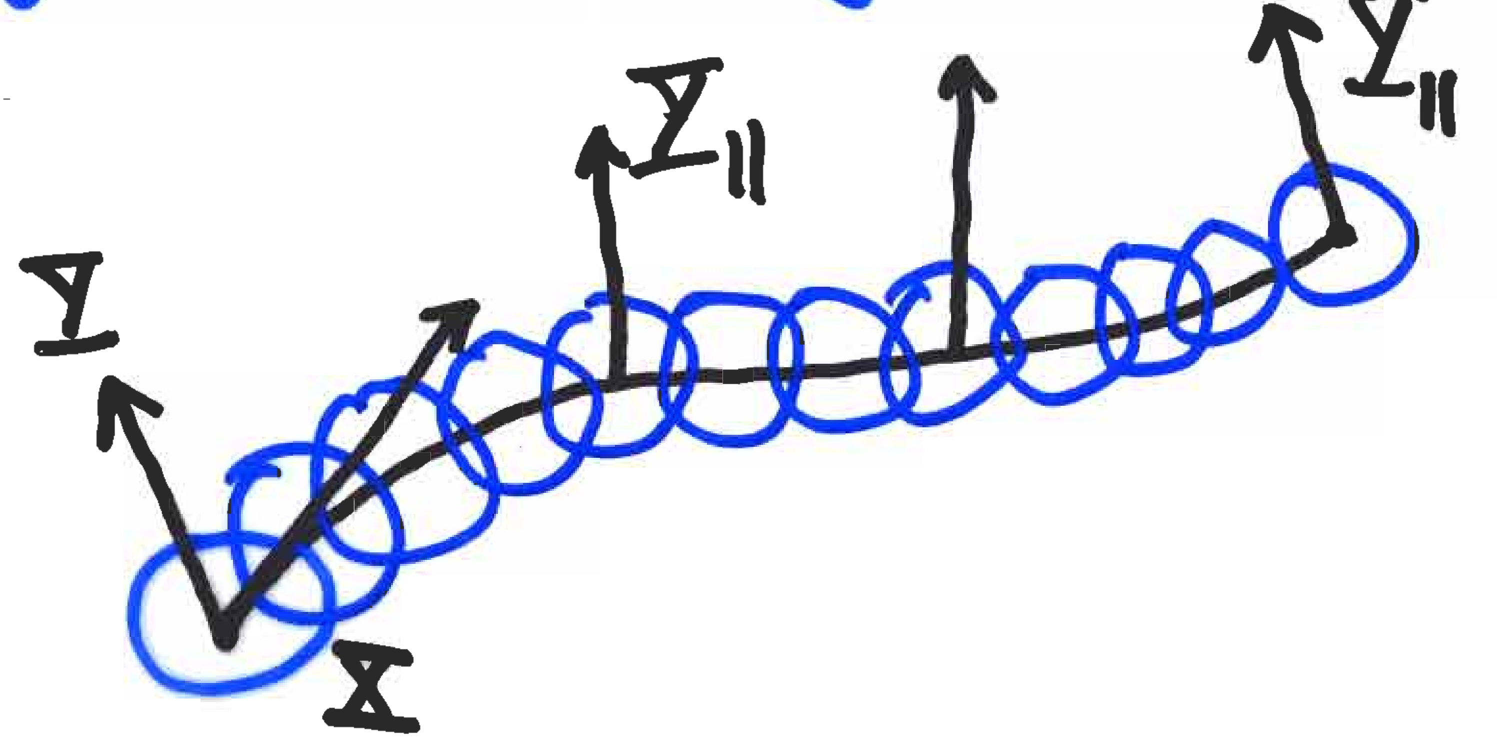
$$g_{ij}(P) = \mathrm{diag}(-1,1,1,1)$$

$$g_{ij,k}(P) = 0$$

- The metric tells how the local frames are tied together
- Conversely — if you have equations that tell you how the local frames are tied together, then you can reconstruct the metric

E.G. ‖-translation & geodesic motion by local frames —

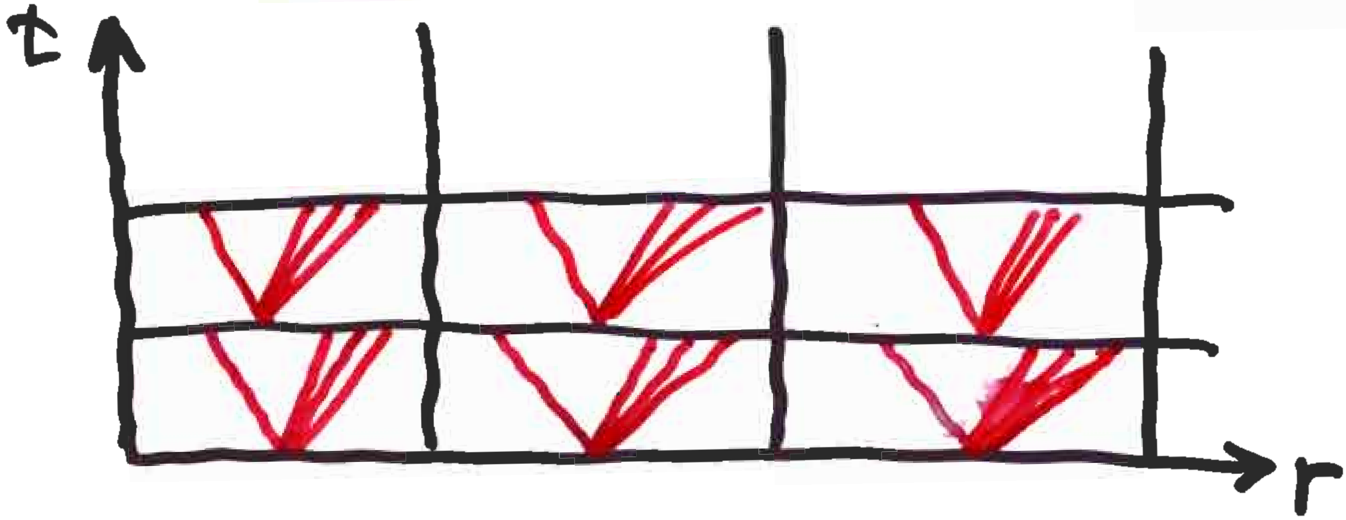
- Physically – $\parallel$-translation/ geodesic motion by local frames



$$\nabla_X Y^i = X(Y)^i + \Gamma^i_{jk} X^j X^k$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma}\left\{-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\right\}$$

- To implement this idea, we look for a **Locally Inertial Glimm Scheme**



- Flat space Compressible Euler Soln's in each grid cell
"locally inertial frame"

- Discontinuities between grid cells that describe how local frames interact $\Rightarrow$ **Curvature**

