Numerical Simulation of GR Shock Waves by a Locally Inertial Godunov Method with Dynamic Time-Dilation

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Wyoming,
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SUMMARY

In this talk I introduce
General Relativity
and the Einstein equations,
and recall the
locally inertial formulation
of the equations
introduced by Jeff Groah and author
to analyze shock waves in
Standard Schwarzschild Coordinates
on spherically symmetric spacetimes.

SUMMARY

I then discusss recent thesis work
of Zeke Vogler
introducing a new numerical method
for computing GR shock-waves
and a new family of initial data
on which he tested the method.
We call the method a
locally inertial Godunov method
with
dynamical time dilation
because clocks are dilated in each grid cell
to simulate effects of spacetime curvature.

SUMMARY

The numerics confirm
shock-wave formation
in forward time,
and
black hole formation
(from a smooth solution)
in backward time
via collapse associated with
an incoming rarefaction wave.

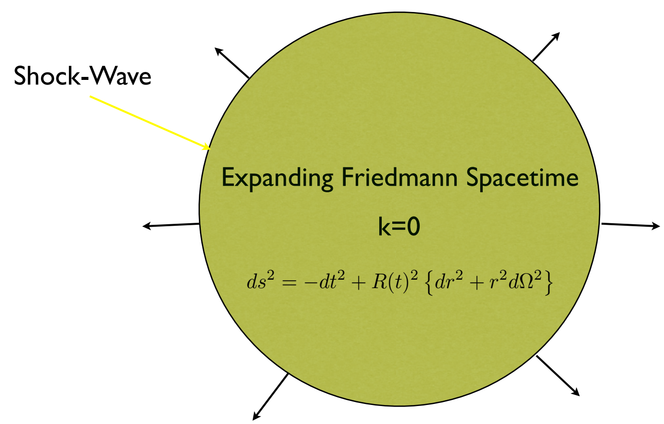
SUMMARY

As far as we know,
there is not yet
a rigorous mathematical proof
of either shock wave formation
or black hole formation
for the Einstein-Euler equations,
starting from smooth initial data.
So we propose these new solutions
as a natural starting point
for rigorous proofs.

COMMENTS

The forward time solutions
can be interpreted as
resolving the
secondary reflected wave,
(an incoming shock wave),
in the Smoller-Temple exact
shock-wave model
for
an explosion into a static,
singular isothermal sphere.

Explosion into a static, singular isothermal sphere



Static Singular Isothermal Spere

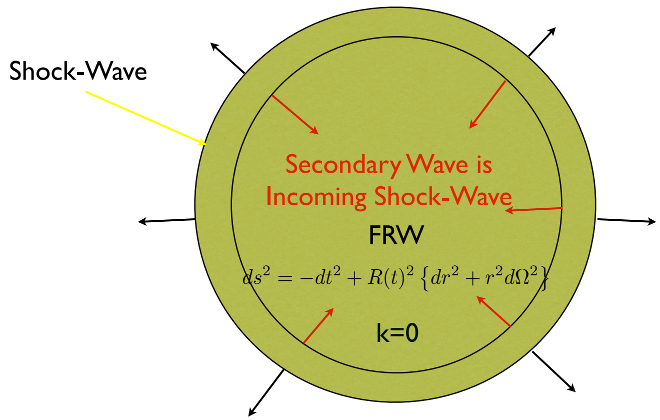
$(\text{p}=\sigma\rho)$

Tolman-Oppenheimer-Volkoff Spacetime

$$ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

Smoller-Temple: Phys. Rev. D, Vol. 51, No. 6, 1995.

Explosion into a static, singular isothermal sphere



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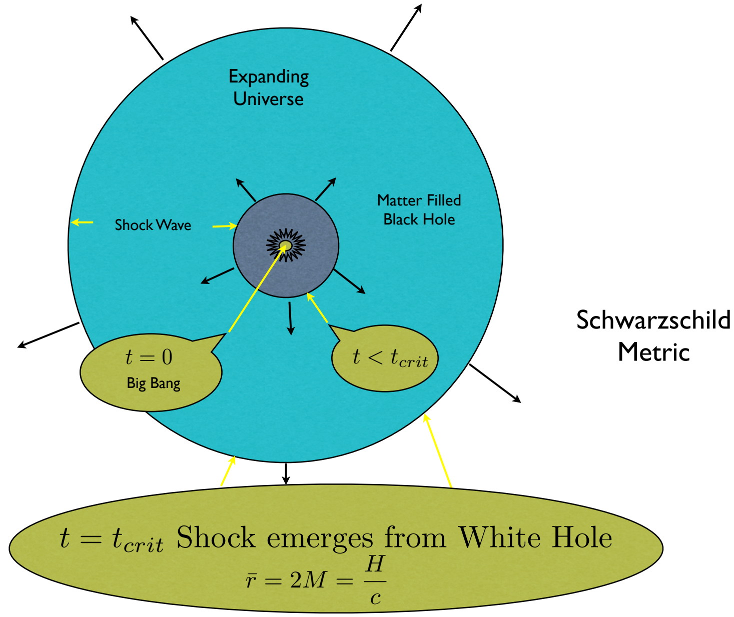
Smoller-Temple: Phys. Rev. D, Vol. 51, No. 6, 1995.

We would like to simulate the secondary reflected wave in our Shock-Wave Cosmology...



Smoller-Temple: PNAS, Vol. 100, no. 20, 2003, pp. 11216-11218.

COMMENTS

Shock wave simulation is complicated by the fact that the

Einstein curvature tensor

is

discontinuous at shock-waves

in

SSC-coordinates

LESSON

"The gravitational metric tensor appears singular at shock waves in coordinates where the analysis and simulation appear feasible..."
Standard Schwarzschild Coordinates

The metric g is only
Lipschitz continuous at shock-waves
The curvature tensor G is
discontinuous at shock-waves

The Einstein equations,
and
Compressible Euler equations,
only hold weakly
at shock waves.

COMMENTS

We are interested to know whether a successful numerical simulation of a fluid dynamical shock wave has been demonstrated before in General Relativity.??

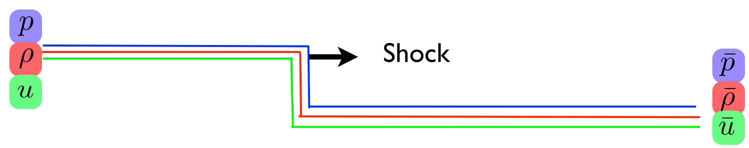
References:

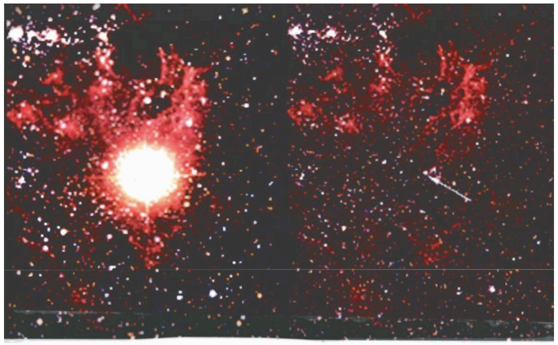
The locally inertial Formulation of the Einstein Equations, (Spherical Symmetry, Standard Schwarzschild Coordinates...)

- *The Numerical Simulation of General Relativistic Shock Waves by a Locally Inertial Godunov Method Featuring Dynamical Time Dilation*, Zeke Vogler, UC-Davis Dissertation, March 2010.
- *A shock-wave formulation of the Einstein equations*, with J. Groah, Meth. and Appl. of Anal., **7**, No. 4,(2000), pp. 793-812.
- *Shock-wave solutions of the Einstein equations: Existence and consistency by a locally inertial Glimm Scheme*, with J. Groah, Memoirs of the AMS, Vol. 172, No. 813, November 2004.
- *Shock Wave Interactions in General Relativity: A Locally Inertial Glimm Scheme for Spherically Symmetric Spacetimes*, with J. Groah and J. Smoller, Springer Monographs in Mathematics, 2007.

Shock Waves
and
General Relativity

- A *blast-wave/shock-wave* marks the leading edge of a classical explosion
- *Shock-wave* $\approx$ *discontinuity* in density and pressure between the explosion and the material beyond the explosion
- An explosion with a finite mass/energy behind it would generate such a *blast-wave*



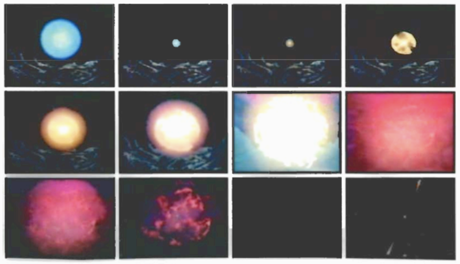


The above two photographs are of the same part of the sky. The photo on the left was taken in 1987 during the supernova explosion of SN 1987A, while the right hand photo was taken beforehand. Supernovae are one of the most energetic explosions in nature, making them like a 1028 megaton bomb (i.e., a few octillion nuclear warheads).

SN1987A
(NASA)

Supernova Images

This is the set of images used to create the supernova inline animation.



One of the most energetic explosive events known is a supernova. These occur at the end of a star's lifetime, when its nuclear fuel is exhausted and it is no longer supported by the release of nuclear energy. If the star is particularly massive, then its core will collapse and in so doing will release a huge amount of energy. This will cause a blast wave that ejects the star's envelope into interstellar space. The result of the collapse may be, in some cases, a rapidly rotating neutron star that can be observed many years later as a radio pulsar.

- Joel Smoller and I wondered whether there could be a wave at the leading edge of the biggest of all explosions--the Big Bang...
- PNAS 2003 we gave a physically plausible model of a Shock-Wave that cuts off the total mass of the Big Bang at a finite value thereby placing our universe of galaxies inside a ``time-reversed Black Hole"...

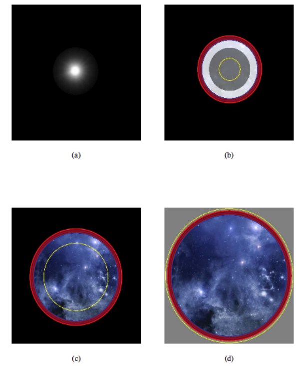


Figure 1. Stages of the Big Bang (a) The beginning (b) The early stage (c) The late stage (d) After the event horizon

- Computer Visualization by Zeke Vogler...
(webpage http://www.math.ucdavis.edu/~temple/)

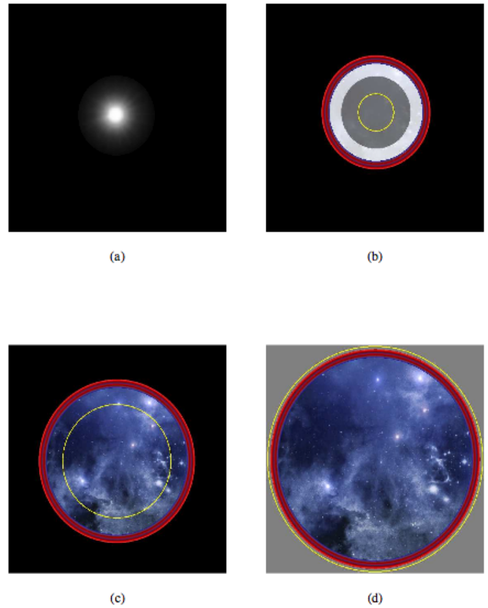
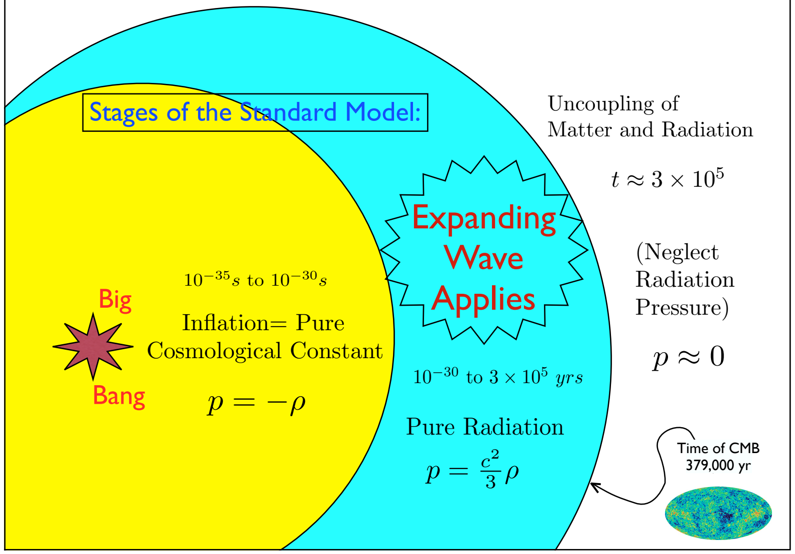
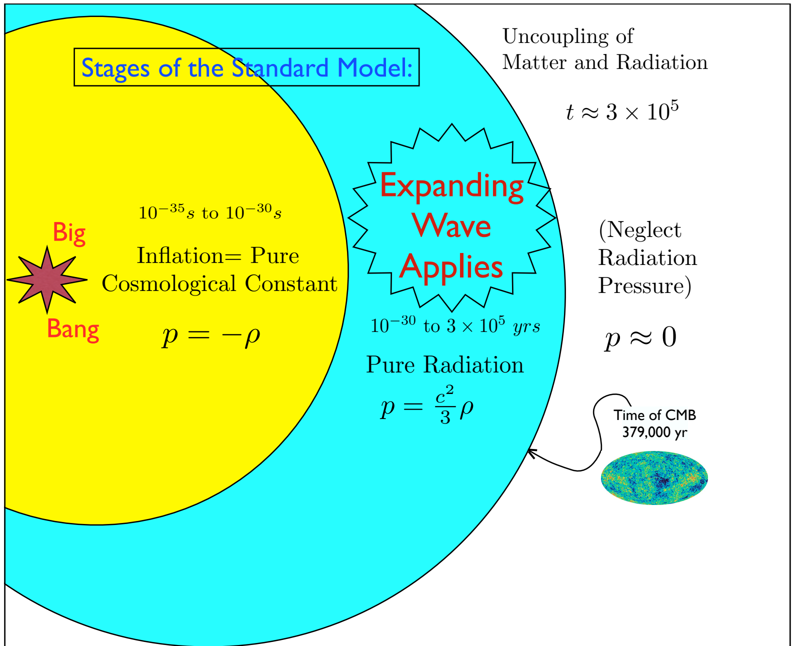


Figure 1. Stages of the Big Bang (a) The beginning (b) The early stage (c) The late stage (d) After the event horizon

- Zeke Volger and I set out to simulate the secondary expansion wave numerically to see if it might account for the anomalous acceleration of the galaxies...when Smoller joined us and we attempted to set up the simulation, we discovered a family of exact self-similar GR expansion waves defined independently of the Shock-Wave...





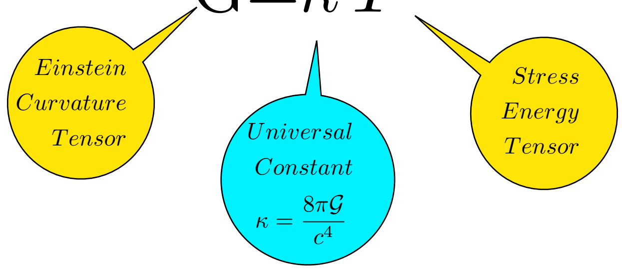
- PNAS 2009 we introduce a family of expanding wave solutions of the Einstein equations that exist during the radiation phase of the expansion...

Introduction
to
General Relativity

Introduction to General Relativity

- GR is the modern theory of the gravitational field
- In 1915, Albert Einstein introduced the Einstein Gravitational Field Equations

$$\mathrm{G} = \kappa T$$



- “Energy-momentum and their fluxes are the sole source of spacetime curvature”
- Especially pleasing because everything converts into energy via (roughly) $E = mc^2$ (1905)

The unknown to be solved for is the gravitational metric tensor **g**

$$ds^2 = g_{ij}dx^i dx^j \equiv \sum_{i,j=0}^{3} g_{ij}dx^i dx^j$$

g gives you time changes along timelike curves and spatial lengths along spacelike curves

- Basic Principle of General Relativity:

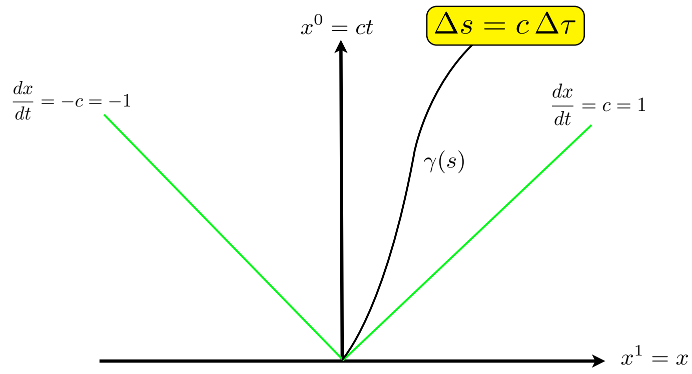
"All properties of the gravitational field are determined by a signature (-1,1,1,1) metric g *defined on the 4-dimensional manifold of events"*

$$\mathcal{M} \equiv Spacetime$$

- Q1: What can you measure from g?
- Q2: What are the constraints that determine the time evolution of g?

