Time-Periodic Solutions of the Compressible Euler Equations

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OUR PROGRAM

- To Explicitly Construct...
- To Understand the Structure of...
- To Give a Mathematical Proof of Existence of...

Time-Periodic Solutions of the Compressible Euler Equations

(Unpublished--Work in Progress)

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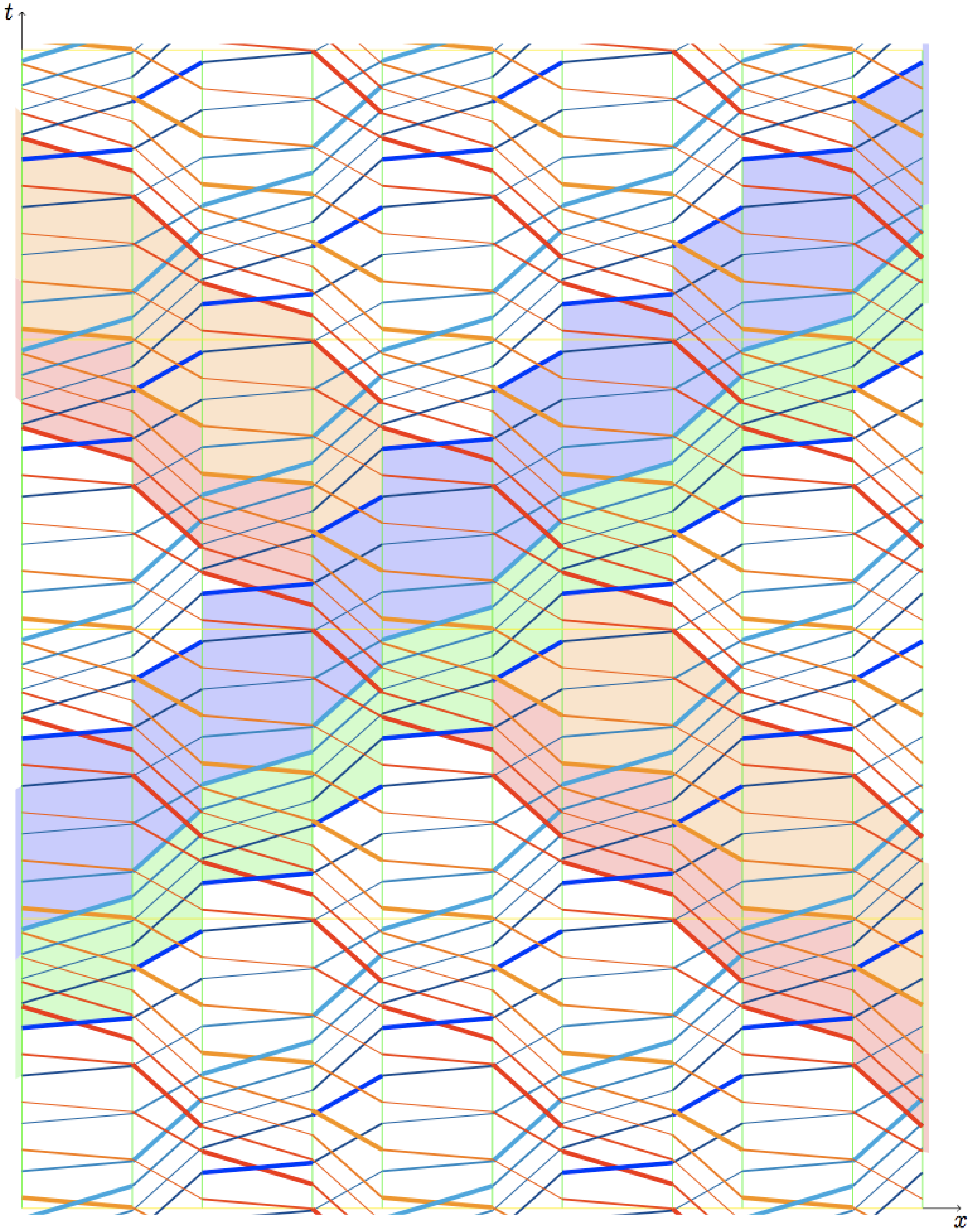
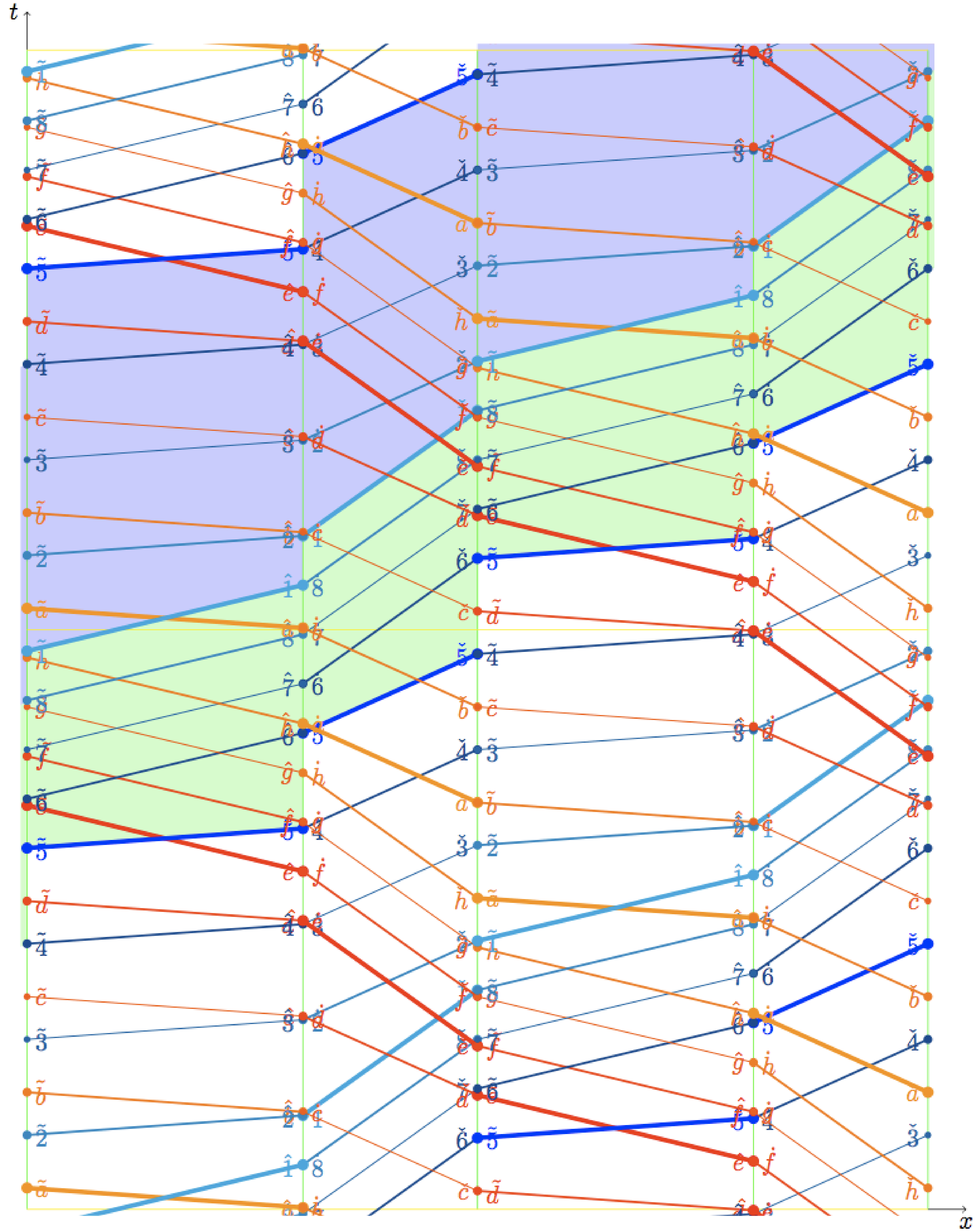
- Q1: By what wave propagation mechanism are time-periodic/shock-free solutions possible?
- Q2: What is the simplest possible structure?



---Our Answer---

The simplest global periodic structure in the xt-plane



Our answer

The Difficulty in a Nutshell

- The compressible Euler Equations in Lagrangian Coordinates: 3-coupled nonlinear conservation laws---

$$\tau_t - u_x = 0$$
$$u_t + p_x = 0$$
$$S_t = 0$$

- Basic warmup problem: scalar Burgers Equation: $u_t + \frac{1}{2}(u^2)_x = 0$

$$u_t + uu_x = 0$$

↓

$$\nabla_{(1,u)}u(x,t) = 0$$

↓

“u=const. along lines of speed u”

↓

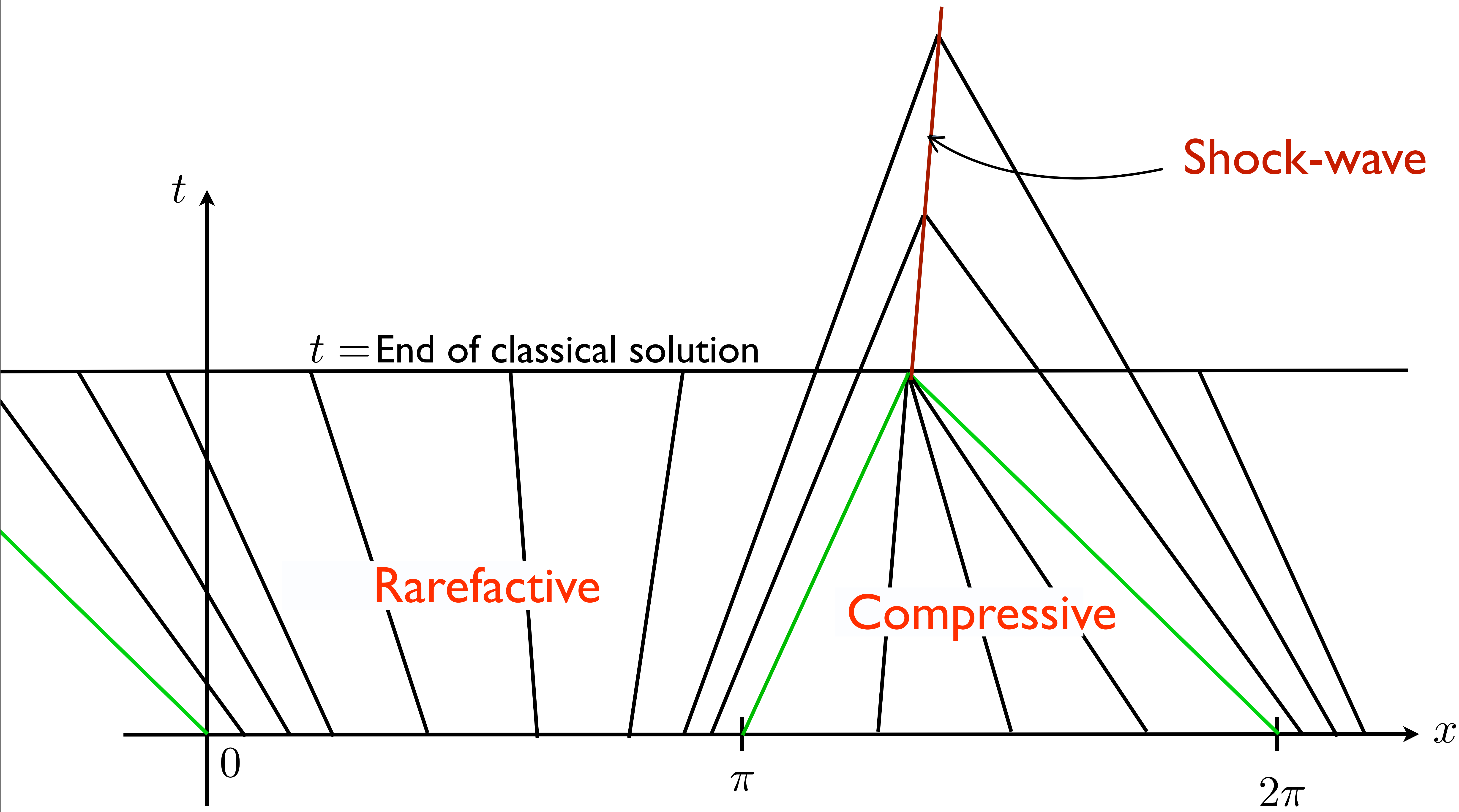
“inconsistent with time-periodic evolution”

The Problem:

- Basic warmup problem: scalar Burgers Equation:

$$u_t + uu_x = 0$$

"inconsistent with time-periodic evolution"



Why we are interested in time-periodic solutions of compressible Euler...

- **Historically** ---The equations were derived by Euler in 1752 as a model for sound wave propagation.

- A first question: Do the nonlinear equations support oscillatory solutions analogous to the linearized theory of sound?
- Riemann had the equations and the problem...
- For most of the last 250 years, experts thought time-periodic solutions were not possible due to shock-wave formation...

- **Scientifically** ---Time-periodic solutions represent dissipation free long distance signaling.

- Could the structure of periodic solutions supply a new paradigm for how sound waves, and other nonlinear waves, really propagate?
- Such waves represent physical waves that travel at a new speed, different from the sound and shock speeds.

- **From a PDE point of view** ---The mechanism requires at least three coupled PDE's

- C.f. work on periodic solutions of the scalar nonlinear Schroedinger Eqn, and scalar nonlinear wave equation.
- Issues of resonances and small divisors analagous to KAM theory arises.
- Diophantime equations and probability theory are involved.

- **Bifurcation theory** ---the bifurcation aspects of the problem have the potential to open the door to all the issues tied to bifurcation theory, such as chaos, period-doubling, etc.

- **Intellectual interest** ---our approach is to guess the solution structure by heuristic reasoning based on nonlinear waves...and this is prerequisite for a rigorous mathematical analysis.

Outline

I. The Compressible Euler Equations

II. History/Prior Results for the Problem

III. Compressive and Rarefactive Waves

IV. The Simplest Possible Periodic Structure that Balances Compression and Rarefaction

V. The Nonlinear Eigenvalue Problem $\approx$ Perturbation of Linear Problem

VI. Exact Linearized Solutions Exhibiting the Simplest Periodic Structure

VII. Isolating Solutions in the Kernel of the Linearized Operator

VIII. Resonances, Small Divisors and Eigenvalues of the Linearized Operator

IX. The Liaunov-Schmidt Method

X. The Bifurcation Equation

XI. The Auxiliary Equation

1. The Compressible Euler Equations

The Compressible Euler Equations

$$\rho_t + div[\rho \mathbf{u}] = 0 \qquad \text{(Ma)}$$

$$(\rho u^i)_t + div[\rho u^i \mathbf{u}] = -\nabla p \qquad \text{(Mo)}$$

$$E_t + div[(E+p)\mathbf{u}] = 0 \qquad \text{(En)}$$

- System (Ma), (Mo), (En) describes the time evolution of a compressible fluid...

$\rho = \frac{mass}{vol}$=density

$\mathbf{u} = (u^1, u^2, u^3)$=velocity

p=pressure

$E = \frac{energy}{vol} = \rho e + \frac{1}{2}\rho \mathbf{u}^2$=total energy

$e = \frac{energy}{mass}$ =specific internal energy

- 5-equations with 6-unknowns $(\rho, u^1, u^2, u^3, p, e)$

An equation of state is required to close the system:

The Entropy:

- Time-irreversibility is measured by the entropy, which evolves according to a derived conservation law:

- The specific entropy S is a state variable obtained by integrating the second law of thermodynamics

$$dS = \frac{de}{T} - p\frac{d\tau}{T} \qquad \text{(2nd Law)}$$

$$\tau = 1/\rho = \text{specific volume}$$

$$S = \frac{entropy}{mass} = \text{specific entropy}$$

- A consequence is the "adiabatic constraint"

$\left(\text{Second Law of Thermodynamics}\right) +$ (Ma),(Mo),(En) ⟹ (Ent)

$$(\rho S)_t + div(\rho S\mathbf{u}) = 0 \qquad \text{(Ent)}$$

The Entropy:

- On smooth solutions:

$$\text{Euler} \Leftrightarrow \begin{cases} \rho_t + div[\rho \mathbf{u}] = 0 \\ (\rho u^i)_t + div[\rho u^i \mathbf{u}] = -\nabla p \\ E_t + div[(E+p)\mathbf{u}] = 0 \end{cases} \Leftrightarrow \begin{array}{c} \rho_t + div[\rho \mathbf{u}] = 0 \\ (\rho u^i)_t + div[\rho u^i \mathbf{u}] = -\nabla p \\ (\rho S)_t + div(\rho S \mathbf{u}) = 0 \end{array}$$

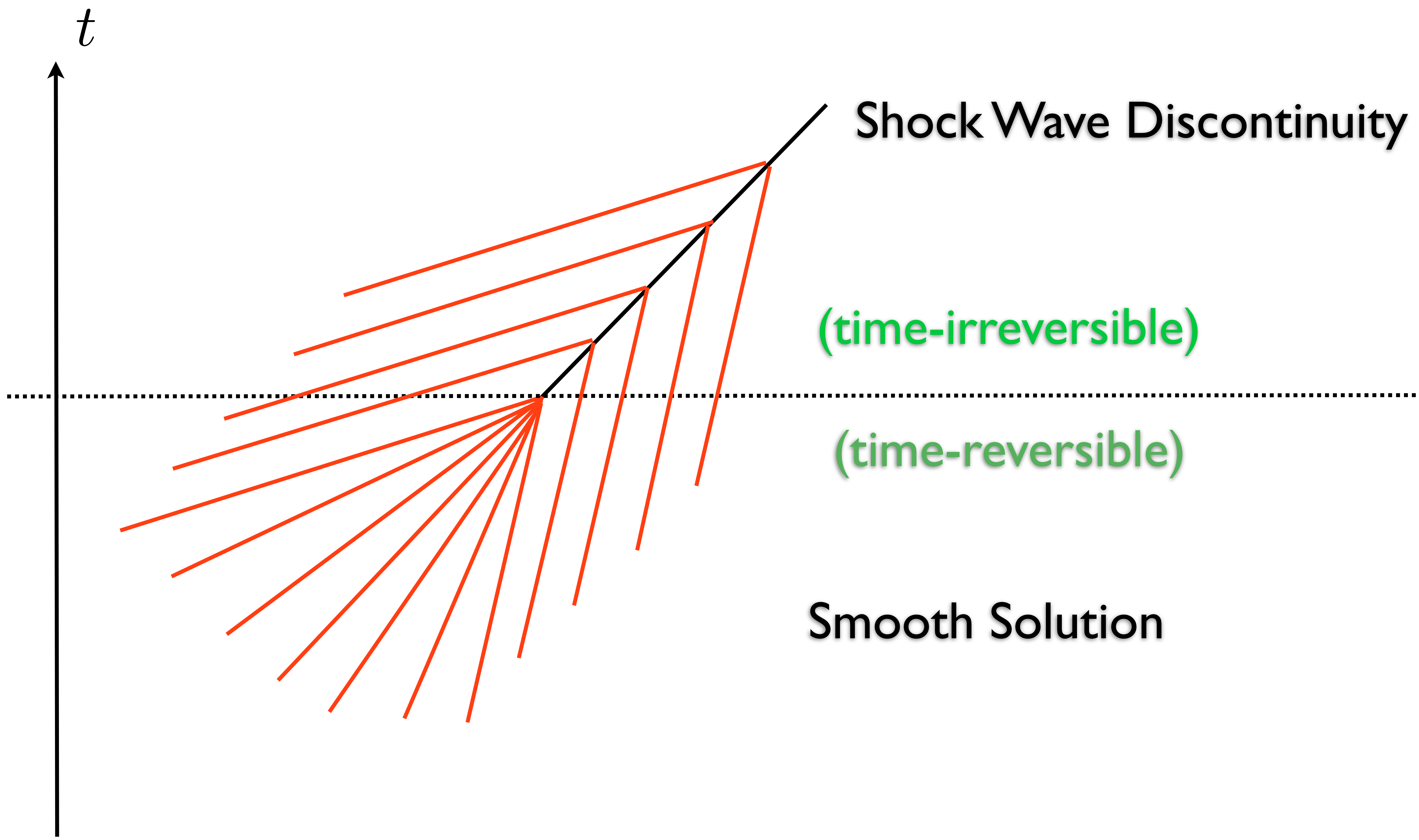
- When shocks are present:

$$\text{Euler} \Leftrightarrow \begin{cases} \rho_t + div[\rho \mathbf{u}] = 0 \\ (\rho u^i)_t + div[\rho u^i \mathbf{u}] = -\nabla p \\ E_t + div[(E+p)\mathbf{u}] = 0 \end{cases} + \quad (\rho S)_t + div(\rho S \mathbf{u}) > 0$$

- Total entropy is strictly increasing in time when shocks are present.

- Conclude: The compressible Euler equations describe the time evolution of a perfect fluid in the limit that all dissipative forces (like friction and heat conduction) are neglected.

- Nevertheless: There is a canonical dissipation present at the zero dissipation limit, and this is encoded in the rate of increase of the entropy at shock waves:



- Conclude: When shock waves are present, the Entropy is a

Strict Liapunov Function

⇓

Time-periodic solutions of the compressible Euler equations must be

Shock-Free

1-Dimensional Wave Propagation:

- For wave propagation in x-direction :

$$\text{Euler} \Longleftrightarrow \begin{cases} \rho_t + (\rho u)_x = 0 & \text{(Ma)} \\ (\rho u)_t + (\rho u^2 + p)_x = 0 & \text{(Mo)} \\ E_t + \{(E+p)u\}_x = 0 & \text{(En)} \end{cases}$$

- 3-conservation laws with a derived (convex) entropy:

$$(\rho S)_t + (\rho S u)_x = 0 \qquad \text{(Ent)}$$

- 5-unknowns $(\rho, p, e, S, u) \Longrightarrow$

An equation of state relating (ρ, p, e, S) is required to close the system.

The Polytropic Equation of State

- The equation of state for a non-interacting gas composed of molecules can be derived from first principles using only the equipartition of energy principle and the second law of thermodynamics, and leads to the fundamental relation for a polytropic, or γ-law gas:

$$e = c_\tau T = c_\tau \left(\frac{1}{\tau}\right)^{\gamma-1} exp\left\{\frac{S}{c_\tau}\right\}$$

where:

$$\gamma = 1 + 2/3r = \text{adiabatic gas constant}$$

$$r = \text{number of atoms in a molecule}$$

- It follows that:

$$\gamma = \text{ratio } c_p/c_v \text{ of specific heats (measurable)}$$

$$p = -\frac{\partial e}{\partial \tau}(S, \tau)$$

The Euler system in Lagrangian coordinates (relative to a frame moving with the fluid)

Assuming:

- Smooth, 1-D motion
- polytropic equation of state

The Euler equations are equivalent to:

$$\boxed{\begin{aligned} \tau_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0 \end{aligned}}$$

$$\text{(Ma)}$$

$$\text{(Mo)}$$

$$\text{(Ent)}$$

- Three coupled nonlinear equations in the three unknowns (τ, u, S)

System closes with the γ-law relation

$$p = K\tau^{-\gamma}e^{S/c_\tau}$$

- 3x3 Lagrangian equations:

$$\tau_t - u_x = 0 \qquad \text{(Ma)}$$

$$u_t + p_x = 0 \qquad \text{(Mo)}$$

$$S_t = 0 \qquad \text{(Ent)}$$

- Note: when S=cost, solution reduces to the 2x2 p-system, a system in which there is sound wave propagation in forward and back directions

$$\tau_t - u_x = 0$$

$$u_t + p(\tau, S)_x = 0$$

- Sound waves: $\boxed{\frac{dx}{dt} = \pm c}$ c= sound speed

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma}\tau^{-\frac{\gamma+1}{2}} e^{S/2c_\tau}. \qquad (1)$$

- Note: there are trivial time-periodic solutions that correspond to an entropy gradient passively transported with the fluid.

$$S(x,0) \equiv S_0(x), \quad \rho(x,t) \equiv \rho_0(x)$$

$$u \equiv u_0 = const., \quad p \equiv p_0 = const.$$

$$p(\rho_0(x), S_0(x)) = p_0$$

- Such solutions transmit no sound-waves

By "time-periodic solutions", we always mean non-trivial solutions with sound-wave propogation

- Note: linearizing about a constant state

$$\begin{aligned} \rho &= \rho_0 \\ S &= S_0 \\ u &= 0 \end{aligned}$$

equations reduce to the wave equation

⬇

Linearized theory of sound

$$\rho_{tt} - c^2 \rho_{xx} = 0$$

II. History/Prior Results

Periodic solutions of Compressible Euler

History/Background

- 1687-- Principia/ Newton attempted to give a continuum version of his laws of motion in order to derive the speed of sound from first principles.
- 1749-- D'Alembert introduced the linear wave equation to describe displacements of a vibrating string.

  The wave equation is the basic equation in which all waves propagate at the same speed, and so it was natural to conjecture that sinusoidal oscillations in the air might account for sound waves. But D'Alembert had no physical derivation of this from first principles.
- 1752-- Euler (building on work of Bernoulli) completed Newton's program by deriving the fully nonlinear theory of sound waves from first principles.

  Euler showed that asymptotically, in the limit of weak signals, the compressible Euler equations reduce to the wave equation in the density, thus demonstrating that sound waves could be described by periodic sinusoidal oscillations in the density.
- This established the framework for the (linear) theory of sound.

Ref: D. Christodoulou, ETH Zurich, 2006/Bulletin, Oct. 2007.

History/Background

- The first question to ask after Euler is:

  Do the fully nonlinear equations admit time-periodic, oscillatory solutions that propagate information like the linear sound waves of the wave equation?

- For most the last 250 years experts have generally thought that such time periodic solutions did not exist, due to the phenomenon of shock wave formation...

History/Background

- 1857-- Riemann showed that shock-wave discontinuities can form from smooth solutions of the compressible Euler equations.

  Introduced Riemann invariants and the Riemann problem to continue the solutions past the time of shock formation

- After Reimann...

- Shock-waves became the cental issue in the study of the compressible Euler equations...

Riemann, B. *Uber die Fortpfanzung ebener Luftwellen von endlicher Schwingungswete*, Abhandlungen der Gesellshaft der Wissenshaften zu Gottingen, Mathematisch-physikalishe Klasse, Vol. 8, 43 (1858-59)

History/Background

- 1964-- Lax proved finite time blow-up in derivatives for 2x2 systems for which the nonlinear fields are "genuinely nonlinear"

  like the p-system.

  Lax's argument is sufficient to imply blow-up in the derivative for space-periodic solutions of the p-system thereby implying the formation of shock-waves---inconsistent with time-periodic evolution.

P.D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, Jour. Math. Physics, Vol. 5, pp. 611-613 (1964).

- 1970-- Glimm and Lax give definitive result for 2x2 systems---shocks must form from periodic initial data for the 2x2 p-system

  Thm: Solutions of the p-system starting from space periodic initial data (small in L^∞) must form shock-waves and decay in the total varation norm at rate ate 1/t.

J. Glimm, P.D. Lax, *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Memoirs Amer. Math Soc. 101(1970).

History/Background

- 1974-97 Blow-up results that extend Lax's result to 3x3 systems were not sufficient to rule out the possibility of time- periodic sound wave propagation in the compressible Euler equations...

F. John, *Formation of singularities in one-dimensional wave propagation*, Comm. Pure Appl. Math., Vol. 27, pp. 377-405 (1974).

T.P. Liu, *Development of singularities in the nonlinear waves for quasi-linear hyperbolic partial differential equations*, J. Diff. Eqns, Vol. 33, pp. 92-111 (1979).

Li Ta-Tsien, Zhou Yi and Kong De-Xing, *Global classical solutions for general quasilinear hyperbolic systems with decay initial data*, Nonlinear. Analysis., Theory., Methods. and Applications., Vol. 28, No. 8, pp. 1299-1332 (1997).

- 1975-- Jim Greenberg produced an example of a time-periodic solution in a 2x2 genuinely nonlinear system--

The example required a degeneracy in wave speeds <u>not present in compressible Euler</u>

History/Background

- 1984-88-- The idea that time periodic solutions may exist was kindled by work of Majda, Rosales and Schonbeck:

A. Majda and R. Rosales, *Resonantly interacting weakly nonlinear hyperbolic waves I. A single variable*, Stud. in Appl. Math., 22, pp. 149-179 (1984).

A. Majda, R. Rosales and M.Schonbeck, *A canonical system of integrodifferential equations arising in resonant nonlinear acoustics*, Stud. in Appl. Math., 79, pp. 205-262 (1988).

- 1988-- Pego produced a periodic solution to an asymptotic model

R.L. Pego, *Some explicit resonating waves in weakly nonlinear gas dynamics*, Studies in Appl. Math., Vol. 79, pp. 263-270 (1988).

History/Background

- 1996-99-- Rosales and two students, Shefter and Vaynblat, produced detailed numerical simulations of the Euler equations starting from periodic initial data, and these numerical studies indicated that periodic solutions of the 3x3 compressible Euler equations do not decay like the 2x2 p-system, and they made observations about the possibility of periodic, or quasi-periodic attractor solutions.

M. Shefter and R. Rosales, *Quasi-periodic solutions in weakly nonlinear gas dynamics*, Studies in Appl. Math., Vol. 103, pp. 279-337 (1999).

D. Vaynblat, *The strongly attracting character of large amplitude nonlinear resonant acoustic waves without shocks. A numerical study.* M.I.T. Dissertation, (1996).

History/Background

- Authors work: how Lie Bracket effects in the full 3x3 Euler system can fundamentally alter wave interactions

B. Temple, R. Young, *The large time existence of periodic solutions for the compressible Euler equations*, Contemporanea Mathematica, Proceedings of the Fourth International Workshop on Partial Differential Equations, IMPA, Brazil, July, 1995.

B. Temple, R. Young, *The large time stability of sound waves*, Commun. Math. Phys, Vol. 179, 417-466 (1996).

- Warmup problems suggesting that periodic solutions may exist were investigated by Young

R. Young, *Periodic solutions for conservation laws*, Contemp. Math., Vol. 225, pp. 239-256 (2000).

R. Young, *Sustained solutions for conservation laws*, Commun. PDE, Vol. 26, pp. 1-32 (2001).

- CONCLUDE: Until now, we do not understand the structure of time periodic solutions, nor the mechanism that can prevent shock formation.
- Moreover, it is difficult to numerically simulate time-periodic solutions by starting with general space periodic data and running the solution until the shock-wave dissipation resolves itself into a periodic configuration...
- ...Errors are difficult to control in large time simulations...
- ... Shock-waves alter the entropy field, and so the background entropy field remains unknown until the shock-wave dissipation is done. The final entropy field to which a general time periodic solution will decay is then pretty much impossible to predict, and hence difficult to simulate without understanding the mechanism for periodic wave propagation at the start.

III. Compressive and Rarefactive waves

Lagrange equations as a System of Conservation Laws

$$\tau_t - u_x = 0$$
$$u_t + p_x = 0$$
$$S_t = 0$$

$\Leftrightarrow$

$$\begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_t + \begin{pmatrix} -u \\ p \\ 0 \end{pmatrix}_x = 0$$

$\Leftrightarrow$

$$U_t + F(U)_x = 0$$

- Sound speed: $$c = \sqrt{-p_\tau} = \sqrt{K\gamma\tau^{-\frac{\gamma+1}{2}} e^{S/2c_\tau}}.$$

- The system supports 3 Wave Families determined by the eigenfamilies (λ_i, R_i) of dF :

1-waves	2-waves	3-waves
$\lambda_1 = -c$	$\lambda_2 = 0$	$\lambda_3 = c$

Simple Waves

- nxn system of conservation laws:

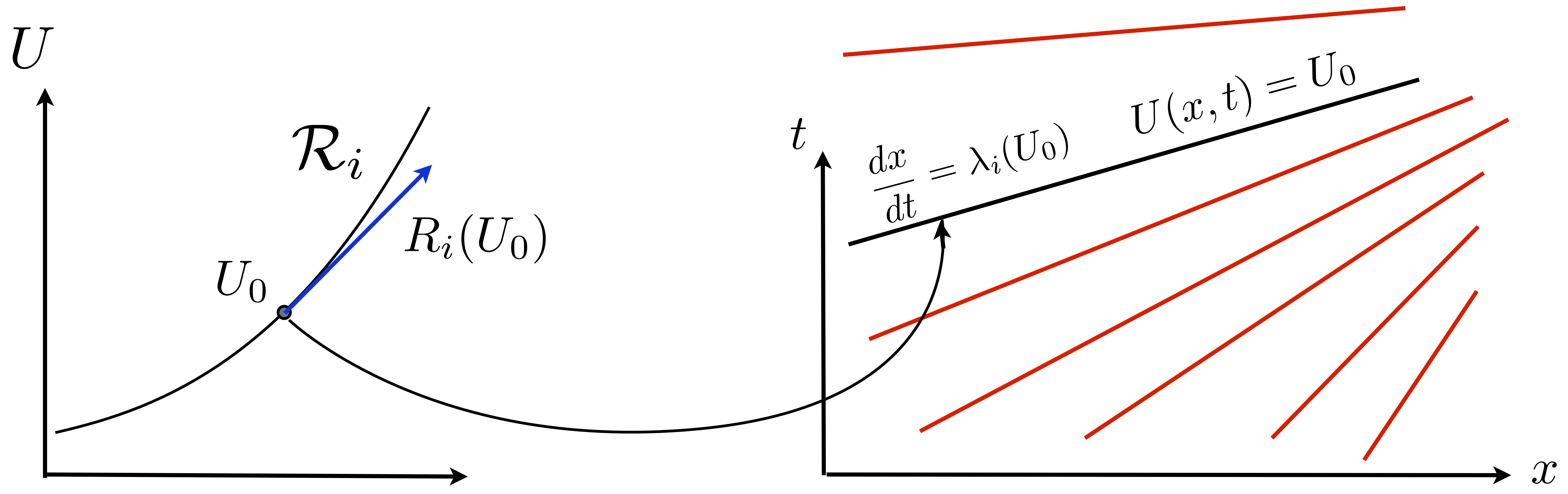
$$U_t + F(U)_x = 0 \quad \Leftrightarrow \quad U_t + dF \cdot U_x = 0$$

- Assume that (λ_i, R_i) is a (smooth) eigen-field for dF :

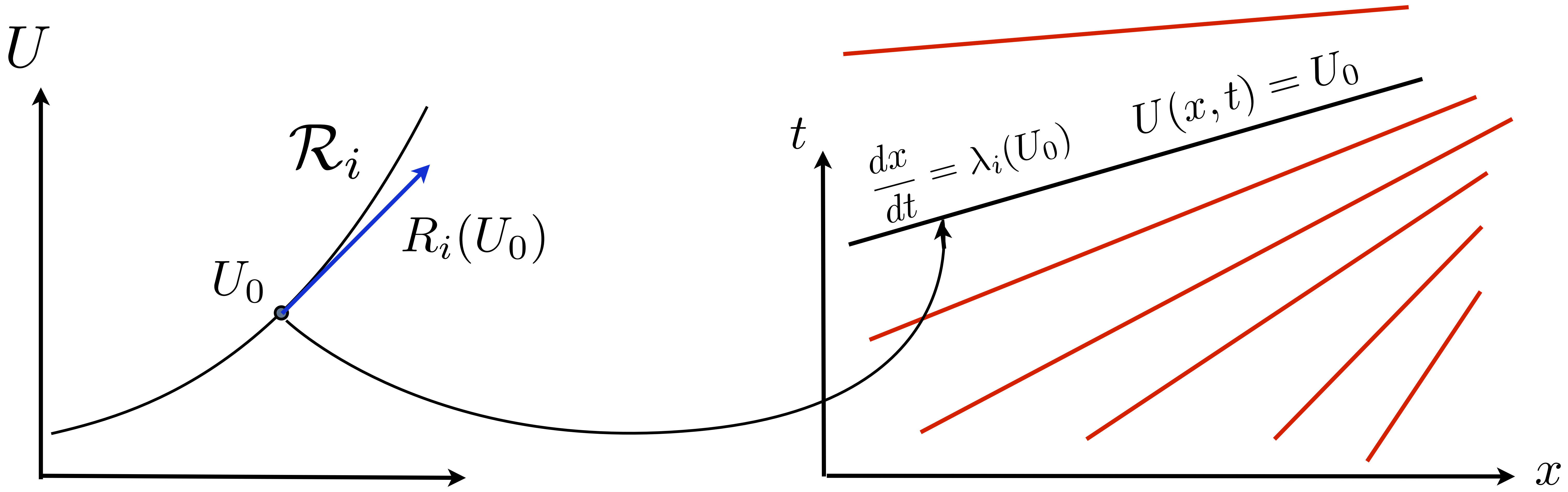
$$(dF - \lambda_i I)\, R_i = 0$$

Let $\mathcal{R}_i$ denote an integral curve of vector field R_i

Letting states U on $\mathcal{R}_i$ propagate with speed $\lambda_i(U)$ defines a 1-parameter family of simple waves



- Letting states U on $\mathcal{R}_i$ propagate with speed $\lambda_i(U)$ defines a 1-parameter family of simple waves



- I.e., assume $\lambda(x,t)$ satisfies $\lambda = const.$ along $\frac{dx}{dt} = \lambda$.

$\nabla_{(x,t)}\lambda \perp \text{curve } \frac{dx}{dt} = \lambda$ ⇒ $-\dfrac{\lambda_t}{\lambda_x} = \lambda$

Then $U(\lambda(x,t))$ satisfies

$$U_t + dF \cdot U_x = U'\lambda_t + dF \cdot U'\lambda_x = \lambda_x \left(dF - \left(-\frac{\lambda_t}{\lambda_x} \right) \right) U' = 0$$

- The three Characteristic families of Euler:

$$U_t + F(U)_x = 0 \quad \Leftrightarrow \quad U_t + dF \cdot U_x = 0$$

$$\begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_t + \begin{pmatrix} -u \\ p \\ 0 \end{pmatrix}_x = 0 \quad \Leftrightarrow \quad \begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_t = \begin{pmatrix} 0 & -1 & 0 \\ p_\tau & 0 & p_S \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_x = 0$$

- Three eigen-families of dF

1-waves	2-waves	3-waves
$\lambda_1 = -c$	$\lambda_2 = 0$	$\lambda_3 = c$
$R_1 = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}$	$R_2 = \begin{pmatrix} -p_S/p_\tau \\ 0 \\ 1 \end{pmatrix}$	$R_3 = \begin{pmatrix} 1 \\ -c \\ 0 \end{pmatrix}$

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma} \left(\frac{1}{\tau}\right)^{\frac{\gamma+1}{2}} e^{S/2c_\tau}$$

- Three eigen-families of dF ...

1-waves	2-waves	3-waves
$\lambda_1 = -c$	$\lambda_2 = 0$	$\lambda_3 = c$
$R_1 = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}$	$R_2 = \begin{pmatrix} -p_S/p_\tau \\ 0 \\ 1 \end{pmatrix}$	$R_3 = \begin{pmatrix} 1 \\ -c \\ 0 \end{pmatrix}$

Conclude:

S is constant through 1,3-waves

u, p are constant through 2-waves

- 3 characteristic families associated with (λ_i, R_i) :

1-waves (back)

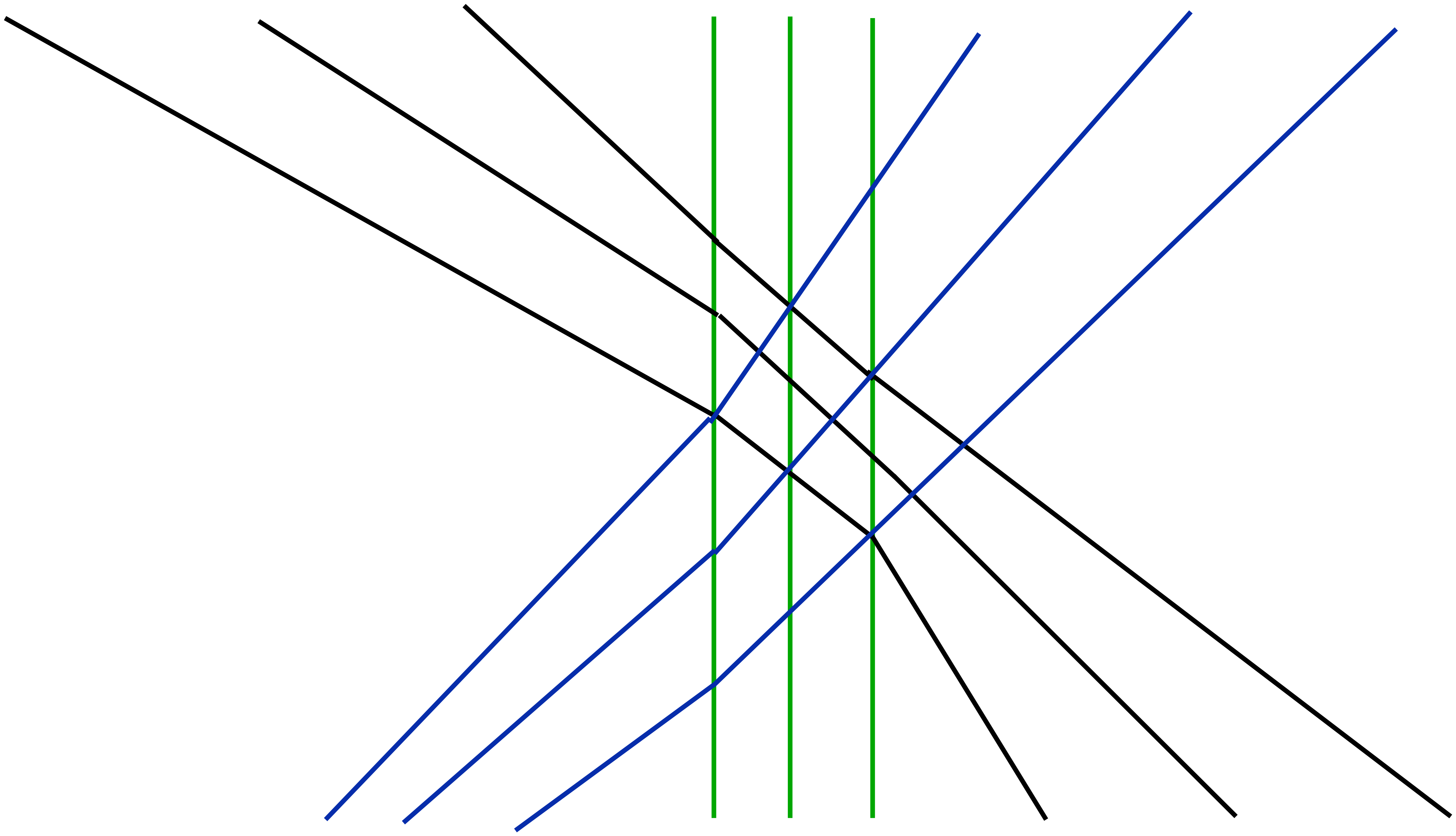
$$\frac{dx}{dt} = \lambda_1 = -c$$

2-waves

$$\frac{dx}{dt} = \lambda_2 = 0$$

3-waves (forward)

$$\frac{dx}{dt} = \lambda_3 = c$$



3-waves

Genuinely Nonlinear

$$\nabla_{R_1} \lambda_1 > 0$$

2-waves

Linearly Degenerate

$$\nabla_{R_2} \lambda_2 \equiv 0$$

1-waves

Genuinely Nonlinear

$$\nabla_{R_3} \lambda_3 > 0$$

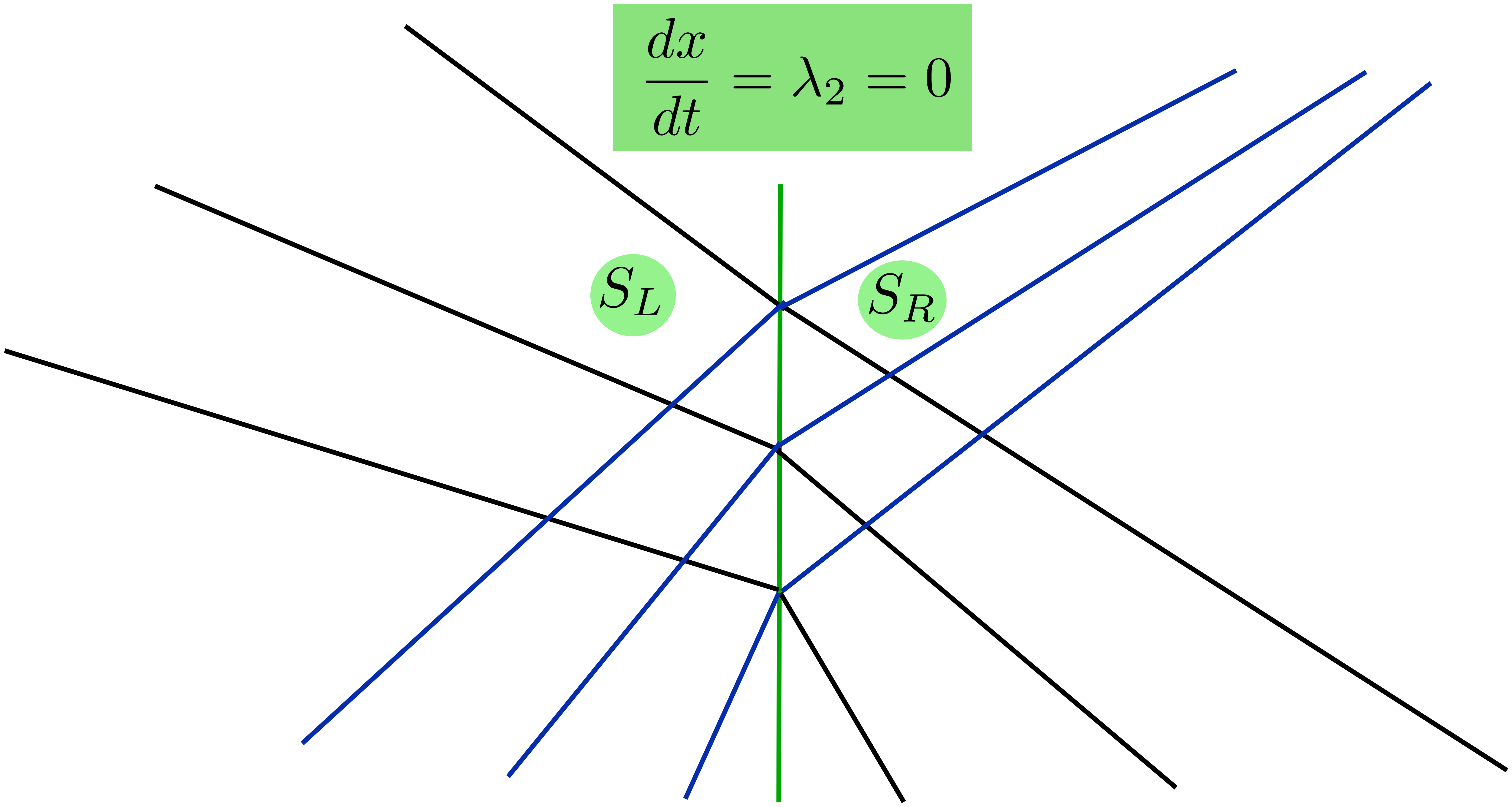
- The 2-field (λ_2, R_2) is Linearly Degenerate:

$$\nabla_{R_2} \lambda_2 \equiv 0$$

2-waves can be rescaled into time-reversible contact discontinuities

2-contact discontinuity

$$\frac{dx}{dt} = \lambda_2 = 0$$



Conclude: time-periodic solutions allow for <u>discontinuities in entropy</u> S

Riemann Invariants (r, s)

- At constant entropy:

$r \equiv const.$ along 1-characteristics

$s \equiv const.$ along 3-characteristics

$S = const.$

$$\tau_t - u_x = 0 \qquad u_t + p_x = 0 \qquad S_t = 0$$

$\Leftrightarrow$

$$r_t - c r_x = 0 \qquad s_t + c s_x = 0$$

$$r = u - \int_\tau^\infty c\,d\tau = \left(\frac{2\sqrt{K\gamma}}{\gamma - 1}\right)\left(\frac{1}{\tau}\right)^{\frac{\gamma-1}{2}} e^{S/2c_\tau}$$

$$s = u + \int_\tau^\infty c\,d\tau = \left(\frac{2\sqrt{K\gamma}}{\gamma - 1}\right)\left(\frac{1}{\tau}\right)^{\frac{\gamma-1}{2}} e^{S/2c_\tau}$$

