Time-Periodic Solutions of the Compressible Euler Equations

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November 20, 2007

A Mechanism for the Progation of Dissipation-Free Long Distance Signaling at the level of the Compressible Euler Equations

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OUR PROGRAM

- To Explicitly Construct...
- To Understand the Structure of...
- To Give a Mathematical Proof of Existence of...

Time-Periodic Solutions of the Compressible Euler Equations

(Unpublished--Work in Progress)

. . .

- Q1: By what wave propagation mechanism are time-periodic/shock-free solutions possible?
- Q2: What is the simplest possible structure?

The compressible Euler equations consist of three coupled nonlinear PDE's that can be interpreted as the continuum version of

Newton's Laws of Motion

(1) Conservation of Mass: (Continuity Equation)

(2) Newton's Force Law: (Continuum Version)

" The time-rate of change of momentum equals minus gradient of the pressure"

(3) Conservation of Energy: (Continuum Version)

Compressible Euler Equations: (1-D Wave Propagation)

- For wave propagation in x-direction:

Compressible Euler $\Longleftrightarrow$
$$\begin{cases} \rho_t + (\rho u)_x = 0 & \text{(Ma)} \\ (\rho u)_t + (\rho u^2 + p)_x = 0 & \text{(Mo)} \\ E_t + \{(E+p)u\}_x = 0 & \text{(En)} \end{cases}$$

- System (Ma), (Mo), (En) describes the time evolution of a compressible fluid...

$\rho = \frac{mass}{vol}$=density

$u =$ velocity

p=pressure

$E = \frac{energy}{vol} = \rho e + \frac{1}{2}\rho u^2$

$e = \frac{energy}{mass}$ =specific internal energy

The Entropy:

- Time-irreversibility is measured by the entropy, which evolves according to a derived conservation law:

- The specific entropy S is a state variable obtained by integrating the second law of thermodynamics

$$dS = \frac{de}{T} - p\frac{d\tau}{T} \qquad \text{(2nd Law)}$$

$$\tau = 1/\rho = \text{specific volume}$$

$$S = \frac{entropy}{mass} = \text{specific entropy}$$

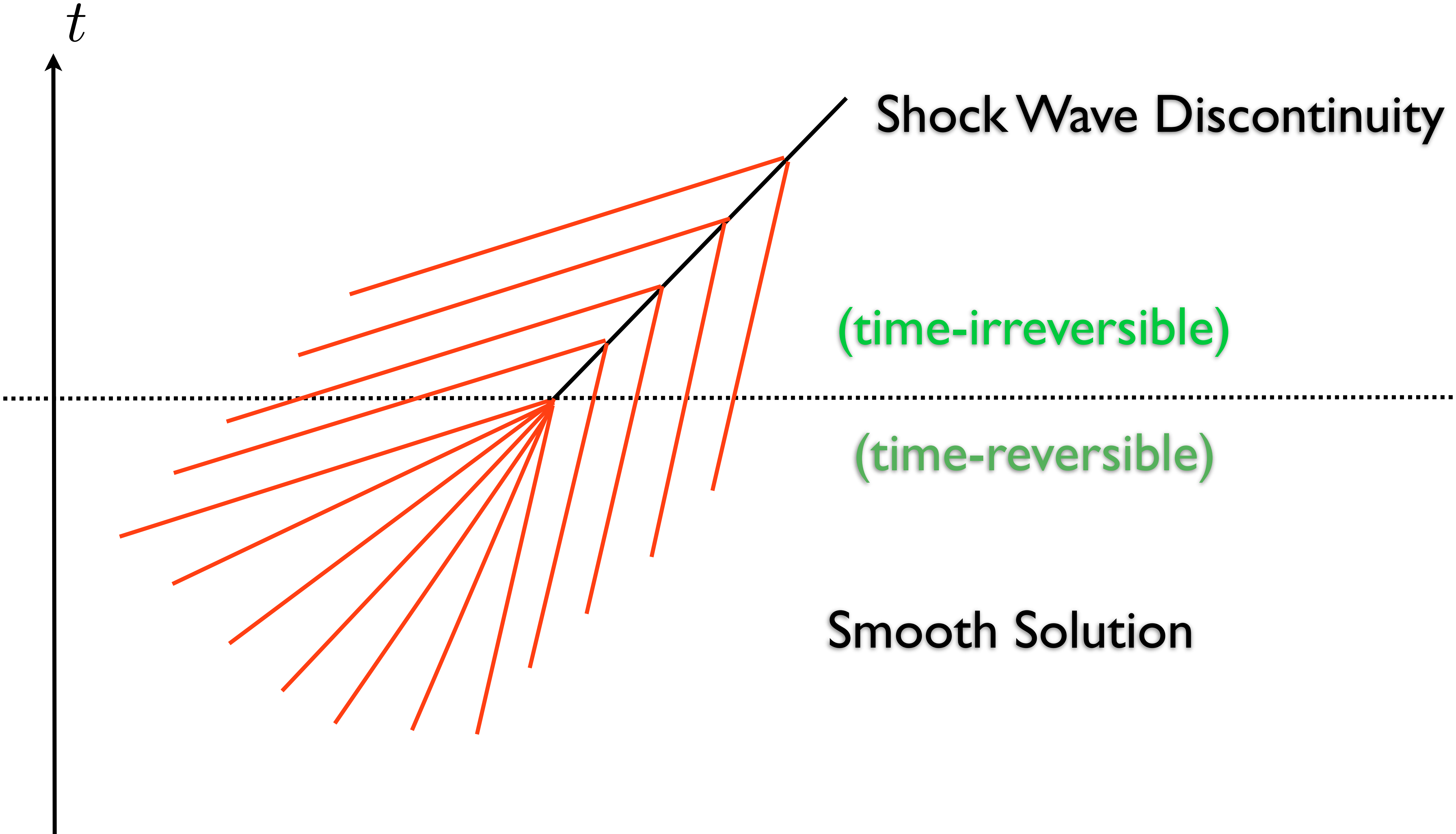
- A consequence is the "adiabatic constraint"

(Second Law of Thermodynamics) + (Ma),(Mo),(En) ⟹ (Ent)

$$\boxed{(\rho S)_t + (\rho S u)_x = 0} \qquad \text{(Ent)}$$

- The compressible Euler equations describe the time evolution of a perfect fluid in the limit that all dissipative forces (like friction and heat conduction) are neglected.

- Nevertheless: There is a canonical dissipation present at the zero dissipation limit, and this is encoded in the rate of increase of the entropy at shock waves:



Compressible Euler Equations:
(The Equation of State)

An equation of state relating (ρ, p, e) is required to close the system.

$$\text{Euler} \iff \begin{cases} \rho_t + (\rho u)_x = 0 & \text{(Ma)} \\ (\rho u)_t + (\rho u^2 + p)_x = 0 & \text{(Mo)} \\ E_t + \{(E+p)u\}_x = 0 & \text{(En)} \end{cases}$$

$$E = \rho e + \frac{1}{2}\rho u^2$$

The Fundamental Equation of State

- The equation of state for a polytropic or γ-law gas :

$$e = k_0 \, T = k_0 \, \rho^{\gamma - 1} \, \exp\left\{ \frac{S}{k_0} \right\}$$

where:

$$\gamma = 1 + 2/3r = \text{adiabatic gas constant}$$

$r =$ number of atoms in a molecule

- It follows that:

$$p = -\frac{\partial e}{\partial \tau}(S, \tau)$$

$$\tau = 1/\rho$$

- The nonlinearities are Analytic Functions

- "A non-interacting gas composed of molecules derived from first principles using only the equipartition of energy principle and the second law of thermodynamics..."

Conclude: the compressible Euler equations with a polytropic equation of state represents

(1) The continuum version of Newton's Laws...

(2) The nonlinear theory of sound waves...

Our question: "Do there exist time-periodic solutions of the compressible Euler equations that transmit sound waves like the linear theory of sound?"

Linearizing the Compressible Euler Equations about constant state

$$\begin{aligned} \rho &= \rho_0 \\ S &= S_0 \\ u &= 0 \end{aligned}$$

Equations reduce to the Wave Equation

Linear Theory of Sound

$$\boxed{\rho_{tt} - c^2 \rho_{xx} = 0}$$

E.g. The sinusoidal solutions of the wave equation are the "modes of vibration" that form the basis for the modern theory of music..."

2π-periodic solutions: $\rho(x,t) = \sin\{n(x \pm ct)\}$

Our question: "Do there exist time-periodic solutions of the compressible Euler equations that transmit sound waves like the linear theory of sound?"

For most of the last 250 years experts have thought that time periodic solutions of the compressible Euler equations were not possible because of the ubiquitous formation of

SHOCK-WAVES

The Difficulty in a Nutshell

- The compressible Euler Equations form a system of 3-coupled nonlinear conservation laws of form---

$$u_t + f(u)_x = 0$$

- Basic warmup problem: scalar Burgers Equation:

$$u_t + \frac{1}{2}(u^2)_x = 0$$

$$u_t + uu_x = 0$$

⬇

$$\nabla_{(1,u)} u(x,t) = 0$$

⬇

"u=const. along lines of speed u"

⬇

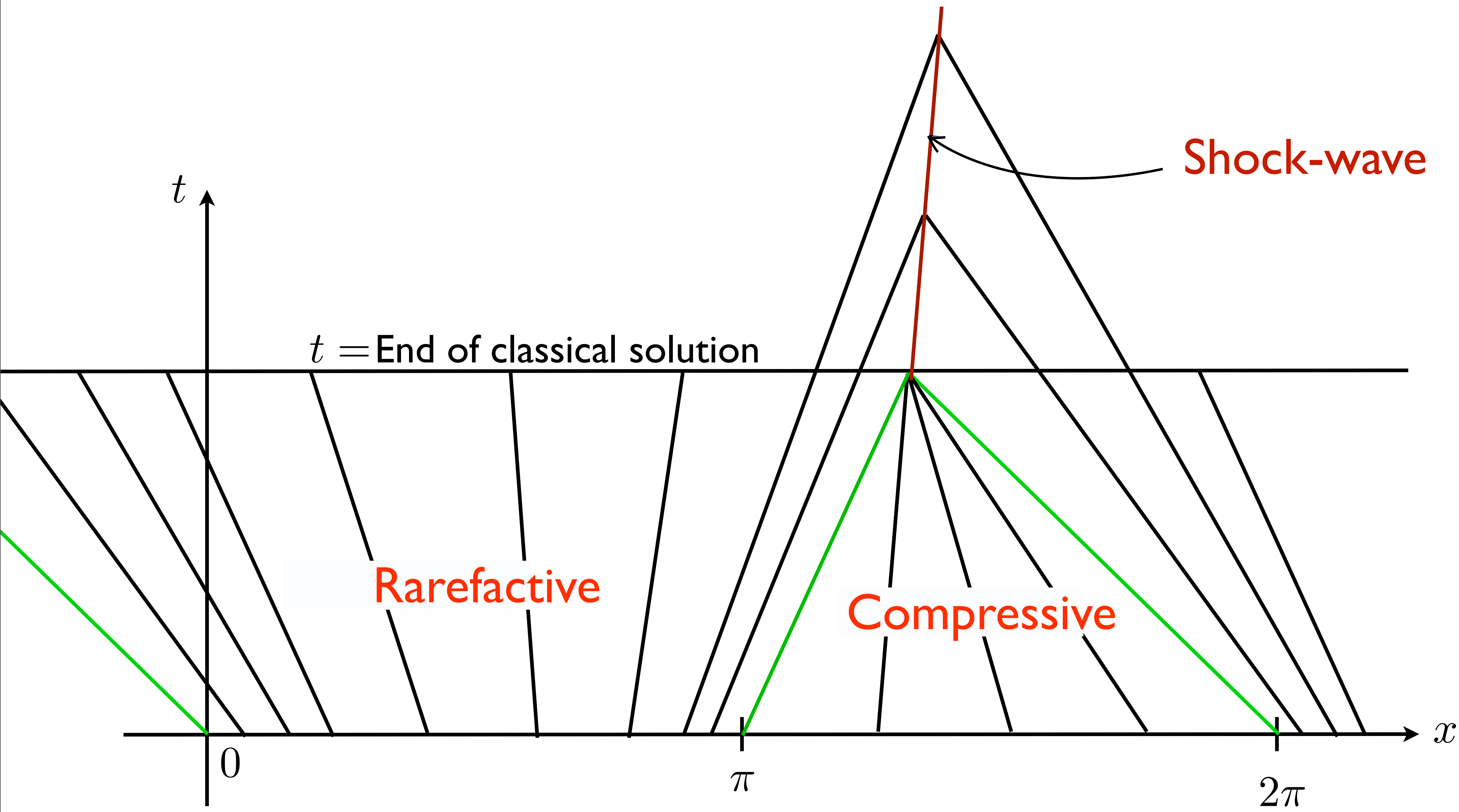
"inconsistent with time-periodic evolution"

The Problem:

- Basic warmup problem: scalar Burgers Equation:

$$u_t + uu_x = 0$$

"inconsistent with time-periodic evolution"



For Example: This always happens in the 2x2 p-system obtained by closing off the first two Euler equations by assuming

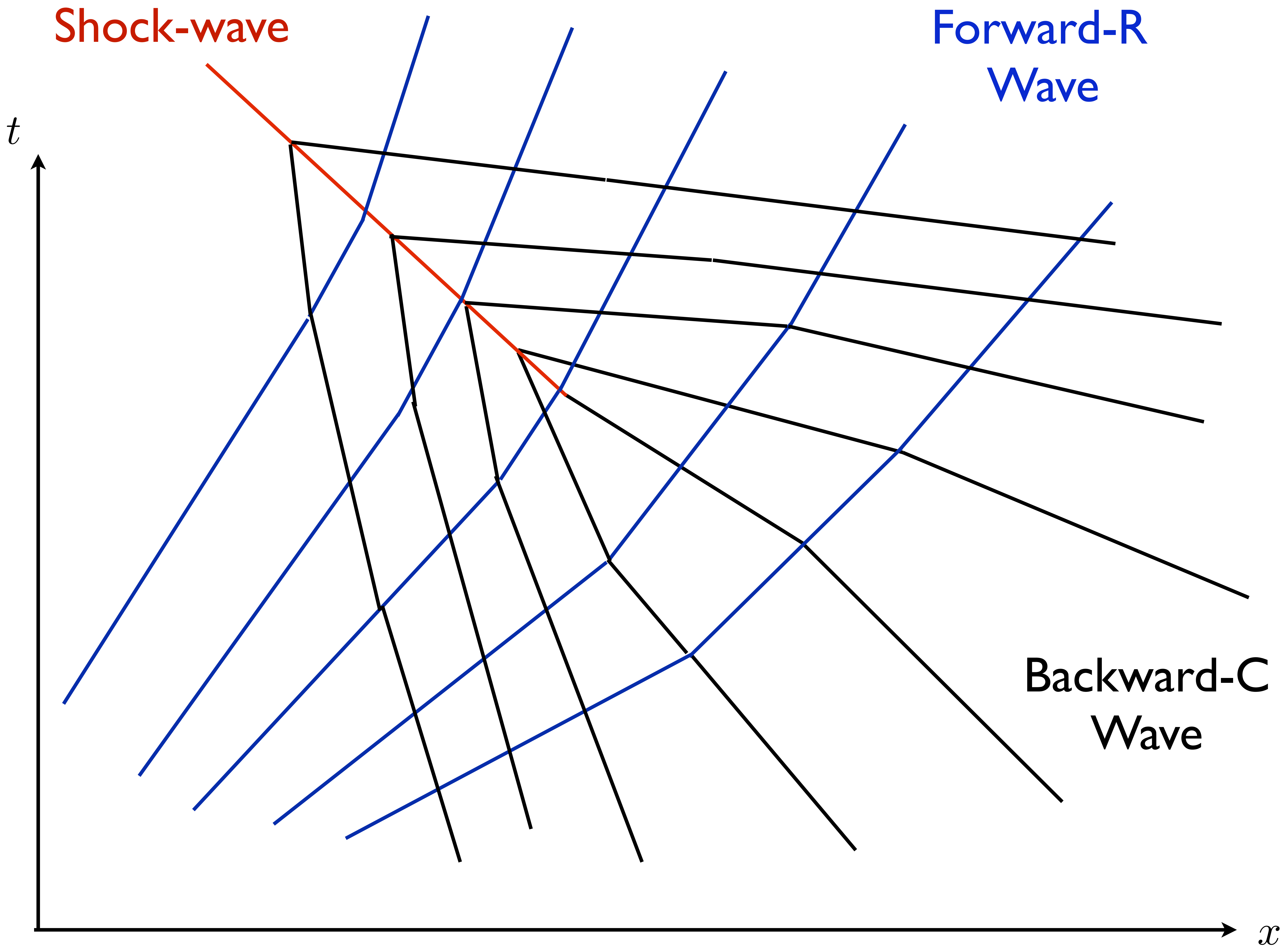
$$\boxed{p = p(\rho)}$$

That is:

$$p\text{-system}\begin{cases} \rho_t + (\rho u)_x = 0 & \text{(Ma)} \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0 & \text{(Mo)} \end{cases}$$

$$E_t + \{(E + p)u\}_x = 0 \qquad \text{(En)}$$

Theorem: (Glimm/Lax 1970) This always happens in 2x2 systems like the p-system...

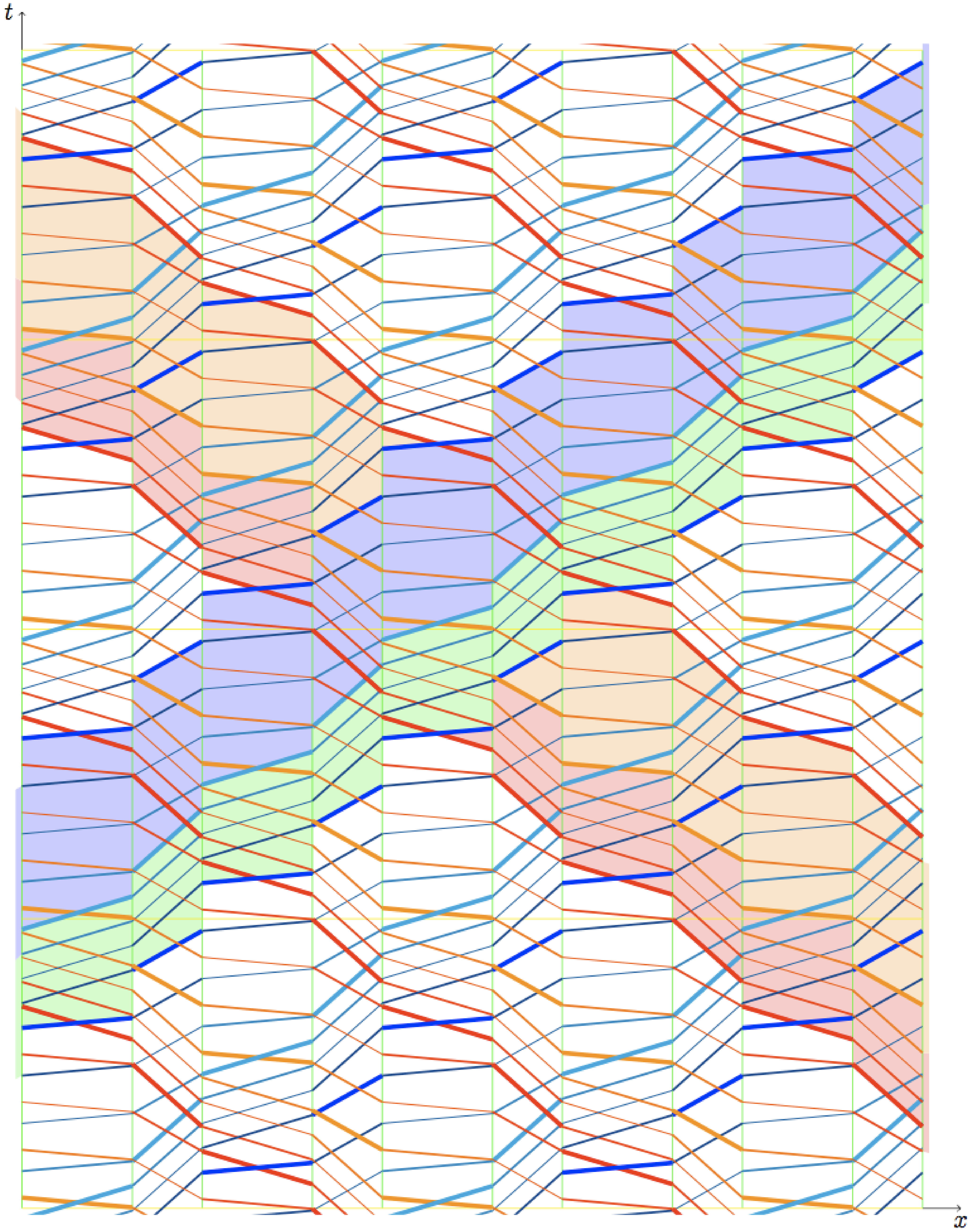


"Periodic solutions decay by shock wave dissipation at rate 1/t"

We are proposing the following picture of the simplest time-periodic solution of the compressible Euler equations:

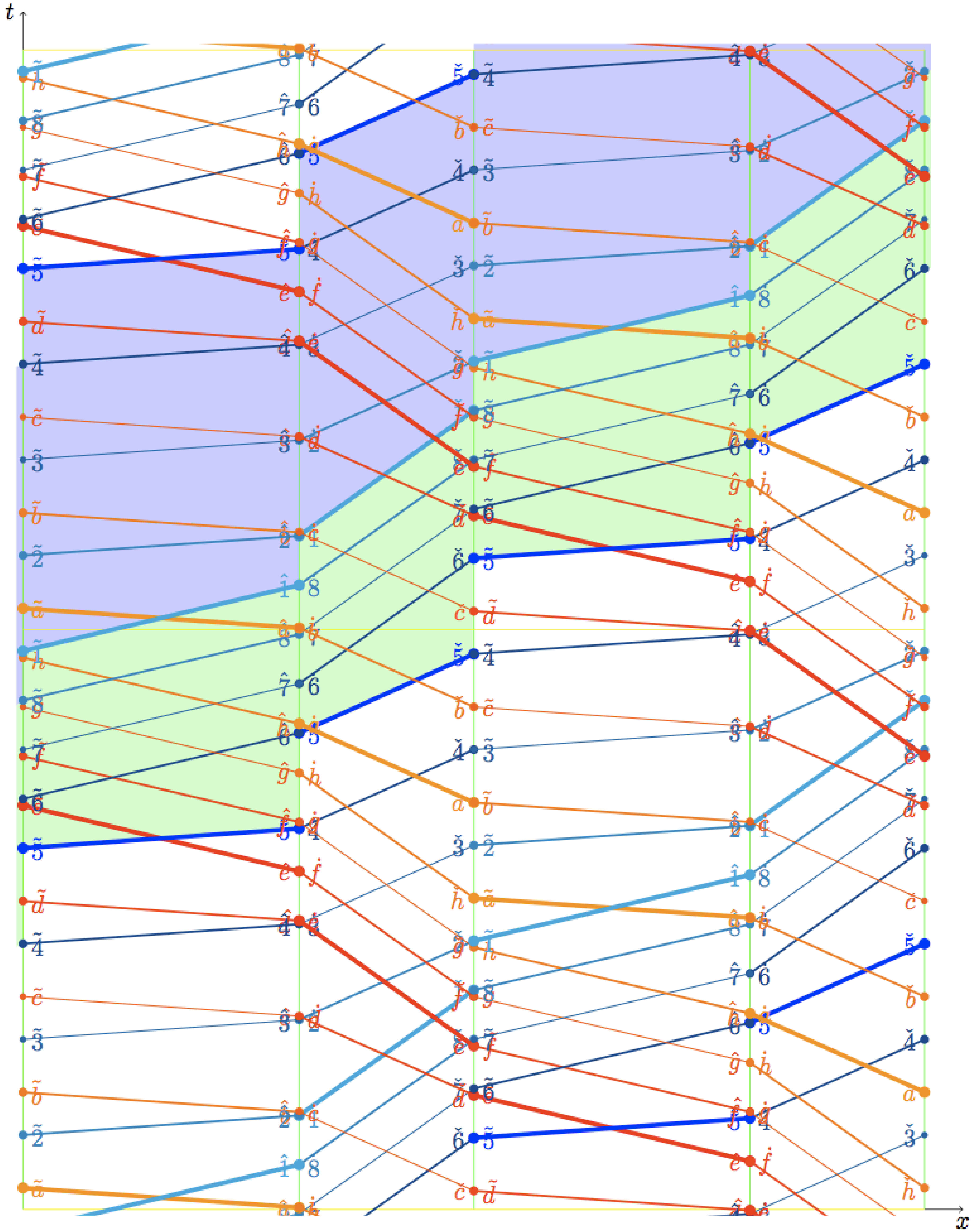
The Mechanism requires at least three coupled equations!

The remainder of the talk is devoted to motivating and/or explaining this picture...



---Our Answer---

The simplest global periodic structure in the xt-plane



Our answer

Why we are interested in time-periodic solutions of compressible Euler...

- Historically ---The equations were derived by Euler in 1752 as a model for sound wave propagation.
- Scientifically ---Time-periodic solutions represent dissipation-free long-distance signaling.
- Could the structure of periodic solutions supply a new paradigm for how sound waves, and other nonlinear waves, really propagate?
- PDE Issues ---Resonances/Small Divisors/KAM theory/ Combinatorics/Diophantine Equations/ Probability Theory
- Bifurcation Theory ---Potential to open the door to issues such as chaos, period-doubling, etc.
- Intellectual Interest ---Our approach is to guess the structure by heuristic reasoning based on nonlinear waves...and this is prerequisite to rigorous analysis.

History/Prior Results

Periodic solutions of Compressible Euler

History/Background

- The first question to ask after Euler is:

  Do the fully nonlinear equations admit time-periodic, oscillatory solutions that propagate information like the linear sound waves of the wave equation?

- For most the last 250 years experts have generally thought that such time periodic solutions did not exist, due to the phenomenon of shock wave formation...

History/Background

- 1687-- Principia/ Newton attempted to give a continuum version of his laws of motion in order to derive the speed of sound from first principles.
- 1749-- D'Alembert introduced the linear wave equation to describe displacements of a vibrating string.

  The wave equation is the basic equation in which all waves propagate at the same speed, and so it was natural to conjecture that sinusoidal oscillations in the air might account for sound waves. But D'Alembert had no physical derivation of this from first principles.
- 1752-- Euler completed Newton's program by deriving the fully nonlinear theory of sound waves from first principles.

  Euler showed that asymptotically, in the limit of weak signals, the compressible Euler equations reduce to the wave equation in the density, thus demonstrating that sound waves could be described by periodic sinusoidal oscillations in the density.
- This established the framework for the (linear) theory of sound.

Ref: D. Christodoulou, ETH Zurich, 2006/Bulletin, Oct. 2007.

History/Background

- 1857-- Riemann showed that shock-wave discontinuities can form from smooth solutions of the compressible Euler equations.

  Introduced Riemann invariants and the Riemann problem to continue the solutions past the time of shock formation

- After Reimann...

- Shock-waves became the cental issue in the study of the compressible Euler equations...

Riemann, B. *Uber die Fortpfanzung ebener Luftwellen von endlicher Schwingungswete*, Abhandlungen der Gesellshaft der Wissenshaften zu Gottingen, Mathematisch-physikalishe Klasse, Vol. 8, 43 (1858-59)

History/Background

- 1964-- Lax proved finite time blow-up in derivatives for 2x2 systems for which the nonlinear fields are "genuinely nonlinear"

  like the p-system.

  Lax's argument is sufficient to imply blow-up in the derivative for space-periodic solutions of the p-system thereby implying the formation of shock-waves--- inconsistent with time-periodic evolution.

P.D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, Jour. Math. Physics, Vol. 5, pp. 611-613 (1964).

- 1970-- Glimm and Lax give definitive result for 2x2 systems--- shocks must form from periodic initial data for the 2x2 p-system

  Thm: Solutions of the p-system starting from space periodic initial data (small in L^∞) must form shock-waves and decay in the total varation norm at rate ate 1/t.

J. Glimm, P.D. Lax, *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Memoirs Amer. Math Soc. 101(1970).

History/Background

- **1974-97** Blow-up results that extend Lax's result to 3x3 systems were not sufficient to rule out the possibility of time- periodic sound wave propagation in the compressible Euler equations...

F. John, *Formation of singularities in one-dimensional wave propagation*, Comm. Pure Appl. Math., Vol. 27, pp. 377-405 (1974).

T.P. Liu, *Development of singularities in the nonlinear waves for quasi-linear hyperbolic partial differential equations*, J. Diff. Eqns, Vol. 33, pp. 92-111 (1979).

Li Ta-Tsien, Zhou Yi and Kong De-Xing, *Global classical solutions for general quasilinear hyperbolic systems with decay initial data*, Nonlinear. Analysis., Theory., Methods. and Applications., Vol. 28, No. 8, pp. 1299-1332 (1997).

History/Background

- 1984-88-- The idea that time periodic solutions may exist was kindled by work of Majda, Rosales, Schonbeck and Pego:

A. Majda and R. Rosales, *Resonantly interacting weakly nonlinear hyperbolic waves I. A single variable*, Stud. in Appl. Math., 22, pp. 149-179 (1984).

A. Majda, R. Rosales and M.Schonbeck, *A canonical system of integrodifferential equations arising in resonant nonlinear acoustics*, Stud. in Appl. Math., 79, pp. 205-262 (1988).