■ What you can measure:

(1)*"Proper time change or aging time, as measured by an observer traversing a timelike curve through spacetime, will equal the arclength as measured by g"*

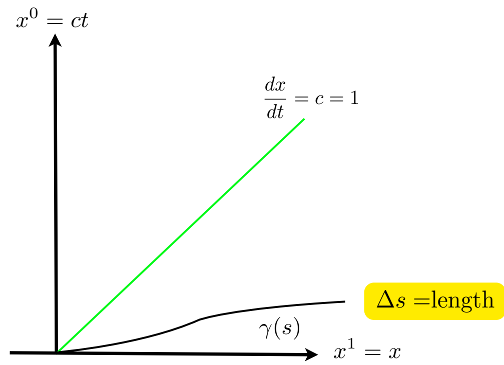


$$ds^2 = g_{ij}dx^i dx^j$$

$$\Delta s = \int ds \equiv \int_a^b \|\dot{\gamma}(\xi)\|\, d\xi$$

What you can measure:

(2) *"<u>Spatial lengths</u> of objects correspond to g-lengths of the curves that define their shape"*

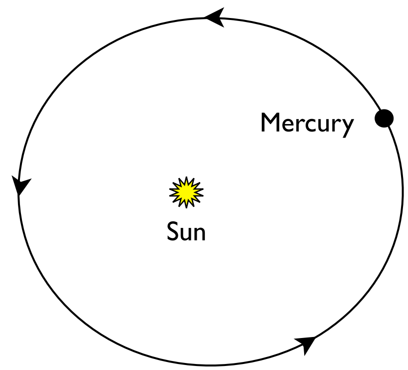


$$ds^2 = g_{ij} dx^i dx^j$$

$$\Delta s = \int ds \equiv \int_a^b \|\dot{\gamma}(\xi)\| \, d\xi$$

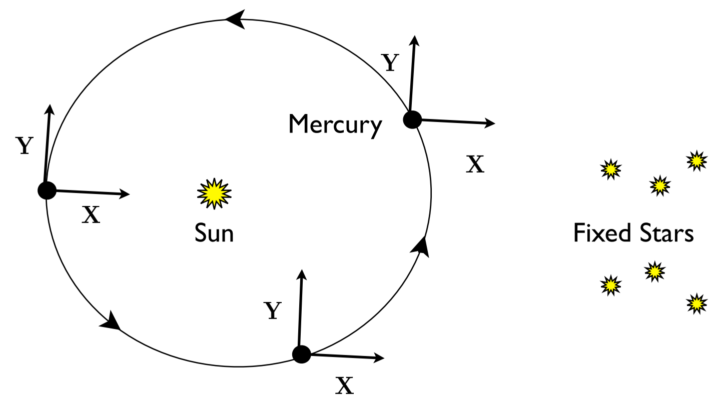
- What you can measure:

(3) *"Freefall paths through the gravitational field are geodesics of the spacetime metric* g"



- What you can measure:

(4) *"Non-rotating vectors (gyroscopes) carried by an observor in freefall are parallel transported by the unique symmetric connection determined by* g*"*



- Geodesics and Parallel translation are determined by the Covariant Derivative:

$$\nabla_{\mathbf{Y}}\mathbf{X} = \mathbf{Y}(\mathbf{X}) + \Gamma^k_{ij} X^i Y^j \frac{\partial}{\partial x^k}$$

Where the Christoffel symbols are given by...

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\sigma} \{ -g_{ij,\sigma} + g_{\sigma i,j} + g_{j\sigma,i} \}$$

For example, if... $\dot{\gamma}(s) = \mathbf{Y} \equiv Y^j \frac{\partial}{\partial x^j}$

Then... $\mathbf{Y}(\mathbf{X}) = \dot{\mathbf{X}} = \frac{d}{ds}\mathbf{X}(\gamma(s)) = Y^j \frac{\partial}{\partial x^j}\mathbf{X}$

- The main point:

"The Covariant Derivative corrects differentiation of vectors to a tensor operation..."

$$\nabla_{\mathbf{Y}}\mathbf{X} = \mathbf{Y}(\mathbf{X}) + \Gamma^k_{ij} X^i Y^j \frac{\partial}{\partial x^k}$$

A tensor!!

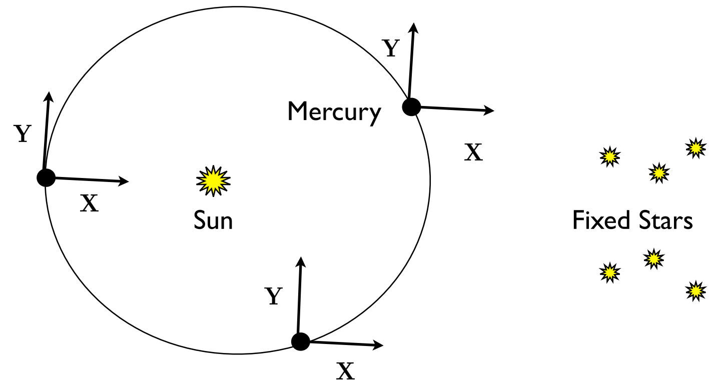
Not a tensor

Not a tensor

The components of $\nabla_{\mathbf{Y}}\mathbf{X}$ transform like a vector under change of coordinates...

$$\mathbf{X}^i = \mathbf{X}^\alpha \frac{\partial x^i}{\partial y^\alpha} \quad \mathbf{Y}^i = \mathbf{Y}^\alpha \frac{\partial x^i}{\partial y^\alpha} \qquad (\nabla_{\mathbf{Y}}\mathbf{X})^i = (\nabla_{\mathbf{Y}}\mathbf{X})^\alpha \frac{\partial x^i}{\partial y^\alpha}$$

Covariant Derivative gives you Geodesics and Parallel Translation



- Geodesics (freefall paths)

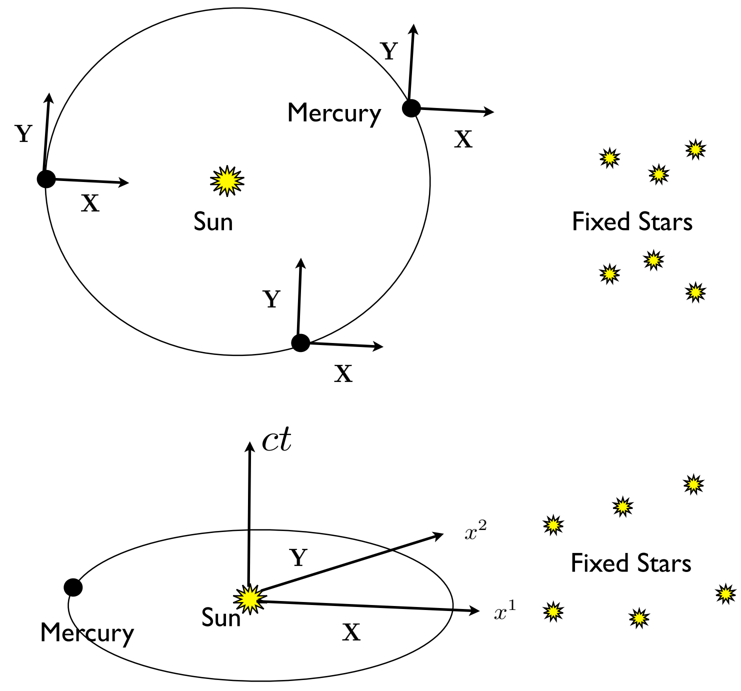
$$\nabla_T T = 0 \quad \Leftrightarrow \quad \mathbf{T} \text{ tangent to the geodesic}$$

- Parallel Translation (the non-rotating frames carried by freefall)

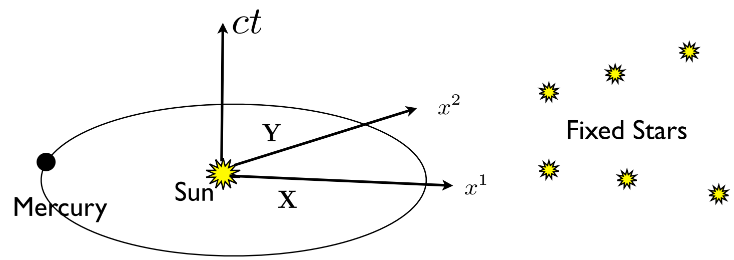
$$\nabla_T X = 0 \quad \nabla_T Y = 0 \quad \Leftrightarrow \quad \mathbf{X}, \mathbf{Y} \text{ parallel in direction } \mathbf{T}$$

...but **T** is the tangent vector in space*time*...

...but $\mathbf{T}$ is the tangent vector in space*time*...



...but $\mathbf{T}$ is the tangent vector in spacetime...



In spacetime the trajectory (world line) of Mercury is...

$$\gamma(s) = (x^0(s), x^1(s), x^2(s))$$

$$\mathbf{T} = \dot{\gamma}(s) = (\dot{x}^0(s), \dot{x}^1(s), \dot{x}^2(s))$$

$$\dot{x}^0 = \frac{d(ct)}{ds} = \frac{d(ct)}{d(c\tau)} \approx 1$$

$$\dot{x}^i = \frac{dx^i}{ds} = \frac{dx^i}{d(c\tau)} \approx \frac{1}{c} << 1$$

$$ds^2 = g_{ij}dx^i dx^j$$

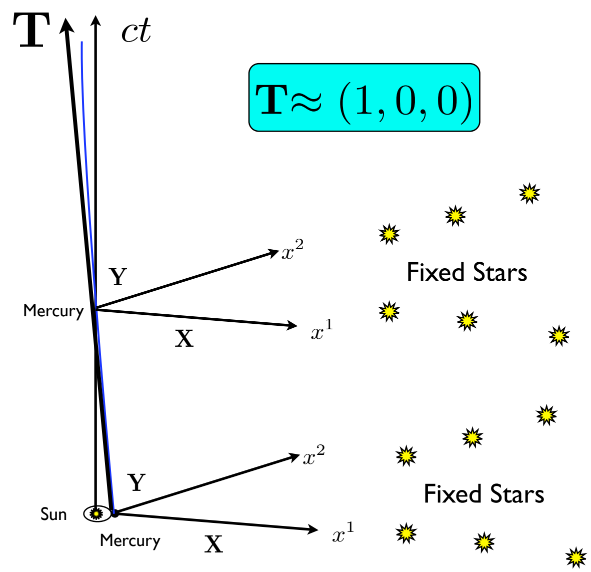
$$ds = c\,d\tau$$

$$dx^0 = c\,dt$$

so...

$$\mathbf{T} \approx (1, 0, 0)$$

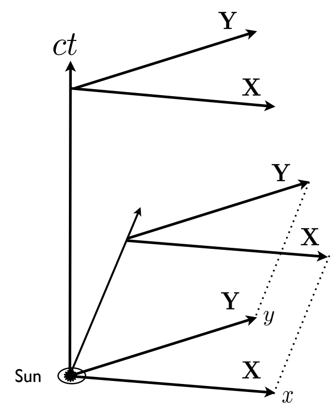
...so the correct picture is...



- The non-rotating frames parallel translated along T remain almost aligned with the fixed stars, with general relativistic corrections determined by how g differs from flat spacetime.

In flat Minkowski space the geodesics are straight lines, and parallel translation is fixed with the coordinate axes

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$



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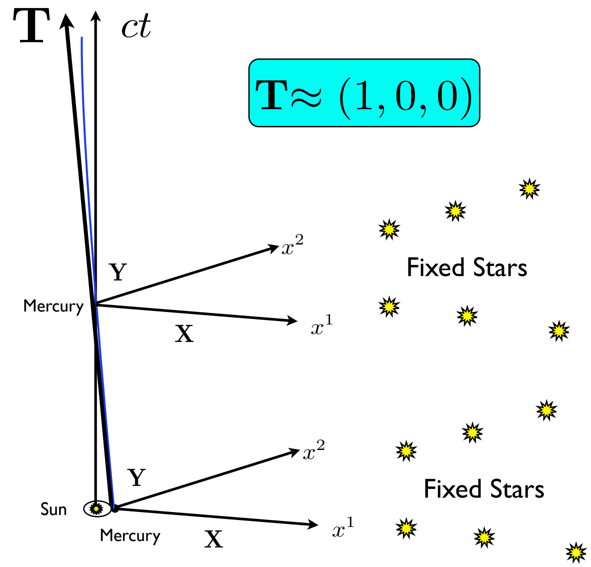
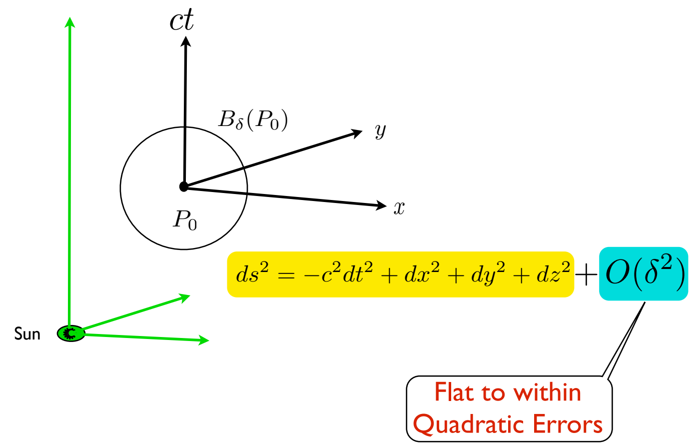


"We are connected to the stars by an almost flat Minkowski spacetime with GR corrections order 1/c."

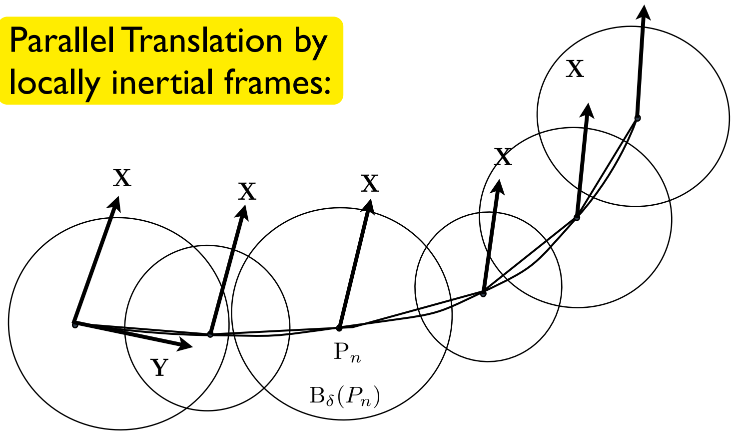
Curved spacetime is locally Minkowskian in the sense that around every point there exist

Locally Inertial coordinates



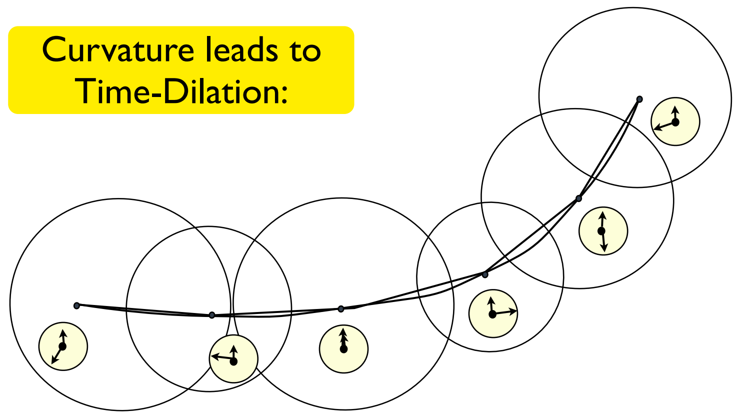
The covariant derivative reflects the locally inertial character of spacetime

Parallel Translation by locally inertial frames:



- Given path $\gamma(s)$ in spacetime, cover it with locally inertial coordinate frames $B_\delta(P_n)$
- Transport components as constant in each inertial frame
- Refine to squeeze out quadratic errors...

- Clocks in different locally inertial frames "run at different rates" and cannot be synchronized by any global time coordinate...

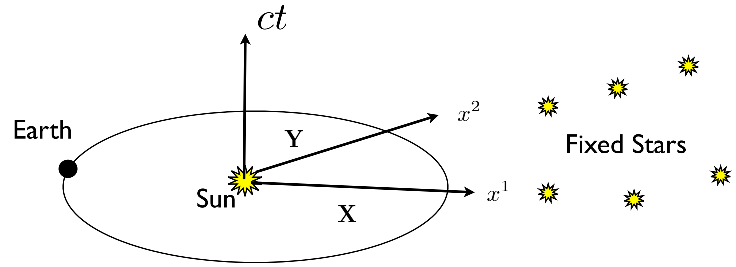


- Conclude: Parallel Translation must agree with $\nabla_{\mathbf{Y}}\mathbf{X} = 0$ in order that spacetime have (locally) the same inertial properties as flat Minkowski Spacetime...

- Reverse it: $\nabla_{\mathbf{Y}}\mathbf{X} = 0$ gives a coordinate independent (covariant) description of Parallel Translation by locally inertial frames

"In General Relativity: Inertial coordinate systems are local properties of spacetime that change from point to point"

- A picture: The earth moves “unaccelerated” through each local inertial frame, but these frames change from point to point, thus producing apparent accelerations in a global coordinate system in which metric components $\neq (-\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$



- A point of view: “One can view the gravitational metric as a sort of book-keeping device for keeping track of the locally inertial coordinate systems as they change from point to point in spacetime”

- Even though the physics is most naturally expressed in a locally inertial coordinate system, the analysis of solutions can only be done in global coordinate systems that hide the locally inertial simplicity.

This motivates our

"locally inertial Godunov method"

for simulating

GR-shock-waves

in spherically symmetric

spacetimes

Zeke Vogler Thesis, UC-Davis March 2010

- There is no global inertial coordinate system in which planetary trajectories are all straight lines...

- This is an expression of the fact that gravitational fields produce non-zero spacetime curvature...