- Problem: r and s depend on the entropy S

A Convenient Change of Variables

- Change Variables: $(\tau, u, S) \longmapsto (z, u, m)$

$$m = e^{S/2c_\tau}$$

m re-scales the entropy S

$$z = \int_\tau^\infty \frac{c}{m} d\tau = \left(\frac{2\sqrt{K\gamma}}{\gamma - 1}\right)\left(\frac{1}{\tau}\right)^{\frac{\gamma-1}{2}}$$

z re-scales the specific vol. $\tau = 1/\rho$

- The transformed Euler equations:

$$\begin{aligned} z_t + \tfrac{c}{m} u_x &= 0 \\ u_t + mcz_x + 2\tfrac{p}{m} m_x &= 0 \\ m_t &= 0 \end{aligned}$$

- At each constant value of the entropy, the system reduces to a transformed version of the 2x2 p=system that depends on the entropy through variable m:

$$
\begin{aligned}
z_t + \frac{c}{m} u_x &= 0 \\
u_t + m c z_x + 2 \frac{p}{m} m_x &= 0 \\
m_t &= 0
\end{aligned}
$$

$$S = \text{const.}$$

$$
\boxed{
\begin{aligned}
z_t + \frac{c}{m} u_x &= 0 \\
u_t + m c z_x &= 0
\end{aligned}
}
$$

- In terms of the Riemann invariants r and s:

$$r = u - mz$$

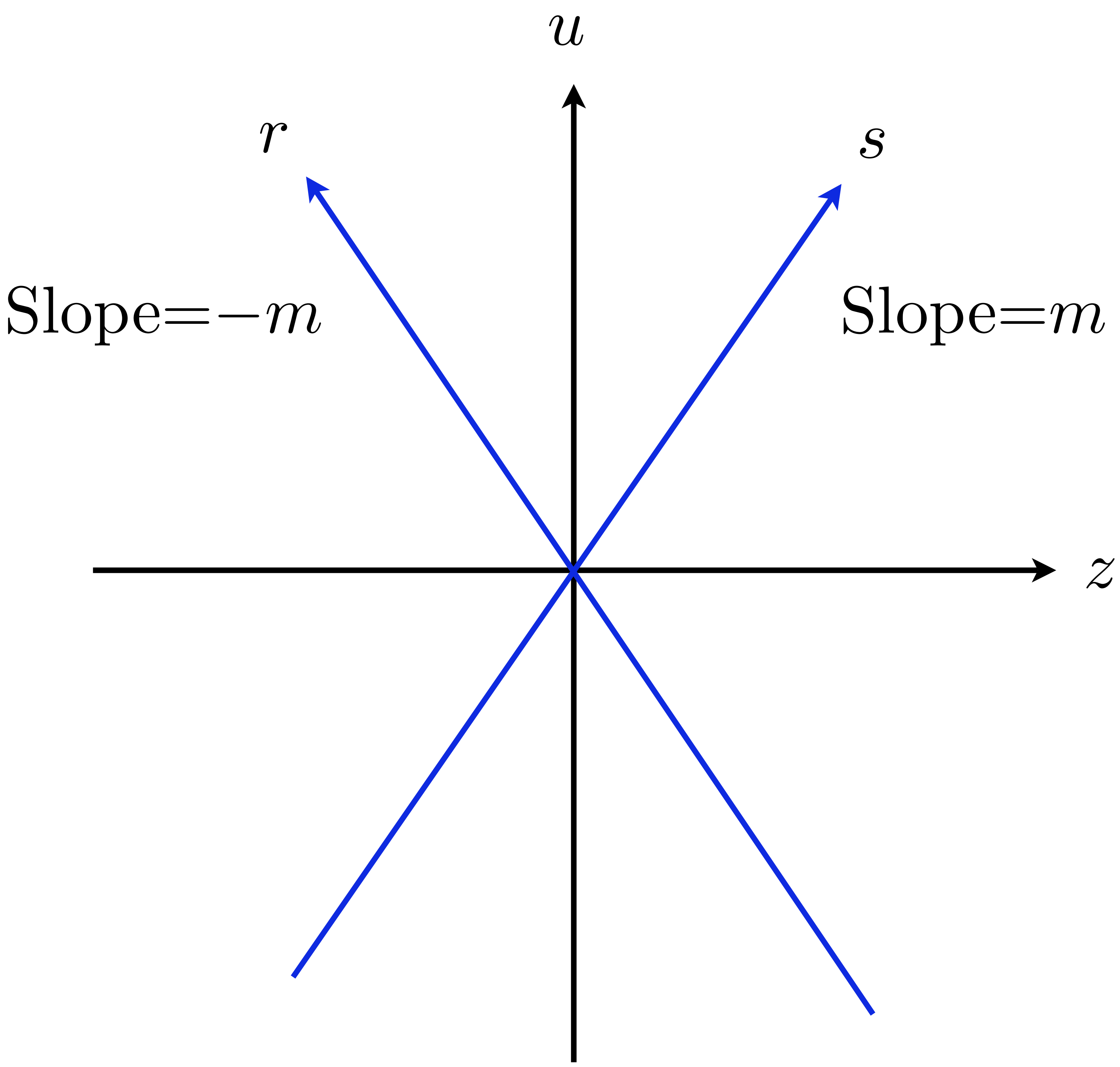
$$s = u + mz$$

$$\boxed{\begin{aligned} z_t + \tfrac{c}{m} u_x &= 0 \\ u_t + mcz_x &= 0 \end{aligned}} \quad \Longleftrightarrow \quad \begin{aligned} r_t - cr_x &= 0 \\ s_t + cs_x &= 0 \end{aligned}$$

(z, u) <u>independent</u> of entropy $\qquad$ (r, s) <u>depend</u> on entropy

- Conclude: Equations in (z, u) isolate the dependence on S in coefficients

Relationship Between Coordinates



Riemann invariant coordinates in (z,u)-plane

$$m = e^{S/2c_\tau}$$

Compressive and Rarefactive Waves (R/C)

Consider 1,3-waves at constant entropy S:

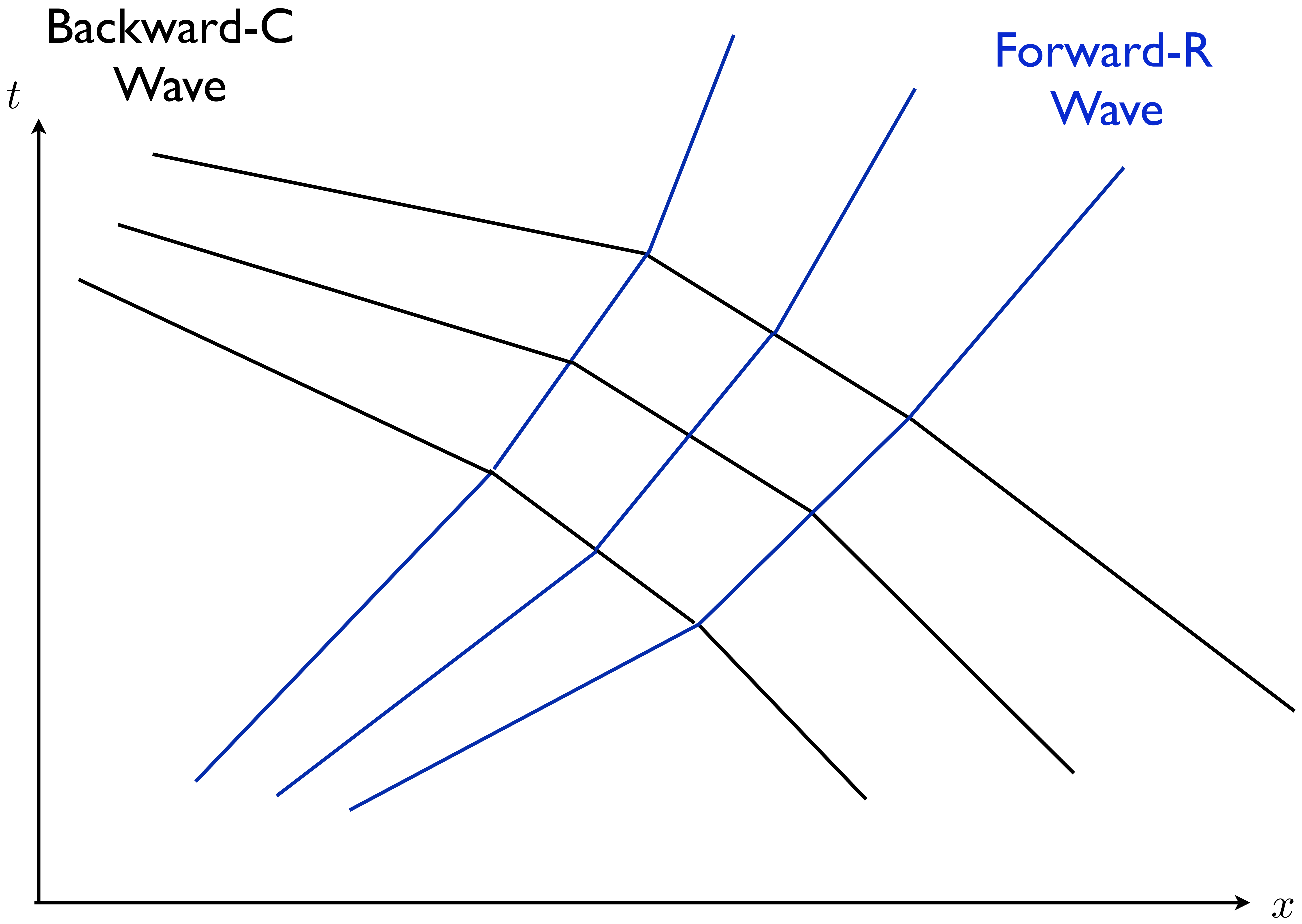
$$1\text{-wave} \equiv \text{“}backward\text{”-wave}$$

$$3\text{-wave} \equiv \text{“}forward\text{”-wave}$$

Definition: The R/C character of a wave in a general smooth solution is defined (pointwise) by:

Forward R	iff	$s_t \leq 0,$
Forward C	iff	$s_t \geq 0,$
Backward R	iff	$r_t \geq 0,$
Backward C	iff	$r_t \leq 0.$

Theorem: R/C character is preserved along backward and forward characteristics at constant entropy:

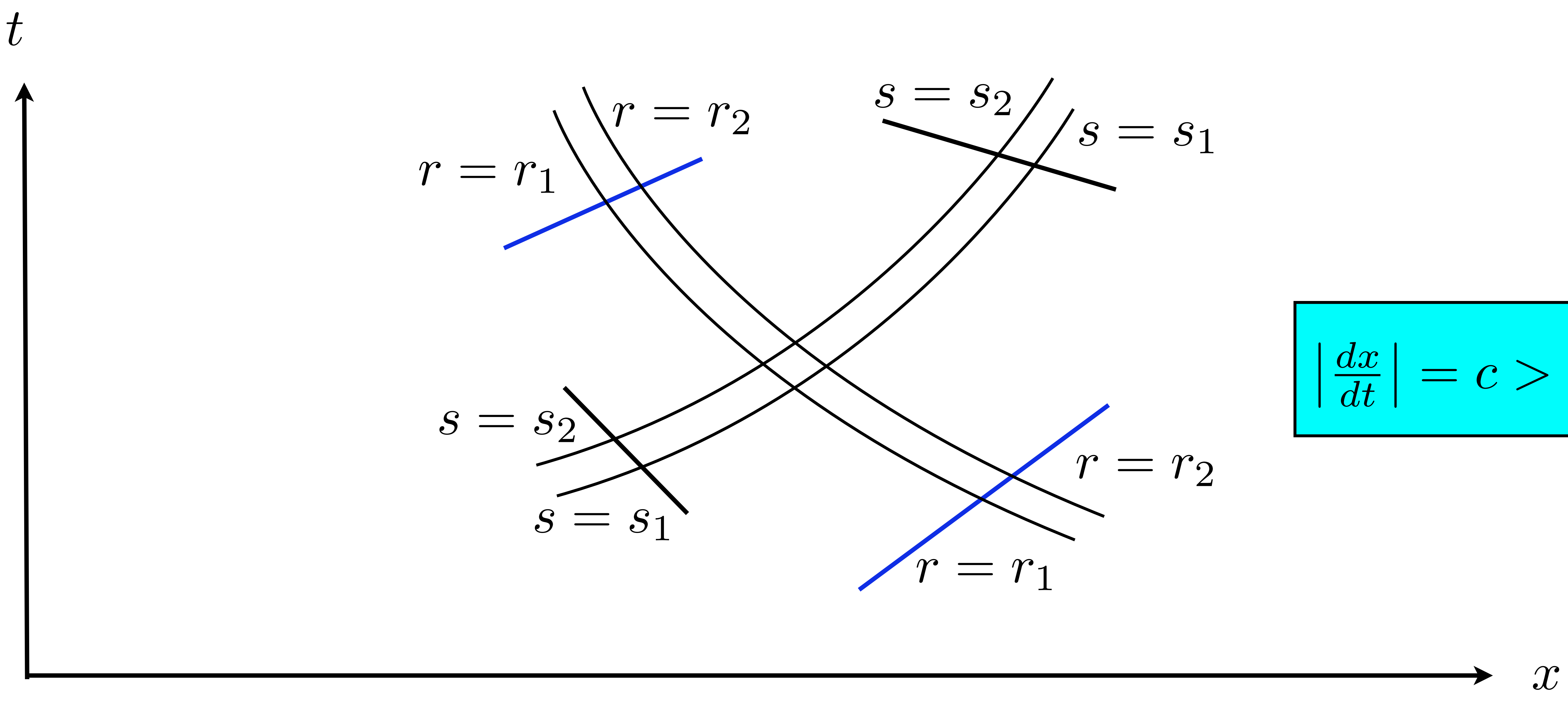


Proof: Direct consequence of... when S=Const...

$$r \equiv const. \text{ along 1-characteristics}$$
$$s \equiv const. \text{ along 3-characteristics}$$

$Sign\{r_t\}$ is constant along 1-characteristics

$Sign\{s_t\}$ is constant along 3-characteristics



$\left|\frac{dx}{dt}\right| = c > 0$

Correctness of Definition of R/C

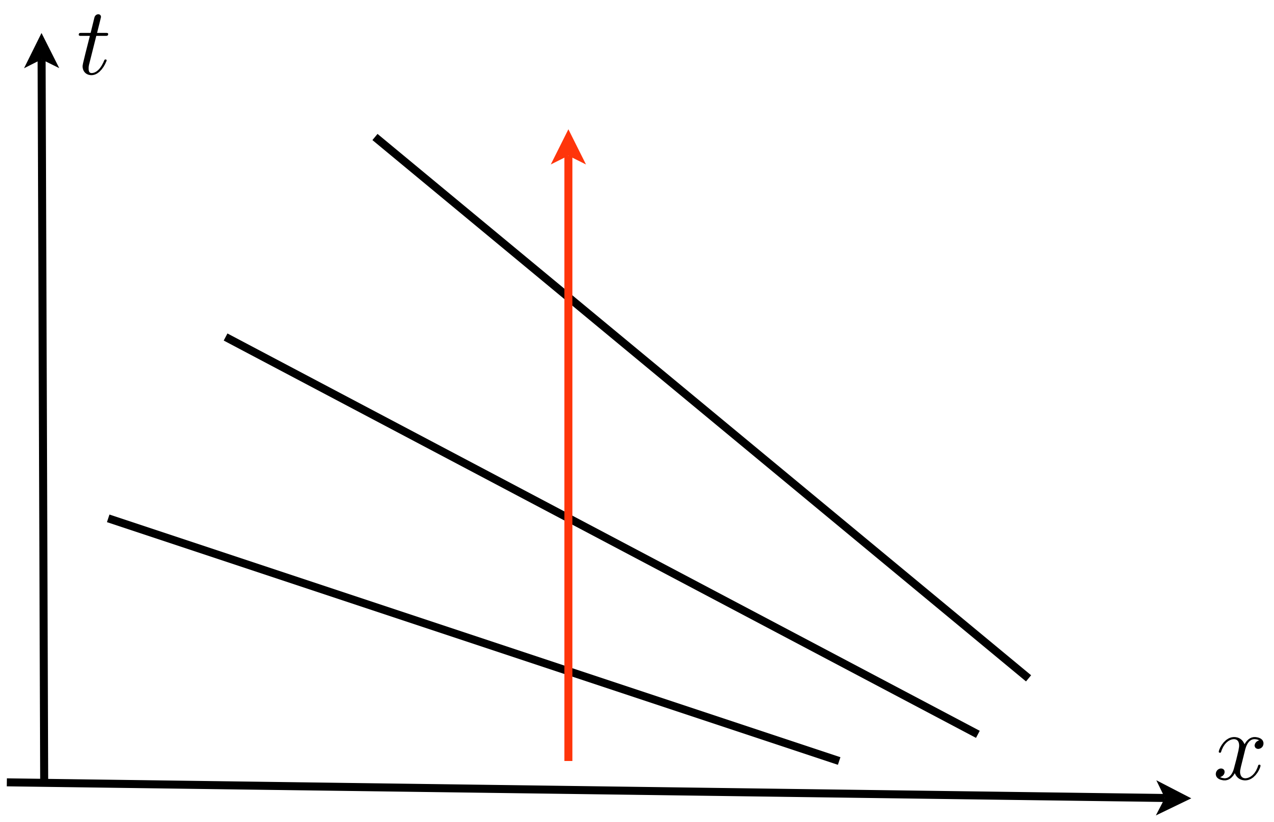
- C.f. the sound speed:

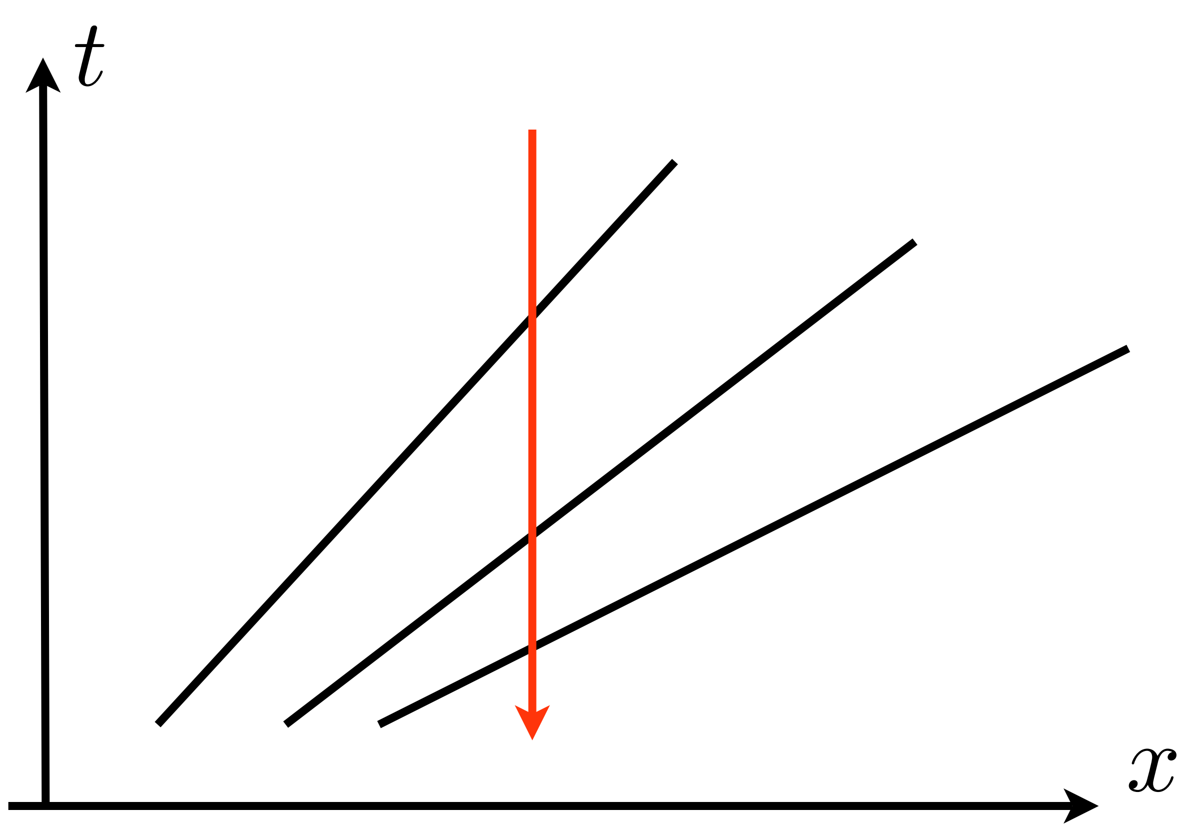
$$c = K_c m z^{\frac{\gamma+1}{\gamma-1}} = K_c m^{\frac{-2}{\gamma-1}} (s-r)^{\frac{\gamma+1}{\gamma-1}}$$

- Conclude: (R=Rarefactive, C=Compressive)

$$\begin{aligned} \text{Backward } R \text{ iff } r_t &\geq 0, \\ \text{Backward } C \text{ iff } r_t &\leq 0 \end{aligned}$$

$$\begin{aligned} \text{Forward } R \text{ iff } s_t &\leq 0, \\ \text{Forward } C \text{ iff } s_t &\geq 0 \end{aligned}$$





$\lambda = -c$ increases ↑

$\lambda = +c$ increases ↓

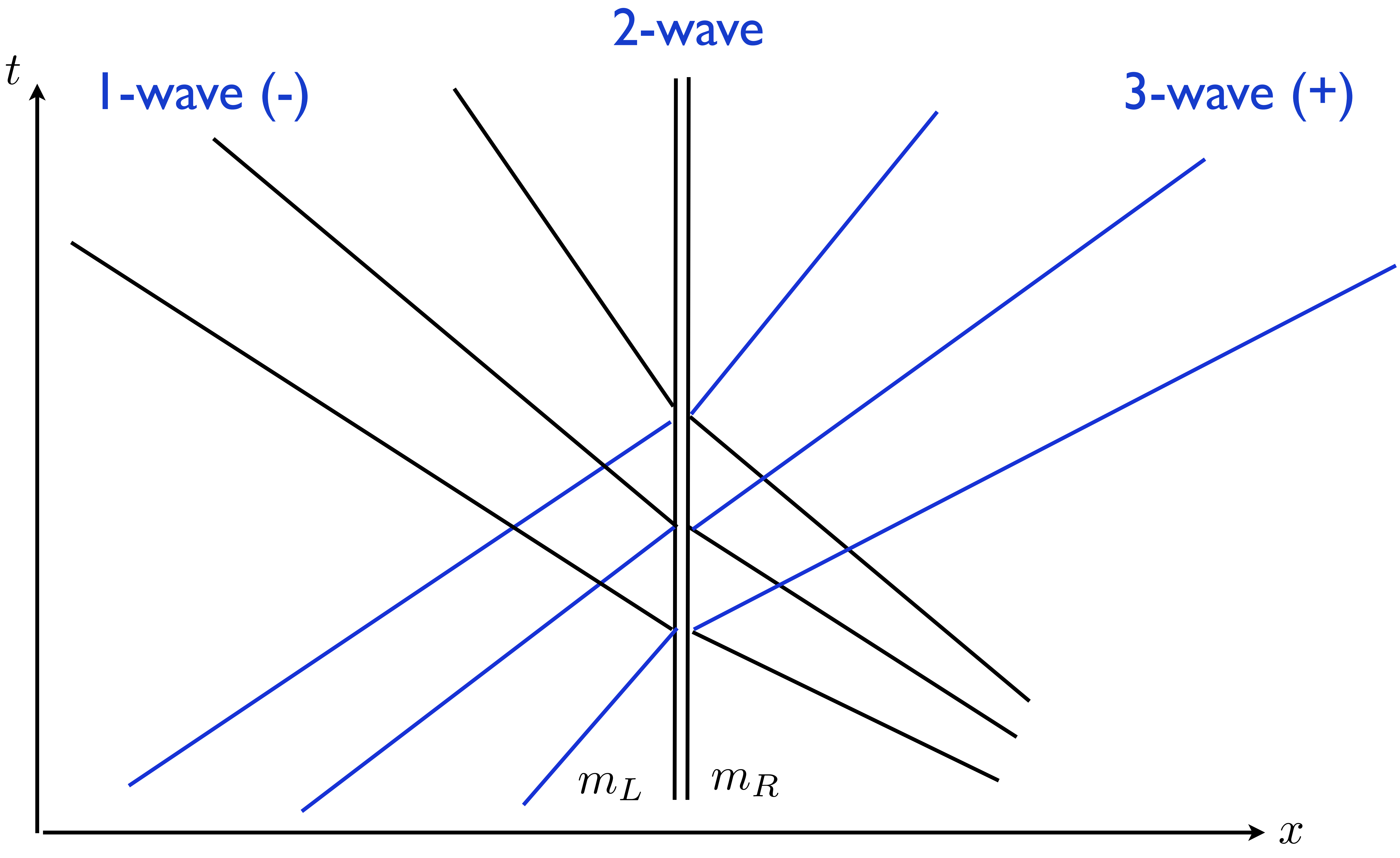
$$-\frac{\partial c}{\partial r} > 0$$

$$\frac{\partial c}{\partial s} > 0$$

E.g., $R \equiv$ wave speed <u>increases</u> from <u>left to right</u> across the wave

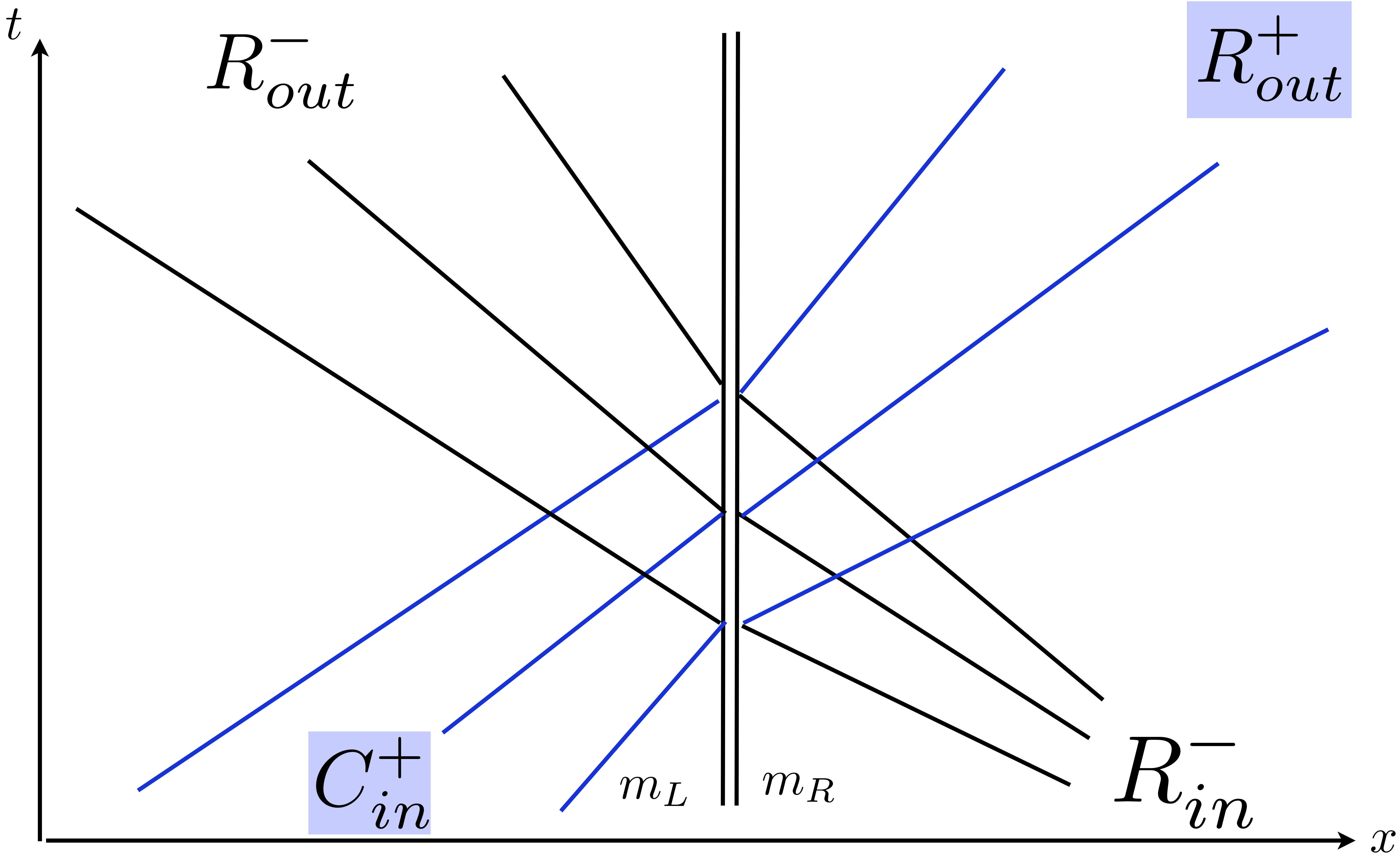
The R/C character of a wave CAN CHANGE at an entropy jump...

For Example:



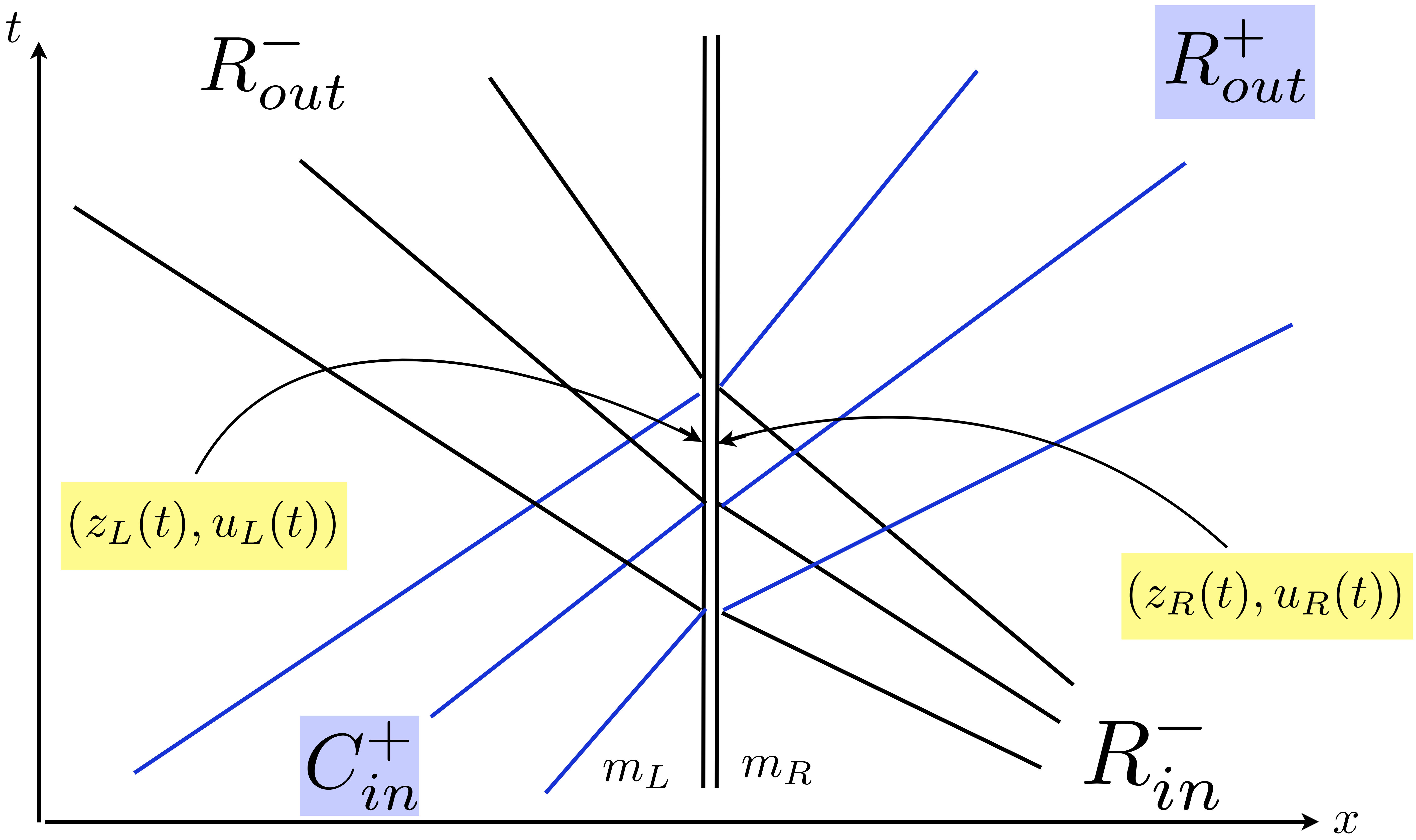
The R/C character of a wave
CAN CHANGE
at an entropy jump...

For Example:



The R/C character of a wave CAN CHANGE at an entropy jump...

For Example:



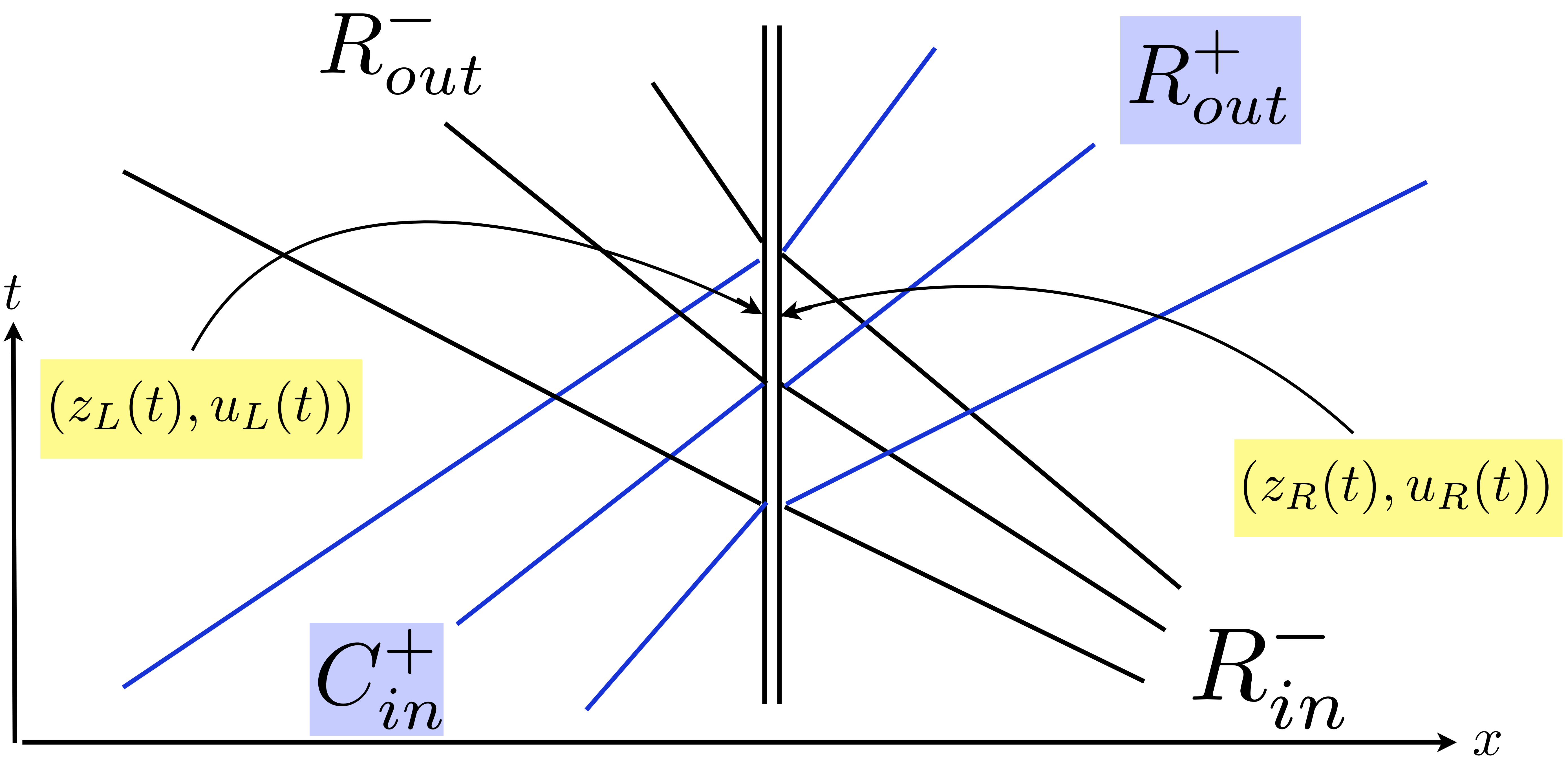
The Rankine-Hugoniot jump conditions characterize how R/C changes at an entropy jump...

Theorem 5. *The following inequalities characterize when a nonlinear wave changes its R/C value at an entropy jump:*

$$
\begin{array}{lll}
R_{in}^{-} \to C_{out}^{-} & \text{iff} & q_L^R m_L \dot z_L < \dot u_L < m_L \dot z_L, \\
C_{in}^{-} \to R_{out}^{-} & \text{iff} & m_L \dot z_L < \dot u_L < q_L^R m_L \dot z_L, \\
R_{in}^{+} \to C_{out}^{+} & \text{iff} & -q_L^R m_L \dot z_L < \dot u_L < -m_L \dot z_L, \\
C_{in}^{+} \to R_{out}^{+} & \text{iff} & -m_L \dot z_L < \dot u_L < -q_L^R m_L \dot z_L.
\end{array}
$$

$$q_L^R = \left(\frac{m_R}{m_L}\right)^{\frac{1}{\gamma}}$$

For Example:



To see this:

Ranking-Hugoniot ⟷ $\sigma[u] = [F]$

$\sigma = 0$

$$[\mathrm{F}]=0$$

$$u_L(t) = u_R(t)$$

$$m_L z_L(t) = m_R z_R(t) q_R^L$$

$$q_R^L = \left(\frac{m_L}{m_R}\right)^{\frac{1}{\gamma}}$$

$$c_R = c_L q_R^L$$

WE HAVE: the backward wave will change its R/C value at the entropy jump iff the sign of

$$\dot{r} = \dot{u} - m\dot{z}$$

changes across the jump, and the forward wave will change R/C character at the entropy jump iff the sign of

$$\dot{s} = \dot{u} + m\dot{z}$$

changes sign across the jump.

THEREFORE: a backward wave changes from C to R across the entropy jump iff

$$\dot{r}_L = \dot{u}_L - m_L\dot{z}_L < 0$$

and

$$\dot{r}_R = \dot{u}_R - m_R\dot{z}_R > 0.$$

- I.e., by R-H:

$$u_R(t) = u_L(t),$$
$$m_R z_R(t) = m_L z_L(t) q_L^R,$$

where

$$q_L^R = \left(\frac{m_R}{m_L}\right)^{\frac{1}{\gamma}}$$

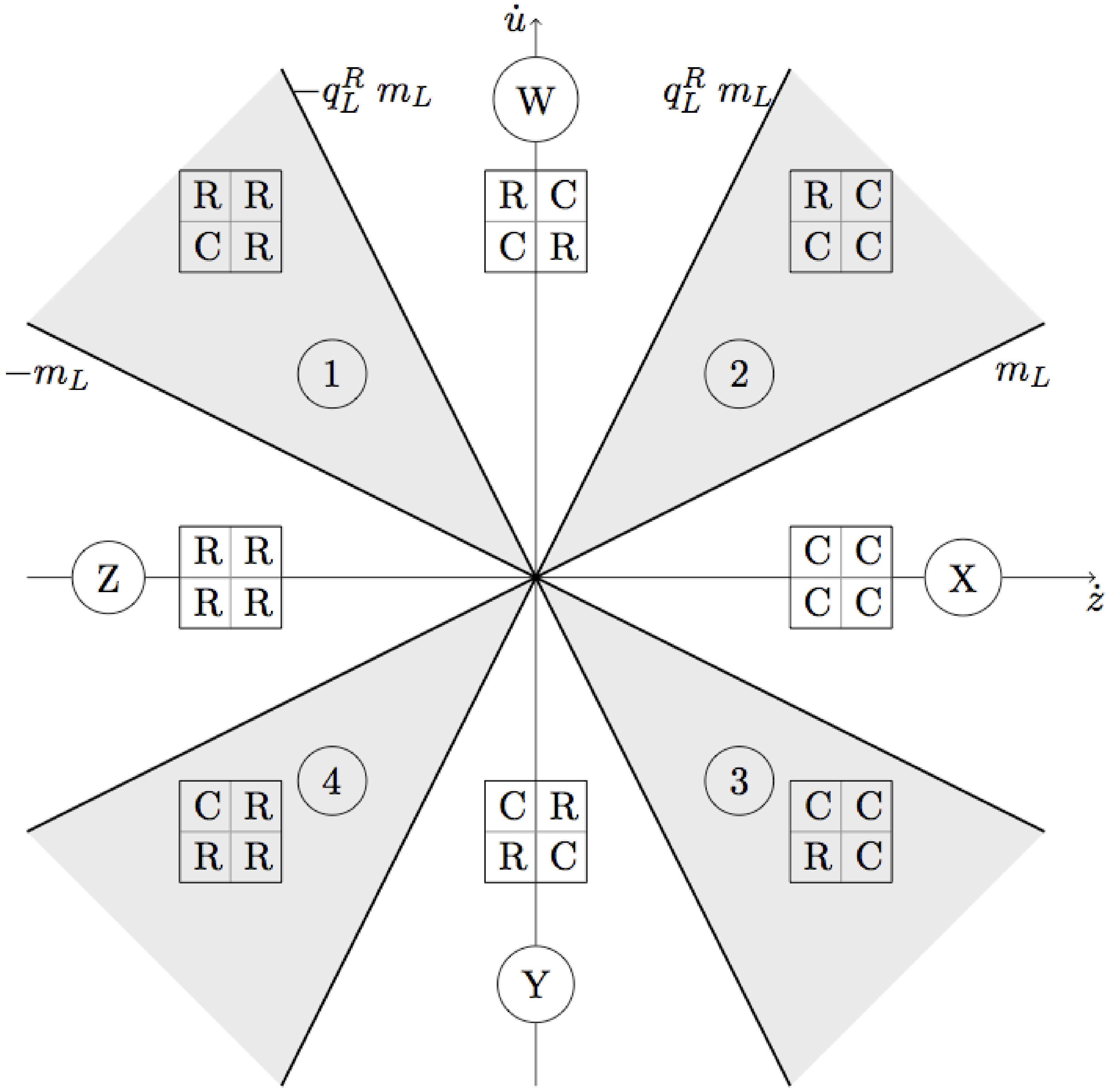
- SO IT FOLLOWS THAT:

$\dot{r}_L = \dot{u}_L - m_L \dot{z}_L < 0$ and $\dot{r}_R = \dot{u}_R - m_R \dot{z}_R > 0$

is equivalent to

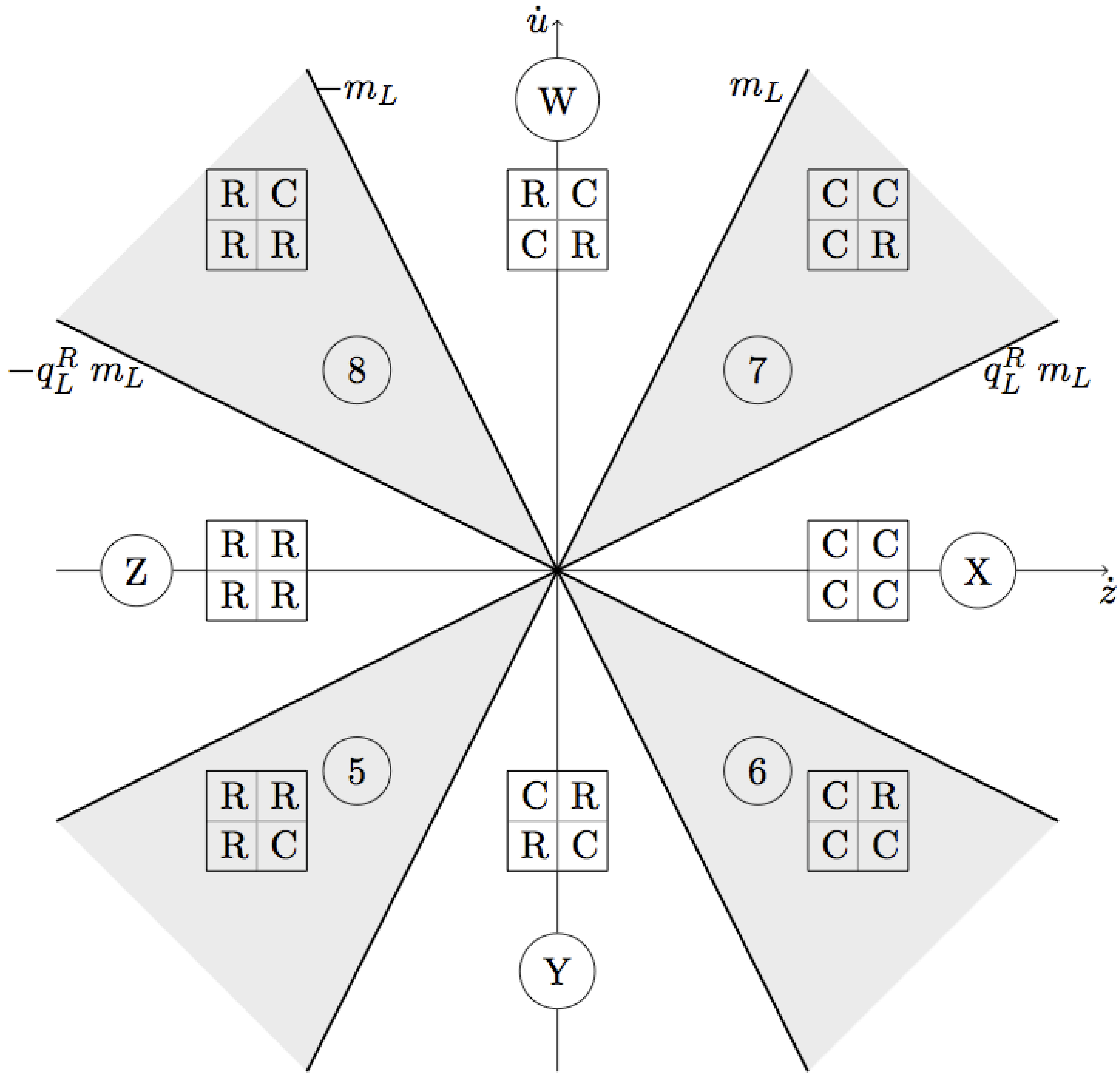
$$q_L^R m_L \dot{z}_L < \dot{u}_L < m_L \dot{z}_L.$$

CONCLUDE: we can determine the R/C changes across the entropy jump from inequalities on the time derivative of the solution at the left hand side of the entropy jump alone. Doing this in all cases yields the following theorem.



