10.1 Sequences

Topics

- What is a sequence, terms of a sequence
- Does the sequence converge (go to a number) or diverge (either go back and forth between 2 numbers or go to plus or minus infinity)
- Finding patterns for sequences
- Applications of sequences in real life situations
A sequence \( \{a_n\} \) is a function whose domain is the set of positive integers. The function values \( a_1, a_2, a_3, \ldots, a_n, \ldots \) are real numbers and are called the terms of the sequence.

Example: \( a_n = \frac{1}{n} \), for \( n = 1, 2, 3, 4 \ldots \)

\[
\begin{align*}
a_1 &= 1/1, \\
a_2 &= 1/2, \\
a_3 &= 1/3, \\
a_4 &= 1/4, \ldots
\end{align*}
\]
Example: $a_n = \frac{2}{1+n}$, for $n = 1, 2, 3, 4...$

$a_1 = 2/(1 + 1) = 1,$
$a_2 = 2/(1 + 2) = 2/3,$
$a_3 = 2/(1 + 3) = 1/2,$
$a_4 = 2/(1 + 4) = 2/5,$ ....
The limit of a sequence

The main question to ask about a sequence is does it converge (i.e. approach a single finite value, called the limit) or does it diverge (i.e. approach $\pm \infty$ OR does it flip around between 2 or more numbers).

- $a_n = \frac{2}{n+1}$ we saw that the first terms of this sequence were 1, 2/3, 1/2, 2/5, ... and if we keep going we know we are dividing 2 by a bigger and bigger number so the sequence converges and the limit will be 0.

- $a_n = n$ here the first few terms are 1, 2, 3, 4, ... so obviously this sequence keeps getting bigger and bigger so it is not approaching a single finite value and hence it diverges.
Limit examples cont.

- \( a_n = 2 + (-1)^n \) here we find the first few terms and see

\[
\begin{align*}
a_1 &= 1 \\
a_2 &= 3 \\
a_3 &= 1 \\
a_4 &= 3,
\end{align*}
\]

So this sequence oscillates between 1 and 3 and so the sequence does not converge to a single finite value and hence diverges.

- \( a_n = \frac{2^n}{2^n+2} \) here we can use tricks we saw before when we learned limits of functions to see that

\[
a_n = \frac{2^n}{2^n + 2} = \frac{2^n}{2^n(1 + \frac{2}{2^n})} = \frac{1}{1 + \frac{2}{2^n}}
\]

Now if we take the limit as \( n \to \infty \) and we know that \( \lim_{n \to \infty} \frac{2}{2^n} = 0 \) then \( \lim_{n \to \infty} a_n = \frac{1}{1 + 0} = 1 \).
We define the expression \( n! = n \cdot (n - 1) \cdot (n - 2) \ldots 2 \cdot 1 \) to be called \( n \)-factorial.

Find the limit of

\[
a_n = \frac{(-1)^{n+1}}{(n + 1)!}
\]

We will go over this in class but the numerator is always \( \pm 1 \) and the denominator gets bigger and bigger so the limit will be zero. Write out the first few terms to see this yourself.
Sometimes you may be given the first few terms of a sequence and be asked to find the pattern. This takes some practice but in simple examples you will be able to figure it out.

Example: Determine an \( n^{th} \) term for the sequence

\[
\frac{1}{2} \quad \frac{4}{6} \quad \frac{9}{24} \quad \frac{16}{120} \quad \ldots
\]

We will go over this in class but the answer is \( a_n = (-1)^{n+1} \frac{n^2}{(n+1)!} \).
Applications

A deposit of $1000 is made in an account that earns 6% interest, compounded monthly. Find a sequence that represents the monthly balances. Each month you receive $6/12 = 0.5\%$ interest. The balance after 1 month is then

$$A_1 = 1000 + (1000)(0.005) = 1000(1.005).$$

After 2 months the balance is

$$A_2 = 1000(1.005)(1.005) = 1000(1.005)^2$$

This leads to the pattern

$$A_n = 1000(1.005)^n.$$

The general formula for this pattern is

$$P [1 + (r/12)]^n$$