A Locally Inertial Glimm Scheme for General Relativity

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Reference:

Shock Wave Solutions of the Einstein Equations with
Perfect Fluid Sources: Existence and Consistency by a
Locally Inertial Glimm Scheme

Jeff Groah & BTemple
(Memoirs of AMS - Nov. 2004 & Springer Lecture Notes -)
Introduction

- 1915 Einstein introduced his Field Equations for GR.
- 1965 Glimm gave his theory of wave-interactions

Our Project: To put these two theories together

Simplest Setting: Spherically Symmetric Spacetimes
The Central Issue:

"The gravitational metric appears to be singular at shock-waves in the coordinates where the analysis appears feasible."
Assume: Spherical Symmetry + Standard Schwarzschild Coordinates

\[ ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega^2 \]  
\[ B = (1 - \frac{2M}{r})^{-1} \]  

Einstein Eqn's:

\[ \frac{A}{r^2 B} \left\{ r \frac{B'}{B} + B - 1 \right\} = kA^2 T^{00} \quad (s_3 p_3 v) \]  
\[ -\frac{B_t}{rB} = kAB T^{0i} \quad (s_3 p_3 v) \]  
\[ \frac{1}{r^2} \left\{ r \frac{A'}{A} - (B - 1) \right\} = kB^2 T^{ii} \quad (s_3 p_3 v) \]  
\[ -\frac{1}{rAB^2} \left\{ B_{tt} - A' + \Phi \right\} = \frac{2kr}{B} T^{22} \quad (s_3 p_3 v) \]

\[ \Phi = -\frac{BA_t B_t}{2AB} - \frac{B(B_t)^2}{2(B/B)} - \frac{A'}{r} + \frac{AB'}{rB} \]
\[ + \frac{A}{2} \left( \frac{A'}{A} \right)^2 + \frac{AA'BB'}{2AB} \]
Note:
- No derivatives on fluid sources
- (1), (2), (3) are 1st order in $A, B$
- (4) is 2nd order in $A, B$

Conclude:
- Shocks $\Rightarrow$ T discont.
- (1), (2), (3) 1st order $\Rightarrow A, B \in C^{0,1}$
- (4) 2nd order $\Rightarrow$ (4) can only be satisfied weakly when $A, B \in C^0$

We Prove: $\exists$ weak soln of (1) - (9) with $A, B \in C^{0,1}$
Q: How smooth should shock wave solutions of $G = kT$ be?

\[ G = kT \]
\[ \frac{\partial^2 g}{\partial T^2} = kT (s, p, u) \]

Shock $\Rightarrow$ jump discontinuity in fluid $\frac{p_R}{s_R} \frac{u_R}{u_L}$

$G = kT$ $\Rightarrow$ jump in 2nd deriv of $g$

$\Rightarrow$ $g \in C^{1,1}$?

$C^{1,1}$ is 1-derivative Lipschitz Cont.
\[ \sum g_{ij} = k T(s, t, u) \]

RHS discontinuous \( \Downarrow \)

LHS has one cont derivative

?? \( g \in C^{1,1} \) ?? ??

\[ g \in C^{1,1} \quad \partial g \in C^{1,1} \quad \partial^2 g \text{ jump discontinuous} \]
Conclude: Our solutions of the Einstein-Euler equations are only $C^{0,1}$ at shocks ↓
Soln's are one degree less smooth than general theory tells us they should be

Open Question: Is there a coordinate transformation that smooths a $C^{0,1}$ weak solution to a $C^{1,1}$ strong solution of $6 = KT$?

For single shock surfaces in $\mathbb{R}^4$, the answer is Yes

Ref: Israel/Smoller-Te
Open Problem: Given a $C^{0,1}$ shock-wave soln of $G = kT$, does $\exists$ coord. trans. $x \rightarrow y$

such that

$$g_{ij}(x) \in C^{0,1}$$

but

$$g_{AB}(y) = \frac{\partial x^i}{\partial y^A} g_{ij} \frac{\partial x^j}{\partial y^B} \in C^{1,1}$$?
For a single shock surface we know it's true

\[ \sum_{i=1}^{3} \in \mathbb{R}^4 \]

Gaussian normal coordinates \( \Rightarrow \)

\[ \text{ge } C^0 \Rightarrow \text{ge } C^{1,3} \]

Ref: Israel, Smoller-tp

Q: what about when 2 shocks cross?

Open Problem

C.F. Hawking & Ellis
Theorem 1 Let $\Sigma$ denote a smooth, 3-dimensional shock surface in spacetime with spacelike normal vector $n$. Assume that the components $g_{ij}$ of the gravitational metric $g$ are smooth on either side of $\Sigma$, (continuous up to the boundary on either side separately), and Lipschitz continuous across $\Sigma$ in some fixed coordinate system. Then the following statements are equivalent:

(i) $[K] = 0$ at each point of $\Sigma$.