Tangent space showing all possible R/C wave structures when

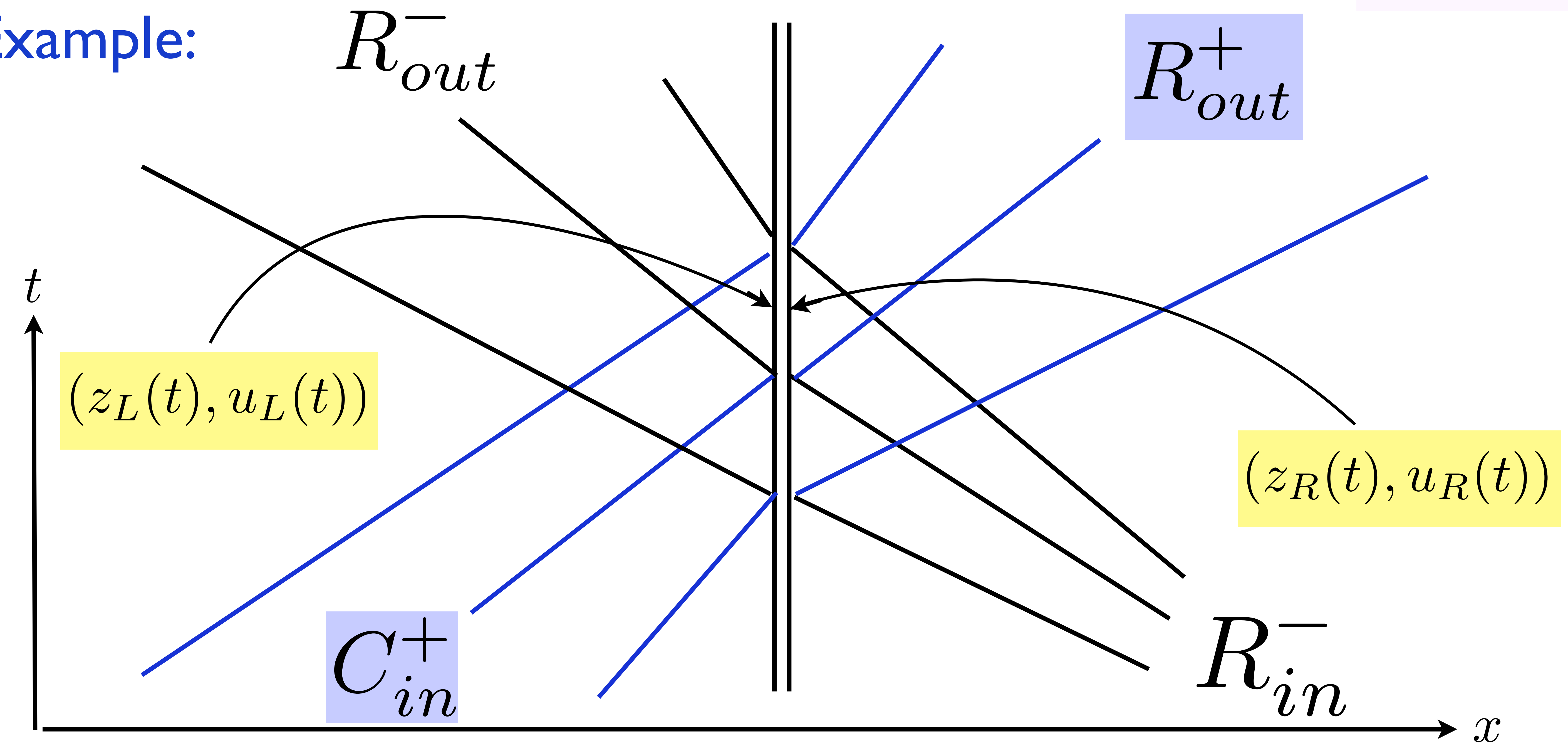
$m_L < m_R$



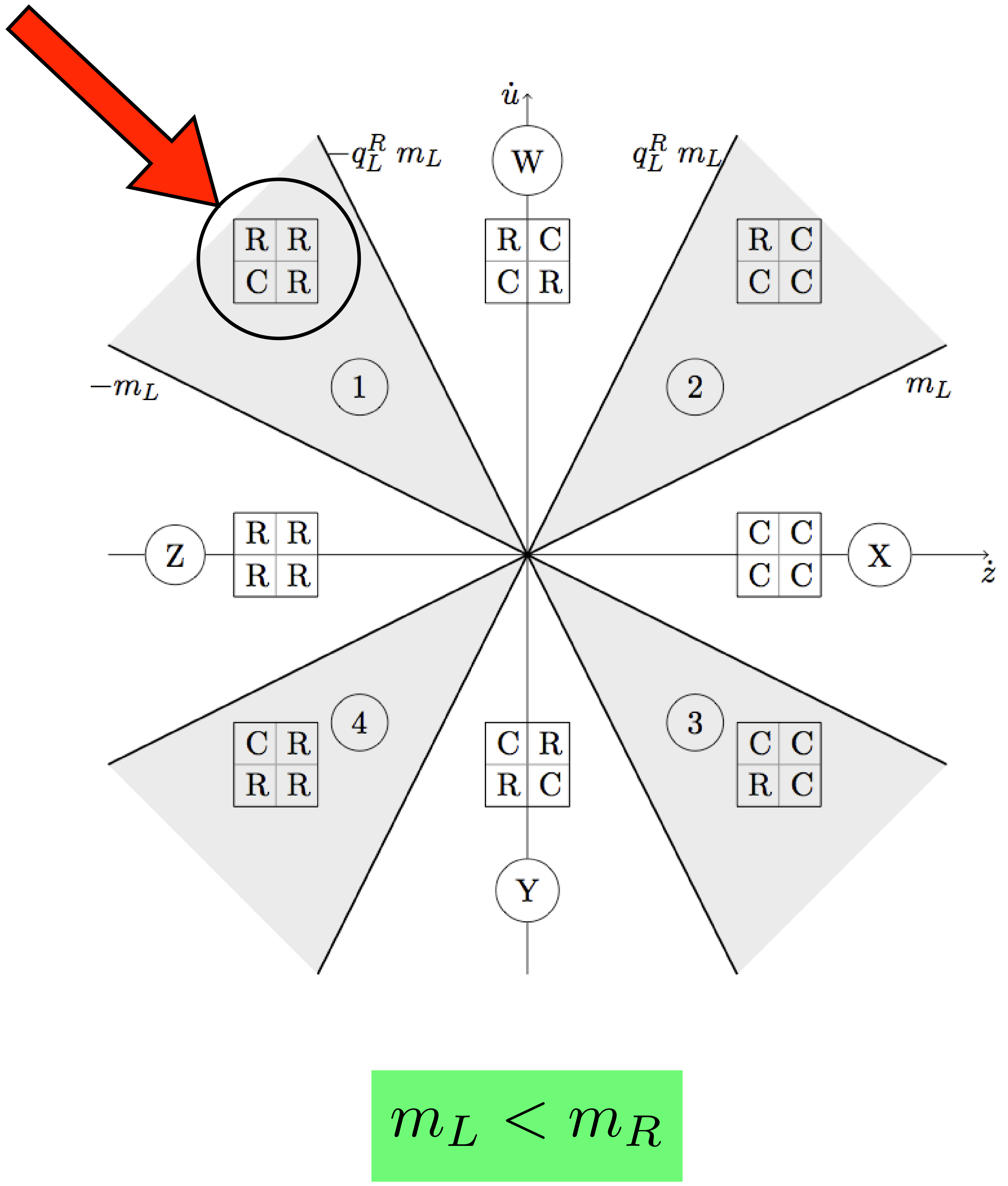
Tangent space showing all possible R/C wave structures when

$m_L > m_R$

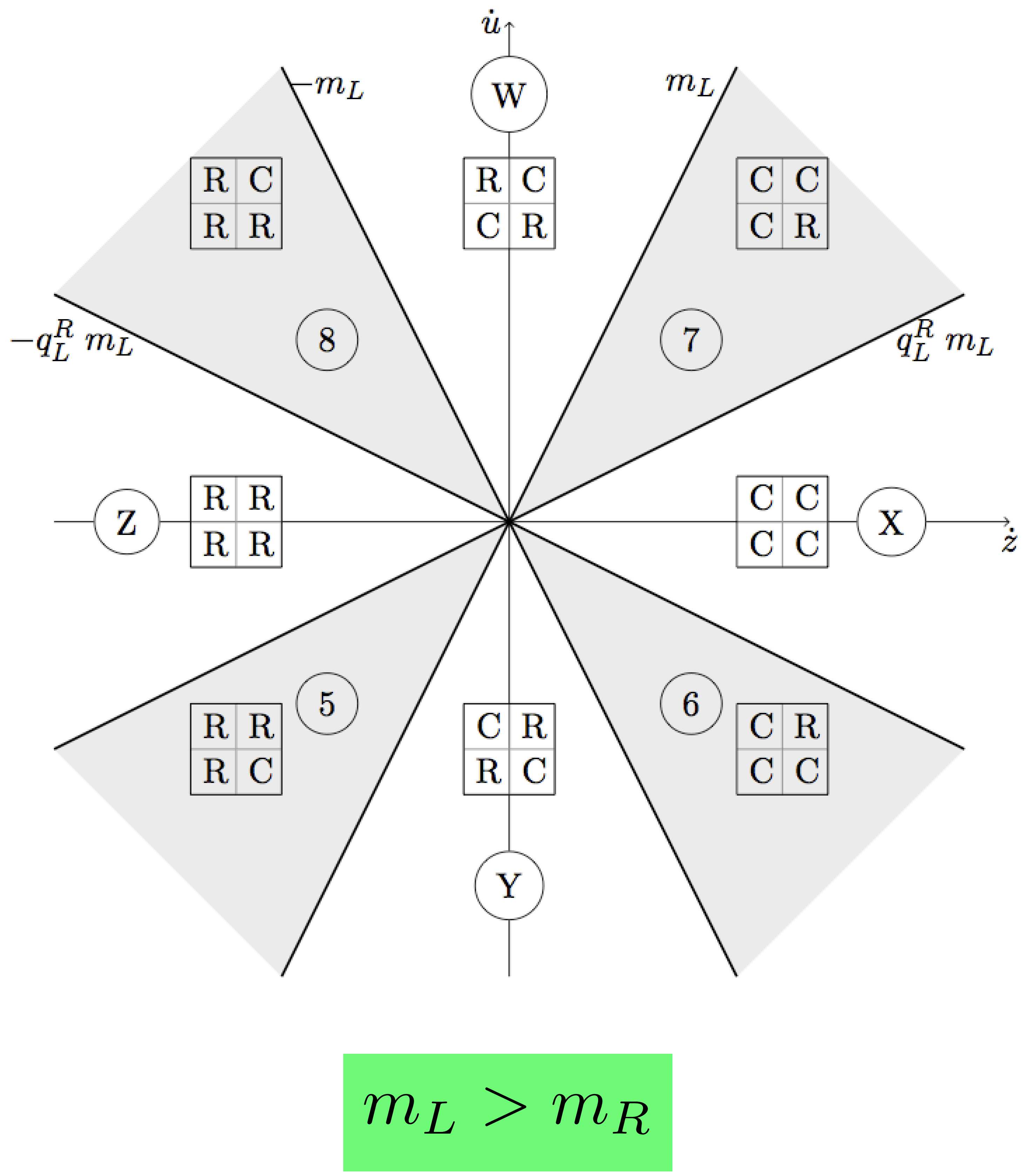
Example:



R^-_{out}
R^+_{out}
t
$(z_L(t), u_L(t))$
$(z_R(t), u_R(t))$
C^+_{in}
R^-_{in}
x



$\dot{u}$
$-q^R_L\, m_L$
$q^R_L\, m_L$
$-m_L$
m_L
W
R R
C R
R C
C R
R C
C C
1
2
Z
R R
R R
C C
C C
X
$\dot{z}$
4
3
C R
R R
C R
R C
C C
R C
Y
$m_L < m_R$



$\dot{u}$
$-m_L$
m_L
$-q^R_L\, m_L$
$q^R_L\, m_L$
W
R C
R R
R C
C R
C C
C R
8
7
Z
R R
R R
C C
C C
X
$\dot{z}$
5
6
R R
R C
C R
R C
C R
C C
Y
$m_L > m_R$

- Note: All 16 possible interaction squares appear EXCEPT ones where R/C value of both waves change simultaneously:

Not possible:

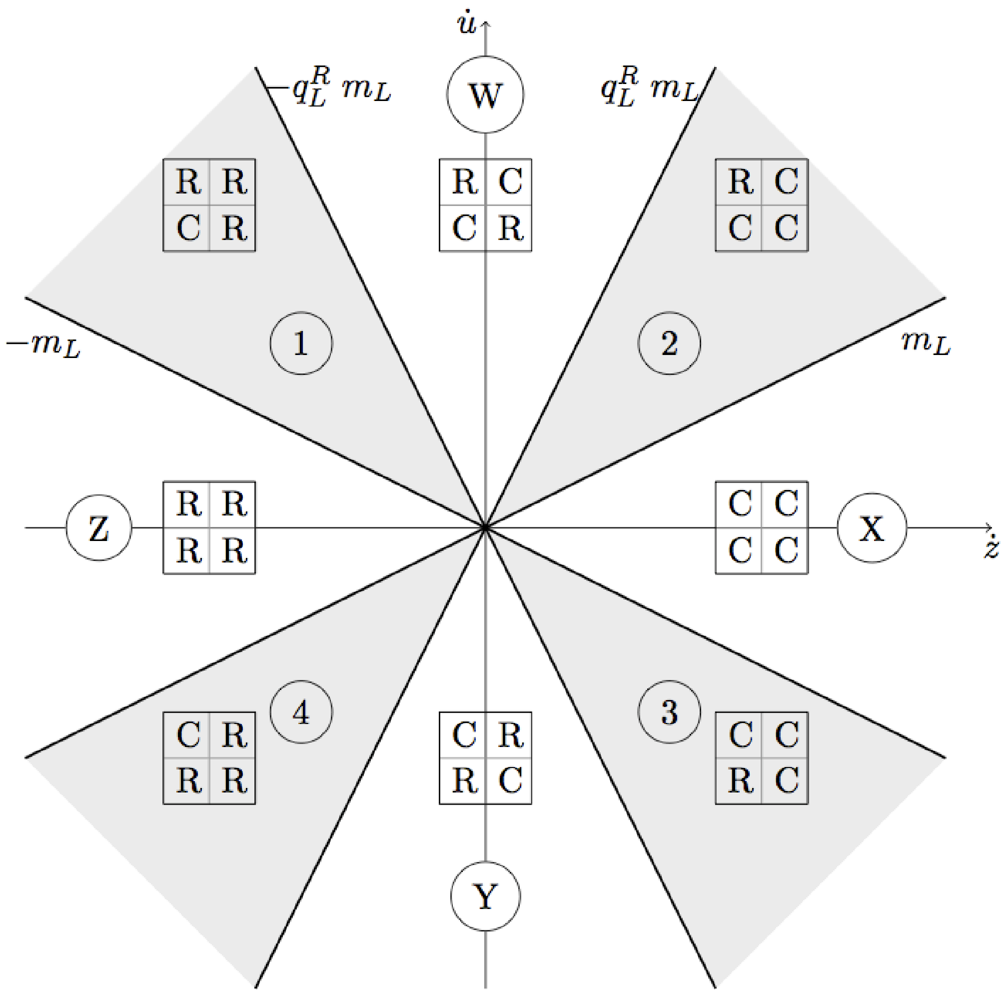
R	R
C	C

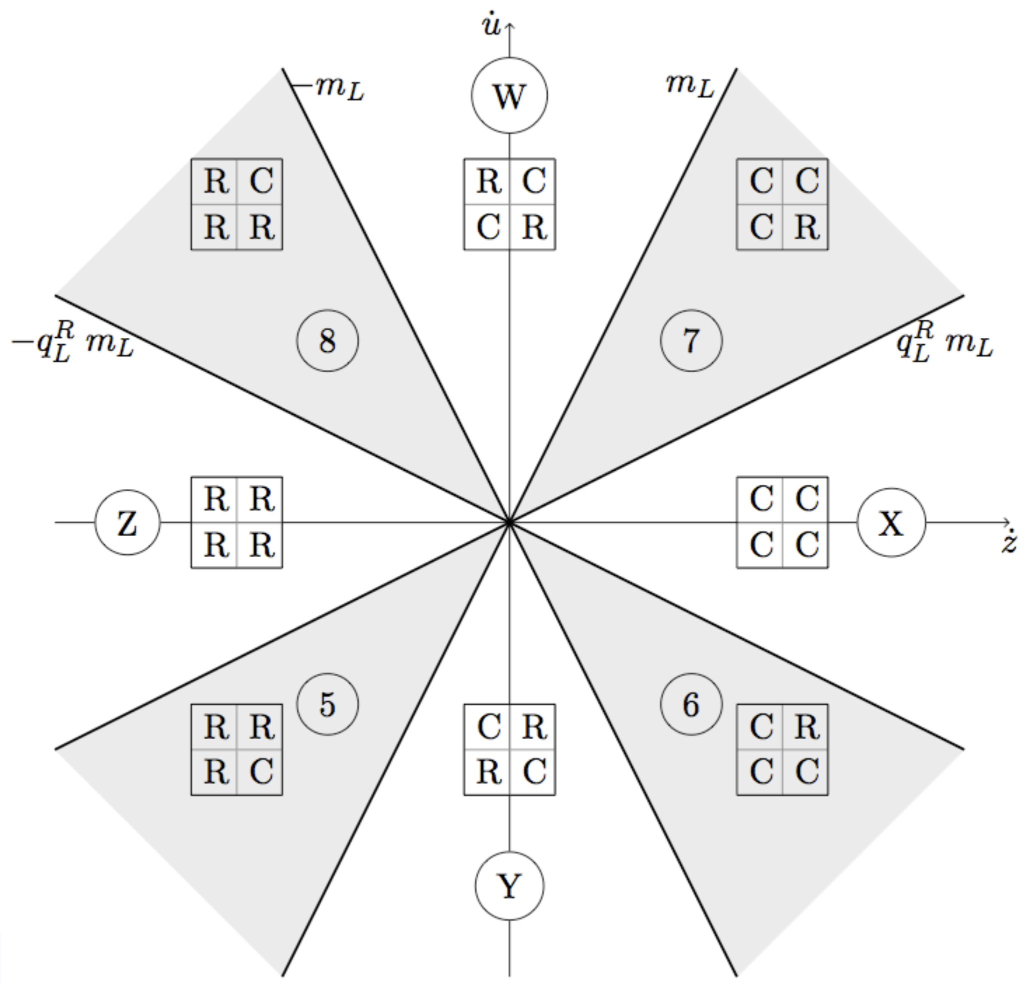
C	C
R	R

R	C
R	C

C	R
C	R

- CONCLUDE: A wave in one family can change its R/C value only in the presence of a wave of the opposite family that transmits its R/C value

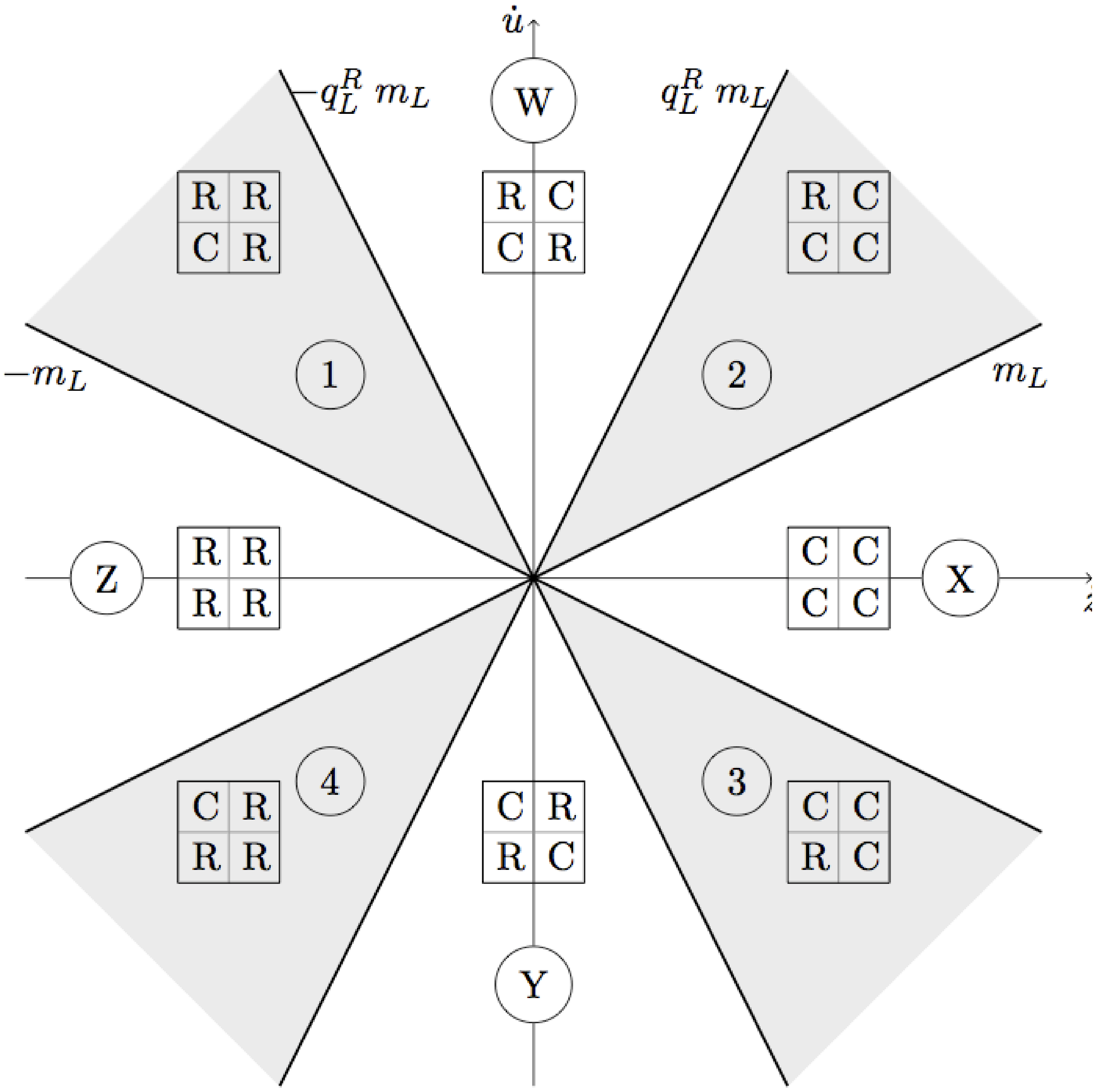


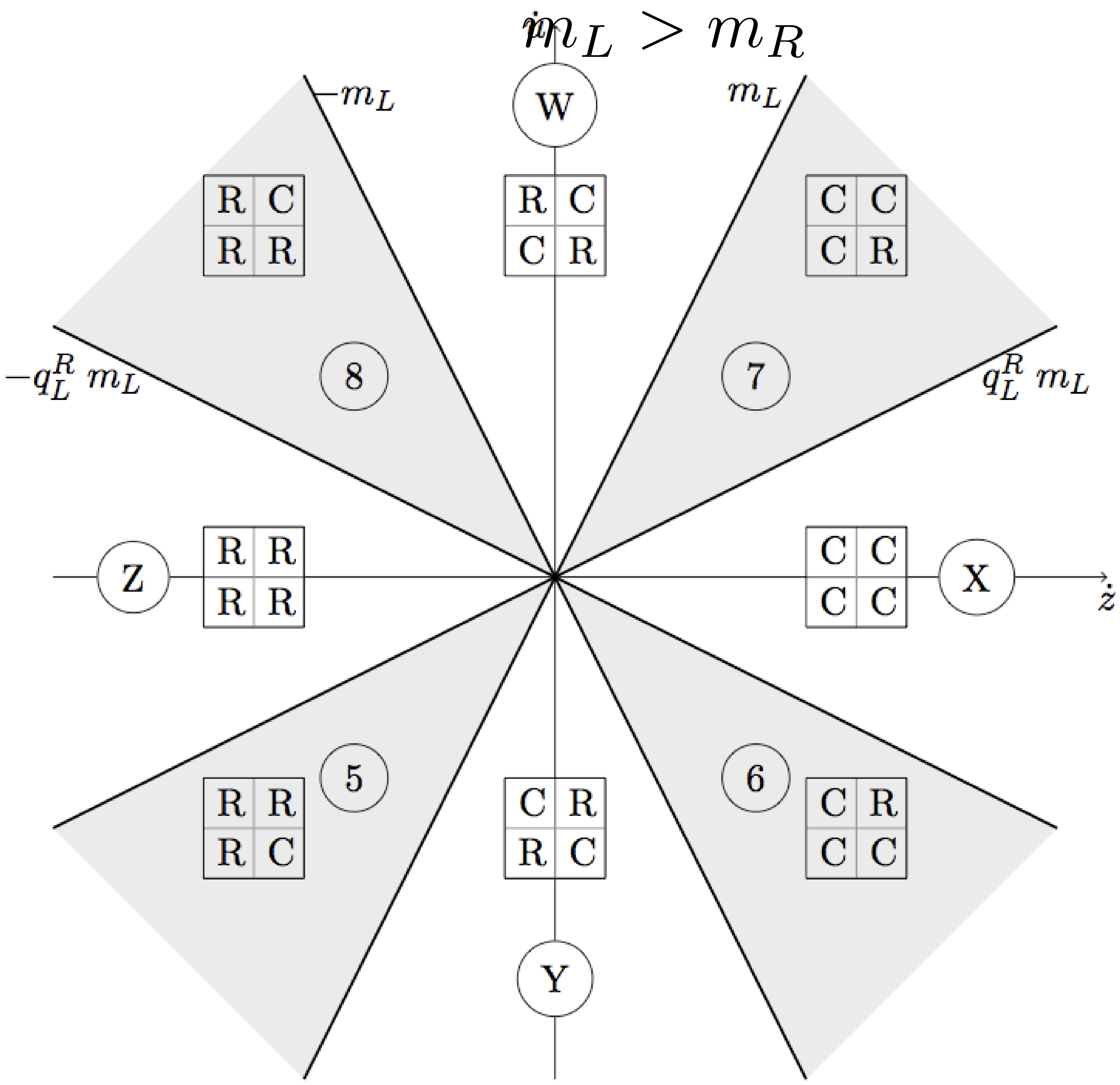


• Note: There is a LR-asymmetry:

The interaction squares for $m_L < m_R$

are different from the squares for $m_L > m_R$





$m_L < m_R$ $m_L > m_R$

IV. The Simplest Possible Periodic Structure that Balances Compression and Rarefaction

DEFN: We we say that a periodic pattern of R's and C's is

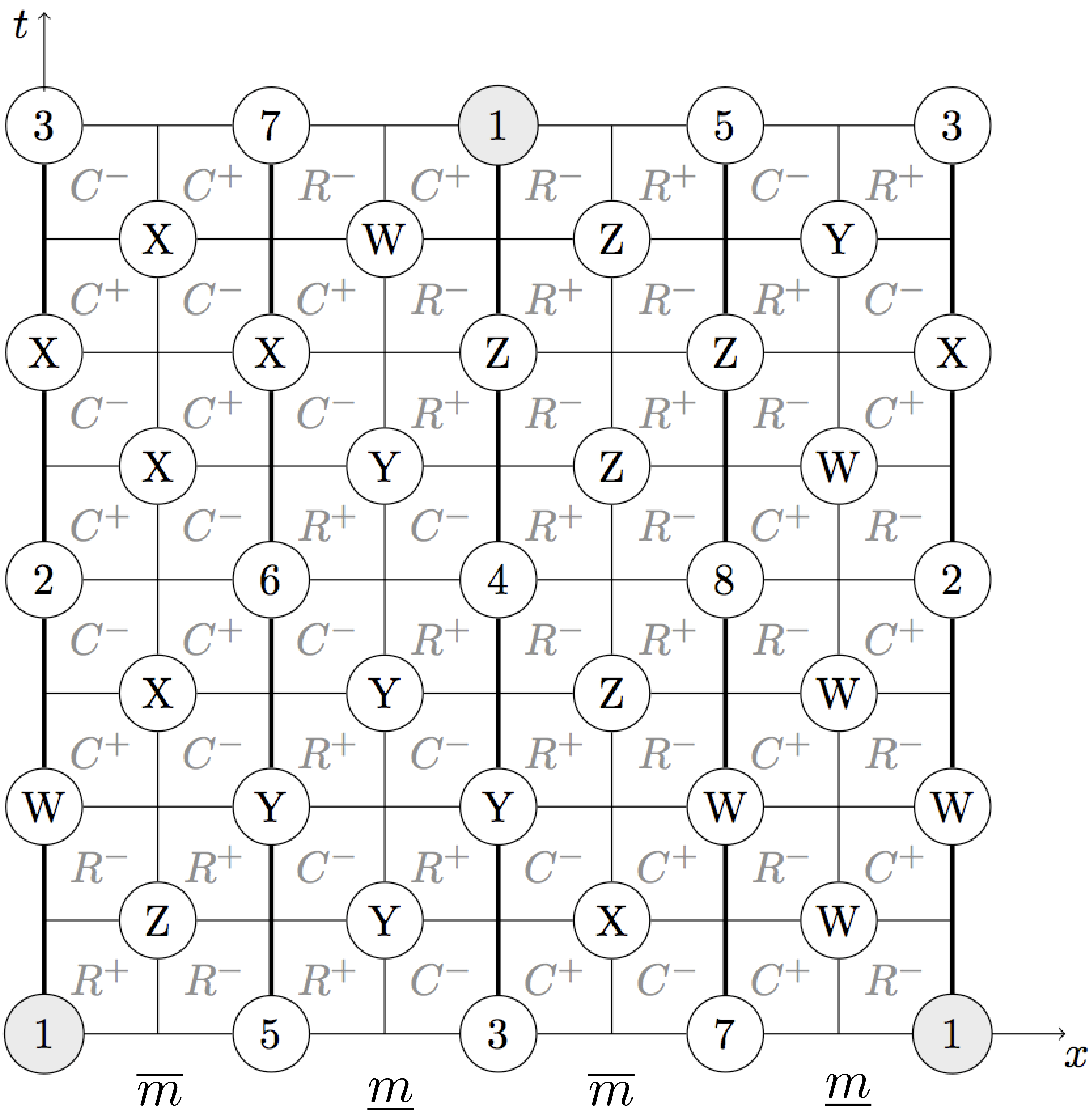
CONSISTENT

IF:

(1) R/C is balanced along every 1,3-characteristic

(2) The interaction squares at entropy jumps $m_L > m_R, m_L > m_R$ are consistent with squares in the $m_L > m_R, m_L > m_R$ R/C diagrams, respectively

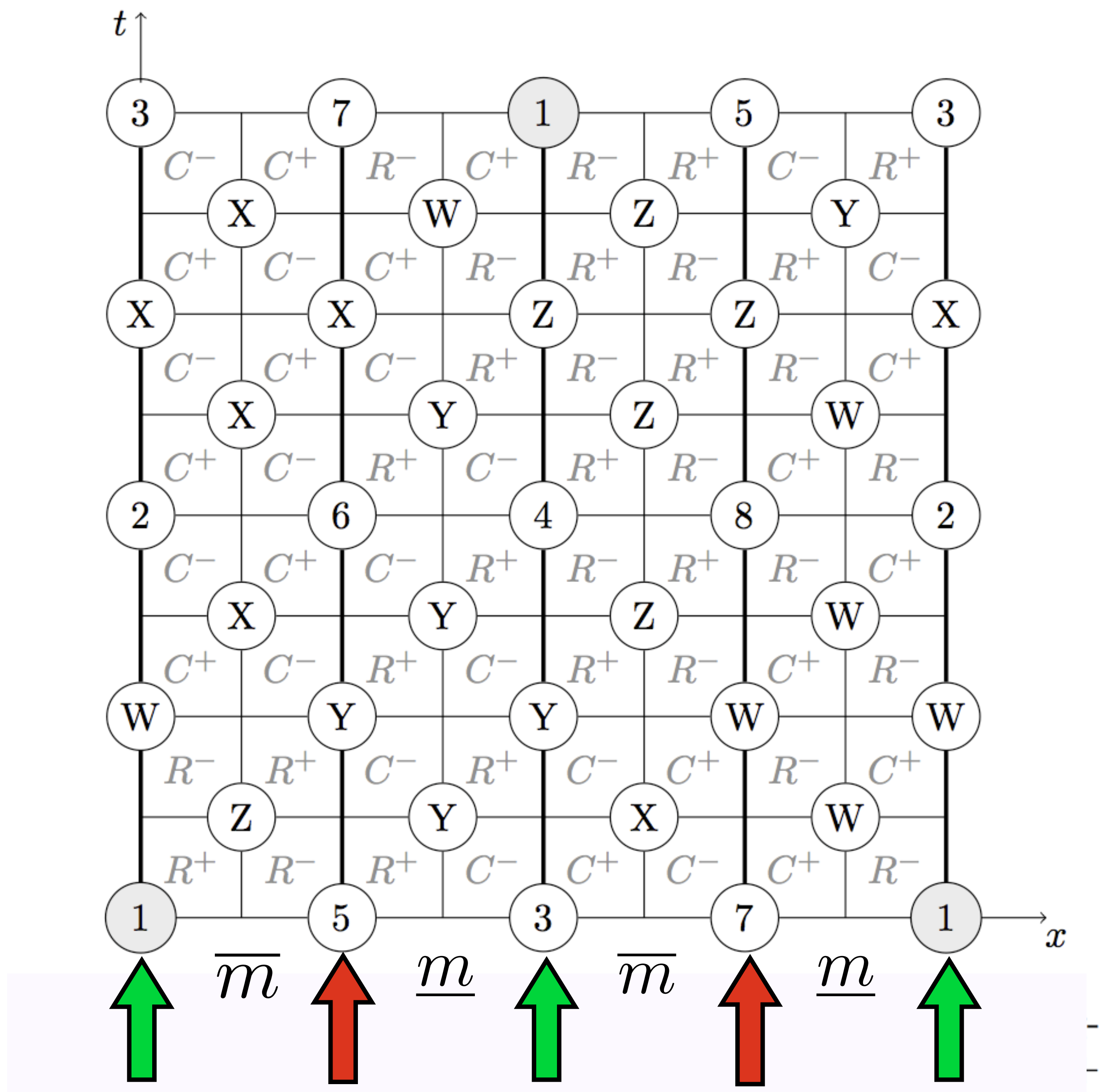
The simpest consistent R/C pattern



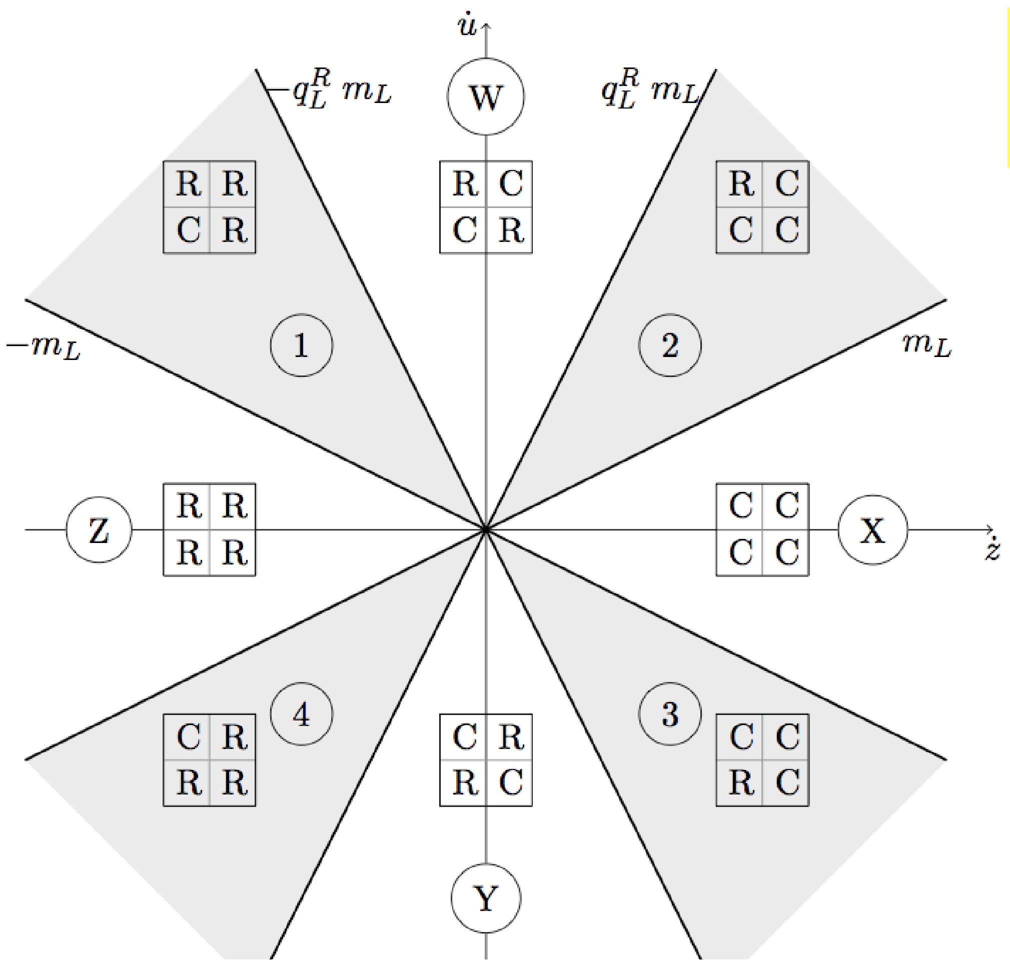
"Extend periodically"

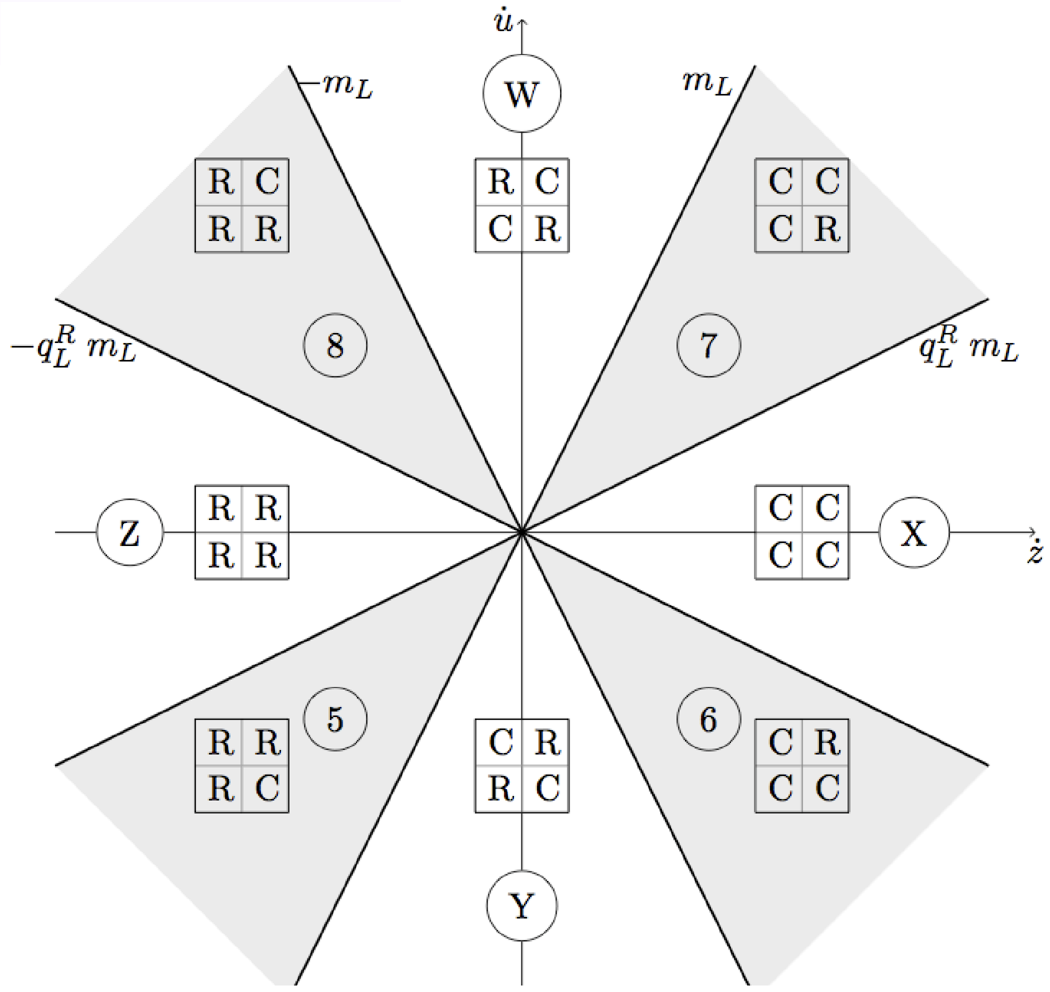
$$\overline{m} > \underline{m}$$

Each number above is consistent with the numbered interaction below



$\overline{m} > \underline{m}$

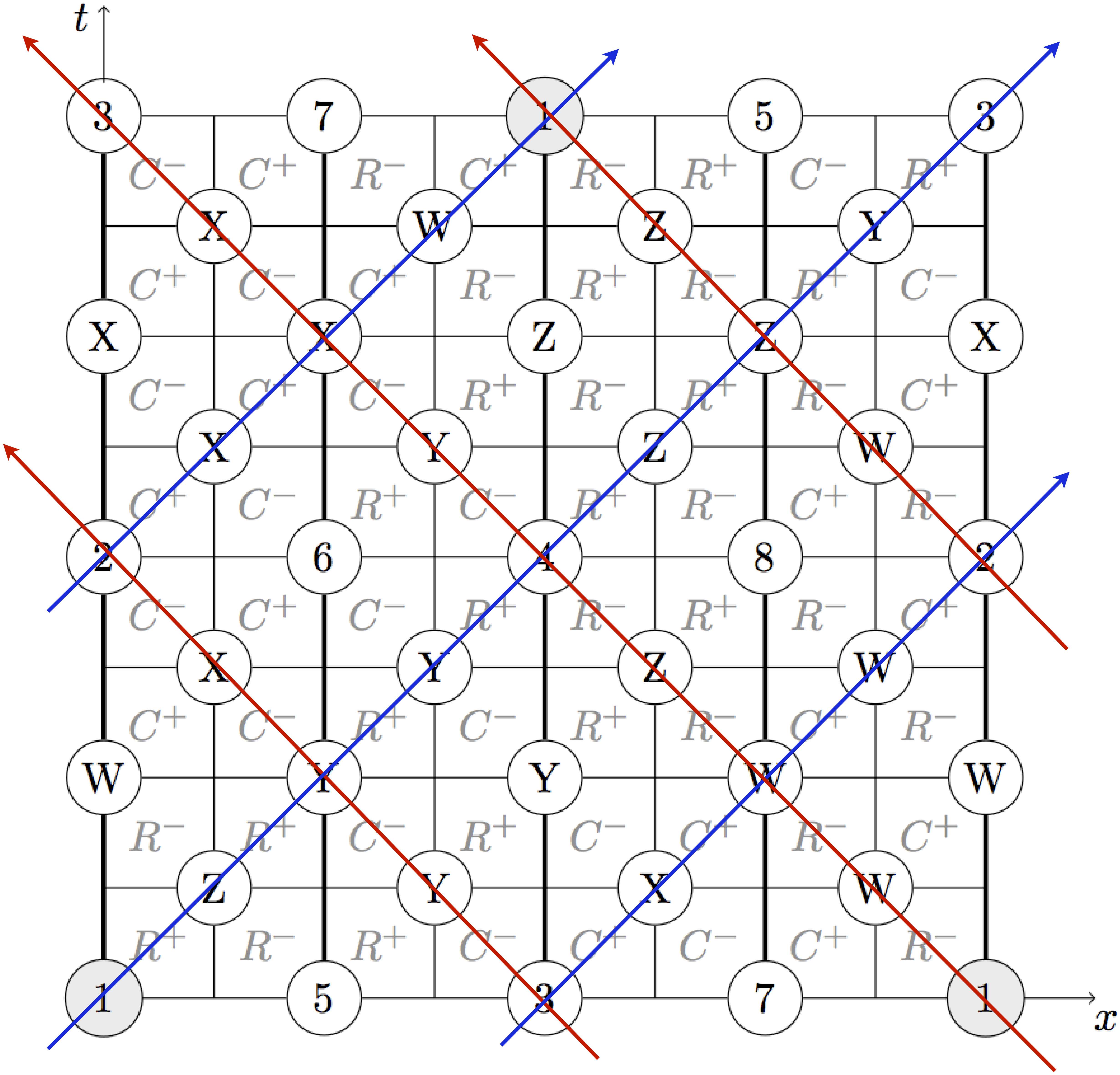




$\underline{m} = m_L < m_R = \overline{m}$

$\overline{m} = m_L > m_R = \underline{m}$

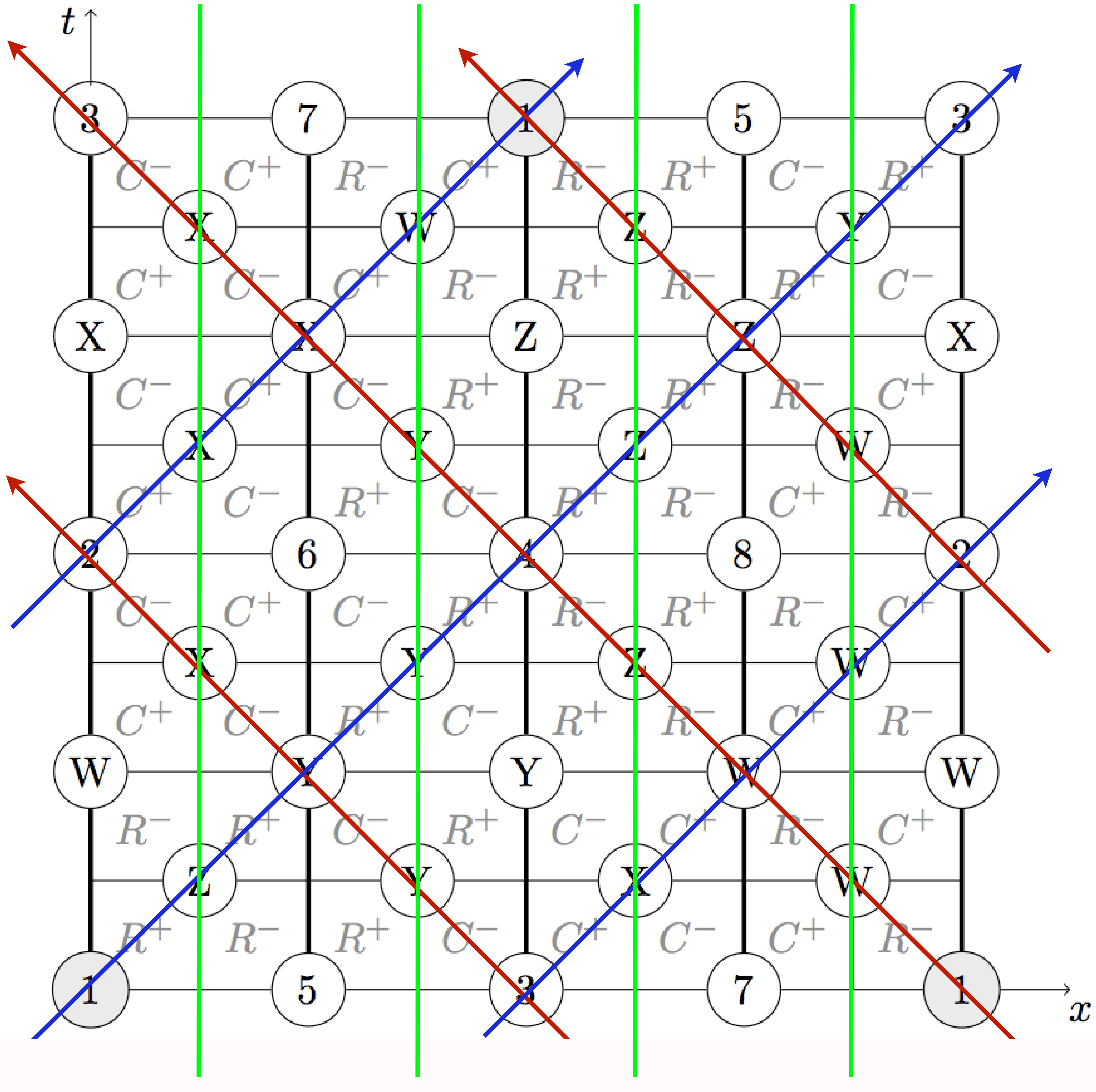
Each 1,3-characteristic traverses 8-C's and 8-R's before returning