■ Compressible Euler in Flat Space

$$G = \kappa T$$

$$\mathrm{div}\, G = 0 \Rightarrow \mathrm{div}\, T = 0$$

Covariant divergence

In an inertial frame $T \to T_M$

$$\mathrm{div}\, T_M = 0 \qquad T_M \equiv T\text{-Minkowski}$$

$$\mathrm{div}\left\{(\rho + p) u^i u^j + p\,\eta^{ij}\right\} = 0$$

■ 1-d Compressible Euler

$$\operatorname{div} T = 0$$

$$\Updownarrow$$

$$(T_M^{00})_t + (T_M^{01})_x = 0$$

$$(T_M^{01})_t + (T_M^{11})_x = 0$$

$$u^0 \equiv T_M^{00} = (p + \rho c^2)\frac{c^2}{c^2 - v^2} - p$$

$$u^1 \equiv T_M^{01} = (p + \rho c^2)\frac{cv}{c^2 - v^2}$$

$$T_M^{11} = (p + \rho c^2)\frac{v^2}{c^2 - v^2} + p$$

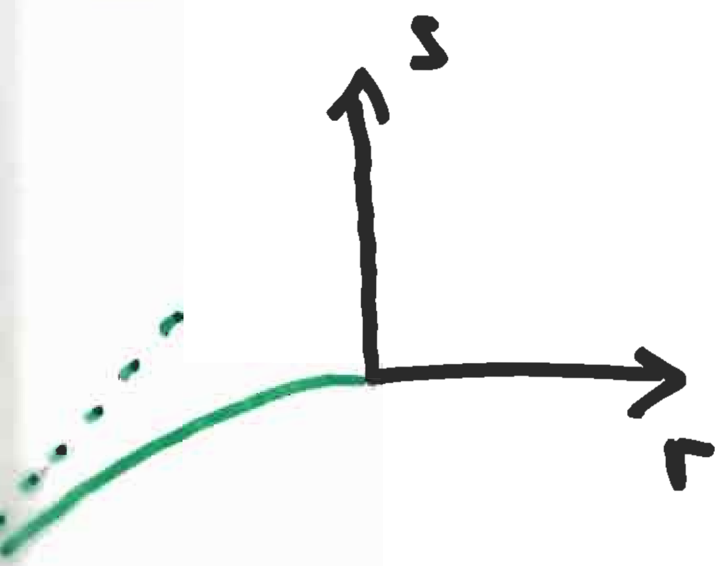
■ Global Existence Theory

(Sm-Te Comm Math Phys 151 (1993))

Assume: $p = \sigma \rho \quad \sigma \equiv \text{const}$

(Eg $p = \frac{c^2}{3}\rho$ Fundamental)

Remarkable Fact – Nishida Property



$$r = \frac{1}{2}\ln\frac{c+v}{c-v} - \frac{K_D}{2}\ln\rho$$

$$s = \frac{1}{2}\ln\frac{c+v}{c-v} + \frac{K_D}{2}\ln\rho$$

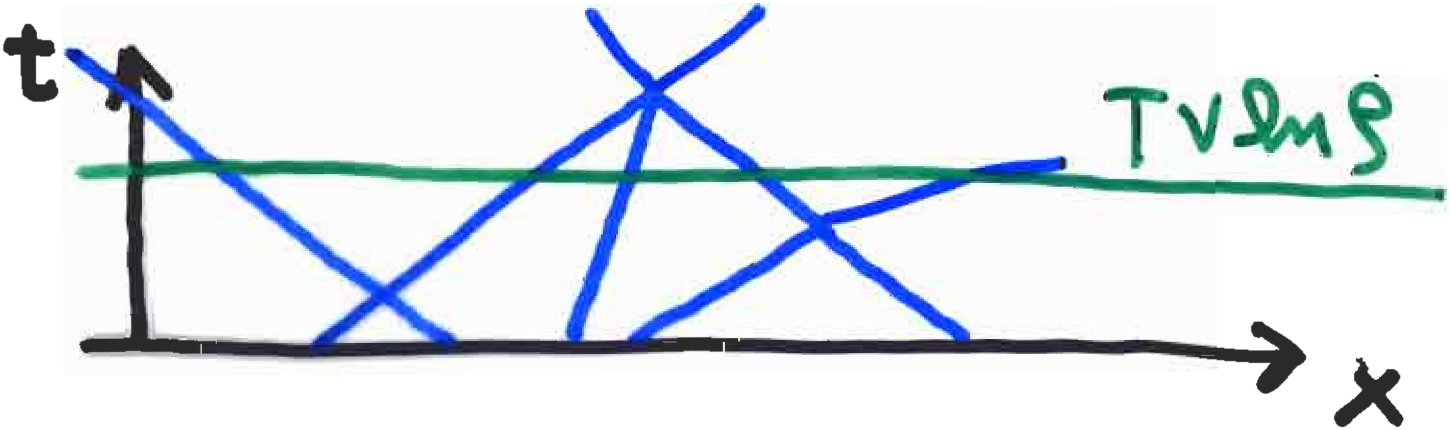
$$K_D = \frac{\sigma c}{c^2+\sigma^2}$$

Shock curves are translation/reflect

Symmetric in rs-plane

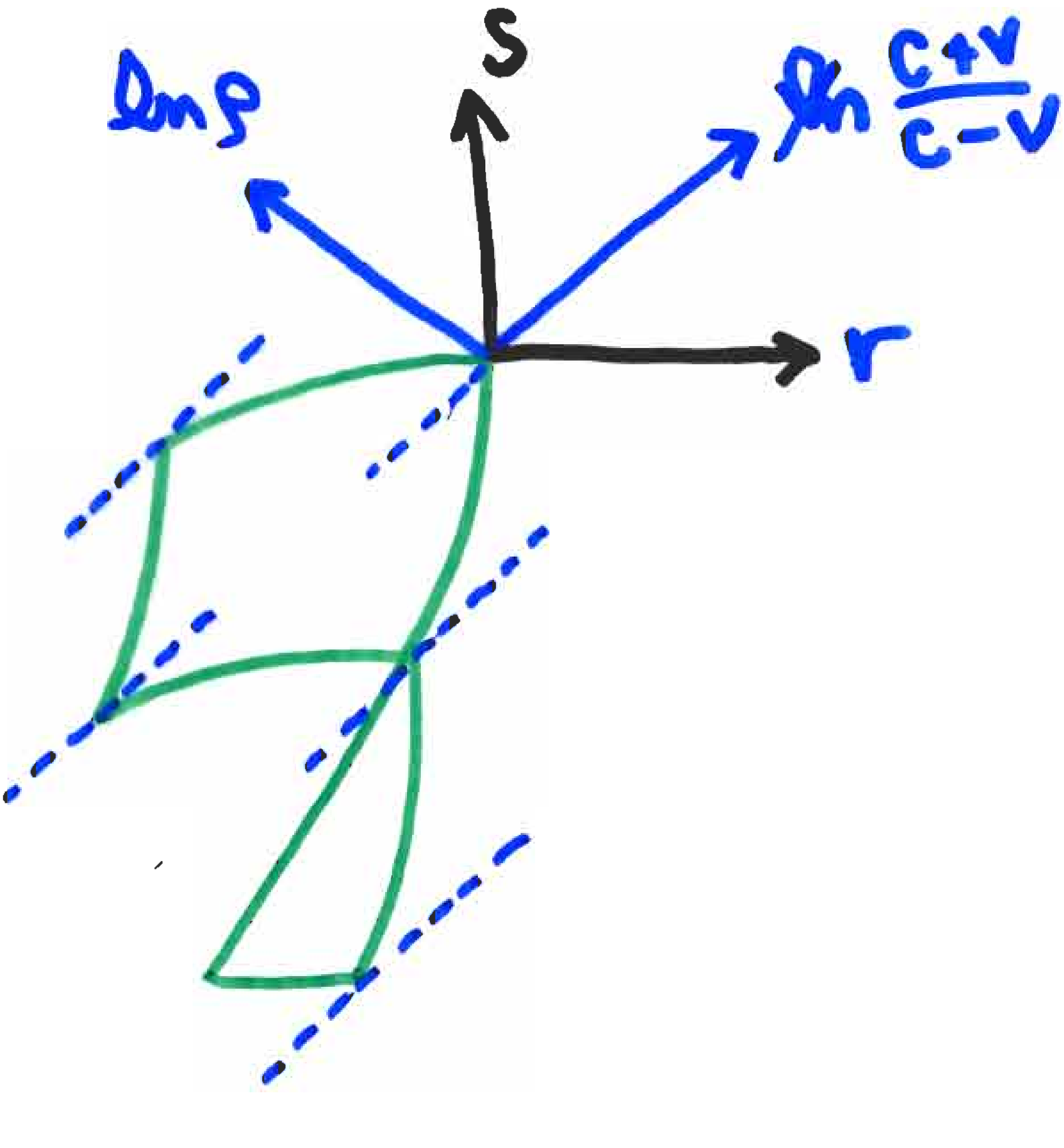
Main Lemma: TV $\ln\rho$ is non-increasing for wave-interactions