(ii) The curvature tensors $R^i_{jk}$ and $G_{ij}$, viewed as second order operators on the metric components $g_{ij}$, produce no delta function sources on $\Sigma$.

(iii) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of $P$, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates.

(iv) For each $P \in \Sigma$, there exists a coordinate frame that is locally Lorentzian at $P$, and can be reached within the class of $C^{1,1}$ coordinate transformations.

\[
\begin{align*}
\left[ G^{-} \right] \cdot n_{\sigma} &= 0 \\
[S] \text{ of Ne ~'94}
\end{align*}
\]
To construct $C^0$ weak solutions of (1)-(4), we introduce a "locally inertial Glimm Scheme".

This is satisfying because Einstein's theory of gravity is fundamentally a locally inertial theory.

Even the physics is most naturally expressed in a locally inertial coordinate system, analysis can only be done in global coordinate systems that hide the locally inertial simplicity.
General Relativity as a locally inertial theory of gravity

(Towards a "Locally Inertial Glimm Scheme"

\[ G = kT \]

\[ \partial^2 g_{ij} = kT(s, \theta, u) \]

\[ x: \mathbb{M} \rightarrow \mathbb{R}^4 \text{ locally inertial } \Theta \]

\[ g_{ij}(P) = \text{diag}(c, \lambda, \lambda, \lambda) \]

\[ g_{ij}, \lambda(P) = 0 \]
- The metric tells how the local frames are tied together.
- Conversely— if you have equations that tell you how the local frames are tied together, then you can reconstruct the metric.

E.g., translation & geodesic motion by local frames —
Physically - II-translation/geodesic motion by local frames

\[ \nabla_X Y^i = X(Y)_i + \Gamma^{i}_{jk} X^j Y^k \]

\[ \Gamma^{i}_{jk} = \frac{1}{2} g^{ir} \left\{ -g_{jr, k} + g_{rj, k} + g_{kr, j} \right\} \]
To implement this idea, we look for a **Locally Inertial Glimm Scheme**.

- Flat space Compressible Euler Solns in each grid cell
  - "locally inertial frame"

- Discontinuities between grid cells that describe how local frames interact \(\Rightarrow\) **Curvature**
Compressible Euler in Flat Space

\[ G = \kappa T \]

\[ \text{div} G = 0 \Rightarrow \text{div} T = 0 \]

Covariant divergence

In an inertial frame \( T \to T_M \)

\[ \text{div} T_M = 0 \quad T_M = T - \text{Minkowski} \]

\[ \text{div}\left\{ (s+p)u^iu^j + p \eta^{ij} \right\} = 0 \]
\[ \text{div } T = 0 \]
\[ (T^0)_t + (T^0)_x = 0 \]
\[ (T^0)_t + (T^0)_x = 0 \]

\[ u^0 = T^0_M = (p + sc^2) \frac{c^2}{c^2 - v^2} - p \]

\[ u^1 = T^1_M = (p + sc^2) \frac{cv}{c^2 - v^2} \]

\[ T^2_M = (p + sc^2) \frac{v^2}{c^2 - v^2} + p \]
Global Existence Theory
(Sm-Te Comm Math Phys 151 (1993))

Assume: \( P = \sigma \, \rho \) \( \sigma \in \text{const} \)

(Eg \( P = \frac{c^2}{3} \rho \) Fundamental)

Remarkable Fact - Nishida Property

\[
\begin{align*}
  r &= \frac{1}{2} \ln \left( \frac{c + v}{c - v} \right) - \frac{K_0}{2} \ln \rho \\
  s &= \frac{1}{2} \ln \left( \frac{c + v}{c - v} \right) + \frac{K_0}{2} \ln \rho
\end{align*}
\]

\[
K_0 = \frac{\sigma - c}{c^2 + \sigma^2}
\]

Shock curves are translation/reflect symmetric in rs-plane
Main Lemma: \( \text{TVd}_{\infty} \) is non-increasing for wave interaction.

\[ \implies \text{NO VACUUM} \]

\[ \text{TVd}_{\infty}(\cdot,t) \leq \text{TVd}_{\infty}(\cdot,0) \]
\[ TV\ln p(t) \leq TV\ln p(\cdot,0) \]
**Theorem:** If \( TV \ln \rho_0 < \infty \) on initial data, then \( \exists \) global weak (shock-wave) soln of compressible Euler

\[
\text{div} \, T_M = 0
\]

in \((1,1)\)-Flat Minkowski Space

**Q:** How to extend to Curved Space-time?
Conclude: Having a Nishida system for the Flat Space Compressible Euler Equations begs the following natural question:

Can we extend this to a theory of strong shock wave interactions in curved spacetime by implementing a Locally Inertial Glimm Scheme?