The lettered interactions at constant entropy jump transmit R/C

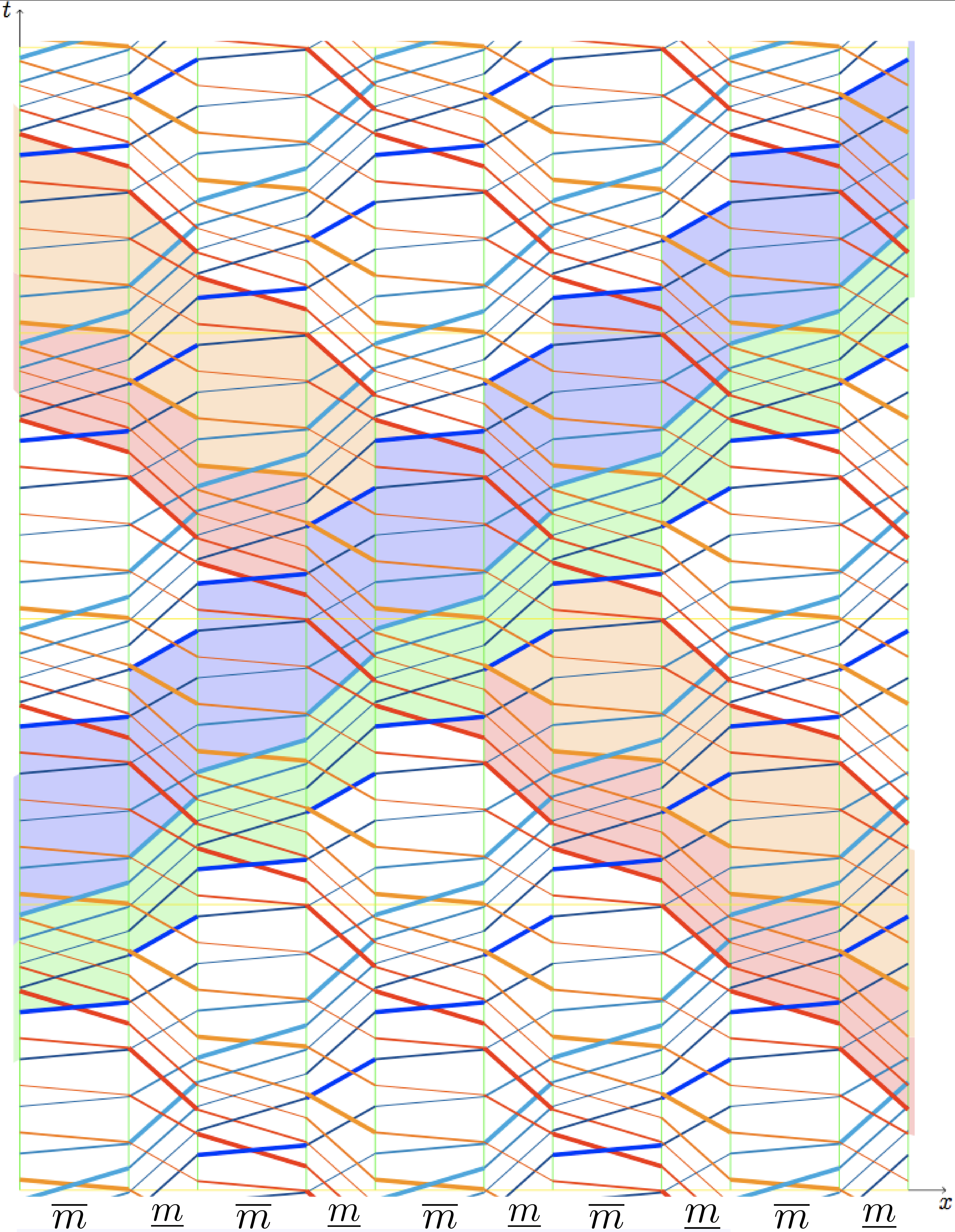
Identifying these

1,3-characteristics traverse 4-C's and 4-R's before returning



- Inspection of the change in $\dot{u},\ \dot{z}$ along the axes indicates that the solution is consistent with elliptical rotation of the solution along the entopy jumps

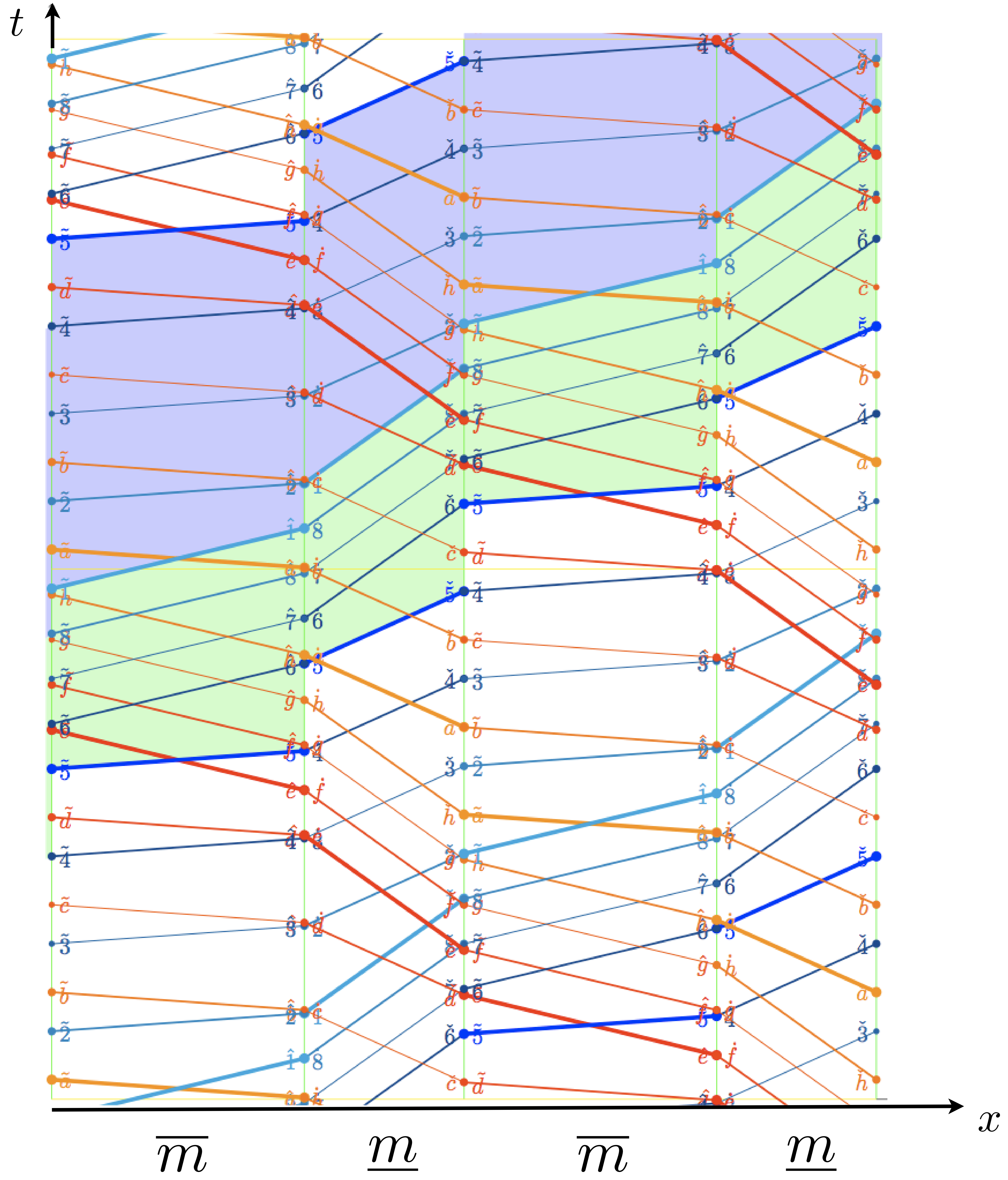
- This leads to the following consistent cartoon of the simplest possible periodic solution



The simplest possible periodic structure

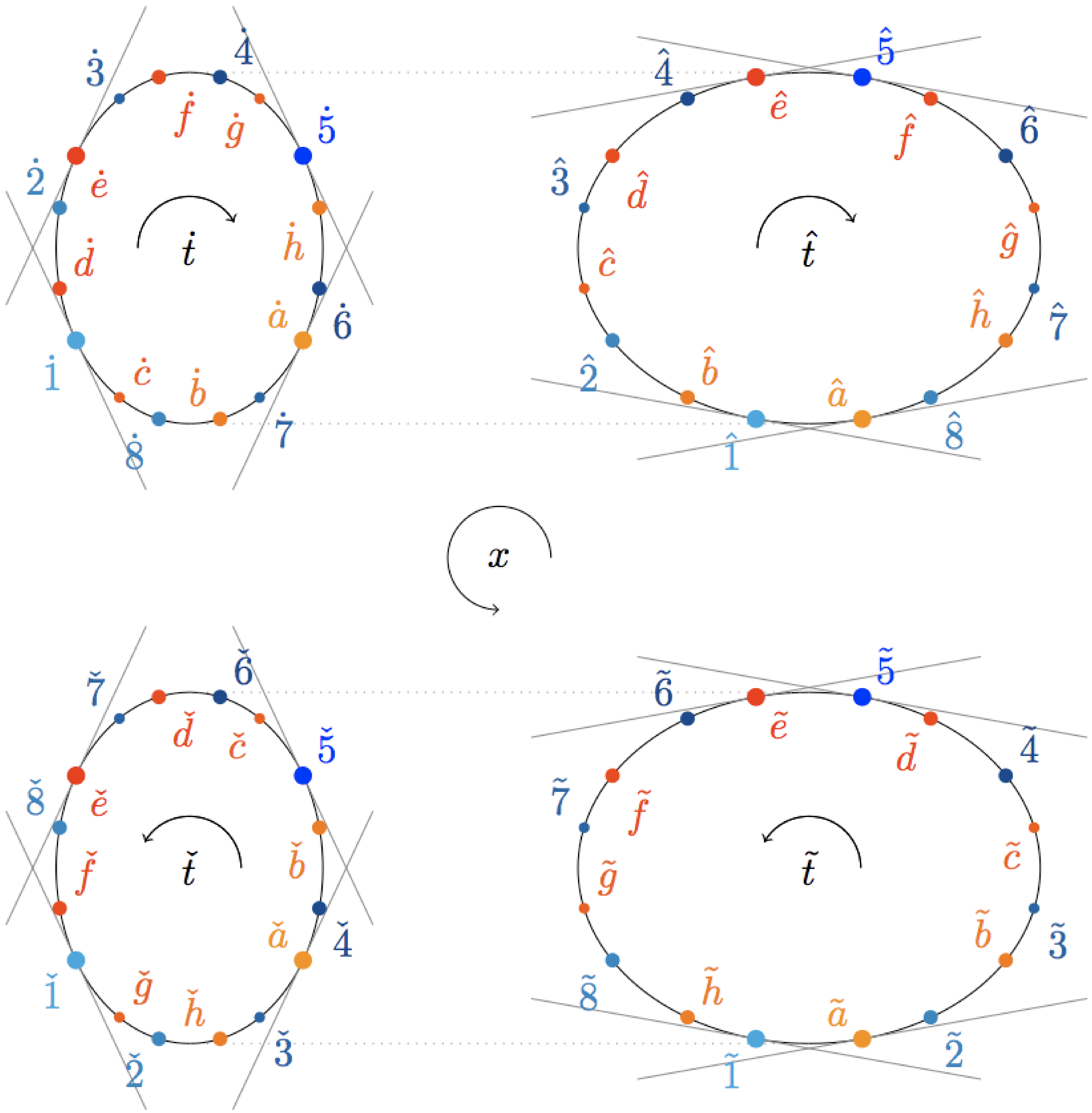
$$\overline{m} > \underline{m}$$

- Labeling the states along the entropy jumps and plotting them according to the change in $\dot{u}, \dot{z}$ indicates that the solution is consistent with elliptical rotation of the solution along the entopy jumps

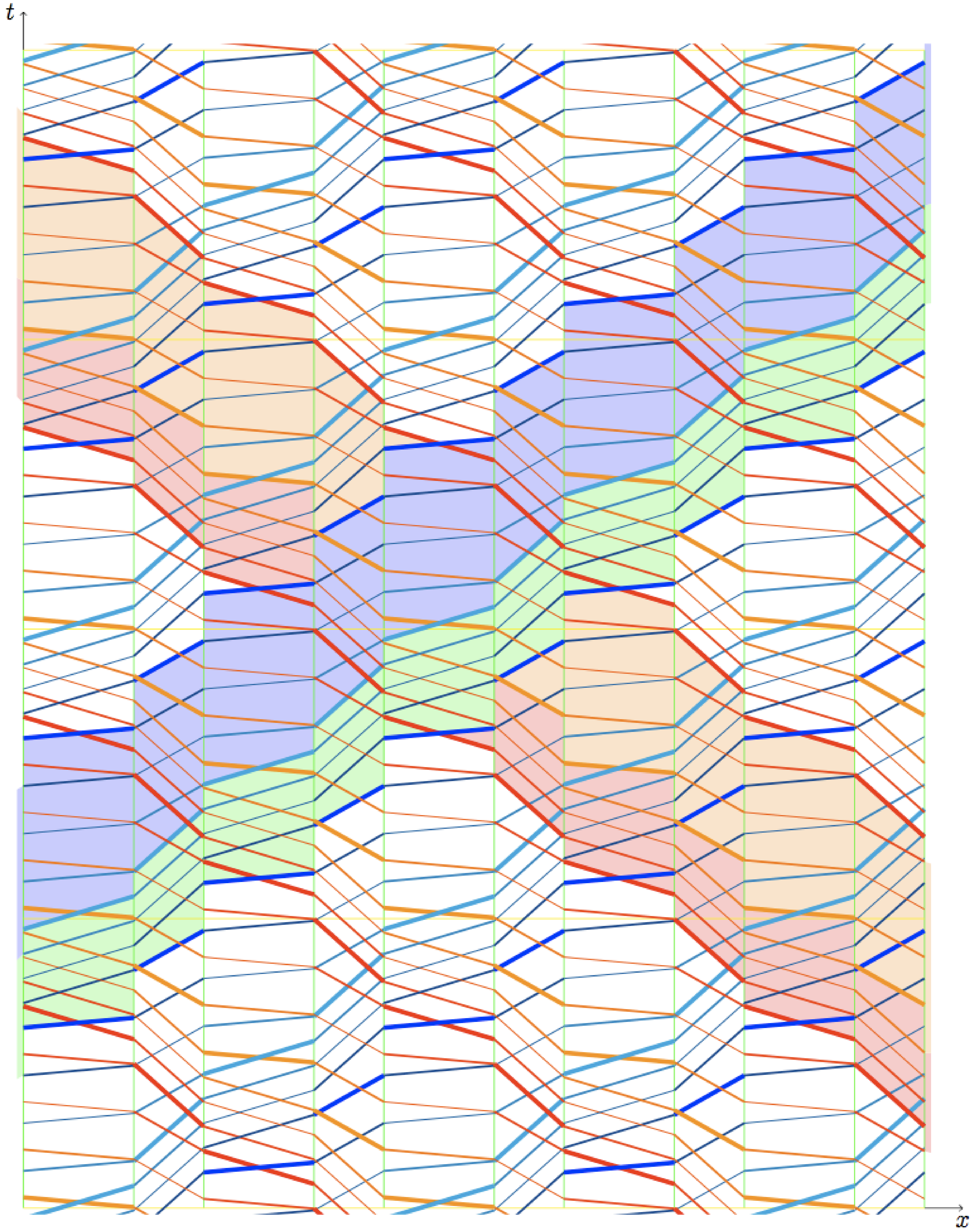


Labeling the states by numbers and letters

$$\overline{m} > \underline{m}$$



Ellipses showing periodicity in (z,u)-plane



The global nonlinear periodic structure

End

First

Talk

Time-Periodic Solutions of the Compressible Euler Equations-II

Blake Temple
University of California, Davis

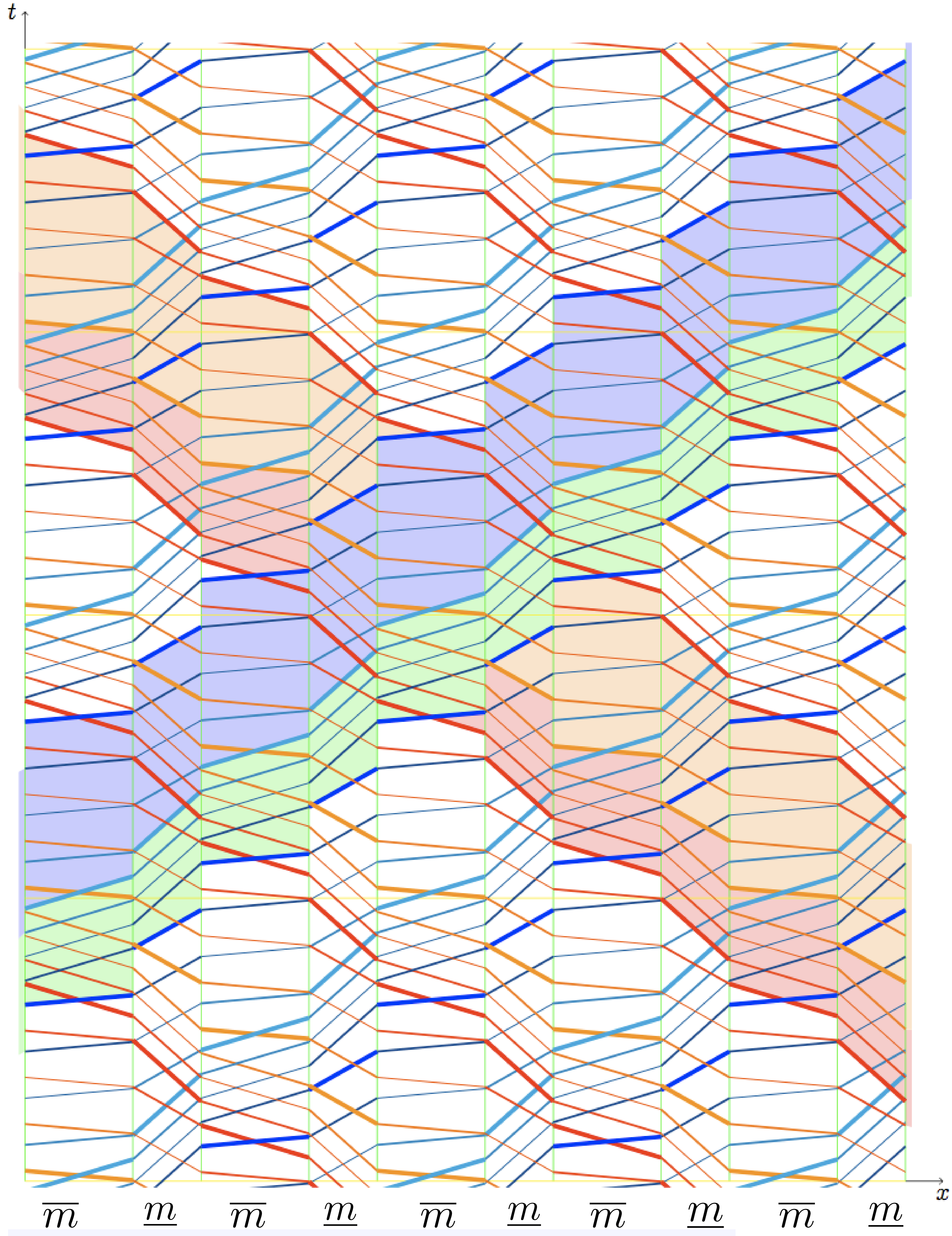
. . .

Robin Young
University of Massachusetts, Amherst

Outline

Recall:

$\overline{m} > \underline{m}$



The simplest possible global nonlinear periodic structure

Recall: A Convenient Change of Variables

- Change Variables: $(\tau, u, S) \longmapsto (z, u, m)$

$$m = e^{S/2c_\tau}$$

m re-scales the entropy S

$$z = \int_\tau^\infty \frac{c}{m} d\tau = \left(\frac{2\sqrt{K\gamma}}{\gamma - 1}\right)\left(\frac{1}{\tau}\right)^{\frac{\gamma-1}{2}}$$

z re-scales the specific vol. $\tau = 1/\rho$

- The transformed Euler equations m=const:

$$\begin{aligned} z_t + \frac{c}{m} u_x &= 0 \\ u_t + m c z_x &= 0 \end{aligned}$$

TO START

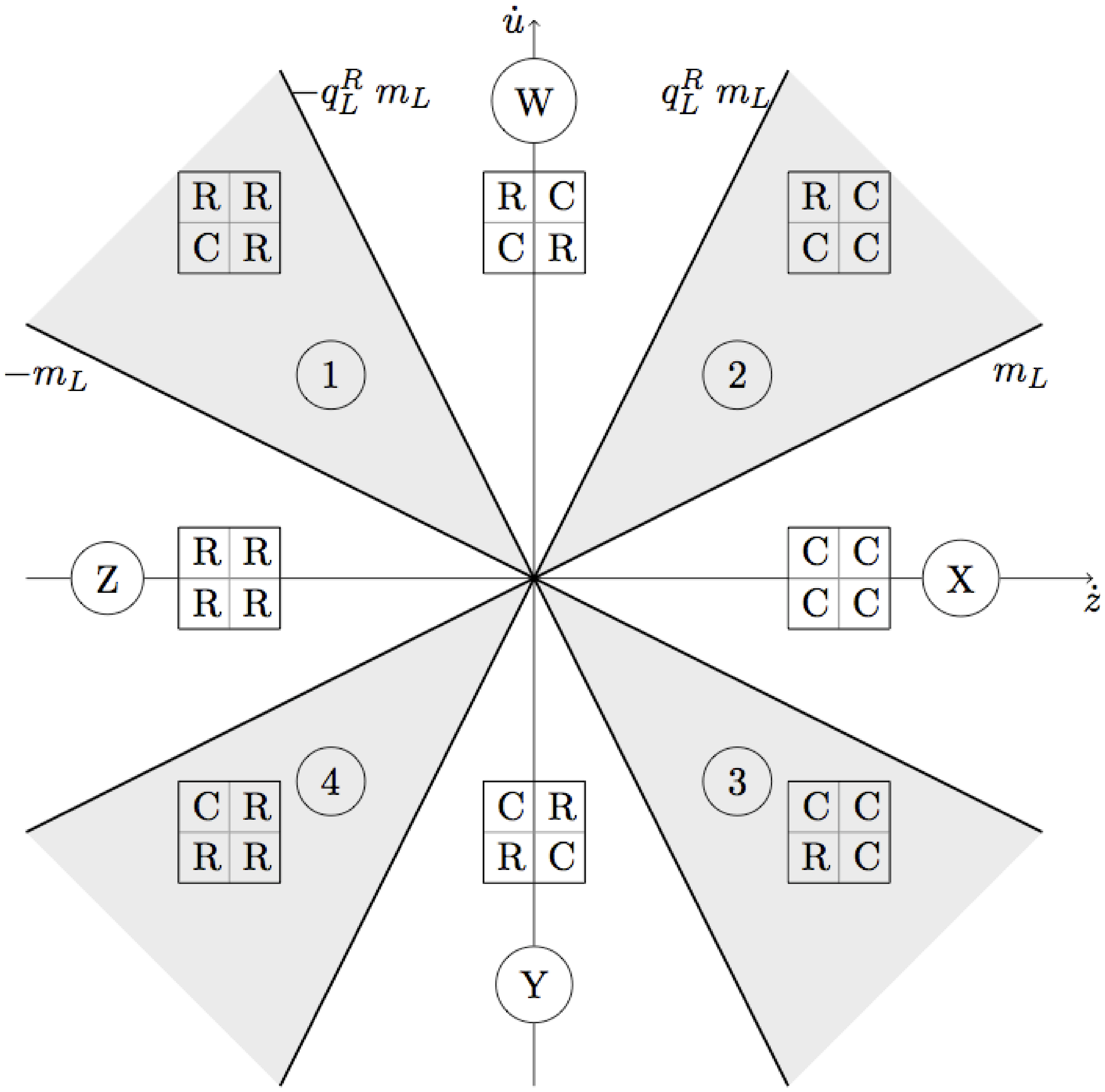
- OBSERVATIONS-- regarding the

Simplest Periodic Structure that balances compression and rarefaction along characteristics

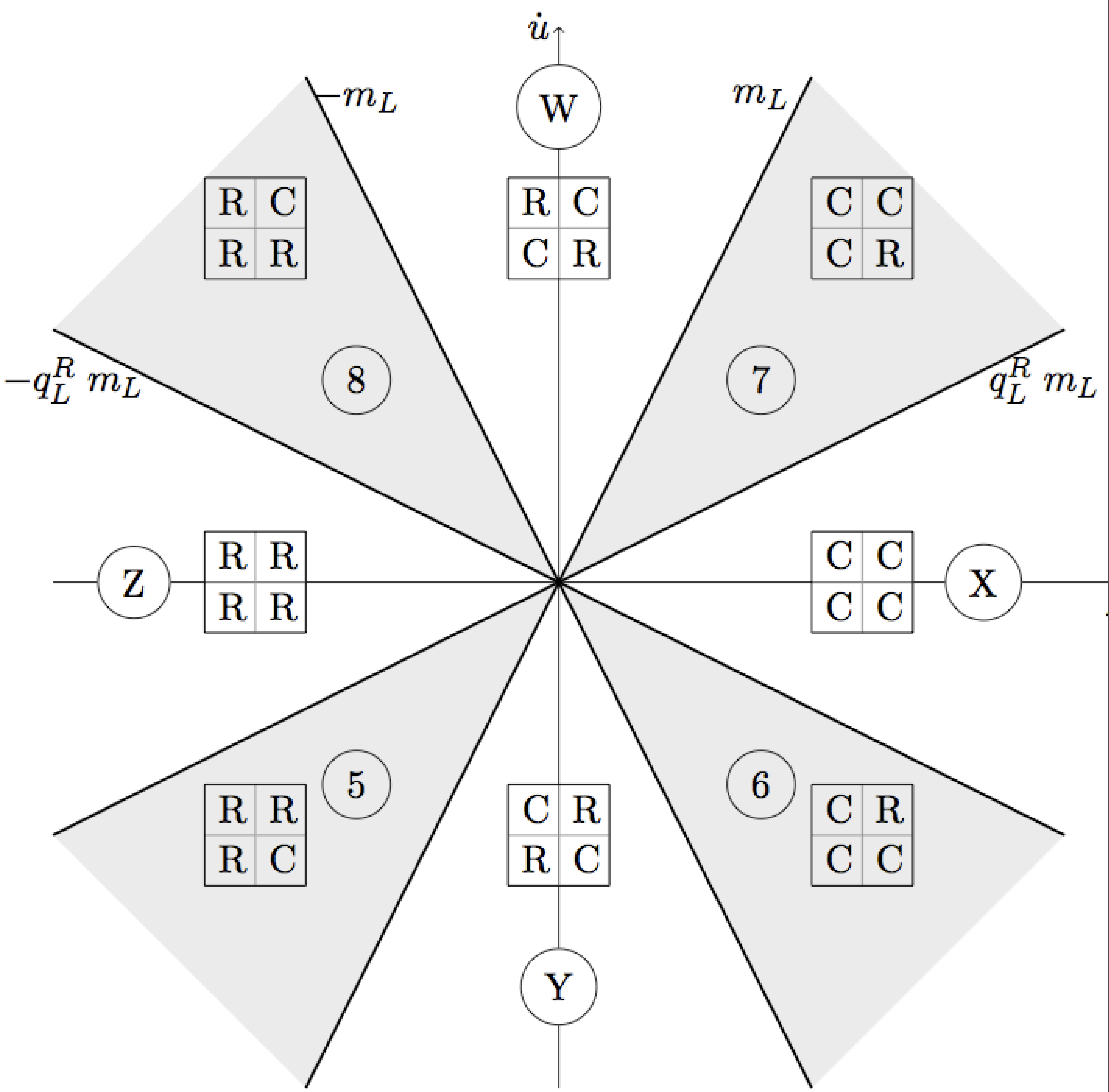
• Recall: There is an LR-asymmetry:

Interaction squares for $m_L < m_R$

differ from squares for $m_L > m_R$



$m_L < m_R$



$m_L > m_R$

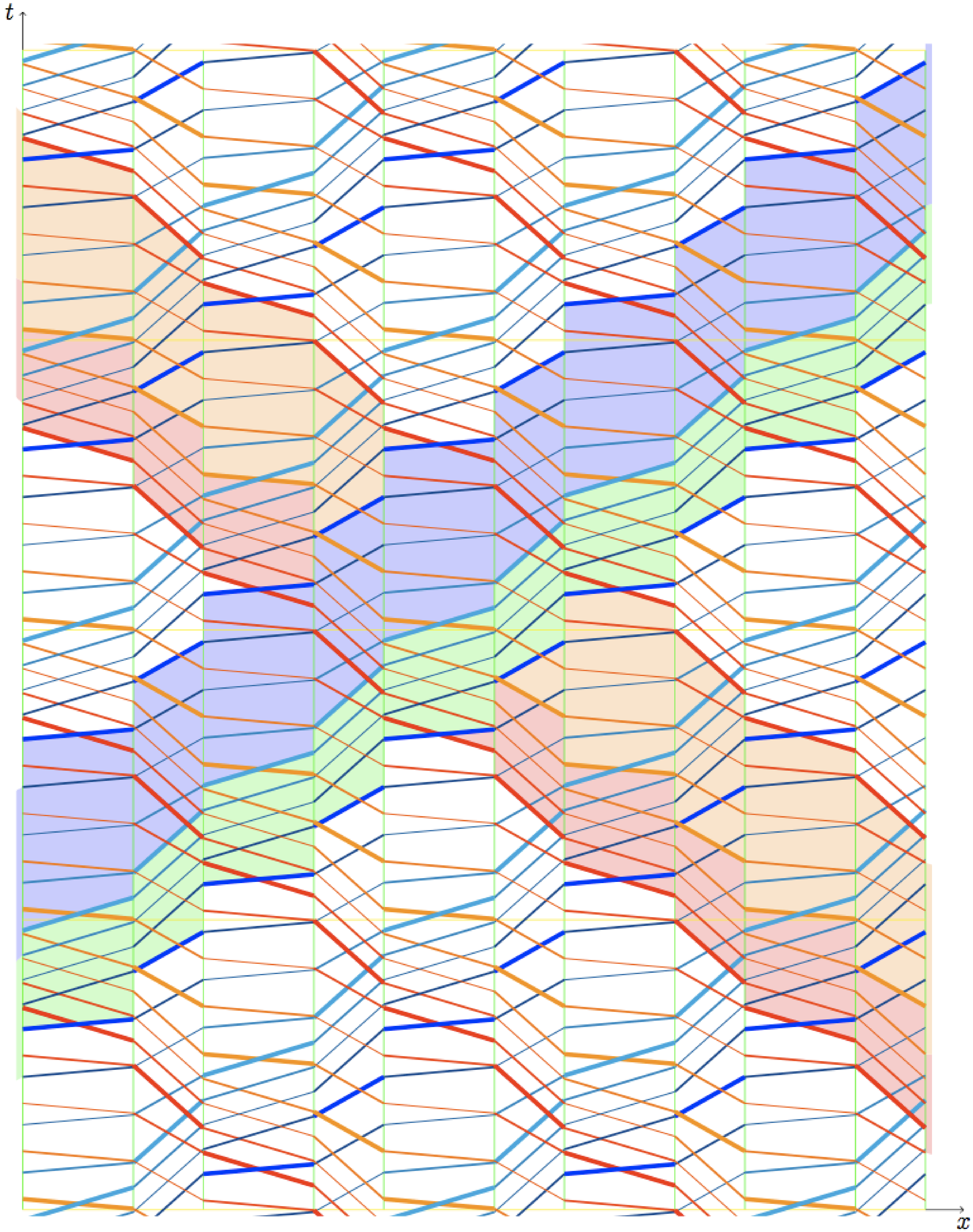
LR-asymmetry

↓

Max/Min Characteristics
always jump
UP
going
OUTWARD

↓

The 1,3-wave crests are
SUBSONIC
NOT
SUPERSONIC



The 1,3-Max/Min-Characteristics jump UP going OUTWARD ➡ SUBSONIC

The speed of the wave crests is like an effective "Group-Velocity"

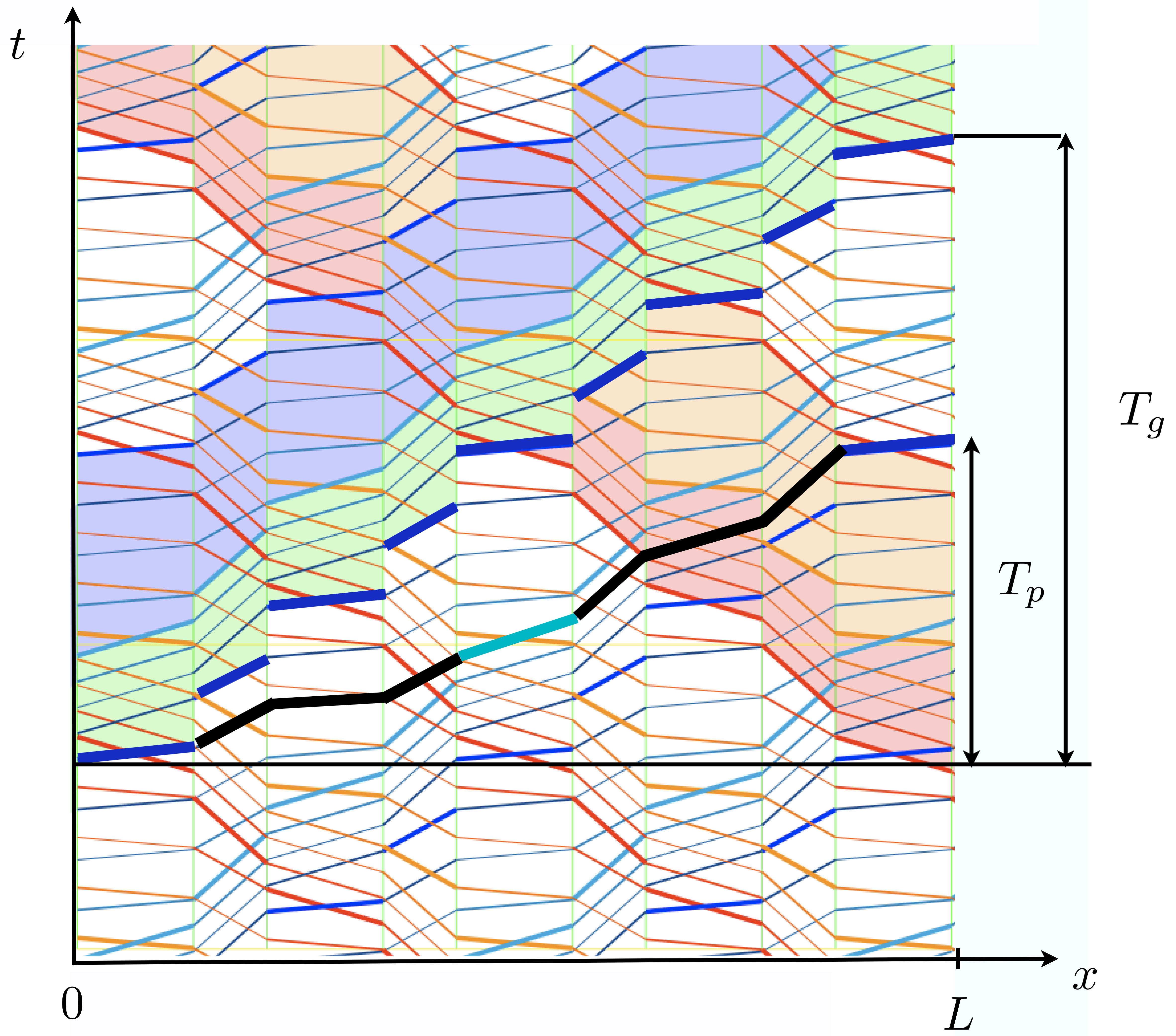
The characteristic=sound speed like a "Phase-Velocity"

.....

Max/Min-Characteristics jump UP

↓

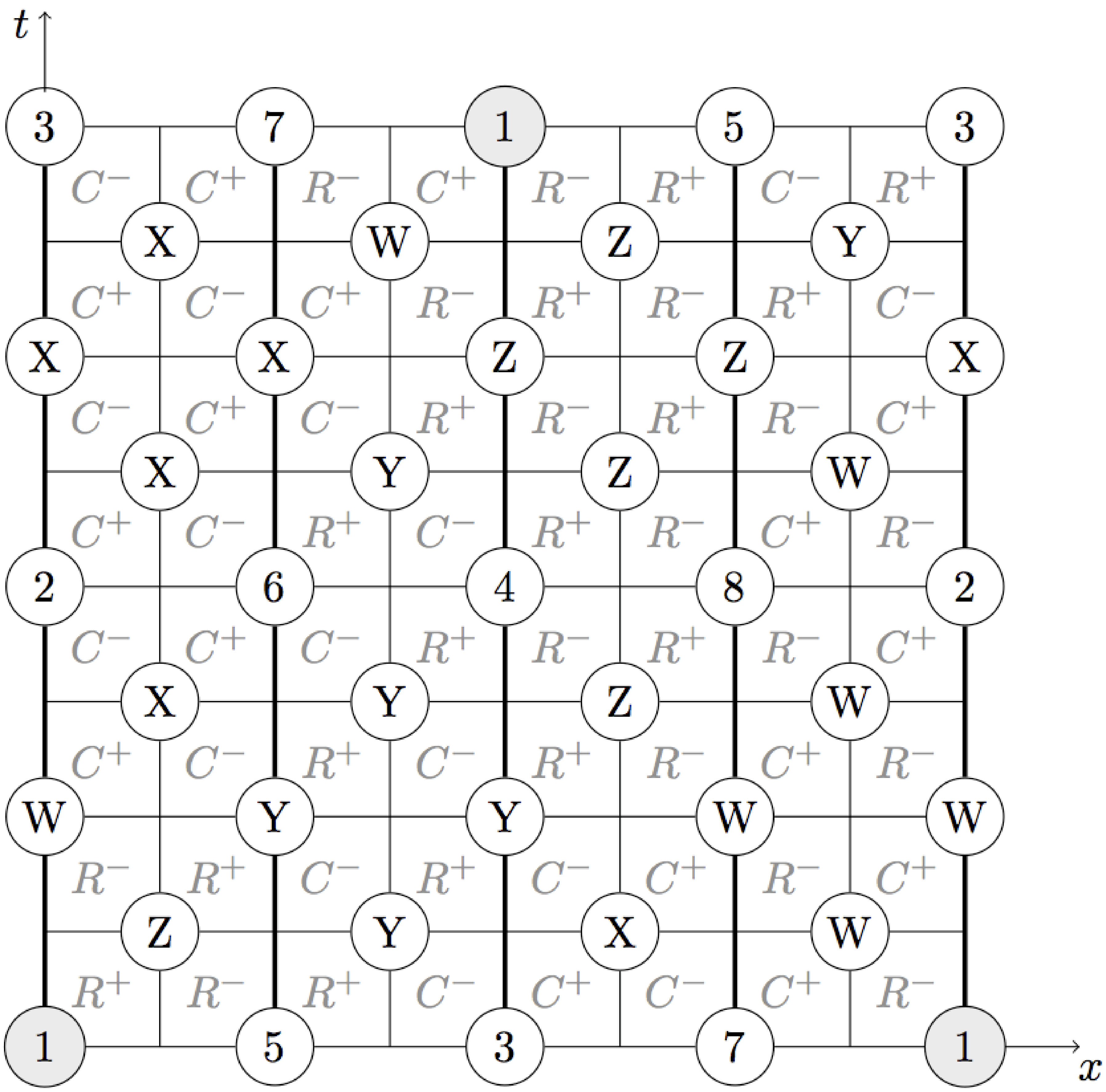
Group-Velocity < Phase-Velocity



v_g = speed of the wave crests<speed of the sound waves = v_p

$$v_g = \frac{L}{T_g} < v_p = \frac{L}{T_p}$$

Note: The simplest periodic R/C pattern is a cartoon that could be realized in a periodic solution in different ways:



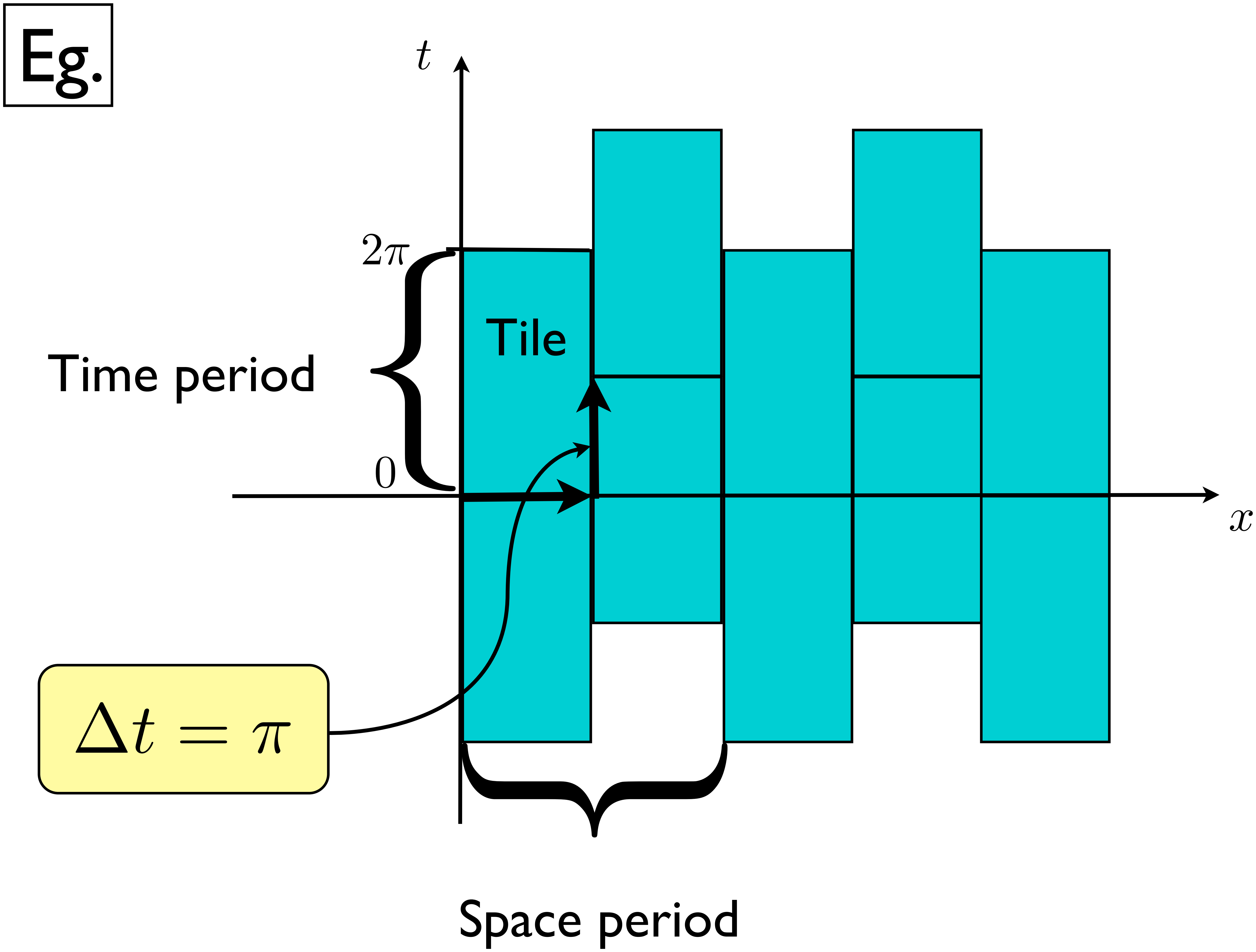
OBSERVE: The simplest periodic structure imposes two special symmetries:

(1) Periodicity in space

(2) Max/Min-characteristics JOIN UP

These are conditions imposed on the tiling that defines the periodic structure in xt-space

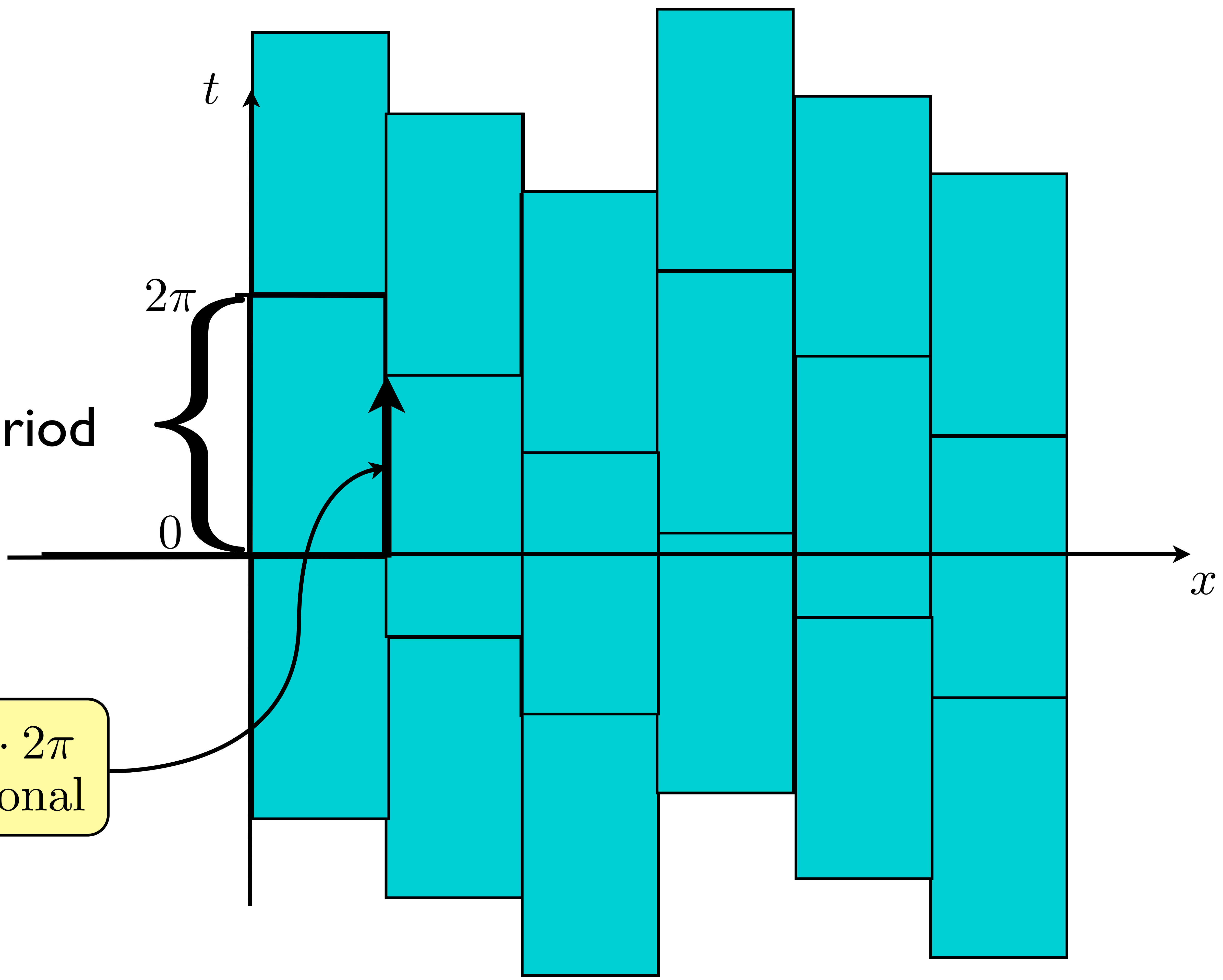
The solution will be periodic in space iff the time shift is commensurate with 2π



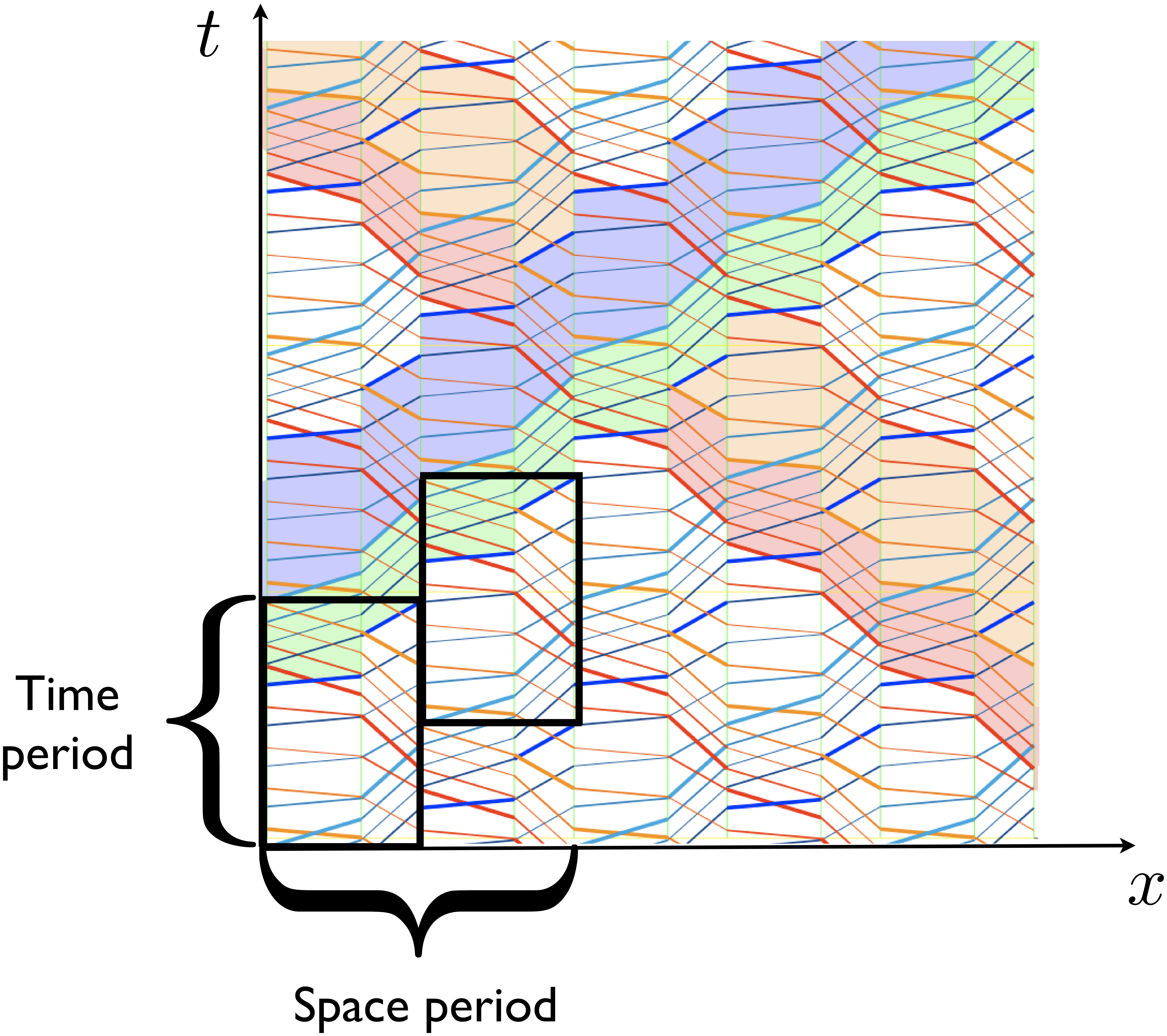
Space-periodic tiling

If the time shift is in-commensurate with 2π then solution will be quasi-periodic in space

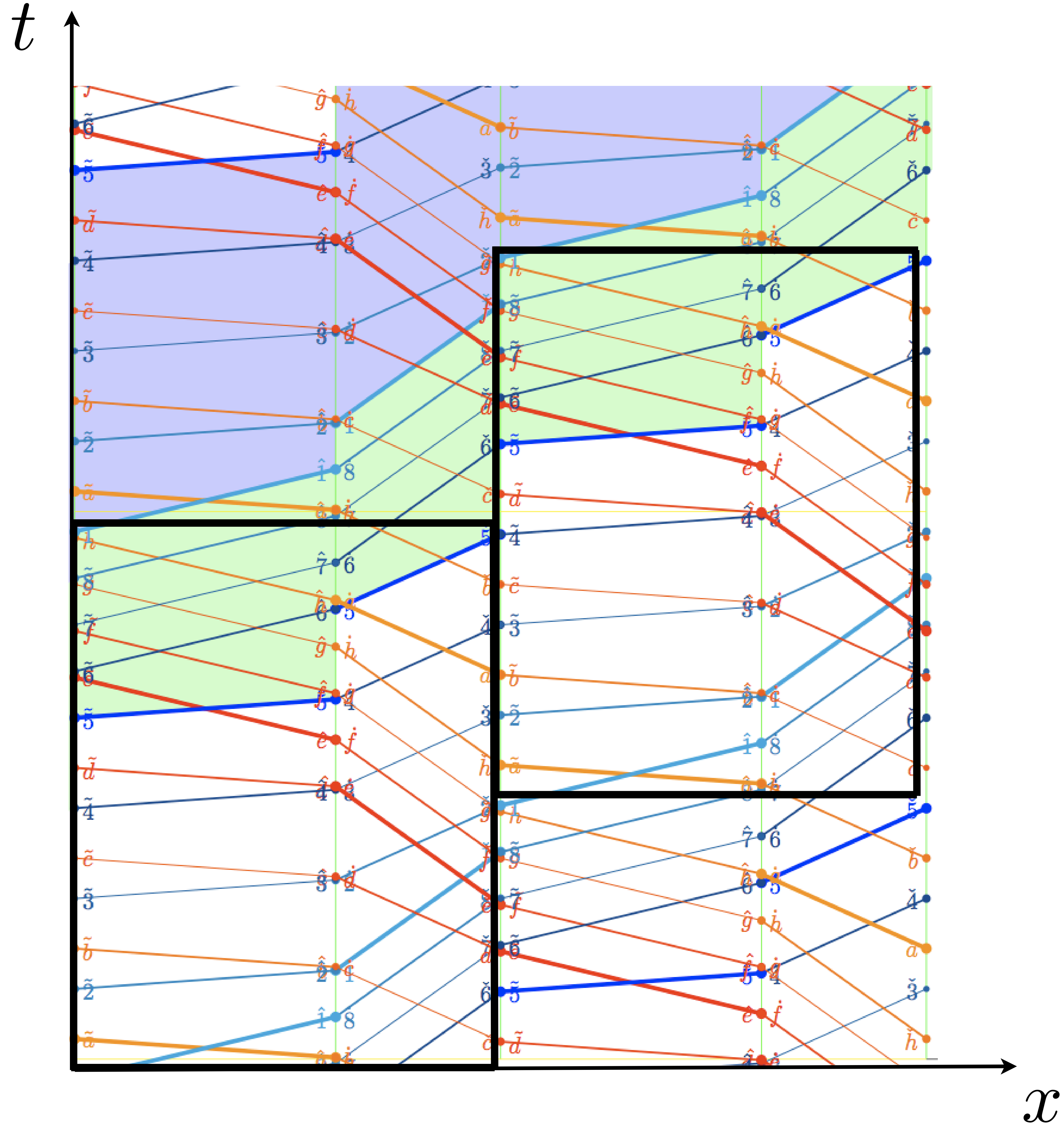
Eg.



