Numerical Refinement of a Finite Mass Shock-Wave Cosmology

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• Exact solution incorporating a shock-wave into the standard FRW metric for cosmology...


References:

- Connecting the shock wave cosmology model with Guth’s theory of inflation...


References:

- The locally inertial Glimm Scheme...


Our Shock Wave Cosmology Solution

Expanding Universe

Matter Filled Black Hole

Shock Wave

$\bar{r} = 2M = \frac{H}{c}$

$t = t_{crit}$ Shock emerges from White Hole

$\bar{r} = 2M = \frac{H}{c}$

$t = 0$

Big Bang

$t < t_{crit}$
The solution can be viewed as a natural generalization of a $k=0$ Oppenheimer-Snyder solution to the case of non-zero pressure, inside the Black Hole----

$2M/r > 1$
Shock-Wave Cosmology Solution

\[ ds^2 = -dt^2 + R(t)^2 \left\{ dr^2 + r^2 d\Omega^2 \right\} \]

**TOV:**

\[ ds^2 = -B(\bar{r}) d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \]

\[ \frac{2M}{\bar{r}} > 1 \]

\( \bar{r} \) is timelike
• In [Smoller-Temple, PNAS] we constructed an exact shock wave solution of the Einstein equations by matching a \((k=0)\)-FRW metric to a TOV metric inside the Black Hole across a subluminal, entropy-satisfying shock-wave, out beyond one Hubble length

• To obtain a large region of uniform expansion at the center consistent with observations we needed \(\frac{2M}{\bar{r}} > 1\)

\[
\frac{2M}{\bar{r}} = 1 \iff \bar{r} = \text{Hubble Length} \equiv \frac{H}{c}
\]
• The TOV metric inside the Black Hole is the simplest metric that cuts off the FRW at a **finite total mass**.

• Approximately like a **classical explosion** of finite mass with a **shock-wave** at the leading edge of the expansion.

• The solution decays time asymptotically to Oppenheimer-Snyder----a finite ball of mass expanding into empty space outside the black hole, something like a gigantic supernova.
• Limitation of the model: TOV density and pressure are determined by the equations that describe the matching of the metrics

\[ p = \frac{c^2}{3} \rho \]

can only be imposed on the FRW side

• Imposing \( p = \frac{c^2}{3} \rho \) on the TOV side introduces secondary waves which can not be modeled in an exact solution

• Question: How to model the secondary waves?
• **OUR QUESTION:** How to refine the model to incorporate the correct TOV equation of state, and thereby model the secondary waves in the problem?

• **OUR PROPOSAL:** Get the initial condition at the end of inflation

• Use the Locally Inertial Glimm Scheme to simulate the region of interaction between the FRW and TOV metrics
DETAILS OF THE EXACT SOLUTION
The Final Equations

\[ S' = 2S(1 + 3u) \{(\sigma - u) + (1 + u)S\} \equiv F(S, u) \]

\[ u' = (1 + u) \{-(1 - 3u)(\sigma - u) + 6u(1 + u)S\} \equiv G(S, u) \]

Big Bang \equiv 0 \leq S \leq 1 \equiv \text{Emerges From White Hole}

Entropy Condition: \( \bar{p} < p, \bar{\rho} < \rho, \bar{p} < \bar{\rho} \)

\[ S < \left(\frac{1 - u}{1 + u}\right) \left(\frac{\sigma - u}{\sigma + u}\right) \equiv E(u) \]
\[ u = \frac{\bar{p}}{\rho} \]

The entropy conditions for an outgoing shock hold when

\[ u_\sigma(S) < E(S) \]

**Entropy Inequalities**

\[ S = 0 \quad \text{Big Bang} \]

\[ S = 1 \quad \text{Solution emerges from White Hole} \]
\[ u = \frac{\bar{p}}{\rho} \]

\[ \sigma < \frac{1}{3} \]

\[ p = \sigma \rho \]

Phase Portrait

\[ u = E(S) \]