Simplest Setting: Spherically Symmetric Space times
THE SIMPLEST SETTING
for
SHOCK-WAVES

• Spherical Symmetry—
Assume Standard Schwarzschild Coordinates:

\[ g_{ij}dx^idx^j = \]
\[ ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2d\Omega^2, \]
\[ d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \]
\[ x = (x^0, ..., x^3) \equiv (t, r, \theta, \phi). \]

• Define the mass function \( M(r,t) \): 

\[ B(r,t) \equiv \left( 1 - \frac{2M(r,t)}{r} \right)^{-1}, \]
• Stress Tensor $T$:

$$T^{ij} = (\rho c^2 + p)w^iw^j + pg^{ij}, \quad i, j = 0, ..., 3$$

\[
\begin{align*}
A \cdot T^{00} &= \frac{c^4 + \sigma^2 v^2}{c^2 - v^2} \rho = T^{00}_M = u^0 \\
\sqrt{AB} \cdot T^{01} &= \frac{c^2 + \sigma^2}{c^2 - v^2} cv\rho = T^{01}_M = u^1 \\
B \cdot T^{11} &= \frac{v^2 + \sigma^2}{c^2 - v^2} \rho c^2 = T^{11}_M
\end{align*}
\]

$\rho c^2 = \text{density,} \quad p = \text{pressure,} \quad v = \text{velocity}$

• Assume Equation of State:

$$p = \sigma^2 \rho$$

$\sigma = \text{sound speed} < c = \text{light speed}$. 

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Plug $T^{ij}$ and $g^{ij}$ into $G = \kappa T$: (MAPLE)

\[
\frac{A}{r^2B} \left\{ \frac{B'}{B} + B - 1 \right\} = \kappa A^2 T^{00} \tag{1}
\]

\[-\frac{B_t}{rB} = \kappa A B T^{01} \tag{2}\]

\[
\frac{1}{r^2} \left\{ \frac{A'}{A} - (B - 1) \right\} = \kappa B^2 T^{11} \tag{3}\]

\[-\frac{1}{rAB^2} \{ B_{tt} - A'' + \Phi \} = \frac{2\kappa r}{B} T^{22} \tag{4}\]

\[
\Phi = -\frac{BA_t B_t}{2AB} - \frac{B}{2} \left( \frac{B_t}{B} \right)^2 - \frac{A'}{r} + \frac{AB'}{rB} + \frac{A}{2} \left( \frac{A'}{A} \right)^2 + \frac{AA' B'}{2AB} \]

- $(1) \equiv M' = \frac{1}{2} \kappa r^2 A T^{00}$, $(2) \equiv \dot{M} = -\frac{1}{2} \kappa r^2 A T^{01}$

- "prime" $\equiv \partial / \partial r$, "dot" $\equiv \partial / \partial t$

\[
B = \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \]
The equation (1) \( M' = \frac{1}{2} \kappa r^2 A T^{00} \) implies:

\[
M(r, t) = M_0 + \frac{\kappa}{2} \int_{r_0}^r T^{00}_M(r, t) r^2 \, dr
\]

- The scalar curvature \( R \) satisfies

\[
R = (c^2 - 3\sigma^2) \rho
\]

- Components of \( T_M \) satisfy:

\[
|T^{01}_M| < T^{00}_M,
\]

\[
\frac{\sigma^2}{c^2 + \sigma^2} T^{00}_M < T^{11}_M < T^{00}_M
\]

- This defines the simplest setting for shock wave propagation in General relativity.
**Assume:**

- Solutions defined outside ball radius \( R_0 \)

\[
\begin{array}{c}
\mathcal{V}_{R_0} = \mathcal{U} \\
\mathcal{M}_{R_0} \\
\mathcal{B}_{R_0} \\
\mathcal{A}_{R_0} \\
R_0
\end{array}
\]

- No Black Hole: \( 1 < B(r,0) \leq \bar{B} < \infty \)

- Finite Total Mass:

\[
M_\infty = \frac{K}{2} \int_{R_0}^{\infty} U^0(r,0) r^2 \, dr < \infty
\]

- Finite Supnorm:

\[
0 < x \| S(x,0) \| = S(x,0) < \infty
\]

\( x \equiv r \)

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Theorem: Assume there exist positive constants $L$, $V$ and $\bar{v}$ such that $v_0(r)$ and $\rho_0(r) > 0$, $r_0 \leq r < \infty$, satisfy

$$TV_{[r,r+L]} \ln \rho_0(\cdot) < V,$$
$$TV_{[r,r+L]} \ln \left(\frac{c+v_0(r)}{c-v_0(r)}\right) < V,$$
$$|v_0(r)| < \bar{v} < c \quad (5)$$

Then a bounded, weak, (shock wave), solution of the initial-boundary value problem for Einstein equations (1)-(4) exists up to some positive time $T > 0$.