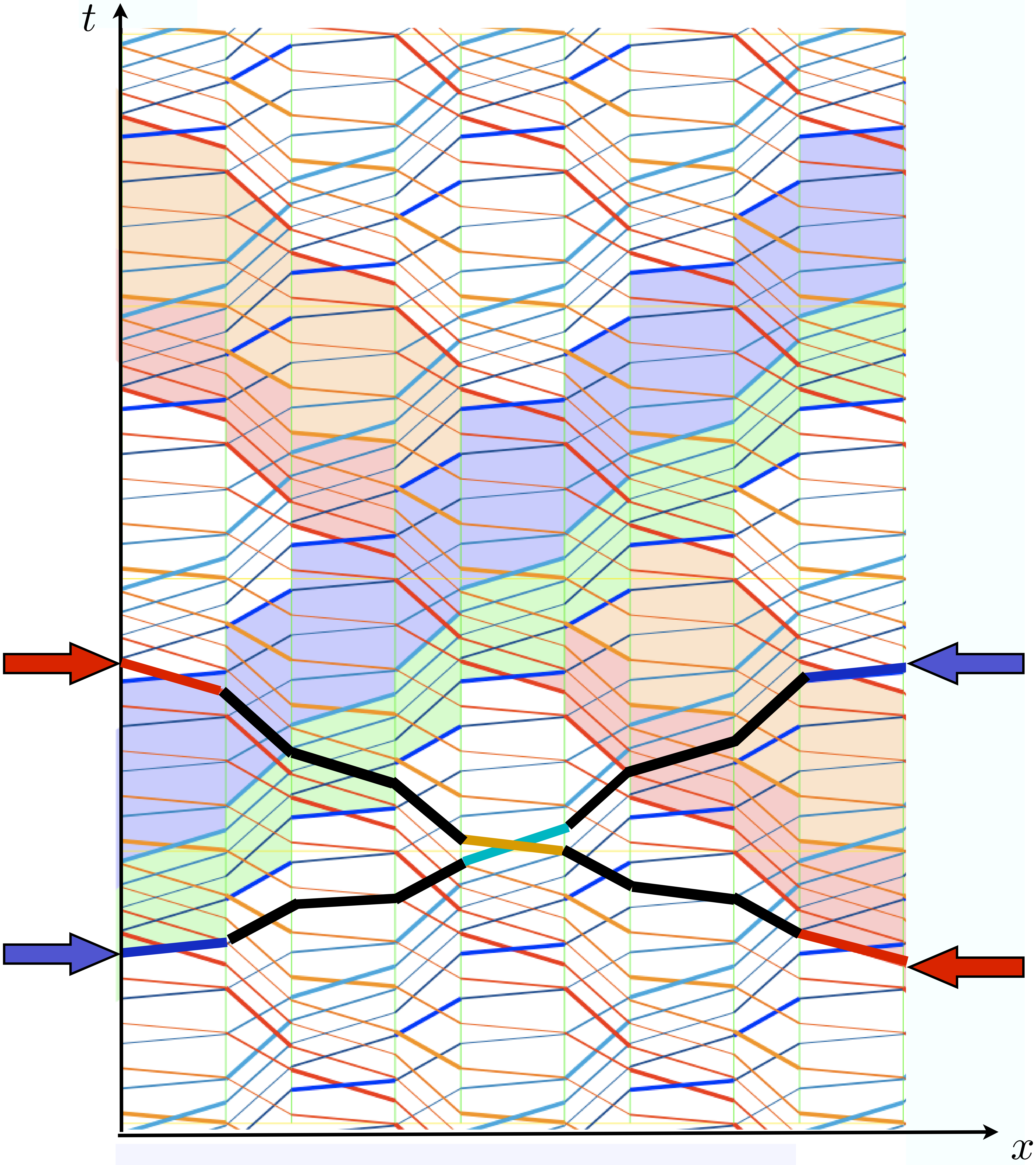
Quasi-periodic in space



(1) Simplest structure is space-periodic



(1) Simplest structure is space-periodic



(2) In the simplest periodic structure
Max/Min-characteristics JOIN UP

The speed of the wave crests is like an effective "Group-Velocity"

The characteristic=sound speed like a "Phase Velocity"

That Max/Min Characteristics Join Up

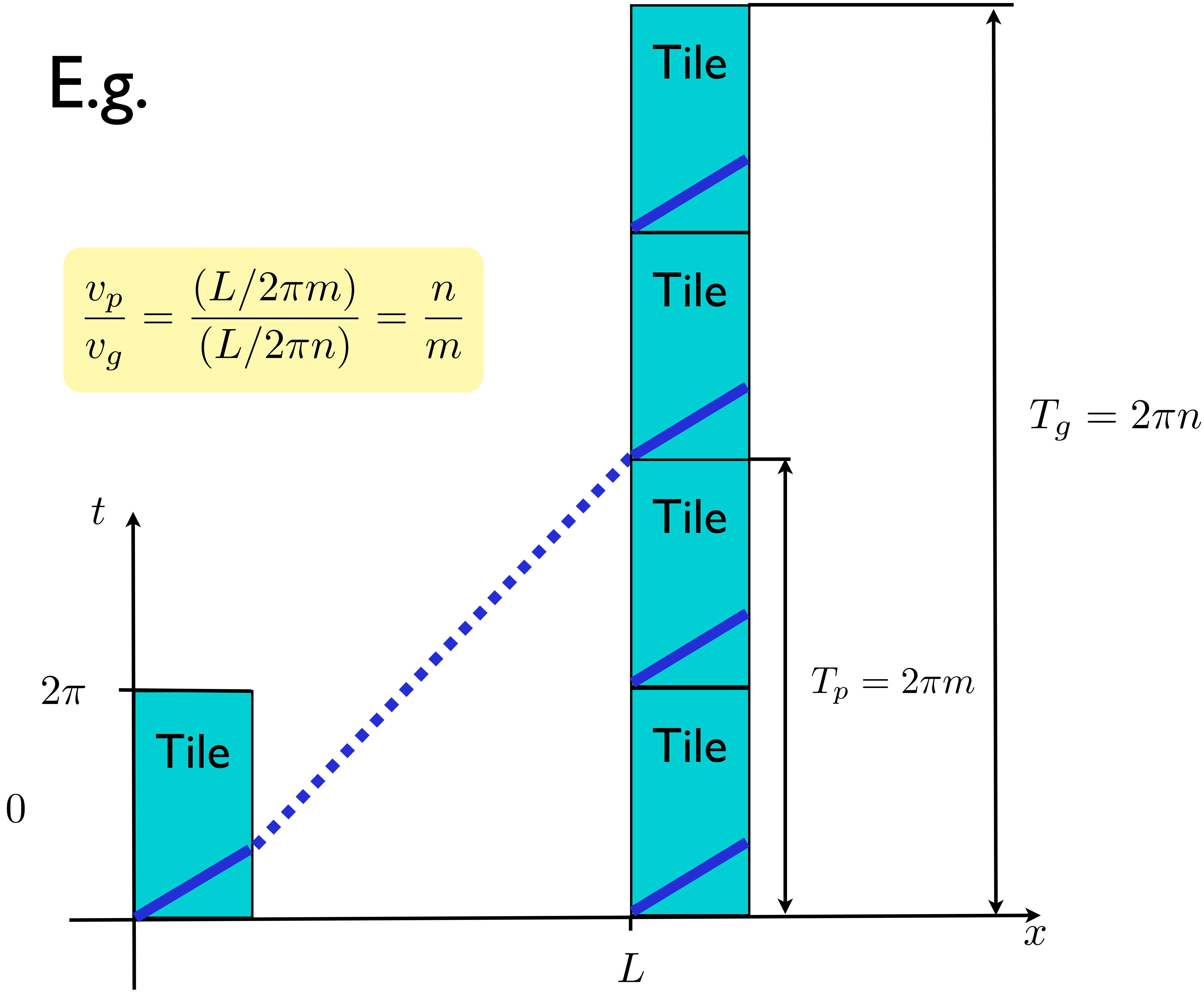
$$\Downarrow$$

Group Velocity is commensurate with Phase Velocity

When max/min characteristics join up v_p is commensurate with v_g

E.g.

$$\frac{v_p}{v_g} = \frac{(L/2\pi m)}{(L/2\pi n)} = \frac{n}{m}$$



V. The nonlinear eigenvalue problem as a perturbation of a linear problem

Outline

I. The Compressible Euler Equations

II. History/Prior Results for the Problem

III. Compressive and Rarefactive Waves

IV. The Simplest Possible Periodic Structure that Balances Compression and Rarefaction

V. The Nonlinear Eigenvalue Problem as a Perturbation of Linear Problem

VI. Exact Linearized Solutions Exhibiting the Simplest Periodic Structure

VII. Isolating Solutions in the Kernel of the Linearized Operator

VIII. Resonances, Small Divisors and Eigenvalues of the Linearized Operator

IX. The Liapunov-Schmidt Method

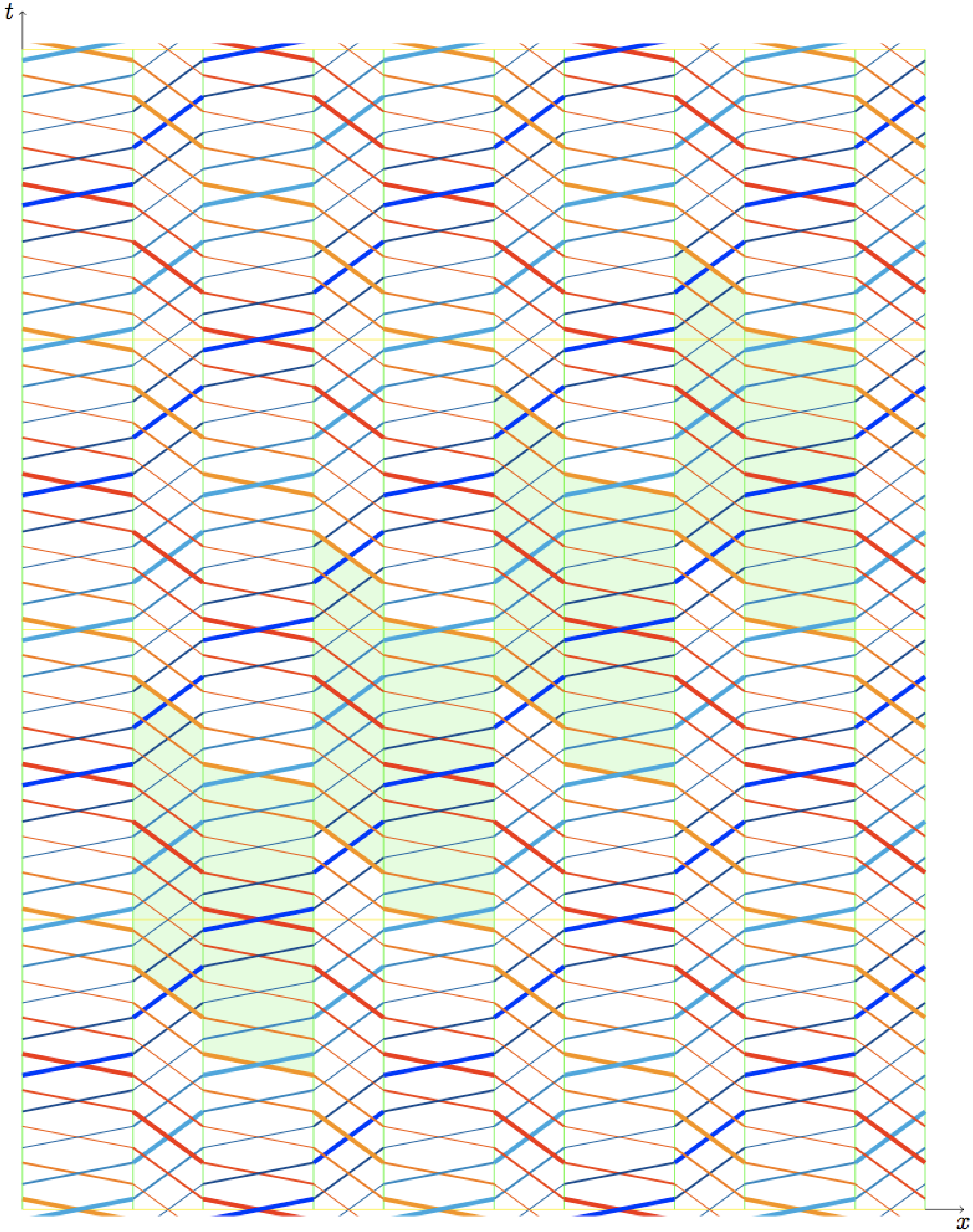
X. The Bifurcation Equation

XI. The Auxiliary Equation

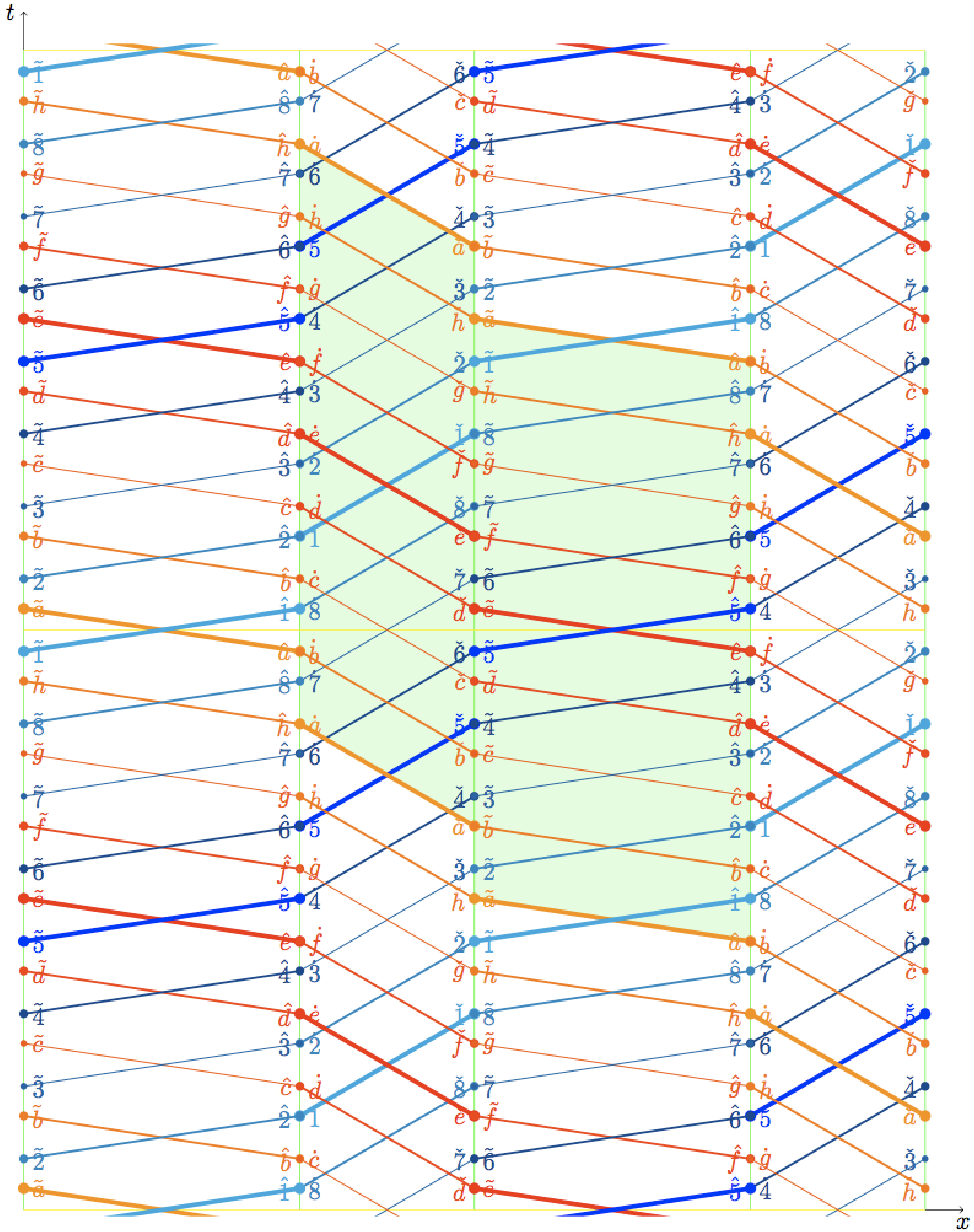
Consider first the "Linearized Problem" in which wave speeds are constant at each entropy level

....

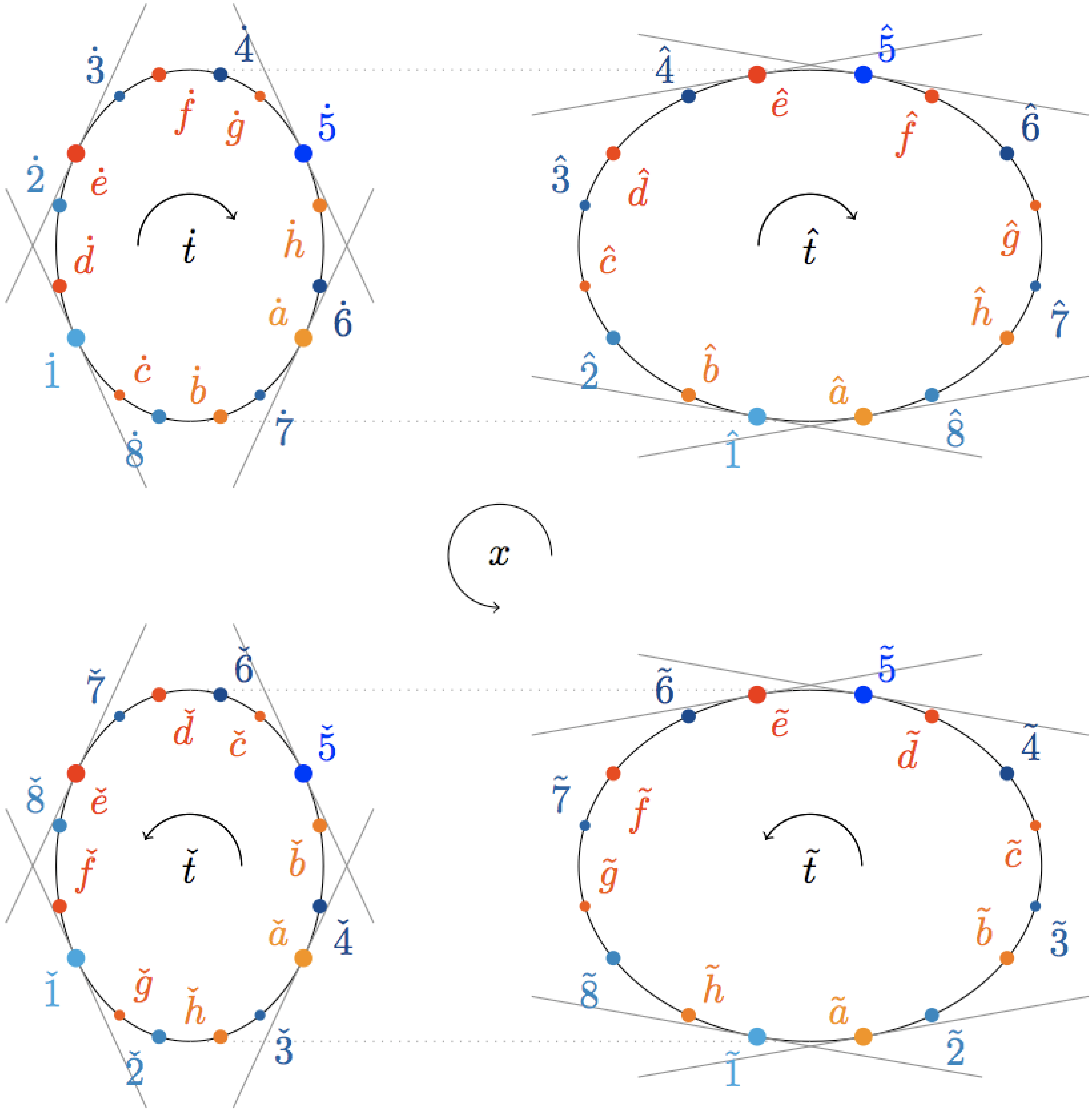
This is the limit as the states at each entropy level oscillate near a constant state



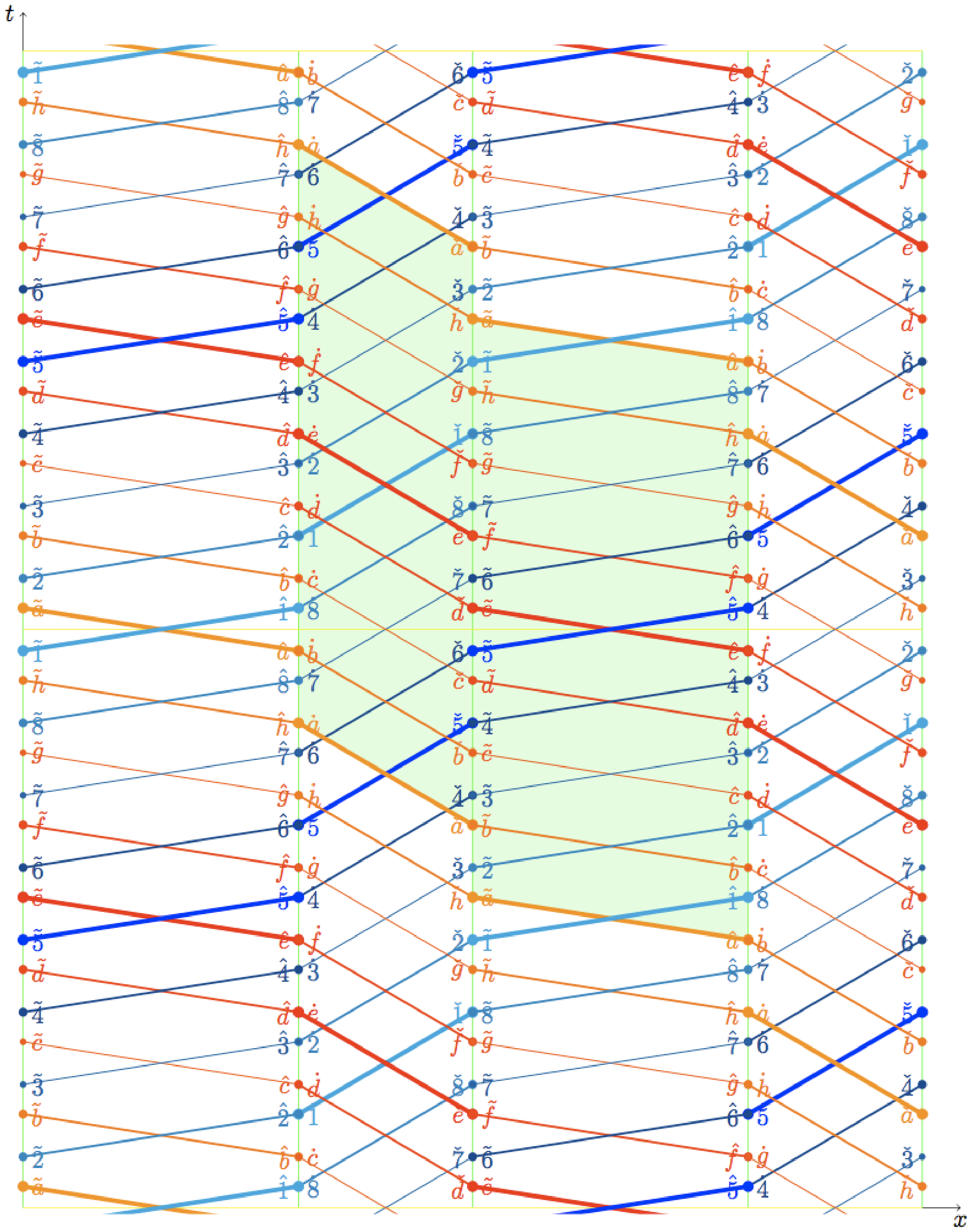
The periodic structure for the LINEARIZED PROBLEM

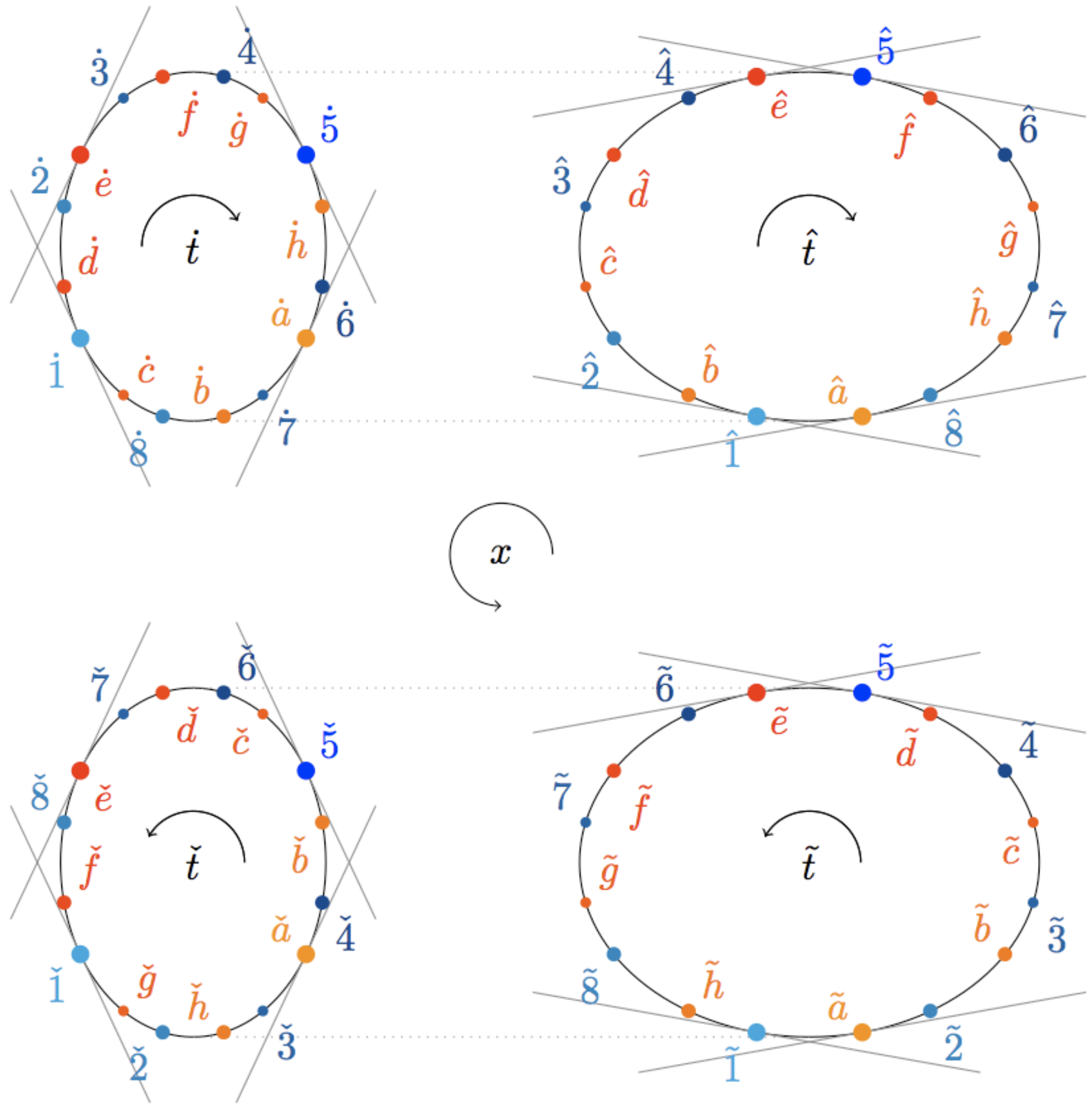


Labeling the states in xt-space



Corresponding states in (z,u)-plane lie on ellipses





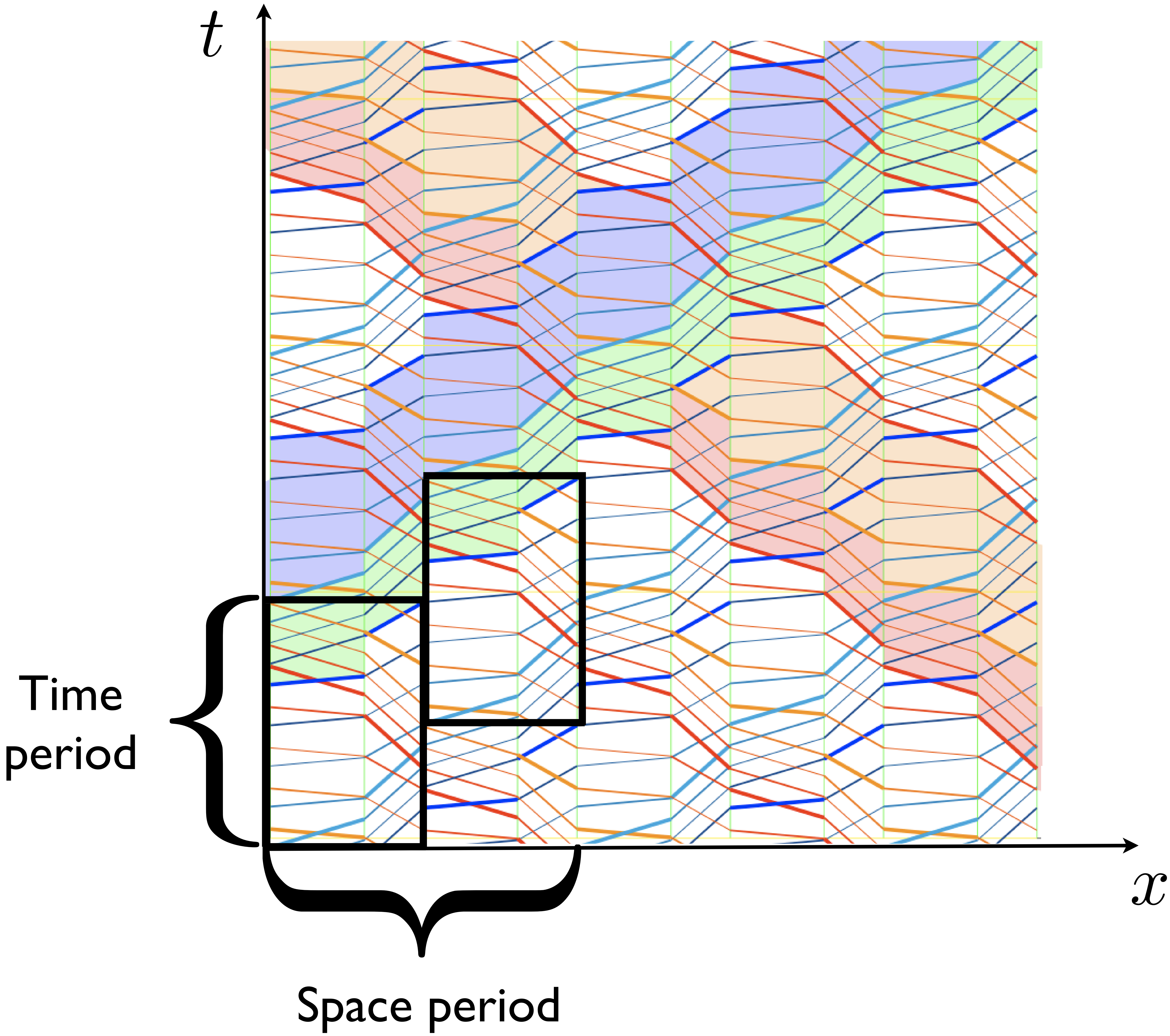
FIRST GOAL: show that in the <u>linearized case</u> this picture is

EXACT

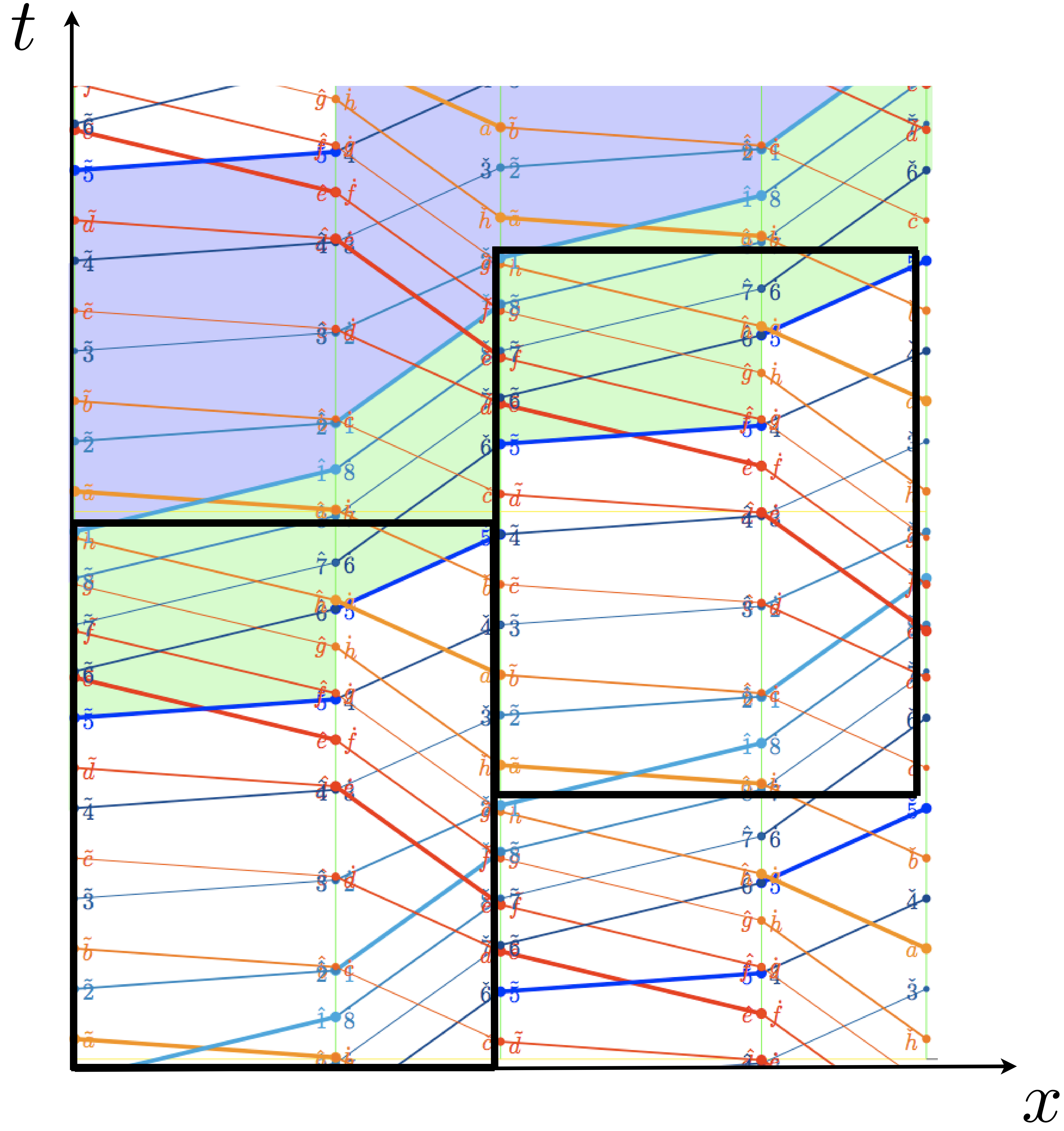
and use this to set up a perturbation problem for the nonlinear problem

CONSIDER AGAIN:

The Nonlinear Problem



(1) Simplest structure is space-periodic



(1) Simplest structure is space-periodic

Inspection of the periodic structure indicates:

- Solution jumps between two entropy levels $\overline{m} > \underline{m}$
- Starting with time-periodic "initial data" U(t) at x=0, solution evolves through five operations before periodic return:

(1) $\overline{\mathcal{E}}:$ Nonlinear evolution at $m = \overline{m}$

(2) $\mathcal{J}:$ Jump from $m = \overline{m}$ to $m = \underline{m}$

(3) $\underline{\mathcal{E}}:$ Nonlinear evolution at $m = \underline{m}$

(4) $\mathcal{J}^{-1}:$ Jump from $m = \underline{m}$ to $m = \overline{m}$

(5) $\mathcal{S}:$ Half period shift

AGAIN: The half period shift

IMPOSES a SYMMETRY

I.e.

