7.8 Double Integrals

A partial integral with respect to $x$:

$$\int 2x^2y \, dx = \frac{2}{3} x^3 y + C(y).$$

Here you are integrating the function $2x^2y$ with respect to $x$, meaning you think of $y$ as a constant. The constant of integration $C(y)$ is now a function of $y$ because if you differentiate any function of $y$ with respect to $x$ you get zero.
The fundamental theorem of calculus still applies and hence

\[
\int_{2}^{3} 2x^2y \, dx = \frac{2}{3} x^3y \bigg|_{x=2}^{x=3} = \frac{2}{3} (3)^3y - \frac{2}{3} (2)^3y.
\]

The new thing here is that the endpoints can also be functions of \( y \), for example:

\[
\int_{1}^{y} 2x^2y \, dx = \frac{2}{3} x^3y \bigg|_{x=1}^{x=y} = \frac{2}{3} (y)^3y - \frac{2}{3} (1)^3y = \frac{2}{3} y^4 - \frac{2}{3} y.
\]

Another new thing is that the answer will be a function of \( y \), not a constant. To get a constant you can now integrate the \( y \) variable out, this a double integral.
Double Integral

Find

\[
\int_1^2 \int_0^x (5x^2y - 2) \, dy \, dx
\]

First do the *inner* integral, meaning the partial integral with respect to \( y \),

\[
\int_0^x (5x^2y - 2) \, dy = \left[ \frac{5}{2}x^2y^2 - 2y \right]_{y=0}^{y=x} = \frac{5}{2}x^4 - 2x.
\]

Now plug this in to the *outer* integral (with respect to \( x \)).

\[
\int_1^2 \frac{5}{2}x^4 - 2x \, dx = \left[ \frac{1}{2}x^5 - x^2 \right]_1^2 = 25/2.
\]
Area with double integrals

Now if we are given a region defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, the area of this region is given by

$$\int_a^b \int_{g_1(x)}^{g_2(x)} 1 \, dy \, dx.$$

Note that this is the same as we saw in single variable calculus because if you do the partial integral with respect to $y$ you get that the above expression is equal to

$$\int_a^b g_2(x) - g_1(x) \, dx$$

which is the area between the curves. Draw a picture.
Alternatively a region may be described as $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$ which has area given by

$$\int_c^d \int_{h_1(y)}^{h_2(y)} 1 \, dx \, dy.$$  

You always have to be sure that your outer integral has constants for endpoints!!

In general you can switch the order of integration
Area example

Find the area bounded between the curves $y = x$ and $y = x^2$. This can be done two ways.

$$A = \int_0^1 \int_{x^2}^x 1 \, dy \, dx = \int_0^1 x - x^2 \, dx$$

$$= \left. x^2/2 - x^3/3 \right|_{x=1}^{x=0} = 1/2 - 1/3 = 1/6.$$

OR

$$A = \int_0^1 \int_y^\sqrt{y} 1 \, dx \, dy$$

Work it out the second way and see that you get the same answer.