For $t < T$, the metric functions $A$ and $B$ are Lipschitz continuous ($C^{0,1}$) functions of $(r,t)$, and (5) continues to hold for $t < T$, with adjusted values for $V$ and $\bar{v}$ that are determined from the analysis.

$$\lim_{r \to \infty} M(r,t) = M_0 \equiv \text{const} \cdot 6$$
The theorem allows for arbitrary numbers of interacting shock waves, of arbitrary strength.

Note that $\lim_{r \to \infty} M(r,t) = M_\infty$ is a non-local condition.

\[ (1) \quad \equiv \quad \frac{A}{r^2 B} \left\{ \frac{r B'}{B} + B - 1 \right\} = \kappa A^2 T^{00} \]
\[ (3) \quad \equiv \quad \frac{1}{r^2} \left\{ \frac{r A'}{A} - (B - 1) \right\} = \kappa B^2 T^{11} \]
\[ (4) \quad \equiv \quad -\frac{1}{rAB^2} \{ B_{tt} - A'' + \Phi \} = \frac{2\kappa r}{B} T^{22} \]

$A(r,t)$ and $B(r,t)$ are at most Lipschitz continuous at shocks.

$(4)$ only satisfied in the weak sense of the theory of distributions.
Note:

- $B \rightarrow \infty \iff \text{Black Hole}$

\[ B = \frac{1}{1-\frac{2M}{r}} \]

- Conclude: Since Black Holes can form, we cannot do better than a local exist. theory

- $\sigma \rightarrow \infty \iff \text{naked singularity (except } \sigma = \frac{1}{2} )$
• **Theorem** (Groah, Te): The following equivalencies hold in weak sense:

\[ G = \kappa T \iff (1) + (2) + (3) + (4) \]
\[ \iff (1) + (3) + \text{Div}T = 0 \]

• Weak equivalence for \( A, B \) Lipschitz continuous, \( T \) bounded measurable.

• For \((1) + (3) + \text{Div}T = 0\), equation \((2)\) holds as a **constraint**: it holds on weak solutions so long as it holds on the boundary \( r = r_0 \).

Corollary: An equivalent system:

\[ \text{New Twist: } u = (u^0, u^1) = (T^0_M, T^1_M) \]

\[
\begin{align*}
    u_t + f(A, u)_x &= g(A, u, x) \iff \text{Div} T = 0 \\
    A' &= h(A, u, x) \iff (1) + (3)
\end{align*}
\]

\[
\begin{align*}
    A &= (A, B) \\
    f(A, u) &= \sqrt{\frac{A}{B}} (T^0_M T^1_M) \\
    g(A, u, x) &= (g^0, g^1) \\
    h(A, u, x) &= (h^0, h^1)
\end{align*}
\]

Note: \( u \) independent of \( A = (A, B) \)
\[ u_t + f(A, u)_x = g(A, u, x) \iff \text{Div}T = 0 \]
\[ A' = h(A, u, x) \iff (1) + (3) \]

\[
g^0 = -\frac{2}{x} \sqrt[3]{\frac{A}{B}} T_M^{01} \]

\[
g^1 = -\frac{1}{2} \sqrt[3]{\frac{A}{B}} \left\{ \frac{4}{x} \frac{T_M^{11}}{2} + \frac{(B - 1)}{x} (T_M^{00} - T_M^{11}) \right.
\quad + 2\kappa x B (T_M^{00} T_M^{11} - (T_M^{01})^2 - 4x T^{22}) \}
\]

\[
h^0 = \frac{(B - 1)A}{x} + \kappa x A B T_M^{11} \]

\[
h^1 = -\frac{(B - 1)B}{x} + \kappa x B^2 T_M^{00}, \quad \zeta = \frac{\sigma^2 c^2}{(c^2 - \sigma^2)^2} \]

\[
\frac{T_M^{11}}{T_M^{00}} = \frac{2\zeta + 1}{2\zeta} \left\{ 1 - \sqrt{1 - \frac{4\zeta}{(2\zeta + 1)^2} \left( \zeta + \left[ \frac{T_M^{01}}{T_M^{00}} \right]^2 \right)} \right\}
\]
Remarkably: Change of variables

\[ T \rightarrow u = T_M \]

\[ \Rightarrow \text{Time derivatives } A_t \text{ and } B_t \text{ cancel out!} \]