Isocline

\[ Q_{\sigma}(S) \]

\[ s_{\sigma}(S) \equiv \text{shock speed} < c \text{ for } 0 < S < 1 \]

\[ s_{\sigma}(S) \to 0 \text{ as } S \to 0 \]

Rest Point

Big Bang

Solution emerges from White Hole

\[ S = 0 \]

\[ S = 1 \]
\[ u = \frac{\bar{p}}{\rho} \]

\[ \sigma = \frac{1}{3} \]

\[ u = E(S) \]

\[ Q_{1/3}(S) \]

\[ u_{\sigma}(S) \]

\[ s_{\sigma}(S) \rightarrow c \text{ as } S \rightarrow 0 \]

\[ \rho = \sigma \rho \]

Phase Portrait

Double Rest Point

Isocline

Big Bang

Solution emerges from White Hole
\[ r = \frac{\bar{r}}{R} \]

Shock Position = \( r(t) \)

\[ r = \frac{c}{H R} \]

= Hubble Length

- The Hubble length catches up to the shock-wave at \( S=1 \), the time when the entire solution emerges from the White Hole.
Kruskall Picture

Black Hole Singularity

White Hole Singularity

$\bar{t} = +\infty$

$\bar{t} = -\infty$

$\bar{t} = t^*$

$\frac{2M}{\bar{r}} = 1$

$\bar{p} \neq 0$

$\bar{\rho} \neq 0$

FRW Universe

TOV

Inside Black Hole

Outside Black Hole

Shock Surface

Schwarzschild Spacetime

p = 0

$\rho = 0$

FrW

Universe

Inside Black Hole

Shock Surface

Schwarzschild Spacetime

p = 0

$\rho = 0$
• We are interested in the case $\sigma = 1/3$
  $\approx$ correct for $t = \text{Big Bang}$ to $t = 10^5\text{yr}$

• $\rho$ and $p$ on the FRW and TOV side
tend to the same values as $t \to 0$

• It is as though the solution is emerging
  from a spacetime of constant density and
  pressure at the Big Bang $\approx$ Inflation
\[
\sigma = 1/3
\]

Double Rest Point

\[
\rho = \sigma \rho
\]

\[
S = 1
\]

Big Bang

\[
S = 0
\]

Solution emerges from White Hole

\[
s_\sigma(S) \to c \text{ as } S \to 0
\]
Conclude: A solution like this would emerge at the end of inflation if the fluid at the end of inflation became co-moving wrt a \((k = 0)\) FRW metric for \(\bar{r} < \bar{r}_0\), and co-moving wrt the simplest spacetime of finite total mass for \(\bar{r} > \bar{r}_0\).

The inflationary diSitter spacetime has all of the symmetries of a vacuuum, and so there is no preferred frame at the end of inflation.
diSitter spacetime in (k=0)-FRW coordinates

No preferred coordinates

\( t = \text{const.} \)

\( u \) co-moving wrt \( t = \text{const.} \)

\((k = 0)\)-FRW

\[ ds^2 = -dt^2 + R(t)^2 \{ dr^2 + r^2 d\Omega^2 \} \]

\text{Inf lation} \quad T_{ij} = -\rho g_{ij}

\text{End of Inf lation} \quad T_{ij} = (\rho + p)u^i u^j + pg_{ij}

Shock

\text{Finite-Mass time-slice at the end of Inflation}

\( \frac{2M}{\bar{r}} > 1 \) and \( \bar{r} \) timelike

\text{diSitter spacetime in TOV-coordinates}

\( M, \bar{r} = \text{const.} \)

\( u \) co-moving wrt \( M = \text{const.} \)

\text{Shock}

\text{TOV}

\[ ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \]
$t = t_0$  

$ds^2 = -B(\tilde{r}, \tilde{t}) d\tilde{t}^2 + \frac{1}{1 - \frac{2M(\tilde{r}, \tilde{t})}{\tilde{r}}} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$

**k=0 FRW metric**

**Standard Schwarzschild coordinates**

**TOV metric inside the black hole**

$\tilde{r} = \tilde{r}_0$

$M = M(\tilde{r}_0)$

$\frac{2M(\tilde{r}_0)}{\tilde{r}_0} > 1$

$t = \text{the end of inflation} \approx 10^{-30} \text{s} = t_0$
A Locally Inertial Method for Computing Shocks
\[ ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