- Theorem: You cannot in general remove the second derivatives of g at the center of a locally intertial coordinate system, and these measure SPACETIME CURVATURE

$$\frac{\partial^2 g}{\partial x^j \partial x^k} \equiv g_{ij,jk} \neq 0$$

Riemann 1854:

- Introduced the Riemann Curvature Tensor

"A tensorial measure of the 2nd derivatives

$g_{ij,kl}$

that cannot be removed by coordinate transformation"

$$R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \left\{\Gamma^\sigma_{jl}\Gamma^i_{\sigma k} - \Gamma^\sigma_{jk}\Gamma^i_{\sigma l}\right\}$$

A Tensor!

Curl + Commutator

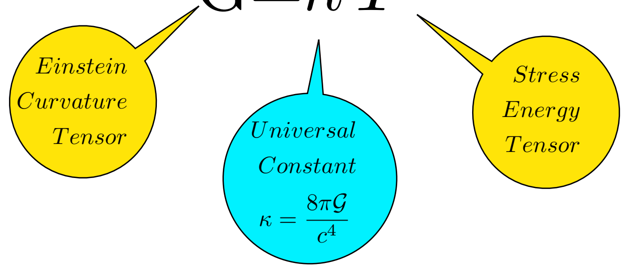
(Not a tensor) (Not a tensor)

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\sigma} \left\{-g_{ij,\sigma} + g_{i,j} + g_{j\sigma,i}\right\}$$

■ Not every spacetime metric can be a gravitational field

The Einstein equations give the constaints on g

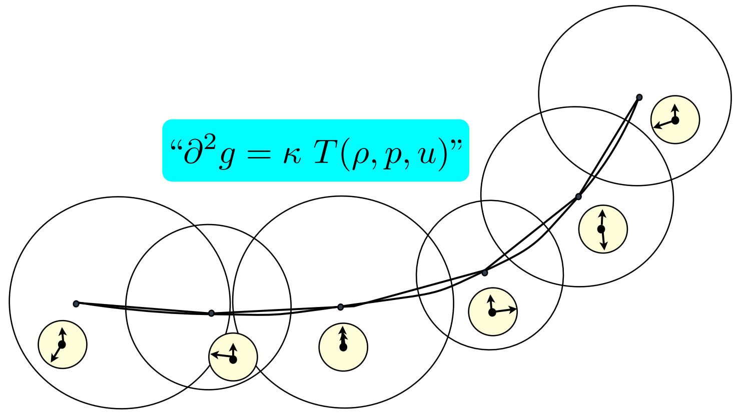
$$\mathrm{G} = \kappa\, T$$



- "Energy-momentum and their fluxes are the sole source of spacetime curvature"

- Said Differently: $G = 8\pi T$ gives the constraints under which locally inertial frames interact and evolve…

$\mathbf{G}{=}8\pi T$: The Constraints on the evolution of g…



$$G = \frac{8\pi\mathcal{G}}{c^4}\, T$$

Einstein Curvature Tensor

Fundamental Constant

Stress Energy Tensor

- In a coordinate system x:

$$G_{ij} \equiv R^{\sigma}_{i\sigma j} - \frac{1}{2} R^{\sigma\tau}_{\sigma\tau} g_{ij}$$

- For a perfect fluid:

$$T_{ij} = (\rho + p) u_i u_j + p g_{ij}$$

$\rho =$ energy density$=\rho c^2$

$u =$ 4-velocity$=\frac{dx}{ds}$

$p =$ pressure

T_{ij} gives the energy and momentum densities and their fluxes for a perfect fluid, the RHS of the Einstein Equations G=kT

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}$$

But: The Einstein equations are constructed so that $Div\ G = 0$ follows as a consequence of the Bianchi identies:

$$R^k_{i[jk,l]=0}$$

As a result: Conservation of energy-momentum is identically satisfied on solutions:

$$Div\ T = 0$$

These reduce to the relativistic compressible Euler equations in each locally inertial frame...

"The Euler equations are a subsystem of the Einstein Equations!!!!...

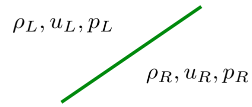
■ Q: How smooth should the metric be at Shock Waves?

$$G = \kappa\, T$$

↕

$$\text{“}\partial^2 g = \kappa\, T(\rho, p, u)\text{”}$$

SHOCK → Jump Discontinuity in fluid



$G = \kappa\, T$ → Jump in 2nd derivative of g

→ $g \in C^{1,1}$??

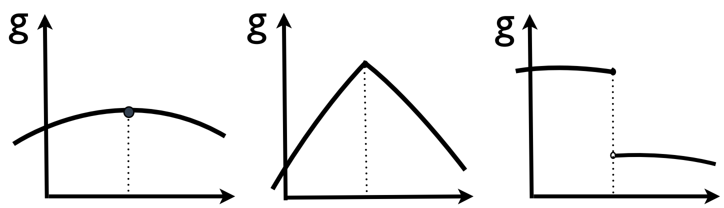
$C^{1,1} \equiv$ 1-derivative Lipschitz continuous

■ Smoothness of Metric at Shocks

$$“\partial^2 g = \kappa\, T(\rho, p, u)”$$

RHS discontinuous

LHS has one continuous derivative



$g \in C^{1,1}$ $\quad$ $\nabla g \in C^{0,1}$ $\quad$ $\nabla^2 g \in C^{0,1}$

Jump Discont.

- BUT: The shock-wave solutions we construct are only $C^{0,1}$ at shock-waves

Ref: TE/GR Memoirs 2004: They are true weak solutions of the Einstein equations

Conclude: Solutions are one degree less smooth than the equations ask for!

Open question: Is there a change of coordinates that smooths the metric components to $C^{1,1}$??

For single shock surfaces the answer is YES. Ref. Israel/TeSm

Thesis Problem-Moritz Rientes: Can the metric be smoothed at points of shock-wave interaction? (OPEN)

Examples of Gravitational Metrics: Exact Solutions of the Einstein Equations

■ Examples:

(1) Schwarzschild Metric:

(Gravitational field outside a star)

$$ds^2 = -\left(1 - \frac{2\mathcal{G}M}{r}\right) dt^2 + \frac{1}{(1 - \frac{2\mathcal{G}M}{r})} dr^2 + r^2 d\Omega^2 \qquad \text{(S)}$$

$\mathcal{G} = \text{Newton's Constant}$

$M = \text{Mass of the Sun at } r = 0$

➡ Planets follow geodesics of (S)

(Schwarzschild Radius= $2\mathcal{G}M$)

Birkoff's Theorem: (S) is the only spherically symmetric gravitational field in empty space.

(2) Tolman-Oppenheimer-Volkoff (TOV) Metric:

(Static fluid sphere)

$$ds^2 = -B(r)dt^2 + \frac{1}{(1 - \frac{2\mathcal{G}M(r)}{r})}dr^2 + r^2 d\Omega^2 \quad \text{(TOV)}$$

$\approx$ Gravitational field inside a star

$$M(r) = \text{"Total mass inside radius r"}$$

Setting for:

Chandresekhar Stability Limit
Buchdahl Stability Limit

(3) Friedmann-Robertson-Walker (FRW) Metric:

(Standard Model of Cosmology)

$$ds^2 = -dt^2 + R(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\} \qquad \text{(FRW)}$$

$\mathrm{R(t)} \equiv$ Cosmological Scale Factor

Big Bang → $0 \leq R(t) \leq 1$ ← Present Universe

$$H = \frac{\dot{R}}{R} = \text{Hubble Constant} \approx h_0 \frac{100\ km}{s\, mps}$$

Galaxies follow geodesics of (FRW)

$k < 0$	⟷	$\Omega_M = \frac{Q_0}{Q_{crit}} < 1$	(Open)
$k = 0$	⟷	$\Omega_M = 1$	(Critical)
$k > 0$	⟷	$\Omega_M > 1$	(Closed)

All 3 are examples of Spherically Symmetric Metrics of the general form

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + E(r,t)dtdr + C(r,t)d\Omega^2$$

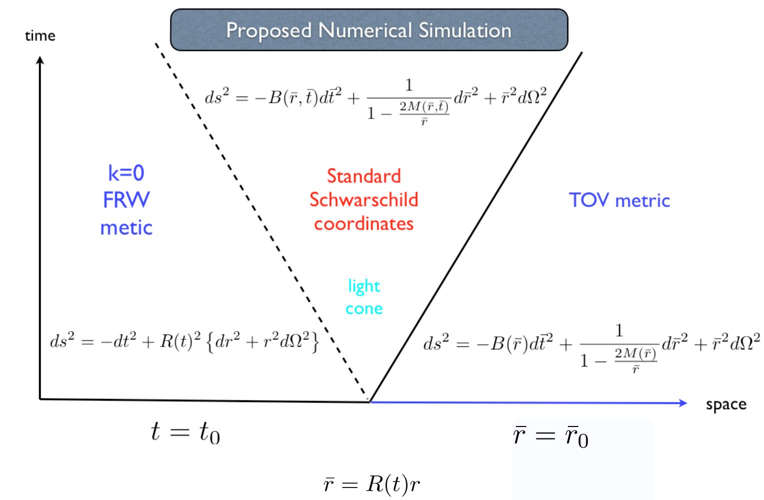
$d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2$ = "line element on unit sphere"

Theorem: Every Spherically Symmetric Spacetime can (generically) be transformed over to Standard Schwarzschild Coordinates (SSC) where the metric takes the simpler form

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega^2$$

To do a numerical simulation of GR-shock waves we match the FRW metric to the TOV metric Lipschitz continuously on an initial surface, and numerically simulate the evolution in SSC coordinates

This requires <u>MAPPING</u> FRW over to SSC coordinates.



Einstein Equations
in
Standard
Schwarzchild
Coordinates

The Simplest Setting for Shock-Waves

- Spherical Symmetry–
  Assume Standard Schwarzschild Coordinates:

$$\begin{aligned} g_{ij}dx^i dx^j &= \\ ds^2 &= -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega^2 \end{aligned}$$

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2,$$
$$\mathbf{x} = (x^0, ..., x^3) \equiv (t, r, \theta, \phi).$$

- Define the mass function $M(r,t)$:

$$B(r,t) \equiv \left(1 - \frac{2M(r,t)}{r}\right)^{-1}$$

- Stress Tensor T :

$$T^{ij} = (\rho c^2 + p) w^i w^j + p g^{ij}, \quad i,j = 0, ..., 3$$

$$\begin{aligned} A \cdot T^{00} &= \frac{c^4 + \sigma^2 v^2}{c^2 - v^2} \rho = T_M^{00} = u^0 \\ \sqrt{AB} \cdot T^{01} &= \frac{c^2 + \sigma^2}{c^2 - v^2} c v \rho = T_M^{01} = u^1 \\ B \cdot T^{11} &= \frac{v^2 + \sigma^2}{c^2 - v^2} \rho c^2 = T_M^{11} \end{aligned}$$

$\rho c^2 = density, \qquad p = pressure, \qquad v = velocity$

- Assume Equation of State:

$$p = \sigma^2 \rho$$

$\sigma = sound\ speed < c = light\ speed.$

Consequences

- The equation (1)$\equiv M' = \frac{1}{2}\kappa r^2 A T^{00}$ implies:

$$M(r,t) = M_{r_0} + \frac{\kappa}{2}\int_{r_0}^{r} T_M^{00}(r,t) r^2 \, dr$$

- The scalar curvature R satisfies

$$R = (c^2 - 3\sigma^2)\rho$$

- Components of T_M satisfy:

$$\left|T_M^{01}\right| < T_M^{00},$$
$$\frac{\sigma^2}{c^2+\sigma^2} T_M^{00} < T_M^{11} < T_M^{00}$$

- This defines the simplest setting for shock wave propagation in General relativity.

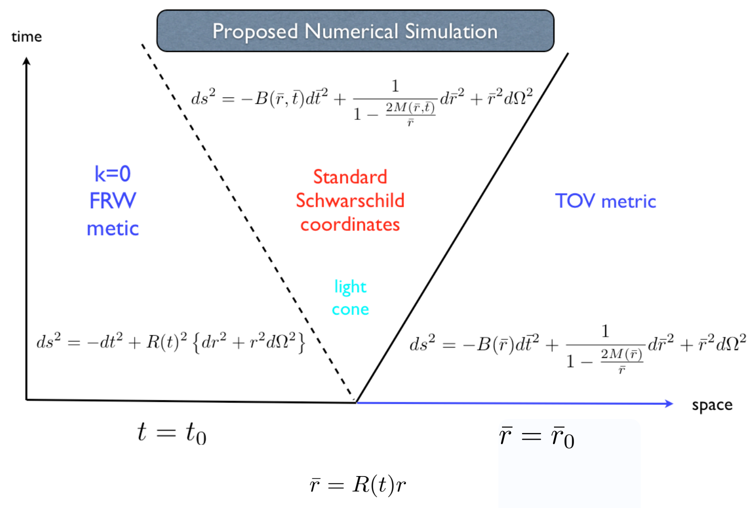
- The Einstein equations require a constraint on the initial data to even get started with a simulation...

- Vogler starts with two solutions FRW and TOV that automatically statisfy constraints.

- Matching Lipschitz continuously is sufficient to (weakly) meet the constraints on the initial data and start the simulation.

- To match them continuously we put initial data into common SSC coordinates.

- The Locally Inertial Godunov Method works in SSC.


Proposed Numerical Simulation
time
space
k=0
FRW
metic
Standard
Schwarschild
coordinates
TOV metric
light
cone
$ds^2 = -B(\bar{r},\bar{t})d\bar{t}^2 + \frac{1}{1-\frac{2M(\bar{r},\bar{t})}{\bar{r}}}d\bar{r}^2 + \bar{r}^2 d\Omega^2$
$ds^2 = -dt^2 + R(t)^2\left\{dr^2 + r^2 d\Omega^2\right\}$
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$t = t_0$
$\bar{r} = \bar{r}_0$
$\bar{r} = R(t)r$

A Locally Inertial Method for Computing Shocks

Einstein Equations-Standard Schwarzschild Coordinates

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2$$

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For Example: $$B(t,r) = \frac{1}{1 - \frac{2\mathcal{G}M(t,r)}{r}}$$

$M(t,r)$=Mass inside radius r at time t

$\mathcal{G}$ =Newton's Gravitational Constant

$2\mathcal{G}M(t,r) = 1$ **Black Hole**

Einstein Equations-Standard Schwarzschild Coordinates

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2$$

$G = 8\pi T$ ⇒

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(MAPLE)

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$$\Phi = -\frac{BA_tB_t}{2AB} - \frac{B}{2}\left(\frac{B_t}{B}\right)^2 - \frac{A'}{r} + \frac{AB'}{rB} + \frac{A}{2}\left(\frac{A'}{A}\right)^2 + \frac{A}{2}\frac{A'}{A}\frac{B'}{B}.$$

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(1)+(2)+(3)+(4) ⟺ (weakly) (1)+(3)+div T=0

(Te-Groah Memoirs 2004)

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- Time derivatives A_t and B_t cancel out under change $T \to \mathbf{u}$

- Good choice because o.w. there is no A_t equation to close $Div\ T = 0$!

$$\frac{A}{r^2 B}\left\{r\frac{B'}{B} + B - 1\right\} = \kappa A^2 T^{00} \quad (1)$$

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Locally Inertial Formulation

$$\left\{T_M^{00}\right\}_{,0} + \left\{\sqrt{\frac{A}{B}}T_M^{01}\right\}_{,1} = -\frac{2}{x}\sqrt{\frac{A}{B}}T_M^{01}, \tag{2}$$

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Flat Space Relativistic Euler

$$T_M^{00} = \frac{c^4 + \sigma^2 v^2}{c^2 - v^2}\rho$$
$$T_M^{01} = \frac{c^2 + \sigma^2}{c^2 - v^2}cv\rho$$
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$$\mathbf{u}_t + f(\mathbf{A}, \mathbf{u})_x = g(\mathbf{A}, \mathbf{u}, x)$$
$$\mathbf{A}' = h(\mathbf{A}, \mathbf{u}, x)$$

$$\mathbf{u} = (T_M^{00}, T_M^{01})$$
$$\mathbf{A} = (A, B)$$

The Locally Inertial Equations

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-----Topic of Moritz Rientes Thesis.