- \( \text{Div} T = 0 \) reads:

\[
0 = T_{,0}^{00} + T_{,1}^{01} + \frac{1}{2} \left( \frac{2A_t}{A} + \frac{B_t}{B} \right) T^{00} \\
\quad + \frac{1}{2} \left( \frac{3A_t'}{A} + \frac{B_t'}{B} + \frac{4}{r} \right) T^{01} + \frac{B_t}{2A} T^{11}
\]

\[
0 = T_{,0}^{01} + T_{,1}^{11} + \frac{1}{2} \left( \frac{A_t}{A} + \frac{3B_t}{B} \right) T^{01} \\
\quad + \frac{1}{2} \left( \frac{A_t'}{A} + \frac{2B_t'}{B} + \frac{4}{r} \right) T^{11} \\
\quad + \frac{A_t'}{2B} T^{00} - 2 \frac{r}{B} T^{22}
\]

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Proof: Fractional Step Glimm Scheme

\[ R_{ii}: A = A_{ii} \]

- Stagger discontinuities in \( A \) with discontinuities in \( U \)
- Solve RP for \( \frac{1}{2} \)-time step
- Solve ODE for \( \frac{1}{2} \)-time step

Ref: Luskin - Te \( \sim 82 \)
Groah - Thesis
PROOF: Fractional Step Glimm Scheme

- Define grid $x_i, t_j$, $i, j = 0, 1, 2, \ldots$, $x_0 = r_0$.

- Stagger discontinuities in $A$ with discontinuities in $u$:
  - Choose: $A = A_{ij}$ in Grid Rectangle $R_{ij}$
    
    $R_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_j, t_{j+1}]$

- Discontinuities in $u$ are set at $(x_i, t_j)$:
  $\Rightarrow$ Riemann problem posed at the bottom center of $R_{ij}$

- Solve $u_t + f(A_{ij}, u)_x = 0$ in $R_{ij}$ for $\frac{1}{2}$-step

- Solve $u_t = g(A_{ij}, u, x) - \nabla_A f \cdot A'$ for $\frac{1}{2}$-step

- Solve $A' = h(A, u, x)$, $A(r_0) = A_{r_0}$ at $t = t_{j+1}$

Ref: Luskin-Te 1982
Groah - Thesis
• Solve RP for $\frac{1}{2}$-timestep

\[ u_t + f(A_{i,j}, u)_x = 0 \]
\[ u = \begin{cases} 
    u_{i-1,j} & x \leq x_i \\
    u_{i,j} & x \geq x_i
\end{cases} \]

• Solve ODE for $\frac{1}{2}$-timestep

\[ u_t = g(A_{i,j}, u, x) - \nabla_x A' \cdot f \cdot A' \]
\[ u(0) = u_{i,j}^{RP} \]
At end of timestep, solve

\[ A' = \lambda(A, u, x) \]
\[ A(r_0) = A_{r_0} \]

\[ \begin{bmatrix}
  u_{10} & A_{11} & u_{12} & A_{13} & \cdots & u_{1i} & A_{1i+1} \\
  i = 0 & i = 1 & i = 2 & \cdots & i = \beta & i = \beta + 1
\end{bmatrix} \]

- Use values from fractional step at \( t = t_{j+1}^- \) to obtain \( A \) at \( t = t_{j+1}^+ \)
- Rediscretize by Glimm’s Random Choice Sequence
The Riemann Problem Step:

- The RP in $R_{ij}$

$$u_t + f(A_{ij}, u)_x = 0,$$

$$u_0(x) = \begin{cases} 
  u_L = u_{i-1,j} & x < x_i, \\
  u_R = u_{ij} & x > x_i,
\end{cases}$$

- $A = A_{ij} \Rightarrow R_{ij}$ defines a "locally inertial coordinate frame"

- Presence of $A_{ij}$ alters speeds but not states
  Riemann Problem $\iff$ Special Relativity

\[ u_t + f(A, u) x = 0 \quad A = (a, b) = \text{const} \]

\[
\begin{align*}
\left( \frac{T^{oo}}{T^o} \right)_t + \sqrt{\frac{A}{B}} \left( \frac{T^{oo}}{T^o} \right)_x & = 0 \\
\end{align*}
\]

\[ \overline{x} = \sqrt{\frac{A}{B}} x \]

\[ \begin{array}{c}
\text{Compressible} \\
\text{Euler} \\
\text{in} \\
\text{Flat Space} \\
\end{array} \]

\[ p = \sigma \rho \]

\[ \Rightarrow \text{The affect of } \sqrt{\frac{A}{B}} \text{ is to change the speeds of waves not the states} \]

\[ \Rightarrow \text{TV} \text{ is non-increasing} \]
• Main Point: $TV \ln \rho(\cdot, t)$ is non-increasing on RP step of method.


• Conclude: Fractional Step Method\equiv
  "Covariant version of Glimm's method"

• The boundaries between the "locally inertial coordinate frames" are the discontinuities in A along sides, top and bottom, of grid rectangles.