$\Rightarrow$ NO VACUUM



$$TV \ln\rho(\cdot,t) \leq TV \ln\rho(\cdot,0)$$

$$TV \ln\rho(\cdot, t) \leq TV \ln\rho(\cdot, 0)$$



Theorem: If $TV \ln \rho_0 < \infty$ on initial data, then $\exists$ global weak (shock-wave) soln of compressible Euler

$$\mathrm{div}\, T_M = 0$$

in (1,1)-Flat Minkowski Space

Q: How to extend to Curved Spacetime?

Conclude: Having a Nishida system for the Flat Space Compressible Euler Equations, Begs the following natural question:

Can we extend this to a theory of strong shock wave interactions in curved spacetime by implementing a

Locally Inertial Glimm Scheme?

Simplest Setting: Spherically Symmetric Spacetimes

THE SIMPLEST SETTING
for
SHOCK-WAVES

- Spherical Symmetry–
  Assume Standard Schwarzschild Coordinates:

$$
\begin{aligned}
g_{ij}dx^i dx^j &= \\
ds^2 &= -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega^2,
\end{aligned}
$$

$$
\begin{gathered}
d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2, \\
\mathbf{x} = (x^0, ..., x^3) \equiv (t, r, \theta, \phi).
\end{gathered}
$$

- Define the mass function $M(r,t)$:

$$B(r,t) \equiv \left(1 - \frac{2M(r,t)}{r}\right)^{-1},$$

- Stress Tensor T :

$$T^{ij} = (\rho c^2 + p)w^i w^j + p g^{ij}, \quad i,j = 0,...,3$$

$$\begin{aligned}
A \cdot T^{00} &= \frac{c^4 + \sigma^2 v^2}{c^2 - v^2}\rho = T_M^{00} = u^0 \\
\sqrt{AB} \cdot T^{01} &= \frac{c^2 + \sigma^2}{c^2 - v^2} c v \rho = T_M^{01} = u^1 \\
B \cdot T^{11} &= \frac{v^2 + \sigma^2}{c^2 - v^2}\rho c^2 = T_M^{11}
\end{aligned}$$

$$\rho c^2 = density, \qquad p = pressure, \qquad v = velocity$$

- Assume Equation of State:

$$p = \sigma^2 \rho$$

$$\sigma = sound\ speed < c = light\ speed.$$

- Plug T^{ij} and g^{ij} into $G = \kappa T$: (MAPLE)

$$
\begin{aligned}
\frac{A}{r^2 B}\left\{ r\frac{B'}{B} + B - 1 \right\} &= \kappa A^2 T^{00} && (1)\\
-\frac{B_t}{rB} &= \kappa A B T^{01} && (2)\\
\frac{1}{r^2}\left\{ r\frac{A'}{A} - (B-1) \right\} &= \kappa B^2 T^{11} && (3)\\
-\frac{1}{rAB^2}\left\{ B_{tt} - A'' + \Phi \right\} &= \frac{2\kappa r}{B} T^{22} && (4)
\end{aligned}
$$

$$
\Phi = -\frac{B A_t B_t}{2AB} - \frac{B}{2}\left(\frac{B_t}{B}\right)^2 - \frac{A'}{r} + \frac{AB'}{rB} + \frac{A}{2}\left(\frac{A'}{A}\right)^2 + \frac{A}{2}\frac{A'}{A}\frac{B'}{B}
$$

- (1)$\equiv$ $M' = \frac{1}{2}\kappa r^2 A T^{00}$, (2)$\equiv$ $\dot{M} = -\frac{1}{2}\kappa r^2 A T^{01}$

- "prime" $\equiv \partial/\partial r$, "dot" $\equiv \partial/\partial t$

$$B = \frac{1}{1 - \frac{2GM}{r}}$$

CONSEQUENCES:

- The equation (1)$\equiv M' = \frac{1}{2}\kappa r^2 AT^{00}$ implies:

$$M(r,t) = M_{r_0} + \frac{\kappa}{2}\int_{r_0}^{r} T_M^{00}(r,t) r^2\, dr$$

- The scalar curvature R satisfies

$$R = (c^2 - 3\sigma^2)\rho$$

- Components of T_M satisfy:

$$\left|T_M^{01}\right| < T_M^{00},$$
$$\frac{\sigma^2}{c^2+\sigma^2} T_M^{00} < T_M^{11} < T_M^{00}$$

- This defines the simplest setting for shock wave propagation in General relativity.

$$T_M^{00} = AT^{00} = u^0$$

Assume:

- Soln's defined <u>outside</u> ball radius r_0

- No Black Hole: $1 \le B(r,0) \le \bar{B} < \infty$

- Finite Total Mass:

$$M_\infty = \frac{\kappa}{2}\int_{r_0}^{\infty} u^0(r,0)\, r^2\, dr < \infty$$

- Finite Supnorm:

$$0 < x\,\rho(x,0) \equiv \tilde{\rho}(x,0) < \infty$$

$$x \equiv r$$

MAIN RESULT:

- Theorem: Assume there exist positive constants L, V and $\bar{v}$ such that $v_0(r)$ and $\rho_0(r) > 0$, $r_0 \leq r < \infty$, satisfy

$$\begin{gathered} TV_{[r,r+L]} ln\, \rho_0(\cdot) < V, \\ TV_{[r,r+L]}\, ln \left(\frac{c+v_0(\cdot)}{c-v_0(\cdot)}\right) < V, \\ |v_0(r)| < \bar{v} < c \end{gathered} \qquad (5)$$

- Then a bounded, weak, (shock wave), solution of the initial-boundary value problem for Einstein equations (1)-(4) exists up to some positive time $T > 0$.

- For $t < T$, the metric functions A and B are Lipschitz continuous ($C^{0,1}$) functions of (r,t), and (5) continues to hold for $t < T$, with adjusted values for V and $\bar{v}$ that are determined from the analysis.

$\lim_{r \to \infty} M(r,t) = M_\infty \equiv const$

COMMENTS:

- The theorem allows for arbitrary numbers of interacting shock waves, of arbitrary strength.