Einstein equations - Spherical Symmetry

\[
\frac{A}{r^2 B} \left\{ r \frac{B'}{B} + B - 1 \right\} = \kappa A^2 T^{00} \\
- \frac{B_t}{rB} = \kappa AB T^{01} \\
\frac{1}{r^2} \left\{ r \frac{A'}{A} - (B - 1) \right\} = \kappa B^2 T^{11} \\
- \frac{1}{rAB^2} \{B_{tt} - A'' + \Phi\} = \frac{2\kappa r}{B} T^{22},
\]

\[ G = 8\pi T \]

\[ B = \frac{1}{1 - \frac{2M}{r}} \]

\[ \Phi = - \frac{B A_t B_t}{2AB} - \frac{B}{2} \left( \frac{B_t}{B} \right)^2 - \frac{A'}{r} + \frac{AB'}{rB} \\
+ \frac{A}{2} \left( \frac{A'}{A} \right)^2 + \frac{A A' B'}{2 \, A \, B}. \]

\[ (1)+(2)+(3)+(4) \quad \leftrightarrow \quad (1)+(3)+\text{div } T=0 \quad \text{(weakly)} \]
References:

- **The locally inertial Glimm Scheme**...


Locally Inertial Glimm/Godunov Method

\[ u_t + f(A, u)_x = g(A, u, x) \]
\[ A' = h(A, u, x) \]

\[ \mathbf{A}_0(t) \]
\[ t = t_j \]
\[ x = r_0 \]
\[ x_{i-1} \quad x_{i-\frac{1}{2}} \quad x_i \quad x_{i+\frac{1}{2}} \quad x_{i+1} \]

Fractional Step Method:
- Solve RP for 1/2 timestep
- Solve ODE for 1/2 time step

\[ A = (A, B) \]
\[ u = (u^1, u^2) \]

Nishida System
\[ p = \frac{c^2}{3} \rho \]

Global Exact Soln of RP, \([\text{Smol,Te}]\)
Remarkable Change of Variables

Equations close under change to Local Minkowski variables:

\[ T \rightarrow u = T_M \]

I.e., \( \text{Div } T = 0 \) reads:

\[
0 = T_{,0}^{00} + T_{,1}^{01} + \frac{1}{2} \left( \frac{2A_t}{A} + \frac{B_t}{B} \right) T^{00} + \frac{1}{2} \left( \frac{3A'}{A} + \frac{B'}{B} + \frac{4}{r} \right) + \frac{B_t}{2A} T^{11}
\]

\[
0 = T_{,0}^{01} + T_{,1}^{11} + \frac{1}{2} \left( \frac{A_t}{A} + \frac{3B_t}{B} \right) T^{01} + \frac{1}{2} \left( \frac{A'}{A} + \frac{2B'}{B} + \frac{4}{r} \right) T^{11} + \frac{A'}{2B} T^{00} - 2 \frac{r}{B} T^{22}
\]

Time derivatives \( A_t \) and \( B_t \) cancel out under change \( T \rightarrow u \)

Good choice because o.w. there is no \( A_t \) equation to close \( \text{Div } T = 0 \! \)!

\[
\frac{A}{r^2 B} \left\{ \frac{r B'}{B} + B - 1 \right\} = \kappa A^2 T^{00} \quad (1)
\]

\[
\frac{B_t}{r B} = \kappa A B T^{01} \quad (2)
\]

\[
\frac{1}{r^2} \left\{ \frac{r A'}{A} - (B - 1) \right\} = \kappa B^2 T^{11} \quad (3)
\]