"Locally Flat Method"
\[ u_i = g - A' \cdot \nabla A f \equiv G(A, u, x), \quad A \equiv A_{ij}, \quad x = x_i \]

\[ G^0 = -\frac{1}{2} \sqrt{\frac{A}{B}} T_{01}^M \left\{ \frac{2(B + 1)}{x} - \kappa B \left( T_{00}^M - T_{11}^M \right) \right\} \]

\[ G^1 = -\frac{1}{2} \sqrt{\frac{A}{B}} \left\{ \frac{4}{x} T_{11}^M + \frac{B - 1}{x} \left( T_{00}^M + T_{11}^M \right) + \kappa B \left[ T_{00}^M T_{11}^M - 2 \left( T_{01}^M \right)^2 + \left( T_{11}^M \right)^2 \right] - 4x T^{22} \right\} \]

- \( g \) accounts for the discontinuities in \textbf{TIME} (along top and bottom of \( R_{ij} \))

- \(-A' \cdot \nabla A f\) acts for discontinuities in \textbf{SPACE} (along sides of \( R_{ij} \))

- Proof of convergence of the residual demonstrates that this interpretation is \textbf{correct}.
The ODE has nice properties in $(\rho, v)$-plane:

\[ u_t = g - A' \cdot \nabla_A f \]

\[
\dot{\rho} = \frac{\kappa \sqrt{AB}}{2} \left[ \frac{(c^2 + \sigma^2)^2 \nu c}{c^4 - \sigma^2 v^2} \right] \rho \{ \rho - \rho_1 \},
\]

\[
\dot{v} = -\frac{\kappa \sqrt{AB}}{2} \left[ \frac{(c^4 - v^4) \sigma^2 c}{c^4 - \sigma^2 v^2} \right] \{ \rho - \rho_2 \},
\]

\[
\rho_1 = \frac{4}{\kappa B(c^2 + \sigma^2)x^2},
\]

\[
\rho_2 = \frac{4v^2 \sigma^2 - (B - 1)(c^4 - \sigma^2 v^2)}{\kappa B(c^2 + v^2)\sigma^2 c^2 x^2},
\]

\[
\rho_2 < \frac{4v^2 \sigma^2}{\kappa B(c^2 + v^2)\sigma^2 c^2 x^2} < \rho_1, \quad -c < v < c
\]
- **Autonomous system at each** $(i,j)$

- $\rho > 0, |v| < c$ is an invariant region

- $\rho \leq \rho_1$ is a bounded invariant region
  $\rho \geq \rho_1$ is an unbounded invariant region:
  $(\rho = \rho_1$ is a solution$)$

- **Solutions exist/bounded for all** $t \geq 0$
  (ODE's are quadratic in $\rho$!)
with(DEtools):

A := 1; B := 1.3; kappa := 1; sigma := 1/sqrt(3); r := 1;
fieldplot([diff(x(t), t) = 1/2*(sigma^2+c^2)*(x(t)*r^2*kappa*B*(sigm^2+c^2)-4)*sqrt(A/B)*x(t)*c*y(t)/(c^4-y(t)^2*sigma^2)*r), diff(y(t), t) = -1/2*sqrt(A/B)*(c^2-y(t)^2)*((B-1)*(c^4-y(t)^2*sigma^2)-4*y(t)^2*sigma^2+kappa*x(t)*r^2*B*c^2*sigma^2*(c^2+y(t)^2))/(c*r*(c^4-t*sigma^2))], [x(t), y(t)], t = -2..2, x = 0..5, y = -c..c, arrows = small)
Bounds for Fractional Step Method:

- RP+ODE ⇒ approx solns defined for all \( t \) so long as the CFL-condition holds

\[
\frac{\Delta x}{\Delta t} \geq \max \left\{ 2\sqrt{\frac{A}{B}} \right\}
\]

- We show: CFL bound depends only on:
  \( \|B\|_\infty, \|S\|_\infty, S \equiv S(x,t) = x\rho(x,t) \).

- We prove: All norms bounded by \( \|B\|_\infty, \|S\|_\infty, \) and \( \|TVL\ln \rho(\cdot,t)\|_\infty \)

- Thm: Solution extends to first time \( T \) at which one of these three norms tends to infinity.
\begin{itemize}
  \item $B \rightarrow \infty \iff \text{Black Hole}$
  \begin{equation}
    B = \frac{1}{1 - \frac{2M}{r}}
  \end{equation}

  \item $\rho \rightarrow \infty \iff \text{Naked Singularity}$,
    \begin{equation}
      (R = \{c^2 - 3\sigma^2\} \rho)
    \end{equation}

  \item \textbf{Open Problem}: Can
    \begin{equation}
      B, \|S\|_\infty, \|TV_L \ln \rho(\cdot,t)\|_\infty \rightarrow \infty
    \end{equation}
    some other way?