- Note that $\lim_{r\to\infty} M(r,t) = M_\infty$ is a *non-local* condition.

-

$$
\begin{aligned}
(1) \quad &\equiv \quad \frac{A}{r^2 B}\left\{r\frac{B'}{B} + B - 1\right\} = \kappa A^2 T^{00} \\
(3) \quad &\equiv \quad \frac{1}{r^2}\left\{r\frac{A'}{A} - (B-1)\right\} = \kappa B^2 T^{11} \\
(4) \quad &\equiv \quad -\frac{1}{rAB^2}\left\{B_{tt} - A'' + \Phi\right\} = \frac{2\kappa r}{B} T^{22}
\end{aligned}
$$

$\Rightarrow A(r,t)$ and $B(r,t)$ are at most Lipschitz continuous at shocks.

$\Rightarrow$(4) only satisfied in the weak sense of the theory of distributions.

Note:

- $B \to \infty \iff$ Black Hole

$$B = \frac{1}{1 - \frac{2M}{r}}$$

- Conclude: Since Black Holes can form, we cannot do better than a local exist. theory

- $\rho \to \infty \iff$ naked singularity (except $\sigma = \frac{1}{3}$)

FORMULATION AS A SYSTEM OF CONSERVATION LAWS WITH SOURCES

- Theorem (Groah, Te): The following equivalencies hold in weak sense:

$$
\begin{aligned}
G = \kappa T \quad &\Longleftrightarrow \quad (1) + (2) + (3) + (4) \\
&\Longleftrightarrow \quad (1) + (3) + DivT = 0
\end{aligned}
$$

- Weak equivalence for A, B Lipschitz continuous, T bounded measurable.

- For $(1) + (3) + DivT = 0$, equation (2) holds as a constraint: it holds on weak solutions so long as it holds on the boundary $r = r_0$.

- Reference: Groah, Te: *A shock-wave formulation of the Einstein equations*, (~~to appear~~), *Methods and Applications of Analysis*

Handbook of Mathematical Fluid Dynamics Springer Book

- Corollary: An equivalent system:
$\left[New\ Twist:\ u=(u^0,u^1)=(T^{00}_M,T^{01}_M)\right]$

$$\begin{aligned} u_t + f(\mathbf{A},u)_x = g(\mathbf{A},u,x) &\quad \Longleftrightarrow \quad DivT=0 \\ \mathbf{A}' = h(\mathbf{A},u,x) &\quad \Longleftrightarrow \quad (1)+(3) \end{aligned}$$

$$\begin{aligned} \mathbf{A} &= (A,B) \\ f(\mathbf{A},u) &= \sqrt{\frac{A}{B}}\left(T^{01}_M,T^{11}_M\right) \\ g(\mathbf{A},u,x) &= \left(g^0,g^1\right) \\ h(\mathbf{A},u,x) &= \left(h^0,h^1\right) \end{aligned}$$

- Note: u independent of $\mathbf{A}=(A,B)$

$$u_t + f(\mathbf{A}, u)_x = g(\mathbf{A}, u, x) \quad \Longleftrightarrow \quad DivT = 0$$

$$\mathbf{A}' = h(\mathbf{A}, u, x) \quad \Longleftrightarrow \quad (1) + (3)$$

$$\begin{aligned}
g^0 &= -\frac{2}{x}\sqrt{\frac{A}{B}}T_M^{01} \\
g^1 &= -\frac{1}{2}\sqrt{\frac{A}{B}}\left\{\frac{4}{x}T_M^{11} + \frac{(B-1)}{x}(T_M^{00} - T_M^{11}) \right. \\
&\qquad \left. + 2\kappa x B(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4xT^{22}\right\},
\end{aligned}$$

$$\begin{aligned}
h^0 &= \frac{(B-1)A}{x} + \kappa x A B T_M^{11} \qquad T^{22} = \frac{p}{x^2} = \frac{\sigma^2\rho}{x^2} \\
h^1 &= -\frac{(B-1)B}{x} + \kappa x B^2 T_M^{00}, \qquad \zeta = \frac{\sigma^2 c^2}{(c^2 - \sigma^2)^2}
\end{aligned}$$

$$\frac{T_M^{11}}{T_M^{00}} = \frac{2\zeta + 1}{2\zeta}\left\{1 - \sqrt{1 - \frac{4\zeta}{(2\zeta+1)^2}\left(\zeta + \left[\frac{T_M^{01}}{T_M^{00}}\right]^2\right)}\right\}$$

- Remarkably: Change of variables

$$T \to u = T_M$$

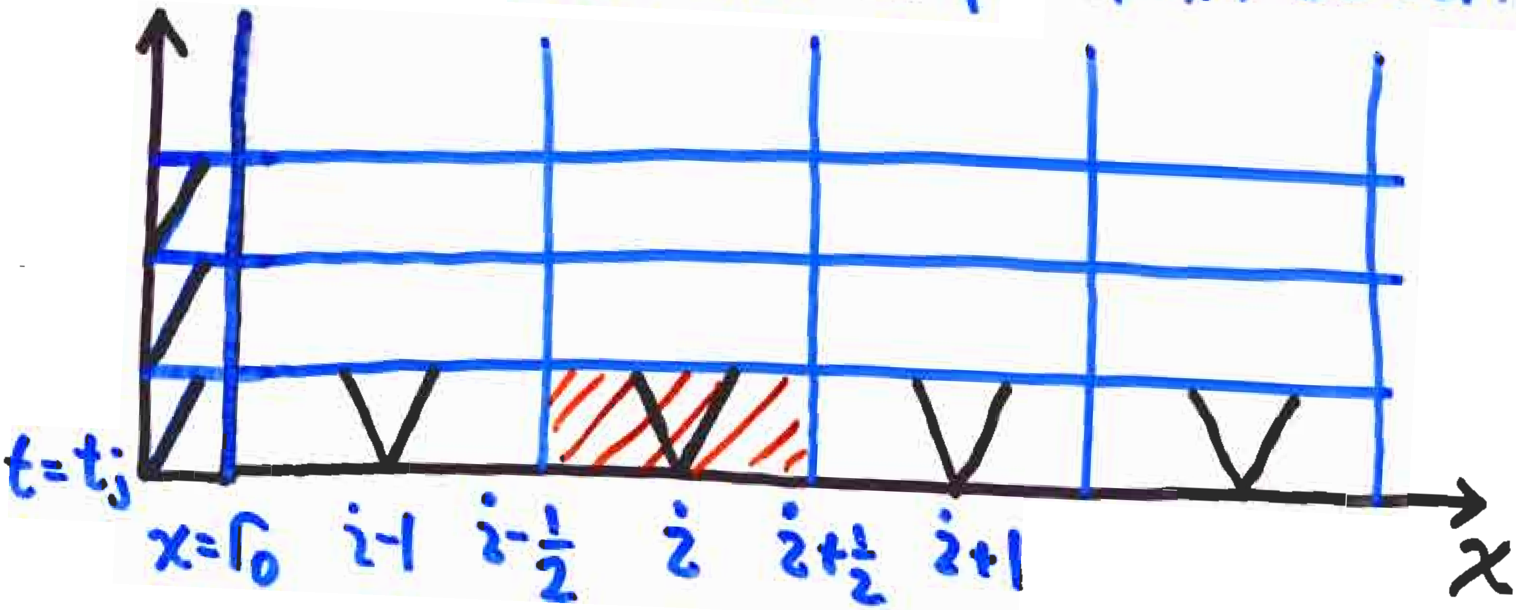
$\Rightarrow$ Time derivatives A_t and B_t *cancel out*!

