Applications of Differential Equations

- **Logistic Growth:** In many situations where there is growth of a population, the growth is bounded above by some maximum. This kind of growth is called *logistic growth* where the growth of a population is proportional to *both* the size of the population and the difference between the size of the population and the maximum.

Let \( y(t) \) represent the size of the population at time \( t \), and suppose \( 0 < y(t) < L \), i.e. \( y \) is always bounded between 0 and \( L \).

\[
\frac{dy}{dt} = ky(L - y).
\]

We can solve this differential equation using separation of variables:

\[
\int \frac{1}{y(L - y)} dy = \int k dt.
\]
Logistic Growth

The general solution to the differential equation

$$y' = ky(L - y),$$

is given by

$$y(t) = \frac{L}{1 + be^{-kLt}}.$$

We will go through this derivation in class. Don’t forget partial fractions:

$$\frac{1}{y(L - y)} = \frac{1}{Ly} + \frac{1}{L(L - y)},$$

Also don’t forget that

$$\int \frac{1}{Ly} dy + \int \frac{1}{L(L - y)} dy = \frac{1}{L} (\ln |y| - \ln |L - y|) = \frac{1}{L} \ln \left( \frac{y}{L - y} \right)$$
Logistic Growth

What does \( y(x) = \frac{L}{1 + be^{-Lkt}} \) look like for different \( L \)?

\[
\frac{L}{1 + be^{-Lkt}}
\]

\( b = 1 \)
\( k = 1 \)
\( L = 1, 5, 10 \)
Logistic Growth

What does \( y(x) = \frac{L}{1 + be^{-kLx}} \) look like for different \( b \)?

\[
\frac{L}{1 + be^{-Lkt}}
\]

- \( b = 0.2, 1, 5 \)
- \( k = 1 \)
- \( L = 1 \)
Logistic Growth

What does \( y(x) = \frac{L}{1 + be^{-kLx}} \) look like for different \( k \)?

\[
\frac{L}{1 + be^{-Lkt}}
\]

- \( b = 1 \)
- \( k = 0.2, 1, 5 \)
- \( L = 1 \)
Example

The state game commission releases 100 deer into a game preserve. During the first 5 years the population increases to 450 deer. Find a model for the population growth assuming logistic growth with a limit of 5000 deer. What does the model predict the size of the population will be in 10 years, 20 years, 30 years?
$L = 5000$ and we use $y(0) = 100$ and $y(5) = 450$ to find $k$ and the constant of integration $b$.

$$y = \frac{5000}{1 + be^{-5000kt}}$$

Using $y(0) = 100$ we get $b = 49$, and using $y(5) = 450$ we find that $k = \frac{1}{10000} = 0.0000678$ or $6.78 \times 10^{-5}$ This gives a solution $y = \frac{5000}{1 + 49e^{-0.0000678t}}$ Plugging in for $t = 10$ we find $y(10) \approx 1897$ deer are in the population. Plugging in for $t = 20$ we find that $y(20) \approx 4741$ and at $t = 30$ $y(30) \approx 4990$. 
During a chemical reaction, substance $A$ is converted into substance $B$ at a rate that is proportional to the square of the amount of $A$. When $t = 0$, 50 grams of $A$ is present, and after 2 hours only 10 grams of $A$ remain unconverted. How much of $A$ is present after 4 hours?

Let $A(t)$ be the amount of unconverted substance $A$ at time $t$. The differential equation we are trying to solve is

$$\frac{dA}{dt} = kA^2.$$

We can solve this by separation of variables and use the conditions $A(0) = 50$, and $A(2) = 10$ to find $k$ and the constant of integration.
\[
\frac{dA}{dt} = kA^2,
\]

has the general solution

\[
A = \frac{-1}{kt + C}.
\]

Using the initial condition we see that \( C = -1/50 \), and using the fact that \( A(2) = 10 \) we find that \( k = -1/25 \). This gives

\[
A = \frac{50}{2t + 1}
\]

So at \( t = 4 \) \( A = \frac{50}{9} \approx 5.56 \) grams of substance \( A \) are remaining.