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- Note: In GR it is usually assumed that metrics are at least $C^{1,1}$, and for example, this is assumed in Hawking-Penrose Theorems

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- **Theorem:** (Groah-Te Memoirs 2004) If for $r \geq r_0 > 0$

  (1) $TV\{\ln\rho(r)\} < V_0 \ln\rho(r)$, $TV\left\{\frac{c-v}{c+v}\right\} < V_0$

  (2) $A_0(r)$, $B_0(r)$ are Lipshitz continuous solutions of (3), (4)

  Then there exists a weak solution of (1)-(4) on $r \geq r_0$, $0 < t \leq T$

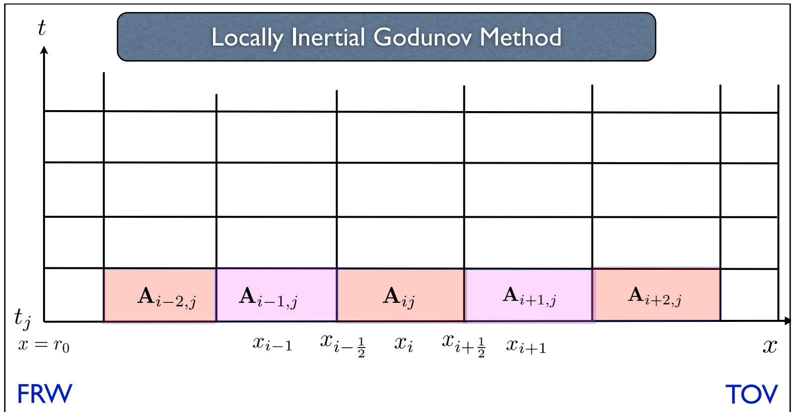
- **Conclude:** The Einstein equations are consistent at the level of arbitrary numbers of interacting shock waves of arbitrary strength

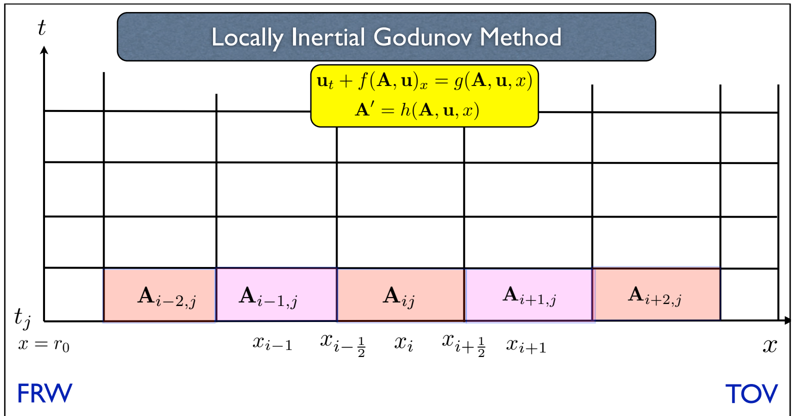
- Any Lipshitz cont. metric that meets constraints (2), (3) suffices

Locally inertial Godunov Method with Dynamic Time-Dilation

Zeke Vogler
UC-Davis
2010

Locally Inertial Godunov Method




Locally Inertial Godunov Method
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t
t_j
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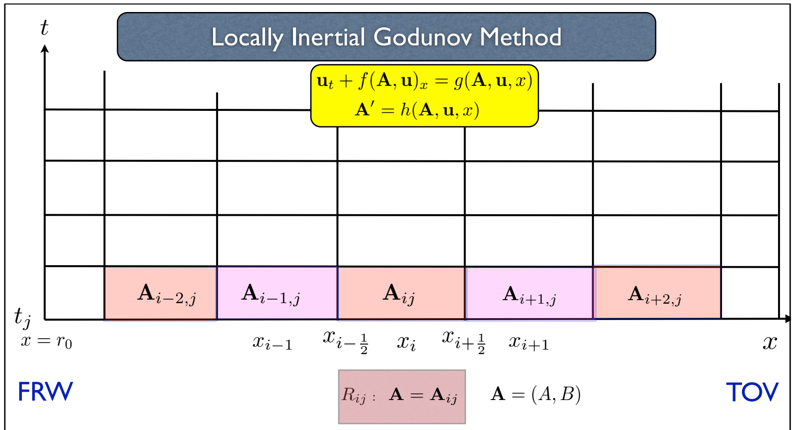
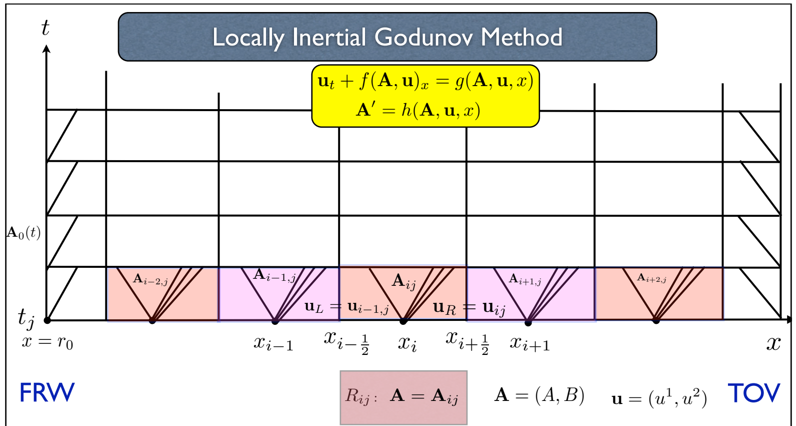
x_{i-1}
$x_{i-\frac{1}{2}}$
x_i
$x_{i+\frac{1}{2}}$
x_{i+1}
x
FRW
TOV

Locally Inertial Godunov Method



- $\mathbf{A}_{ij} = const.$ ➡ Locally Flat in each Grid Cell

Locally Inertial Godunov Method

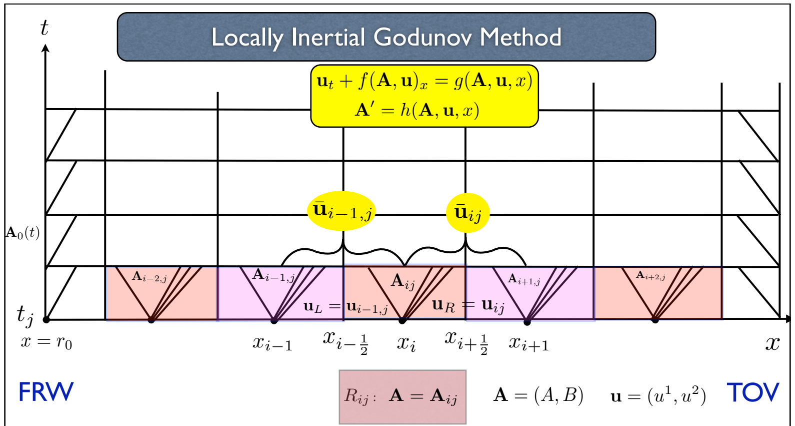


- $\mathbf{A}_{ij} = const.$ ➡ Locally Flat in each Grid Cell
- Stagger fluid discontinuities with metric discontinuities and solve Riemann Problem for 1/2 time-step

$$\mathbf{u}_t + f(\mathbf{A}_{ij},\mathbf{u})_x = 0$$

$$\mathbf{u} = \begin{cases} \mathbf{u}_{i-1,j} & x \le x_i \\ \mathbf{u}_{ij} & x > x_i \end{cases} \longrightarrow u_{ij}^{RP}$$

Locally Inertial Godunov Method



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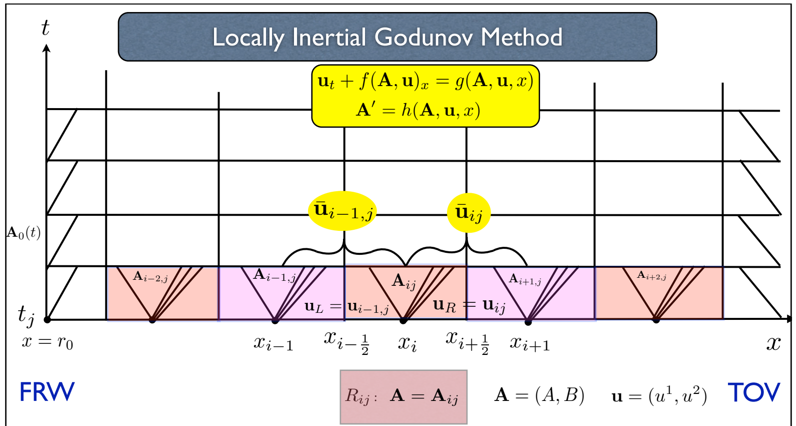
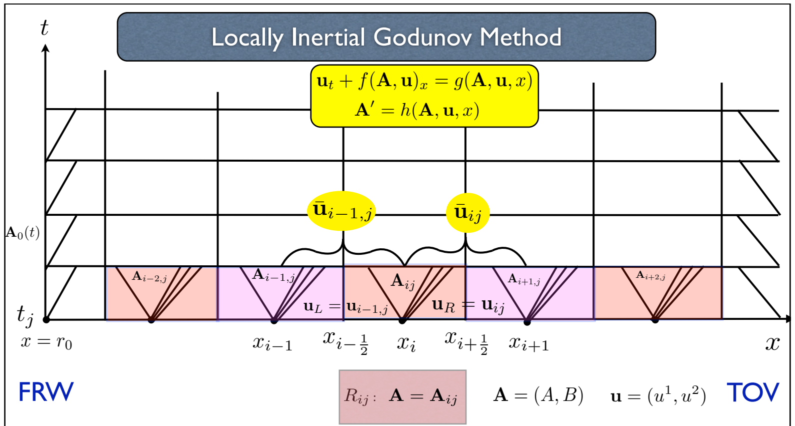


- $\mathbf{A}_{ij} = const.$ ➡ Locally Flat in each Grid Cell
- Stagger fluid discontinuities with metric discontinuities and solve Riemann Problem for 1/2 time-step
- Take Godunov Averages
- Solve Source-ODE for 1/2 time-step (operator splitting)

$$\mathbf{u}_t = g(\mathbf{A}_{ij},\mathbf{u},x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$$

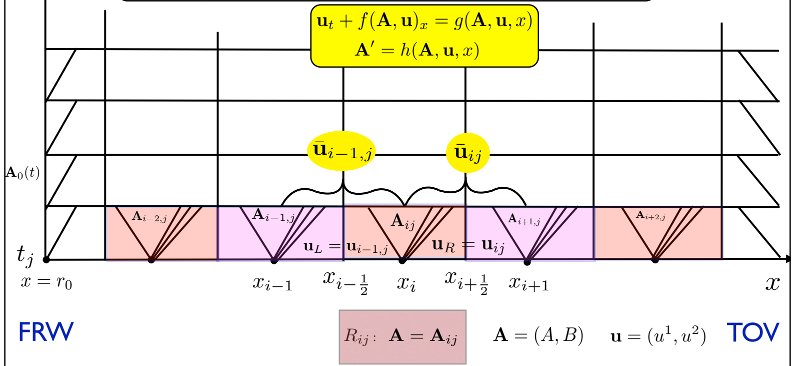
$$\mathbf{u}(0) = \bar{\mathbf{u}}_{ij}$$

Locally Inertial Godunov Method



- $\mathbf{A}_{ij} = const.$ ➡ Locally Flat in each Grid Cell
- Stagger fluid discontinuities with metric discontinuities and solve Riemann Problem for 1/2 time-step
- Take Godunov Averages
- Solve Source-ODE for 1/2 time-step (operator splitting)
- Update the metric: $\mathbf{A}' = h(\mathbf{A},\mathbf{u},x)$, $\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$

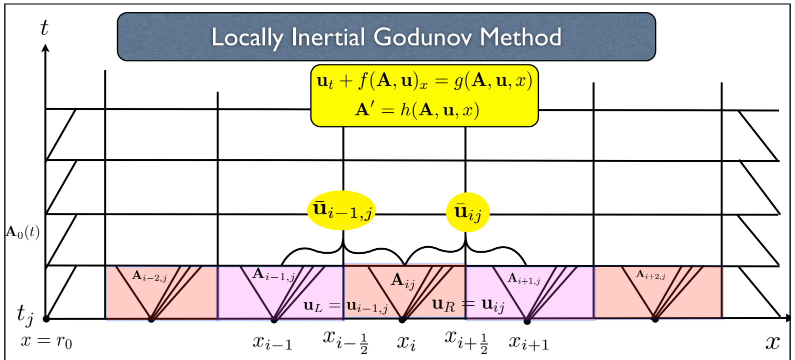
Locally Inertial Godunov Method



- $\mathbf{A}_{ij} = const.$ ➡ Locally Flat in each Grid Cell

Fractional Step Method: — Solve RP for 1/2 timestep / Solve ODE for 1/2 time step ➡ Sample or Average ➡ $\mathbf{A}' = h(\mathbf{A},\mathbf{u},x)$, $\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$

Locally Inertial Godunov Method



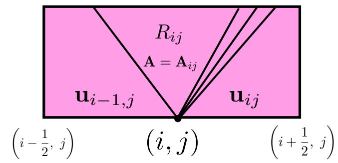
FRW $\quad R_{ij}: \ \mathbf{A} = \mathbf{A}_{ij} \quad \mathbf{A} = (A, B) \quad \mathbf{u} = (u^1, u^2)$ TOV

- $\mathbf{A}_{ij} = const.$ ➡ Locally Flat in each Grid Cell

Fractional Step Method: — Solve RP for 1/2 timestep; Solve ODE for 1/2 time step ➡ Sample or Average ➡ $\mathbf{A}' = h(\mathbf{A},\mathbf{u},x)$, $\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$

$p = \frac{c^2}{3}\rho$ $\Longrightarrow$ Nishida System $\Longrightarrow$ Global Exact Soln of RP, [Smol,Te]

Grid Rectangle



- Solve RP for $\frac{1}{2}$-timestep

$$\mathbf{u}_t + f(\mathbf{A}_{ij}, \mathbf{u})_x = 0$$
$$\mathbf{u} = \begin{cases} \mathbf{u}_{i-1,j} & x \le x_i \\ \mathbf{u}_{ij} & x > x_i \end{cases} \longrightarrow u_{ij}^{RP}$$

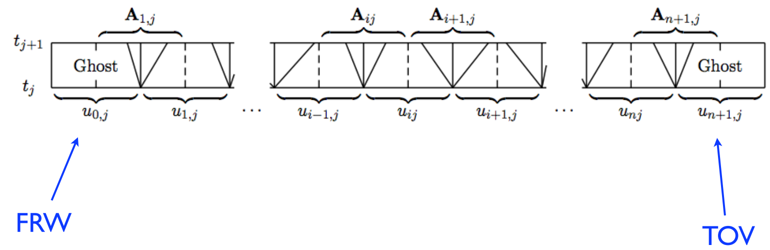
- Solve ODE for $\frac{1}{2}$-timestep

$$\mathbf{u}_t = g(\mathbf{A}_{ij}, \mathbf{u}, x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$$
$$\mathbf{u}(0) = \mathbf{u}_{ij}^{RP}$$

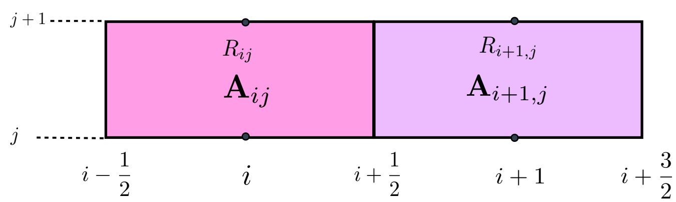
- Sample/Average then update $\mathbf{A}$ to time t_{j+1}

$$\mathbf{A}' = h(\mathbf{A}, \mathbf{u}, x)$$
$$\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$$

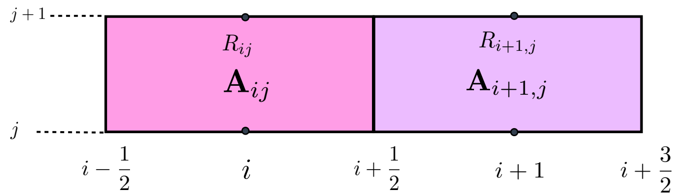
Staggering the metric A and solution u



A=Constant
in
Consecutive Grid Rectangles:

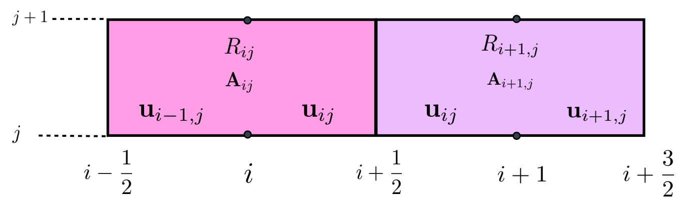


A=Constant
in
Consecutive Grid Rectangles:

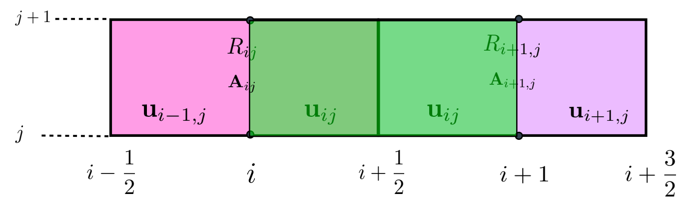


A=constant in each grid cell ➡ "Locally Inertial"

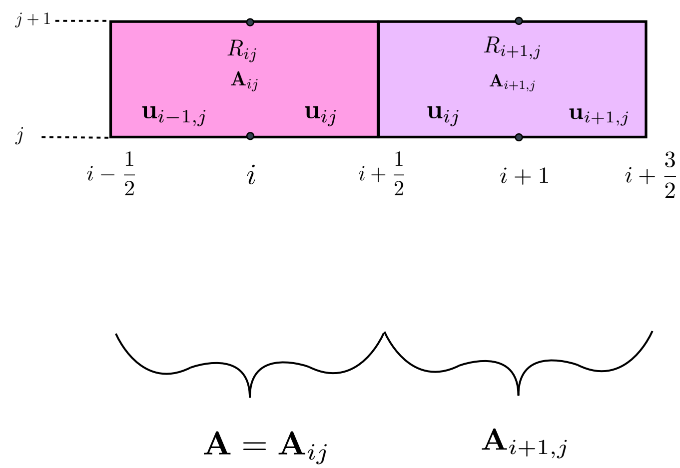
Stagger the Fluid Discontinuities with the Metric Discontinuities