- $DivT = 0$ reads:

$$\begin{aligned} 0 &= T^{00}_{,0} + T^{01}_{,1} + \frac{1}{2}\left(\frac{2A_t}{A} + \frac{B_t}{B}\right) T^{00} \\ &\quad + \frac{1}{2}\left(\frac{3A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) T^{01} + \frac{B_t}{2A} T^{11} \end{aligned}$$

$$\begin{aligned} 0 &= T^{01}_{,0} + T^{11}_{,1} + \frac{1}{2}\left(\frac{A_t}{A} + \frac{3B_t}{B}\right) T^{01} \\ &\quad + \frac{1}{2}\left(\frac{A'}{A} + \frac{2B'}{B} + \frac{4}{r}\right) T^{11} \\ &\quad + \frac{A'}{2B} T^{00} - 2\frac{r}{B} T^{22} \end{aligned}$$

☒ PROOF: Fractional Step Glimm Scheme



$$\underbrace{}_{R_{ij}:\ \underset{\sim}{A} = \underset{\sim}{A}_{ij}}$$

- Stagger Discontinuities in $\underset{\sim}{A}$ with Discontinuities in u

- Solve RP for $\frac{1}{2}$ - timestep

- Solve ODE for $\frac{1}{2}$ - time step

Ref: Luskin - Te ~82
Groah - Thesis

PROOF: Fractional Step Glimm Scheme

- Define grid $x_i, t_j, \quad i,j = 0,1,2,\ldots, \quad x_0 = r_0$.

- Stagger discontinuities in $\mathbf{A}$ with discontinuities in u:

  - Choose: $\mathbf{A} \equiv \mathbf{A}_{ij}$ in Grid Rectangle R_{ij}
  - $$R_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_j, t_{j+1}]$$

- Discontinuities in u are set at (x_i, t_j):
  $\Rightarrow$ Riemann problem posed
  at the bottom center of R_{ij}

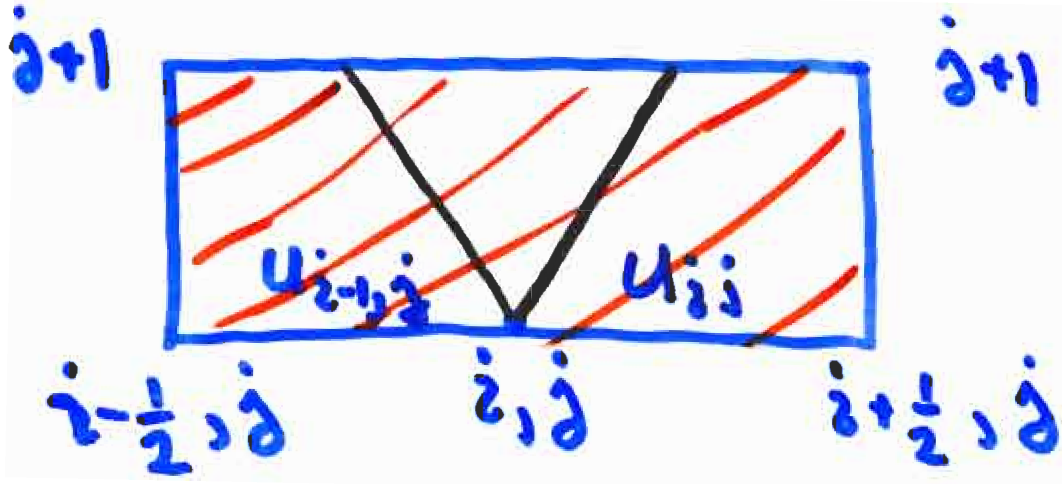
- Solve $u_t + f(\mathbf{A}_{ij}, u)_x = 0$ in R_{ij} for $\frac{1}{2}$-step

- Solve $u_t = g(\mathbf{A}_{ij}, u, x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$ for $\frac{1}{2}$-step

- Solve $\mathbf{A}' = h(\mathbf{A}, u, x), \; \mathbf{A}(r_0) = \mathbf{A}_{r_0}$, *at* $t = t_{j+1}$

Ref: Luskin-Te 1982
Groah - Thesis

- Grid Rectangle R_{ij} $\underset{\sim}{A} = A_{ij}$



- Solve RP for $\frac{1}{2}$- timestep

$$u_t + f(\underset{\sim}{A}_{ij}, u)_x = 0$$

$$u = \begin{cases} u_{i-1,j} & x \leq x_i \\ u_{ij} & x \geq x_i \end{cases}$$

- Solve ODE for $\frac{1}{2}$- timestep

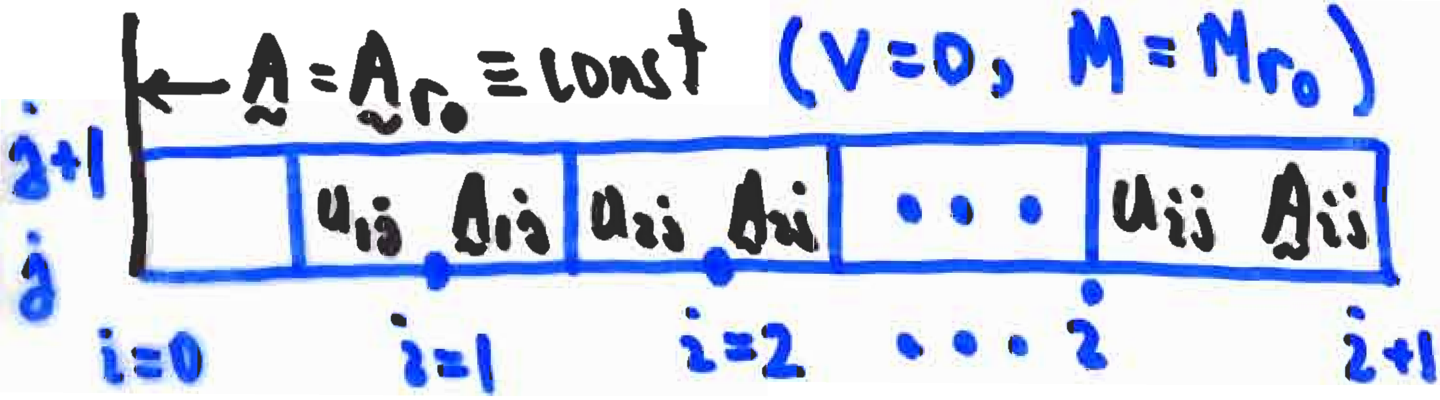
$$u_t = g(\underset{\sim}{A}_{ij}, u, x) - \nabla_{\underset{\sim}{A}} f \cdot A'$$

$$u(0) = u_{ij}^{RP}$$

- At end of timestep, Solve

$$\underset{\sim}{A}' = h(\underset{\sim}{A}, u, x)$$

$$\underset{\sim}{A}(r_0) = \underset{\sim}{A}_{r_0}$$



- Use values from fractional step @ $t = t_{j+1}^-$ to obtain $\underset{\sim}{A}$ @ $t = t_{j+1}^+$

- Rediscretize by Glimm's Random Choice Sequence

The Riemann Problem Step:

- The RP in R_{ij}

$$u_t + f(\mathbf{A}_{ij}, u)_x = 0,$$

$$u_0(x) = \begin{cases} u_L = u_{i-1,j} & x < x_i, \\ u_R = u_{ij} & x > x_i, \end{cases}$$

- $\mathbf{A} = \mathbf{A}_{ij} \Rightarrow R_{ij}$ defines a "locally inertial coordinate frame"

- Presence of $\mathbf{A}_{ij}$ alters speeds but not states
  Riemann Problem $\Longleftrightarrow$ Special Relativity

- REF: J. Smoller and B. Temple, *Global solutions of the relativistic Euler equations*, Comm. Math. Phys., **157**(1993), p.67-99.