R.L. Pego, *Some explicit resonating waves in weakly nonlinear gas dynamics*, Stud. in Appl. Math., 79, pp. 263-270 (1988).

(Scalar/Asymptotic Models)

History/Background

- 1996-99-- Rosales and two students, Shefter and Vaynblat, produced detailed numerical simulations of the Euler equations starting from periodic initial data, and these numerical studies indicated that periodic solutions of the 3x3 compressible Euler equations do not decay like the 2x2 p-system, and they made observations about the possibility of periodic, or quasi-periodic attractor solutions.

M. Shefter and R. Rosales, *Quasi-periodic solutions in weakly nonlinear gas dynamics*, Studies in Appl. Math., Vol. 103, pp. 279-337 (1999).

D. Vaynblat, *The strongly attracting character of large amplitude nonlinear resonant acoustic waves without shocks. A numerical study.* M.I.T. Dissertation, (1996).

- CONCLUDE: Until now, we do not understand the structure of time periodic solutions, nor the mechanism that can prevent shock formation.
- Moreover, it is difficult to numerically simulate time-periodic solutions by starting with general space periodic data and running the solution until the shock-wave dissipation resolves itself into a periodic configuration...
- ...Errors are difficult to control in large time simulations...
- ... Shock-waves alter the entropy field, and so the background entropy field remains unknown until the shock-wave dissipation is done. The final entropy field to which a general time periodic solution will decay is then pretty much impossible to predict, and hence difficult to simulate without understanding the mechanism for periodic wave propagation at the start.

PART I. Derive the simplest periodic structure by analyzing how waves can change from

Compressive to Rarefactive

at entropy jumps.

"Derive the simplest pattern of

R's and C's such that

Compression (C)

& Rarefaction (R)

are balanced along

characteristics (sound waves)".

PART II. Realize solutions with this simplest structure at a Linearized Level.

Set up a Perturbation Problem to prove the existence of nearby nonlinear solutions by the Implicit Function Theorem in Banach Spaces

Compressive
and
Rarefactive waves

The Euler system in Lagrangian coordinates (relative to a frame moving with the fluid)

Assuming:

- Smooth, 1-D motion
- polytropic equation of state

The Euler equations are equivalent to:

$$\boxed{\begin{aligned} \tau_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0 \end{aligned}}$$

$$\text{(Ma)}\quad \text{(Mo)}\quad \text{(Ent)}$$

- Three coupled nonlinear equations in the three unknowns (τ, u, S)

System closes with the γ-law relation

$$p = K\tau^{-\gamma}e^{S/c_\tau}$$

Lagrangian Formulation

- In a frame moving with the fluid, the Euler equations are equivalent to:

$\tau = 1/\rho$

$$\tau_t - u_x = 0 \qquad \text{(Ma)}$$

$$u_t + p_x = 0 \qquad \text{(Mo)}$$

$$S_t = 0 \qquad \text{(Ent)}$$

- S=Const → 2x2 p-system

$$\tau_t - u_x = 0$$

$$u_t + p(\tau, S)_x = 0$$

- Sound waves: $\boxed{\frac{dx}{dt} = \pm c}$ c= Sound speed

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma}\tau^{-\frac{\gamma+1}{2}} e^{S/2c_\tau}.$$

Lagrange equations as a System of Conservation Laws

$$\tau_t - u_x = 0$$
$$u_t + p_x = 0$$
$$S_t = 0$$

$$\Leftrightarrow \quad \begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_t + \begin{pmatrix} -u \\ p \\ 0 \end{pmatrix}_x = 0 \quad \Leftrightarrow \quad U_t + F(U)_x = 0$$

- Sound speed: $c = \sqrt{-p_\tau} = \sqrt{K\gamma}\tau^{-\frac{\gamma+1}{2}} e^{S/2c_\tau}.$

- The system supports 3 Wave Families determined by the eigenfamilies (λ_i, R_i) of dF :

1-waves	2-waves	3-waves
$\lambda_1 = -c$	$\lambda_2 = 0$	$\lambda_3 = c$

- 3 characteristic families associated with (λ_i, R_i) :

1-waves (back)

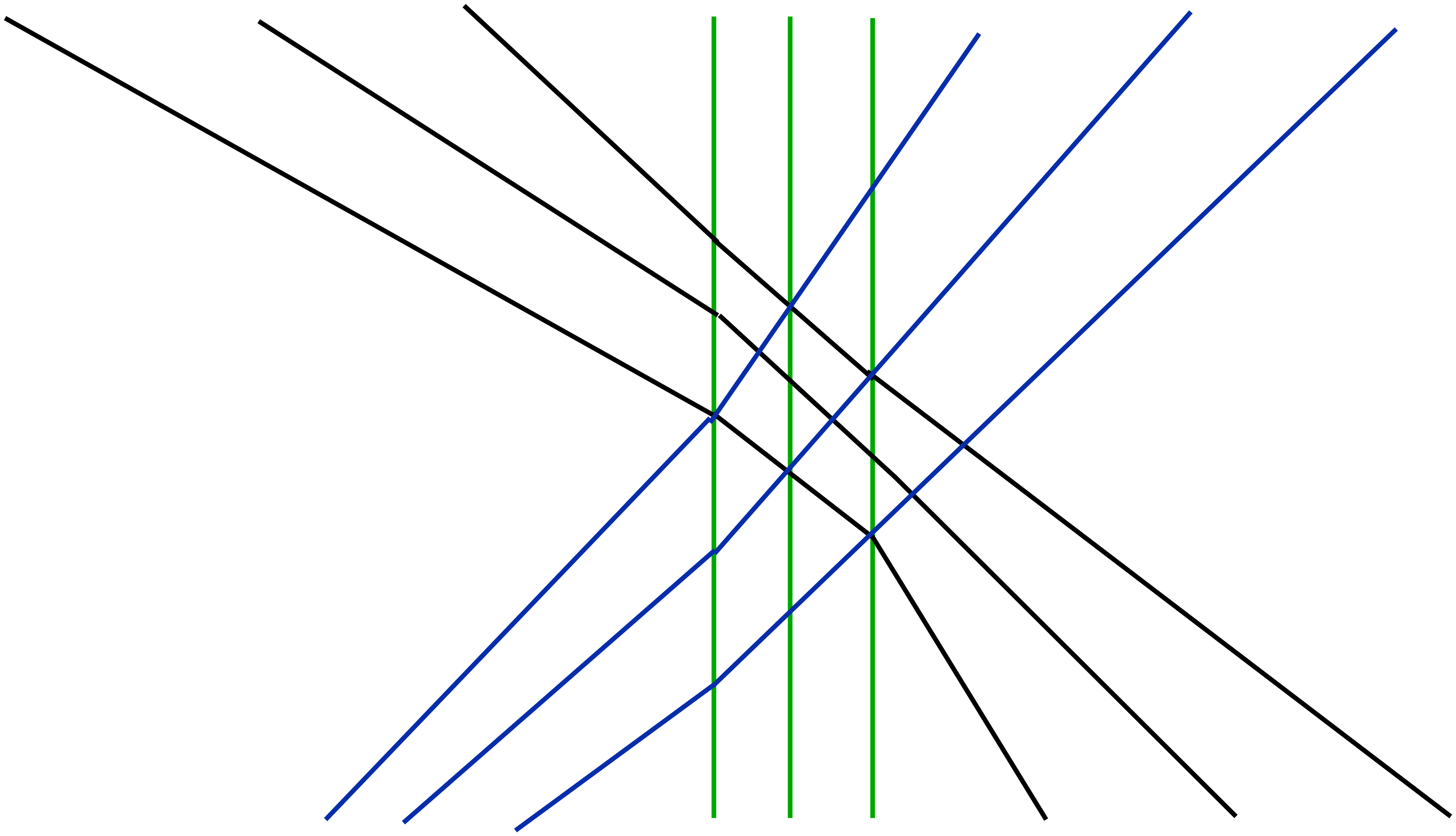
$$\frac{dx}{dt} = \lambda_1 = -c$$

2-waves

$$\frac{dx}{dt} = \lambda_2 = 0$$

3-waves (forward)

$$\frac{dx}{dt} = \lambda_3 = c$$



3-waves

Genuinely Nonlinear

$\nabla_{R_1} \lambda_1 > 0$

2-waves

Linearly Degenerate

$\nabla_{R_2} \lambda_2 \equiv 0$

1-waves

Genuinely Nonlinear

$\nabla_{R_3} \lambda_3 > 0$

- The three Characteristic Families of Euler:

$$U_t + F(U)_x = 0 \quad \Leftrightarrow \quad U_t + dF \cdot U_x = 0$$

$$\begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_t + \begin{pmatrix} -u \\ p \\ 0 \end{pmatrix}_x = 0 \quad \Leftrightarrow \quad \begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_t = \begin{pmatrix} 0 & -1 & 0 \\ p_\tau & 0 & p_S \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ u \\ S \end{pmatrix}_x = 0$$

- Three eigen-families of dF

1-waves	2-waves	3-waves
$\lambda_1 = -c$	$\lambda_2 = 0$	$\lambda_3 = c$
$R_1 = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}$	$R_2 = \begin{pmatrix} -p_S/p_\tau \\ 0 \\ 1 \end{pmatrix}$	$R_3 = \begin{pmatrix} 1 \\ -c \\ 0 \end{pmatrix}$

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma} \left(\frac{1}{\tau}\right)^{\frac{\gamma+1}{2}} e^{S/2c_\tau}$$

- Three eigen-families of dF ...

1-waves	2-waves	3-waves
$\lambda_1 = -c$	$\lambda_2 = 0$	$\lambda_3 = c$
$R_1 = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}$	$R_2 = \begin{pmatrix} -p_S/p_\tau \\ 0 \\ 1 \end{pmatrix}$	$R_3 = \begin{pmatrix} 1 \\ -c \\ 0 \end{pmatrix}$

Conclude:

S is constant through 1,3-waves

u, p are constant through 2-waves

Riemann Invariants (r, s)

- At constant entropy:

$$\tau_t - u_x = 0$$
$$u_t + p_x = 0$$
$$S_t = 0$$

$\Leftrightarrow$

$$r_t - c r_x = 0$$
$$s_t + c s_x = 0$$

$S = const.$

$$r \equiv const. \text{ along 1-characteristics}$$
$$s \equiv const. \text{ along 3-characteristics}$$

- Problem: r and s depend on the entropy S

Simple Waves

- nxn system of conservation laws:

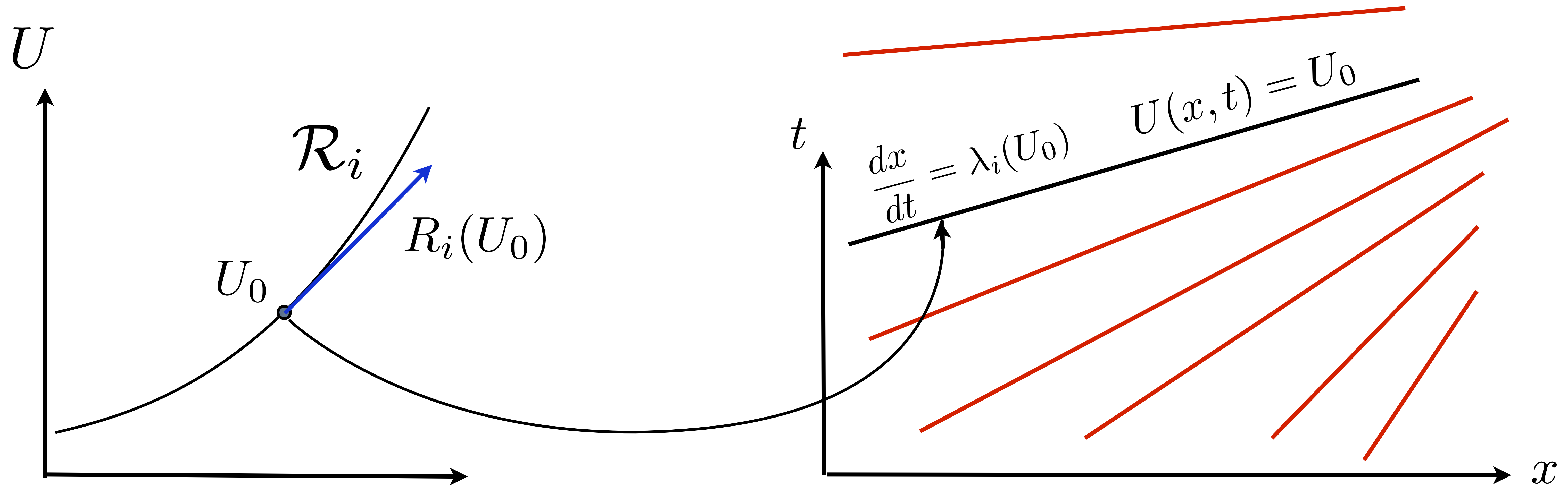
$$U_t + F(U)_x = 0 \quad \Leftrightarrow \quad U_t + dF \cdot U_x = 0$$

- Assume that (λ_i, R_i) is a (smooth) eigen-field for dF :

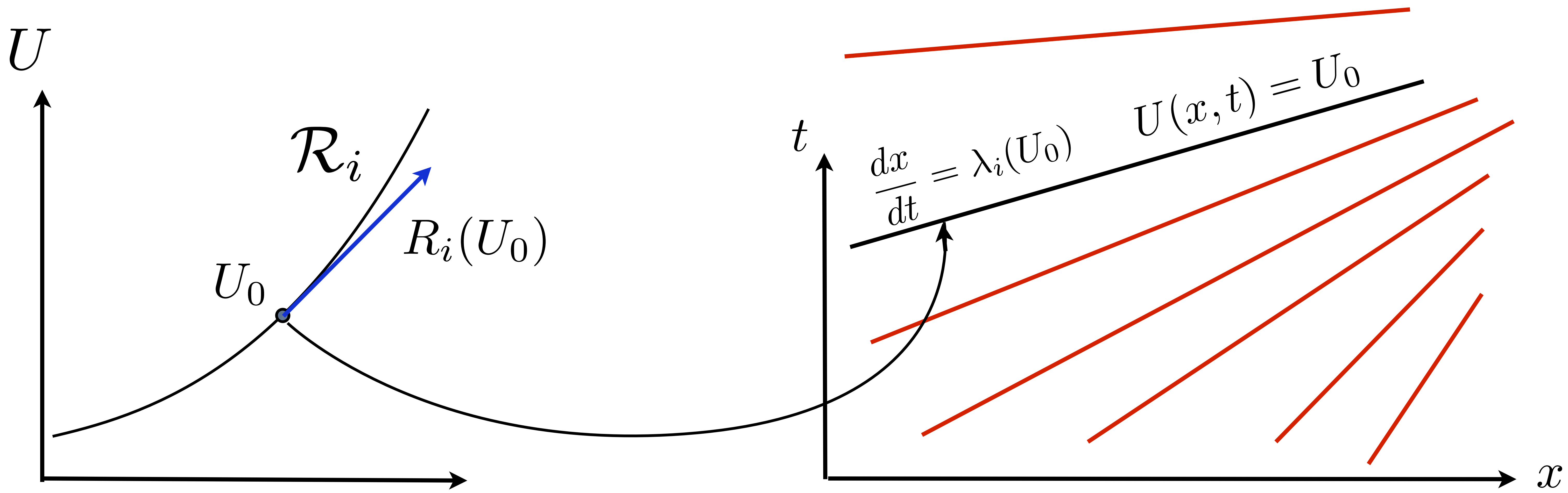
$$(dF - \lambda_i I)\, R_i = 0$$

Let $\mathcal{R}_i$ denote an integral curve of vector field R_i

Letting states U on $\mathcal{R}_i$ propagate with speed $\lambda_i(U)$ defines a 1-parameter family of simple waves



- Letting states U on $\mathcal{R}_i$ propagate with speed $\lambda_i(U)$ defines a 1-parameter family of simple waves



1,3-Simple waves are either
Compressive (C) or Rarefactive (R)

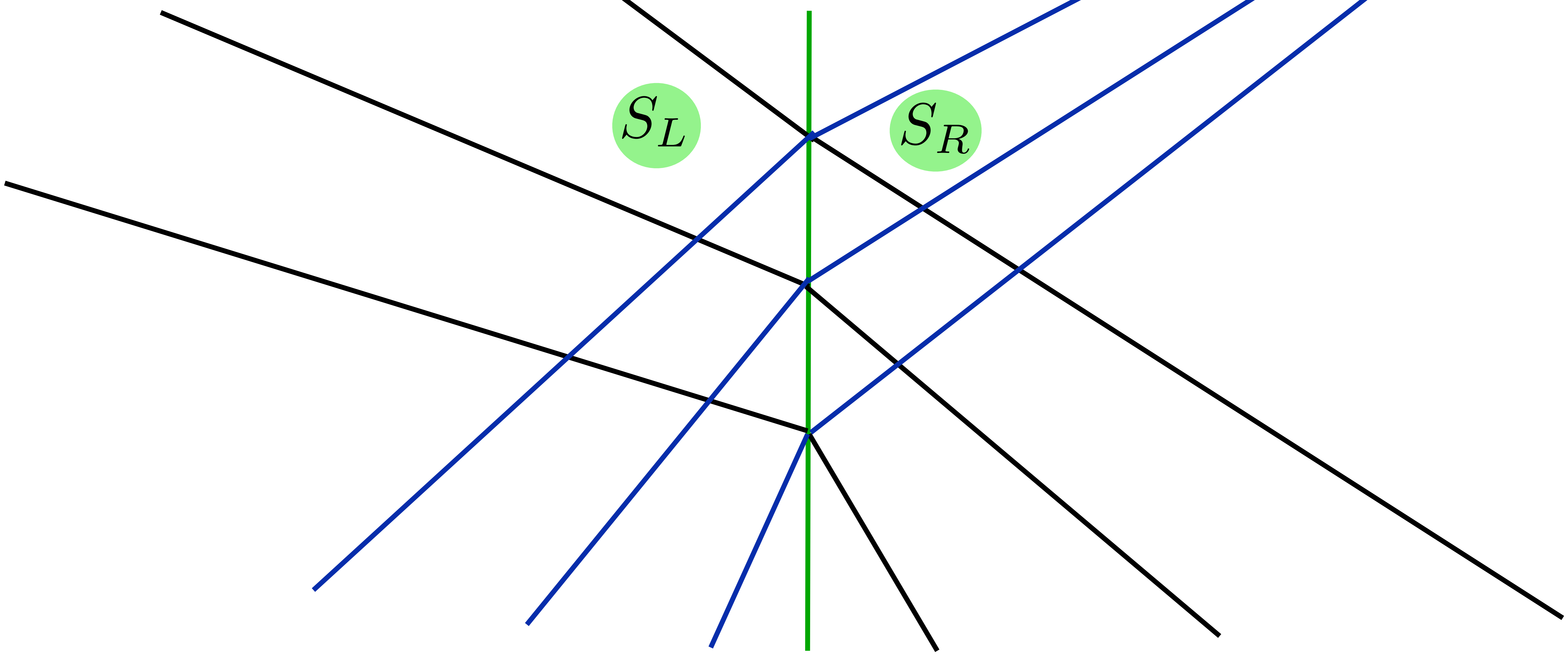
- The 2-field (λ_2, R_2) is Linearly Degenerate:

$$\nabla_{R_2} \lambda_2 \equiv 0$$

2-waves can be rescaled into time-reversible contact discontinuities

2-contact discontinuity

$$\frac{dx}{dt} = \lambda_2 = 0$$



Conclude: time-periodic solutions allow for discontinuities in entropy S

A Convenient Change of Variables

- Use z and m instead of ρ and S:

$$m = e^{S/2c_\tau}$$

$$z = \int_\tau^\infty \frac{c}{m} d\tau = \left(\frac{2\sqrt{K\gamma}}{\gamma - 1} \right) \rho^{\frac{\gamma - 1}{2}}$$

- At each constant value of the entropy, the system reduces to a transformed version of the 2x2 p=system that depends on the entropy through variable m:

$$
\begin{aligned}
z_t + \frac{c}{m} u_x \qquad\qquad &= 0 \\
u_t + mcz_x + 2\frac{p}{m} m_x \quad &= 0 \\
m_t \qquad\qquad &= 0
\end{aligned}
$$

S=const.

$$
\boxed{\begin{aligned}
z_t + \frac{c}{m} u_x \quad &= 0 \\
u_t + mcz_x \quad &= 0
\end{aligned}}
$$

- In terms of the Riemann invariants r and s:

$$r = u - mz$$
$$s = u + mz$$

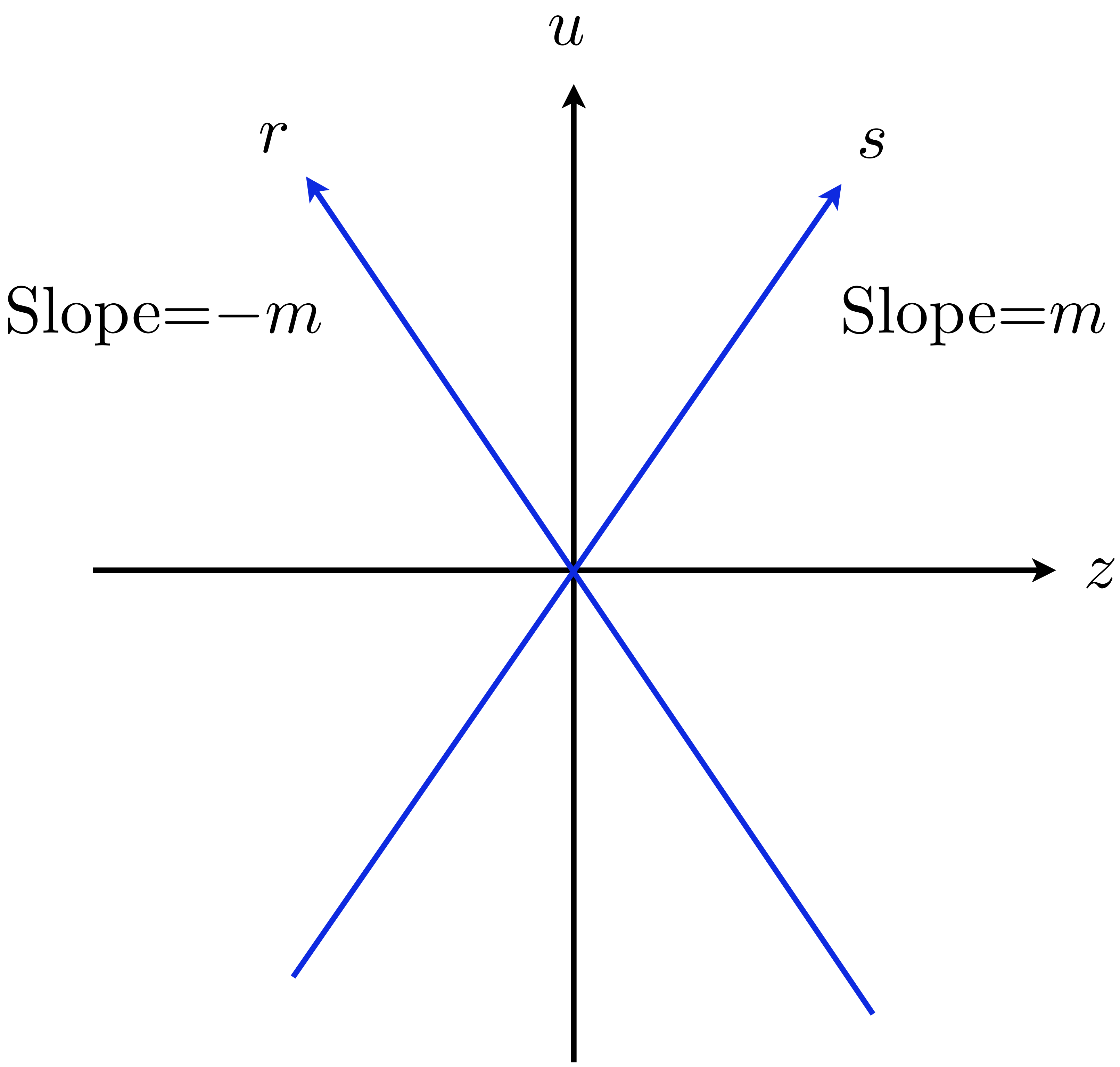
$$\boxed{\begin{aligned} z_t + \tfrac{c}{m} u_x &= 0 \\ u_t + mcz_x &= 0 \end{aligned}} \quad \Longleftrightarrow \quad \begin{aligned} r_t - cr_x &= 0 \\ s_t + cs_x &= 0 \end{aligned}$$

(z, u) <u>independent</u> of entropy

(r, s) <u>depend</u> on entropy

- Conclude: Equations in (z, u) isolate the dependence on S in coefficients

Relationship Between Coordinates



Riemann invariant coordinates in (z, u)-plane

$$m = e^{S/2c_\tau}$$

Compressive and Rarefactive Waves (R/C)

Consider 1,3-waves at constant entropy S:

$$1\text{-wave} \equiv \text{“}backward\text{”-wave}$$

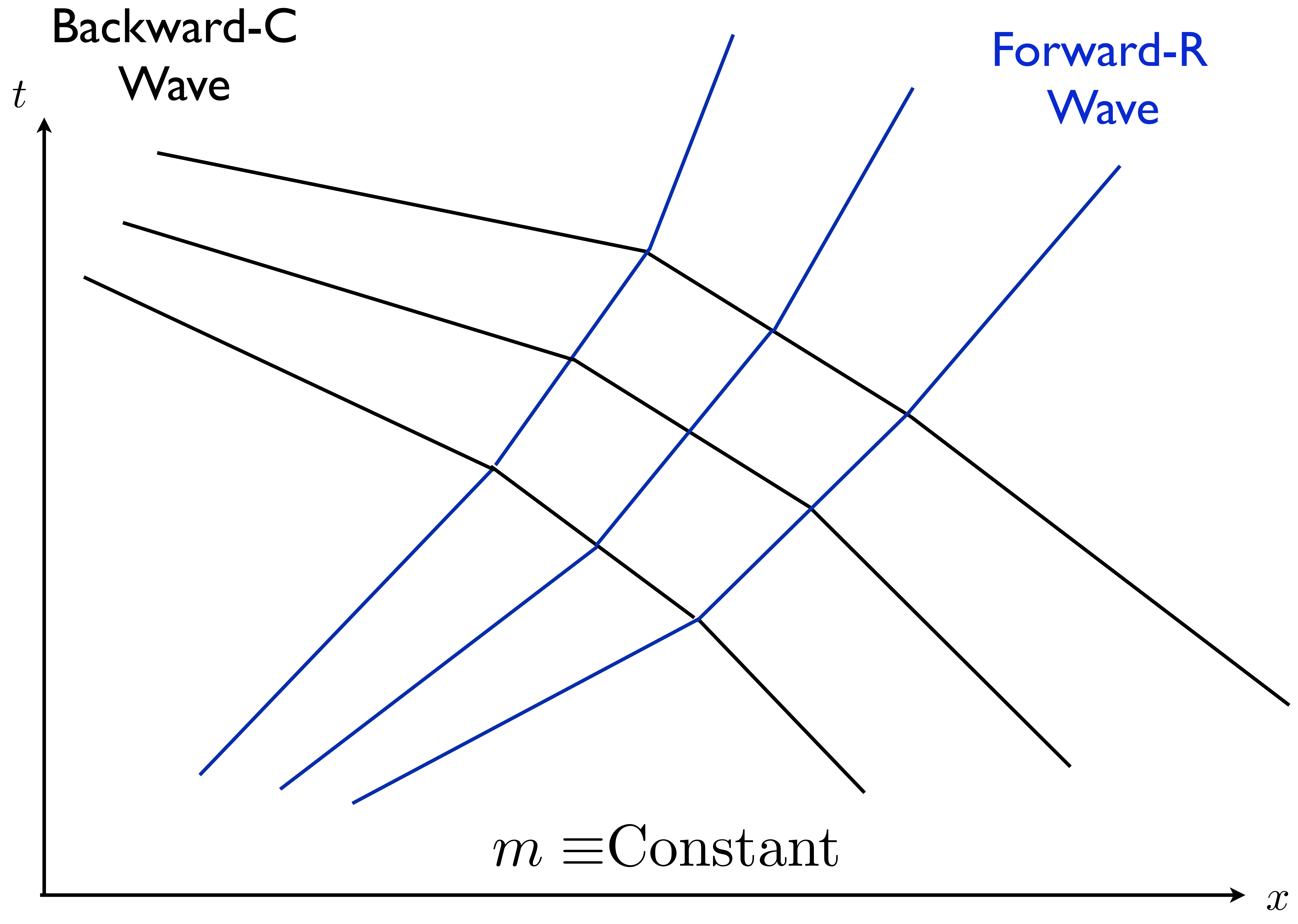
$$3\text{-wave} \equiv \text{“}forward\text{”-wave}$$

Definition: The R/C character of a wave in a general smooth solution is defined (pointwise) by:

$$\begin{array}{rcl} \text{Forward } R & \text{iff} & s_t \leq 0, \\ \text{Forward } C & \text{iff} & s_t \geq 0, \\ \text{Backward } R & \text{iff} & r_t \geq 0, \\ \text{Backward } C & \text{iff} & r_t \leq 0. \end{array}$$

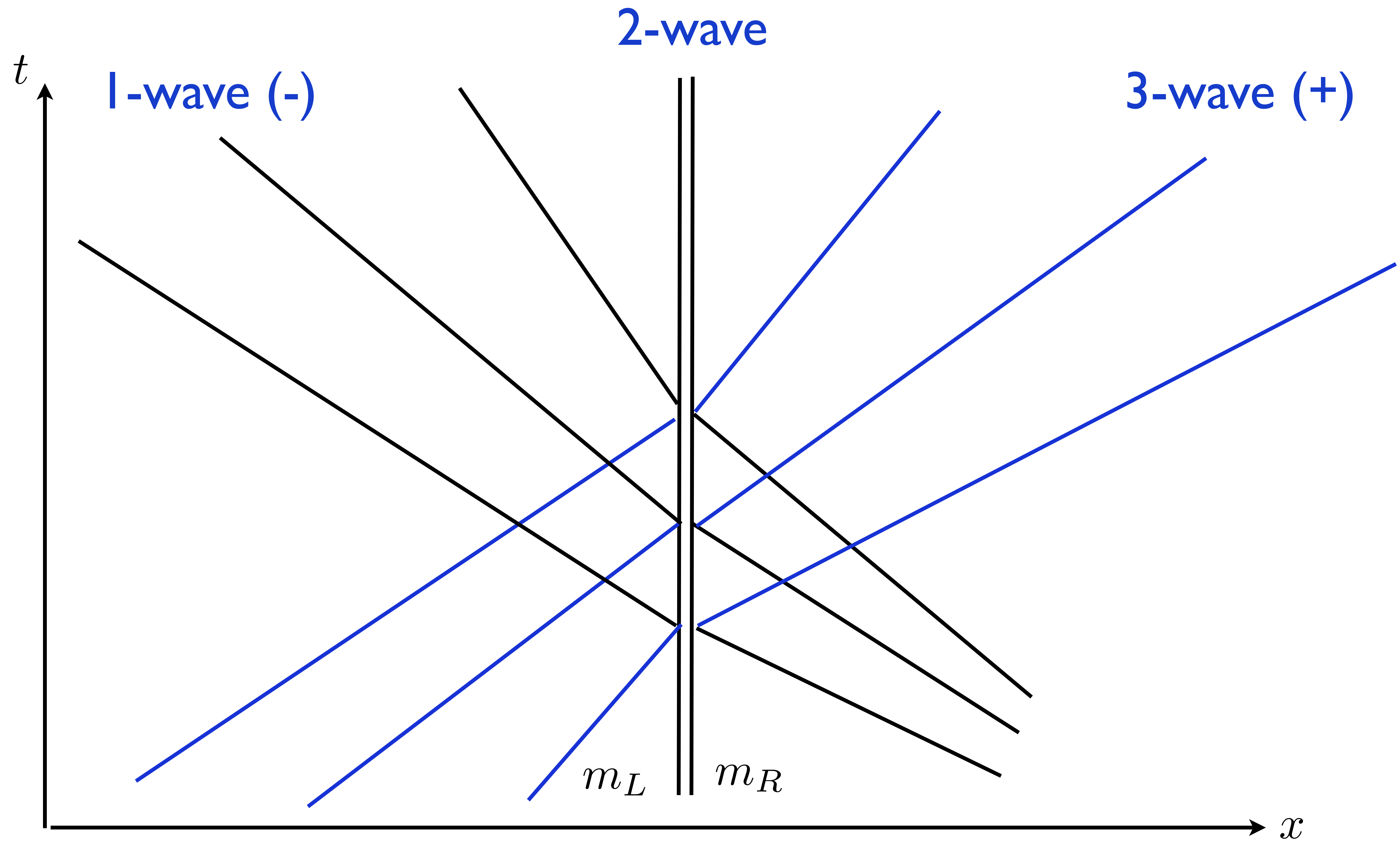
When the ENTROPY is CONSTANT...