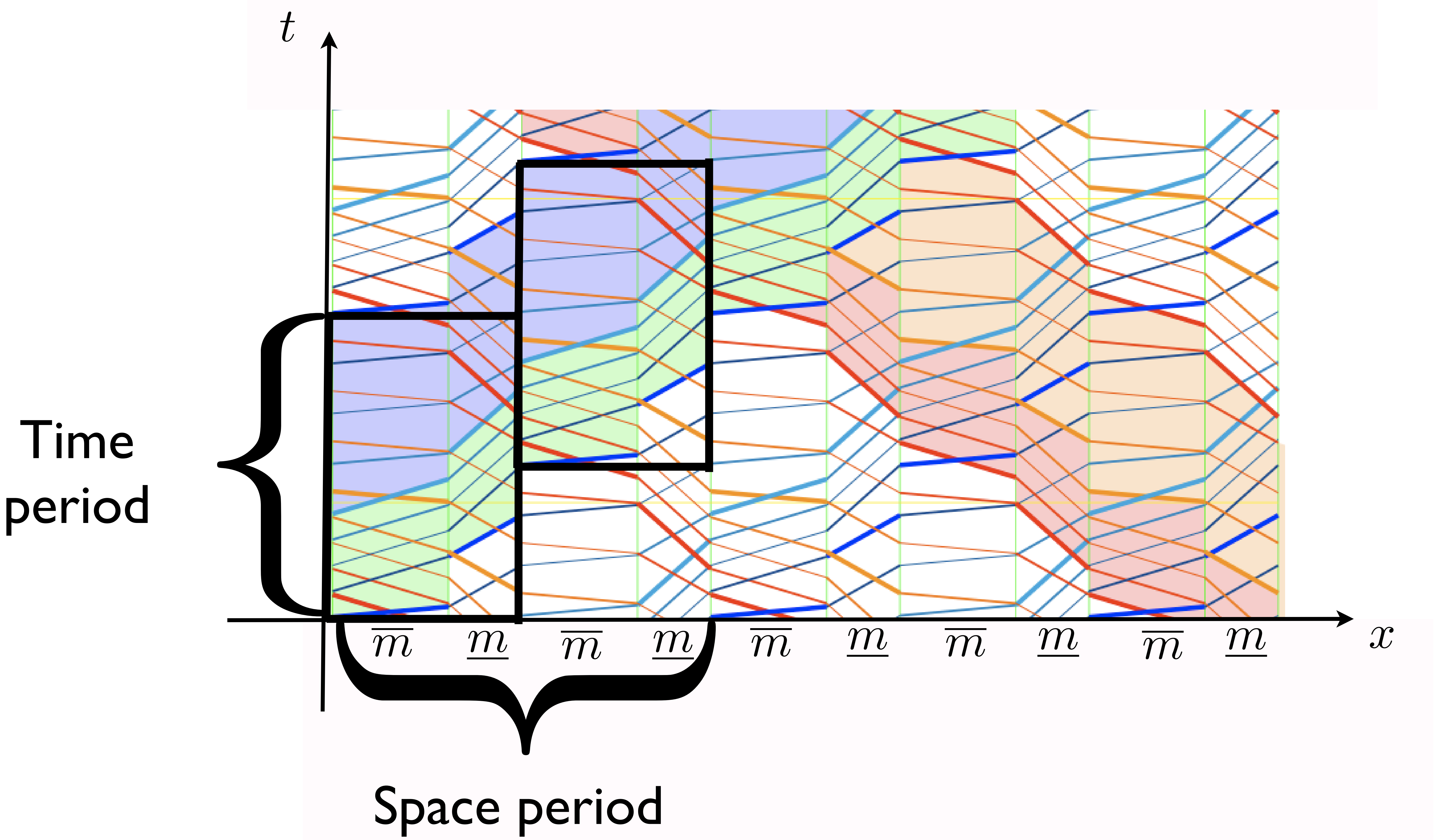
(1) $\overline{\mathcal{E}}$: Nonlinear evolution at $m = \overline{m}$

(2) $\mathcal{J}$: Jump from $m = \overline{m}$ to $m = \underline{m}$

(3) $\underline{\mathcal{E}}$: Nonlinear evolution at $m = \underline{m}$

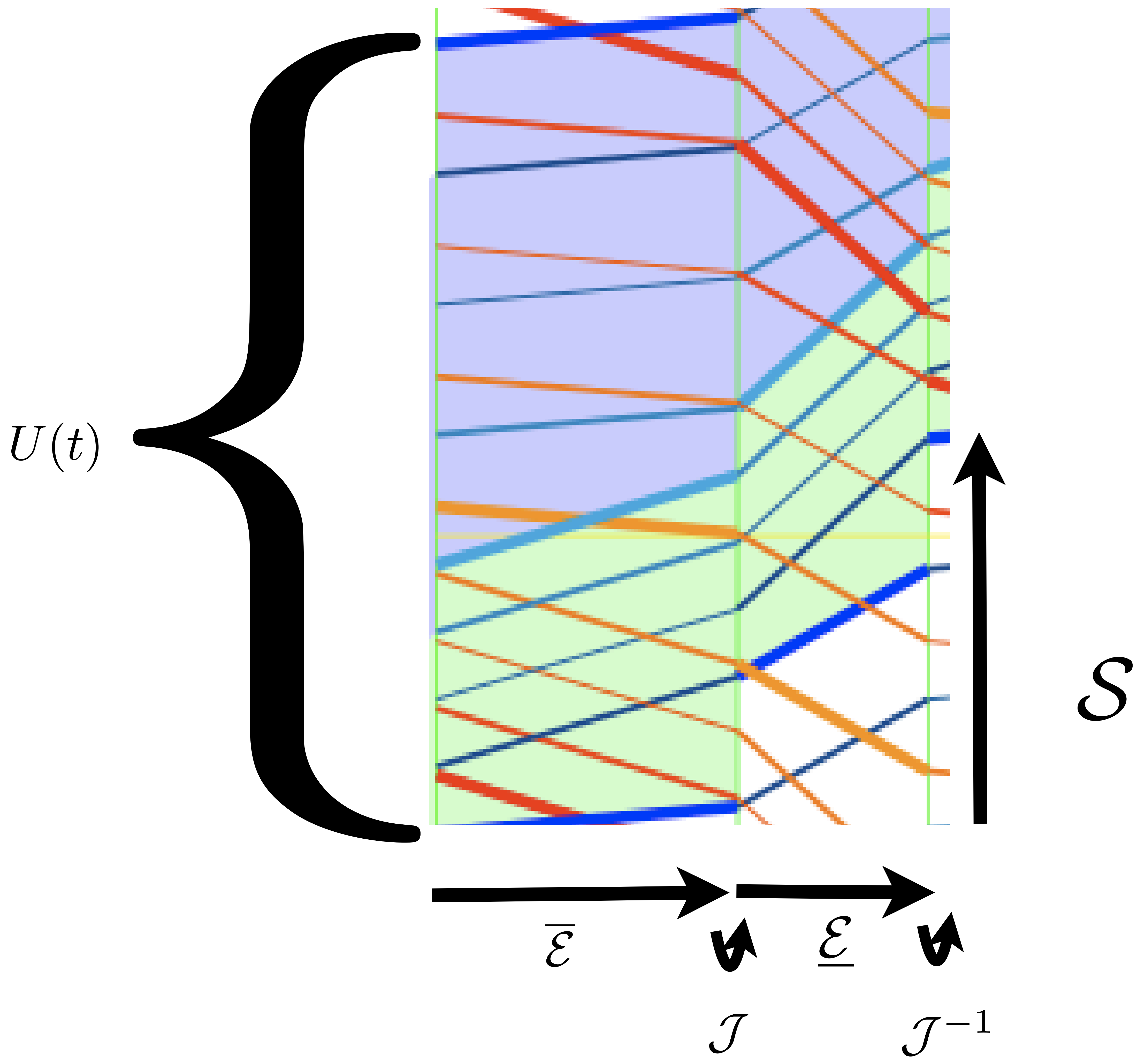
(4) $\mathcal{J}^{-1}$: Jump from $m = \underline{m}$ to $m = \overline{m}$

(5) $\mathcal{S}$: Half period shift

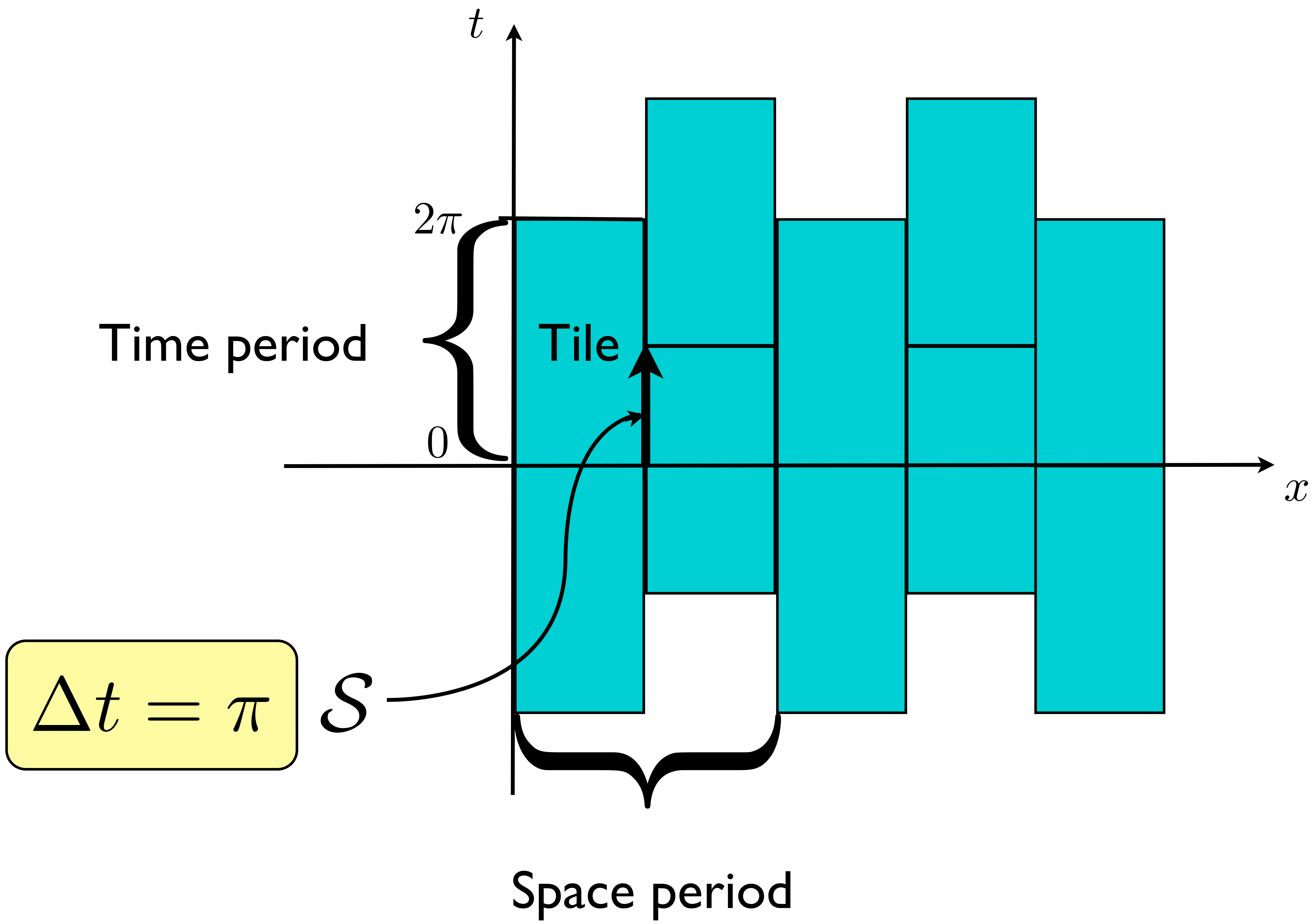


The periodicity condition

$$\mathcal{S}\cdot\mathcal{J}^{1}\cdot\underline{\mathcal{E}}\cdot\mathcal{J}\cdot\overline{\mathcal{E}}\;[U(\cdot)] = U(\cdot)$$



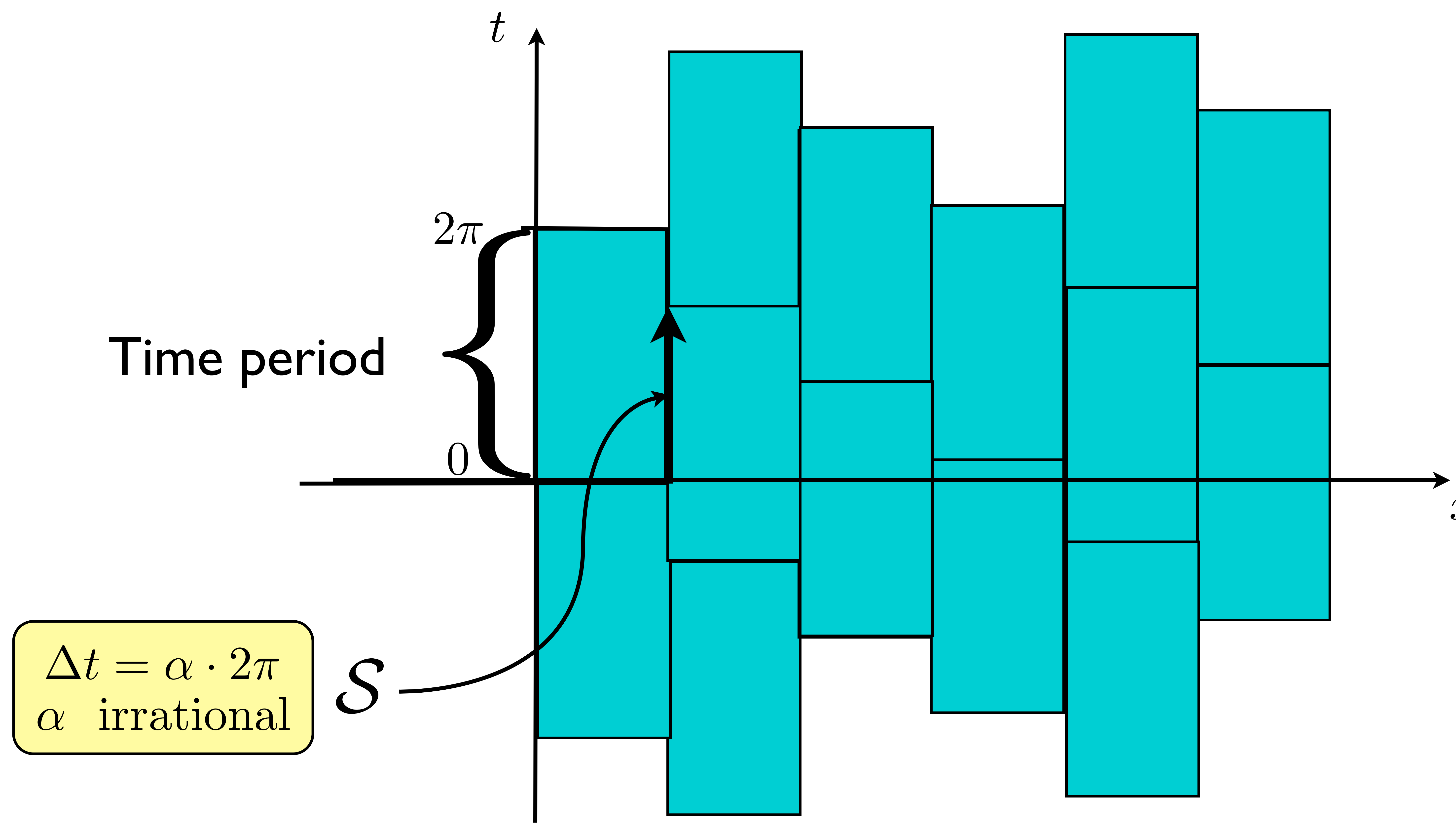
- Note: The half period shift $\mathcal{S}$ ensures solution is periodic in space



- If $\mathcal{S}$ were incommensurate with π, then the solution will be only quasi-periodic in space.

SIDE COMMENT

Our constructions appear robust enough to construct linearized solutions for any shift $\mathcal{S}$

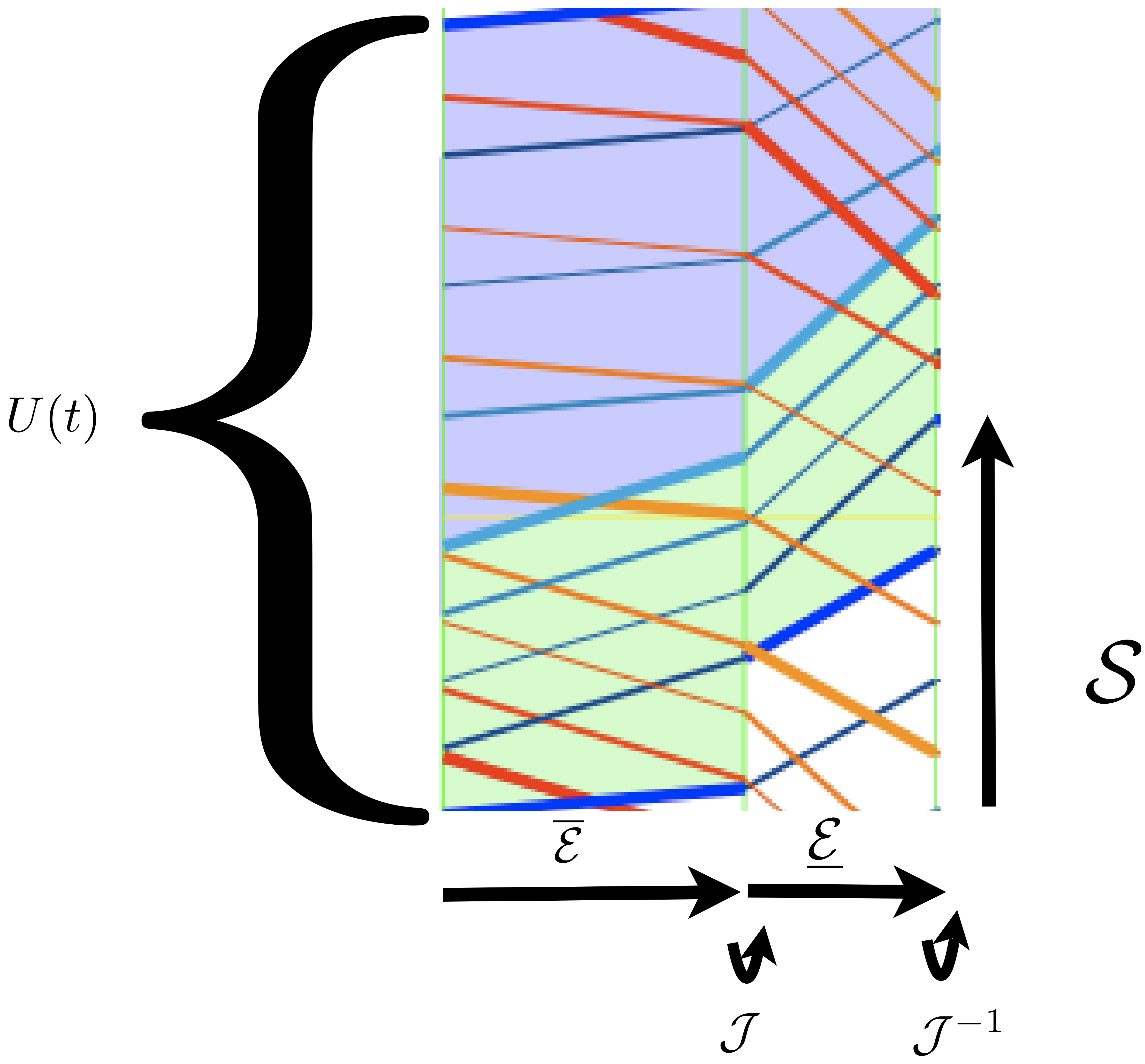


Quasi-periodic in space

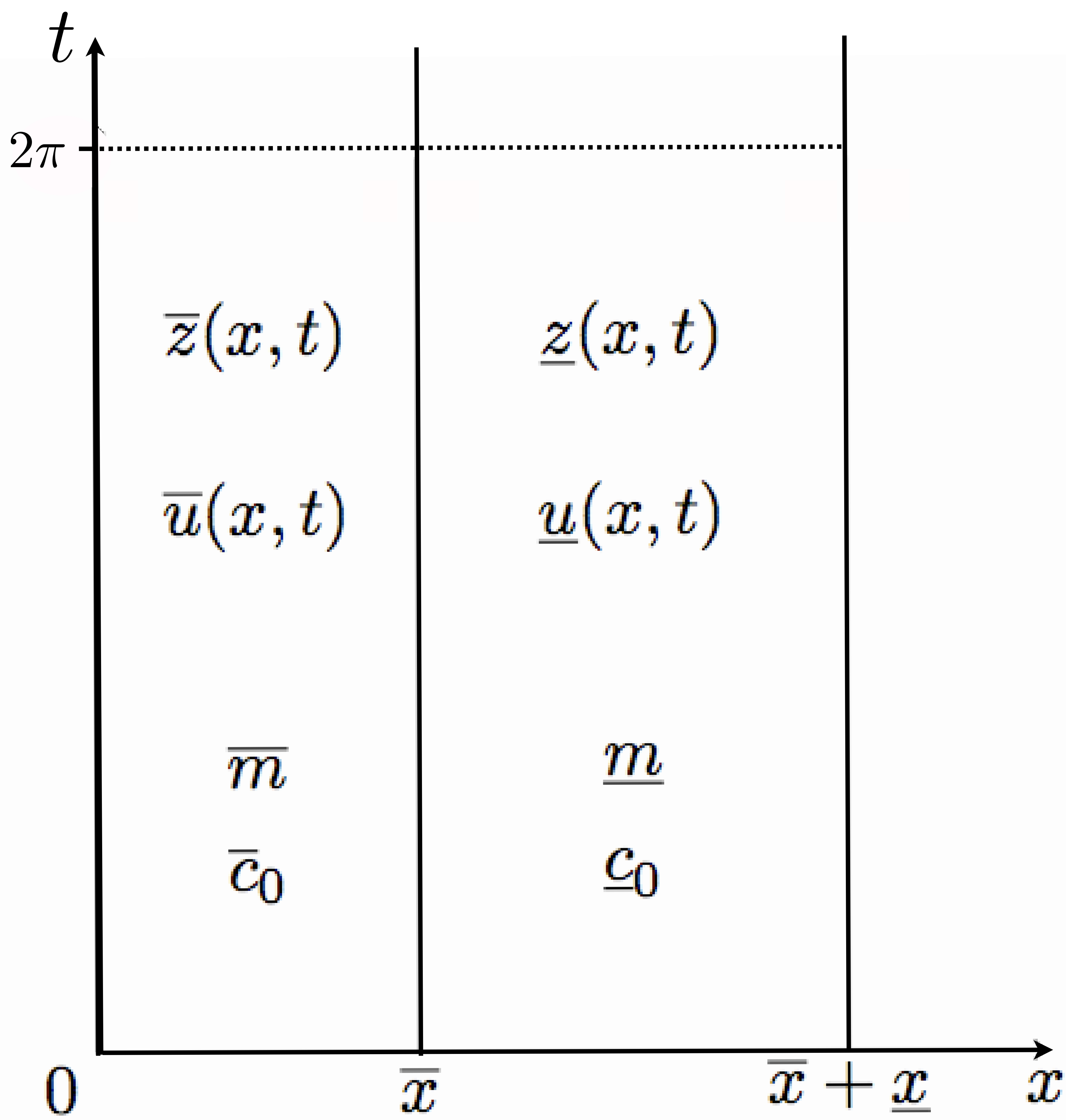
$\Big($ The speed of the tile is the "group velocity", determined by $\mathcal{S}$ $\Big)$

We restrict to the simplest periodicity condition

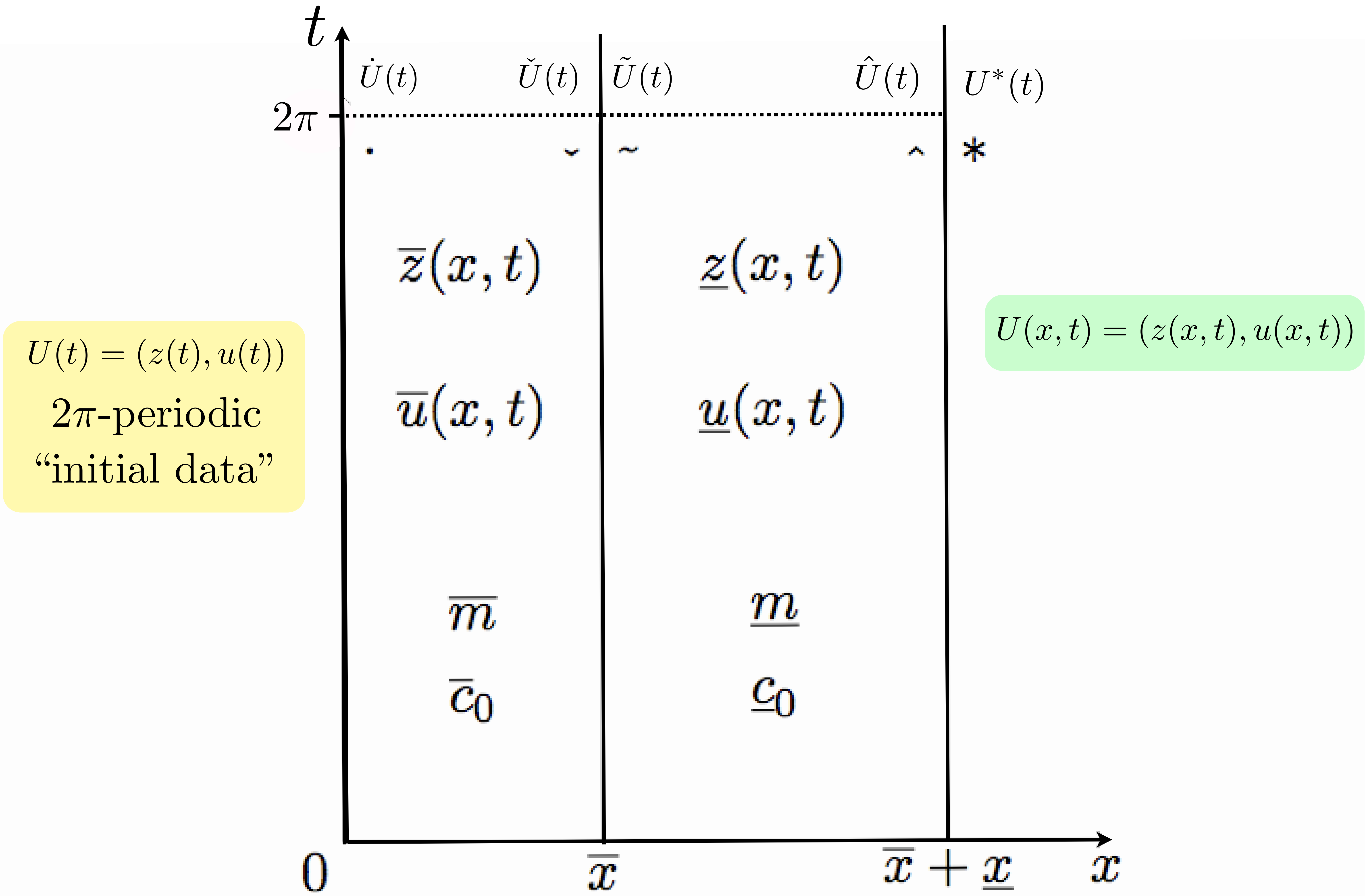
$$\mathcal{S} \cdot \mathcal{J}^{-1} \cdot \overline{\mathcal{E}} \cdot \mathcal{J} \cdot \underline{\mathcal{E}}\,[U(\cdot)] = U(\cdot)$$



- A periodic tile in xt-space:



- Label the stages of $U(\cdot)$ evolution by $\bullet \;\vee\; \sim \;\wedge\; *$

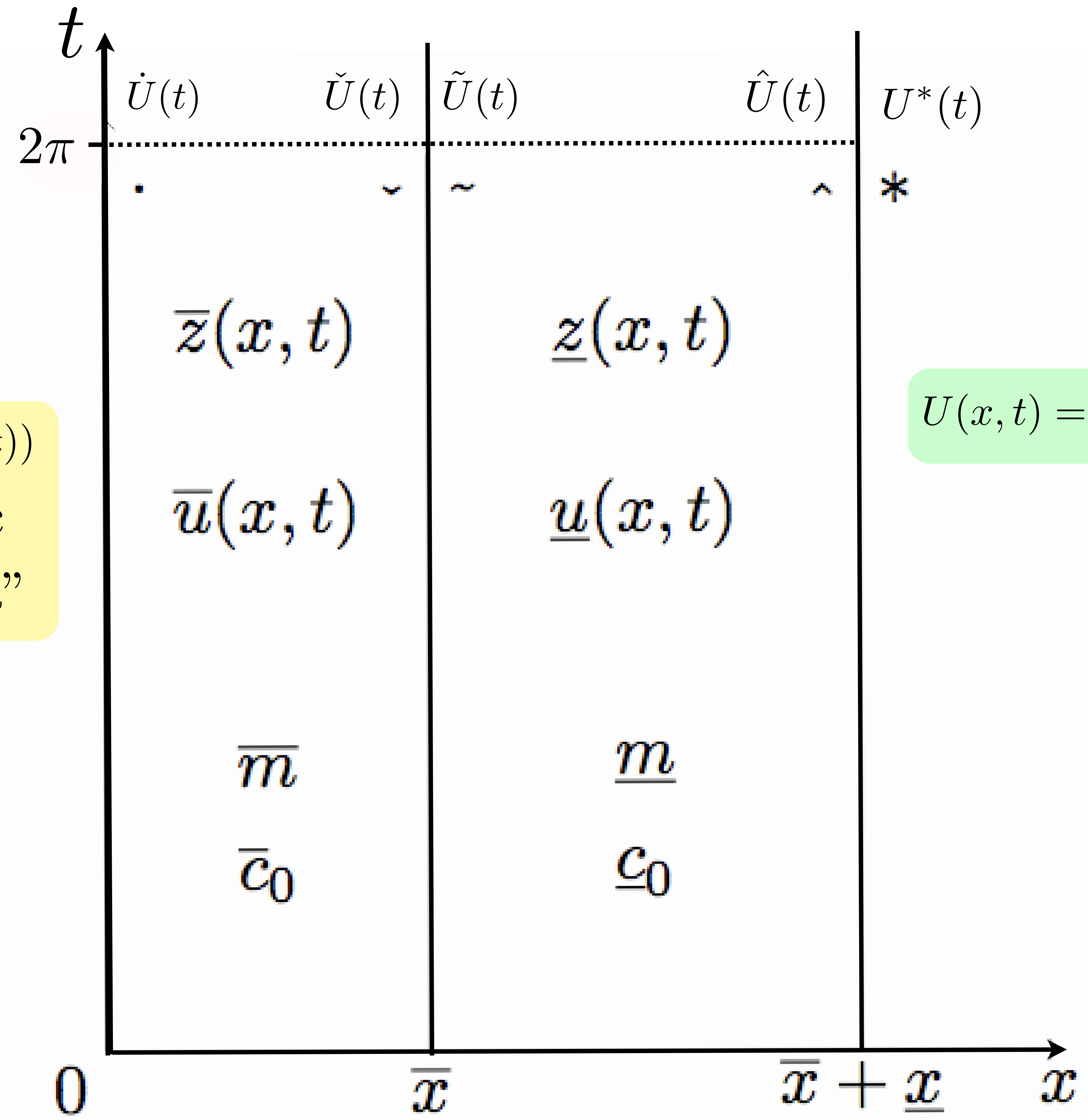


- Label the stages of $U(\cdot)$ evolution by $\bullet\ \smile\ \sim\ \wedge\ *$

$$\hat{U}(t) = \underline{\mathcal{E}}[\tilde{U}(t)] \leftarrow \tilde{U}(t) = \mathcal{J}[\check{U}(t)] \leftarrow \check{U}(t) = \overline{\mathcal{E}}[\dot{U}(t)] \leftarrow \dot{U}(t) = U(t)$$

$$\downarrow$$

$$U^*(t) = \mathcal{J}^{-1}[\hat{U}(t)] \rightarrow U^{\#}(t) = \mathcal{S}[U^*(t)]$$



$U(t) = (z(t), u(t))$
2π-periodic
"initial data"

$U(x,t) = (z(x,t), u(x,t))$

Non-dimensionalize the Problem

- Choose base states $m_0, \; z_0, \; u_0$

  With corresponding sound speed

$$c_0 = c(z_0, m_0) = K_c m z^{\frac{\gamma+1}{\gamma-1}}$$

- Let x_0 be the starting point for evolution in x

Dimensionless Variables

$m \equiv m_0$

- Give time and space the same dimension by defining y through the relation

$$y - y_0 = \frac{x - x_0}{c_0}$$

- Define the dimensionless variables

$$\begin{aligned} w &= \frac{z}{z_0} \\ v &= \frac{u - u_0}{m_0 z_0} \end{aligned}$$

- Equations convert to the dimensionless form

$$\begin{aligned} w_y + \sigma(w) v_t = 0 \\ v_y + \sigma(w) w_t = 0 \end{aligned}$$

$$\sigma(w) = w^{-d}$$

$$d \equiv \frac{\gamma + 1}{\gamma - 1}$$

- To see this...

$$z_t - \frac{c}{m_0} u_x = 0$$
$$u_t + m_0 c z_x = 0$$

$$m \equiv m_0$$

$$w = \frac{z}{z_0}, \qquad v = \frac{u - u_0}{m_0 z_0}$$

$$z = z_0 w, \qquad u = u_0 + m_0 z_0 v$$

$$z_0 w_t + \frac{c}{m_0} m_0 z_0 v_x = 0$$
$$m_0 z_0 v_t + m_0 c z_0 w_x = 0$$

$$y - y_0 = \frac{x - x_0}{c_0}$$

$$\frac{\partial}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = \frac{1}{c_0}\frac{\partial}{\partial y}$$

$$w_t + \left(\frac{c}{c_0}\right) v_y = 0$$
$$v_t + \left(\frac{c}{c_0}\right) w_y = 0$$

$$c = K_c m z^{\frac{\gamma+1}{\gamma-1}}$$

$$\sigma(w) = w^{-d}$$

$$w_y + \sigma(w) v_t = 0$$
$$v_y + \sigma(w) w_t = 0$$

Conclude: The nonlinear evolution equations take the non-dimensional form:

$$w_y + \sigma(w)v_t = 0$$
$$v_y + \sigma(w)w_t = 0$$

where:

$$\sigma \equiv \sigma(w) = w^{-d}$$

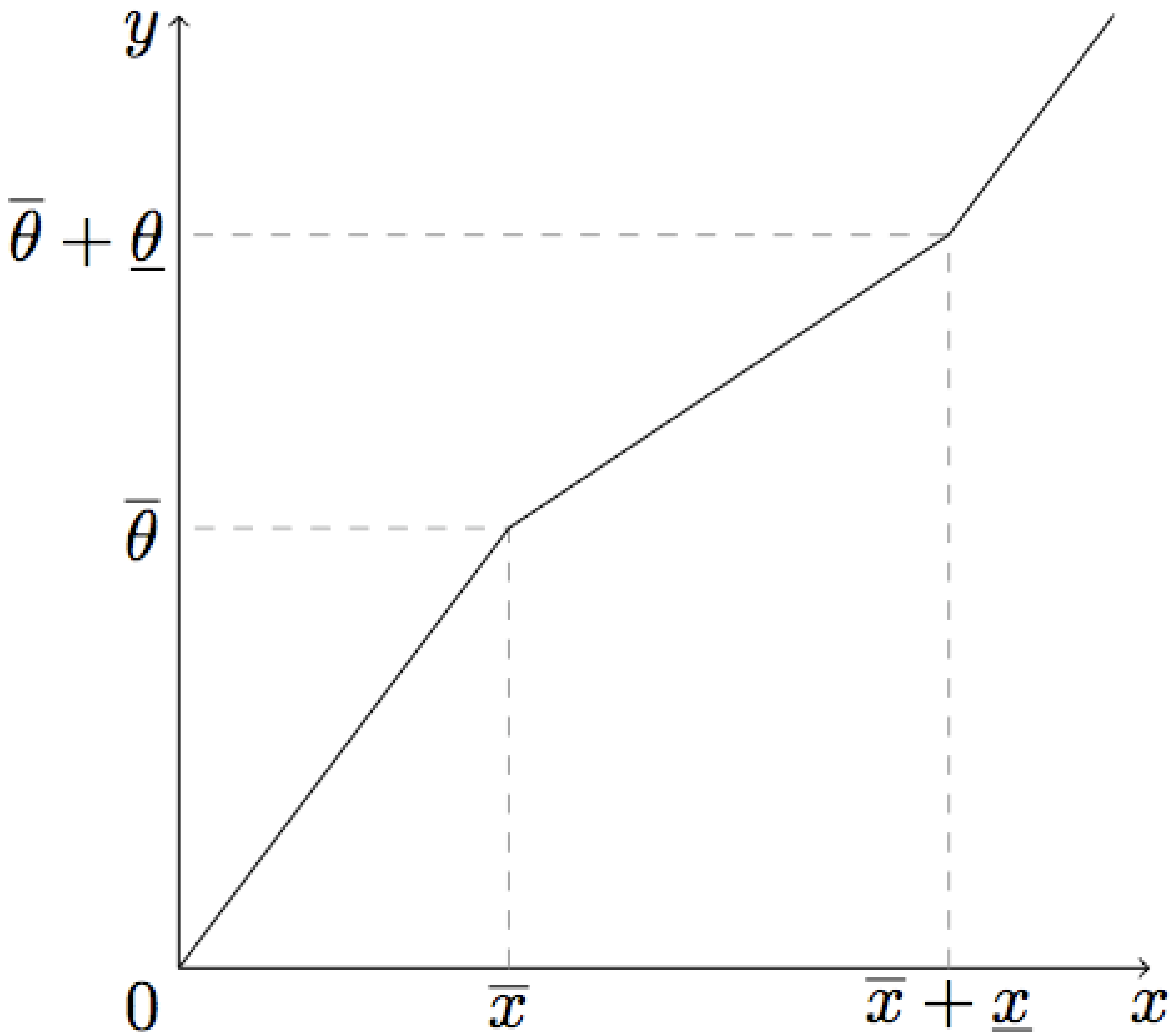
$$d \equiv \frac{\gamma+1}{\gamma-1}$$

Remarkable Fact:

The equations are independent of base states!

I.e., independent of m_0, z_0, u_0

The Transformed Problem

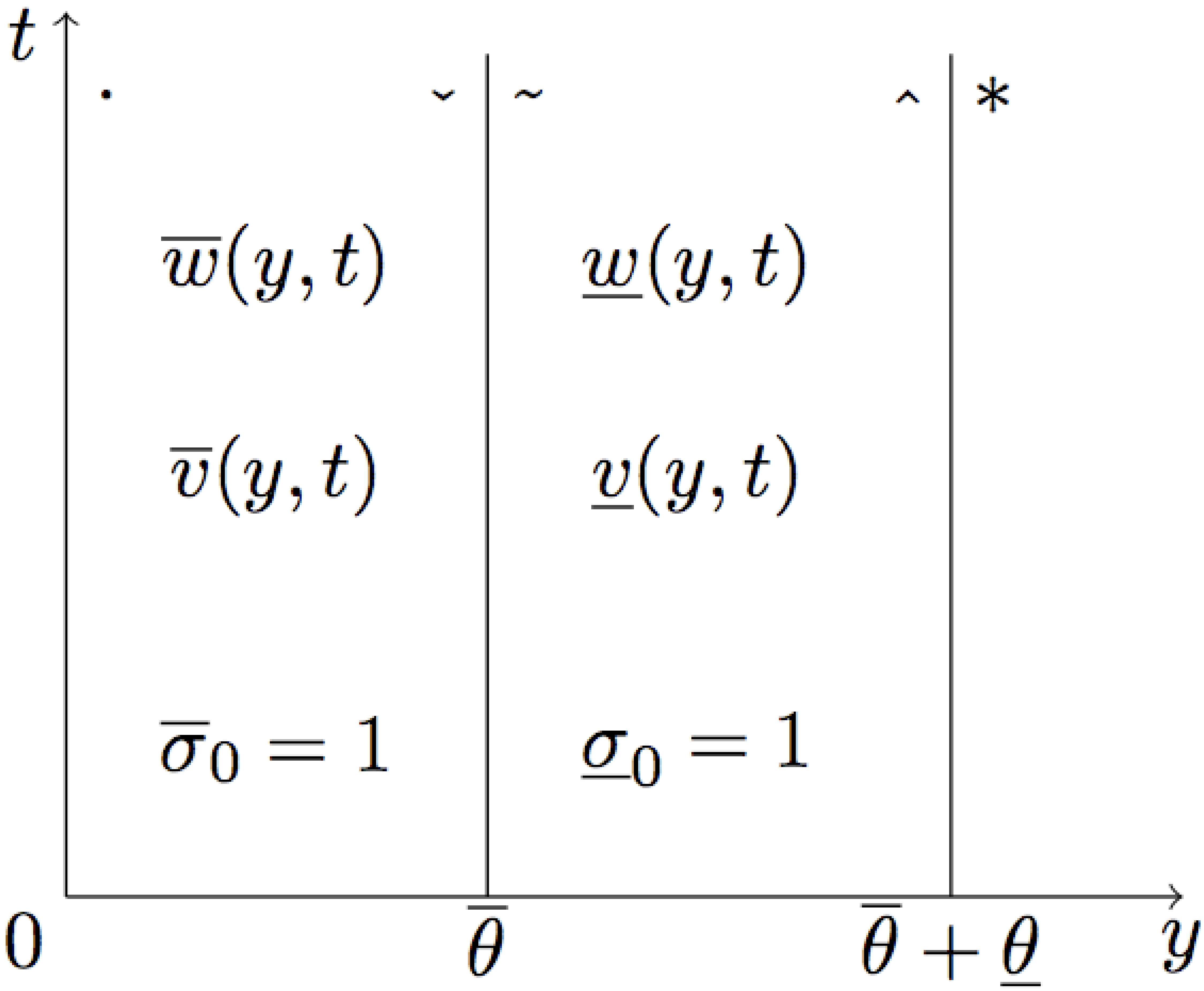


$$\overline{\theta} = \frac{c(\overline{z}_0)}{\overline{x}}$$

$$\underline{\theta} = \frac{\underline{x}}{c(\underline{z}_0)}$$

The mapping $x \to y$, $\overline{x} \to \overline{\theta}$, $\underline{x} \to \underline{\theta}$

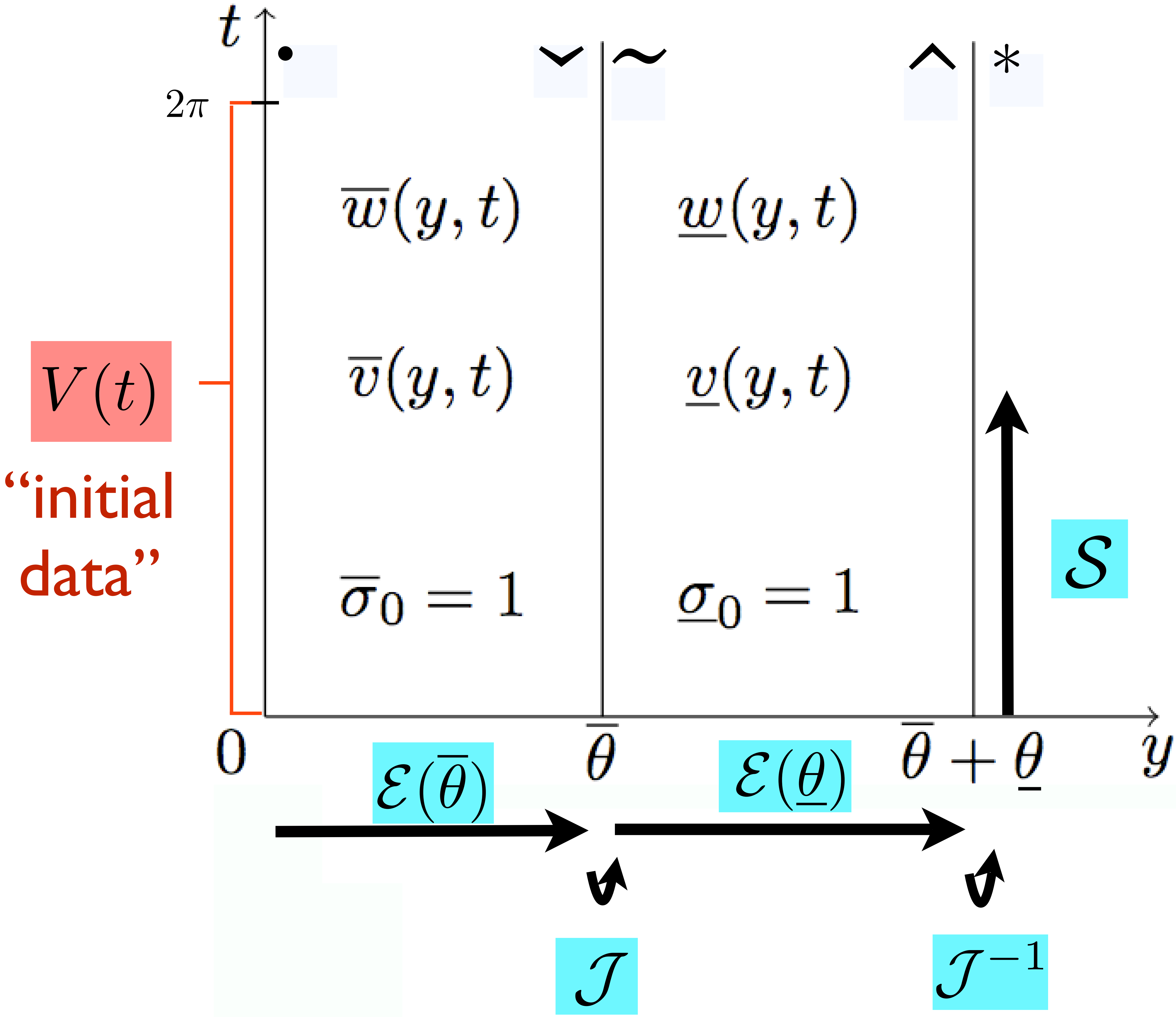
The Transformed Problem



Periodic tile consisting of two constant entropy levels $\overline{m}$, $\underline{m}$ in the (y,t)-plane

The Transformed Problem

$$\mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[V(\cdot)] = V(\cdot)$$



The Nonlinear Evolution Operator

DEFINE:

$$V(y,\cdot) = \mathcal{E}(y)[V(\cdot)]$$

to be evolution by system

$$w_y + \sigma(w)v_t = 0$$
$$v_y + \sigma(w)w_t = 0$$

through interval $[0, y]$ starting from “initial data”

$$V(0,t) = V(t)$$

The R-H Jump Conditions in (w,v)-Coordinates

Theorem: R-H Jump Conditions in (w,v)-variables at an entropy jump are:

$$\overline{w} = \underline{w}$$

$$\overline{m}^{\frac{d-1}{d+1}}\,\overline{v} = \underline{m}^{\frac{d-1}{d+1}}\,\underline{v}$$

.....

COROLLARY: In (w, v)-coordinates, w and $\sigma(w)$ are continous while v is discontinuous at entropy jumps

The R-H Jump Conditions in (w,v)-Coordinates

Theorem: R-H Jump Conditions in (w,v)-variables at an entropy jump are:

$$\overline{w} = \underline{w}$$

$$\overline{m}^{\frac{d-1}{d+1}}\overline{v} = \underline{m}^{\frac{d-1}{d+1}}\underline{v}$$

• • • • •

Proof: R-H in (z,u)-coordinates are...

$$[u] = \overline{m z_0 v} - \underline{m z_0 v} = 0$$

$$[p] = \overline{m}^2 \overline{z}^{d+1} - \underline{m}^2 \underline{z}^{d+1} = 0$$

from which it follows by substitution.