Riemann Problem Step

$u_t + f(\underset{\sim}{A}, u)_x = 0 \quad \underset{\sim}{A} = (A, B) = \text{const}$

$$\begin{pmatrix} T_M^{00} \\ T_M^{01} \end{pmatrix}_t + \sqrt{\frac{A}{B}} \begin{pmatrix} T_M^{01} \\ T_M^{11} \end{pmatrix}_x = 0$$

Const

$\bar{x} = \sqrt{\frac{A}{B}}\, x$

$$\Leftrightarrow \quad \begin{pmatrix} T_M^{00} \\ T_M^{01} \end{pmatrix}_t + \begin{pmatrix} T_M^{01} \\ T_M^{11} \end{pmatrix}_{\bar{x}} = 0$$

Compressible Euler in Flat Space

$p = \sigma \rho$

$\Rightarrow$ The affect of $\sqrt{\frac{A}{B}}$ is to change the speeds of waves not the states

$\Rightarrow$ TV $\ln \rho$ non-increasing

- Main Point: $TV \ln \rho(\cdot, t)$ is non-increasing on RP step of method.

  REF: Nishida, *Global solution for an initial boundary value problem of a quasilinear hyperbolic system*, Proc. Jap. Acad., **44**(1968), pp. 642-646.

- Conclude: Fractional Step Method≡ "Covariant version of Glimm's method"

- The boundaries between the "locally inertial coordinate frames" are the discontinuities in $\mathbf{A}$ along sides, top and bottom, of grid rectangles.

"Locally Flat Method"

The ODE Step:

$$u_t = g - \mathbf{A}' \cdot \nabla_{\mathbf{A}} f \equiv G(\mathbf{A}, u, x), \quad \mathbf{A} \equiv \mathbf{A}_{ij}, \ x = x_i$$

$$\begin{aligned}
G^0 &= -\frac{1}{2}\sqrt{\frac{A}{B}} T_M^{01} \left\{ \frac{2(B+1)}{x} - \kappa x B \left(T_M^{00} - T_M^{11} \right) \right\} \\
G^1 &= -\frac{1}{2}\sqrt{\frac{A}{B}} \left\{ \frac{4}{x} T_M^{11} + \frac{B-1}{x} \left(T_M^{00} + T_M^{11} \right) + \right. \\
&\qquad \left. \kappa x B \left[T_M^{00} T_M^{11} - 2 \left(T_M^{01} \right)^2 + \left(T_M^{11} \right)^2 \right] - 4 x T^{22} \right\}
\end{aligned}$$

- g accounts for the dicontinuities in TIME (along top and bottom of R_{ij})

- $-\mathbf{A}' \cdot \nabla_{\mathbf{A}} f$ accts for discontinuties in SPACE (along sides of R_{ij})

- Proof of convergence of the residual demonstrates that this interpretation is correct.

The ODE has nice properties in (ρ, v)-plane:

$$u_t = g - \mathbf{A}' \cdot \nabla_{\mathbf{A}} f$$

$$\begin{aligned}
\dot{\rho} &= \frac{\kappa\sqrt{AB}x}{2}\left[\frac{(c^2+\sigma^2)^2 vc}{c^4-\sigma^2 v^2}\right]\rho\left\{\rho-\rho_1\right\}, \\
\dot{v} &= -\frac{\kappa\sqrt{AB}x}{2}\left[\frac{(c^4-v^4)\sigma^2 c}{c^4-\sigma^2 v^2}\right]\left\{\rho-\rho_2\right\},
\end{aligned}$$

$$\rho_1 = \frac{4}{\kappa B(c^2+\sigma^2)x^2},$$

$$\rho_2 = \frac{4v^2\sigma^2 - (B-1)(c^4-\sigma^2 v^2)}{\kappa B(c^2+v^2)\sigma^2 c^2 x^2},$$

$$\rho_2 < \frac{4v^2\sigma^2}{\kappa B(c^2+v^2)\sigma^2 c^2 x^2} < \rho_1, \quad -c < v < c$$

ODE Phase Portrait:

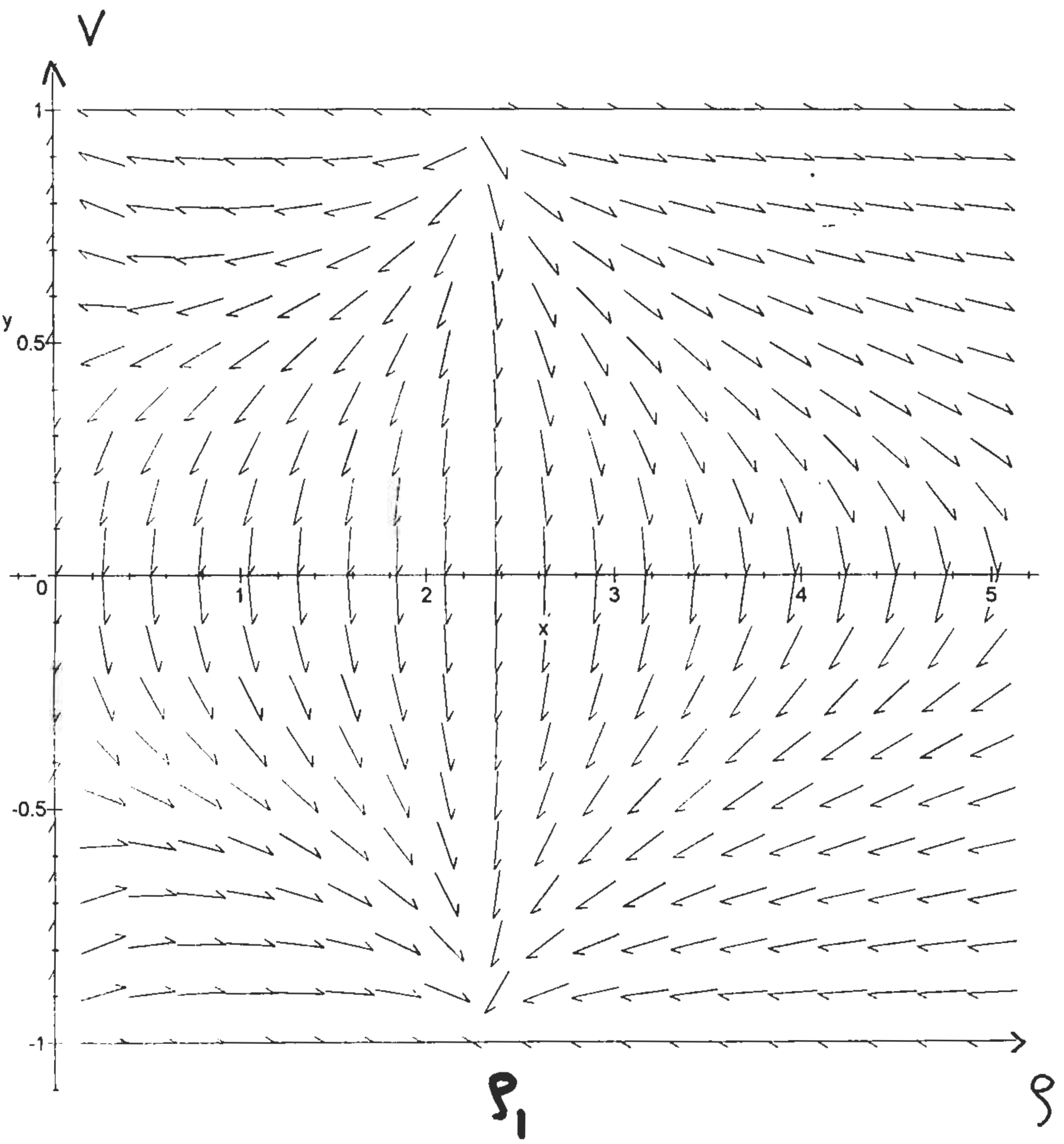
- Autonomous system at each (i,j)