Theorem: R/C character is preserved along backward and forward characteristics



The R/C character of a wave CAN CHANGE at an entropy jump...

For Example:



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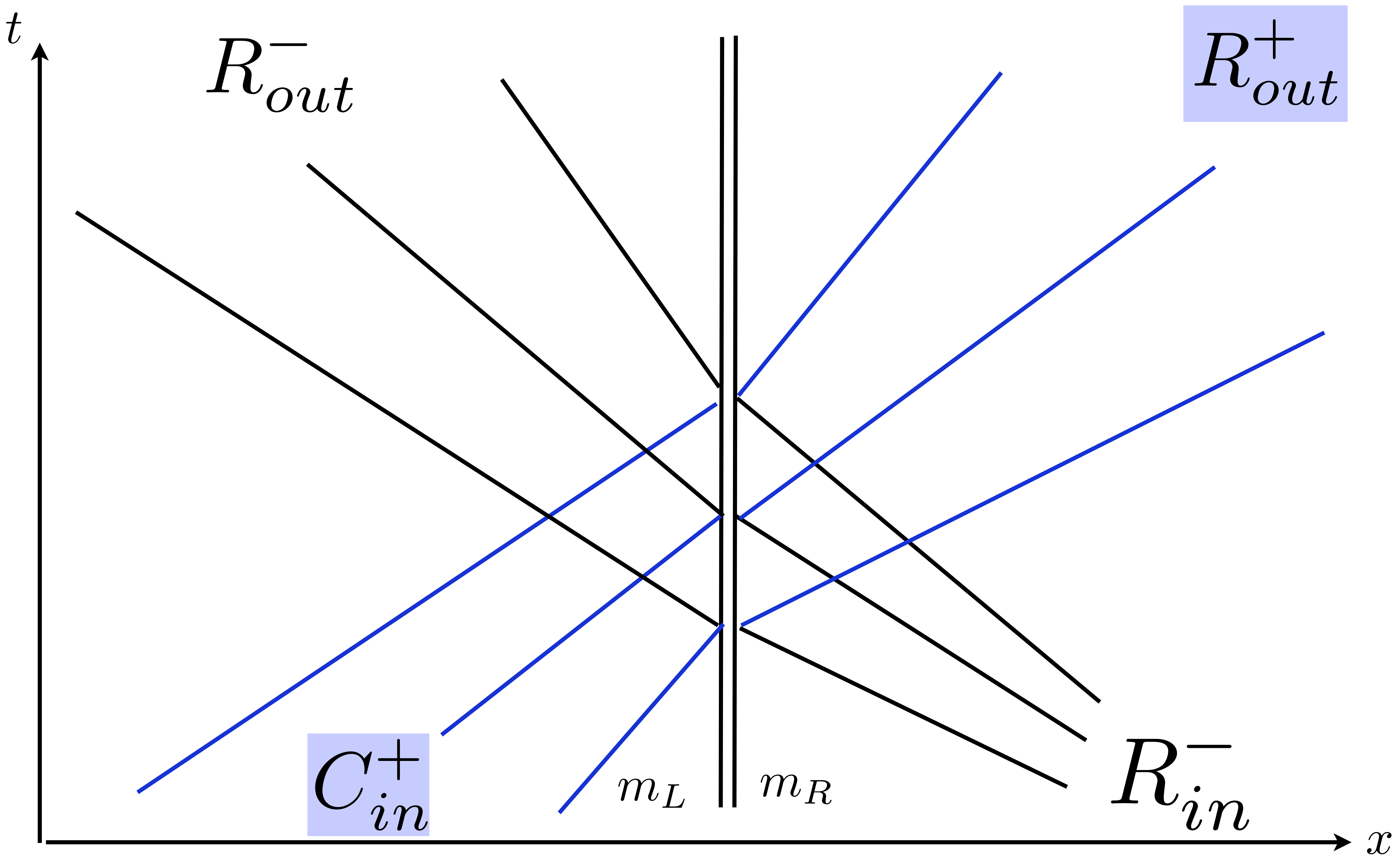

The R/C character of a wave CAN CHANGE at an entropy jump...

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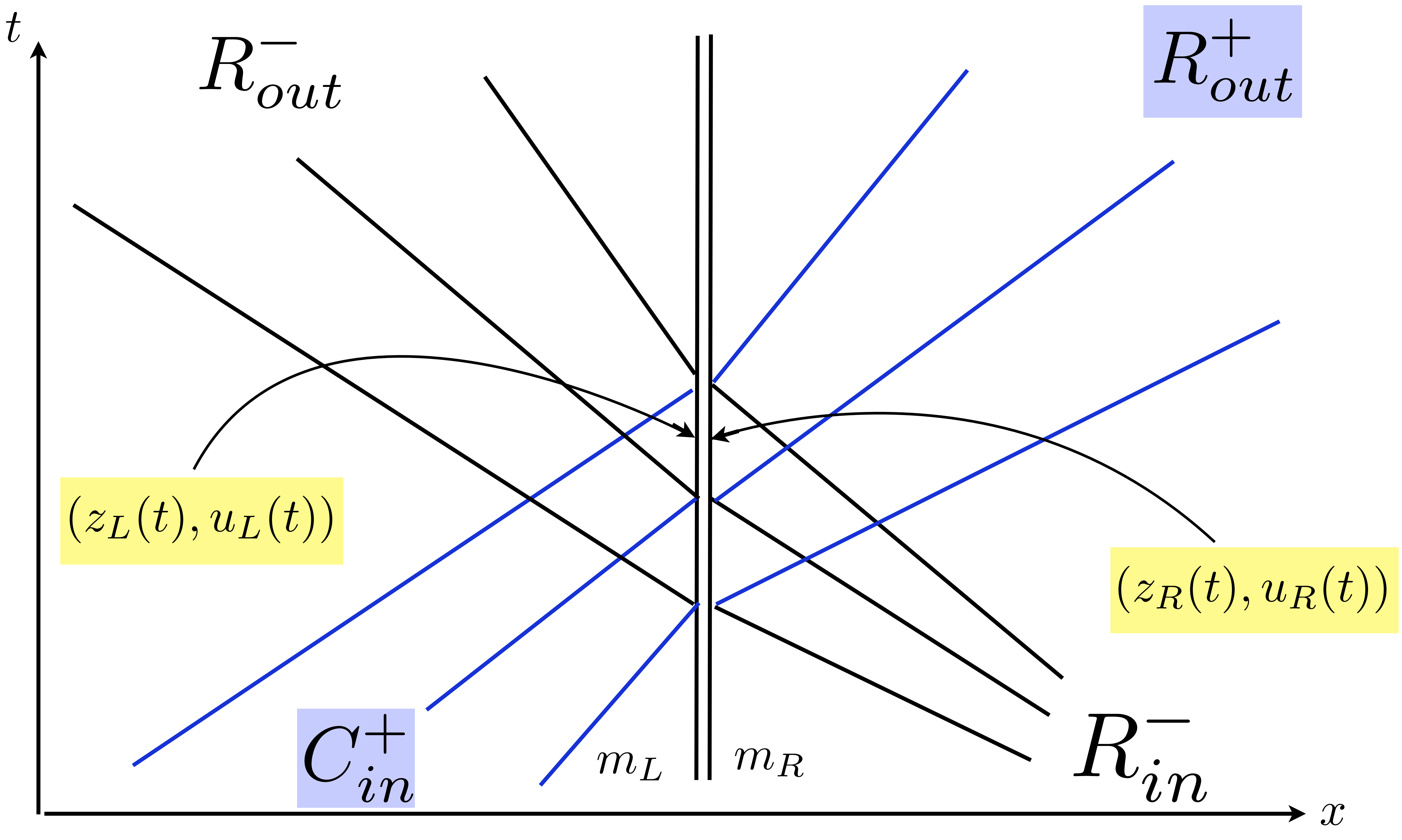


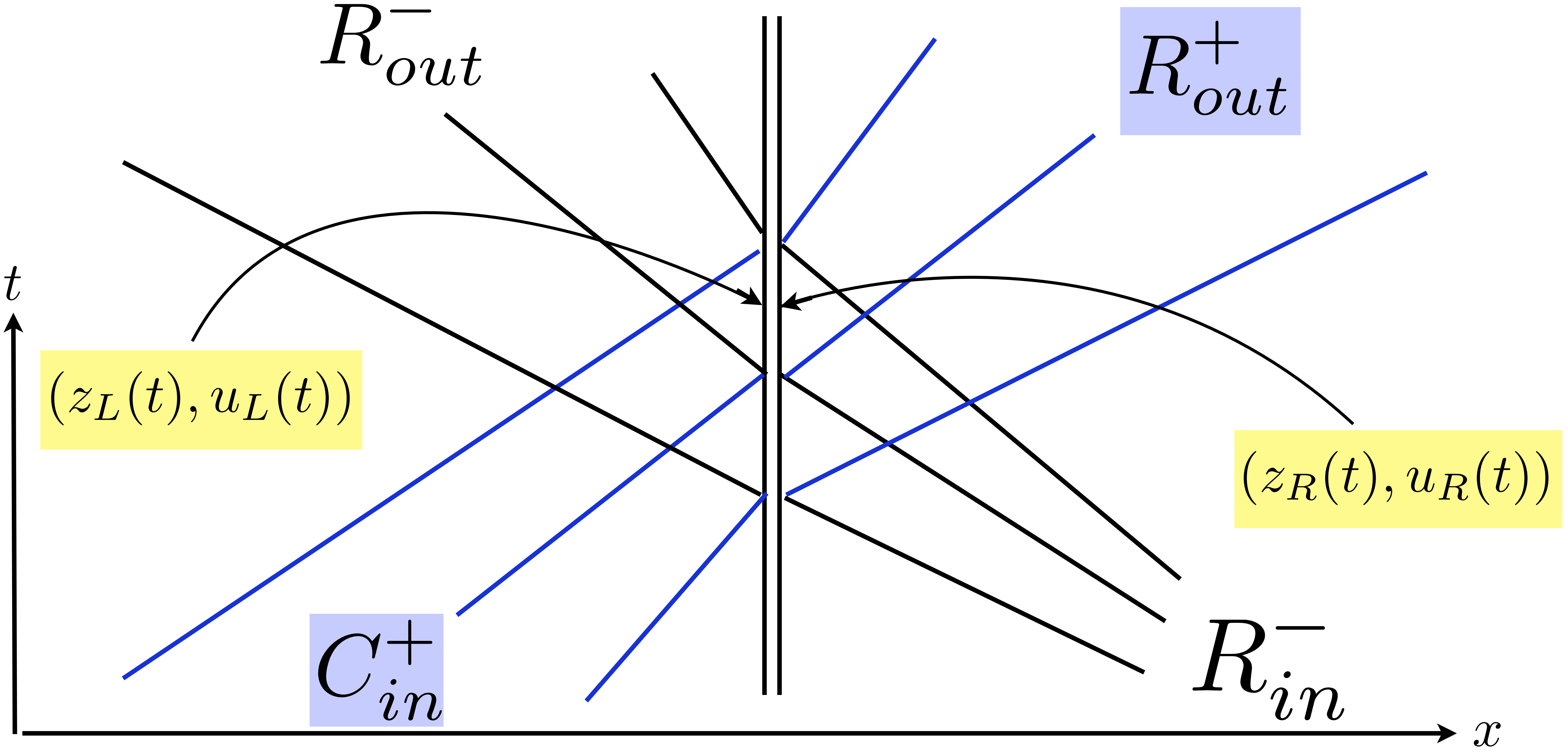
The Rankine-Hugoniot jump conditions characterize how R/C changes at an entropy jump...

Theorem 5. *The following inequalities characterize when a nonlinear wave changes its R/C value at an entropy jump:*

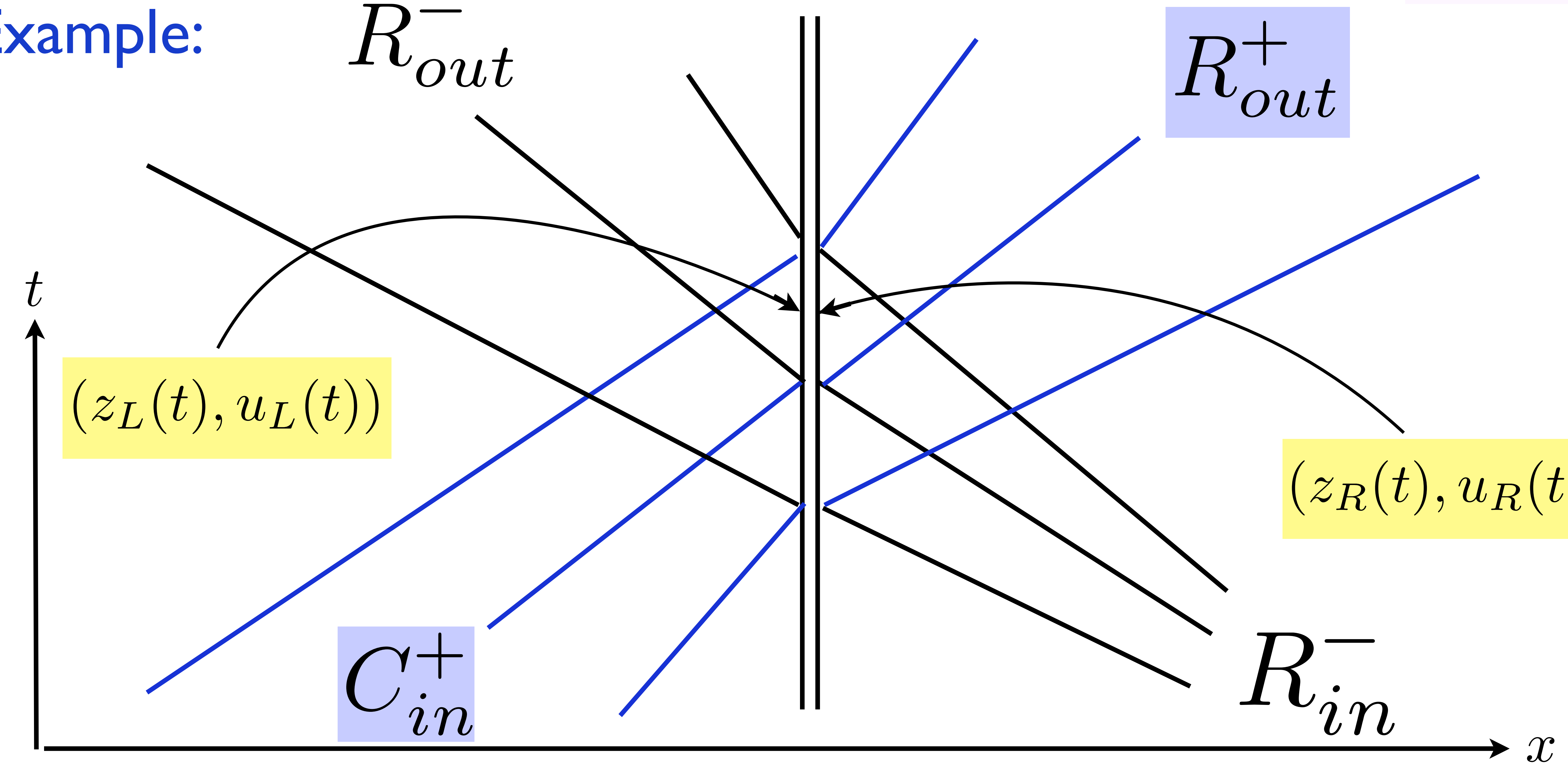
$$\begin{aligned}
R^-_{in} \to C^-_{out} \quad &\text{iff} \quad q^R_L m_L \dot{z}_L < \dot{u}_L < m_L \dot{z}_L, \\
C^-_{in} \to R^-_{out} \quad &\text{iff} \quad m_L \dot{z}_L < \dot{u}_L < q^R_L m_L \dot{z}_L, \\
R^+_{in} \to C^+_{out} \quad &\text{iff} \quad -q^R_L m_L \dot{z}_L < \dot{u}_L < -m_L \dot{z}_L, \\
C^+_{in} \to R^+_{out} \quad &\text{iff} \quad -m_L \dot{z}_L < \dot{u}_L < -q^R_L m_L \dot{z}_L.
\end{aligned}$$

$$q^R_L = \left(\frac{m_R}{m_L}\right)^{\frac{1}{\gamma}}$$

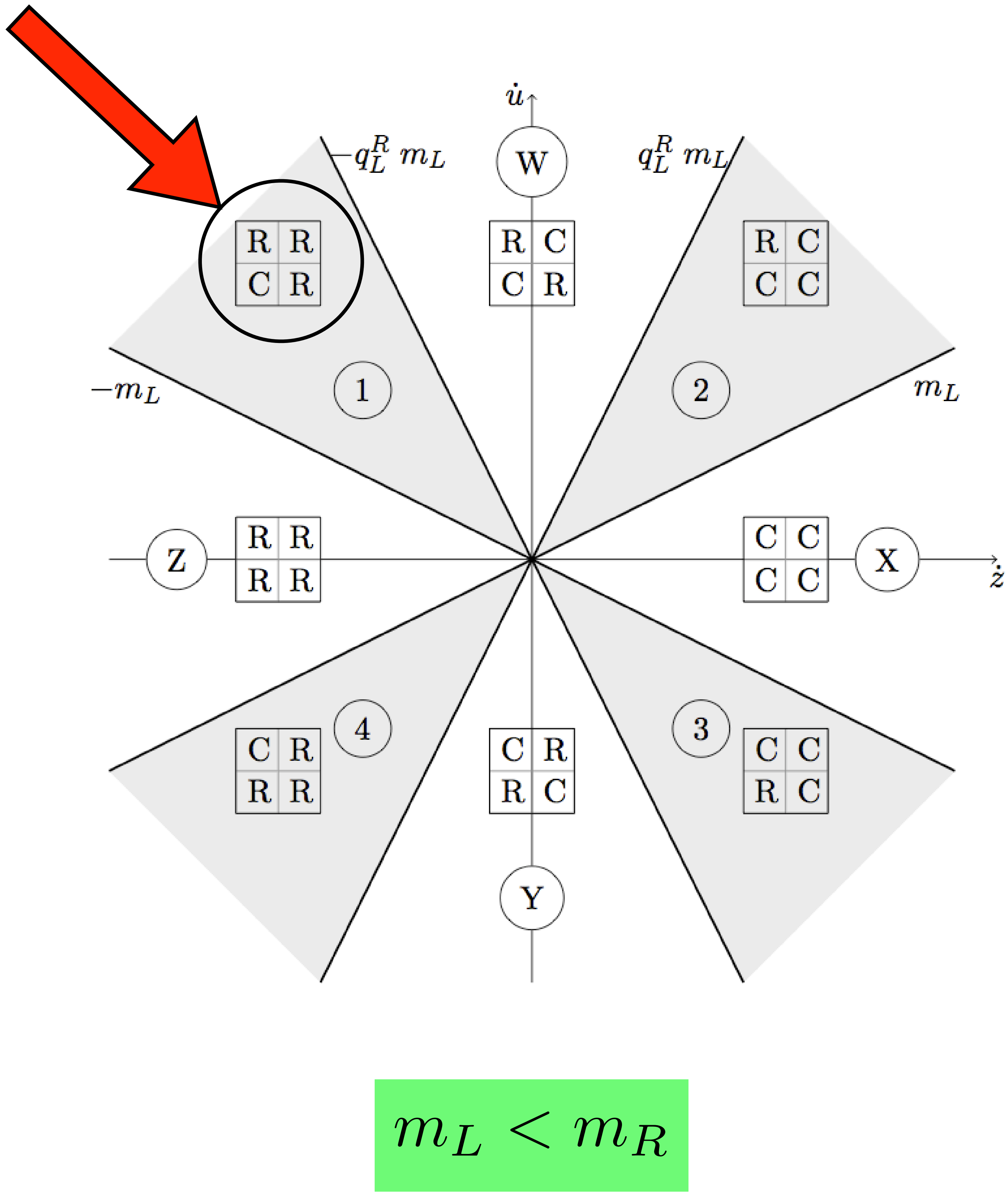
For Example:



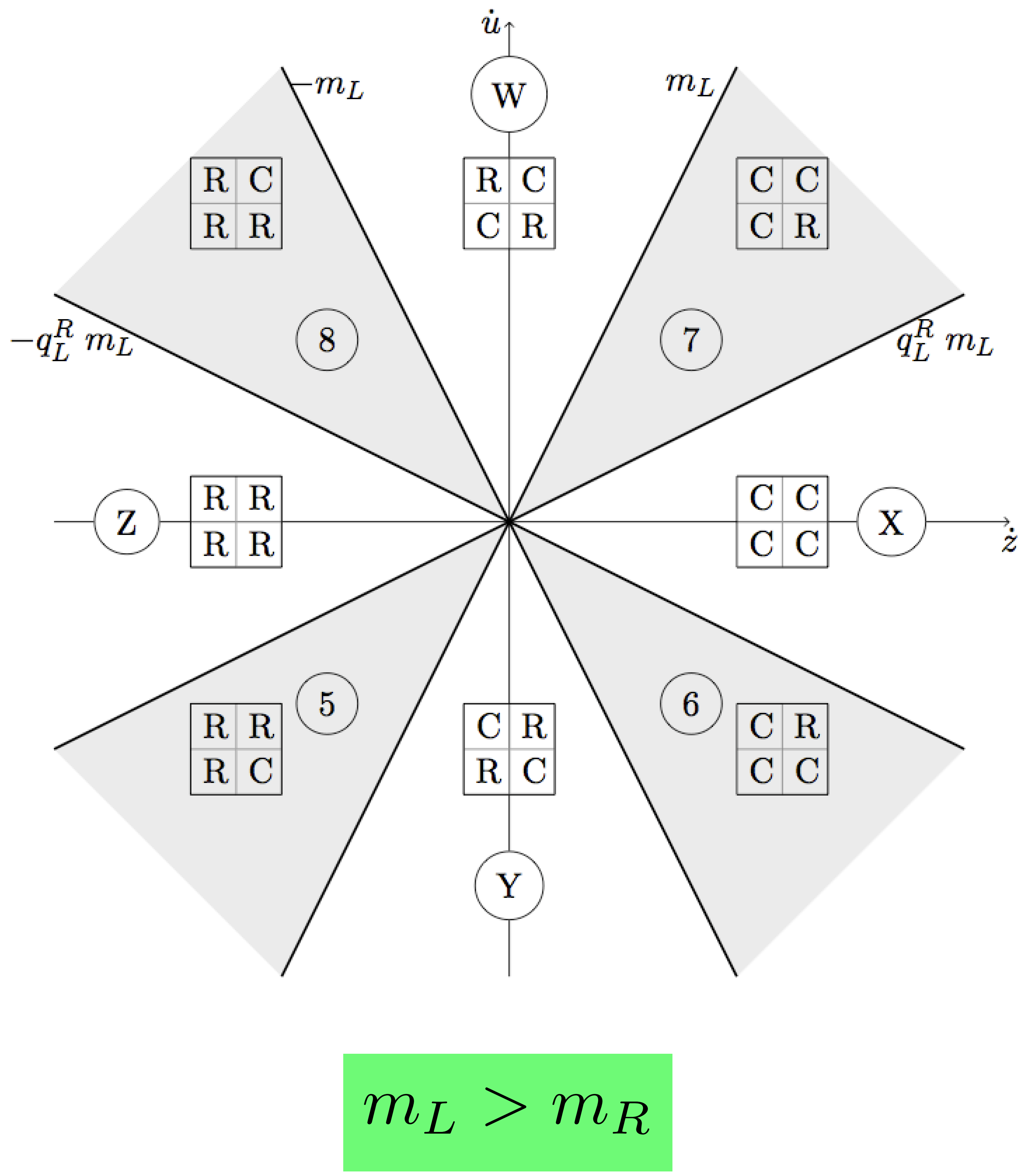
Example:



R^-_{out}
R^+_{out}
t
$(z_L(t), u_L(t))$
$(z_R(t), u_R(t))$
C^+_{in}
R^-_{in}
x

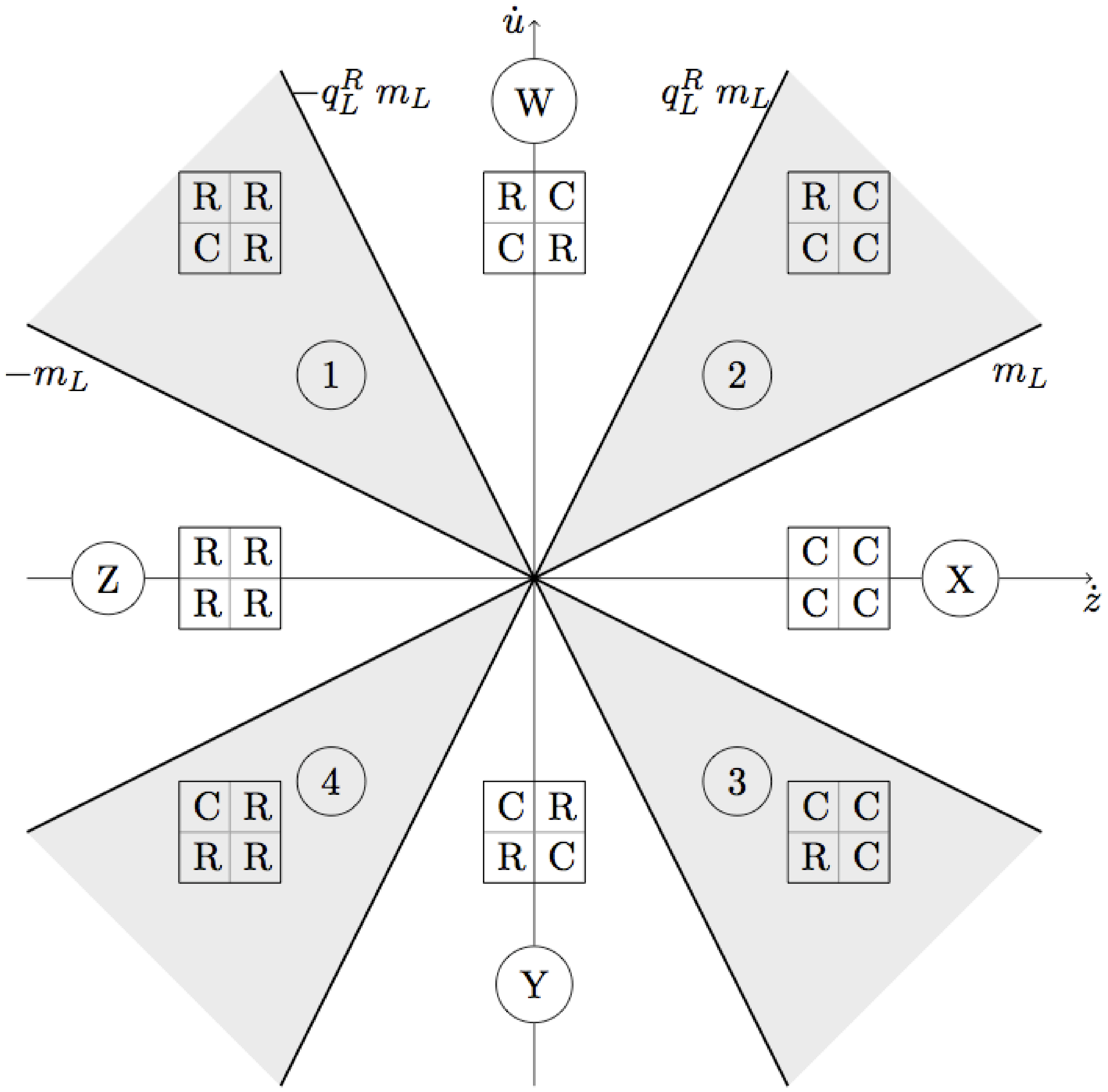


$\dot{u}$
$-q_L^R\, m_L$
$q_L^R\, m_L$
$-m_L$
m_L
W
X
Y
Z
1
2
3
4
R R / C R
R C / C R
R C / C C
R R / R R
C C / C C
C R / R R
C R / R C
C C / R C
$\dot{z}$
$m_L < m_R$



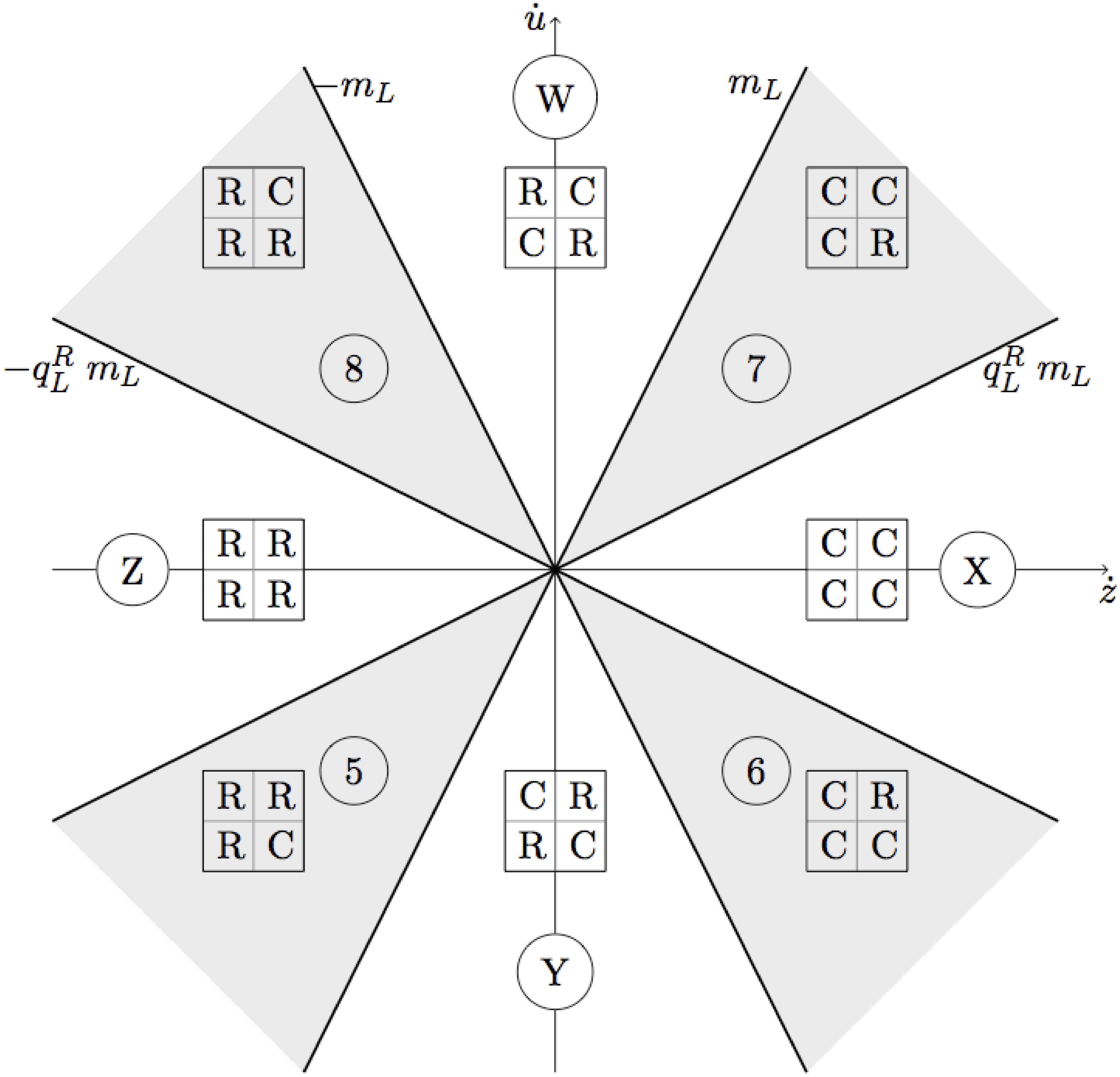
$\dot{u}$
$-m_L$
m_L
$-q_L^R\, m_L$
$q_L^R\, m_L$
W
X
Y
Z
5
6
7
8
R C / R R
R C / C R
C C / C R
R R / R R
C C / C C
R R / R C
C R / R C
C R / C C
$\dot{z}$
$m_L > m_R$

CONCLUDE: we can determine the R/C changes across the entropy jump from inequalities on the time derivative of the solution at the left hand side of the entropy jump alone. Doing this in all cases yields the following theorem.



Tangent space showing all possible R/C wave structures when

$m_L < m_R$



Tangent space showing all possible R/C wave structures when

$m_L > m_R$

- Note: All 16 possible interaction squares appear **EXCEPT** ones where R/C value of both waves change simultaneously:

Not possible:

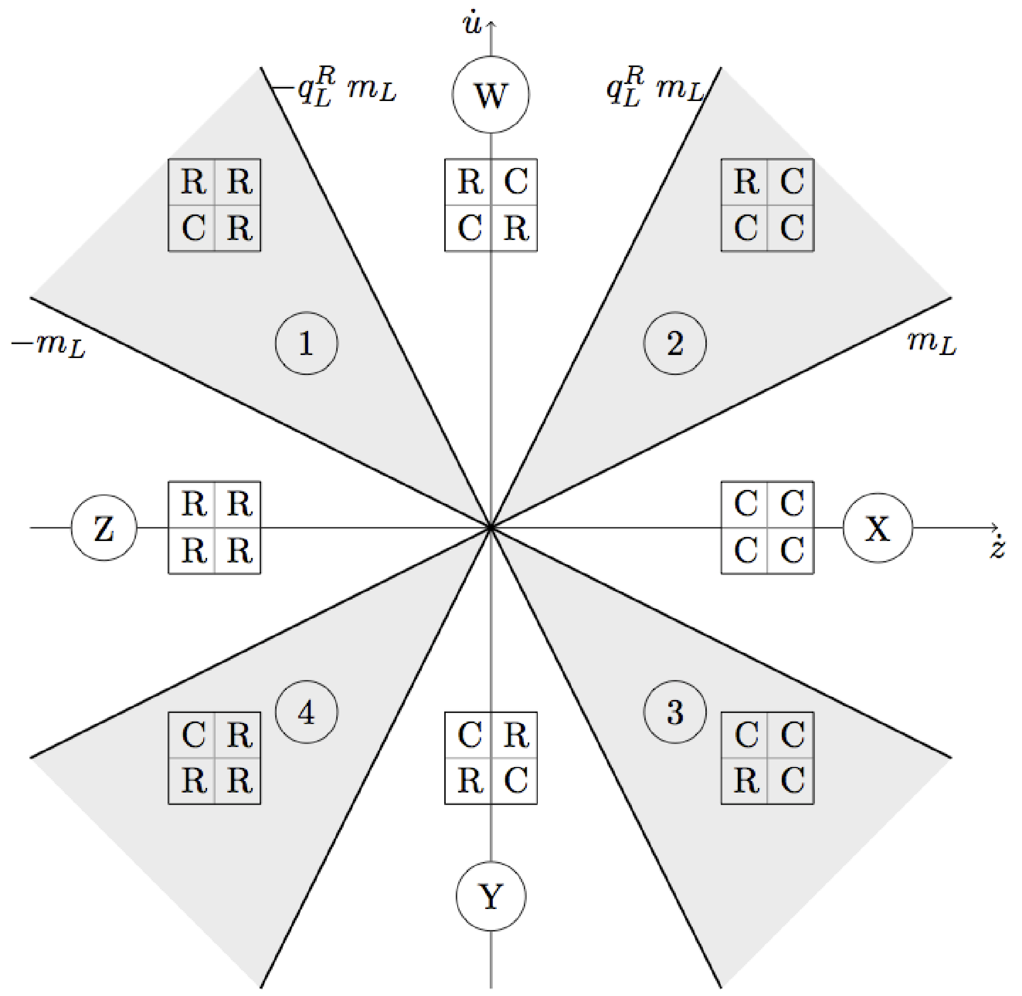
R	R
C	C

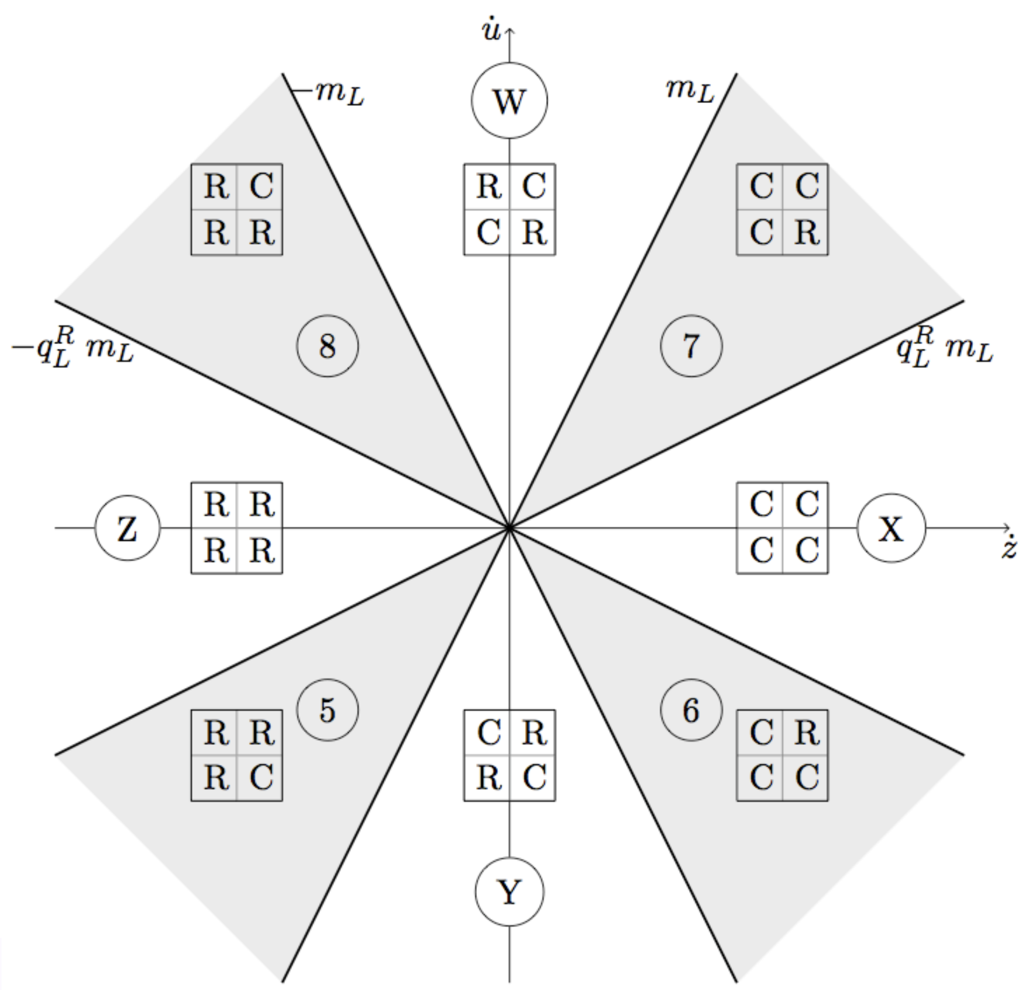
C	C
R	R

R	C
R	C

C	R
C	R

- CONCLUDE: A wave in one family can change its R/C value only in the presence of a wave of the opposite family that transmits its R/C value





The Simplest Possible Periodic Structure that Balances Compression and Rarefaction

DEFN: We we say that a periodic pattern of R's and C's is

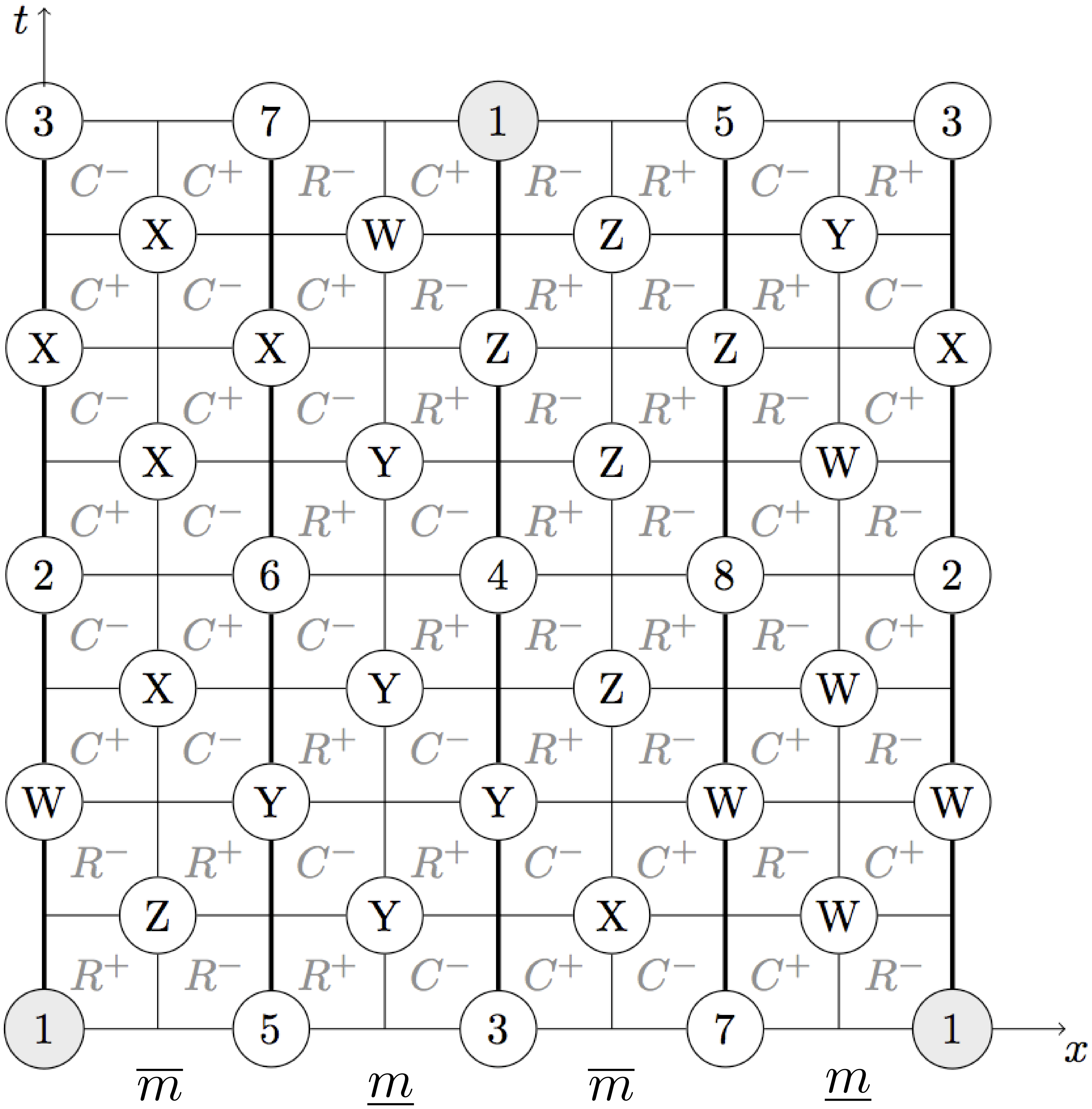
CONSISTENT

IF:

(1) R/C is balanced along every 1,3-characteristic

(2) The interaction squares at entropy jumps $m_L > m_R, m_L > m_R$ are consistent with squares in the $m_L > m_R, m_L > m_R$ R/C diagrams, respectively

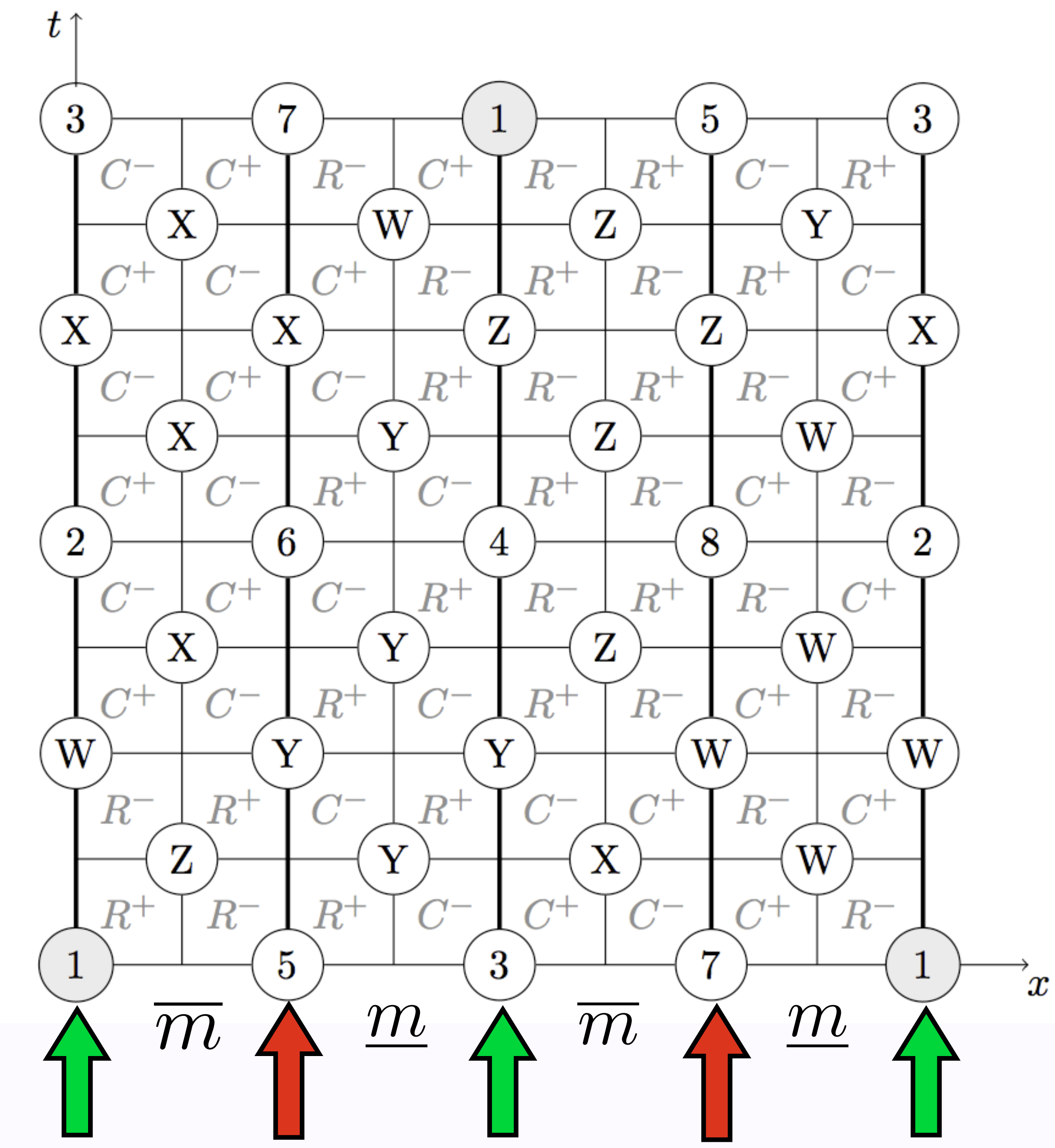
The simpest consistent R/C pattern



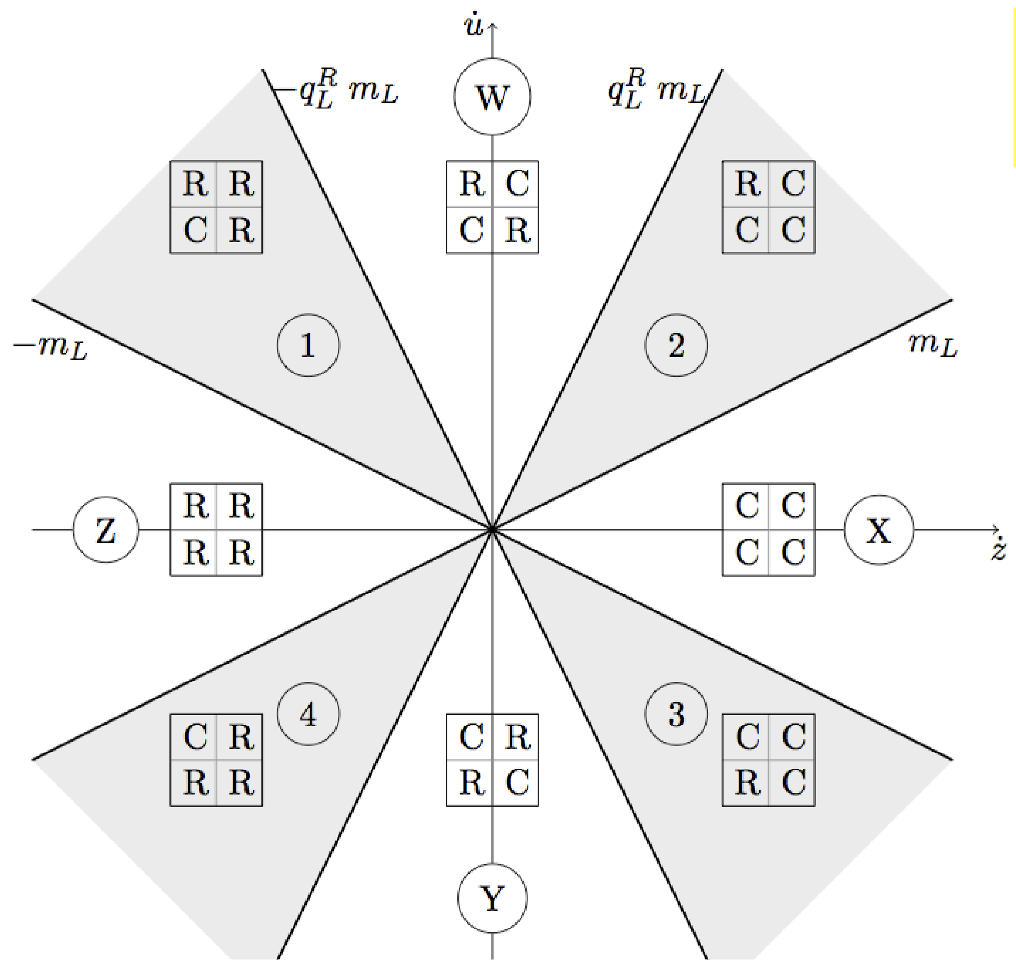
“Extend periodically”