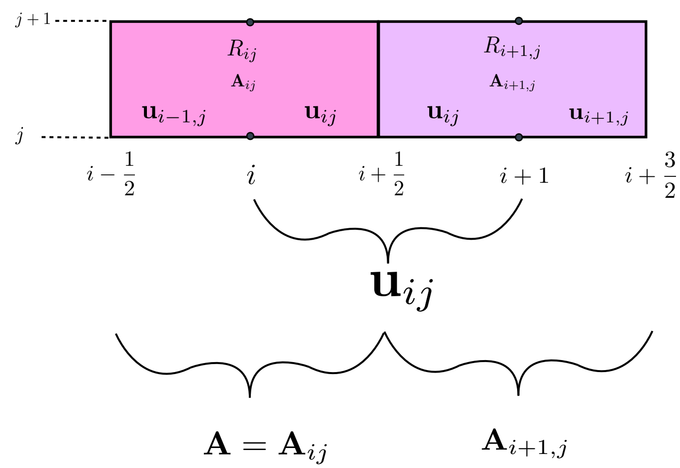
Stagger the Fluid Discontinuities with the Metric Discontinuities



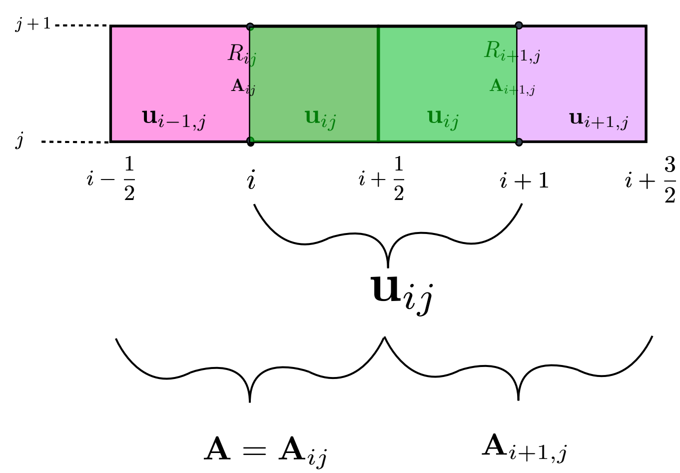
Stagger the Fluid Discontinuities with the Metric Discontinuities



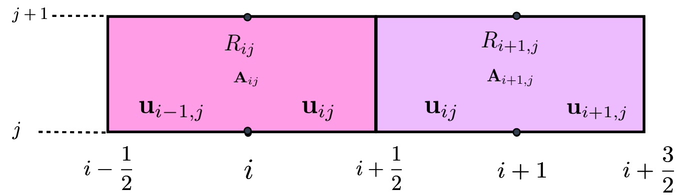
Stagger the Fluid Discontinuities with the Metric Discontinuities



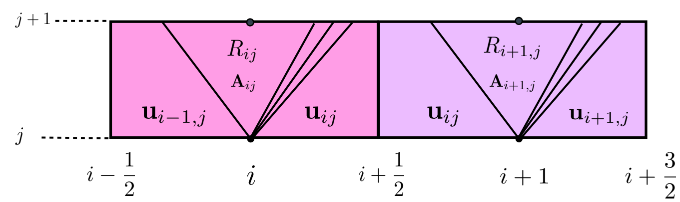
Stagger the Fluid Discontinuities with the Metric Discontinuities



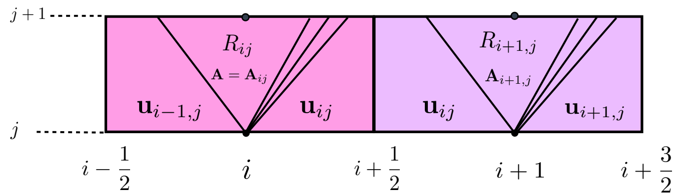
Solve the Riemann Problem in Each Grid Cell



Solve the Riemann Problem in Each Grid Cell



Solve the Riemann Problem in Each Grid Cell

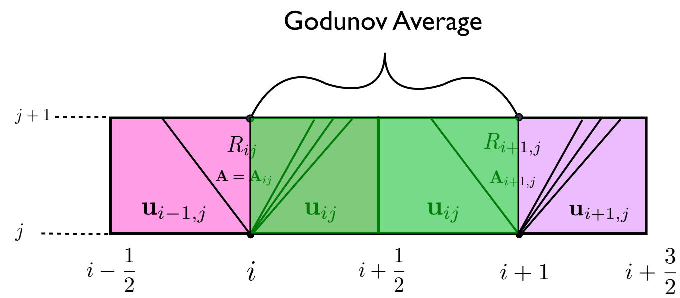


Solve the Riemann Poblem for half a time step in R_{ij}

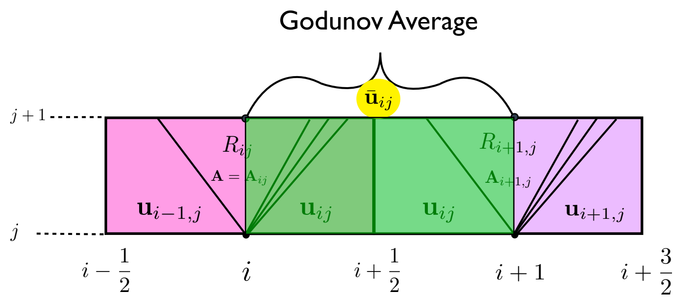
$$\mathbf{u}_t + f(\mathbf{A}_{ij}, \mathbf{u})_x = 0$$

$$\mathbf{u} = \begin{cases} \mathbf{u}_{i-1,j} & x \leq x_i \\ \mathbf{u}_{ij} & x > x_i \end{cases} \longrightarrow u_{ij}^{RP}$$

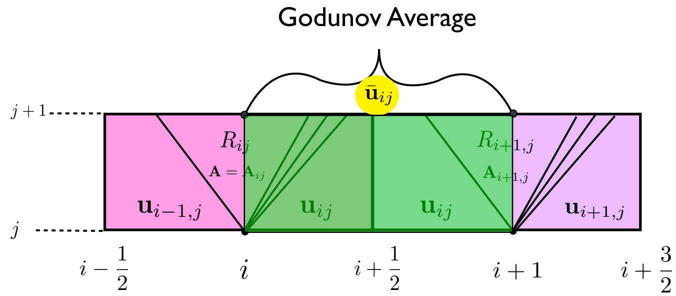
Average Across Two Half-cells



Average Across Two Half-cells



Run the Godunov Average under the ODE step for half a time-step



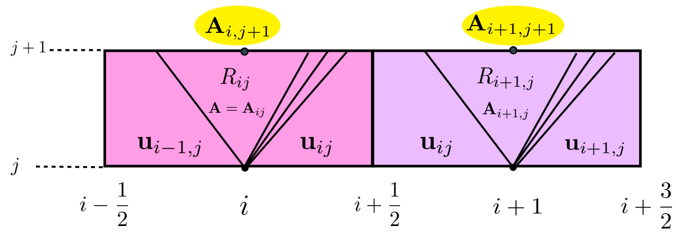
Solve ODE for $\frac{1}{2}$-timestep

$$\mathbf{u}_t = g(\mathbf{A}_{ij}, \mathbf{u}, x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$$

$$\mathbf{u}(0) = \bar{\mathbf{u}}_{ij}$$

(Operator Splitting)

Update A by solving ODE



Update $\mathbf{A}$ to time t_{j+1}

$$\mathbf{A}' = h(\mathbf{A}, \mathbf{u}, x)$$

$$\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$$

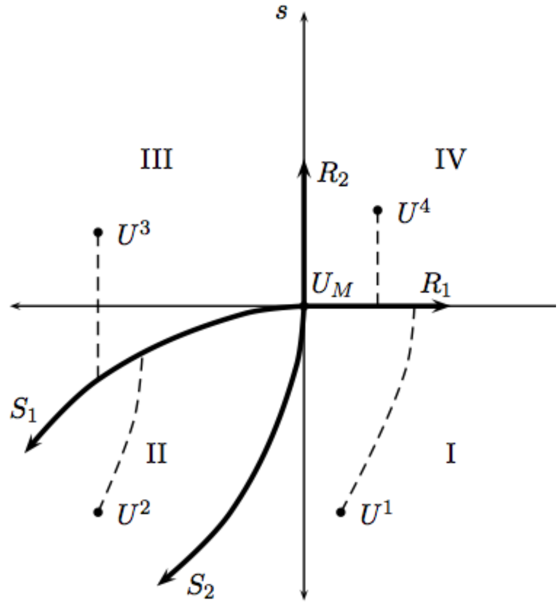
(Left Hand Data from FRW)

For the Riemann Problem we employ the special structure when

$$p = \sigma\rho$$

- Relativistic Nishida System: Sm-Te 1994
- Exact Formulas for Shock and Rarefaction Curves
- Global Solution of RP--No Vacuum
- Extreme Relativistic Limit of Free Particles
  Pure Radiation:

$$p = \frac{c^2}{3}\rho$$



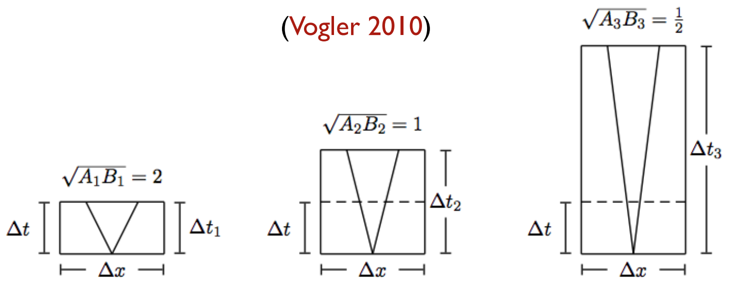
The shock curves are rigid translates of one another in the plane of Riemann Invariants (r,s)
(Sm-Te 1993)

The value $\mathbf{A}_{ij} = (A_{ij}, B_{ij})$ in each grid cell determines the "local time dilation"

- Metric: $$ds^2 = -Bdt^2 + \frac{1}{A}dr^2 + r^2 d\Omega^2$$
- Light Ray: $ds = 0,\ d\Omega = 0$ ➡ $-Bdt^2 + \frac{1}{A}dr^2 = 0$
- Speed of Light: $$c = \frac{dr}{dt} = \sqrt{AB}$$

Local Time Dilation

(Vogler 2010)



$\sqrt{AB}$ determines the *Time-Dilation* factor in each grid

- $\Delta t = \Delta t_1 \equiv \Delta t_{min}$
- Δt_j=Riemann problem time for light ray to hit grid boundary.
- Δt_j^*=fraction of the Riemann problem that evolves by time Δt.
- $\Delta t_j^* = \frac{\sqrt{A_j B_j}}{\sqrt{A_1 b_1}} \Delta t_j$

Convergence (Vogler)

Theorem: *Let $u_{\Delta x}(t,x)$ and $\mathbf{A}_{\Delta x}(t,x)$ be the approximate solution generated by the locally inertial Godunov method starting from the initial data $u_{\Delta x}(t_0,x)$ and $\mathbf{A}_{\Delta x}(t_0,x)$ for $t_0 > 0$. Assume these approximate solutions exist up to some time $t_{end} > t_0$ and converge to a solution $(u_{\Delta x}, \mathbf{A}_{\Delta x}) \to (u, \mathbf{A})$ as $\Delta x \to 0$ along with a total variation bound at each time step t_j*

$$T.V._{[r_{min},r_{max}]}\{u_{\Delta x}(t_j,\cdot)\} < V, \tag{5.3}$$

where $T.V._{[r_{min},r_{max}]}\{u_{\Delta x}(t_j,\cdot)\}$ represents the total variation of the function $u_{\Delta x}(t_j,x)$ on the interval $[r_{min}, r_{max}]$. Assume the total variation is independent of the time step t_j and the mesh length Δx. Then the solution $(u, \mathbf{A})$ is a weak solution to the Einstein equations (2.9)-(2.12).

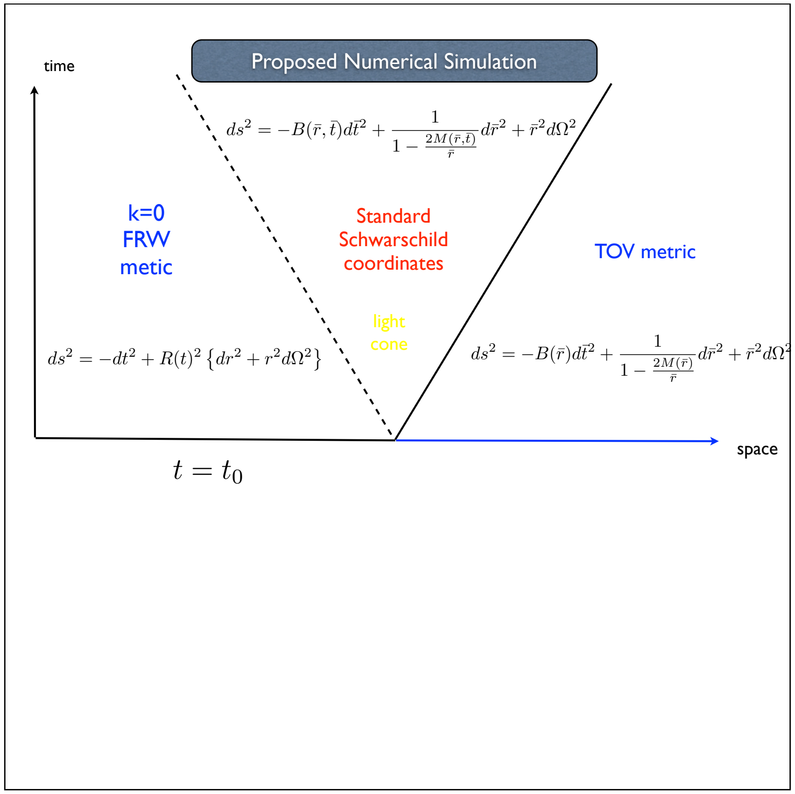
(1) Total Variation Bound
(2) L^1 - Convergence

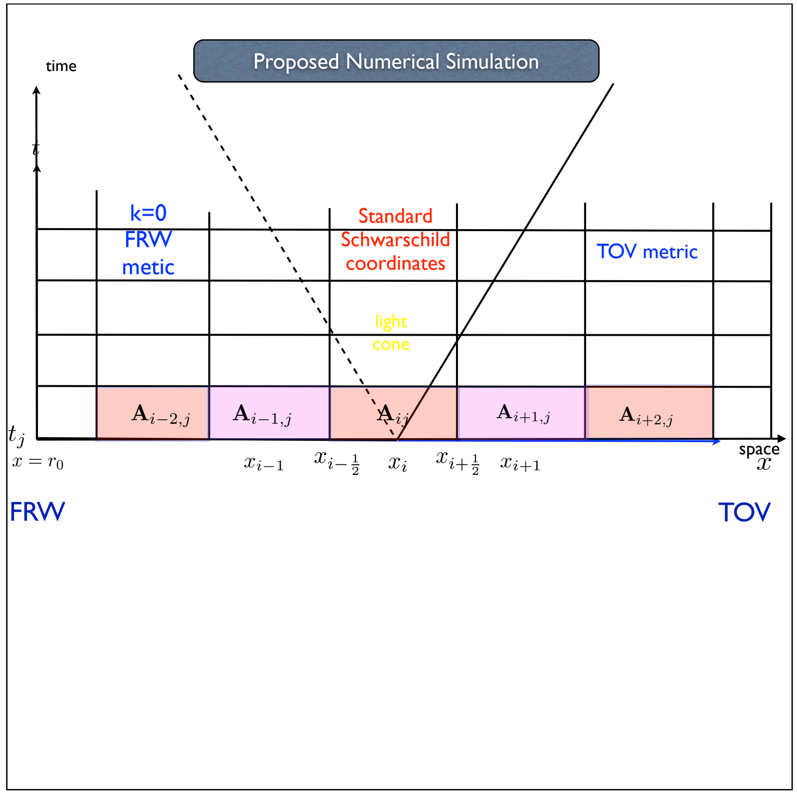
➡ Weak Solution of Einstein Equations

Vogler verifies (1) and (2) Numerically...