- $\rho > 0$, $|v| < c$ is an invariant region

- $\rho \leq \rho_1$ is a bounded invariant region
  $\rho \geq \rho_1$ is an unbounded invariant region:
  ($\rho = \rho_1$ is a solution)

- **Solutions** exist/bounded **for all** $t \geq 0$
  (ODE's are quadratic **in** ρ!)

```
ith(DEtools):
:=1: A:=1: B:=1.3: kappa:=1: sigma:=1/sqrt(3): r:=1:
fieldplot([diff(x(t),t)=1/2*(sigma^2+c^2)*(x(t)*r^2*kappa*B*(sigm
^2+c^2)-4)*sqrt(A/B)*x(t)*c*y(t)/((c^4-y(t)^2*sigma^2)*r),diff(y(
),t)=-1/2*sqrt(A/B)*(c^2-y(t)^2)*((B-1)*(c^4-y(t)^2*sigma^2)-4*y(
)^2*sigma^2+kappa*x(t)*r^2*B*c^2*sigma^2*(c^2+y(t)^2))/(c*r*(c^4-
(t)^2*sigma^2))],[x(t),y(t)],t=-2..2,x=0..5,y=-c..c,arrows=small)
```




```
with(DEtools):
```

Bounds for Fractional Step Method:

- RP+ODE $\Rightarrow$ approx solns defined for all t *so long as* the CFL-condition holds

$$\frac{\Delta x}{\Delta t} \geq Max\left\{2\sqrt{\frac{A}{B}}\right\}$$

- We show: CFL bound depends only on: $\|B\|_\infty$, $\|S\|_\infty$, $S \equiv S(x,t) = x\rho(x,t)$.

- We prove: All norms bounded by $\|B\|_\infty$, $\|S\|_\infty$, and $\|TV_L \ln \rho(\cdot,t)\|_\infty$

- Thm: Solution extends to first time T at which one of these three norms tends to infinity.

Interpretation:

- $B \to \infty \iff$ *Black Hole*

$B = \dfrac{1}{1-\frac{2M}{r}}$

- $\rho \to \infty \iff$ *Naked Singularity*,

$$(R = \{c^2 - 3\sigma^2\}\rho)$$

- Open Problem: Can
$$B,\ \|S\|_\infty,\ \|TV_L \ln \rho(\cdot, t)\|_\infty \to \infty$$
some *other way*?

- Open problem: Do there exist coordinate transformations that smooth the metric components of these solutions from the smoothness class $C^{0,1}$ up to the class $C^{1,1}$?

- Such a transformation would map weak solutions $\Rightarrow$ strong solutions.

Strategy:

- We have uniform estimates for RP and ODE steps *separately*, but not under interation.

- Basic idea: RP step preserves $TV_L \ln \rho \Rightarrow$ need $\Delta TV_L \ln \rho$ for ODE is $O(\Delta t) \Rightarrow$ compactness by Oleinik/Glimm compactness argument.

- Main Technical Problem: Growth of $TV_L \ln \rho$ coupled to growth of M_∞

$$M_\infty = \frac{\kappa}{2} \int_{r_0}^{\infty} u(r,t) r^2 dr$$

a *non-local* condition.

- Difficulty in keeping track of order of choice of constants.

Main Estimate:

- Assume that for $t < T_0$, $u_{\Delta x}, \mathbf{A}_{\Delta x}$ satisfy

$$\begin{aligned} M_{\Delta x}(x,t_j) &\leq \bar{M} \\ B_{\Delta x}(x,t_j) &\leq \bar{B} \\ 0 < S_{\Delta x}(x,t_j) &\leq \bar{S} \\ \left|v_{\Delta x}(x,t_j)\right| &\leq \bar{v} \end{aligned}$$

- Assume that there exists constants L, V_0 such that $|x_{i_2} - x_{i_1}| \leq L \Rightarrow$

$$\sum_{i_1 \leq i \leq i_2,\ p=1,2} |\gamma_{i0}^p| < V_0$$

Conclude: (A) Total variation bound:

$$\sum_{i_1 \leq i \leq i_2,\ p=1,2} |\gamma_{ij}^p| < 2\bar{V}_*,$$

$$t_j \leq T_2 = \left(\frac{1}{G_2}\right) \frac{\bar{V}_*}{\{2\bar{V}_* + H(2\bar{V}_*)\}}$$

(B) Conclude: L^1_{loc} bounds: $t_j \leq \text{Min}\{T_0, T_2\}$

$$\int_{x_{i_1}}^{x_{i_2}} \|\underset{\sim}{z}_{\Delta x}(x, t_{j_2}) - \underset{\sim}{z}_{\Delta x}(x, t_{j_1})\| dx \leq G_2 |t_{j_2} - t_{j_1}|$$

$\underset{\sim}{z} = (z, w) \equiv$ plane of R.I.'s

(C) Conclude: Supnorm bounds: $t_j \leq \text{Min}\{T_0, T_2\}$

$$\| z_{ij} - z_{i;j,0} \| \leq F_0 (G_2 \cdot t_j)$$

Main Point: Dependence on $\bar{M}, \bar{B}, \bar{S}, \bar{V}$ is only thru

$$G_2 = G_2(\bar{M}, \bar{B}, \bar{S}, \bar{V})$$

Idea: estimates are strong enough to control $\bar{M}, \bar{B}, \bar{S}, \bar{v}$ for short time

- **The Technique** = Successively replace Supnorm bound assumptions by Small time assumptions: (in order)

$$V \le \bar{V} \Leftarrow t < T_V(\bar{M}, \bar{B}, \bar{S}, \bar{V})$$
$$S \le \bar{S} \Leftarrow t < T_S(\bar{M}, \bar{B}, \bar{S}, \bar{V})$$
$$M \le \bar{M} \Leftarrow t < T_M(\quad " \quad)$$
$$B \le \bar{B} \Leftarrow t < T_B(\quad " \quad)$$