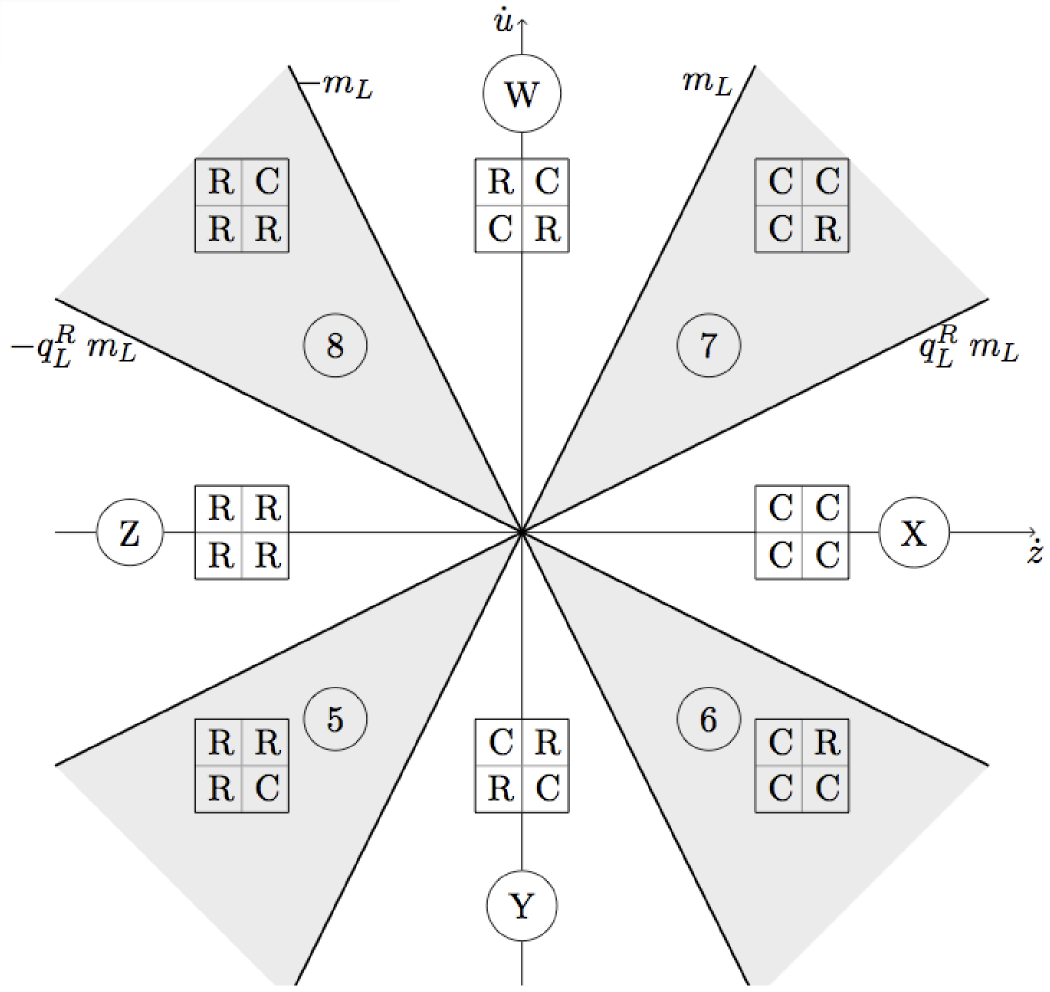
$\overline{m} > \underline{m}$

Each number above is consistent with the numbered interaction below



$\overline{m} > \underline{m}$

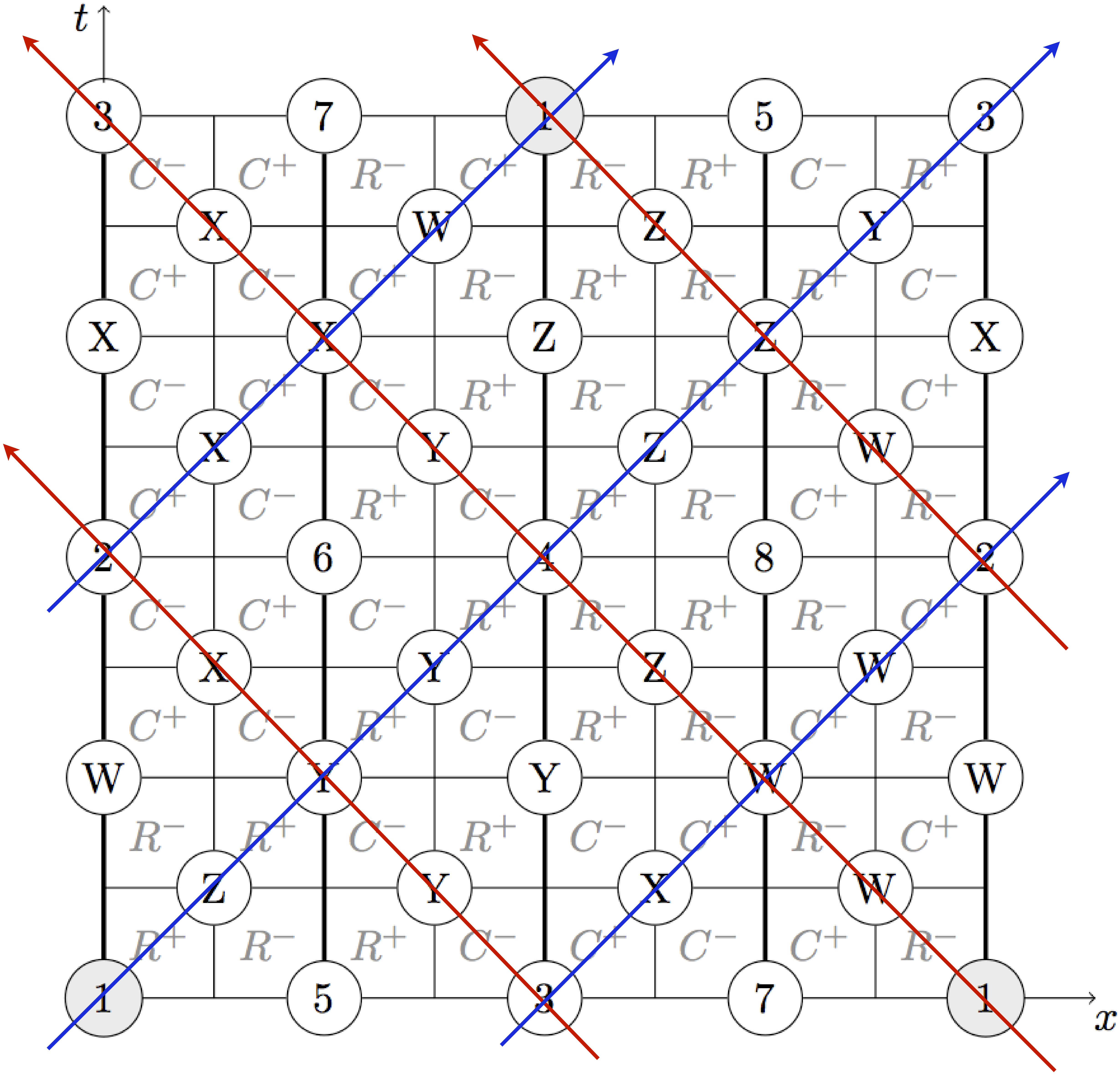




$\underline{m} = m_L < m_R = \overline{m}$

$\overline{m} = m_L > m_R = \underline{m}$

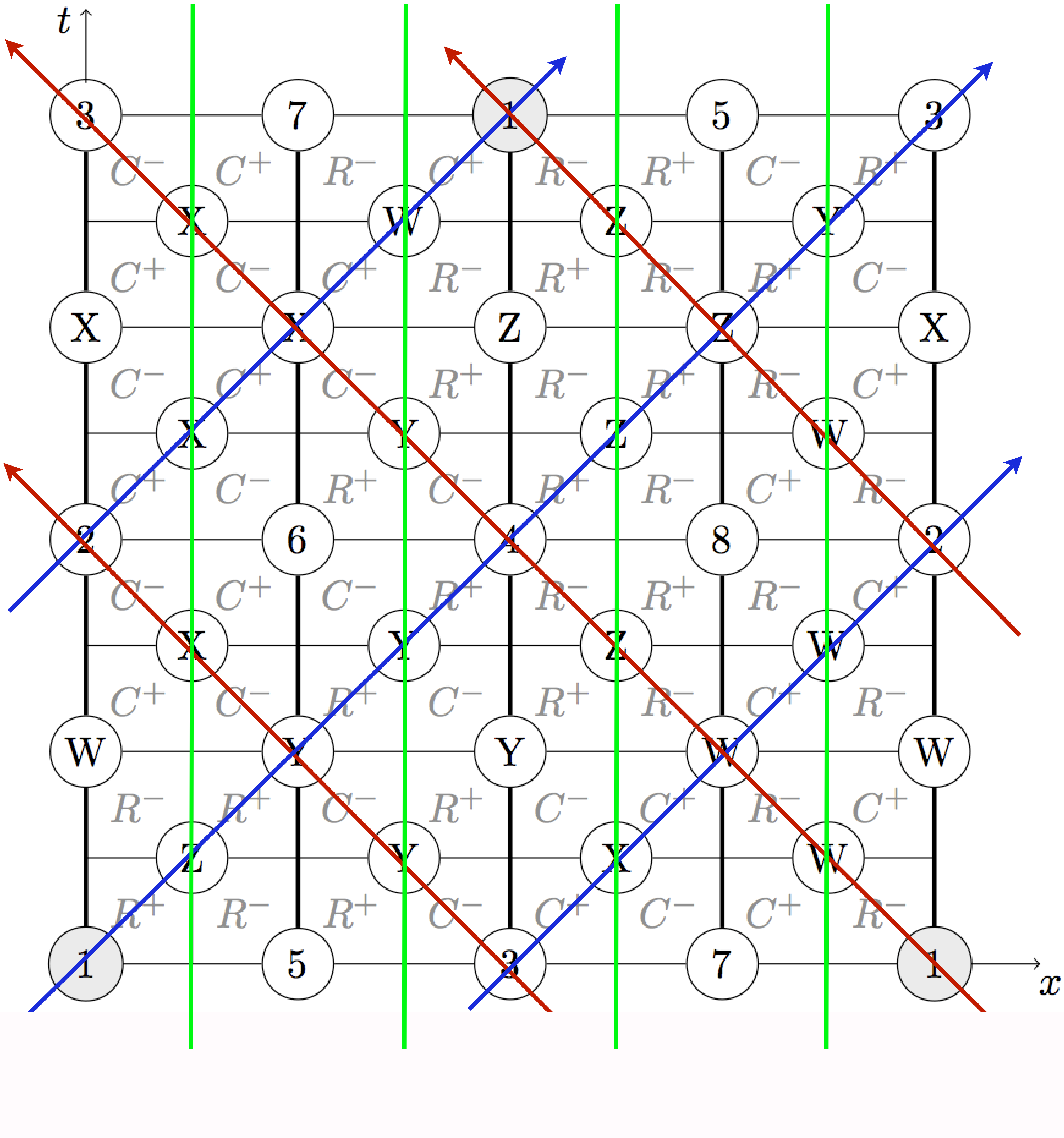
Each 1,3-characteristic traverses 8-C's and 8-R's before returning



The lettered interactions at constant entropy jump transmit R/C

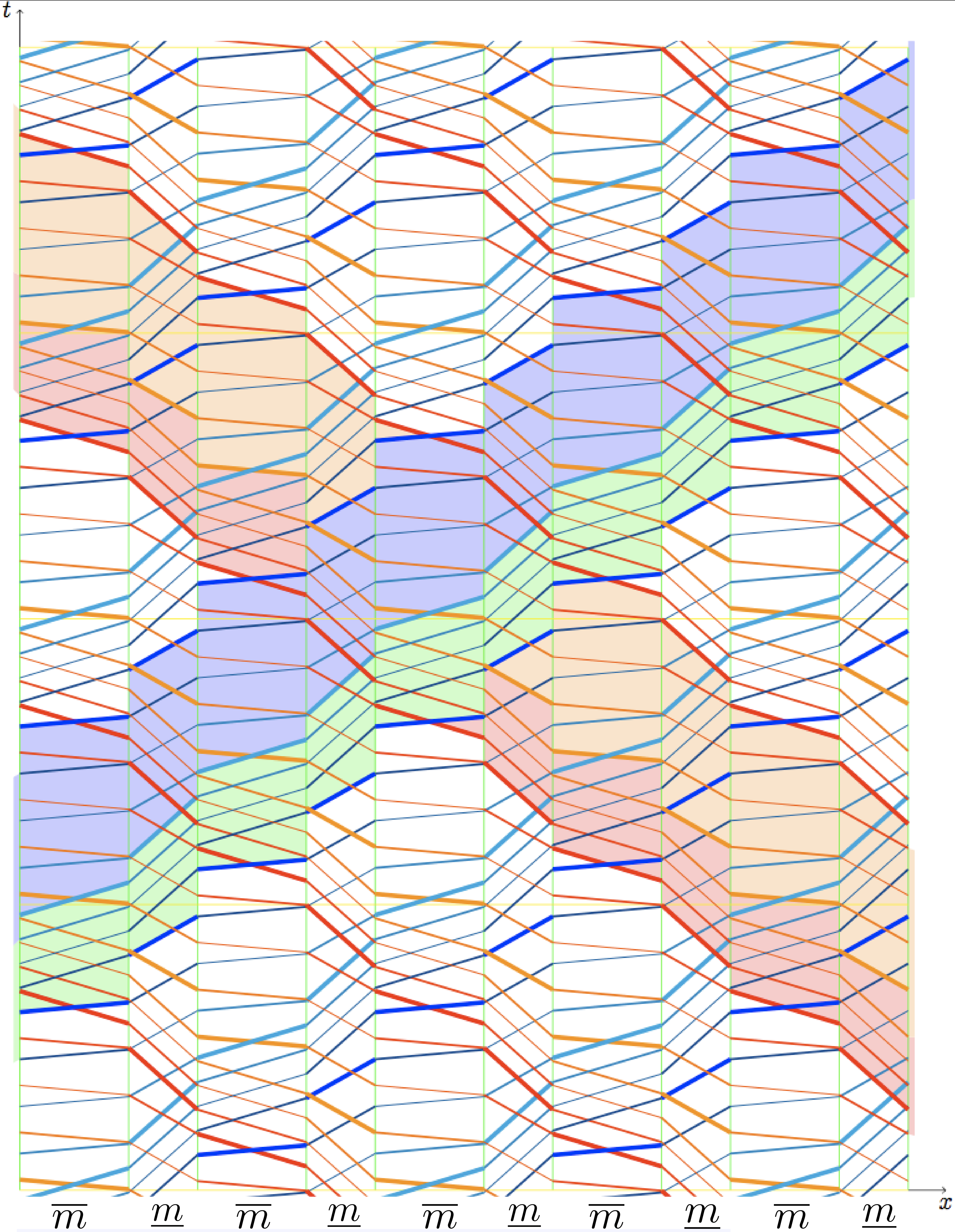
Identifying these

1,3-characteristics traverse 4-C's and 4-R's before returning



- Inspection of the change in $\dot{u}, \dot{z}$ along the axes indicates that the solution is consistent with elliptical rotation of the solution along the entopy jumps

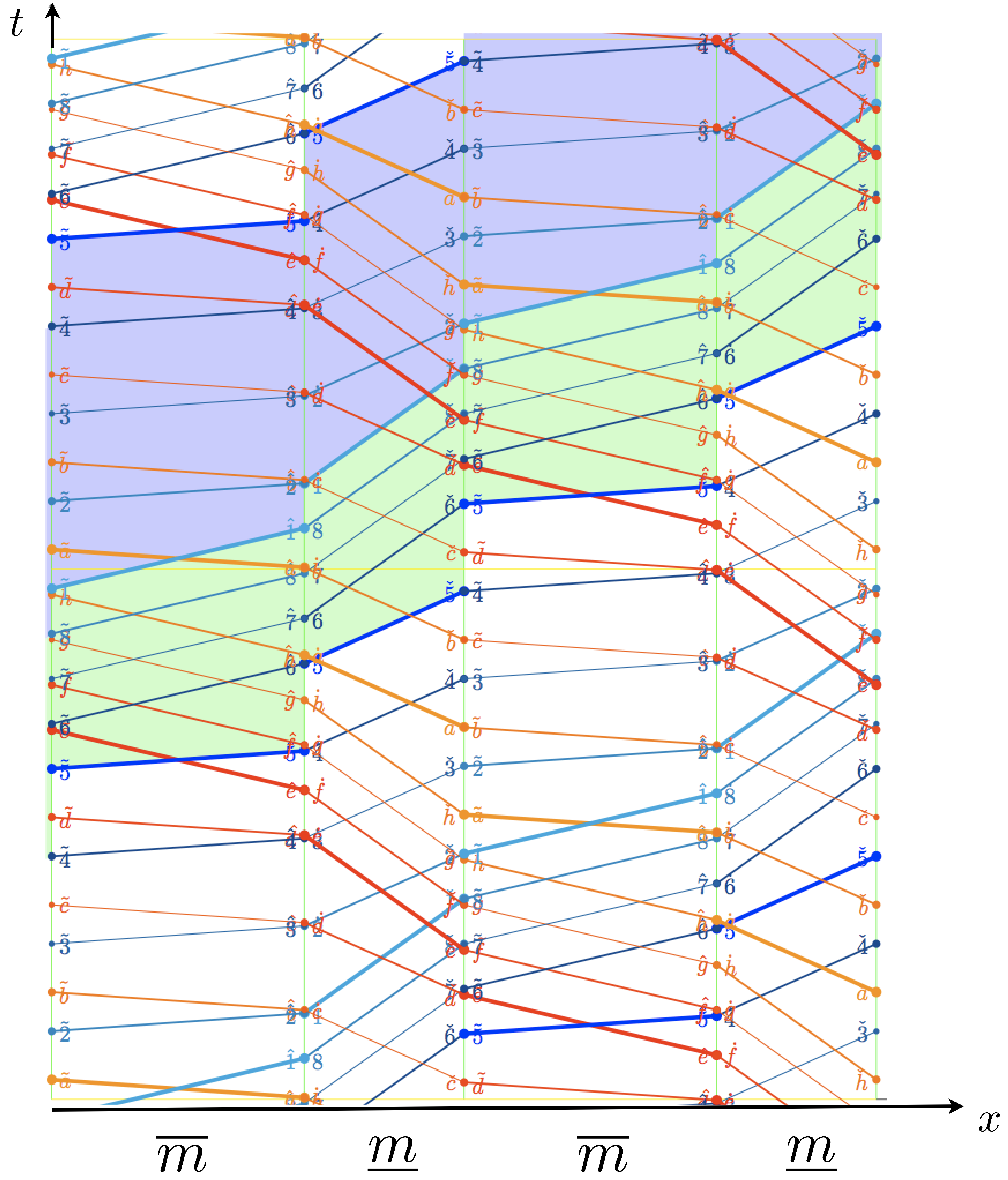
- This leads to the following consistent cartoon of the simplest possible periodic solution



The simplest possible periodic structure

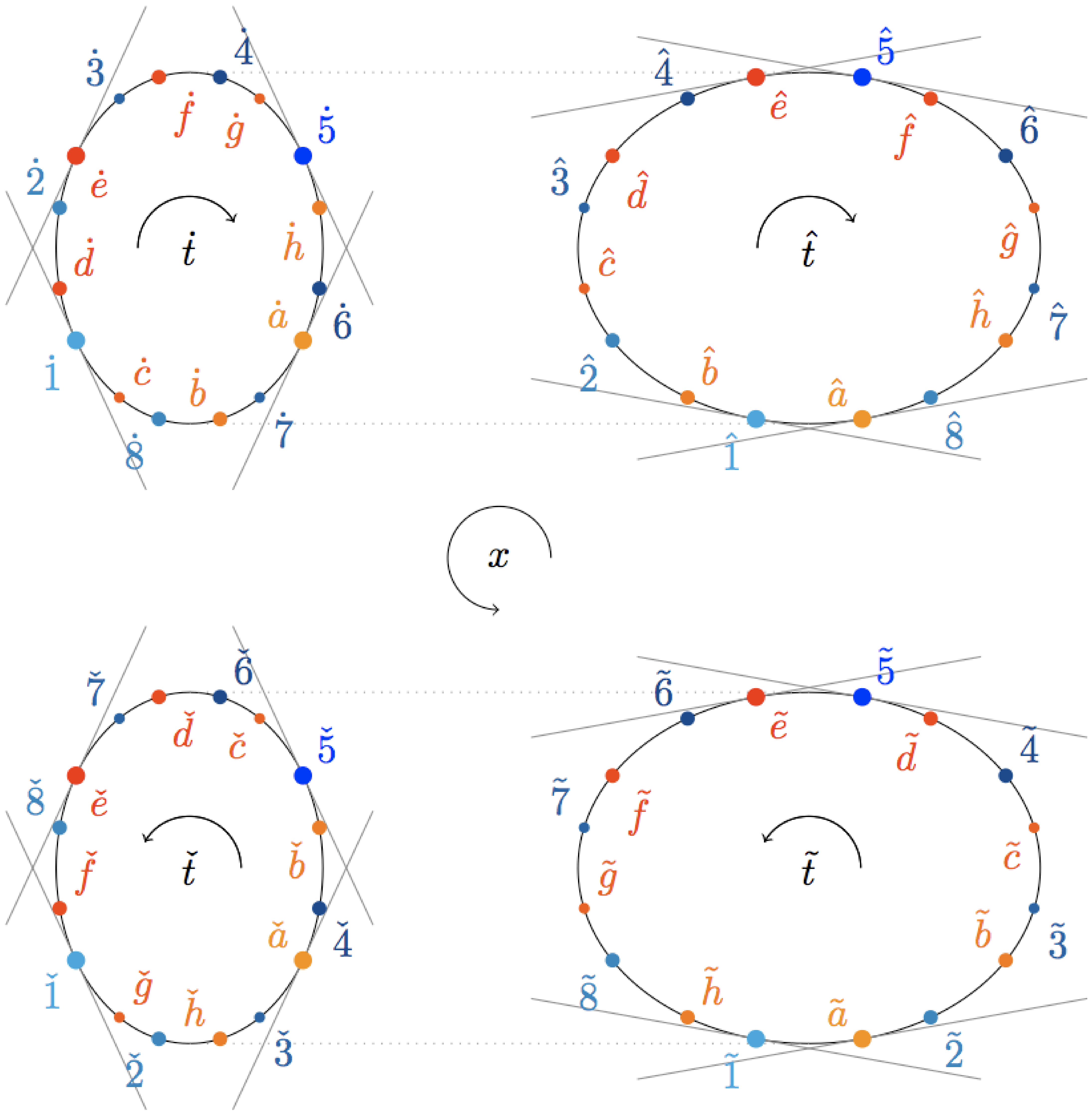
$$\overline{m} > \underline{m}$$

- Labeling the states along the entropy jumps and plotting them according to the change in $\dot{u}, \dot{z}$ indicates that the solution is consistent with elliptical rotation of the solution along the entopy jumps

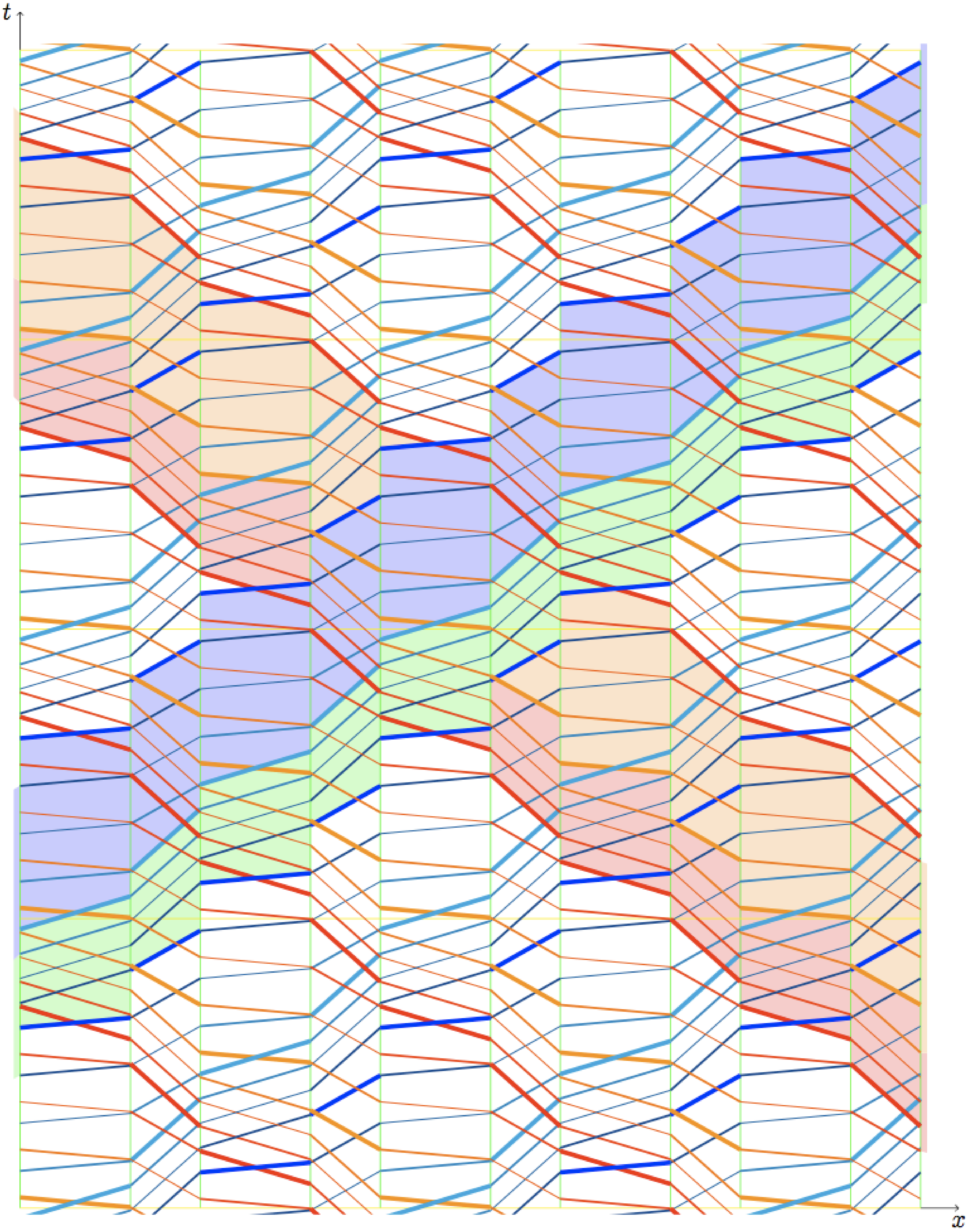


Labeling the states by numbers and letters

$\overline{m} > \underline{m}$



Ellipses showing periodicity in (z,u)-plane

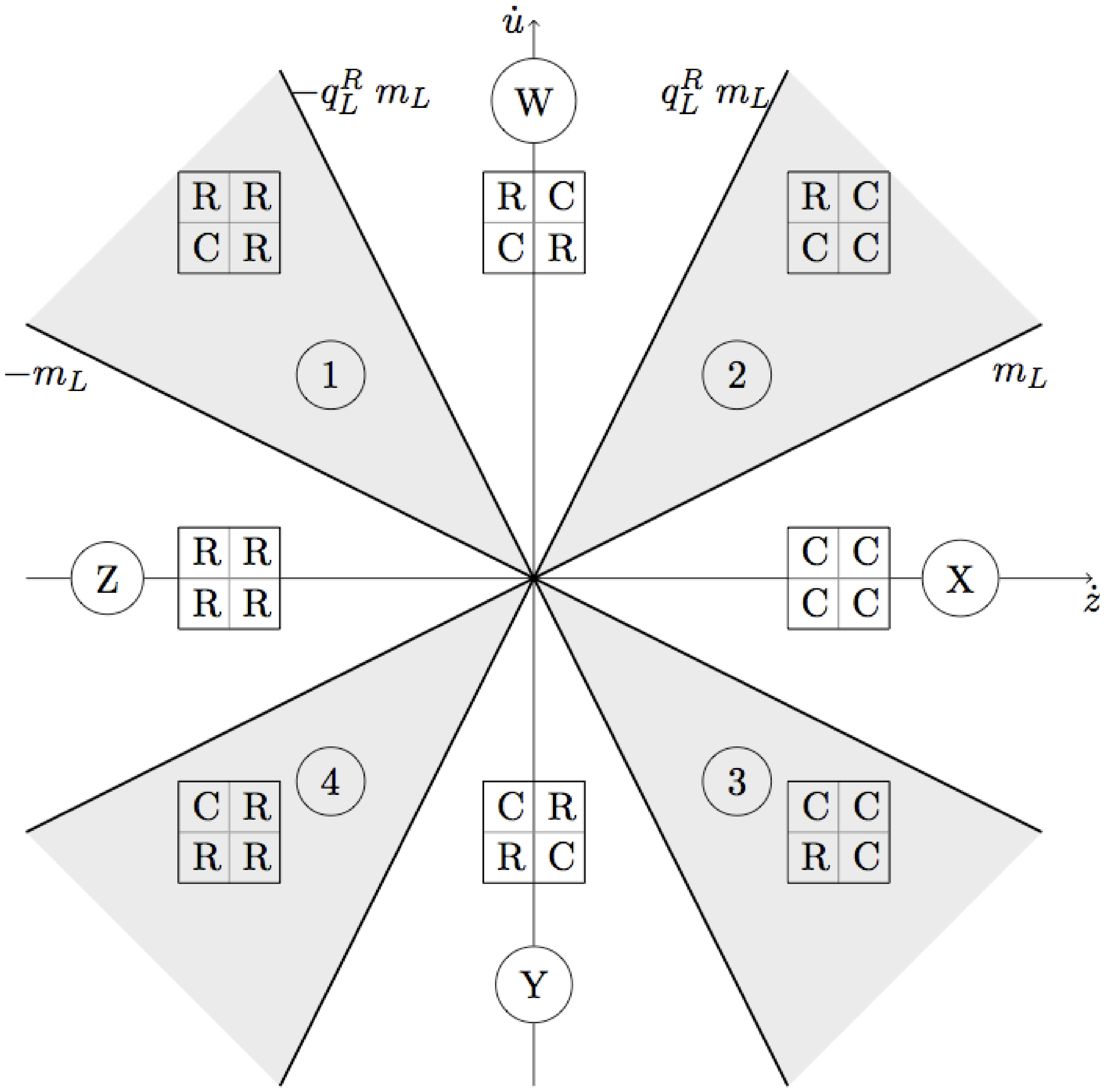


The global nonlinear periodic structure

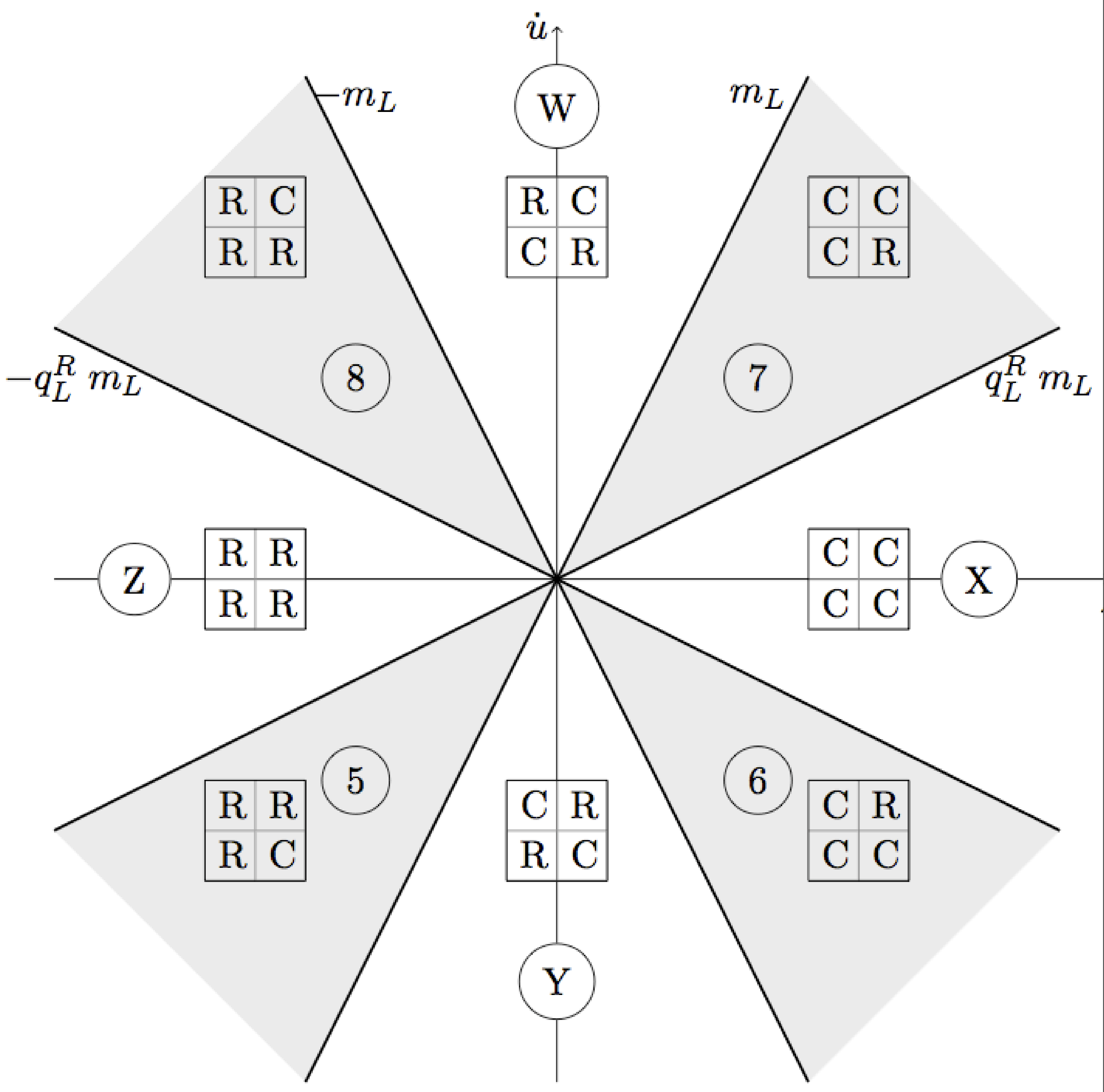
- NOTE: There is an LR-asymmetry:

Interaction squares for $m_L < m_R$

differ from squares for $m_L > m_R$



$m_L < m_R$



$m_L > m_R$

LR-asymmetry

↓

Max/Min Characteristics

always jump

UP

going

OUTWARD

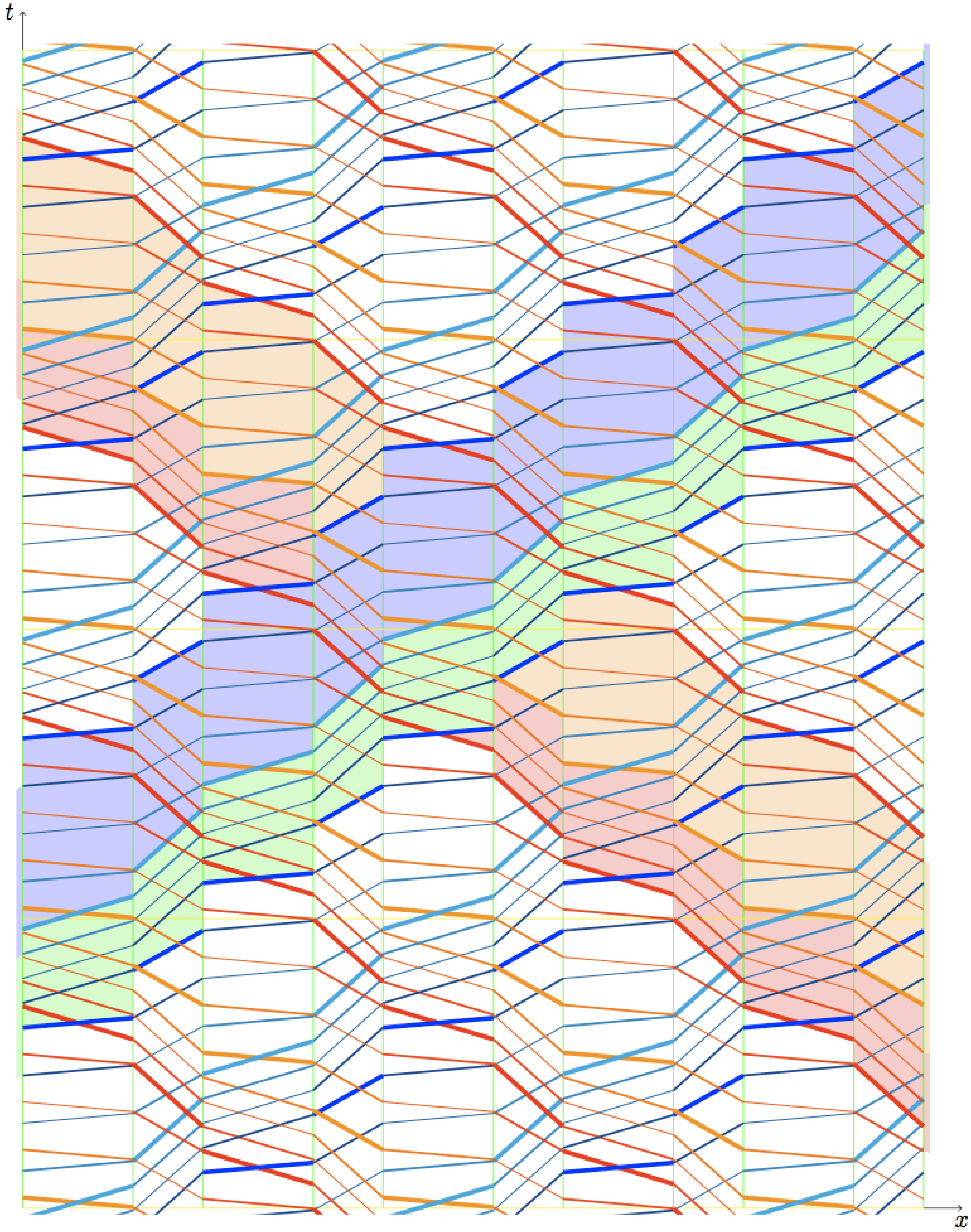
↓

The 1,3-wave crests are

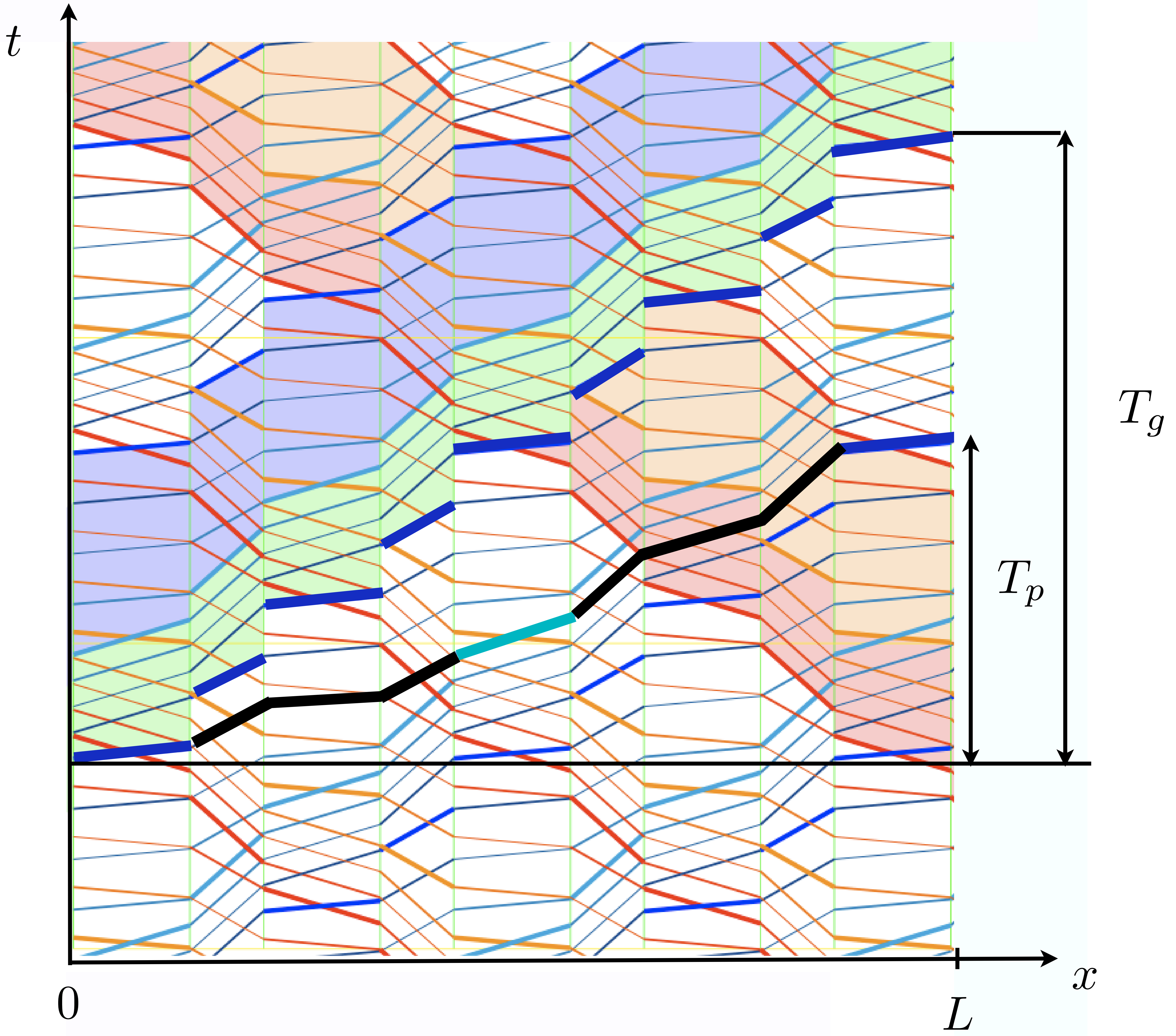
SUBSONIC

NOT

SUPERSONIC



The 1,3-Max/Min-Characteristics jump UP going OUTWARD ➡ SUBSONIC



v_g = speed of the wave crests<speed of the sound waves = v_p

$$v_g = \frac{L}{T_g} < v_p = \frac{L}{T_p}$$

The speed of the wave crests is
like an effective
"Group-Velocity"

The characteristic=sound speed like a
"Phase-Velocity"

.....

Max/Min-Characteristics jump UP

↓

Group-Velocity < Phase-Velocity

PART II. Realize solutions with this simplest structure at a Linearized Level

Set up a Perturbation Problem to prove the existence of nearby nonlinear solutions by the Implicit Function Theorem in Banach Spaces

Problem: Resonances/Small Divisors

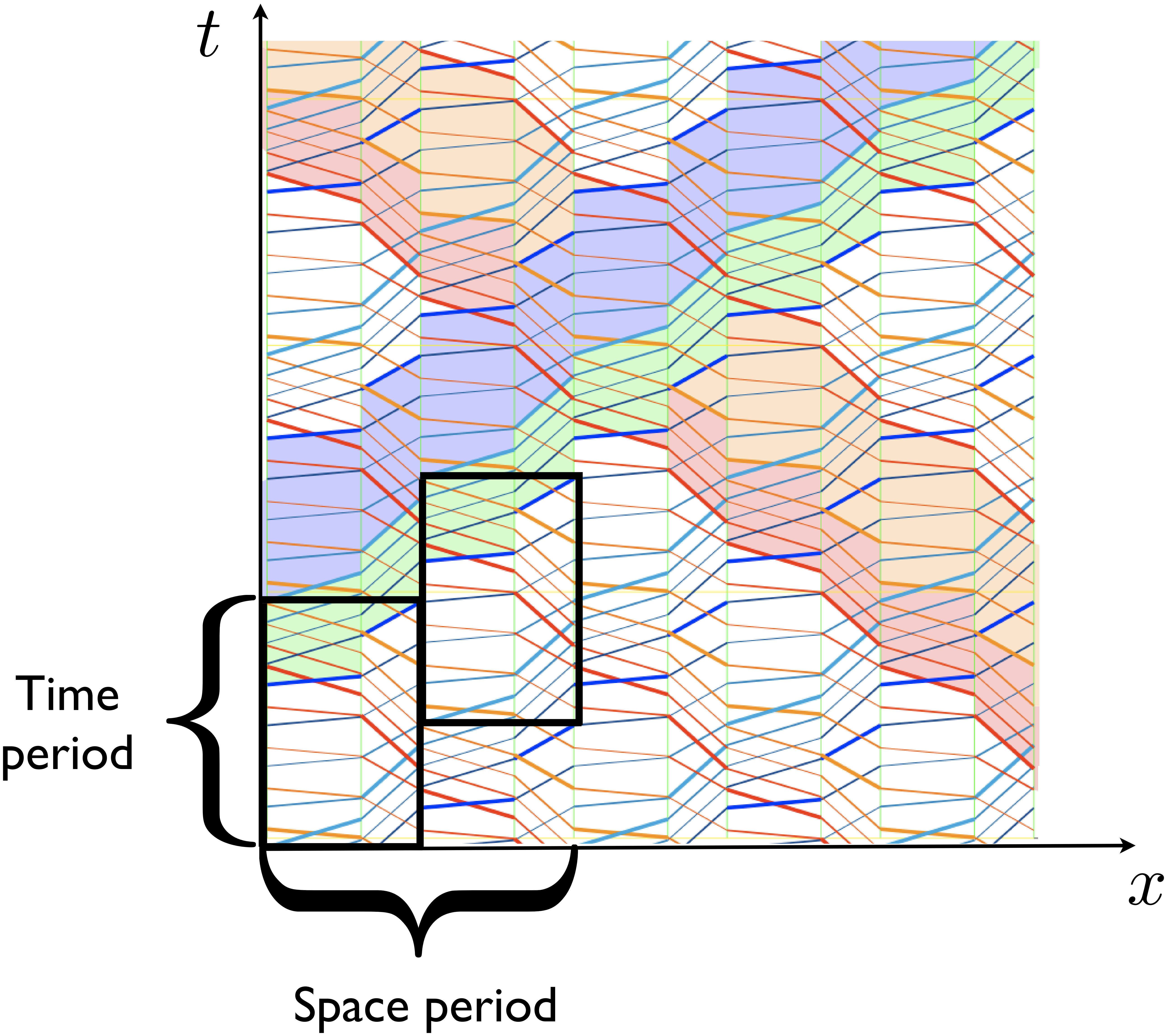
The nonlinear eigenvalue problem as a perturbation of a linear problem

OBSERVE: The simplest periodic structure imposes two special symmetries:

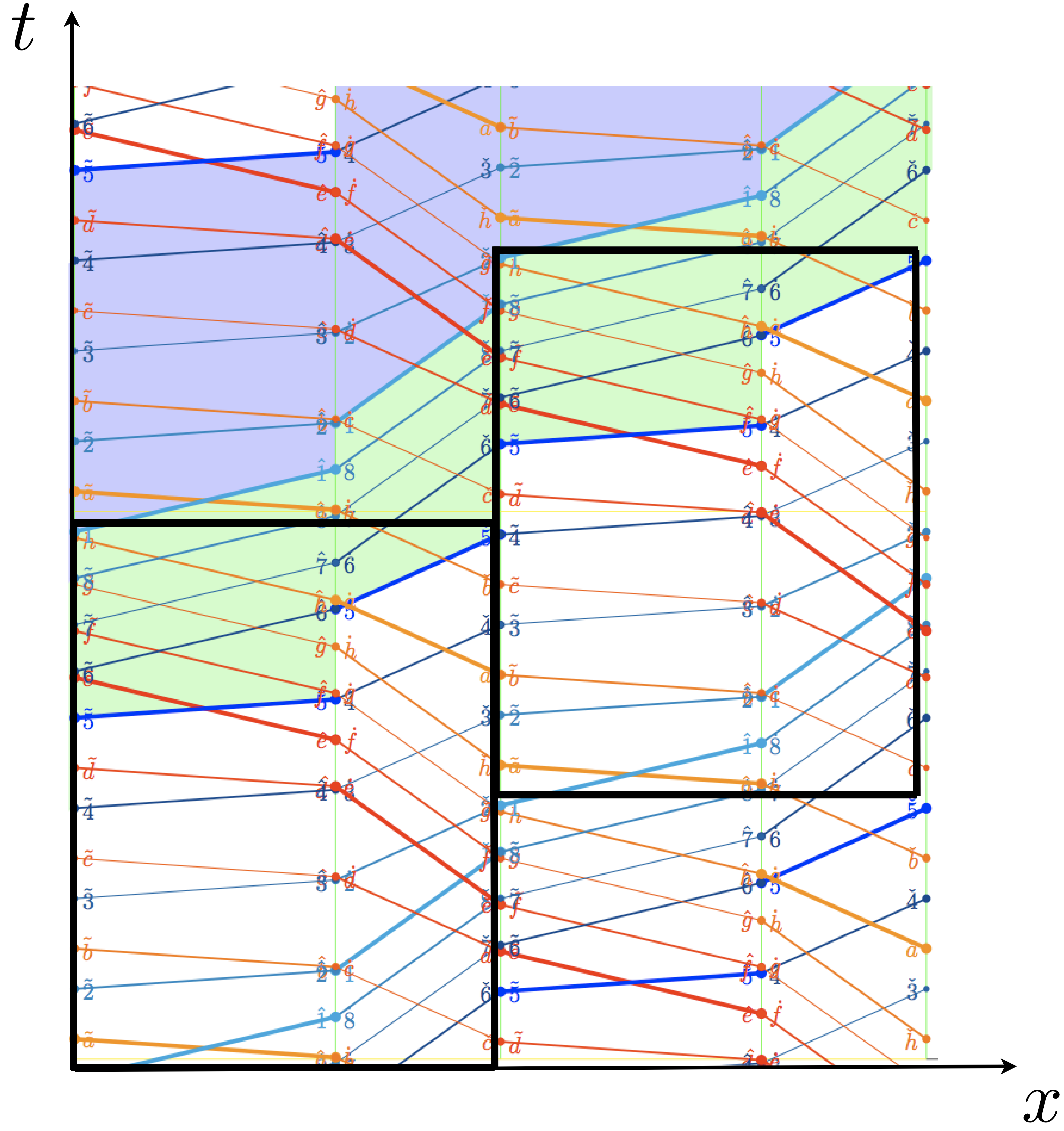
(1) Periodicity in space

(2) Max/Min-characteristics JOIN UP

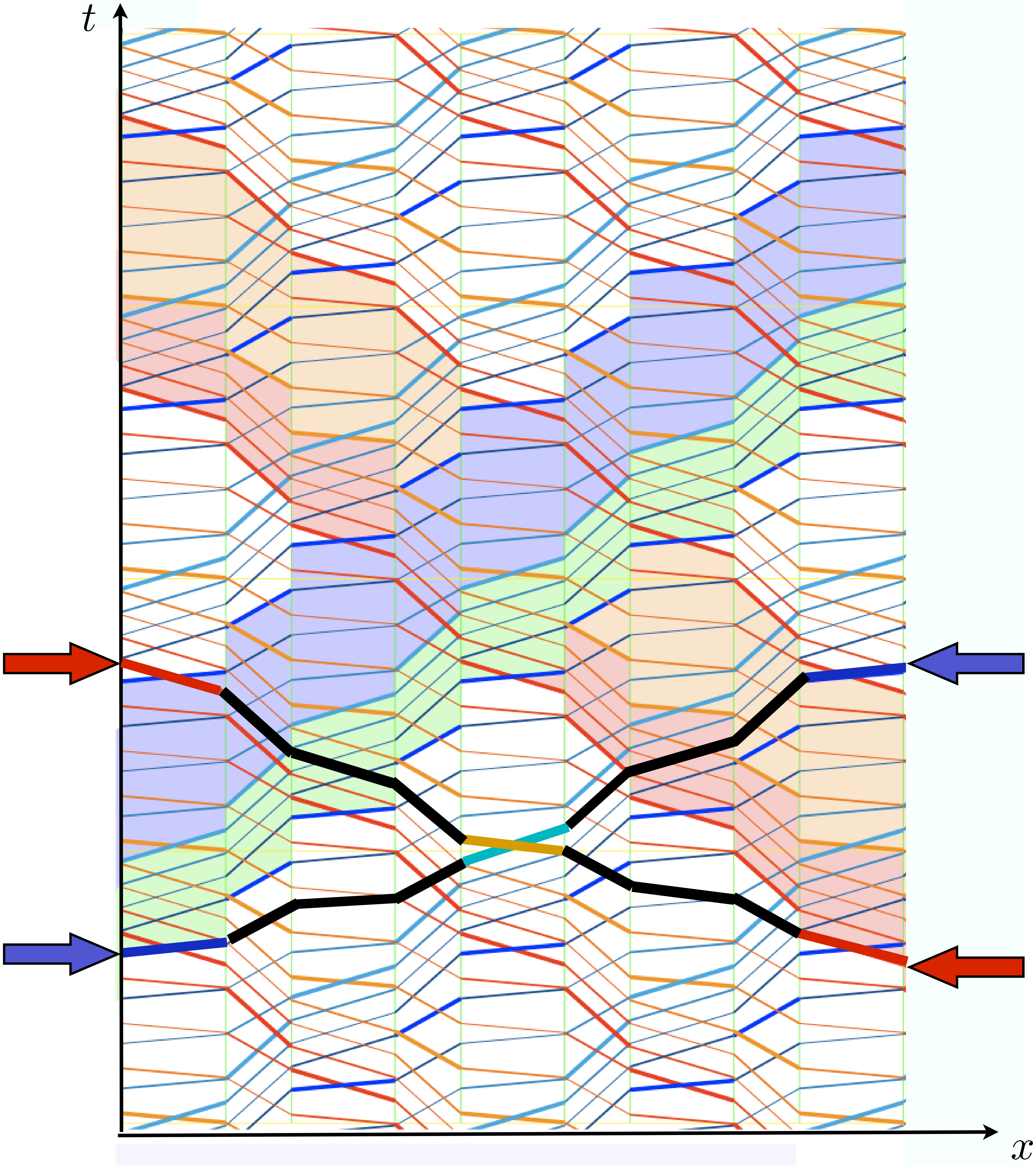
These are conditions imposed on the tiling that defines the periodic structure in xt-space



(1) Simplest structure is space-periodic



(1) Simplest structure is space-periodic



(2) In the simplest periodic structure

Max/Min-characteristics JOIN UP

Inspection of the periodic structure indicates:

- Solution jumps between two entropy levels $\overline{m} > \underline{m}$
- Starting with time-periodic "initial data" U(t) at x=0, solution evolves through five operations before periodic return:

(1) $\overline{\mathcal{E}}:$ Nonlinear evolution at $m = \overline{m}$

(2) $\mathcal{J}:$ Jump from $m = \overline{m}$ to $m = \underline{m}$

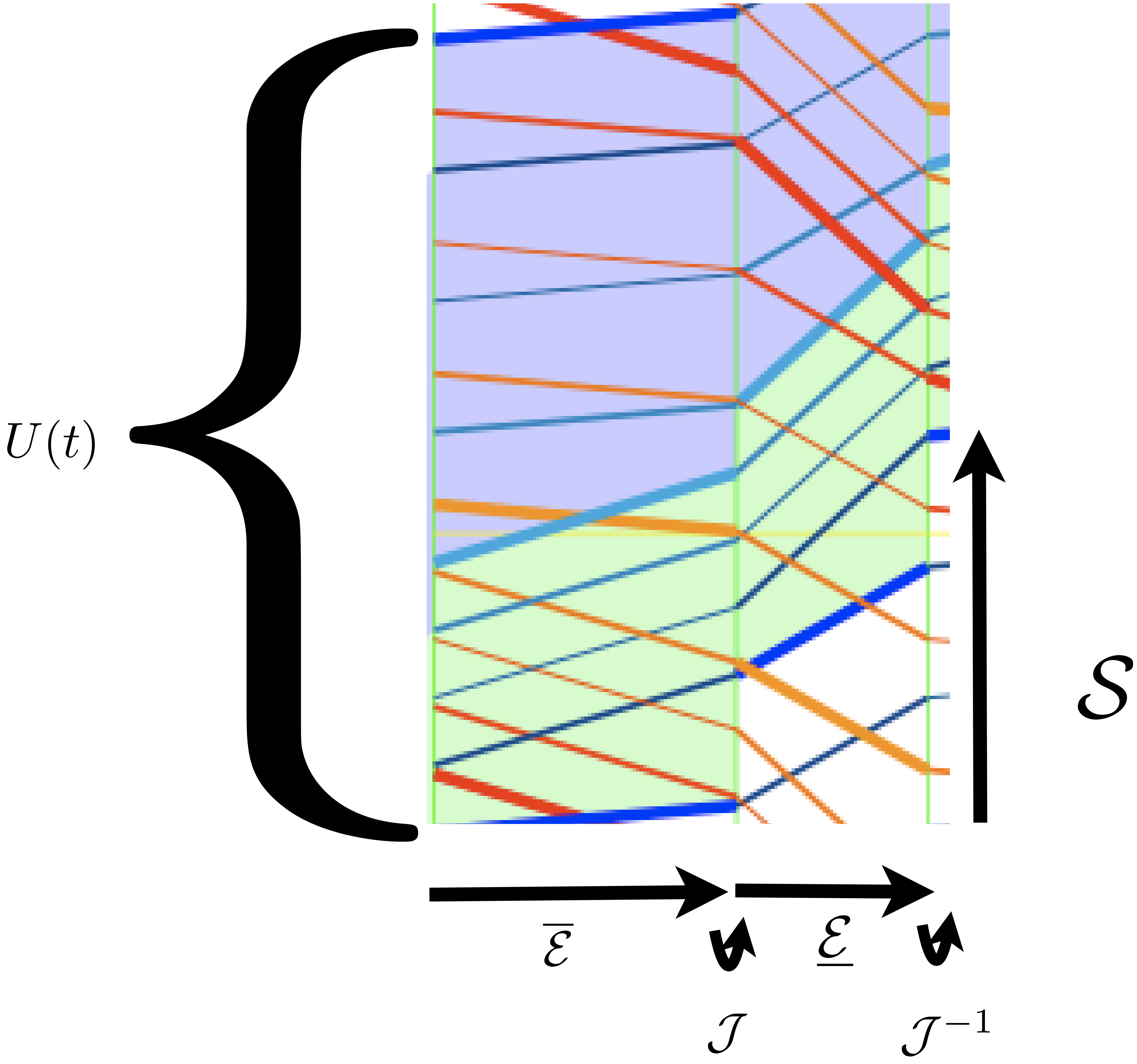
(3) $\underline{\mathcal{E}}:$ Nonlinear evolution at $m = \underline{m}$

(4) $\mathcal{J}^{-1}:$ Jump from $m = \underline{m}$ to $m = \overline{m}$

(5) $\mathcal{S}:$ Half period shift

The periodicity condition

$$\mathcal{S} \cdot \mathcal{J}^1 \cdot \underline{\mathcal{E}} \cdot \mathcal{J} \cdot \overline{\mathcal{E}} \, [U(\cdot)] = U(\cdot)$$



I.e.