  \item \textbf{Open Problem}: Do there exist coordinate transformations that smooth the metric components of these solutions from the smoothness class $C^{0,1}$ up to the class $C^{1,1}$?

  \item Such a transformation would map weak solutions $\Rightarrow$ strong solutions.
\end{itemize}
- We have uniform estimates for RP and ODE steps separately, but not under interaction.

- Basic idea: RP step preserves $TV_L \ln \rho \Rightarrow$ need $\Delta TV_L \ln \rho$ for ODE is $O(\Delta t) \Rightarrow$ compactness by Oleinik/Glimm compactness argument.

- Main Technical Problem: Growth of $TV_L \ln \rho$ coupled to growth of $M_\infty$

$$M_\infty = \frac{\kappa}{2} \int_{r_0}^{\infty} u(r,v)r^2 dr$$

a non-local condition.

- Difficulty in keeping track of order of choice of constants.
Main Estimate:

- Assume that for $t < T_0$, $u_{\Delta x}, A_{\Delta x}$ satisfy
  \[
  M_{\Delta x}(x, t_j) \leq \bar{M} \\
  B_{\Delta x}(x, t_j) \leq \bar{B} \\
  0 < S_{\Delta x}(x, t_j) \leq \bar{S} \\
  |v_{\Delta x}(x, t_j)| \leq \bar{v}
  \]

- Assume that there exists constants $L, V_0$ such that $|x_{i_2} - x_{i_1}| \leq L \Rightarrow$
  \[
  \sum_{i_1 \leq i \leq i_2, \ p = 1, 2} |\eta_{i_0}^p| < V_0
  \]

Conclude: (A) Total variation bound:

\[
\sum_{i_1 \leq i \leq i_2, \ p = 1, 2} |\eta_{ij}^p| < 2\bar{V}_s, \\

 t_j \leq T_2 = \left( \frac{1}{G_2} \right) \frac{\bar{V}_s}{\{2\bar{V}_s + H(2\bar{V}_s)\}}
\]
(B) Conclude: $L^1_{loc}$ bounds: $t_j \leq \min \{T_0, T_3\}$

$$\int_{x_{i_2}}^{x_{i_2}} \| \mathcal{Z}_{Ax} (x, \tau_{i_2}) - \mathcal{Z}_{Ax} (x, \tau_{i_2}) \| dx \leq G_2 |t_j - t_{i_1}|$$

$\mathcal{Z} = (z, \eta) \subset$ plane of R.I. 's

(C) Conclude: Supnorm bounds: $t_j \leq \min \{T_0, T_3\}$

$$\| \mathcal{Z}_{i_2} - \mathcal{Z}_{i_3} \| \leq F_0 (G_2 \cdot t_j)$$

Main Point: Dependence on $\bar{M}, \bar{B}, \bar{S}, \bar{V}$ is only thru $G_2 = G_2 (\bar{M}, \bar{B}, \bar{S}, \bar{V})$

Idea: estimates are strong enough to control $\bar{M}, \bar{B}, \bar{S}, \bar{V}$ for short times
Not so easy as you might think.

\[
\begin{align*}
&\forall t > T(t, \varepsilon) \Rightarrow B \in B \\
&\forall t > T(m, \varepsilon) \Rightarrow M \in M \\
&\forall t > T(a, b, \varepsilon) \Rightarrow S \in S \\
&\forall t > T(A, B, \varepsilon) \Rightarrow \Delta \in \Delta
\end{align*}
\]

Small time assumptions: (in order)

- Support bound assumptions by
- The Teisingue-Successively Replace
Main Issue: Control Total Mass

\[ M_{ij} = M_0 + \frac{k}{2} \int_{r_0}^{x_i} v_0^0 (r; t_j) r^2 dr \]

- total mass between \( r_0 \) & \( x_i \) at \( t = t_j \)

\[ B_{i,j} = \frac{1}{1 - \frac{2M_{ij}}{x}} \]

Need: \( M_{ij} \leq \bar{M}, B_{ij} \leq \bar{B} \)
Step 0: Bound $M_{ij}$

- On Cons. Law Step
  \[(TV \ln P)_{i+1} \leq (TV \ln P)_i \] (TV)