- Not so easy as you might think -

- **Main Issue:** Control Total Mass

$$M_{ij} = M_{r_0} + \frac{k}{2}\underbrace{\int_{r_0}^{x_i} u^0_{\Delta x}(r,t_j)\, r^2\, dr}_{\text{total mass between } r_0 \,\&\, x_i \text{ at } t=t_j}$$

$$B_{ij} = \frac{1}{1 - \frac{2M_{ij}}{x}}$$

Need: $M_{ij} \leq \bar{M}, \; B_{ij} \leq \bar{B}$

Step ① Bound $M_{\infty j}$

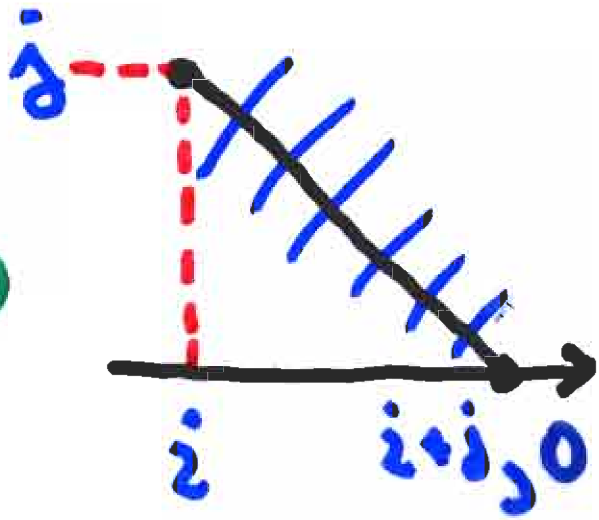
- On Cons. Law Step

$$(TV\ln\rho)_{j+1} \leq (TV\ln\rho)_j \quad (TV)$$

- Use (TV) + order Δt growth on ODE step to get

$$\ln\rho_{ij} - \ln\rho_{i+j,0} \leq F_0(G_0 \cdot t)$$

$$\Rightarrow \rho_{ij} \leq F_0(G_0 t)\rho_{i+j,0}$$



$$\rho_{ij} \le F_0(G_0 t)\,\rho_{i+j,0}$$

$$F_0(\xi) = 2\left(1+\frac{4\xi}{L}\right)\bar{V}_0 + H\left(2\left(1+\frac{4\xi}{L}\right)\bar{V}_0\right) + \xi$$

Curve that relates
Δr to Δs along shock

$$G_0 = G_0(\bar{V}_0, \bar{M}, \bar{B}, \bar{S}, \hat{v})$$

$\Rightarrow$ for t suff small, $F_0(G_0 t)$ can be bded by a LARGE constant indept of $\bar{M}, \bar{B}, \bar{S}, \hat{v}$

- u^0 related to (ρ, v) thru a regular transformation

$$u^0 = u^0(v, \rho)$$

$$\Rightarrow \quad u^0_{ij} \leq F_1(G, t)\, u_{i+j,0}$$

$$\Rightarrow M_{\infty,j} = M_{r_0} + F_1(G,t) \int_{r_0}^{\infty} u^0(r,0)\, r^2\, dr$$

$$\leq F_1(G, t)\, M_{\infty,0}$$

- Conclude —

$$M_{\infty,j} \leq F_1(G_1 \cdot t)\, M_{\infty,0}$$

$F_1(\xi)$ depends only on ∇_0, L

$G_1 = G_1(\nabla_0, \bar{M}, \bar{B}, \bar{S}, \bar{v})$

$\therefore$ choose $\bar{M} = 2 F_1(0)\, M_{\infty,0}$

$\Rightarrow$ for small time $t \in T_M$

$$M_{\infty,j} \leq F_1(G_1 t)\, M_{\infty,0} \leq \bar{M}$$

<u>Problem:</u> $\bar{M} \gg M_{i0} \Rightarrow$ No

control over $B_{ij} = \dfrac{1}{1 - \frac{M_{ij}}{x}}$

Step 2 Improve Estimate for M_{ij}

$$|M_{ij_2} - M_{ij_1}| \le \frac{K}{2}\int_{r_0}^{R} |u^0_{\Delta x}(r,t_{j_2}) - u^0_{\Delta x}(r,t_{j_1})| r^2 dr$$

$$+ \underbrace{\int_{R}^{\infty} |\cdots| r^2 dr}_{\delta}$$

Glimm's Lipschitz cont estimate for $u^0_{\Delta x}$ $\Rightarrow$ $O(|t_{j_2}-t_{j_1}|)$ bds for 1st term, & use $\rho_{ij} \le F(G_0 \cdot t)\rho_{i+j,0}$ to bound the 2nd term

$\Rightarrow \forall \delta > 0 \ \exists R(\delta)$ s.t.

$$|M_{ij_2} - M_{ij_1}| \leq \frac{K}{2} G_2 F(G_2 t) \bar{S} R(\delta)^2 |t_{j_2} - t_{j_1}| + \delta$$

- From this we obtain

$$B_{ij} = \frac{1}{1 - \frac{2M_{ij}}{x}} \leq \bar{B}$$

for $t \leq t_B$

Also $\Rightarrow$ Lipschitz cont of M in time

- Step 3 Argue Constant $M_{\infty j}$

- $M(\infty,t) = M_\infty \quad t \in [0,T]$

Proof:

- $u_{\Delta x} \to u$ a weak soln
- Thm: (2) holds if $(2)_{r_0}$ holds
- $(2) \equiv \dot{M} = -\frac{1}{2} K r^2 \sqrt{\frac{A}{B}}\, T^{01}_M$
- Estimates $\Rightarrow$

$$r^2 \sqrt{\frac{A}{B}}\, T^{01}_M \xrightarrow{r \to \infty} 0$$

- $\dot{M} \to 0$ unif on $[0,T]$ as $r \to \infty$

$$\Rightarrow \quad M(r,t) \equiv M_\infty \equiv \text{const}$$

- Conclusion:

"The Einstein Equations are consistent at the level of shock waves".

For Strong Shocks

(First PDE proof of this)

Conclusion:

- Covariant Glimm Scheme

Q: Can this be applied in other coordinate systems?

- Soln's only $C^{0,1}$

Q: can they be smoothed by coordinate transformation

(Yes or No interesting!)

- Can we improve estimates?

Naked Singularities?

Black Holes?

- **Bigger** **Questions**

- Can every $C^{0,1}$ shock-wave solution in 4-d spacetime be smoothed to $C^{1,1}$ by coordinate transformation?

- Are solutions of Einstein's equations smoother than $C^{0,1}$ with arbitrary sources?

E.g. Relativistic Boltzmann?

- Note $G = 8\pi T$ hyperbolic

- Hawking-Penrose assume $C^{1,1}$

- "If such a transformation does NOT exist, then soln's are one degree less smooth than previously assumed"

- "If such a transformation does exist, then the Rankine-Hugoniot jump conditions & the theory of distributions need not be imposed as extra conditions on the compressible Euler eqn's, but rather follow from Einstein Equations BY THEMSELVES

• <u>Final</u> <u>Question</u>: Could the correct norms for shock-wave solutions in multi-dimensions be more naturally phrased in terms of the metric components instead of the fluid variables?

- Reference: M. Luskin, B. Temple *The existence of a global weak solution of the waterhammer problem*, Comm. Pure Appl. Math. Vol. 35, 1982, pp. 697-735.

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