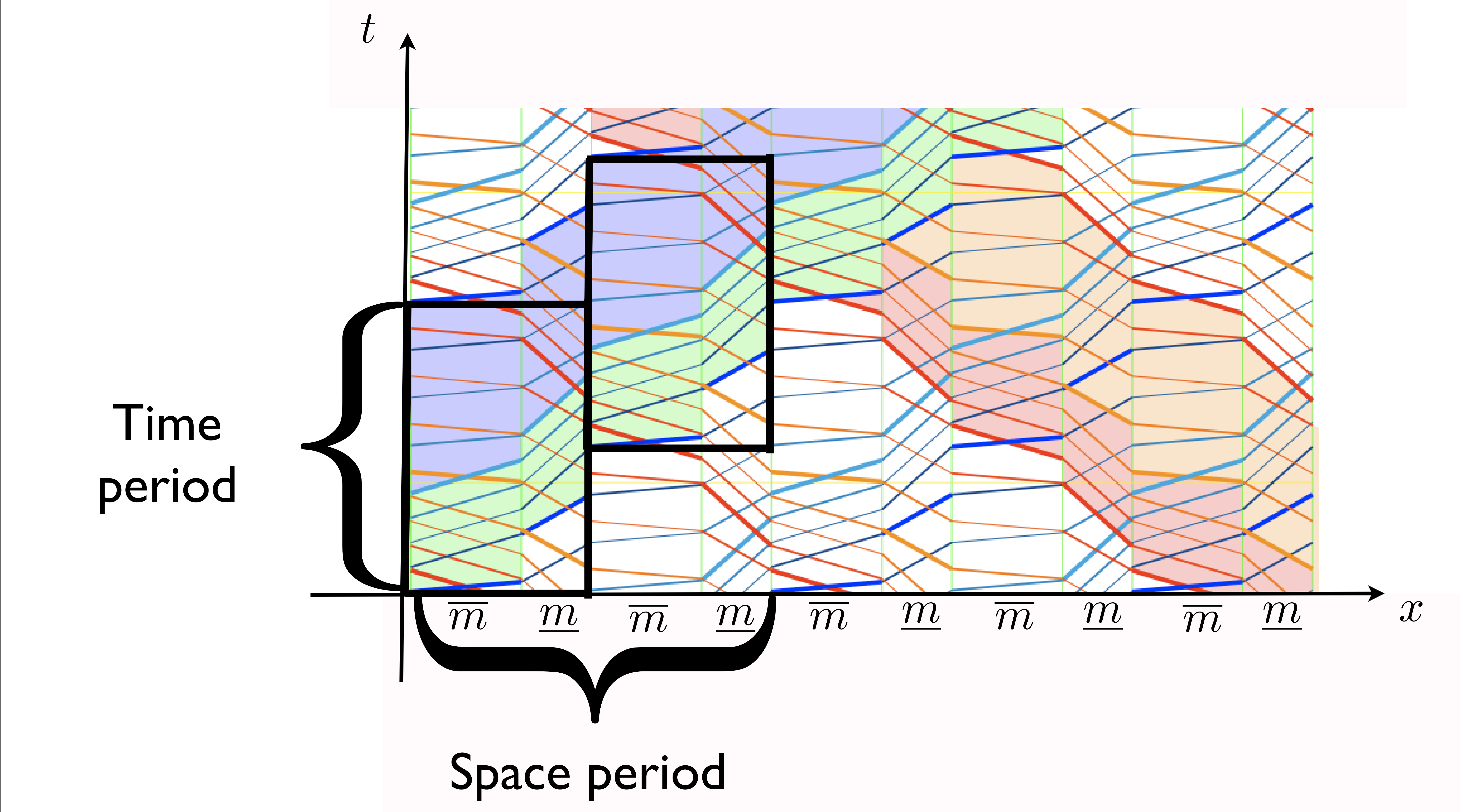
(1) $\overline{\mathcal{E}}$: Nonlinear evolution at $m = \overline{m}$

(2) $\mathcal{J}$: Jump from $m = \overline{m}$ to $m = \underline{m}$

(3) $\underline{\mathcal{E}}$: Nonlinear evolution at $m = \underline{m}$

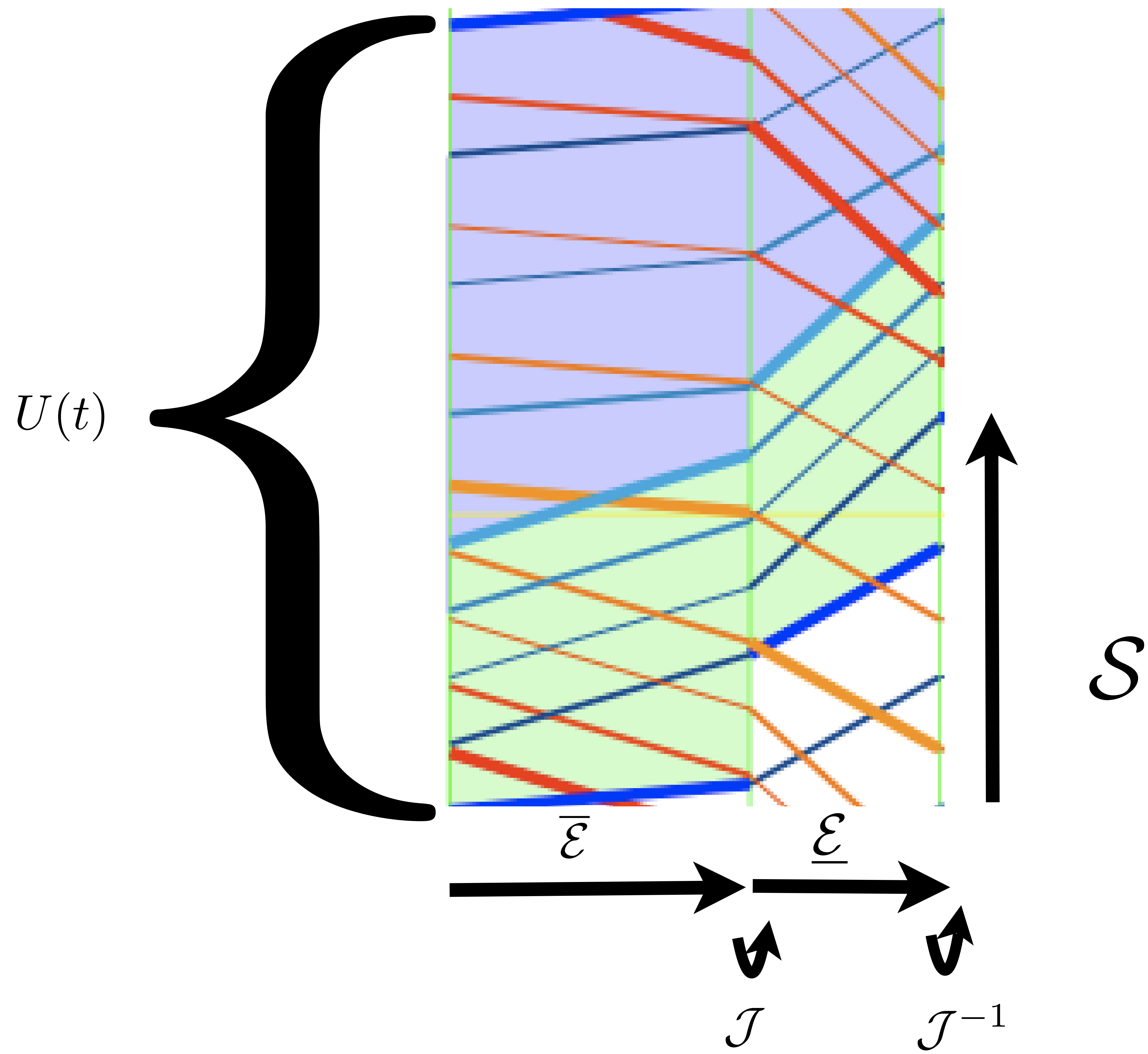
(4) $\mathcal{J}^{-1}$: Jump from $m = \underline{m}$ to $m = \overline{m}$

(5) $\mathcal{S}$: Half period shift

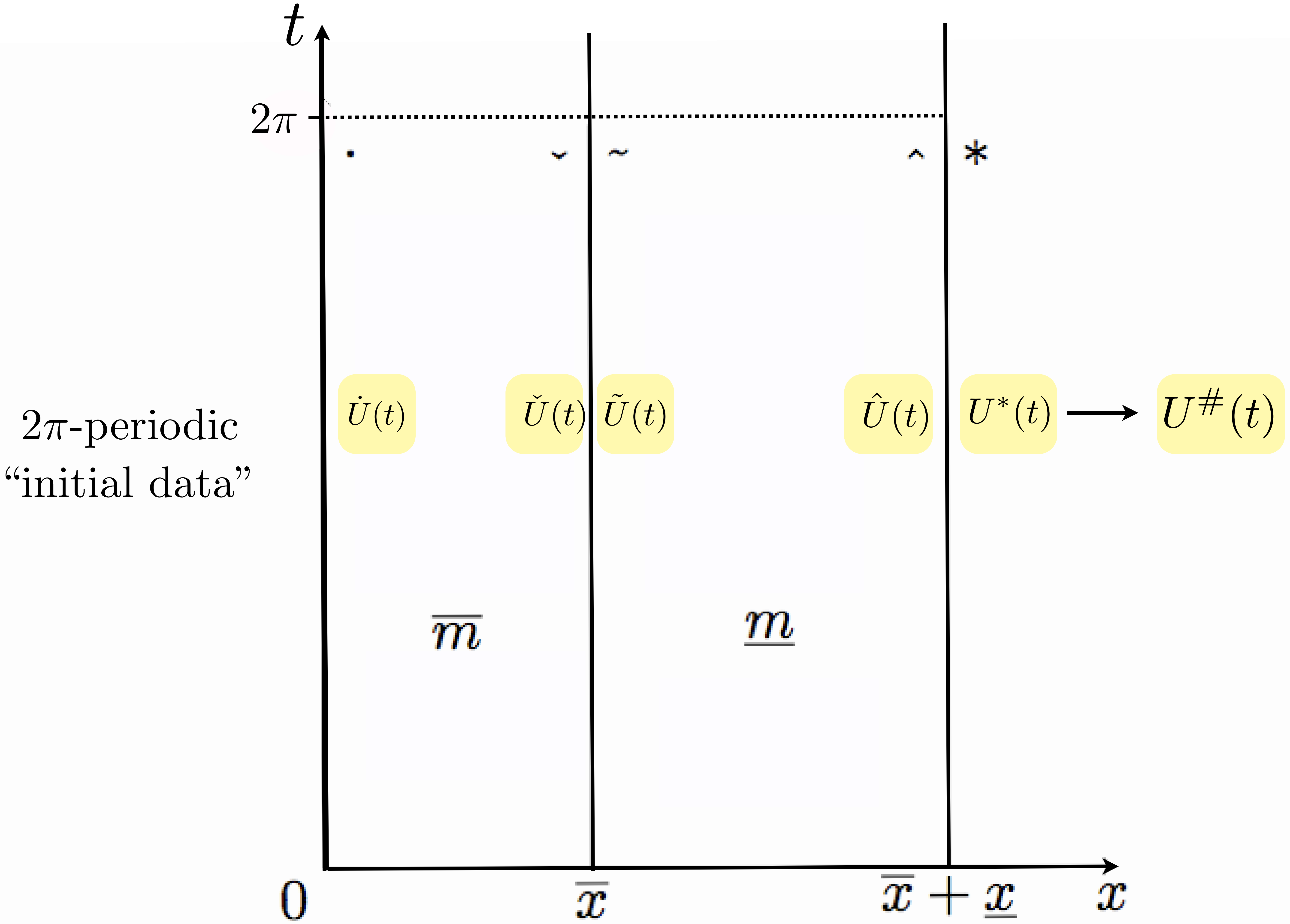


The simplest periodicity condition

$$\mathcal{S} \cdot \mathcal{J}^{-1} \cdot \overline{\mathcal{E}} \cdot \mathcal{J} \cdot \underline{\mathcal{E}}\, [U(\cdot)] = U(\cdot)$$



- Label the stages of $U(\cdot)$ evolution by $\cdot$ $\smile$ $\sim$ $\wedge$ $*$



Dimensionless Variables

$m \equiv m_0$

- Give time and space the same dimension by defining y through the relation

$$y - y_0 = \frac{x - x_0}{c_0}$$

- Define the dimensionless variables

$$\begin{aligned} w &= \frac{z}{z_0} \\ v &= \frac{u - u_0}{m_0 z_0} \end{aligned}$$

- Equations convert to the dimensionless form

$$\begin{aligned} w_y + \sigma(w) v_t = 0 \\ v_y + \sigma(w) w_t = 0 \end{aligned}$$

$$\sigma(w) = w^{-d}$$

$$d \equiv \frac{\gamma + 1}{\gamma - 1}$$

The nonlinear evolution equations take the non-dimensional form:

$$w_y + \sigma(w) v_t = 0$$
$$v_y + \sigma(w) w_t = 0$$

where:

$$\sigma \equiv \sigma(w) = w^{-d}$$

$$d \equiv \frac{\gamma+1}{\gamma-1}$$

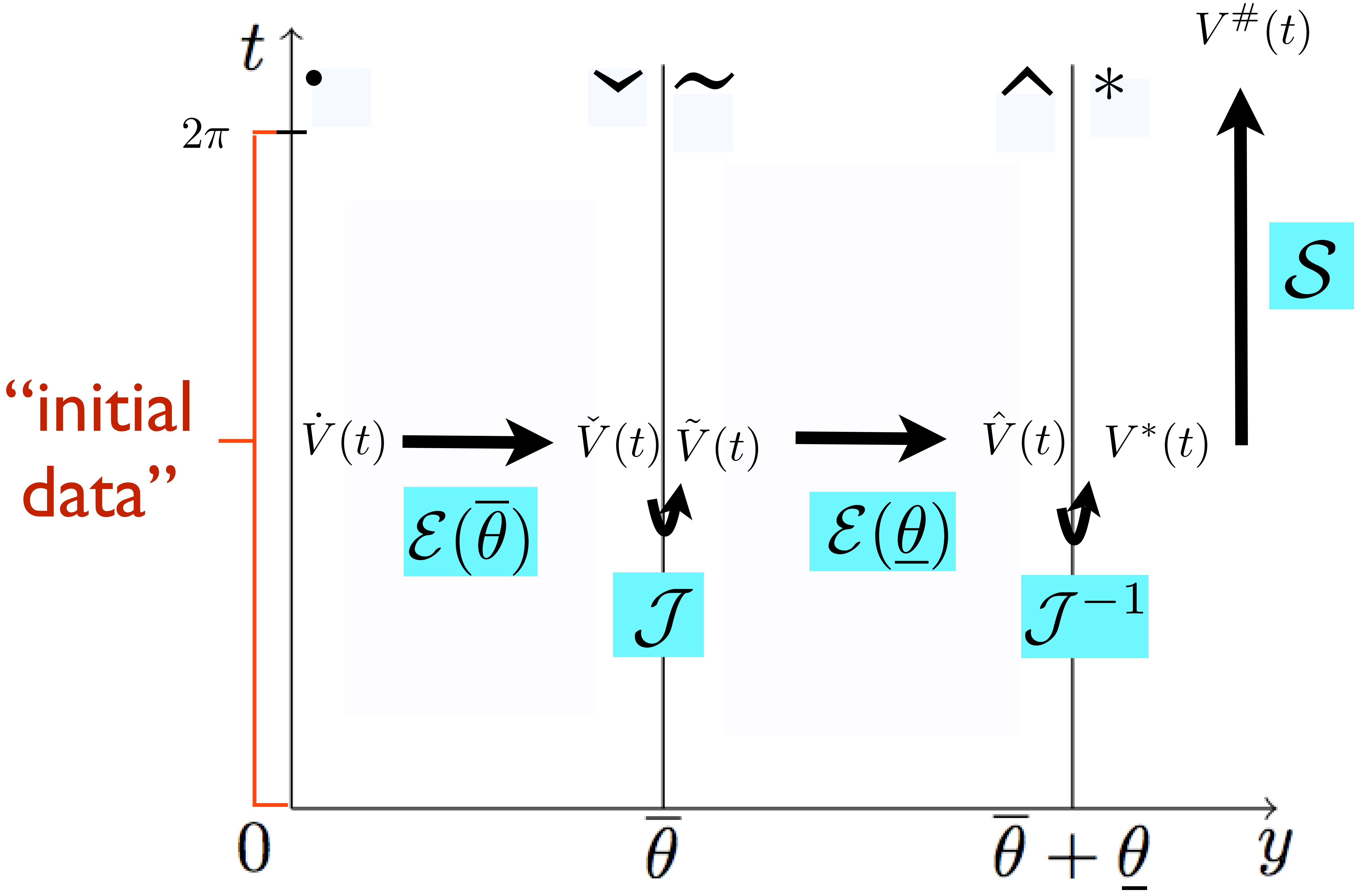
Remarkable Fact:

The equations are independent of base states!

I.e., independent of m_0, z_0, u_0

The Transformed Problem

$$\mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[V(\cdot)] = V(\cdot)$$



$$V(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}$$

The Nonlinear Evolution Operator

DEFINE:

$$V(y,\cdot) = \mathcal{E}(y)[V(\cdot)]$$

to be evolution by system

$$w_y + \sigma(w) v_t = 0$$
$$v_y + \sigma(w) w_t = 0$$

through interval $[0, y]$ starting from “initial data”

$$V(0,t) = V(t)$$

The Entropy Jump Operator in (w,v)-coords

Define: the entropy jump operator $\mathcal{J}$ acting on $V(\cdot)$ pointwise by

$$\mathcal{J}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}\begin{bmatrix} w \\ v \end{bmatrix}$$

where

$$J = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}}$$

$$V = (w, v)$$

NOTE: $\mathcal{J}$ is LINEAR

The Shift Operator

Define the shift operator $\mathcal{S}$ acting on V by

$$\mathcal{S}V(t) = V(t + \pi)$$

NOTE: $\mathcal{S}$ is LINEAR

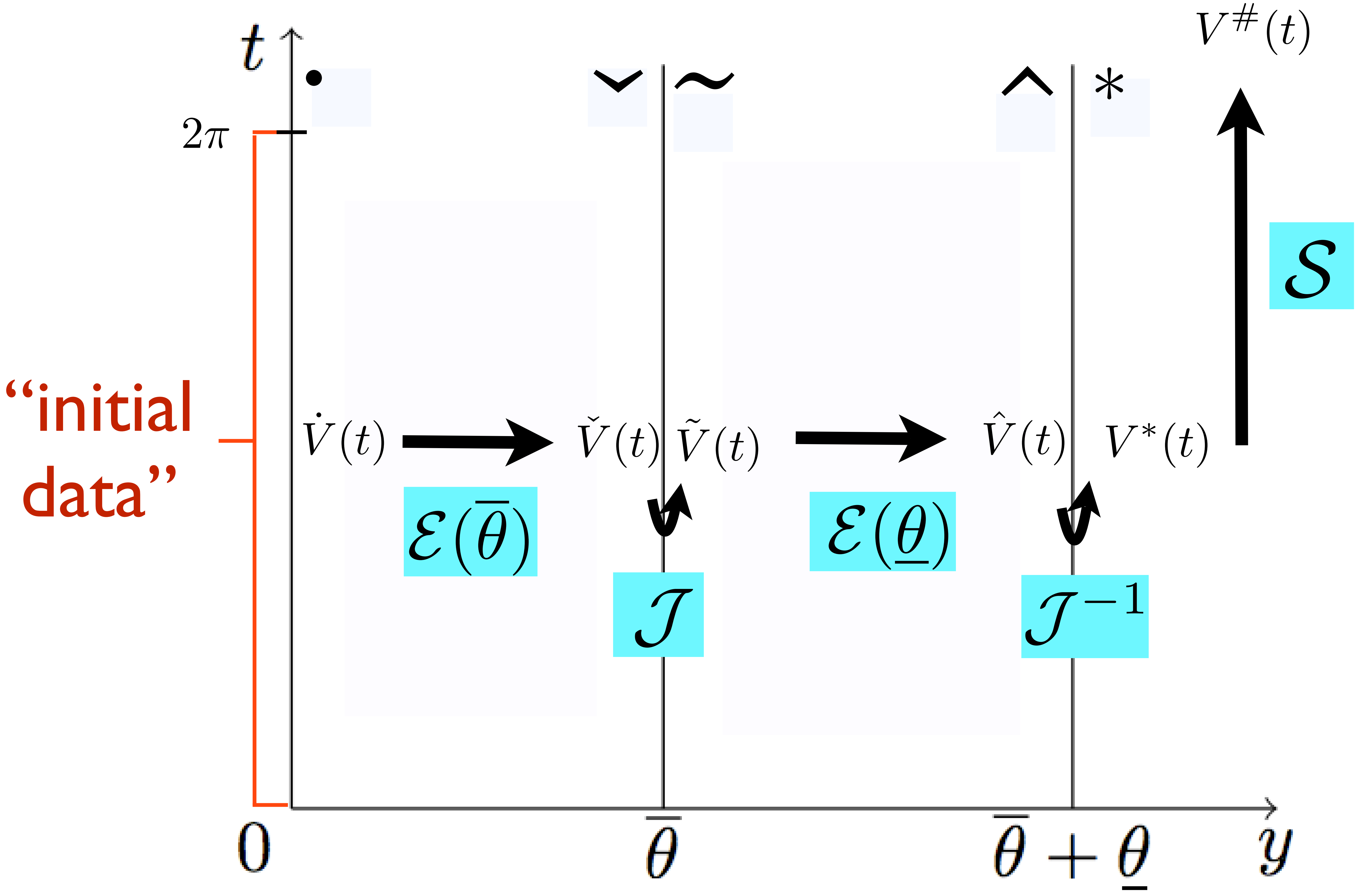
The Periodicity Condition for the Nonlinear Problem in V=(w,v)-space

$$\mathcal{N}[\dot{V}(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[\dot{V}(t)] = \dot{V}(t)$$

$$\mathcal{N} \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})$$

The Periodicity Condition

$$\mathcal{N}[V(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[V] = V(t)$$



C.f. The Nonlinear/Linearized Problem:

The Nolinear Problem:

$$\mathcal{N}[V(t)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{E}(\overline{\theta})\,[V] = V(t)$$

$V(y, \cdot) = \mathcal{E}(y)[V(\cdot)]$ Evolution by

$$\begin{aligned} w_y + \sigma(w) v_t &= 0 \\ v_y + \sigma(w) w_t &= 0 \end{aligned}$$

The Linearized Problem:

$$\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\overline{\theta})[V(\cdot)] = V(\cdot)$$

$V(y, t) = \mathcal{L}(\theta)[V(t)]$ Evolution by

$$\begin{aligned} w_y + v_t &= 0 \\ v_y + w_t &= 0 \end{aligned}$$

$\sigma(w_0) = \sigma(1) = 1$

For the
“Linearized Problem”
wave speeds are
constant at each
entropy level

....

This is the limit as the
states at each entropy
level oscillate near a
constant state

The L^2-Space

Define: the space of periodic functions
even in w, odd in v

$$\Delta = \bigoplus_{n=0}^{+\infty} \Delta_n$$

$$\Delta_n = \left\{ V(t) = \begin{bmatrix} a_n \cos nt \\ b_n \sin nt \end{bmatrix} : a_n, b_n \in \mathbb{R} \right\}$$

$$V(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$$

LEMMA: If $V(t) = (w(t), v(t)) \in \Delta$ is 2π-periodic, sufficiently smooth and sufficiently small, then both $\mathcal{M}[V(\cdot)](t)$ and $\mathcal{N}[V(\cdot)](t)$ are well defined smooth functions, and

$\mathcal{M}[V(\cdot)](y) \in \Delta$ and $\mathcal{N}[V(\cdot)](y) \in \Delta$

for all $0 \le y \le \overline{\theta} + \underline{\theta}$

E.g. $\mathcal{N}[V(\cdot)](y)$ denotes the function y-units through the evolution of $\mathcal{N}$

The Perturbation Problem:

Define:

$$\mathcal{F}_\epsilon = \mathcal{G}_\epsilon - \mathcal{I}$$

where

$$\mathcal{G}_\epsilon[V] = \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

so

$$\begin{aligned} \mathcal{F}(\epsilon, V) \equiv \mathcal{F}_\epsilon[V] &= G_\epsilon[V] - V \\ &= \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} - V \end{aligned}$$

LEMMA 1: If $V \in \Delta$ solves

$$\mathcal{F}_\epsilon[V] = 0$$

for $\epsilon \neq 0$, then

$$W = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V$$

defines a periodic solution of the nonlinear compressible Euler equations.

Proof:

$$\boxed{\mathcal{F}_\epsilon[V] = 0}$$

$$\Downarrow$$

$$\boxed{\mathcal{G}_\epsilon[V] = \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = V}$$

$$\Downarrow$$

$$\boxed{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V}$$ ✓

LEMMA 2: In the limit $\epsilon \longrightarrow 0$ we recover the linear problem:

$$\lim_{\epsilon \longrightarrow 0} \mathcal{G}_\epsilon[V] = \mathcal{M}[V],$$

$$\begin{aligned} \mathcal{F}(\epsilon, V) = G_\epsilon[V] - V &= \frac{1}{\epsilon}\left\{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} - V \\ &= \mathcal{M}[V] - V + O(\epsilon^2) \end{aligned}$$

Proof (Formally):

$$\mathcal{G}_\epsilon[V] = \frac{1}{\epsilon}\left\{\underbrace{\mathcal{N}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon V\right]} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = V$$

Tends to evolution at $\sigma = 1$ with error $O(\epsilon^2)$

Exact Linearized Solutions Exhibiting the Simplest Periodic Structure

Theorem:

(1) There exists a unique solution of the linearized equations in the 1-Fourier mode

(2) For almost every choice of periods, this solution is isolated in the kernel of the linearized operator $\mathcal{M} - \mathcal{I}$

(3) The eigenvalues of the linearized operator can be bounded away from zero by

$$|\lambda_n - 1| \geq \frac{1}{n^r}$$

("The starting estimate you need to apply the hard implicit function in the presence of small divisors")

Proof: $\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\overline{\theta})[V(\cdot)]$

- Consider first linear evolution:

$$V(y,t) = \mathcal{L}(\theta)[V(t)]$$

Evolution by $V(t) = (w, v)$

$$w_y + v_t = 0$$
$$v_y + w_t = 0$$

$$\begin{pmatrix} w_n \\ v_n \end{pmatrix}' + n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} w_n \\ v_n \end{pmatrix} = 0$$

$$\begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = R(ny) \begin{pmatrix} w_n(0) \\ v_n(0) \end{pmatrix}$$

Conclude: counterclockwise rotation $R(n\theta)$ represents $\mathcal{L}(\theta)$ in the n'th F-mode

- Consider next the linear jump operator:

$$\mathcal{J}\left[\begin{array}{c} w \\ v \end{array}\right] = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}\left[\begin{array}{c} w \\ v \end{array}\right] \qquad J = \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{d-1}{d+1}}$$

Then

$$\mathcal{J}: \quad D\begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = \begin{pmatrix} w_n(y) \\ J\, v_n(y) \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$$

Conclude: D represents $\mathcal{J}$ in the n'th F-mode

- Consider next the linear shift operator:

$$\mathcal{S}[V(t)] = V(t+\pi) \qquad V = (w, v)$$

Then

$$V(y,t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n(y)\ \cos nt \\ v_n(y)\ \sin nt \end{pmatrix}$$

$$\mathcal{S}[V(y,t)] = \sum_{n=0}^{\infty} \begin{bmatrix} w_n(y)\ \cos(nt+n\pi) \\ v_n(y)\ \sin(nt+n\pi) \end{bmatrix} = \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} w_n(y)\ \cos nt \\ v_n(y)\ \sin nt \end{pmatrix}$$

$$\mathcal{S} : \qquad S\begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix} = (-1)^n \begin{pmatrix} w_n(y) \\ v_n(y) \end{pmatrix}$$

Conclude: multiplication by $(-1)^n$ represents $\mathcal{S}$ in the n'th F-mode

- CONCLUDE: the linear operator $\mathcal{M}$ is represented by matrix multiplication in each Fourier-mode:

$$\mathcal{M}[V(\cdot)] \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\underline{\theta}) \cdot \mathcal{J} \cdot \mathcal{L}(\bar{\theta})[V(\cdot)]$$

$$V(t) = \sum_{n=0}^{\infty} \begin{pmatrix} w_n \cos nt \\ v_n \sin nt \end{pmatrix}$$

⬇

$$\begin{pmatrix} w_n \\ v_n \end{pmatrix} \longmapsto (-1)^n \cdot D^{-1} \cdot R(n\underline{\theta}) \cdot D \cdot R(n\bar{\theta}) \cdot \begin{pmatrix} w_n \\ v_n \end{pmatrix} \equiv M_n \begin{pmatrix} w_n \\ v_n \end{pmatrix}$$

$$\boxed{M_n = (-1)^n \cdot D^{-1} \cdot R(n\underline{\theta}) \cdot D \cdot R(n\bar{\theta})}$$

✷ Conclude: M_n represents $\mathcal{M}$ in the n'th F-mode

AS A RESULT: The condition for periodicity in the n'th Fourier-mode is:

$$M_n \begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{pmatrix} w_n \\ v_n \end{pmatrix}$$

$$(-1)^n \cdot D^{-1} \cdot R(n\underline{\theta}) \cdot D \cdot R(n\overline{\theta}) \cdot \begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{pmatrix} w_n \\ v_n \end{pmatrix}$$

THUS: we look for values of $(\overline{\theta}, \underline{\theta}, J)$ such that the corresponding operator $\mathcal{M}$ isolates a periodic solution in the 1-mode; I.e. we find $\mathbf{q} = (q_1, q_2) = (w_1, v_1)$ such that

(1) $$M_1 \mathbf{q} = \mathbf{q}$$

(2) $$M_n \begin{pmatrix} w \\ v \end{pmatrix} \neq \begin{pmatrix} w \\ v \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} w \\ v \end{pmatrix} \in \mathbb{R}^2$$

THEOREM: Assume that $J > 1$, $\overline{\theta} > 0$, $\underline{\theta} > 0$ and

$$\overline{\theta} + \underline{\theta} < \pi.$$

Then $V(t) = (q_1 \cos t, q_2 \sin t) \in \Delta_1$ is a solution of $\mathcal{M}[V] = V$ if and only if

$$J = \cot(\overline{\theta}/2)\cot(\underline{\theta}/2)$$

and $q = (q_1, q_2) \in \mathrm{Span}\,\{\mathbf{q}\}$, where

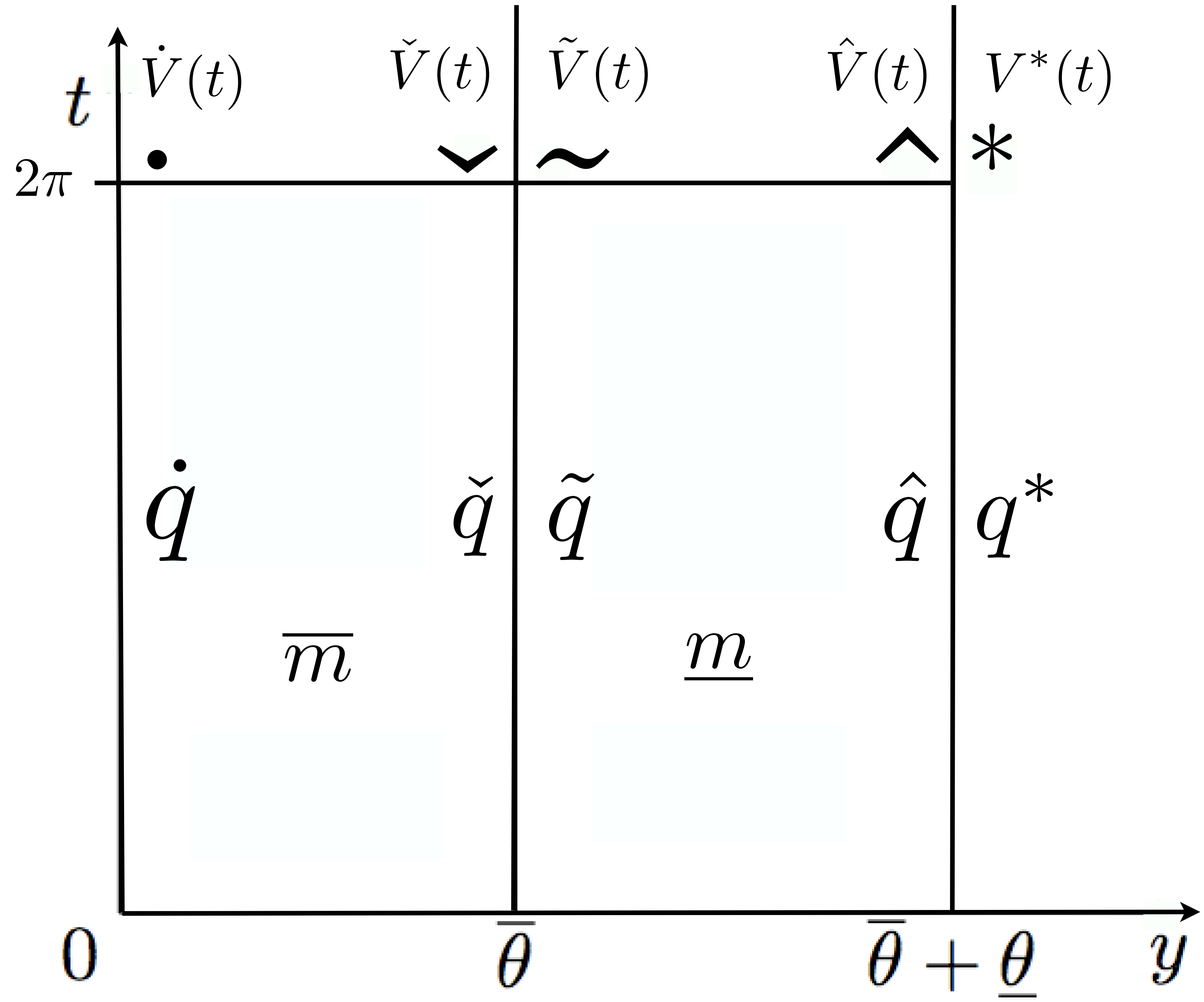
$$\mathbf{q} = (\cos(\overline{\theta}/2), -\sin(\overline{\theta}/2)).$$

