Finding a Maximum Production Level

A manufacturer's production is modeled by the Cobb-Douglas function

$$f(x, y) = 100x^{3/4}y^{1/4}$$

where x represents the units of labor and y represents the units of capital. Each labor unit costs \$200 and each capital unit costs \$250. The total expenses for labor and capital cannot exceed \$50,000. Find the maximum production level.

Maximum production level cont.

The constraint in this problem comes from the sentence: *The total expenses for labor and capital cannot exceed $50,000* which can be translated as

$$200x + 250y = 50,000$$

We write this as a Lagrange multiplier problem, i.e. find the critical values of

$$F(x, y, \lambda) = 100x^{3/4}y^{1/4} - \lambda(200x + 250y - 50,000).$$

Maximum production level cont.

Set the partial derivatives of the function equal to zero

$$F_x(x, y, \lambda) = 75x^{-1/4}y^{1/4} - 200\lambda = 0$$

$$F_y(x, y, \lambda) = 25x^{3/4}y^{-3/4} - 250\lambda = 0$$

$$F_\lambda(x, y, \lambda) = -200x - 250y + 50,000 = 0.$$

Solve this system of three equations and three unknowns. To begin solve for λ in the first equation, substitute it in to the second equation to solve for x and substitute that into the final equation to solve for y. We will go over this in detail in class.

Marginal Productivity of Money

In the previous example, the Lagrange multiplier represents the percent of each additional dollar spent on capital that will turn into production.

$$\lambda = .375x^{-1/4}y^{1/4} = .375(187.5)^{-1/4}(50)^{1/4} \approx 0.27$$

So if an additional $20,000 were spent on capital then that would translate to an additional

$$\lambda \times \$20,000 \approx 5400$$

units of production.

Another example

Minimize $f(x, y) = x^2 - 8x + y^2 - 12y + 48$ subject to the constraint $x + y = 8$. You may assume x, y are all nonnegative.

We will go over the answer in detail in class.

What do you do with 2 constraints?

Maximize $f(x, y, z) = xy + yz$ subject to the constraints $x + 2y = 6$ and $x - 3z = 0$.
Here we write

$$F(x, y, z, \lambda, \mu) = xy + yz - \lambda(x + 2y - 6) - \mu(x - 3z),$$

and now we have to find critical values of F so we compute $F_x, F_y, F_z, F_\lambda, \text{ and } F_\mu,$ and set each one equal to zero and solve the resulting system for x, y, z.

We will go over the details of this in class.