The Entropy Jump Operator in (w,v)-coords

Define: the entropy jump operator $\mathcal{J}$ acting on $V(\cdot)$ pointwise by

$$\mathcal{J}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} \begin{bmatrix} w \\ v \end{bmatrix}$$

where

$$J = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}}$$

$$V = (w, v)$$

Theorem: $\mathcal{J}$ encodes the R-H jump from entropy level $\overline{m}$ on left to $\underline{m}$ on right

and $\mathcal{J}^{-1}$ is the reverse jump

The Entropy Jump Operator in (w,v)-coords

Define: the entropy jump operator $\mathcal{J}$ acting on $V(\cdot)$ pointwise by

$$\mathcal{J}\left[\begin{array}{c} w \\ v \end{array}\right] = \left(\begin{array}{cc} 1 & 0 \\ 0 & J \end{array}\right)\left[\begin{array}{c} w \\ v \end{array}\right]$$

where

$$J = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}}$$

$$V = (w, v)$$

NOTE: $\mathcal{J}$ is LINEAR

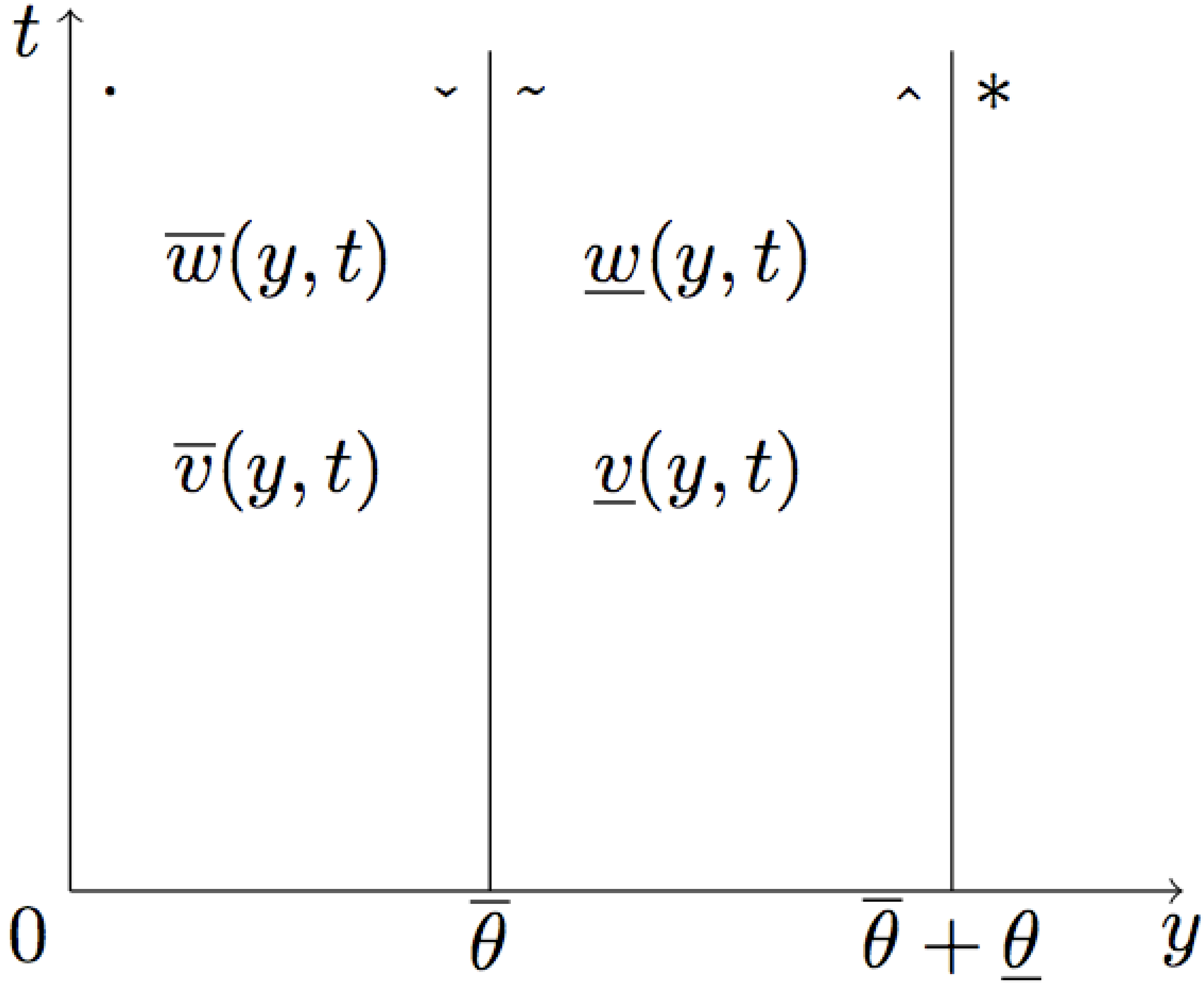
The Shift Operator

Define the shift operator $\mathcal{S}$ acting on V by

$$\mathcal{S}V(t) = V(t + \pi)$$

NOTE: $\mathcal{S}$ is LINEAR

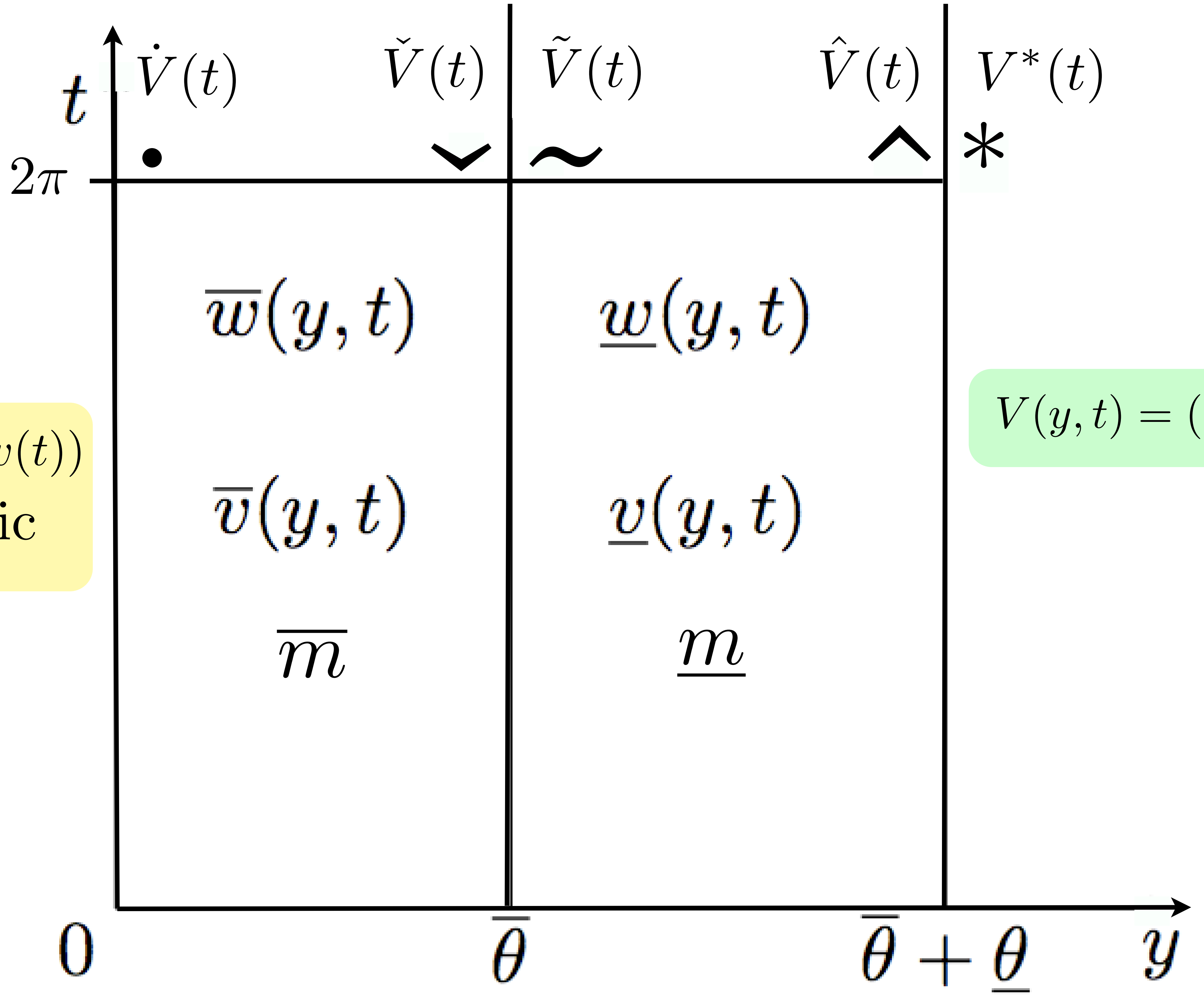
The Transformed Problem



Periodic tile consisting of two constant entropy levels $\overline{m}$, $\underline{m}$ in the (y,t)-plane

- Label the stages of $V(\cdot)$ evolution by $\bullet \;\; \check{} \;\; \sim \;\; \hat{} \;\; *$

$V(t) = (v(t), w(t))$
2π-periodic



$V(y,t) = (v(y,t), w(y,t))$

Periodic tile consisting of two constant entropy levels $\overline{m}$, $\underline{m}$ in the (y,t)-plane

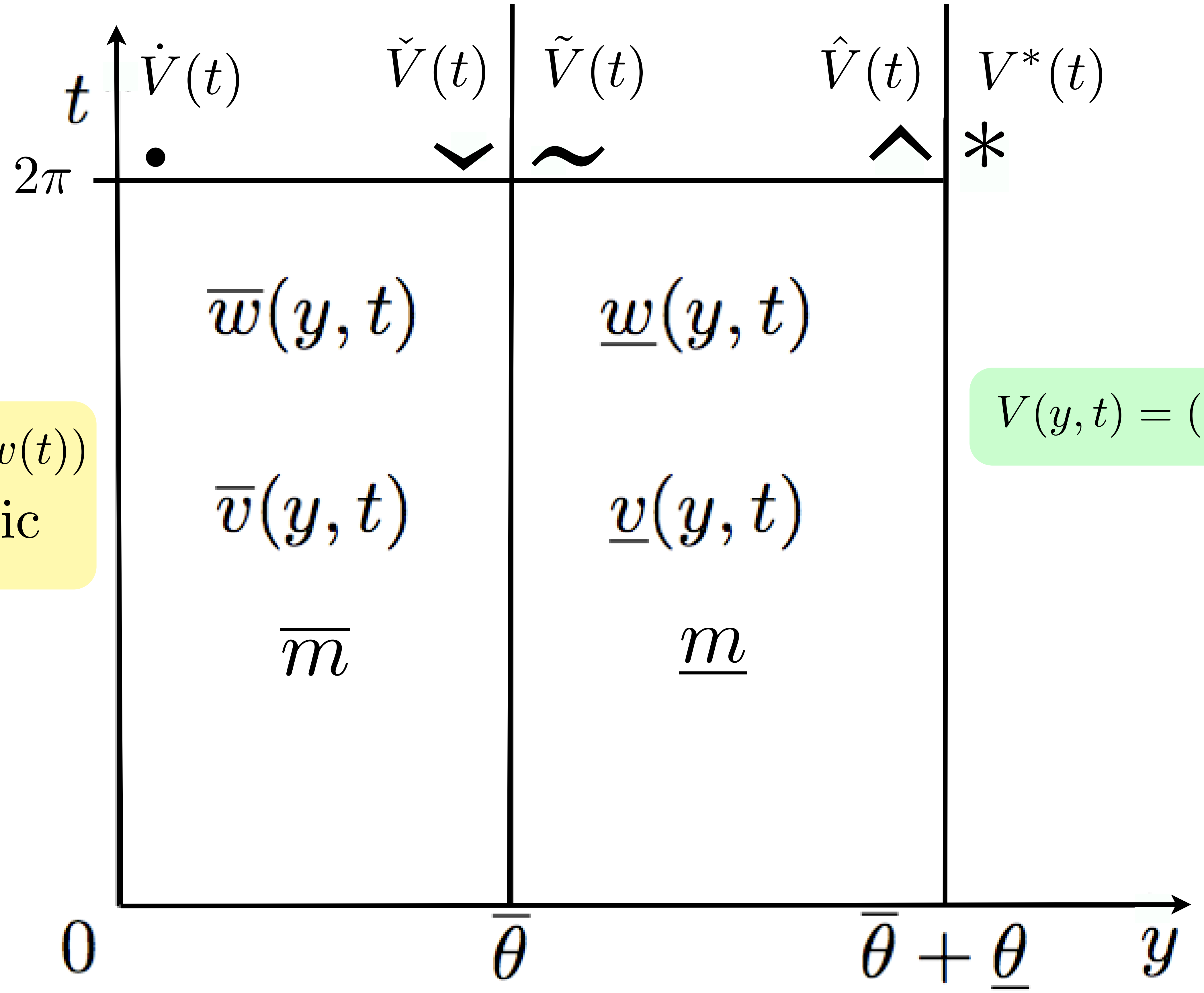
- Label the stages of $V(\cdot)$ evolution by $\bullet \;\; \smile \;\; \sim \;\; \wedge \;\; *$

$$\hat{V}(t) = \mathcal{E}(\underline{\theta})[\tilde{V}(t)] \leftarrow \tilde{V}(t) = \mathcal{J}[\check{V}(t)] \leftarrow \check{V}(t) = \mathcal{E}(\overline{\theta})[\dot{V}(t)] \leftarrow \dot{V}(t) = V(t)$$

$$\downarrow$$

$$V^*(t) = \mathcal{J}^{-1}[\hat{V}(t)] \rightarrow V^{\#}(t) = \mathcal{S}[V^*(t)]$$

$V(t) = (v(t), w(t))$
2π-periodic



$V(y,t) = (v(y,t), w(y,t))$

$$V^{\#}(t) = \mathcal{N}[\dot{V}(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[\dot{V}(t)]$$

$\hat{V}(t) = \mathcal{E}(\underline{\theta})[\tilde{V}(t)]$ ← $\tilde{V}(t) = \mathcal{J}\,[\check{V}(t)]$ ← $\check{V}(t) = \mathcal{E}(\overline{\theta})[\dot{V}(t)]$ ← $\dot{V}(t) = V(t)$

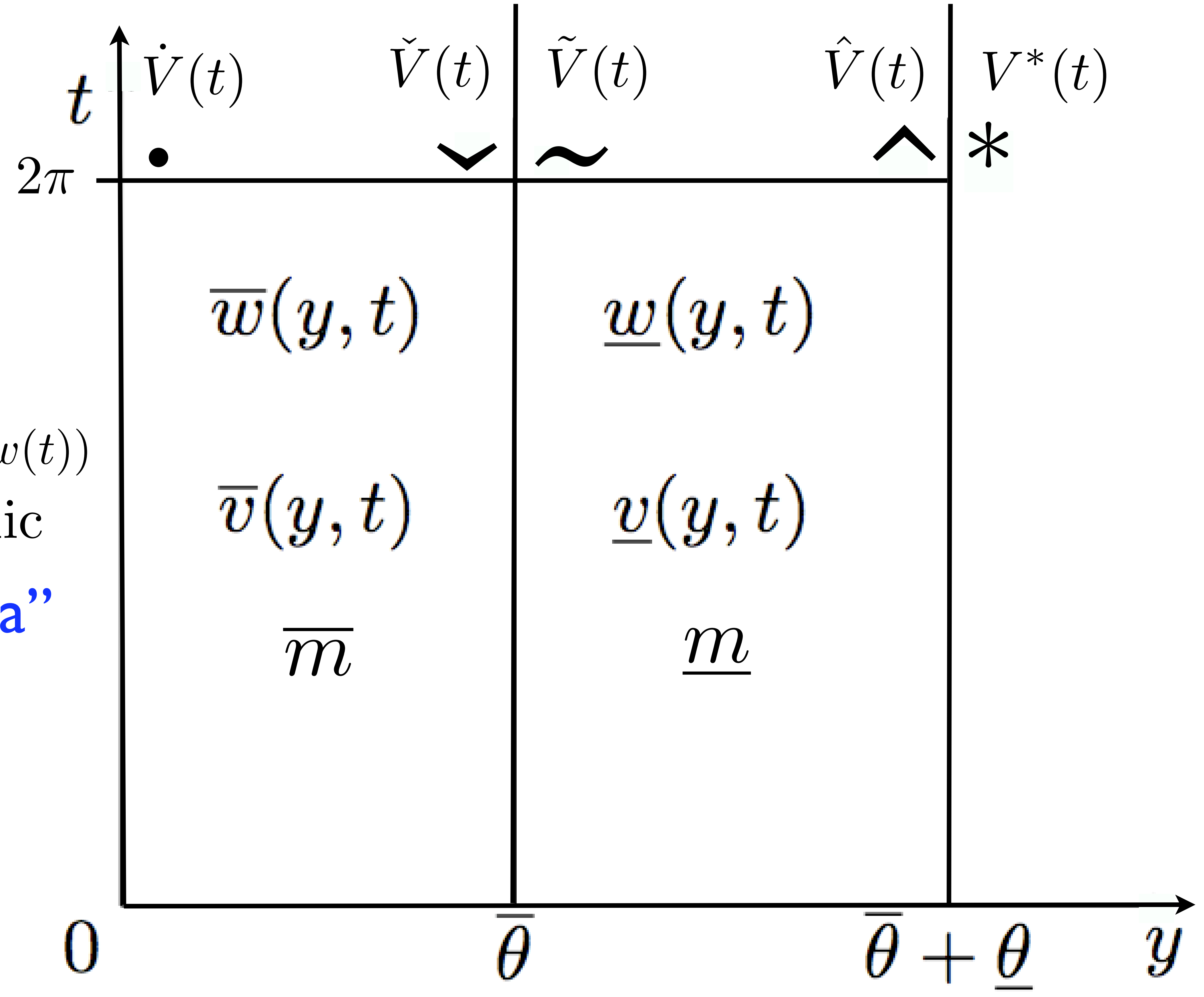
↓

$V^*(t) = \mathcal{J}^{-1}[\hat{V}(t)]$ → $V^{\#}(t) = \mathcal{S}[V^*(t)]$

$V(t) = (v(t), w(t))$

2π-periodic

"initial data"



The Periodicity Condition for the Nonlinear Problem in V=(w,v)-space

$$\mathcal{N}[\dot{V}(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[\dot{V}(t)] = \dot{V}(t)$$

$$\mathcal{N} \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})$$

THEOREM: For fixed positive real numbers $\overline{\theta}$, $\underline{\theta}$ and J, define the nonlinear operator $\mathcal{N} \equiv \mathcal{N}(\overline{\theta}, \underline{\theta}, J)$ by

$$\mathcal{N} \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta}),$$

and let $V(t) = (w(t), v(t))$ denote any smooth solution of

$$\mathcal{N} V(\cdot) = V(\cdot),$$

that satisfies the average one and zero average conditions

$$w_0 \equiv \frac{1}{2\pi} \int_0^{2\pi} w(t) dt = 1,$$

and

$$v_0 \equiv \frac{1}{2\pi} \int_0^{2\pi} v(t) dt = 0,$$

respectively. Then given any base state $U_0 = (\overline{z}_0, \overline{u}_0)$ and entropy state $\overline{m}$, there is a periodic solution $U(x,t) = (z(x,t), u(x,t)$, determined uniquely by $V(t)$, with average values

$$\frac{1}{2\pi} \int_0^{2\pi} z(0,t) dt = \overline{z}_0,$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} u(0,t) dt = \overline{u}_0.$$

- “Proof”: define

$$\begin{aligned} z &= w z_0 \\ u &= u_0 + m z_0 v \end{aligned}$$

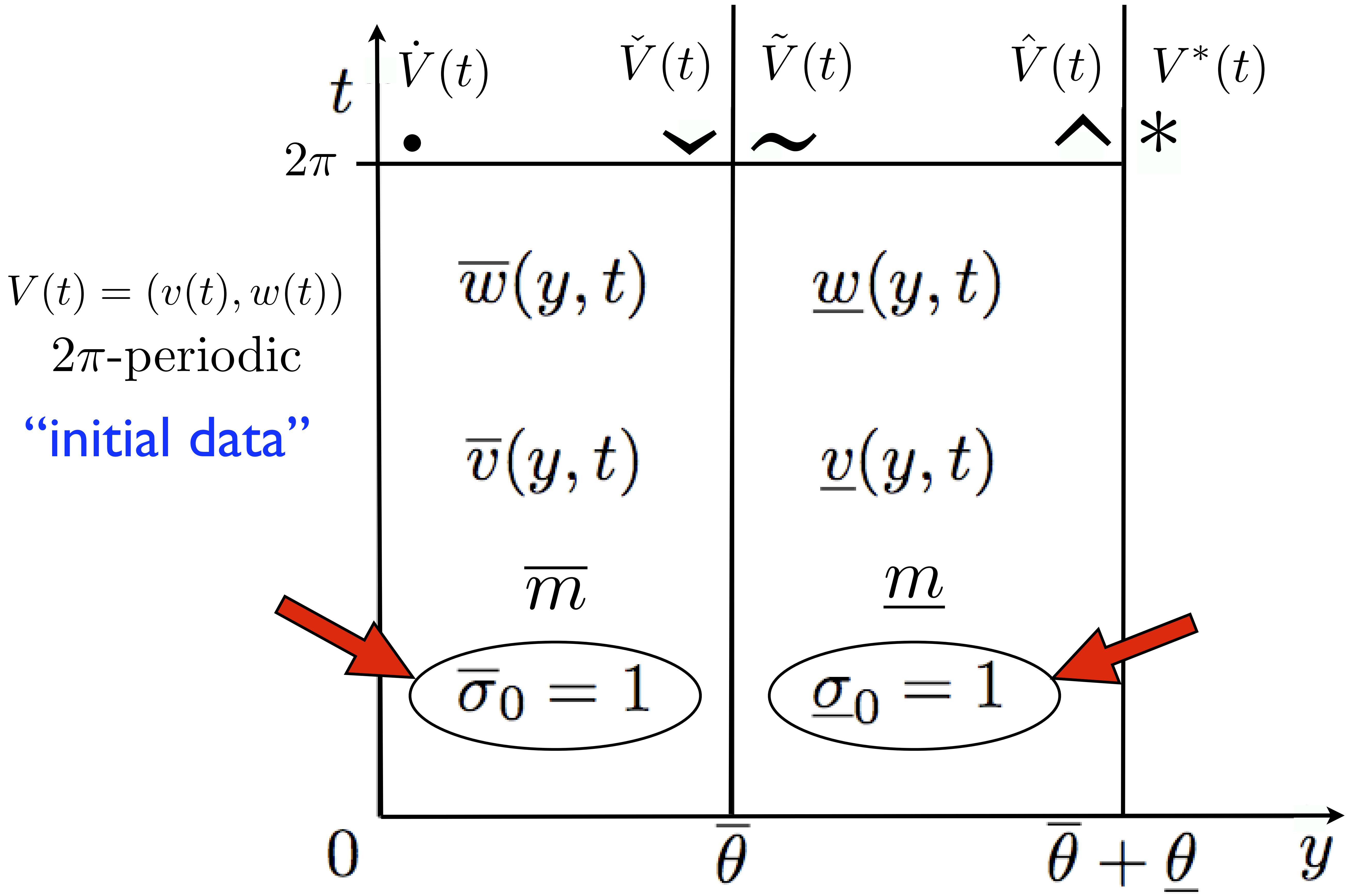
and substitute...

- Conclude: wlog we can assume

$$w_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} w(t)dt = 1$$

$$v_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} v(t)dt = 0$$

$$V^{\#}(t) = \mathcal{N}[\dot{V}(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[\dot{V}(t)]$$



WLOG assume $\sigma_0 \equiv \sigma(w_0) = \sigma(1) = 1$

For the Linearized Problem take:

$$\sigma_0 \equiv 1$$

(I.e., linearize around the nonlinear solution that takes constant states at each entropy level.)

C.f. The Nonlinear/Linearized Problem:

The Nolinear Problem:

$$\mathcal{N}[V(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[V] = V(t)$$

$V(y, \cdot) = \mathcal{E}(y)[V(\cdot)]$ Evolution by

$$\begin{aligned} w_y + \sigma(w) v_t &= 0 \\ v_y + \sigma(w) w_t &= 0 \end{aligned}$$

The Linearized Problem:

$$\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\overline{\theta})[V(\cdot)] = V(\cdot)$$

$V(y, t) = \mathcal{L}(\theta)[V(t)]$ Evolution by

$$\begin{aligned} w_y + v_t &= 0 \\ v_y + w_t &= 0 \end{aligned}$$

$\sigma(w_0) = \sigma(1) = 1$

The L^2-Space

Define: the space of periodic functions
even in w, odd in v

$$\Delta = \bigoplus_{n=0}^{+\infty} \Delta_n$$

$$\Delta_n = \left\{ V(t) = \begin{bmatrix} a_n \cos nt \\ b_n \sin nt \end{bmatrix} \; : \; a_n, b_n \in \mathbb{R} \right\}$$

$$V(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$$

LEMMA: If $V(t) = (w(t), v(t)) \in \Delta$ is 2π-periodic, sufficiently smooth and sufficiently small, then both $\mathcal{M}[V(\cdot)](t)$ and $\mathcal{N}[V(\cdot)](t)$ are well defined smooth functions, and

$\mathcal{M}[V(\cdot)](y) \in \Delta$ and $\mathcal{N}[V(\cdot)](y) \in \Delta$

for all $0 \leq y \leq \overline{\theta} + \underline{\theta}$

E.g. $\mathcal{N}[V(\cdot)](y)$ denotes the function y-units through the evolution of $\mathcal{N}$

PROOF: By the regularity of smooth solutions for the 2×2 systems of conservation laws, together with the fact that $\mathcal{J}$ and $\mathcal{S}$ are linear operators, it follows that $\mathcal{M}[V(\cdot)](t)$ and $\mathcal{N}[V(\cdot)](t)$ are well defined functions in C^k or H^s for $V(t)$ sufficiently small in C^k or H^s, respectively, $s, k \geq 2$, (c.f. Majda, Theorem 2.2, page 46). Thus to verify the zero average in v, it suffices to show that for such $V(t) = V(0,t)$ in the domain of $\mathcal{M}$ and $\mathcal{N}$, if $V(y,t) = (w(y,t), v(y,t))$ is *even* in w and *odd* in v at $y = 0$, then it is *even* in w and *odd* in v for all $0 \leq y \leq \overline{\theta} + \underline{\theta}$, where

$$V(y,t) = \begin{cases} \mathcal{E}(y)V(\cdot), & 0 < y < \overline{\theta}, \\ \mathcal{E}(y-\underline{\theta})\mathcal{J}\mathcal{E}(\overline{\theta})V(\cdot), & \overline{\theta} < y < \overline{\theta} + \underline{\theta}, \\ \mathcal{S}\mathcal{J}^{-1}\mathcal{E}(\underline{\theta})\mathcal{J}\mathcal{E}(\overline{\theta})V(\cdot), & y = \overline{\theta} + \underline{\theta}. \end{cases}$$

But the property *even* in w *odd* in v is clearly preserved by operators $\mathcal{J}$, $\mathcal{J}^{-1}$ and $\mathcal{S}$, so it suffices to show that *even* in w *odd* in v is preserved by the nonlinear evolution $\mathcal{E}$. Since solutions of the equations defining $\mathcal{E}$ are invariant under the mapping $w(y,t) \to w(y,-t)$ and $v(y,t) \to -v(y,-t)$, we can extend a solution $V(y,t)$ from $t \geq 0$ to $t \leq 0$ by the reflection

$$V(y,-t) = (w(t), -v(t)).$$

By the uniqueness of continuous solutions for smooth initial data, we need only show that the matched solution is continuous at $t = 0$ to conclude it is unique, and hence *even* in w *odd* in v by construction. Continuity in w at $t = 0$ is guaranteed by $w(t) = w(-t)$. For continuity of v at $t = 0$, we need to show that $v(y,0) = 0$ for all $0 \leq y \leq \overline{\theta} + \underline{\theta}$. For this the only real issue is to show that the nonlinear evolution in (w,v) preserves $v(y,0) = 0$. To verify this, transform the (w,v) equations to Riemann invariant coordinates $r = v - w$, $s = v + w$, leading to the equivalent system

$$\begin{aligned} r_y - \sigma r_t &= 0, \\ s_y + \sigma s_t &= 0. \end{aligned}$$

It follows that r, s are constant along characteristics $dt/dy = -\sigma$, $dt/dy = \sigma$, respectively. Tracing the characteristics back from point $(w(y,0), v(y,0)$ to points $(w(0,\pm t), v(0,\pm t))$ and using *even* in w *odd* in v at $y = 0$ gives

$$\begin{aligned} v(y,0) + w(y,0) &= -v(0,t) + w(0,t), \\ v(y,0) - w(y,0) &= v(0,t) - w(0,t), \end{aligned}$$

which upon adding leads to $v(y,0) = 0$ as claimed. $\square$

The Perturbation Problem:

Define:

$$\mathcal{F}_\epsilon = \mathcal{G}_\epsilon - \mathcal{I}$$

where

$$\mathcal{G}_\epsilon[V] = \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

so

$$\begin{aligned} \mathcal{F}(\epsilon, V) \equiv \mathcal{F}_\epsilon[V] &= G_\epsilon[V] - V \\ &= \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} - V \end{aligned}$$

LEMMA 1: If $V \in \Delta$ solves

$$\mathcal{F}_\epsilon[V] = 0$$

for $\epsilon \neq 0$, then

$$W = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V$$

defines a periodic solution of the nonlinear compressible Euler equations.

Proof:

$$\boxed{\mathcal{F}_\epsilon[V] = 0}$$

$$\Downarrow$$

$$\boxed{\mathcal{G}_\epsilon[V] = \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = V}$$

$$\Downarrow$$

$$\boxed{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V}$$

✓

LEMMA 2: In the limit $\epsilon \longrightarrow 0$ we recover the linear problem:

$$\lim_{\epsilon \longrightarrow 0} \mathcal{G}_\epsilon[V] = \mathcal{M}[V],$$

$$\begin{aligned} \mathcal{F}(\epsilon, V) = G_\epsilon[V] - V &= \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} - V \\ &= \mathcal{M}[V] - V + O(\epsilon^2) \end{aligned}$$

Proof (Formally):

$$\mathcal{G}_\epsilon[V] = \frac{1}{\epsilon}\left\{\underbrace{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right]} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = V$$

Tends to evolution at $\sigma = 1$ with error $O(\epsilon^2)$

VI. Exact Linearized Solutions Exhibiting the Simplest Periodic Structure

THEOREM: The linearized operator $\mathcal{M}$ satisfies

$$\boxed{\mathcal{M} : \Delta \to \Delta}$$

Moreover, each Δ_n is an invariant subspace for $\mathcal{M}$

$$\boxed{\mathcal{M} : \Delta_n \to \Delta_n}$$

Recall:

$$\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\overline{\theta})[V(\cdot)]$$

$$V(y,t) = \mathcal{L}(\theta)[V(t)]$$

Evolution by

$V(t) = (w, v)$

$$w_y + v_t = 0$$

$$v_y + w_t = 0$$

$$\Delta = \oplus_{n=0}^{+\infty} \Delta_n$$

$$\Delta_n = \left\{ V(t) = \begin{bmatrix} a_n \cos nt \\ b_n \sin nt \end{bmatrix} : a_n, b_n \in \mathbb{R} \right\}$$

THEOREM: Assume that $J > 1$, $\overline{\theta} > 0$, $\underline{\theta} > 0$ and

$$\overline{\theta} + \underline{\theta} < \pi.$$

Then $V(t) = (q_1 \cos t, q_2 \sin t) \in \Delta_1$ is a solution of $\mathcal{M}[V] = V$ if and only if

$$J = \cot(\overline{\theta}/2)\cot(\underline{\theta}/2)$$

and $q = (q_1, q_2) \in \mathrm{Span}\{\mathbf{q}\}$, where

$$\mathbf{q} = (\cos(\overline{\theta}/2), -\sin(\overline{\theta}/2)).$$

Furthermore, if $\dot{q} = \mathbf{q}$, then also

$$\begin{aligned}
\check{q} &= (\cos(\overline{\theta}/2), \sin(\overline{\theta}/2)) \\
\tilde{q} &= (\cos(\overline{\theta}/2), J\sin(\overline{\theta}/2)) \\
&= \rho(\cos(\pi/2 - \underline{\theta}/2), \sin(\pi/2 - \underline{\theta}/2)), \\
\hat{q} &= (-\cos(\overline{\theta}/2), J\sin(\overline{\theta}/2)) \\
&= \rho(-\cos(\pi/2 - \underline{\theta}/2), \sin(\pi/2 - \underline{\theta}/2)), \\
q^* &= (-\cos(\overline{\theta}/2), \sin(\overline{\theta}/2) = -q,
\end{aligned}$$

where we have set $\rho = \|\tilde{q}\|$.

Proof: $\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\overline{\theta})[V(\cdot)]$

- Consider first linear evolution:

$$V(y,t) = \mathcal{L}(\theta)[V(t)]$$

Evolution by $V(t) = (w, v)$

$$w_y + v_t = 0$$
$$v_y + w_t = 0$$

$V(t) \in \Delta$ ⇒ $V(t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n \cos nt \\ v_n \sin nt \end{pmatrix}$

Look for a solution:

$$V(y,t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n(y) \cos nt \\ v_n(y) \sin nt \end{pmatrix} \in \Delta \quad \forall y \in [0, \theta]$$

Plug $V(y,t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n(y) \cos nt \\ v_n(y) \sin nt \end{pmatrix}$ into

$$w_y + v_t = 0$$
$$v_y + w_t = 0$$

$$\sum_{n=0}^{\infty} w_n'(y) \cos nt + n v_n'(y) \cos nt = 0$$

$$\sum_{n=0}^{\infty} v_n'(y) \sin nt - n w_n'(y) \sin nt = 0$$

$$\begin{pmatrix} w_n \\ v_n \end{pmatrix}' + n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} w_n \\ v_n \end{pmatrix} = 0$$

$$\begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = e^{-ny \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \begin{pmatrix} w_n(0) \\ v_n(0) \end{pmatrix}$$

$$\begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = R(ny) \begin{pmatrix} w_n(0) \\ v_n(0) \end{pmatrix}$$

Conclude: counterclockwise rotation $R(n\theta)$ represents $\mathcal{L}(\theta)$ in the n'th F-mode

- Consider next the linear jump operator:

$$\mathcal{J}\left[\begin{array}{c} w \\ v \end{array}\right] = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} \left[\begin{array}{c} w \\ v \end{array}\right] \qquad V = (w, v)$$

Then

$$V(y,t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n(y) \cos nt \\ v_n(y) \sin nt \end{pmatrix}$$

$$\Downarrow$$

$$\mathcal{J}[V(y,t)] = \sum_{n=0}^{\infty} \begin{pmatrix} w_n(y) \cos nt \\ J\, v_n(y) \sin nt \end{pmatrix} \qquad J = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}}$$

$$\Downarrow$$

$$\mathcal{J}: \quad D \begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = \begin{pmatrix} w_n(y) \\ J\, v_n(y) \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$$

Conclude: D represents $\mathcal{J}$ in the n'th F-mode

- Consider next the linear shift operator:

$$\mathcal{S}[V(t)] = V(t+\pi) \qquad V = (w, v)$$

Then

$$V(y,t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n(y)\ \cos nt \\ v_n(y)\ \sin nt \end{pmatrix}$$

⬇

$$\mathcal{S}[V(y,t)] = \sum_{n=0}^{\infty} \begin{bmatrix} w_n(y)\ \cos(nt+n\pi) \\ v_n(y)\ \sin(nt+n\pi) \end{bmatrix} = \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} w_n(y)\ \cos nt \\ v_n(y)\ \sin nt \end{pmatrix}$$

⬇

$$\mathcal{S}: \qquad S \begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = (-1)^n \begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix}$$

Conclude: multiplication by $(-1)^n$ represents $\mathcal{S}$ in the n'th F-mode

- **CONCLUDE: the linear operator $\mathcal{M}$ is represented by matrix multiplication in each F-mode:**

$$\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\overline{\theta})[V(\cdot)]$$

$$V(t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n \cos nt \\ v_n \sin nt \end{pmatrix}$$

⬇

$$\begin{pmatrix} w_n \\ v_n \end{pmatrix} \longmapsto (-1)^n \cdot D^{-1} \cdot R(n\underline{\theta}) \cdot D \cdot R(n\overline{\theta}) \cdot \begin{pmatrix} w_n \\ v_n \end{pmatrix} \equiv M_n \begin{pmatrix} w_n \\ v_n \end{pmatrix}$$

$$M_n = (-1)^n \cdot D^{-1} \cdot R(n\underline{\theta}) \cdot D \cdot R(n\overline{\theta})$$

$$R(n\theta) = \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix} = R(\theta)^n \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$$

Conclude: M_n represents $\mathcal{M}$ in the n'th F-mode

AS A RESULT: The condition for periodicity in the n'th F-mode is:

$$M_n \begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{pmatrix} w_n \\ v_n \end{pmatrix}$$

$$(-1)^n \cdot D^{-1} \cdot R(n\underline{\theta}) \cdot D \cdot R(n\overline{\theta}) \cdot \begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{pmatrix} w_n \\ v_n \end{pmatrix}$$

THUS: we look for values of $(\overline{\theta}, \underline{\theta}, J)$ such that the corresponding operator $\mathcal{M}$ isolates a periodic solution in the 1-mode; I.e. we find $\mathbf{q} = (q_1, q_2) = (w_1, v_1)$ such that

(1) $$M_1 \mathbf{q} = \mathbf{q}$$

(2) $$M_n \begin{pmatrix} w \\ v \end{pmatrix} \neq \begin{pmatrix} w \\ v \end{pmatrix} \quad \text{for all} \begin{pmatrix} w \\ v \end{pmatrix} \in \mathbb{R}^2$$

FOR PART (1).....

THUS: we look for values of $(\overline{\theta}, \underline{\theta}, J)$ such that the corresponding operator $\mathcal{M}$ isolates a periodic solution in the 1-mode; I.e. we find $\mathbf{q} = (q_1, q_2) = (w_1, v_1)$ such that

(1) $$M_1 \mathbf{q} = \mathbf{q}$$

(2) $$M_n \begin{pmatrix} w \\ v \end{pmatrix} \neq \begin{pmatrix} w \\ v \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} w \\ v \end{pmatrix} \in \mathbb{R}^2$$

THEOREM: Assume that $J > 1$, $\overline{\theta} > 0$, $\underline{\theta} > 0$ and

$$\overline{\theta} + \underline{\theta} < \pi.$$

Then $V(t) = (q_1 \cos t, q_2 \sin t) \in \Delta_1$ is a solution of $\mathcal{M}[V] = V$ if and only if

$$J = \cot(\overline{\theta}/2)\cot(\underline{\theta}/2)$$

and $q = (q_1, q_2) \in \mathrm{Span}\,\{\mathbf{q}\}$, where

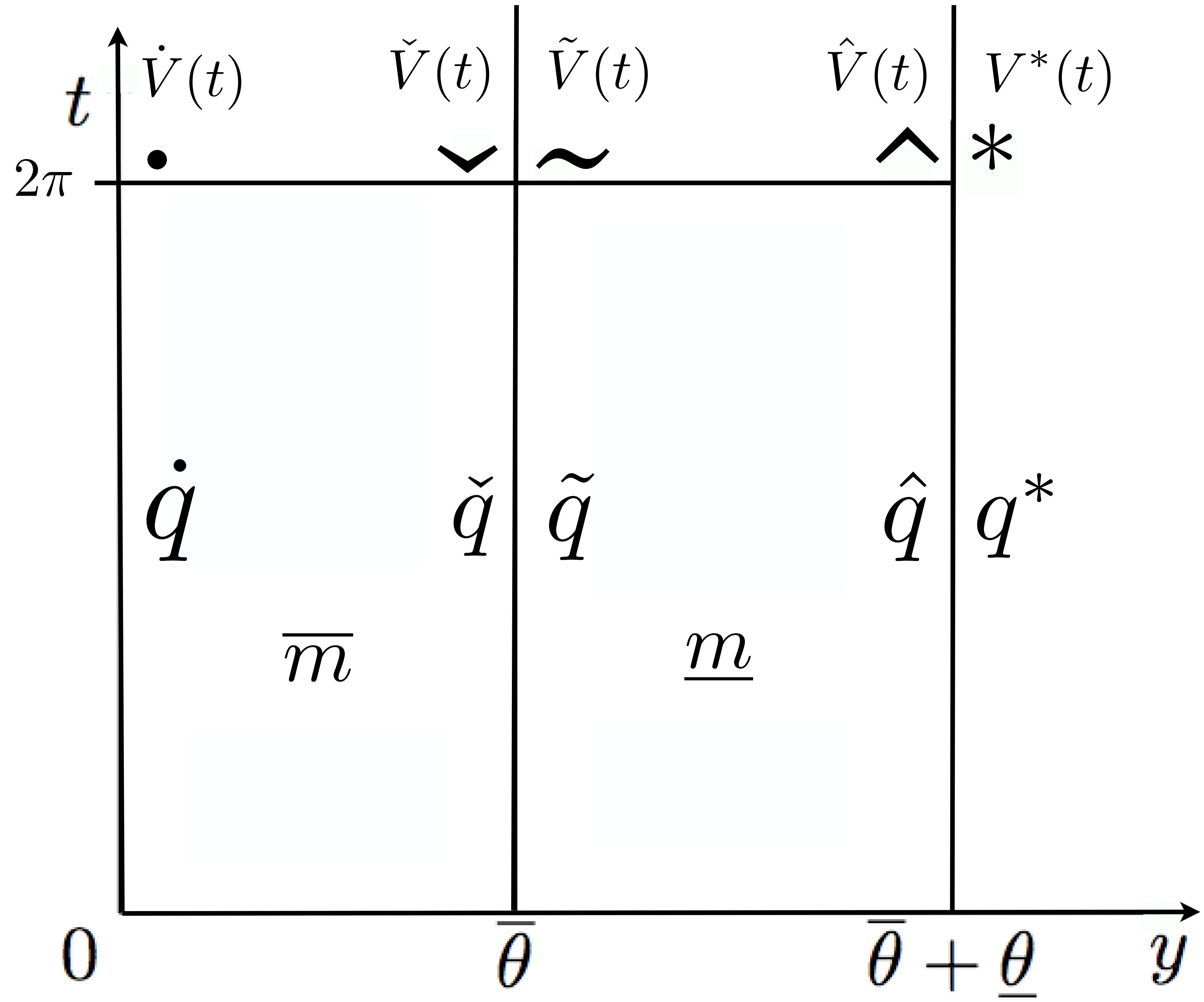
$$\mathbf{q} = (\cos(\overline{\theta}/2), -\sin(\overline{\theta}/2)).$$

Furthermore, if $\dot{q} = \mathbf{q}$, then also

$$\begin{aligned}
\check{q} &= (\cos(\overline{\theta}/2), \sin(\overline{\theta}/2)) \\
\tilde{q} &= (\cos(\overline{\theta}/2), J\sin(\overline{\theta}/2)) \\
&= \rho(\cos(\pi/2 - \underline{\theta}/2), \sin(\pi/2 - \underline{\theta}/2)), \\
\hat{q} &= (-\cos(\overline{\theta}/2), J\sin(\overline{\theta}/2)) \\
&= \rho(-\cos(\pi/2 - \underline{\theta}/2), \sin(\pi/2 - \underline{\theta}/2)), \\
q^* &= (-\cos(\overline{\theta}/2), \sin(\overline{\theta}/2) = -q,
\end{aligned}$$

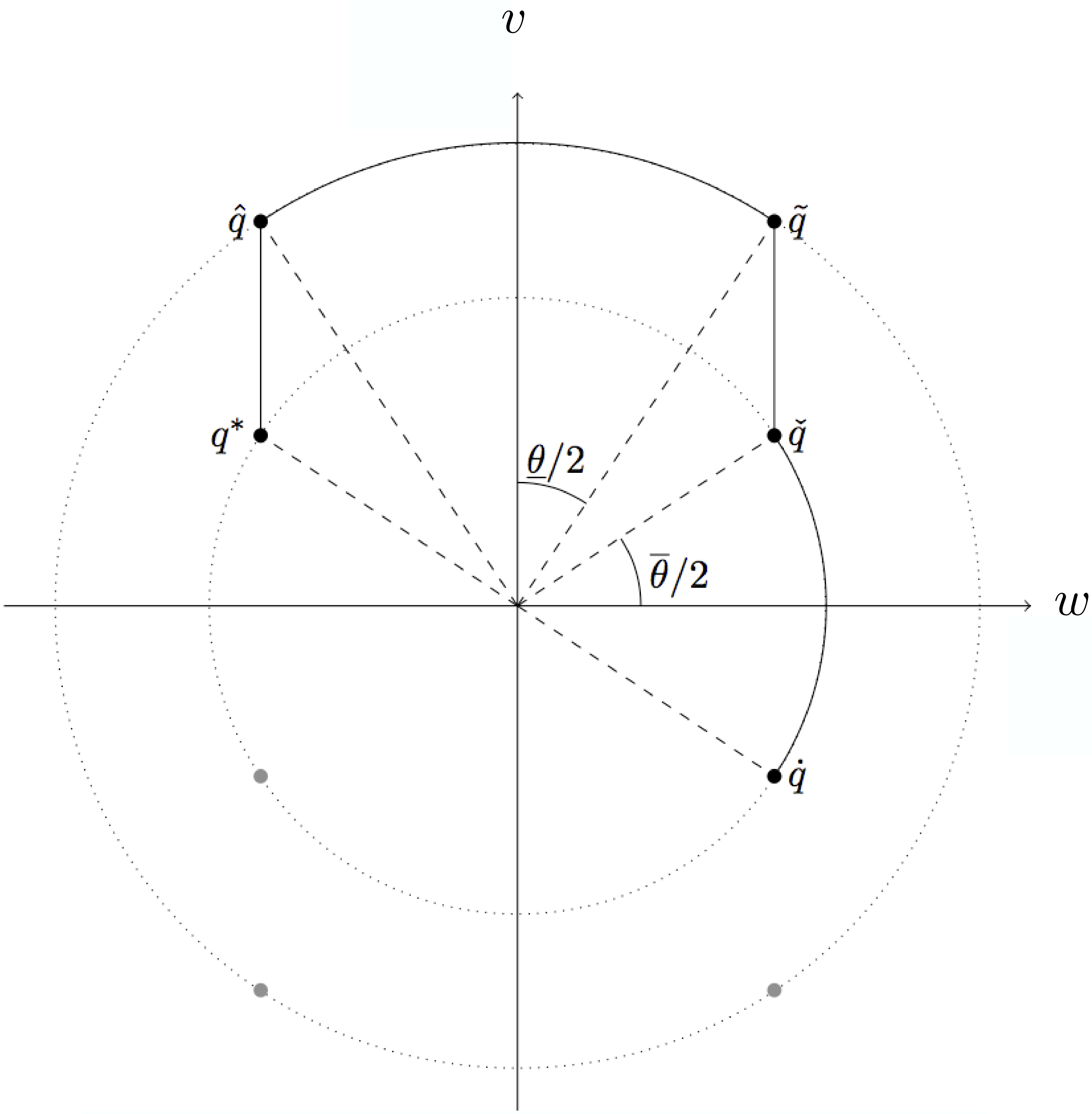
where we have set $\rho = \|\tilde{q}\|$.

RECALL.....



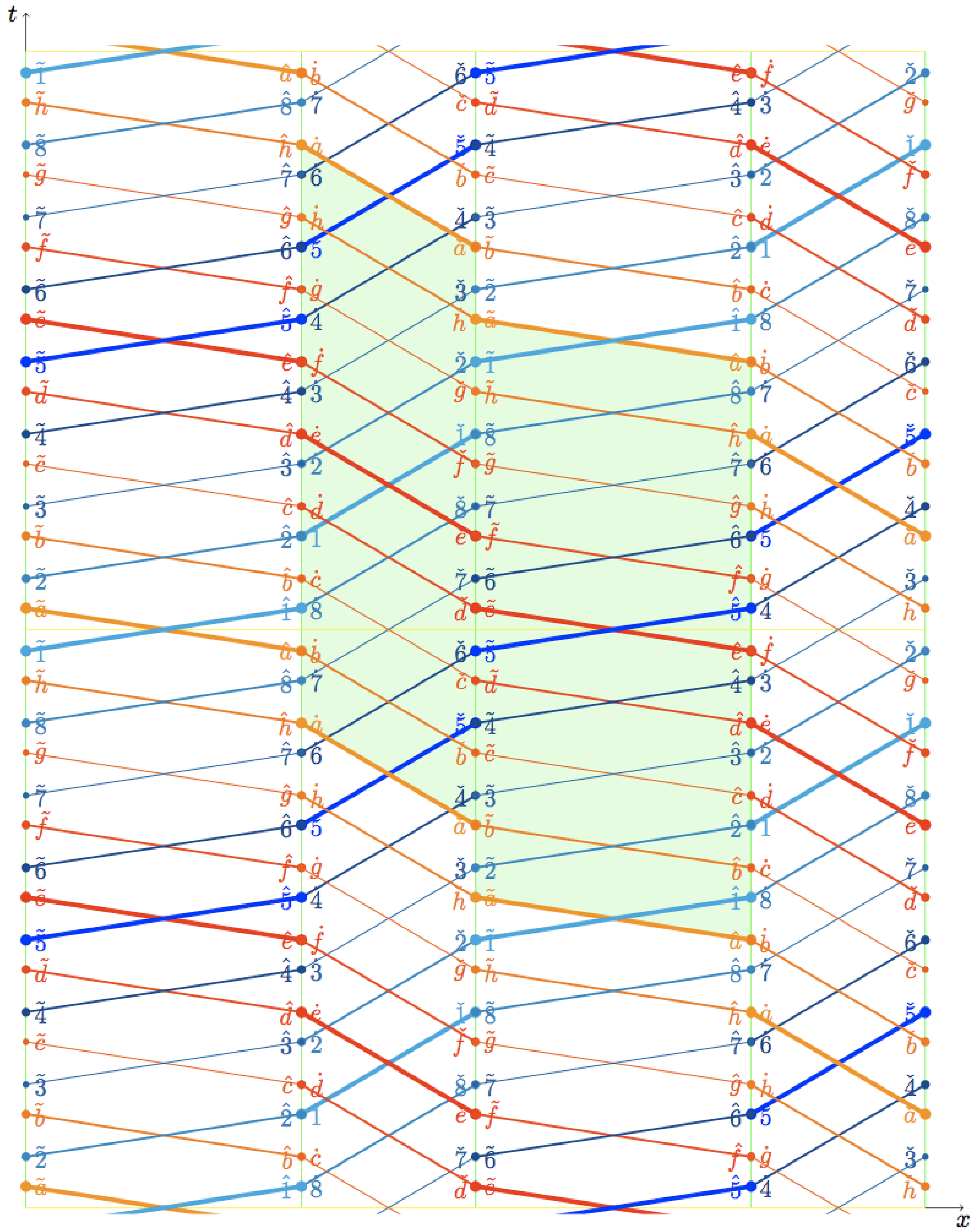
t
$\dot{V}(t)$
$\check{V}(t)$
$\tilde{V}(t)$
$\hat{V}(t)$
$V^*(t)$
2π
•
˘
~
^
*
$\dot{q}$
$\check{q}$
$\tilde{q}$
$\hat{q}$
q^*
$\overline{m}$
$\underline{m}$
0
$\overline{\theta}$
$\overline{\theta} + \underline{\theta}$
y

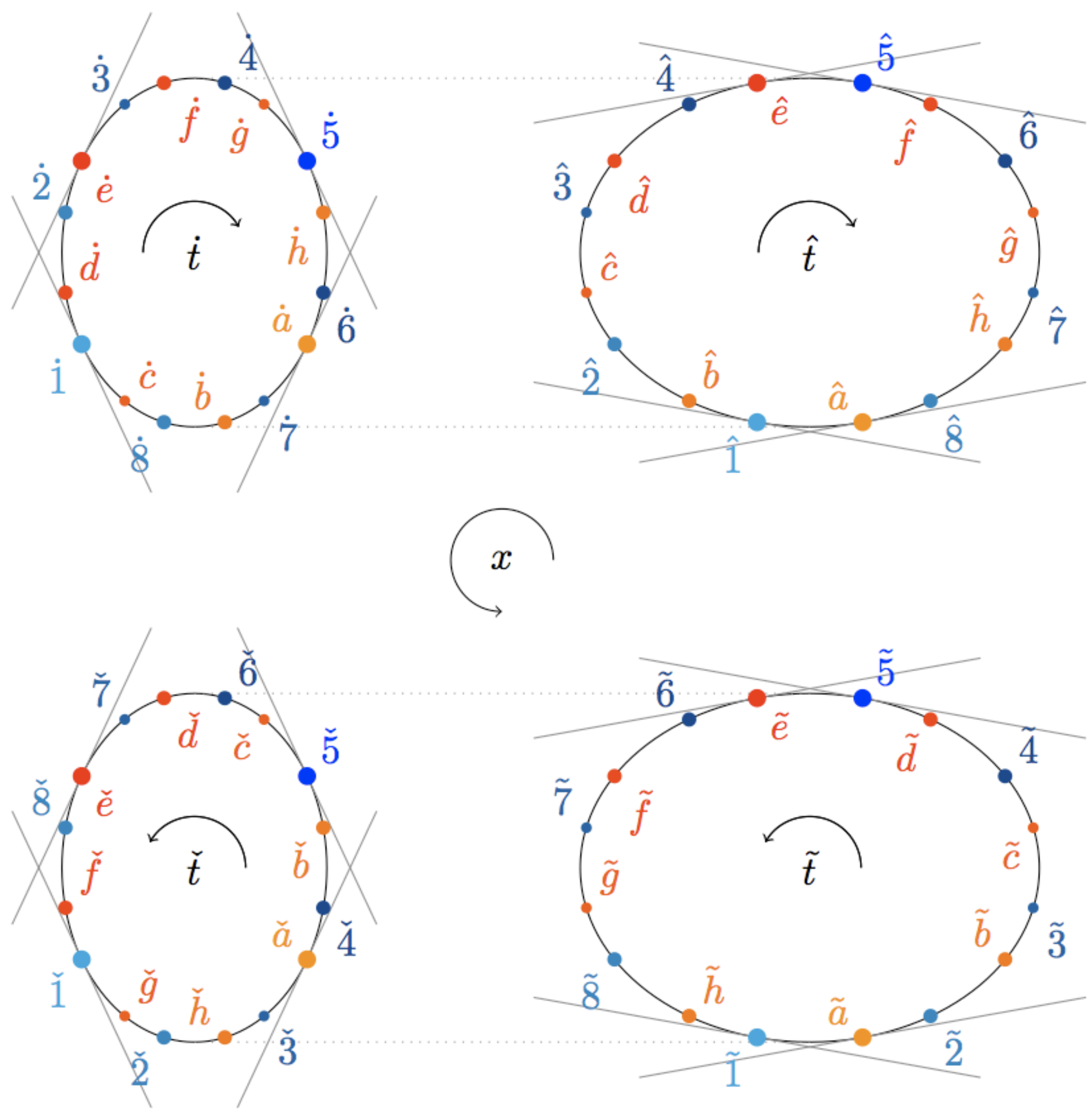
You can solve for $\mathbf{q}$ geometrically:



$$
\begin{aligned}
\dot{q} &= q, \\
\check{q} &= R(\overline{\theta})\dot{q}, \\
\tilde{q} &= D\check{q}, \\
\hat{q} &= R(\underline{\theta})\tilde{q}, \\
q^* &= D^{-1}\hat{q},
\end{aligned}
$$

The states $\dot{q}, \check{q}, \tilde{q}, \hat{q}, q^*$ for $\mathbf{q} \in \Delta_1$





CHECK: The solution in the 1-mode kernel has the structure that balances compression and rarefaction in the nonlinear problem

Linear solutions should perturb to exact solutions of the nonlinear problem

COMMENT: For more general shift operators our solutions will not stay in Δ, so we cannot reduce linear evolution to rotation directly. But we show that in complex Fourier space, linear evolution can be represented by rotations.

