10.5: Taylor Polynomials

Recall that

\[ \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2 \]

What this means is that the limit of the partial sums of the series is 2, in other words

\[ S_1 = 1 \]
\[ S_2 = 1 + \frac{1}{2} \]
\[ S_3 = 1 + \frac{3}{4} \]
\[ S_4 = 1 + \frac{7}{8}... \]

the partial sums are getting closer and closer to 2. In the same way, a Taylor Series is an approximation of a function whose partial sums will get closer and closer to the function near the center.
Example: \( f(x) = e^{-x^2} \) expanded at \( x = 0 \)

We know that

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots
\]

So using this we can write out the Taylor Series for \( e^{-x^2} \)

\[
e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \ldots
\]
Example: \( f(x) = e^{-x^2} \)

We can use this to find the Taylor Polynomials:

\[
S_0 = 1 \\
S_2 = 1 - x^2 \\
S_4 = 1 - x^2 + \frac{x^4}{2} \\
S_6 = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} \\
S_8 = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}
\]

Each of these does a better and better job approximating the function \( e^{-x^2} \) near the origin. See the graphs on the following pages.
Example: \( f(x) = e^{-x^2} \)
Find the Taylor polynomial of degree 4 (centered at the origin) for each of the following

- \( f(x) = \ln(x + 1) \)
- \( f(x) = \frac{1}{(x + 1)^2} \)
- \( f(x) = xe^x \)
Example

Find the Taylor polynomial of degree 4 (centered at the origin) for each of the following

1. $f(x) = \ln(x + 1)$

Since $\ln(x) = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$ is the Taylor series for $\ln x$ centered at $x = 1$, we can do a change of variables and let $y + 1 = x$ to get a Taylor series of $\ln(y + 1)$ centered at $y = 0$:

$$\ln(y + 1) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$$

Therefore

$$S_4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$
Example

Find the Taylor polynomial of degree 4 (centered at the origin) for each of the following

\[ f(x) = \frac{1}{(x + 1)^2} \]

We will use the Taylor series at \( x = 0 \) for

\[ \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \ldots \]

Note that \( \frac{1}{(1+x)^2} = \frac{1}{1+x} \cdot \frac{1}{1+x} \) so we have to multiply these two series and keep all terms up to fourth order
Example

\[
\frac{1}{(1 + x)^2} = (1 - x + x^2 - x^3 + x^4 - ...) \cdot (1 - x + x^2 - x^3 + x^4 - ...)
\]

\[
1 - x + x^2 - x^3 + x^4
\]

\[
- x + x^2 - x^3 + x^4
\]

\[
+ x^2 - x^3 + x^4
\]

\[
- x^3 + x^4
\]

\[
+ x^4
\]

\[
= 1 - 2x + 3x^2 - 4x^3 + 5x^4.
\]

Therefore

\[
S_4 = 1 - 2x + 3x^2 - 4x^3 + 5x^4
\]
Example

Find the Taylor polynomial of degree 4 (centered at the origin) for each of the following

- \( f(x) = xe^x \)

Since \( e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! + \ldots \) is the Taylor series for \( e^x \) centered at \( x = 0 \), we can multiply this series by \( x \) to get the series for \( xe^x \) centered at \( x = 0 \):

\[
xe^x = x + x^2 + x^3/2 + x^4/3!
\]

Therefore

\[
S_4 = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}.
\]