Furthermore, if $\dot{q} = \mathbf{q}$, then also

$$\begin{aligned}
\check{q} &= (\cos(\overline{\theta}/2), \sin(\overline{\theta}/2)) \\
\tilde{q} &= (\cos(\overline{\theta}/2), J\sin(\overline{\theta}/2)) \\
&= \rho(\cos(\pi/2 - \underline{\theta}/2), \sin(\pi/2 - \underline{\theta}/2)), \\
\hat{q} &= (-\cos(\overline{\theta}/2), J\sin(\overline{\theta}/2)) \\
&= \rho(-\cos(\pi/2 - \underline{\theta}/2), \sin(\pi/2 - \underline{\theta}/2)), \\
q^* &= (-\cos(\overline{\theta}/2), \sin(\overline{\theta}/2) = -q,
\end{aligned}$$

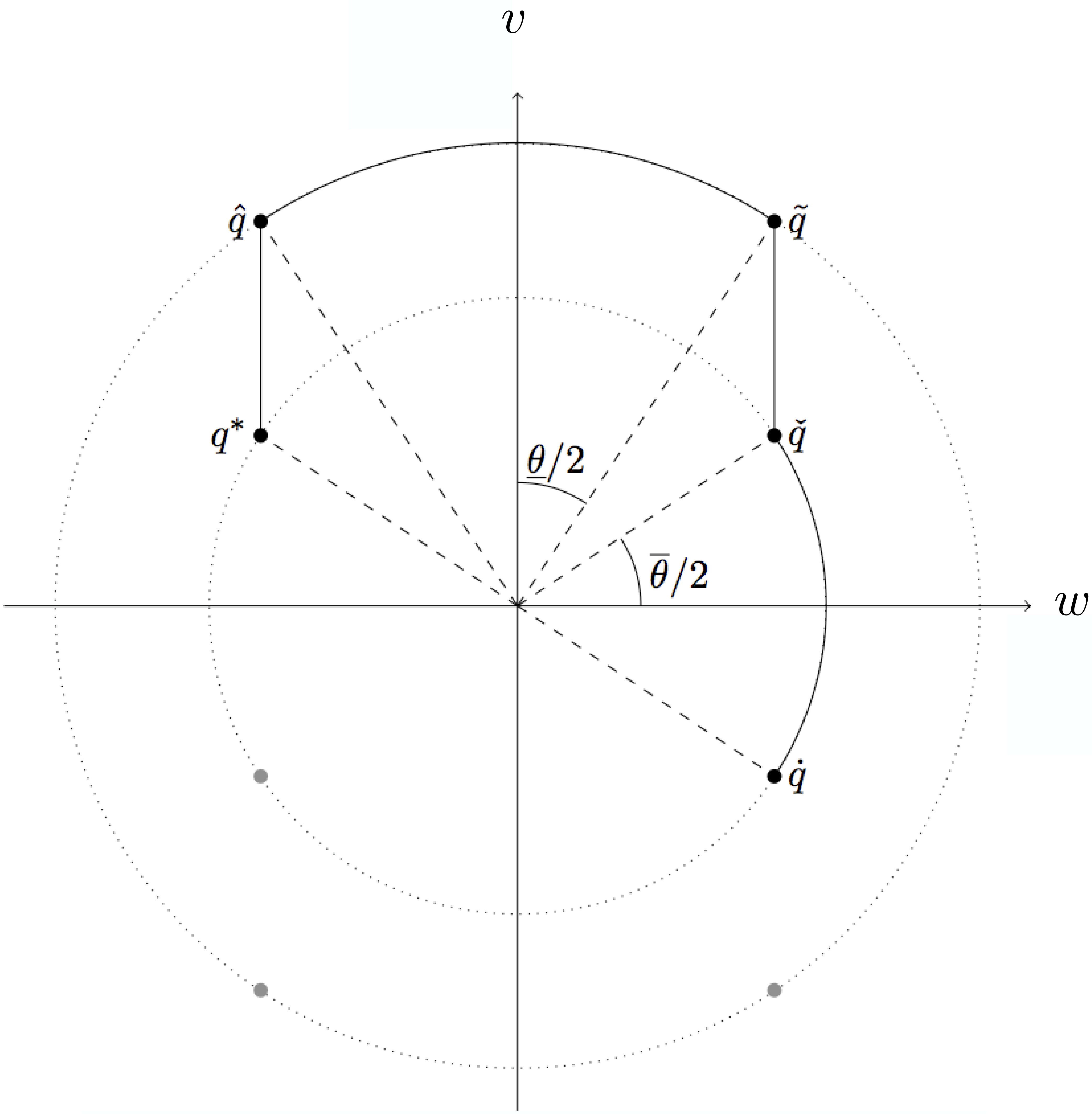
where we have set $\rho = \|\tilde{q}\|$.

RECALL.....



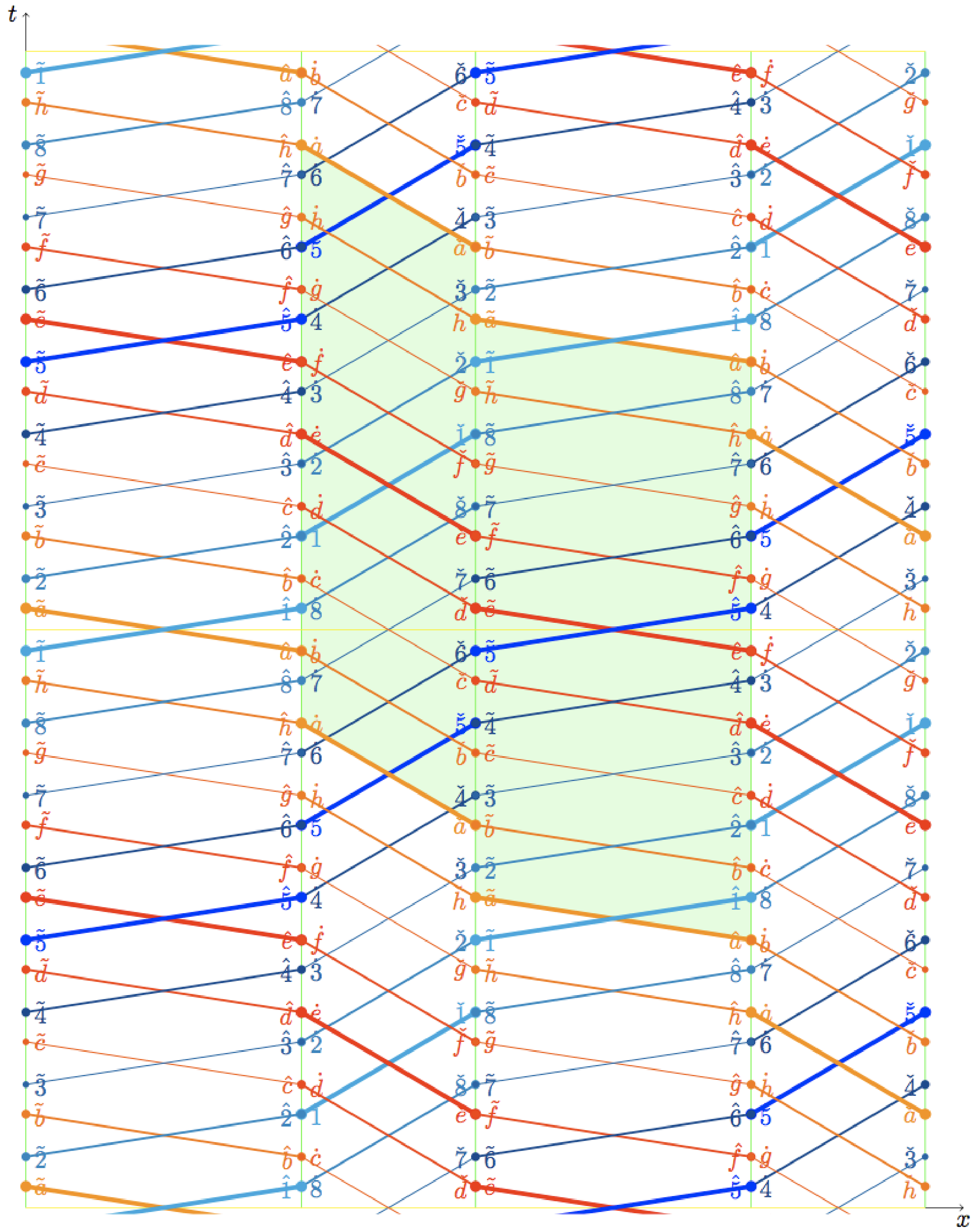
t
$\dot{V}(t)$
$\check{V}(t)$
$\tilde{V}(t)$
$\hat{V}(t)$
$V^*(t)$
2π
$\dot{q}$
$\check{q}$
$\tilde{q}$
$\hat{q}$
q^*
$\overline{m}$
$\underline{m}$
0
$\overline{\theta}$
$\overline{\theta} + \underline{\theta}$
y

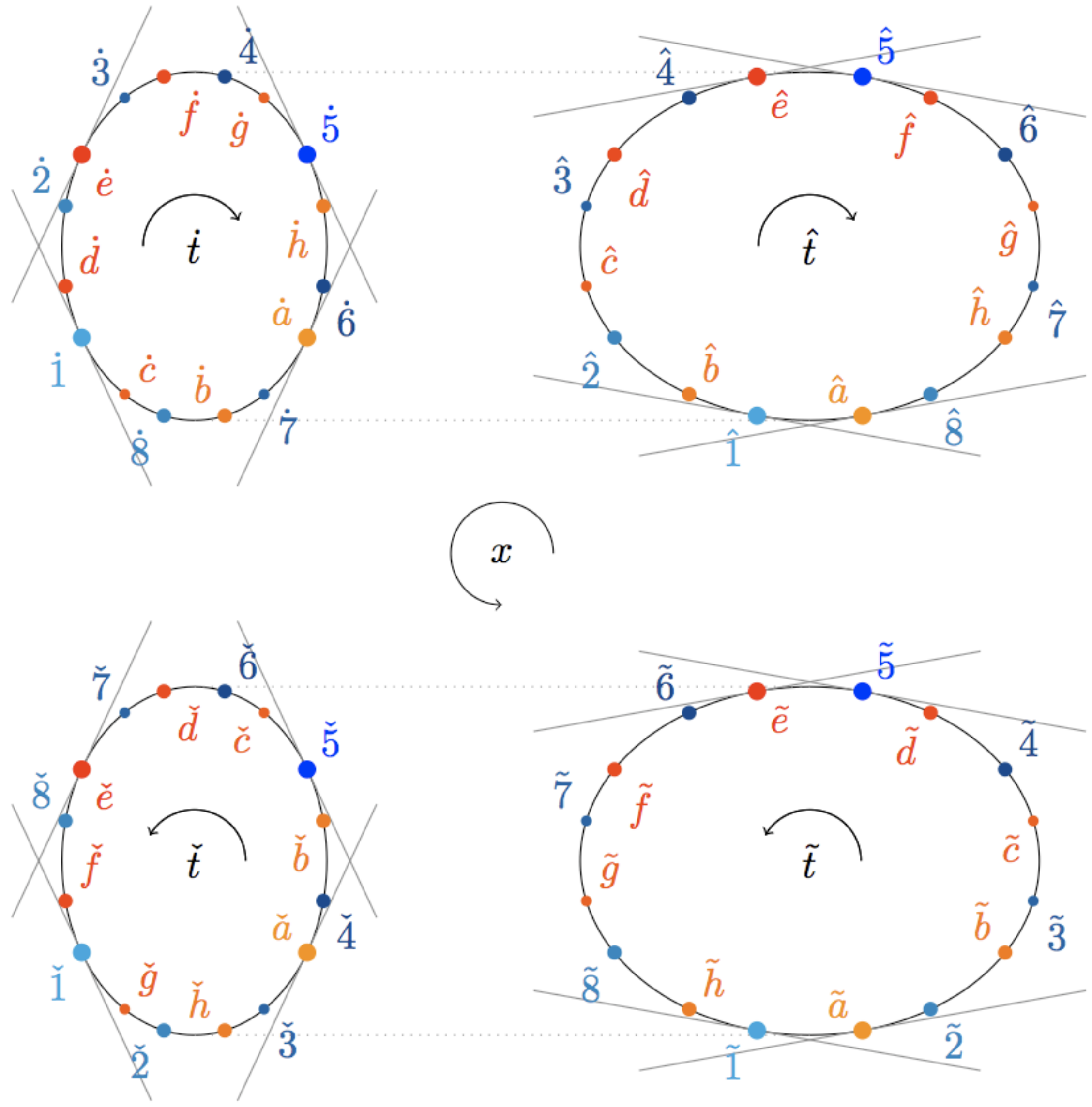
You can solve for **q** geometrically:



$$
\begin{aligned}
\dot{q} &= q, \\
\check{q} &= R(\overline{\theta})\dot{q}, \\
\tilde{q} &= D\check{q}, \\
\hat{q} &= R(\underline{\theta})\tilde{q}, \\
q^* &= D^{-1}\hat{q},
\end{aligned}
$$

The states $\dot{q}, \check{q}, \tilde{q}, \hat{q}, q^*$ for $\mathbf{q} \in \Delta_1$





CHECK: The solution in the 1-mode kernel has the structure that balances compression and rarefaction in the nonlinear problem

Linear solutions should perturb to exact solutions of the nonlinear problem

THEOREM: Let

$$E \equiv \left\{ \Theta = (\overline{\theta}, \underline{\theta}) \;:\; \overline{\theta}, \underline{\theta} > 0,\; 0 < \overline{\theta} + \underline{\theta} < \pi \right\}.$$

Then there exists a subset E^* of full measure in E such that, if $\Theta \in E^*$, then Θ is *non-resonant* in the sense that if J is given in terms of Θ by (**), then the eigenvalues $\lambda_n^{\pm} - 1$ of the linearized operator $\mathcal{M} - I$ are nonzero for all $n \geq 2$.

We now impose a further symmetry and use this to obtain explicit bounds for the eigenvalues of M_n.

THEOREM: Assume the symmetric case,

$$\overline{\theta} = \underline{\theta} \equiv \theta, \quad 0 < \theta < \pi/2.$$

Then there is a set of full measure $\mathcal{A} \subset (0, \pi/2)$ such that, if $\theta \in \mathcal{A}$, then there is a positive constant C and exponent $r \geq 1$ such that the eigenvalues of the linearized operator $\mathcal{M} - I$ satisfy the estimate

$$|\lambda_n^{\pm} - 1| \geq \frac{C}{n^r},$$

for all $n \geq 2$. In particular, if $\frac{\pi - 2\theta}{2\pi}$ is the irrational root of a quadratic equation, we can take $r = 1$.

- “Proof”: Define the transformation

$$\begin{aligned} \phi &= \frac{\pi - \underline{\theta} + \overline{\theta}}{2} \\ \psi &= \frac{\pi - \underline{\theta} - \overline{\theta}}{2} \end{aligned}$$

And apply the theory of Liouville numbers in transformed variables assuming $\overline{\theta} = \underline{\theta} = \theta$ ...

- "Proof": That is, we first prove

$$|\lambda_n^{\pm} - 1| \geq \frac{C}{n^r} \quad \text{iff} \quad |\sin(n\psi)| \geq \frac{C}{n^r}$$

- DEFN: $\xi = \psi/\pi$ is NOT a Liouville Number if $\exists C > 0, r \geq 2$ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{C}{q^r} \qquad \forall\ p/q \in \mathbb{Q}.$$

THM: Non-Liouville numbers form a set of full measure

- Choose $q = n$:

$$\left|\frac{n\psi}{\pi} - p\right| > \frac{C}{n^{r-1}}$$

for all $n, p \in \mathbb{Z}$.

- “Proof”: So...

$$\left|\frac{n\psi}{\pi} - p\right| > \frac{C}{n^{r-1}}$$

⇓

$$\text{Dist}\left\{\frac{n\psi}{\pi}, \mathbb{Z}\right\} > \frac{C}{n^{r-1}}$$

⇓

$$|\sin n\psi| > \frac{C}{n^{r-1}}$$

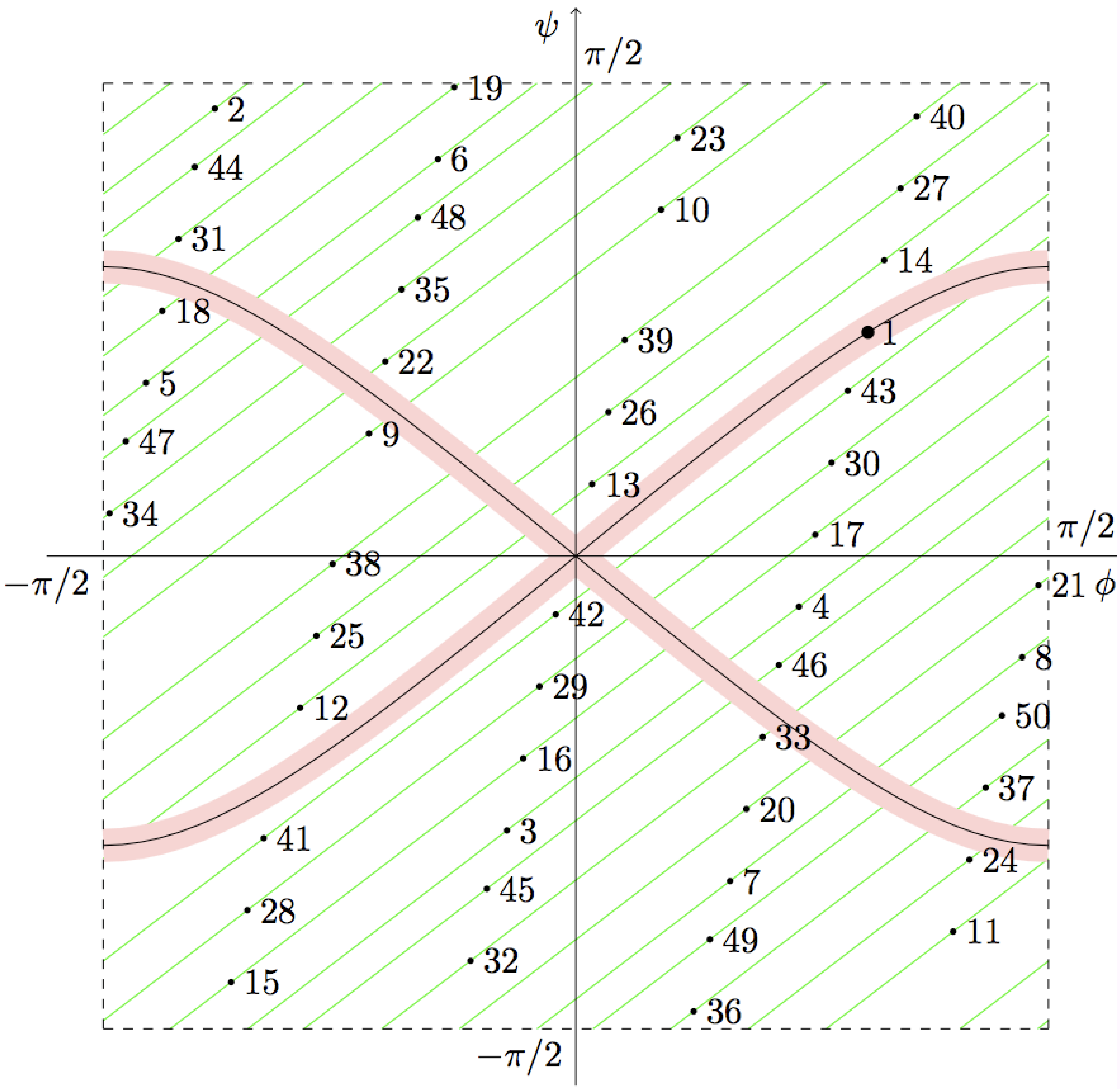
⇓

$$|\lambda_n^{\pm} - 1| \geq \frac{C}{n^{r-1}}$$

- Theorem: If ξ is the irrational root of a rational quadratic polynomial, we can take $r = 2$ (best case)

$$|\lambda_n^{\pm} - 1| \geq \frac{C}{n^r}$$

Numerical Plot of First 50 Eigenvalues–Case $\overline{\theta} \neq \underline{\theta}$



Bifurcation to Nonlinear Solutions

- It remains to prove that the linearized solutions perturb to solutions of the nonlinear equations.

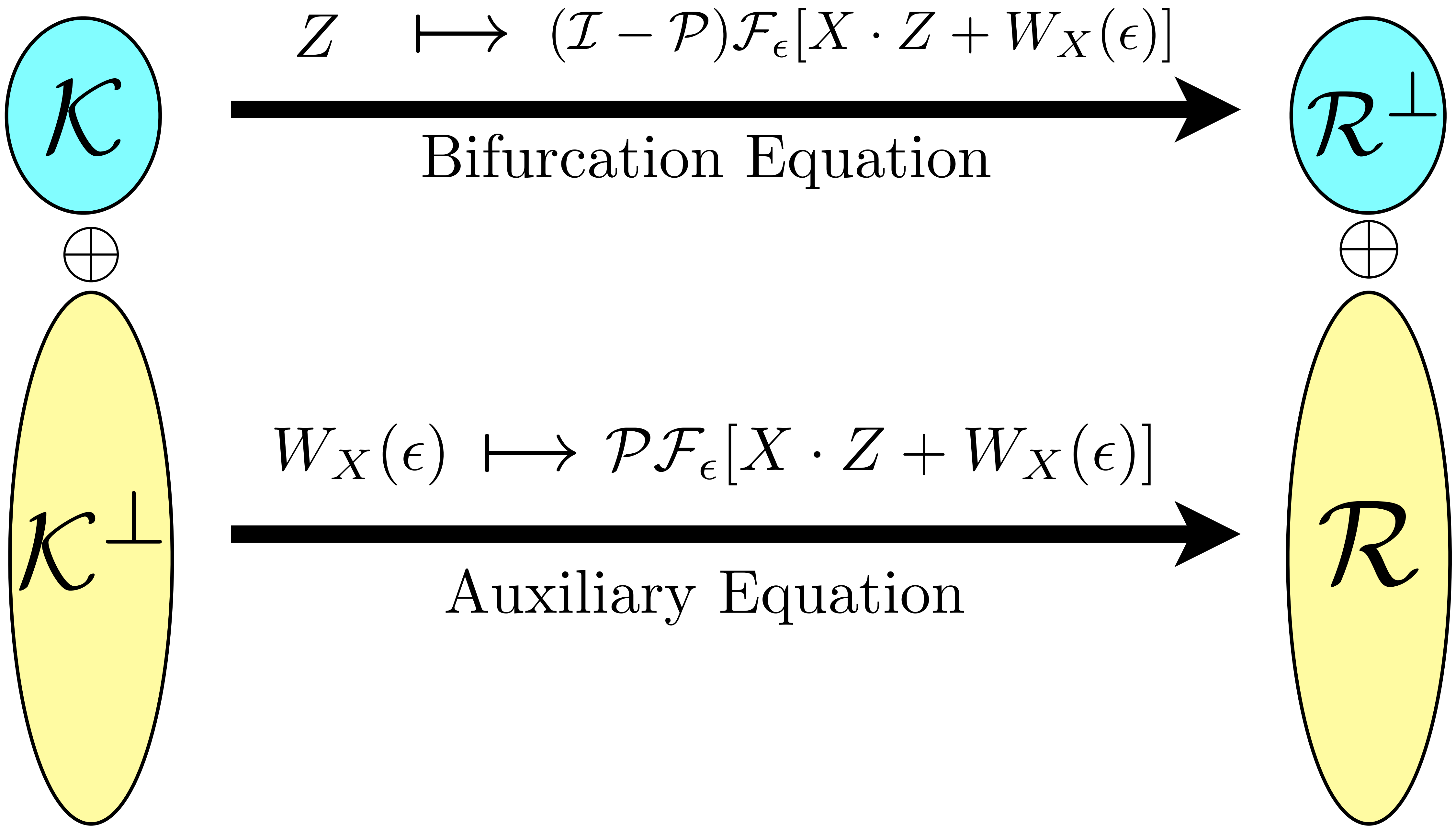
  Liapunov-Schmidt decomposes the nonlinear problem by coordinates natural for the linearized problem:

Picture: L-S Decomposition

- Decompose the nonlinear problem by the RANGE $\mathcal{R}$ and KERNEL $\mathcal{K}$ of the LINEAR OPERATOR $\mathcal{M} - \mathcal{I}$:

$$\mathcal{P} \equiv \text{Projection onto } \mathcal{R}$$

$$\mathcal{I} - \mathcal{P} \equiv \text{Projection onto } \mathcal{R}^{\perp}$$



CONCLUSION

- We have solved the Bifurcation Equation:
- It remains to solve the Auxiliary Equation:

$$\text{AUXILIARY EQUATION:} \quad \mathcal{P}\cdot\mathcal{F}_\epsilon[X\cdot Z+W_X(\epsilon)]=0$$

$$\{W_X(\epsilon)\in\mathcal{K}^\perp\}\longmapsto\mathcal{P}\cdot\mathcal{F}_\epsilon[X\cdot Z+W_X(\epsilon)]\in\mathcal{R}$$

- The map is 1-1 invertible, but the eigenvalues are not bounded away from zero, which leads to issues of small-divisors analogous to KAM theory.

- NOTE: The Auxiliary Equation poses an abstract Implicit Function Theorem problem: "Everything special" about the periodic problem has been removed at this stage.

- NOTE: If the eigenvalues were uniformly bounded away from zero, the standard Implicit Function Theorem for Banach Spaces would directly apply.

We are currently working on this!

Ref's:

15.6 A 'Hard' Implicit Function Theorem. The following result provides the abstract framework for some of the problems mentioned in § 15.4. This will be made evident afterwards, by means of an example dealing with a so-called 'small divisor problem'.

Theorem 15.8. *Let (X_λ), (Y_λ), (Z_λ) be scales of Banach spaces with $\lambda \in \Lambda = [0, 1]$ and $|\cdot|_\mu \leqq |\cdot|_\lambda$ for $\mu \leqq \lambda$ on all scales. Let $(x_*, y_*) \in X_1 \times Y_1$ (the smallest spaces) and, denoting balls with respect to $|\cdot|_\lambda$ by B^λ, let*

$$\Omega_r^\lambda = B_r^\lambda(x_*) \times B_r^\lambda(y_*) \subset X_\lambda \times Y_\lambda .$$

Consider $F: \Omega_r^0 \to Z_0$ with $F(x_*, y_*) = 0$ and assume the existence of constants $M \geqq 1$, $\alpha \geqq 0$ and $\gamma > 0$ such that F satisfies the following three conditions:

(a) *$F: \Omega_r^\lambda \to Z_\lambda$ is continuous, for every $\lambda \in [0, 1]$;*
(b) *for $\lambda \in (0, 1]$ and $\mu < \lambda$, $F: \Omega_r^\lambda \to Z_\mu$ is differentiable in y and*

$$|F(x, y) - F(x, \bar{y}) - F_y(x, \bar{y})(y - \bar{y})|_\mu \leqq \frac{M}{(\lambda - \mu)^{2\alpha}} |y - \bar{y}|_\lambda^2 ;$$

(c) *$F_y(\cdot, \cdot)$ has an approximate right inverse, i.e. for $0 \leqq \mu < \lambda$ there exists $T(x, y) \in L(Z_\lambda, Y_\mu)$ such that*

$$|T(x, y)|_{L(Z_\lambda, Y_\mu)} \leqq \frac{M}{(\lambda - \mu)^\gamma}$$

and

$$|F_y(x, y)\, T(x, y) - I|_{L(Z_\lambda, Z_\mu)} \leqq \frac{M}{(\lambda - \mu)^{2(\alpha + \gamma)}} |F(x, y)|_\lambda \quad on \quad \Omega_r^\lambda .$$

Then to $\lambda \in (0, 1]$ there exist a radius $\varrho(\lambda) \leqq r$ and a map $S_\lambda: \Omega_{\varrho(\lambda)}^\lambda \to Y_{\lambda/2}$ such that $F(x, S_\lambda(x, y)) = 0$ on $\Omega_{\varrho(\lambda)}^\lambda$.

From: Nonlinear Functional Analysis, Deimling

E. Zehnder, *Generalized implicit function theorems with applications to some small divisor problems*, Comm. Pure Appl. Math., Vol. 28, pp. 91-140 (1975).

Other References:

W.Craig and G. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. on Pure Appl. Math., Vol 66, pp. 1409-1498 (1993).

R.Hamilton, *The Inverse Function Theorem of Nash and Moser*, Bull. Am. Math. Soc. Vol. 7,(1982).

END