\[
- \frac{1}{r A B^2} \left\{ B_{tt} - A'' + \Phi \right\} = \frac{2\kappa r}{B} T^{22} \quad (4)
\]
\[
\begin{align*}
\{ T^{00}_M \},_0 + \left\{ \sqrt{\frac{A}{B}} T^{01}_M \right\},_1 &= -\frac{2}{x} \sqrt{\frac{A}{B}} T^{01}_M, \quad (1) \\
\{ T^{01}_M \},_0 + \left\{ \sqrt{\frac{A}{B}} T^{11}_M \right\},_1 &= -\frac{1}{2} \sqrt{\frac{A}{B}} \left( \frac{4}{x} T^{11}_M + \frac{(B - 1)}{x} (T^{00}_M - T^{11}_M) + 2\kappa x B (T^{00}_M T^{11}_M - (T^{01}_M)^2) - 4x T^{22} \right), \quad (2)
\end{align*}
\]

\[
\begin{align*}
\frac{B'}{B} &= -\frac{(B - 1)}{x} + \kappa x B T^{00}_M, \quad (3) \\
\frac{A'}{A} &= \frac{(B - 1)}{x} + \kappa x B T^{11}_M. \quad (4)
\end{align*}
\]

\[
\begin{align*}
\mathbf{u} &= (T^{00}_M, T^{01}_M) \\
\mathbf{A} &= (A, B)
\end{align*}
\]

\[
\begin{align*}
T^{00}_M &= \frac{c^4 + \sigma^2 v^2}{c^2 - v^2 \rho} \\
T^{01}_M &= \frac{c^2 + \sigma^2}{c^2 - v^2 \rho} c v \rho \\
T^{11}_M &= \frac{v^2 + \sigma^2}{c^2 - v^2 \rho} \rho c^2
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}_t + f(A, \mathbf{u})_x &= g(A, \mathbf{u}, x) \\
A' &= h(A, \mathbf{u}, x)
\end{align*}
\]

Locally Inertial Formulation
Flat Space
Relativistic
Euler
\[ u_t + f(A_{ij}, u)_x = 0 \]
\[ u = \begin{cases} 
  u_{i-1,j} & x \leq x_i \\
  u_{ij} & x > x_i 
\end{cases} \]
\[ u(0) = u_{ij}^{RP} \]
\[ A'(r_0, t_{j+1}) = A_0(t_{j+1}) \]
$t = t_0$

$ds^2 = -B(\bar{r}, \bar{t}) d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r},\bar{t})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2$

$\bar{r} = \bar{r}_0$

$M = M(\bar{r}_0)$

$\frac{2M(\bar{r}_0)}{\bar{r}_0} > 1$

$t = \text{the end of inflation} \approx 10^{-30} \text{s} = t_0$
Speculative Question: Could the anomalous acceleration of the Galaxies and Dark Energy be explained within Classical GR as the effect of looking out into a wave?

This model represents the simplest simulation of such a wave.
Standard Model for Dark Energy

- Assume Einstein equations with a cosmological constant:
  \[ G_{ij} = 8\pi T_{ij} + \Lambda g_{ij} \]

- Assume \( k = 0 \) FRW:
  \[ ds^2 = -dt^2 + R(t)^2 \{ dr^2 + r^2 d\Omega^2 \} \]

- Leads to:
  \[ H^2 = \frac{\kappa}{3} \rho + \frac{\kappa}{3} \Lambda \]

- Divide by \( H^2 = \rho_{\text{crit}} \):
  \[ 1 = \Omega_M + \Omega_\Lambda \]

- Best data fit leads to \( \Omega_\Lambda \approx .73 \) and \( \Omega_M \approx .27 \)

- Implies: The universe is 73 percent dark energy
Could the Anomalous acceleration be accounted for by an expansion behind the Shock Wave?

\[ ds^2 = -dt^2 + R(t)^2 \{ dr^2 + r^2 d\Omega^2 \} \]

**TOV:**

\[ ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \]

\[ \frac{2M}{\bar{r}} > 1 \quad \Rightarrow \quad \bar{r} \text{ is timelike} \]
We think this numerical proposal represents a natural mathematical starting point for numerically resolving the secondary waves neglected in the exact solution.

Also a possible starting point for investigating whether the anomalous acceleration/”Dark Energy” could be accounted for within classical GR with classical sources?