Numerical Simulation of GR Shock-Waves

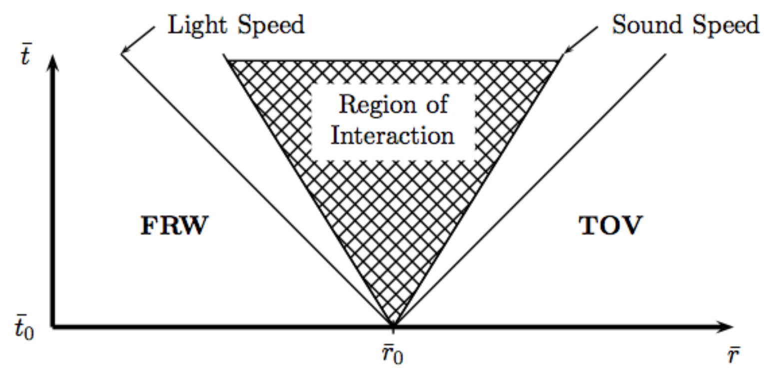
Zeke Vogler
UC-Davis
2010

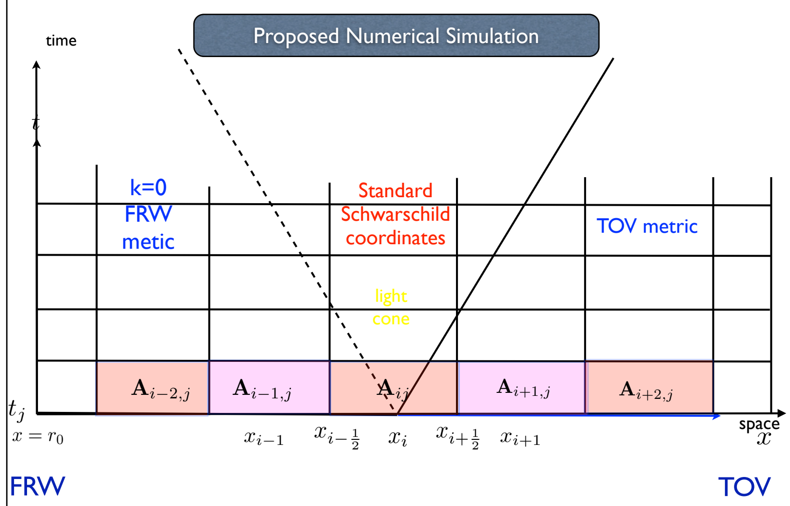

Proposed Numerical Simulation
time
space
k=0
FRW
metic
Standard
Schwarzschild
coordinates
light
cone
TOV metric
$ds^2 = -B(\bar{r},\bar{t})d\bar{t}^2 + \frac{1}{1-\frac{2M(\bar{r},\bar{t})}{\bar{r}}}d\bar{r}^2 + \bar{r}^2 d\Omega^2$
$ds^2 = -dt^2 + R(t)^2 \left\{dr^2 + r^2 d\Omega^2\right\}$
$ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1-\frac{2M(\bar{r})}{\bar{r}}}d\bar{r}^2 + \bar{r}^2 d\Omega^2$
$t = t_0$



Proposed Numerical Simulation
time
t
k=0
FRW
metic
Standard
Schwarzschild
coordinates
TOV metric
light
cone
$\mathbf{A}_{i-2,j}$
$\mathbf{A}_{i-1,j}$
$\mathbf{A}_{ij}$
$\mathbf{A}_{i+1,j}$
$\mathbf{A}_{i+2,j}$
t_j
$x = r_0$
x_{i-1}
$x_{i-\frac{1}{2}}$
x_i
$x_{i+\frac{1}{2}}$
x_{i+1}
space
x
FRW
TOV

Intertaction region between FRW and TOV





Problem: We need to express FRW in SSC coordinates

Theorem: Assume $p = \frac{c^2}{3}\rho$, $k = 0$. Then the FRW metric

$$ds^2 = -dt^2 + R(t)^2 dr^2 + \bar{r}^2 d\Omega^2,$$

under the mapping

$$\begin{aligned} \bar{r} &= R(t)r, \\ \bar{t} &= \left\{1 + \left[\frac{R(t)r}{2t}\right]^2\right\} t, \end{aligned}$$

goes over to the SSC-metric

$$ds^2 = -\frac{d\bar{t}^2}{1 - v(\xi)^2} + \frac{d\bar{r}^2}{1 - v(\xi)^2} + \bar{r}^2 d\Omega^2,$$

where

$$\xi \equiv \frac{\bar{r}}{\bar{t}} = \frac{2v}{1 + v^2}$$

(Sm-Te 2009)

FRW in Self-Similar SSC Form

- FRW in comoving coordinates

$$ds^2 = -dt^2 + R(t)^2 \left\{ dr^2 + r^2 d\Omega^2 \right\}$$

- FRW in SSC coordinates

$$ds^2 = -B(\xi)d\bar{t}^2 + \frac{1}{A(\xi)}dr^2 + \bar{r}^2 d\Omega^2$$

$$\xi = \frac{\bar{r}}{\bar{t}} \qquad \bar{r} = R(t)r$$

- EXACT FRW Solution when $p = \frac{c^2}{3}\rho$

$$A(\xi) = 1 - v(\xi)^2, \qquad B(\xi) = \frac{1}{1 - v(\xi)^2},$$

$$\rho(\xi, \bar{r}) = \frac{3v(\xi)^2}{\kappa \bar{r}^2}, \qquad v(\xi) = \frac{1 - \sqrt{1 - \xi^2}}{\xi},$$

For TOV we take Static Isothermal Sphere

- Assume TOV

$$ds^2 = -B(\bar{r})d\bar{t}^2 + \left(\frac{1}{1 - \frac{2\mathcal{G}M(\bar{r})}{\bar{r}}}\right) d\bar{r}^2 + \bar{r}^2 d\Omega^2,$$

(Special case when A and B are time independent)

- TOV Solves the Oppenheimer-Volkoff Equations (1936):

$$\frac{dM}{d\bar{r}} = 4\pi\bar{r}^2\rho$$

$$\frac{d\bar{p}}{d\bar{r}} = -\frac{\mathcal{G}M\rho}{\bar{r}^2}\left(1 + \frac{p}{\rho}\right)\left(1 + \frac{4\pi\bar{r}^3 p}{M}\right)\left(1 - \frac{2\mathcal{G}M}{\bar{r}}\right)^{-1}$$

- Exact TOV Solution when $p = \sigma\rho$

$$\rho = \frac{\gamma}{r^2} \qquad A = 1 - 8\pi\mathcal{G}\gamma \qquad M = 4\pi\gamma\bar{r} \qquad B = \bar{r}^{\frac{4\sigma}{1+\sigma}}$$

$$\gamma = \frac{1}{2\pi\mathcal{G}}\left(\frac{\sigma}{1 + 6\sigma + \sigma^2}\right)$$

Vogler's Simulation

- Take $p = \frac{c^2}{3}\rho$
- Start with exact FRW and TOV solutions in SSC coordinates
- For initial data it suffices to match FRW to TOV Lipschitz continuously at t=const

$$\frac{B'}{B} = -\frac{(B-1)}{x} + \kappa x B T_M^{00},$$
$$\frac{A'}{A} = \frac{(B-1)}{x} + \kappa x B T_M^{11}.$$

Einstein Constraint Equations

- This poses discontinuous density and velocity jumps at a Lipshitz matching of the metric
- There is a one parameter family of such initial data generating qualitatively different solutions
- Models a GR explosion into a static isothermal sphere

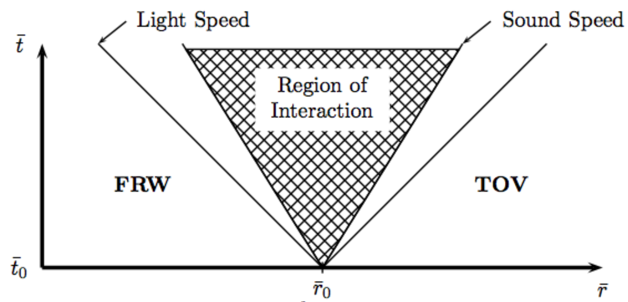
Match the metrics Lipshitz continuously
along the initial data
with discontinuities
in density and velocity → Shock waves!

(FRW) $$ds^2 = -B(\xi)d\bar{t}^2 + \frac{1}{A(\xi)}dr^2 + \bar{r}^2 d\Omega^2$$

↕ ↕

(TOV) $$ds^2 = -B(r)d\bar{t}^2 + \frac{1}{1 - \frac{2\mathcal{G}M(\bar{r})}{\bar{r}}} + \bar{r}^2 d\Omega^2$$

Vogler's One parameter family of intial data depending r_0



$$A_{FRW}(\bar{t}_0,\bar{r}_0) = 1 - v\left(\frac{\bar{r}_0}{\bar{t}_0}\right)^2 = 1 - 8\pi\mathcal{G}\gamma = A_{TOV}(\bar{t}_0,\bar{r}_0).$$

$$B_{TOV}(\bar{t}_0,\bar{r}_0) = B_0(\bar{r}_0)^{\frac{4\sigma}{1+\sigma}} = \frac{1}{1-v_0^2} = B_{FRW}(\bar{t}_0,\bar{r}_0).$$

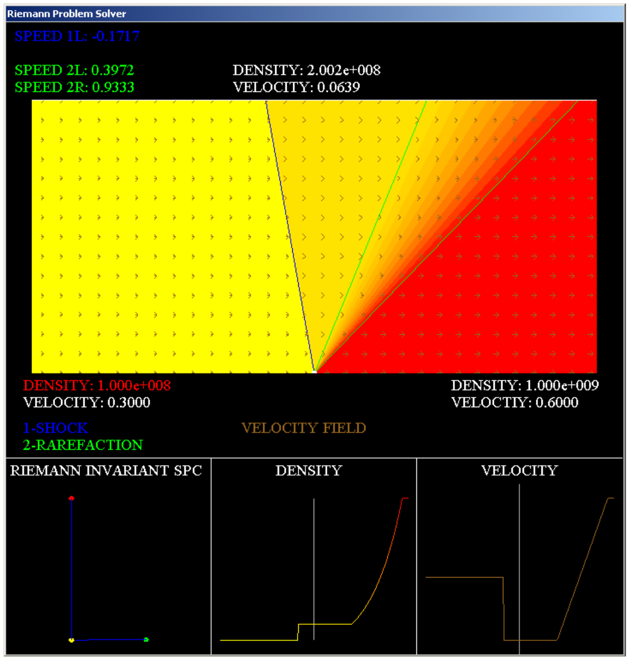
$$\frac{\bar{r}_0}{\bar{t}_0} = \frac{2v_0}{1+v_0^2}.$$

$$A_{init}(\bar{r}) = \begin{cases} 1 - v_{init}^2 & \bar{r} < \bar{r}_0 \\ 1 - 8\pi\mathcal{G}\gamma & \bar{r} > \bar{r}_0. \end{cases} \qquad B_{init}(\bar{r}) = \begin{cases} \frac{1}{1-v_{init}^2} & \bar{r} < \bar{r}_0 \\ B_0(\bar{r})^{\frac{4\sigma}{1+\sigma}} & \bar{r} > \bar{r}_0 \end{cases} \qquad \rho_{init}(\bar{r}) = \begin{cases} \frac{3v_{init}^2}{\kappa\bar{r}^2} & \bar{r} < \bar{r}_0 \\ \frac{\gamma}{\bar{r}^2} & \bar{r} > \bar{r}_0. \end{cases}$$

Matching FRW to TOV leads to one parameter family of initial data that meets the constraints of the Einstein equations

To start--Vogler wrote a Riemann Solver for Relativistic Compressible Euler

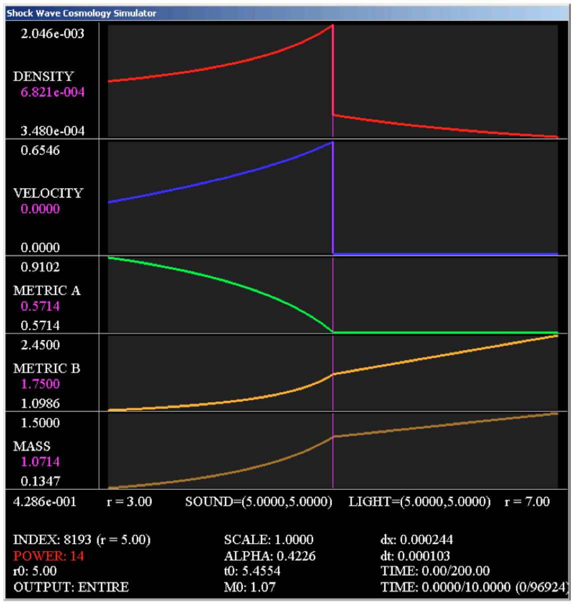
$$p = \frac{c^2}{3}\rho$$



Solution of the Riemann problem in Special relativity

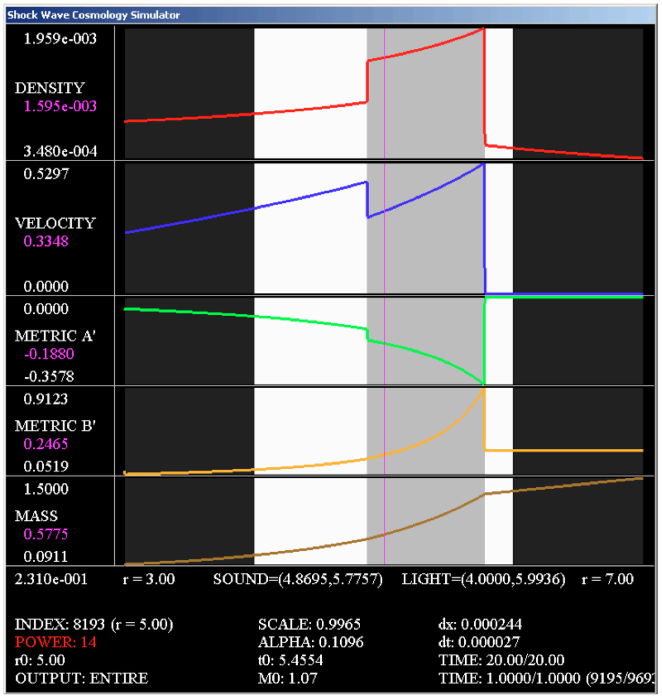
$$p = \frac{c^2}{3}\rho$$

Numerical Simulation of the Matched FRW-TOV Spacetimes

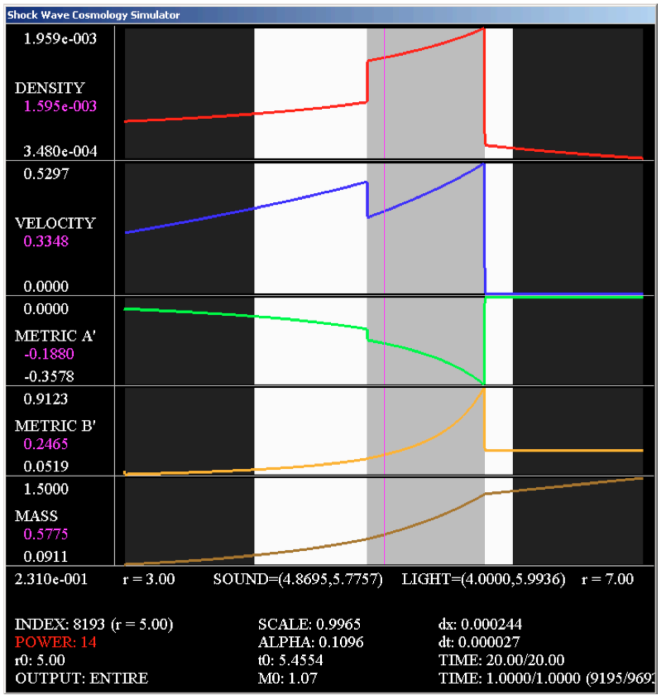


The Initial Profile:

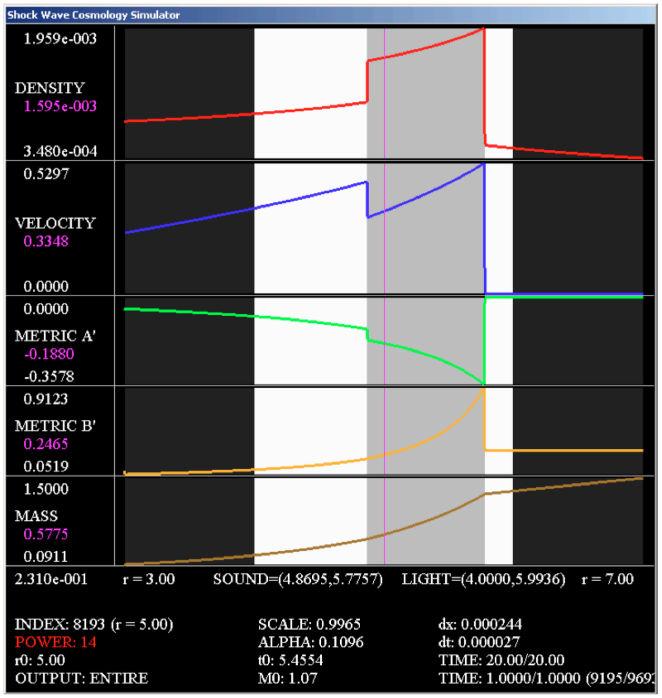
- Note discontinuity in fluid variables
- Lipschitz matching of the metric components



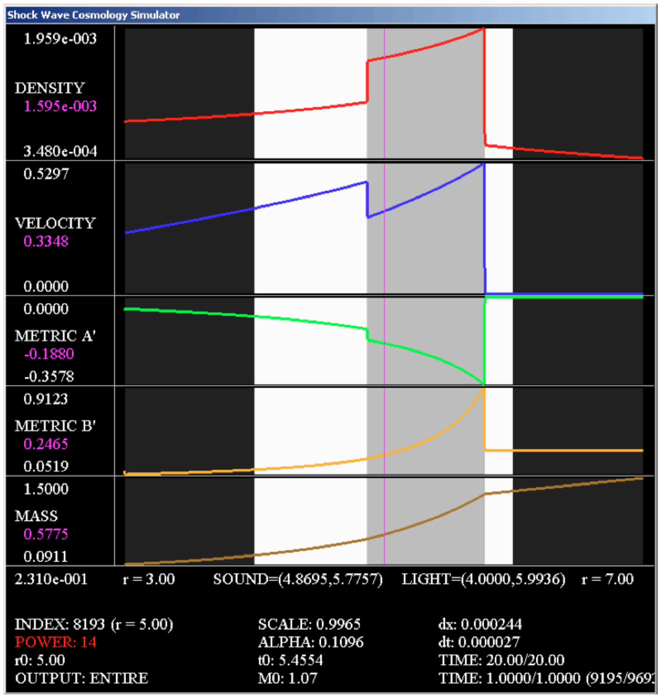
End Time of Simulation:



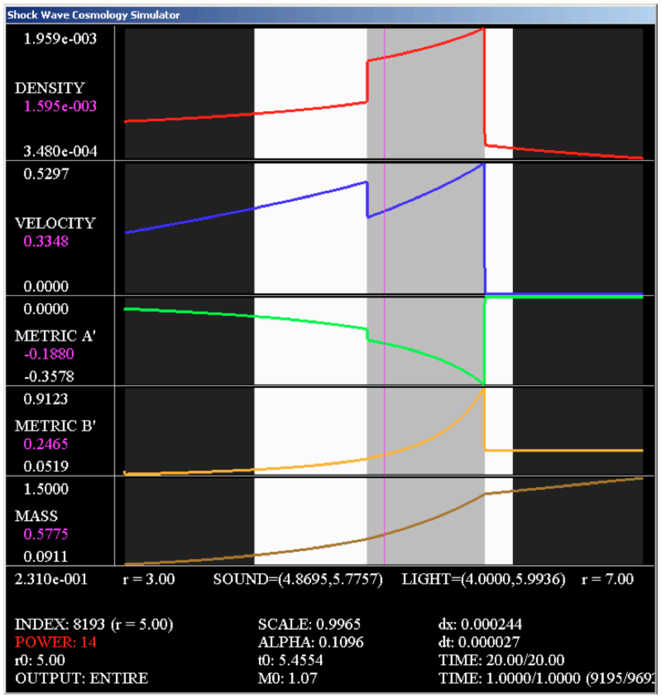
Note: Discontinuities in density and velocity



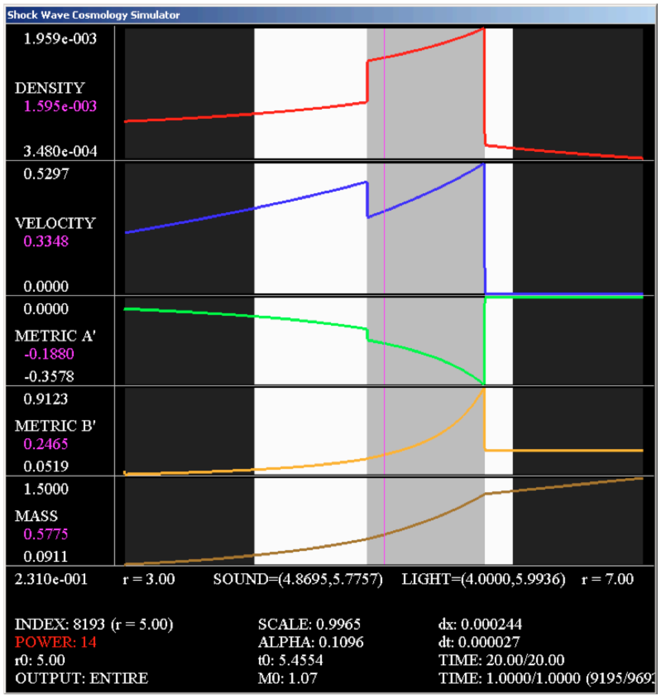
Note: Metric no better than Lipschitz continuous



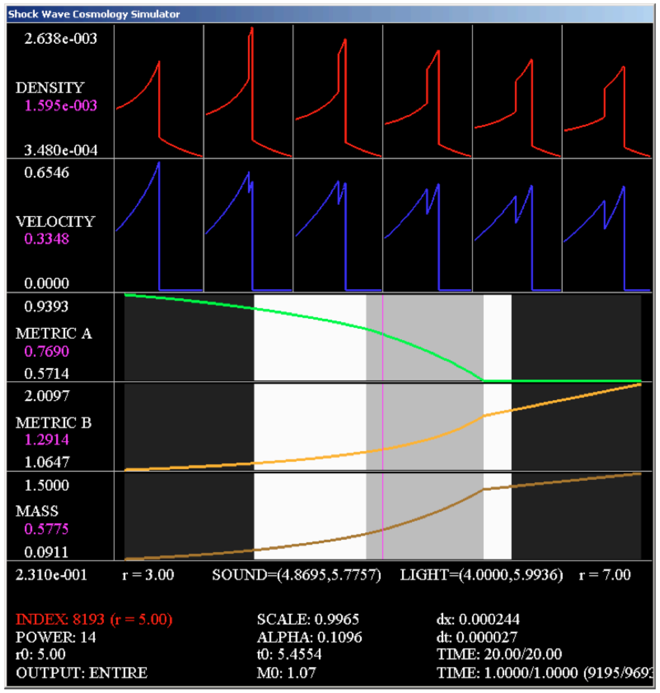
Note: Interaction region inside the cone of sound



Note: Convergence to Exact solutions on either side



Run Time---Simulation 1 Day ---Convergence: 2-3 Days



Time Evolution of the simulated shock waves…
Note interaction creates a region of higher density

Number	ρ		v		A		B	
Gridpoints	Error	Rate	Error	Rate	Error	Rate	Error	Rate
64	1.143e-004	N/A	4.657e-002	N/A	7.051e-003	N/A	3.146e-002	N/A
128	8.490e-005	0.43	3.463e-002	0.43	3.710e-003	0.93	1.557e-002	1
256	5.970e-005	0.51	2.414e-002	0.52	1.817e-003	1	7.704e-003	1
512	4.000e-005	0.58	1.596e-002	0.6	9.243e-004	0.98	2.889e-003	1.4
1024	2.470e-005	0.7	9.741e-003	0.71	4.334e-004	1.1	1.974e-003	0.55
2048	1.410e-005	0.81	5.502e-003	0.82	2.568e-004	0.76	5.160e-004	1.9
4096	7.470e-006	0.92	2.866e-003	0.94	1.232e-004	1.1	4.172e-004	0.31
8192	3.740e-006	1	1.420e-003	1	7.100e-005	0.8	1.111e-004	1.9
16384	1.870e-006	1	7.063e-004	1	3.300e-005	1.1	1.024e-004	0.12

Numerical Convergence: (First order method)

- To test convergence Vogler uses successive mesh refinement...
- Error: measures the L^1-difference between the current mesh refinement and the previous...
- The Rate is the log base 2 of the ratio of successive errors (current divided by previous)...

Number Gridpoints	ρ		v		A		B	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
64	1.143e-004	N/A	4.657e-002	N/A	7.051e-003	N/A	3.146e-002	N/A
128	8.490e-005	0.43	3.463e-002	0.43	3.710e-003	0.93	1.557e-002	1
256	5.970e-005	0.51	2.414e-002	0.52	1.817e-003	1	7.704e-003	1
512	4.000e-005	0.58	1.596e-002	0.6	9.243e-004	0.98	2.889e-003	1.4
1024	2.470e-005	0.7	9.741e-003	0.71	4.334e-004	1.1	1.974e-003	0.55
2048	1.410e-005	0.81	5.502e-003	0.82	2.568e-004	0.76	5.160e-004	1.9
4096	7.470e-006	0.92	2.866e-003	0.94	1.232e-004	1.1	4.172e-004	0.31
8192	3.740e-006	1	1.420e-003	1	7.100e-005	0.8	1.111e-004	1.9
16384	1.870e-006	1	7.063e-004	1	3.300e-005	1.1	1.024e-004	0.12

Numerical Convergence:

- To test convergence he uses successive mesh refinement...
- Error: measures the L^1-difference between the current mesh refinement and the previous.
- The Rate is the log base 2 of the ratio of successive errors (current divided by previous)...
- Ideally, First order method should half the error as you double the number of grid points, so a rate of 1 is ideal...
  Less than one implies convergence slower than expected...
  Greater than one means faster than expected...
  Vogler gets numbers .43 up to 1.9

Number Gridpoints	ρ		v		A		B	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
64	1.143e-004	N/A	4.657e-002	N/A	7.051e-003	N/A	3.146e-002	N/A
128	8.490e-005	0.43	3.463e-002	0.43	3.710e-003	0.93	1.557e-002	1
256	5.970e-005	0.51	2.414e-002	0.52	1.817e-003	1	7.704e-003	1
512	4.000e-005	0.58	1.596e-002	0.6	9.243e-004	0.98	2.889e-003	1.4
1024	2.470e-005	0.7	9.741e-003	0.71	4.334e-004	1.1	1.974e-003	0.55
2048	1.410e-005	0.81	5.502e-003	0.82	2.568e-004	0.76	5.160e-004	1.9
4096	7.470e-006	0.92	2.866e-003	0.94	1.232e-004	1.1	4.172e-004	0.31
8192	3.740e-006	1	1.420e-003	1	7.100e-005	0.8	1.111e-004	1.9
16384	1.870e-006	1	7.063e-004	1	3.300e-005	1.1	1.024e-004	0.12

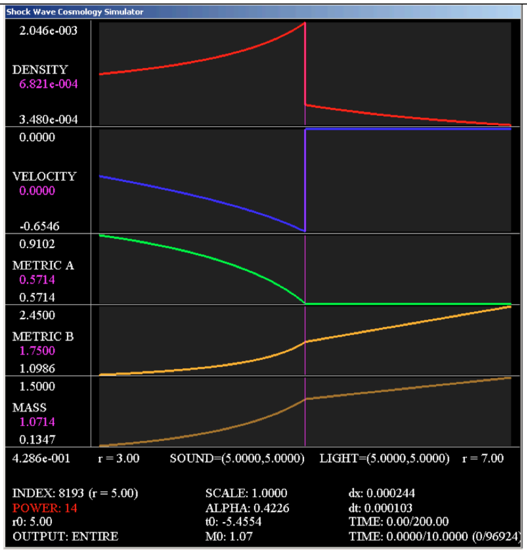
Conclusions:

- Fluid variables start out converging slower than expected, but head toward one under mesh refinement.
- A stays around one.
- B does a high-low swing, but on average has a rate of one. (We think it has to do with the integration of B across the whole simulation space).

Vogler's Conclusions:

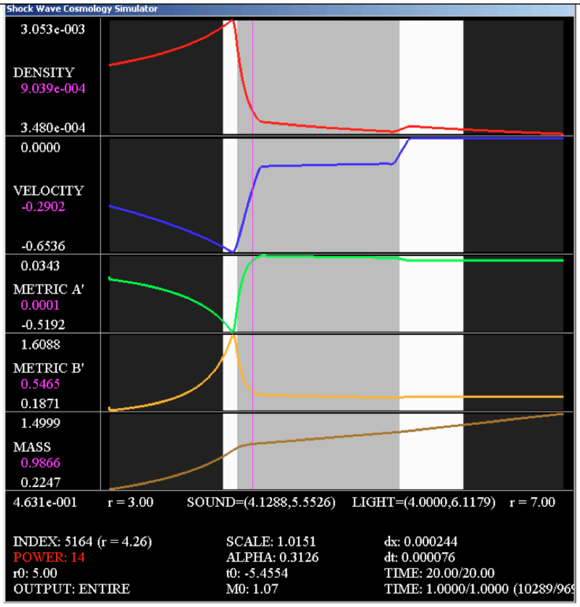
- Shock-waves converge to the cone of sound.
- Numerical convergence to FRW and TOV on outside of the interaction region
- Simulation determines uniquely the time-rescaling function of the TOV spacetime---implies continuity of B comes out of the method
- The simulation is tested with a different SSC representation of the FRW metric and confirms convergence to the same solution.
- The one parameter family of initial data is explored and produces qualitatively different solutions, but always results in two shock-waves

Black Hole Formation From Smooth Initial Data



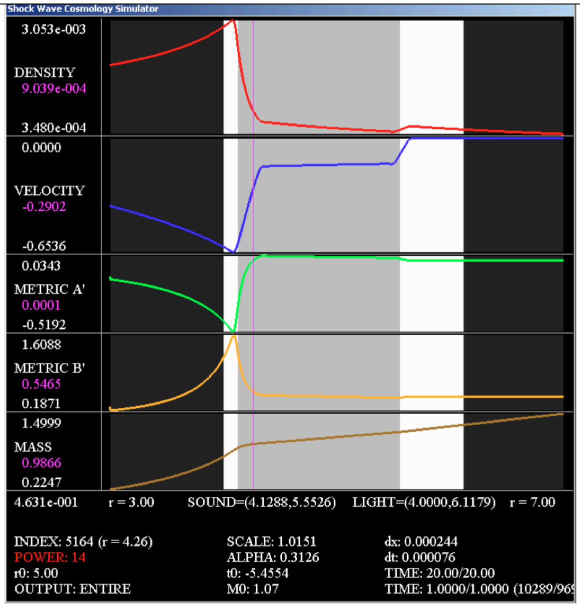
- Initial Data for the Time-Reversed Problem leading to Black Hole formation by two compressive rarefaction waves (smooth data)

(The only difference is the negative velocity...)

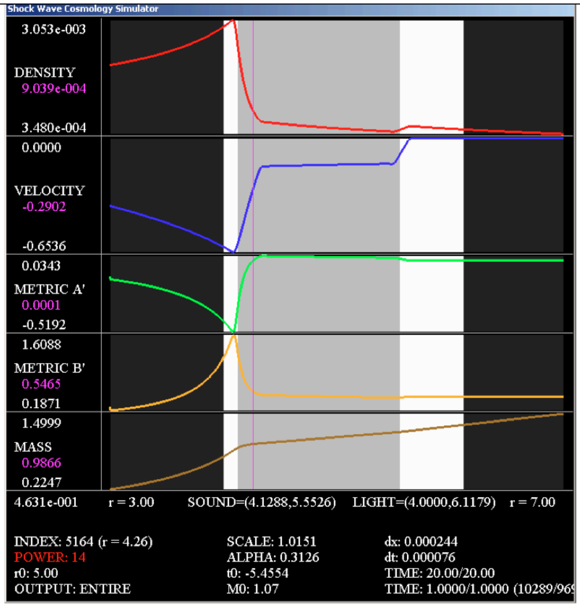


- End time simulation of the time-reversed evolution far from a Black Hole...two rarefaction waves

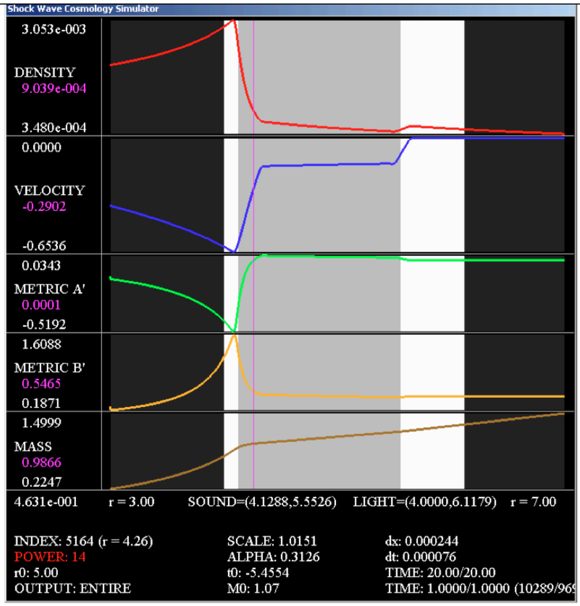
(Not EXACT rarefaction waves because of curvature...)



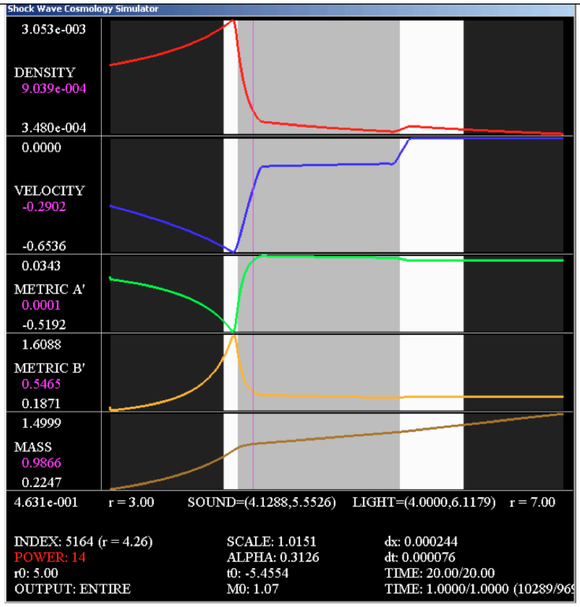
- We believe Black Hole will form under continuation of the time, and this is explored.



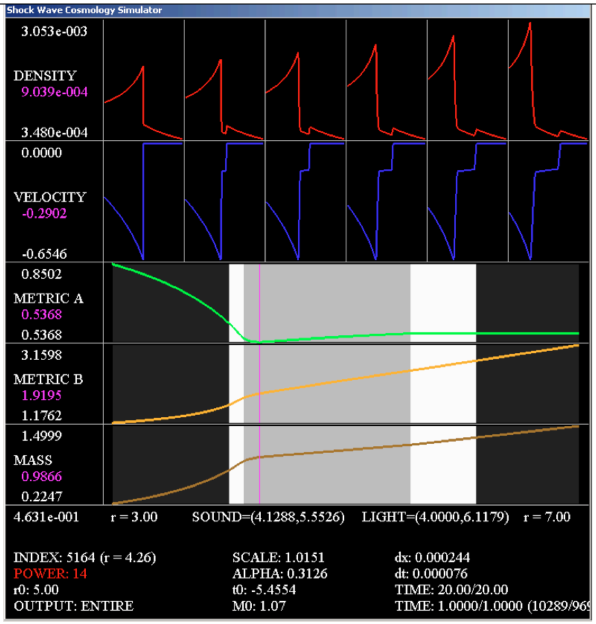
- Note the discontinuities in fluid variables and derivative of the metric are gone, so we have strong solution of the Einstein equations...



- Convergence is slow near the black hole due to time dilation...
- You can't simulate all the way into the Black hole in SSC coordinates because of infinite time -dilation...



- Vogler argues for Blak Hole formation by demonstrating solutions evolve inside 9/8 Schwarzschild Radius, the Buchdahl Stability Limit beyond which no static configuation has sufficient pressure to hold the solution up...



- Time evolution in the Black Hole simulation: (To time t=1)
- Note: interaction creates a region of Lower Denisty approx like interaction of two rarefaction waves.

