Newton’s Method is an application of Taylor Polynomials for finding roots of functions.

In general solving an equation $f(x) = 0$ is not easy, though we can do it in simple cases like find roots of quadratics. If the function is complicated we can approximate the solution using an iterative procedure also known as a numerical method. One simple method is called Newton’s Method.
Newton’s Method (follow these steps)

Suppose that \( x = c \) is an (unknown) zero of \( f \) and that \( f \) is differentiable in an open interval that contains \( c \). To approximate \( c \):

1. Make an initial approximation \( x_1 \) close to \( c \).
2. Determine a new approximation using the formula

\[
  x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

3. If \( |x_n - x_{n+1}| \) is less than the desired accuracy (which will be specified), let \( x_{n+1} \) serve as the final approximation. Otherwise, return to step 2 and calculate a new approximation.

Each calculation of a successive approximation is called an iteration.
**Example**

Use three iterations of Newton’s Method to approximate a zero of $f(x) = x^2 - 2$. Use $x = 1$ as the initial guess.

We need to know $f'(x) = 2x$, and we now can use the formula (we are given that $x_0 = 1$).

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^2 - 2}{2} = 1 + 1/2 = 1.5
\]

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{(1.5)^2 - 2}{2(1.5)} = 1.5 - (2.25 - 2)/3 = 1.42
\]

\[
x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.42 - \frac{(1.42)^2 - 2}{2(1.42)} = 1.41
\]

Notice that the answer is getting closer to the correct answer $x^2 - 2 = 0 \Rightarrow x = \sqrt{2} = 1.41$
If the iterations are getting closer and closer to the correct answer the method is said to converge. However, Newton’s method will not converge if

1. If $f'(x_n) = 0$ for some $n$

2. If $\lim_{n \to \infty} x_n$ does not exist
Finding points of intersection

Use Newton’s Method to estimate the point of intersection of $y = e^{-x^2}$ and $y = x$.

We need to find the point where $e^{-x^2} = x$ or $x - e^{-x^2} = 0$, therefore we apply Newton’s method to the function

$$f(x) = x - e^{-x^2}.$$  

We compute $f'(x) = 1 - 2xe^{-x^2}$, and then the iterative formula becomes

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n^2}}{1 + 2x_ne^{-x_n^2}}.$$
Use Newton’s Method and continue until two successive approximations differ by less than 0.0001. We have

\[ x_{n+1} = x_n - \frac{x_n - e^{-x_n^2}}{1 + 2x_ne^{-x_n^2}} \]

It helps to make a table:

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<th>(n)</th>
<th>(x_n)</th>
<th>(f(x_n))</th>
<th>(f'(x_n))</th>
<th>(\frac{f(x_n)}{f'(x_n)})</th>
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