Thus the construction should apply to general shifts and multiple entropy jumps.

Outline

I. The Compressible Euler Equations

II. History/Prior Results for the Problem

III. Compressive and Rarefactive Waves

IV. The Simplest Possible Periodic Structure that Balances Compression and Rarefaction

V. The Nonlinear Eigenvalue Problem $\approx$ Perturbation of Linear Problem

VI. Exact Linearized Solutions Exhibiting the Simplest Periodic Structure

VII. Isolating Solutions in the Kernel of the Linearized Operator

VIII. Resonances, Small Divisors and Eigenvalues of the Linearized Operator

IX. The Liapunov-Schmidt Method

X. The Bifurcation Equation

XI. The Auxiliary Equation

It Remains to Verify (2).....

THUS: we look for values of $(\overline{\theta}, \underline{\theta}, J)$ such that the corresponding operator $\mathcal{M}$ isolates a periodic solution in the 1-mode; I.e. we find $\mathbf{q} = (q_1, q_2) = (w_1, v_1)$ such that

(1) $$M_1 \mathbf{q} = \mathbf{q}$$

(2) $$M_n \begin{pmatrix} w \\ v \end{pmatrix} \neq \begin{pmatrix} w \\ v \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} w \\ v \end{pmatrix} \in \mathbb{R}^2$$

THEOREM: Let

$$E \equiv \left\{\Theta = (\overline{\theta}, \underline{\theta}) \ : \ \overline{\theta}, \underline{\theta} > 0, \ 0 < \overline{\theta} + \underline{\theta} < \pi\right\}.$$

Then there exists a subset E^* of full measure in E such that, if $\Theta \in E^*$, then Θ is *non-resonant* in the sense that if J is given in terms of Θ by (**), then the eigenvalues $\lambda_n^{\pm} - (-1)^n$ of the linearized operator $\mathcal{M} - I$ are nonzero for all $n \geq 2$.

$$\mathcal{J} = \cot\left(\overline{\theta}/2\right)\cot\left(\underline{\theta}/2\right) \qquad (**)$$

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✸ XI. The Auxiliary Equation

We now impose a further symmetry and use this to obtain explicit bounds for the eigenvalues of M_n.

THEOREM: Assume the symmetric case,

$$\overline{\theta} = \underline{\theta} \equiv \theta, \quad 0 < \theta < \pi/2.$$

Then there is a set of full measure $\mathcal{A} \subset (0, \pi/2)$ such that, if $\theta \in \mathcal{A}$, then there is a positive constant C and exponent $r \geq 1$ such that the eigenvalues of the linearized operator $\mathcal{M} - I$ satisfy the estimate

$$|\lambda_n^{\pm} - (-1)^n| \geq \frac{C}{n^r},$$

for all $n \geq 2$. In particular, if $\frac{\pi - 2\theta}{2\pi}$ is the irrational root of a quadratic equation, we can take $r = 1$.

- “Proof”: Define the transformation

$$\begin{aligned}\phi &= \frac{\pi - \underline{\theta} + \overline{\theta}}{2}\\ \psi &= \frac{\pi - \underline{\theta} - \overline{\theta}}{2}\end{aligned}$$

And apply the theory of Liouville numbers in transformed variables assuming $\overline{\theta} = \underline{\theta} = \theta$ ...

- "Proof": That is, we first prove

$$|\lambda_n^{\pm} - (-1)^n| \geq \frac{C}{n^r} \quad \text{iff} \quad |\sin(n\psi)| \geq \frac{C}{n^r}$$

- Theorem: If $\xi = \psi/\pi$ is NOT a Liouville Number, then $\exists C > 0, r \geq 2$ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{C}{q^r}$$

for all $p/q \in \mathbb{Q}$.
(Non-Liouville numbers form a set of full measure)

- Choose $q = n$:

$$\left|\frac{n\psi}{\pi} - p\right| > \frac{C}{n^{r-1}}$$

for all $n, p \in \mathbb{Z}$.

- “Proof”: So...

$$\left|\frac{n\psi}{\pi} - p\right| > \frac{C}{n^{r-1}}$$

⬇

$$\mathrm{Dist}\left\{\frac{n\psi}{\pi}, \mathbb{Z}\right\} > \frac{C}{n^{r-1}}$$

⬇

$$|\sin n\psi| > \frac{C}{n^{r-1}}$$

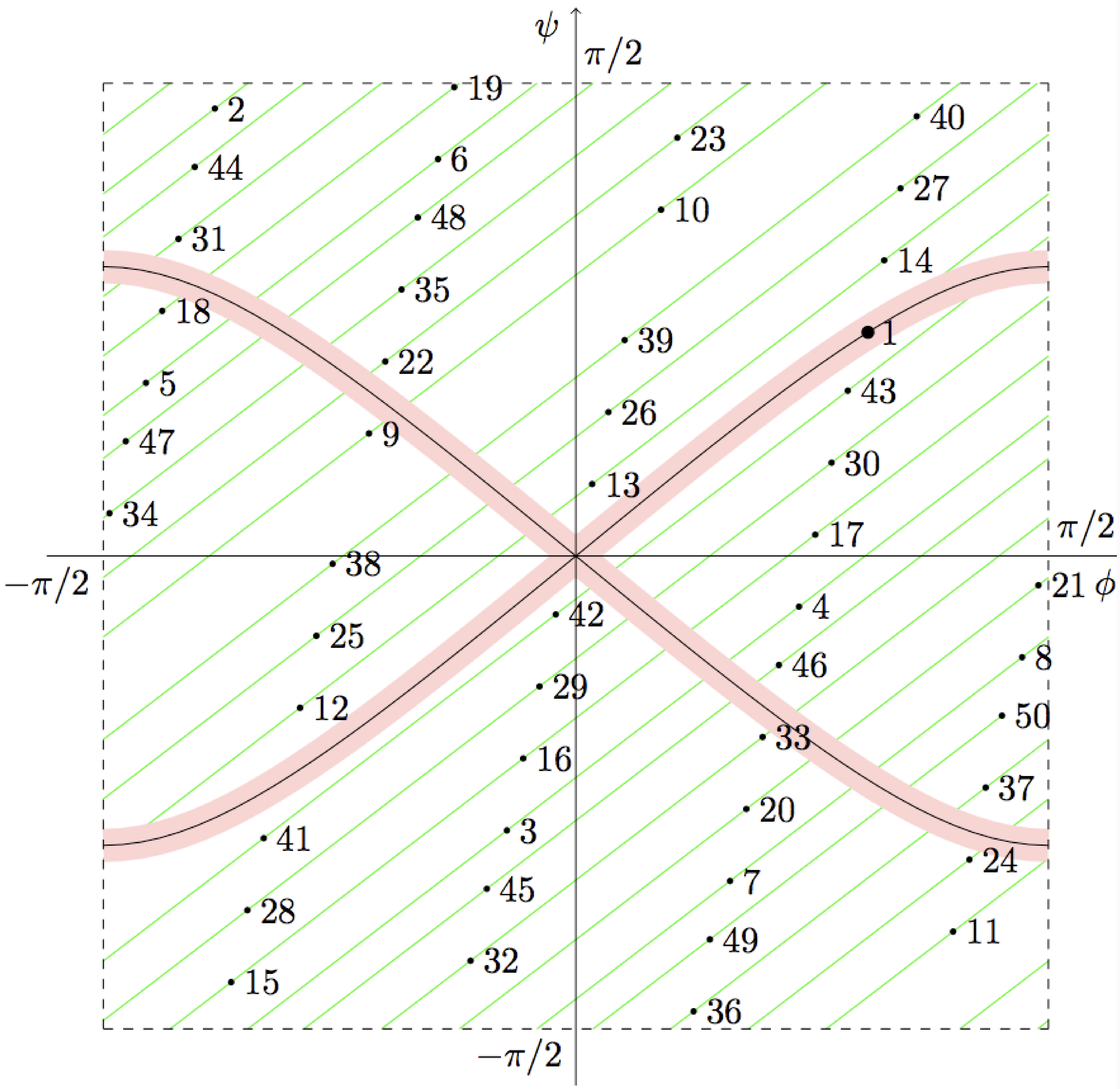
⬇

$$|\lambda_n^{\pm} - (-1)^n| \geq \frac{C}{n^{r-1}},$$

- Theorem: If ξ is the irrational root of a rational quadratic polynomial, we can take $r = 2$ (best case)

$$|\lambda_n^{\pm} - (-1)^n| \geq \frac{C}{n}$$

Numerical Plot of First 50 Eigenvalues–Case $\overline{\theta} \neq \underline{\theta}$



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Bifurcation to Nonlinear Solutions

- It remains to prove that the linearized solutions perturb to solutions of the nonlinear equations.

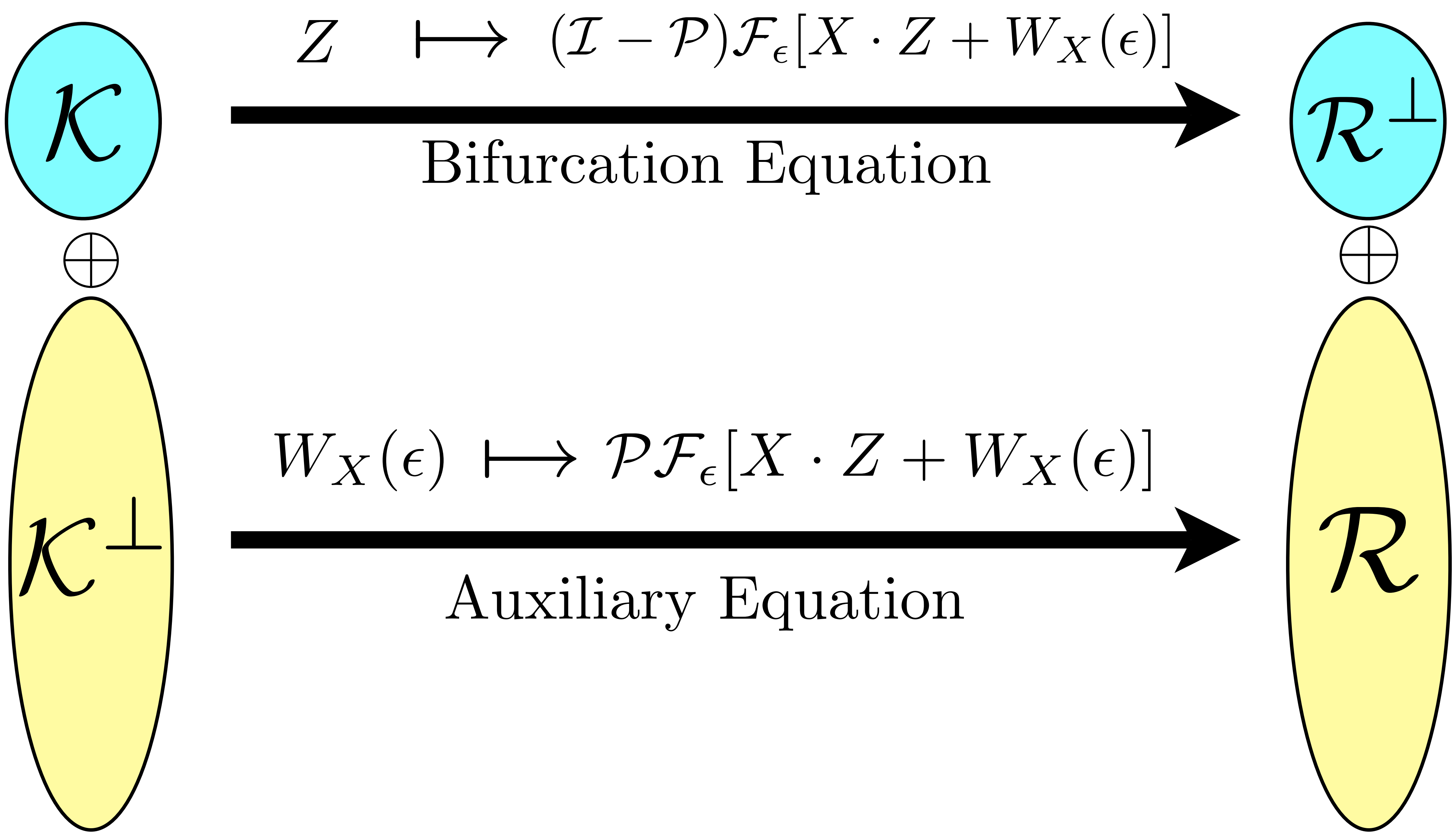
  Liapunov-Schmidt decomposes the nonlinear problem by coordinates natural for the linearized problem:

Picture: L-S Decomposition

- Decompose the nonlinear problem by the RANGE $\mathcal{R}$ and KERNEL $\mathcal{K}$ of the LINEAR OPERATOR $\mathcal{M} - \mathcal{I}$:

$$\mathcal{P} \equiv \text{Projection onto } \mathcal{R}$$

$$\mathcal{I} - \mathcal{P} \equiv \text{Projection onto } \mathcal{R}^{\perp}$$



Liapunov-Schmidt Decomposition

- Decompose the nonlinear problem by the RANGE $\mathcal{R}$ and KERNEL $\mathcal{K}$ of the LINEAR OPERATOR $\mathcal{M} - \mathcal{I}$:

$$\mathcal{P} \equiv \text{Projection onto } \mathcal{R}$$

$$Z \in \mathcal{K}$$

$$\mathcal{I} - \mathcal{P} \equiv \text{Projection onto } \mathcal{R}^{\perp}$$

- For each $X \in \mathbb{R}$ and $Z \in \mathcal{K}$ solve uniquely for $W_X(\epsilon)$:

$$\text{AUXILIARY EQUATION:} \quad \mathcal{P} \cdot \mathcal{F}_{\epsilon}[X \cdot Z + W_X(\epsilon)] = 0$$

$$\{W_X(\epsilon) \in \mathcal{K}^{\perp}\} \longmapsto \mathcal{P} \cdot \mathcal{F}_{\epsilon}[X \cdot Z + W_X(\epsilon)] \in \mathcal{R}$$

- GIVEN: $W_X(\epsilon)$ solve uniquely for $X(\epsilon)$:

$$\text{BIFURCATION EQUATION:} \quad (\mathcal{I} - \mathcal{P})\mathcal{F}_{\epsilon}[X(\epsilon) \cdot Z + W_X(\epsilon)] = 0$$

- THEREBY OBTAIN solution to the nonlinear problem:

$$\mathcal{F}_{\epsilon}[X(\epsilon) \cdot Z + W_X(\epsilon)] = 0 \qquad \epsilon > 0$$

CONCLUSION

- We have solved the Bifurcation Equation:
- It remains to solve the Auxiliary Equation:

AUXILIARY EQUATION: $\mathcal{P} \cdot \mathcal{F}_\epsilon[X \cdot Z + W_X(\epsilon)] = 0$

$$\{W_X(\epsilon) \in \mathcal{K}^\perp\} \longmapsto \mathcal{P} \cdot \mathcal{F}_\epsilon[X \cdot Z + W_X(\epsilon)] \in \mathcal{R}$$

- The map is 1-1 invertible, but the eigenvalues are not bounded away from zero, which leads to issues of small-divisors analogous to KAM theory.

- NOTE: The Auxiliary Equation poses an abstract Implicit Function Theorem problem: “Everything special” about the periodic problem has been removed at this stage.

- NOTE: If the eigenvalues were uniformly bounded away from zero, the standard Implicit Function Theorem for Banach Spaces would directly apply.

We are currently working on this!

Ref's:

15.6 A 'Hard' Implicit Function Theorem. The following result provides the abstract framework for some of the problems mentioned in § 15.4. This will be made evident afterwards, by means of an example dealing with a so-called 'small divisor problem'.

Theorem 15.8. *Let (X_λ), (Y_λ), (Z_λ) be scales of Banach spaces with $\lambda \in \Lambda = [0, 1]$ and $|\cdot|_\mu \leqq |\cdot|_\lambda$ for $\mu \leqq \lambda$ on all scales. Let $(x_*, y_*) \in X_1 \times Y_1$ (the smallest spaces) and, denoting balls with respect to $|\cdot|_\lambda$ by B^λ, let*

$$\Omega_r^\lambda = B_r^\lambda(x_*) \times B_r^\lambda(y_*) \subset X_\lambda \times Y_\lambda .$$

Consider $F: \Omega_r^0 \to Z_0$ with $F(x_*, y_*) = 0$ and assume the existence of constants $M \geqq 1$, $\alpha \geqq 0$ and $\gamma > 0$ such that F satisfies the following three conditions:

(a) *$F: \Omega_r^\lambda \to Z_\lambda$ is continuous, for every $\lambda \in [0, 1]$;*
(b) *for $\lambda \in (0, 1]$ and $\mu < \lambda$, $F: \Omega_r^\lambda \to Z_\mu$ is differentiable in y and*

$$|F(x, y) - F(x, \bar{y}) - F_y(x, \bar{y})(y - \bar{y})|_\mu \leqq \frac{M}{(\lambda - \mu)^{2\alpha}} |y - \bar{y}|_\lambda^2 ;$$

(c) *$F_y(\cdot, \cdot)$ has an approximate right inverse, i.e. for $0 \leqq \mu < \lambda$ there exists $T(x, y) \in L(Z_\lambda, Y_\mu)$ such that*

$$|T(x, y)|_{L(Z_\lambda, Y_\mu)} \leqq \frac{M}{(\lambda - \mu)^\gamma}$$

and

$$|F_y(x, y)\, T(x, y) - I|_{L(Z_\lambda, Z_\mu)} \leqq \frac{M}{(\lambda - \mu)^{2(\alpha + \gamma)}} |F(x, y)|_\lambda \quad \text{on} \quad \Omega_r^\lambda .$$

Then to $\lambda \in (0, 1]$ there exist a radius $\varrho(\lambda) \leqq r$ and a map $S_\lambda: \Omega_{\varrho(\lambda)}^\lambda \to Y_{\lambda/2}$ such that $F(x, S_\lambda(x, y)) = 0$ on $\Omega_{\varrho(\lambda)}^\lambda$.

From: Nonlinear Functional Analysis, Deimling

E. Zehnder, *Generalized implicit function theorems with applications to some small divisor problems*, Comm. Pure Appl. Math., Vol. 28, pp. 91-140 (1975).

Other References:

W.Craig and G. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. on Pure Appl. Math., Vol 66, pp. 1409-1498 (1993).

R.Hamilton, *The Inverse Function Theorem of Nash and Moser*, Bull. Am. Math. Soc. Vol. 7,(1982).

END

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Theorem 5. *The following inequalities characterize when a nonlinear wave changes its R/C value at an entropy jump:*

$$R_{in}^{-} \to C_{out}^{-} \quad \text{iff} \quad q_L^R m_L \dot{z}_L < \dot{u}_L < m_L \dot{z}_L, \tag{65}$$

$$C_{in}^{-} \to R_{out}^{-} \quad \text{iff} \quad m_L \dot{z}_L < \dot{u}_L < q_L^R m_L \dot{z}_L, \tag{66}$$

$$R_{in}^{+} \to C_{out}^{+} \quad \text{iff} \quad -q_L^R m_L \dot{z}_L < \dot{u}_L < -m_L \dot{z}_L, \tag{67}$$

$$C_{in}^{+} \to R_{out}^{+} \quad \text{iff} \quad -m_L \dot{z}_L < \dot{u}_L < -q_L^R m_L \dot{z}_L. \tag{68}$$

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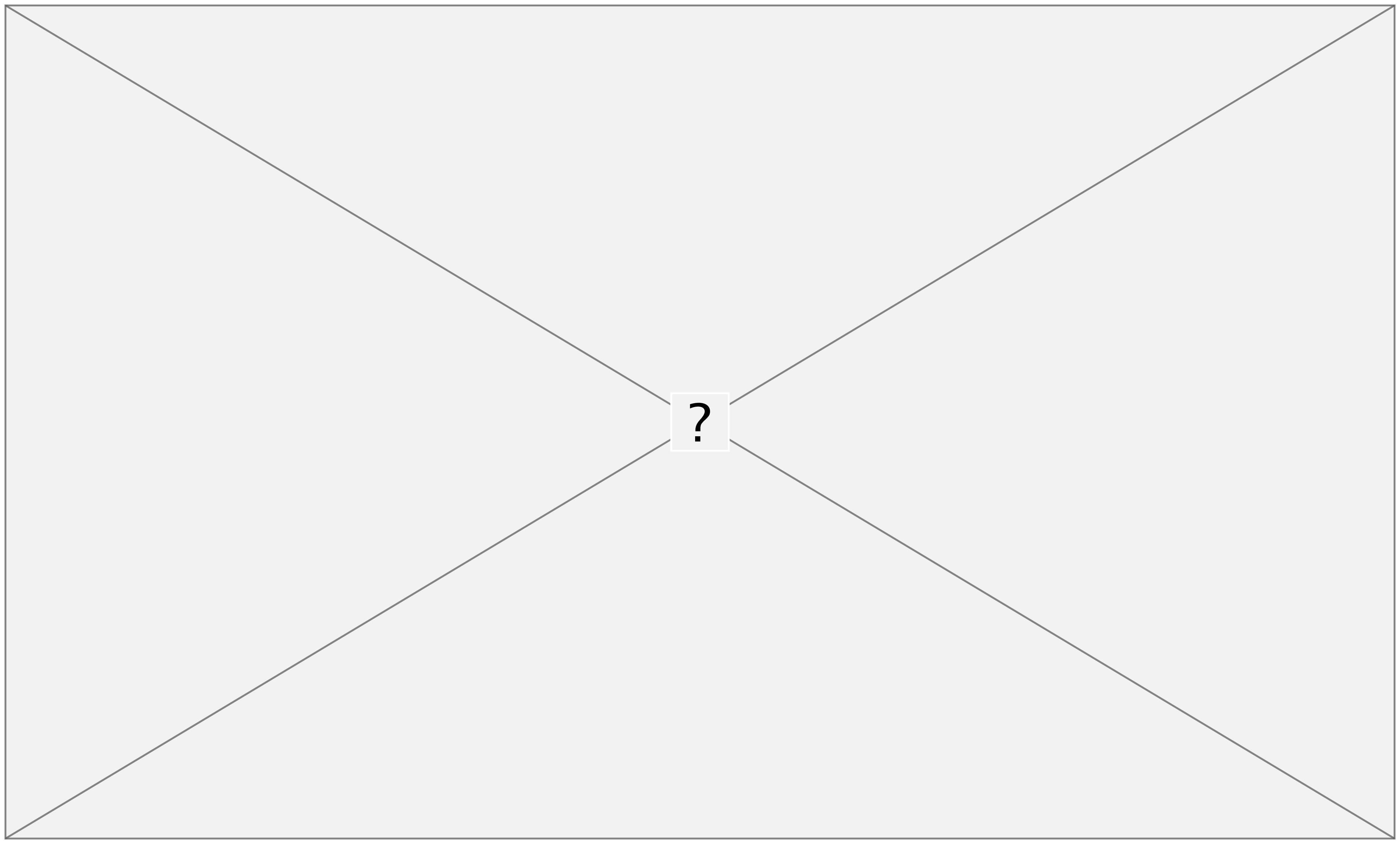

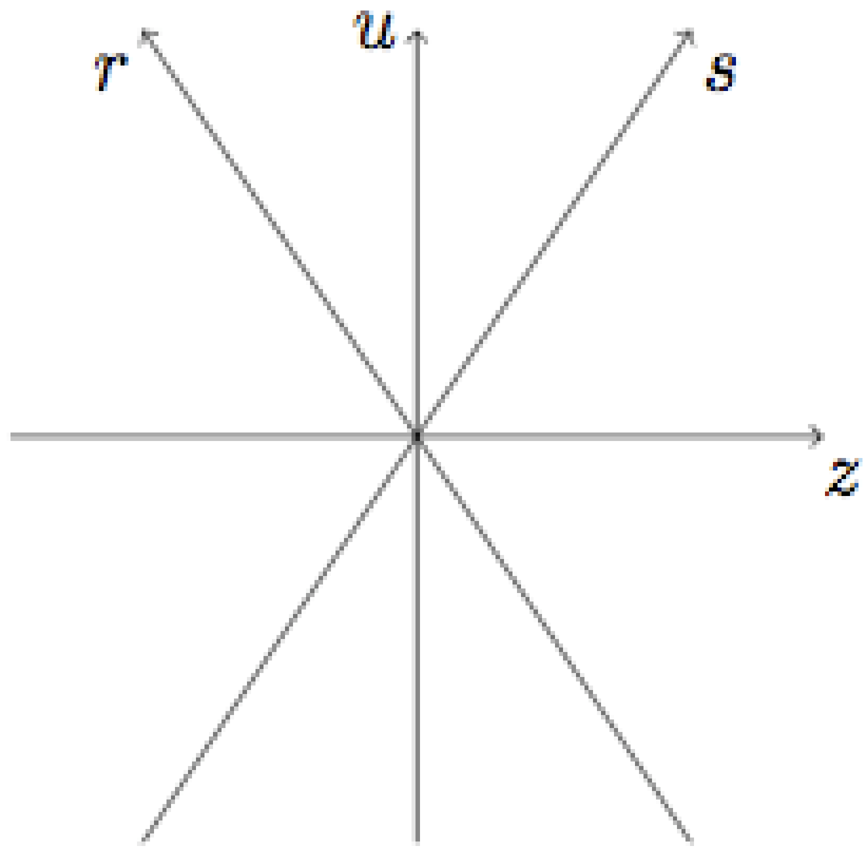

FIGURE 1. Riemann invariant coordinates.

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Edited Slides

Why we are interested in time-periodic solutions of compressible Euler...

- **Historically** ---The equations were derived by Euler in 1752 as a model for sound wave propagation.

- A first question: Do the nonlinear equations support oscillatory solutions analogous to the linearized theory of sound?

- Riemann had the equations and the problem..

- For most of the last 250 years, experts thought time-periodic solutions were not possible due to shock-wave formation...

- **Scientifically** ---Time-periodic solutions represent dissipation free long distance signaling.

- Could the structure of periodic solutions supply a new paradigm for how sound waves, and other nonlinear waves, really propagate?

- Such waves represent physical waves that travel at a new speed, different from the sound and shock speeds.

- **From a PDE point of view** ---The mechanism requires at least three coupled PDE's

- C.f. work on periodic solutions of the scalar nonlinear Schroedinger Eqn, and scalar nonlinear wave equation.

- Issues of resonances and small divisors analagous to KAM theory arises.

- Diophantime equations and probability theory are involved.

- **Bifurcation theory** ---the bifurcation aspects of the problem have the potential to open the door to all the issues tied to bifurcation theory, such as chaos, period-doubling, etc.

- **Intellectual interest** ---our approach is to guess the solution structure by heuristic reasoning based on nonlinear waves...and this is prerequisite for a rigorous mathematical analysis.

History-References

- 1752-- Preliminary version of his equations presented to the Berlin Academy.
- 1757-- The general compressible Euler equations first appeared in published form:

Euler, L. *Principes generaux du mouvement des fluides*, Mémoires de L'Academie des Sciences de Berlin, Vol. 11, pp. 274-315 (1757)

- 1772-- "[Euler] studied compressible flows in the linear approximation, treating the generation and propagation of sound waves" ([Chr])

Euler, L. *Sectio quarta de motu aeris in tubis*, Novi Commentarii Academiae Scientiarum Petropolitanae, Vol. 16, pp. 281-425 (1772)

"...the system of equations at the time of Euler, which consisted of the momentum equations together with the equation of continuity, was underdetermined except in the incompressible limit. The additional equation was supplied by Laplace in 1816 in the form of what was later to be called the adiabatic condition, and allowed him to make the first correct calculation of the speed of sound." ([Chr])

History/Background

- 1857-- Riemann showed that shock-wave discontinuities can form from smooth solutions of the compressible Euler equations.

  Introduced Riemann invariants and the Riemann problem to continue the solutions past the time of shock formation

  (Riemann incorrectly used the adiabatic constraint instead of the energy equation for the weak formulation.)

- 1865-- "Clausius introduced the concept of entropy into theoretical physics" ([Chr])

  (Since periodic solutions do not involve shocks, the equations and the problem appear to have been fully available to Riemann)

Riemann, B. *Uber die Fortpfanzung ebener Luftwellen von endlicher Schwingungswete*, Abhandlungen der Gesellshaft der Wissenshaften zu Gottingen, Mathematisch-physikalishe Klasse, Vol. 8, 43 (1858-59)

Clausius, R. *Über verschiedene fü die Anwendung bequeme Formen der Hauptgleichungen der mechanischen Wärmetheorie*, Annalen der Physik und Chemie, Vol. 125, pp. 353-400, (1865)

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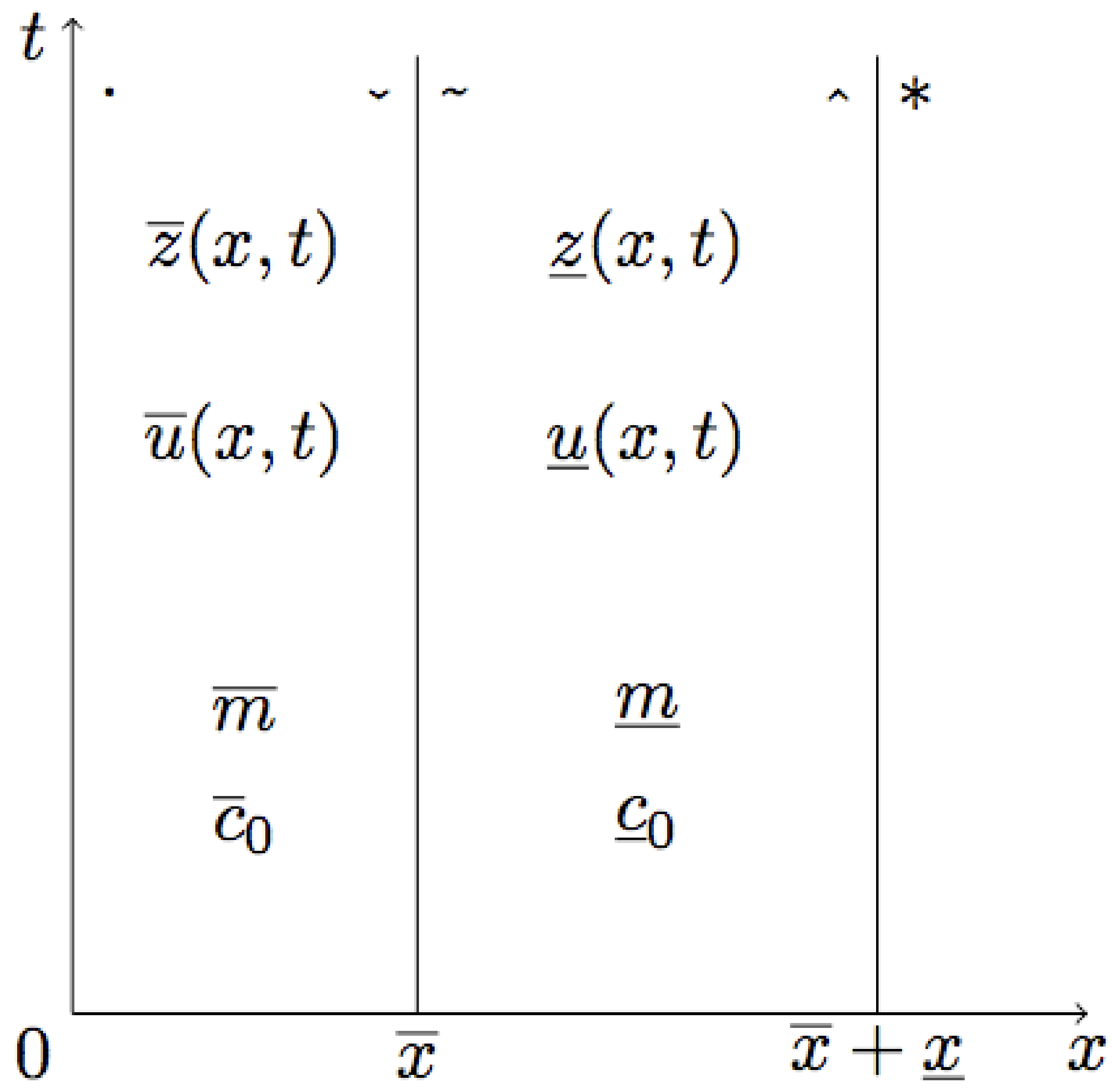

FIGURE 12. Solutions at two constant entropy levels $\overline{S}, \underline{S}$ in the (x,t)-plane.

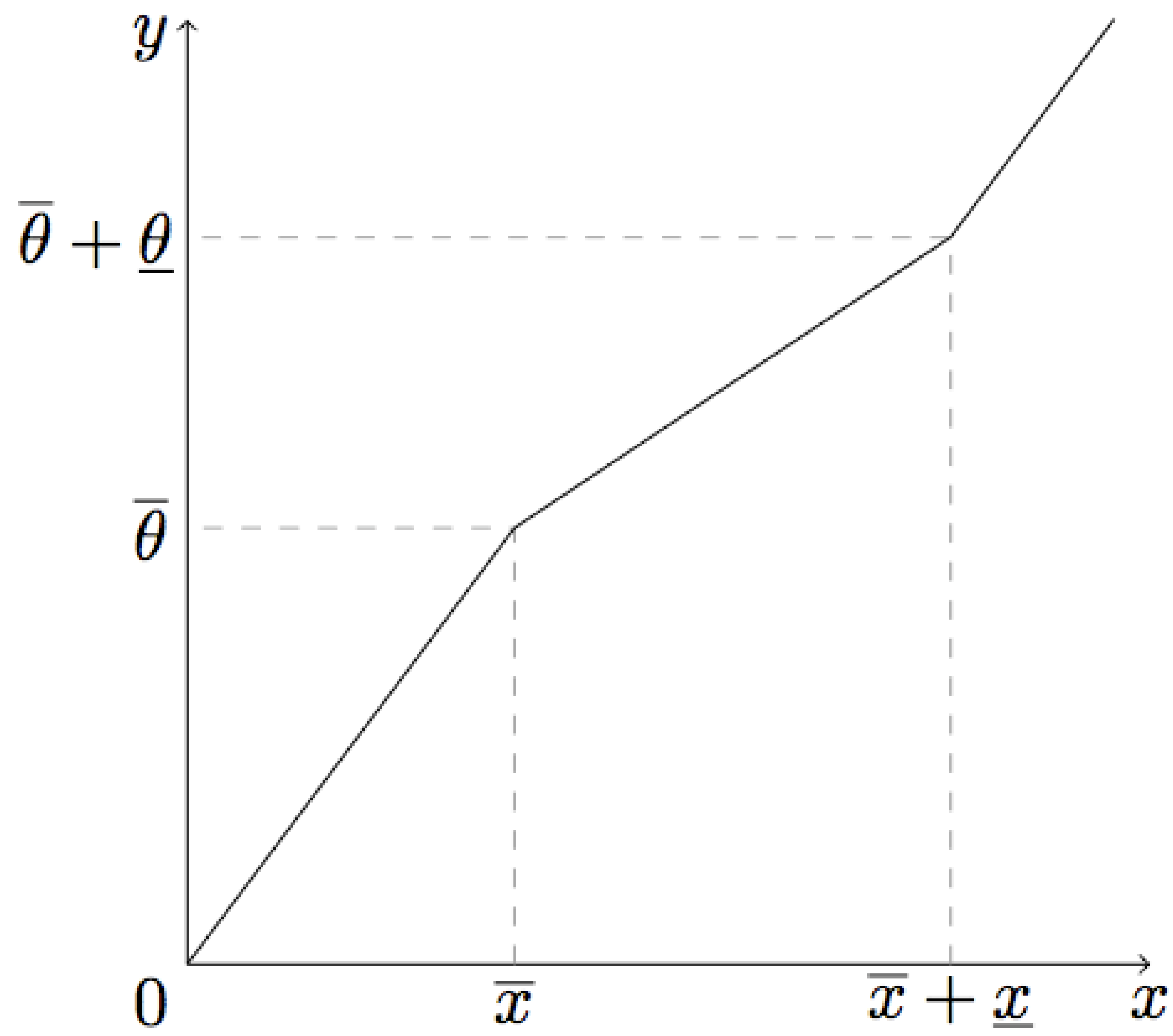

FIGURE 13. The mapping $x \to y$, $\overline{x} \to \overline{\theta}$, $x \to \theta$

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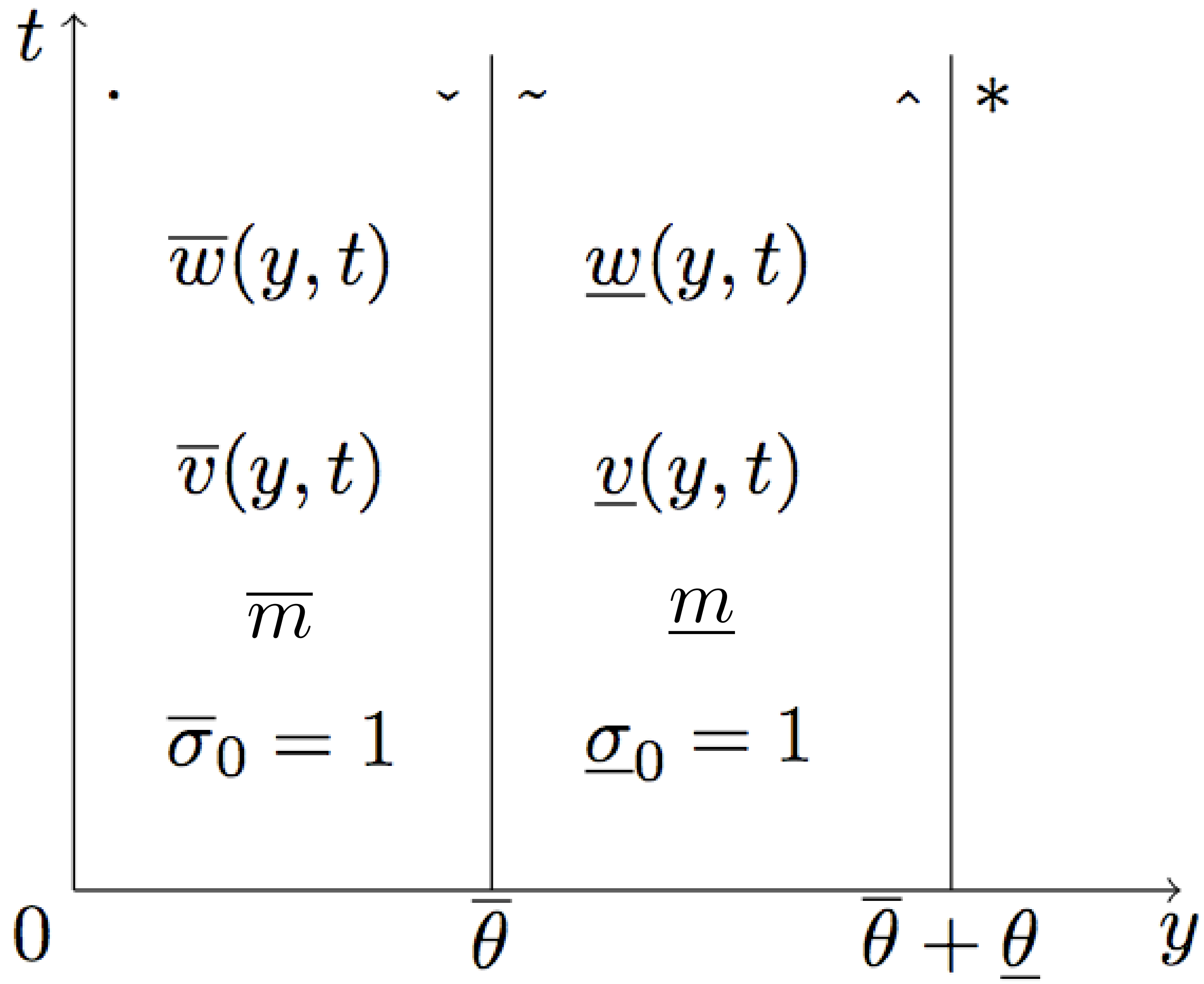

FIGURE 14. Solutions at two constant entropy levels $\overline{S}, \underline{S}$ in the (y,t)-plane.

Solutions at two constant entropy levels $\overline{m}$, $\underline{m}$ in the (y,t)-plane

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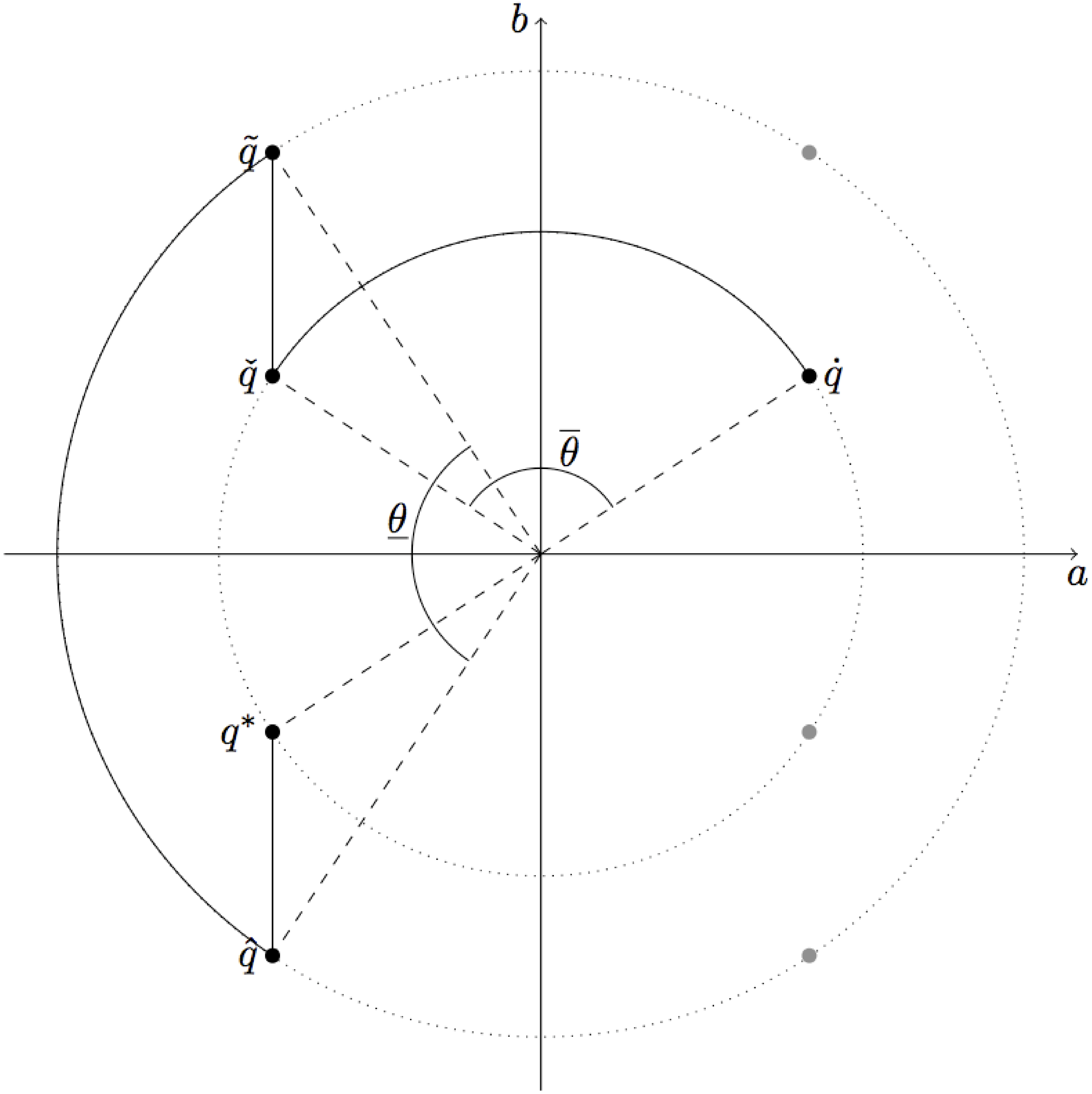
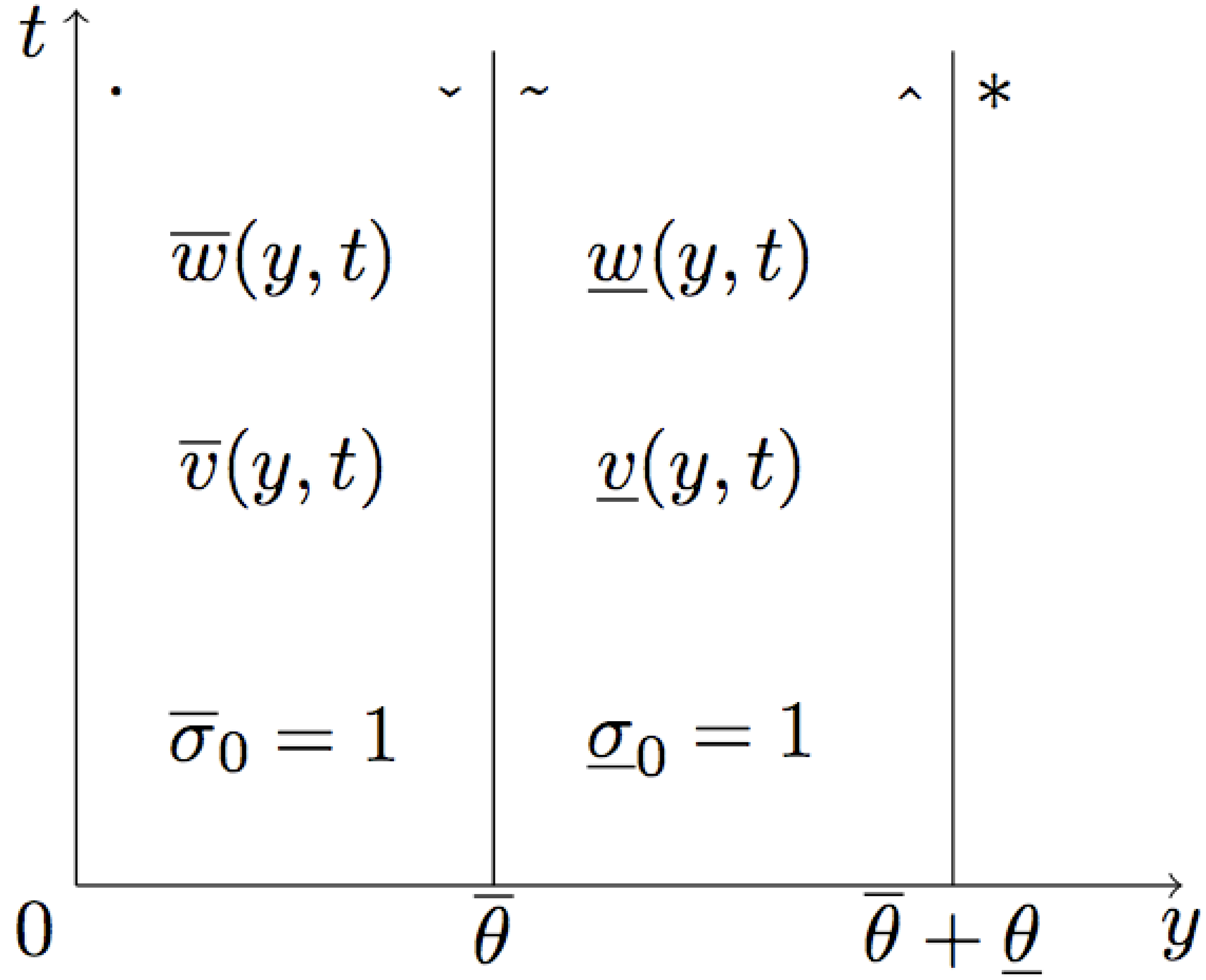


FIGURE 16. The states with $\dot{\mu} > 0$, showing $\underline{\theta} + \overline{\theta} > \pi$.



GURE 14. Solutions at two constant entropy levels $\overline{S}, \underline{S}$
e (y,t)-plane.