Number Gridpoints	ρ		v		A		B	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
64	1.472e-004	N/A	5.334e-002	N/A	2.707e-002	N/A	9.879e-002	N/A
128	1.126e-004	0.39	3.815e-002	0.48	1.346e-002	1	5.532e-002	0.84
256	8.210e-005	0.46	2.694e-002	0.5	6.751e-003	1	3.299e-002	0.75
512	5.850e-005	0.49	1.889e-002	0.51	4.063e-003	0.73	2.632e-002	0.33
1024	4.090e-005	0.52	1.301e-002	0.54	2.042e-003	0.99	1.592e-002	0.73
2048	2.790e-005	0.55	8.770e-003	0.57	8.794e-004	1.2	8.348e-003	0.93
4096	1.860e-005	0.58	5.764e-003	0.61	4.801e-004	0.87	5.295e-003	0.66
8192	1.220e-005	0.62	3.685e-003	0.65	2.705e-004	0.83	3.312e-003	0.68
16384	7.750e-006	0.65	2.294e-003	0.68	1.622e-004	0.74	2.210e-003	0.58

Numerical Convergence:

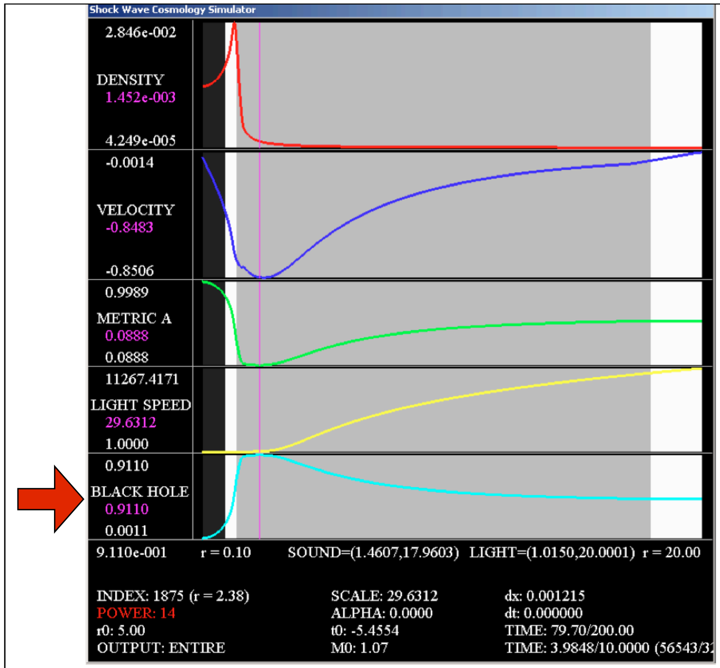
- Convergence under successive mesh refinement...
- L^1 -convergence tending (slowly) to one.
- Convergence rate slower due to time-dilation near Black Hole...
- No High-Low swings in B convergence rate...

Vogler's Conclusions (Same as before):

- Rarefaction-waves converge to the cone of sound.
- Numerical convergence to FRW and TOV on outside of the interaction region
- Simulation determines uniquely the time-rescaling function of the TOV spacetime---implies continuity of B comes out of the method
- The one parameter family of initial data is explored and produces qualitatively different solutions, but always results in two rarefaction-waves

Continuing Simulation Time into Black Hole

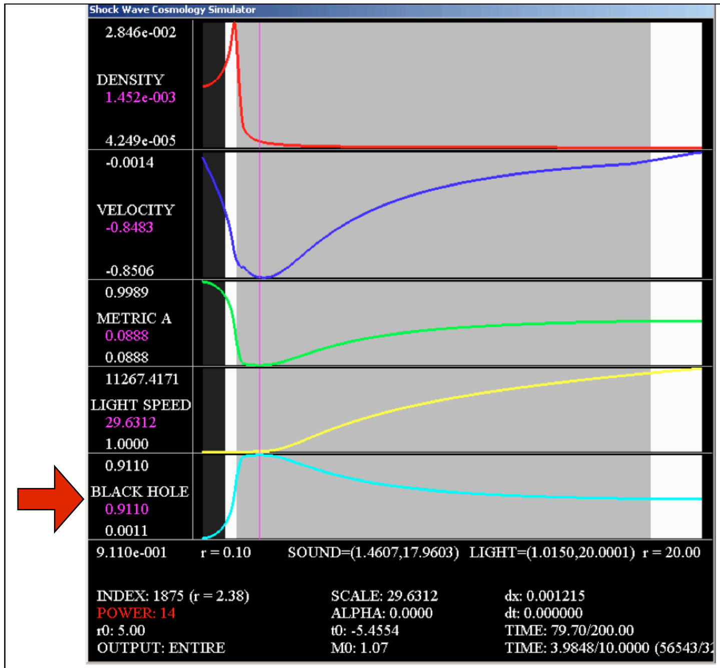
(Simulation time 1-week)



Continuing the Evolution Into Black Hole...

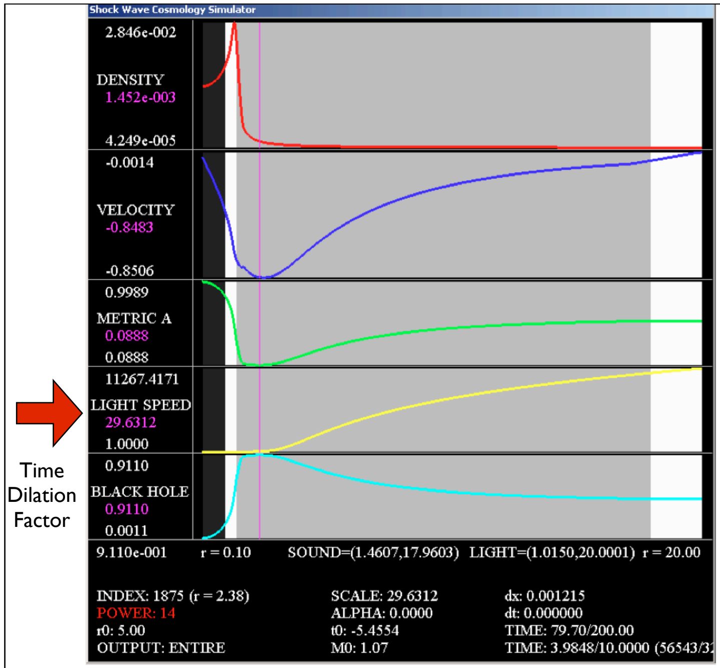
- Density rises by a factor of about 15...

(To time t=3.98)

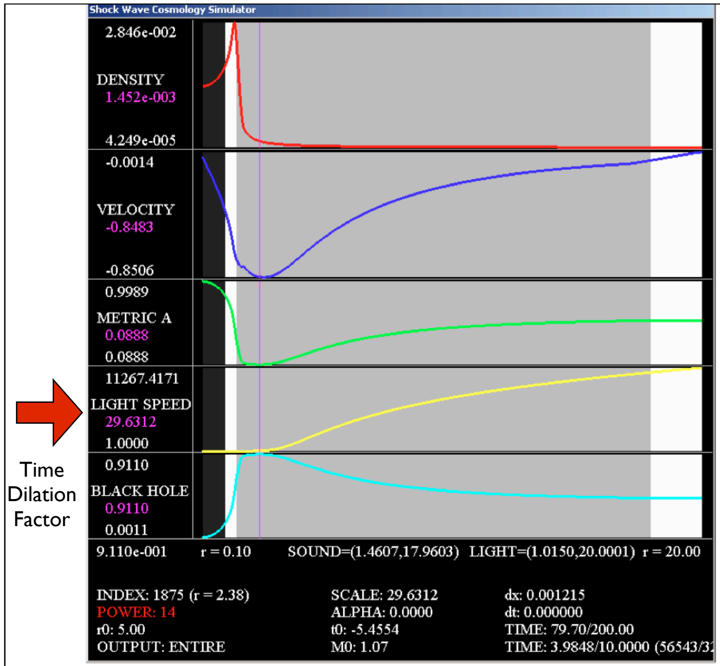


- Black Hole Number $\frac{2M}{r} \approx .911 > .888 =$ Buchdahl Limit

$$r \approx 2.38$$

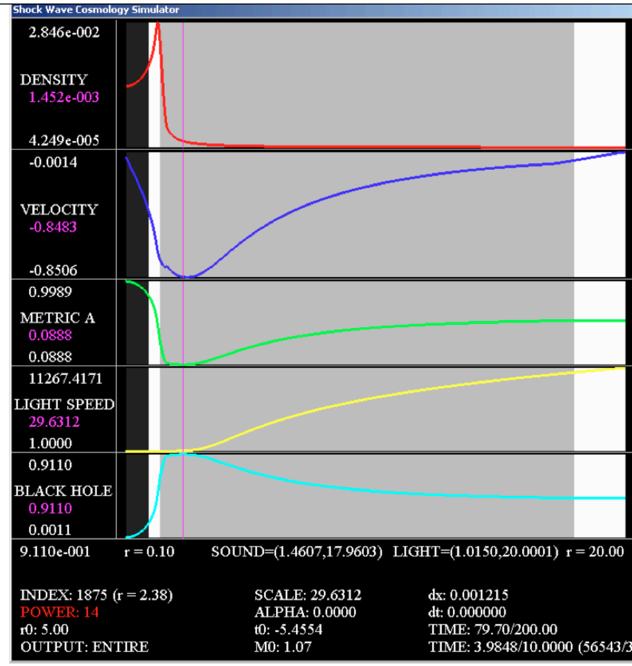


- Time-dilation factor $\sqrt{AB}$ rises to 11,000 implying extreme relativistic effects...

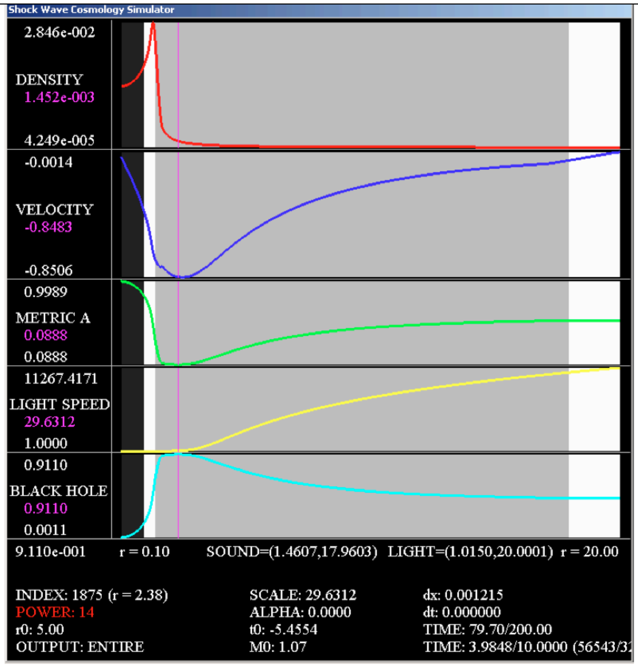


- Time step tends to zero as you approach the Black Hole, making it difficult to continue simulation...

$$\Delta t \propto \frac{1}{11,000}$$



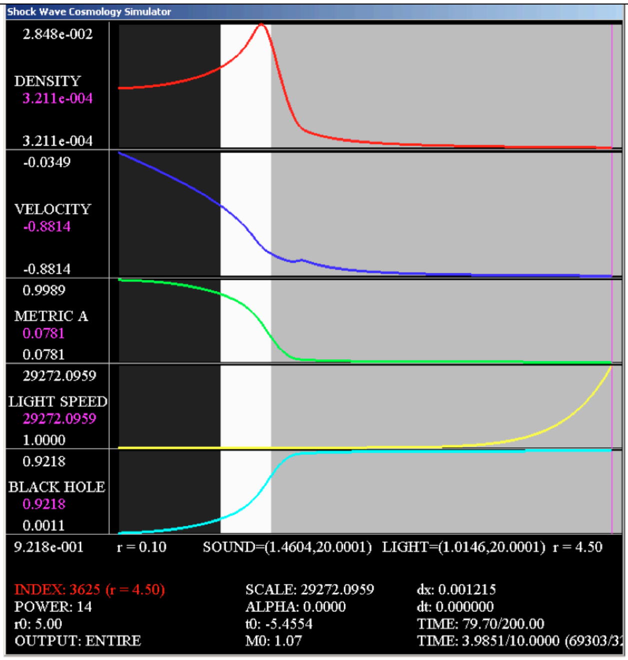
- Zeroing in on the Black Hole, Vogler gets the Black Hole number up to .922...



- Note the Hump in the Black Hole Number where localized formation is occuring...
  (Black Hole developing over an interval simultaneously??)

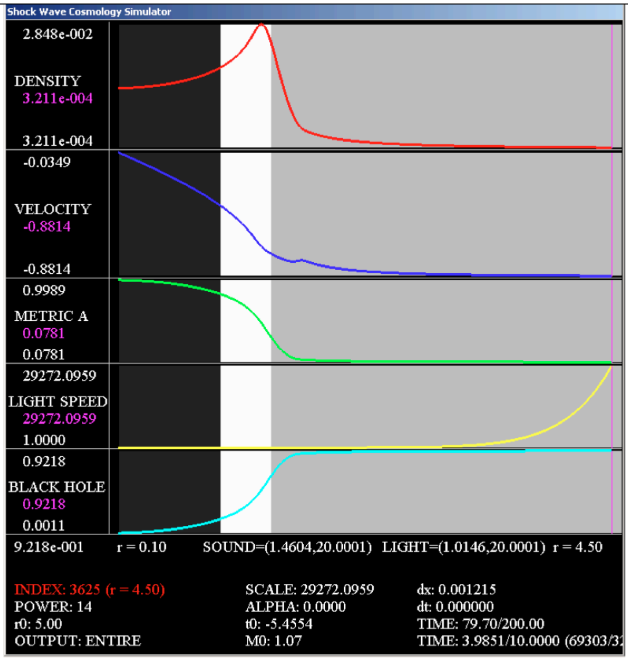
Pushing Simulation to Black Hole Number

$$\frac{2\mathcal{G}M}{r} \approx .922$$

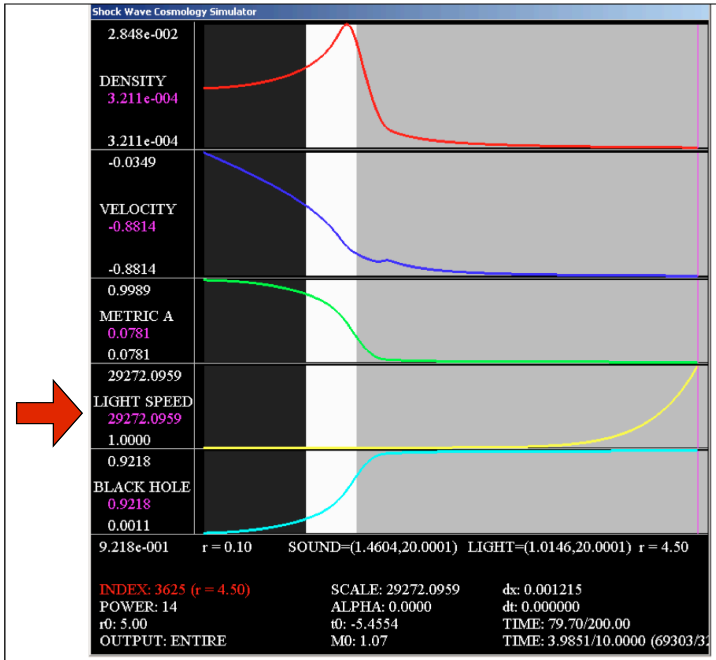


Zooming in on the Black Hole by chopping off the RHS

- Black Hole Number .922 at radius r=4.5 (1-week simulation)



- Region where the solution beats the Buchdahl limit spreads out over larger and larger radii, ranging from roughly r=2 to r=7...



- Time-dilation factor $\sqrt{AB}$ goes from 1 to 29,272
- Looks like exponential growth in dilation factor $\sqrt{AB}$

The solutions get dimensions upon setting the scale of the mass

$$
\begin{aligned}
0.09\ M_\odot &< M < 1.5\ M_\odot, \\
4.43\ \text{km} &< \bar{r}_* < 10.37\ \text{km}, \\
2.69 \times 10^{-5}\ \text{sec} &< \bar{t}_* < 3.18 \times 10^{-5}\ \text{sec}, \\
1.08 \times 10^{-4} M_\odot/\text{km}^3 &< \rho_* < 5.95 \times 10^{-4} M_\odot/\text{km}^3, \\
0\ \text{km/sec} &< v < 1.59 \times 10^5\ \text{km/sec}.
\end{aligned}
$$

Solar Scale

$$
\begin{aligned}
1.62 \times 10^{10}\ M_\odot &< M < 2.7 \times 10^{11}\ M_\odot, \\
0.084\ \text{light-years} &< \bar{r}_* < 0.2\ \text{light-years}, \\
56\ \text{days} &< \bar{t}_* < 66\ \text{days}, \\
1.94 \times 10^{7} M_\odot/\text{km}^3 &< \rho_* < 1.07 \times 10^{8} M_\odot/\text{km}^3, \\
0\ \text{km/sec} &< v < 1.59 \times 10^5\ \text{km/sec}.
\end{aligned}
$$

Galactic Scale

Future Directions:

- PROVE Shock-Wave formation from smooth initial data?
- PROVE Black Hole formation for perfect fluid?
- Continue Black Hole formation beyond Schwarzschild radius in Edington-Finkelstein/ Kruskal coordinates?
- Explore other phenomenon from other initial data?
- Can you smooth the metric at points of shock wave interaction (Moritz Rientes)?
- Multi-dimensional version of a locally inertial method?
- Simulate secondary wave in shock wave cosmology model where $2GM/r>1$?

END