- Use (TV) + order at growth on one step to get

\[\ln P_{ij} - \ln P_{i+1,j} \leq F_0(G_0 \cdot t)\]

\[\Rightarrow P_{ij} \leq F_0(G_0 \cdot t)P_{i+1,j} \]
\[ p_{i,j} \leq F_0(G_0, t) \rho_{i,j,0} \]

\[ F_0(\xi) = 2 \left( 1 + \frac{4\xi}{L} \right) V_0 + H(2(H(\frac{4\xi}{L}) V_0) + 3 \]

\[ G_0 = G_0(V_0, \bar{M}, \bar{B}, \bar{S}, \bar{S}) \]

\[ \Rightarrow \text{for } t \text{ suff small, } F_0(G_0, t) \text{ can be bded by a LARGE constant indept of } \bar{M}, \bar{B}, \bar{S}, \bar{V} \]
\* \( u^0 \) related to \((\tilde{s}, v)\) thru a regular transformation

\[ u^0 = u^0 (v, \tilde{s}) \]

\[ \Rightarrow \]

\[ u^0_{\ell, 0} \leq F_1 (G, t) \ U_{\ell, 0} \]

\[ \Rightarrow \]

\[ M_{\omega, 0} \leq R_0 + F_1 (G, t) \int_{r_0}^{r_0} u^0 (r, 0) r^2 dr \]

\[ \leq F_1 (G, t) \ M_{\omega, 0} \]
Conclude -

$$M_{0,0} \leq F_i (G_i, t) M_{0,0,0}$$

$F_i (s)$ depends only on $V_0, L$

$$G_i \equiv G_i (V_0, \bar{M}, \bar{B}, \bar{S}, \bar{\gamma})$$

:. choose $\bar{M} = 2 F_i (0) M_{0,0,0}$

$\Rightarrow$ for small time $t \leq T_m$

$$M_{0,0} \leq F_i (G_i, t) M_{0,0,0} \leq \bar{M}$$

Problem: $\bar{M} \gg M_{i0} \Rightarrow$ No control over $B_{ij} = \frac{1}{M_{i0}}$
Step 2: Improve Estimate for $M_{i;j}$

$$|M_{i;j} - M_{i;j}| \leq \frac{\epsilon}{2} \int_{r_0}^R |u^0_{\alpha x}(r, t_{i;2}) - u^0_{\alpha x}(r, t_{i;1})|^2 dr$$

$$+ \int_{R}^{\infty} (1 - \log r) r^2 dr$$

Glimm's Lipschitz cont estimate for $u^0_{\alpha x} \gg 0(t_{i;2} - t_{i;1})$ bads for 1st term, & use $p_{i;j} \leq F(G_0 \cdot t) p_{i;j;0}$ to bound the 2nd term.
$\Rightarrow \forall \delta > 0 \ \exists \ R(\delta) \ st. \ 6$

$|M_{i_2,j_2} - M_{i_1,j_1}| \leq \frac{k}{2} G_2 F(G_2 t) \overline{S} \ R(\delta)^2 |t_{j_2} - t_{j_1}| + \delta$

• From this we obtain

$B_{i,j} = \frac{1}{1 - \frac{2M_{i,j}}{x}} \leq \overline{B}$

for $t \leq t_B$

Also $\Rightarrow$ Lipschitz cont. of $M$ in time

• Step (3) Argue $\underline{\text{Constant}} \ M_{i,j}$
\[ M(\infty, t) = M_\infty \quad t \in [0, T] \]

Proof:

- \[ u_\Delta \rightarrow u \] a weak soln
- \textbf{Thm:} (2) holds if (2) \( r_0 \) holds

- (2) \[ \dot{M} = -\frac{1}{2} kr^2 \sqrt{\frac{A}{B}} T_m^{01} \]

- Estimates \[ \Rightarrow \]
  \[ r^2 \sqrt{\frac{A}{B}} T_m^{01} \xrightarrow{r \to \infty} 0 \]

- \[ \dot{M} \rightarrow 0 \] unit on \([0, T]\) as \( r \rightarrow \infty \)

\[ \Rightarrow \quad M(r, t) \equiv M_\infty \text{ const} \]
Conclusions:

"The Einstein Equations are consistent at the level of shock waves."

*For Strong Shocks*

(First PDE proof of this)
Conclusion:

- **Covariant Glimm Scheme**
  - Q: Can this be applied in other coordinate systems?

- Soln's only $C^{0,1}$
  - Q: Can they be smoothed by coordinate transformation (Yes or No interesting!)

- Can we improve estimates?
  - Naked Singularities?
  - Black Holes?
Bigger Questions

- Can every $C^{0,1}$ shock-wave solution in 4-d spacetime be smoothed to $C^{1,1}$ by coordinate transformation?

- Are solutions of Einstein's equations smoother than $C^{0,1}$ with arbitrary sources? E.g. Relativistic Boltzmann?

- Note $G = 8\pi T$ hyperbolic

- Hawking-Penrose assume $C^{1,1}$
"If such a transformation does not exist, then soln's are one degree less smooth than previously assumed."

"If such a transformation does exist, then the Rankine-Hugoniot jump conditions & the theory of distributions need not be imposed as extra conditions on the compressible Euler eqns, but rather follow from Einstein Equations BY THEMSELVES."
Final Question: Could the correct norms for shock-wave solutions in multi-dimensions be more naturally phrased in terms of the metric components instead of the fluid variables?


• J. Groah: Doctoral Thesis