1. (a) Put the following equation into standard form (hint: complete the square in $x, y, z$)

$$x^2 - 4x + y^2 + 2y + z^2 = -4$$

Solution:
Since $x^2 - 4x = (x-2)^2 - 4$ and $y^2 + 2y = (y+1)^2 - 1$ we substitute these in to obtain the answer

$$(x-2)^2 + (y+1)^2 + z^2 = 1$$

(b) What type of surface is given in part (a)
Solution:
This is a sphere of radius $r = 1$ centered at $(2, -1, 0)$

(c) Sketch the $(x, y)$ trace of the surface on the axes below.
Solution:
2. Graph the following plane on the axes below, be sure to include all intercepts.

\[ 2x + 3y - 4z = 12 \]

Solution:
The intercepts occur at \((6, 0, 0), (0, 4, 0), \) and \((0, 0, -3)\) see the graph drawn in class or just connect the dots on your picture to create a triangle which outlines part of the plane.

3. Find the general solution to the following differential equations. You must show your work and you **must check your answer**.

(a) \( \frac{dx}{dt} = 3(x + 1) \)

Solution: First separate variables and then integrate

\[
\int \frac{dx}{x + 1} = \int 3 dt
\]

This gives

\[
\ln |x + 1| = 3t + C \Rightarrow x + 1 = Ae^{3t} \Rightarrow x = -1 + Ae^{3t}.
\]

Checking this we take the derivative \( \frac{dx}{dt} = 3Ae^{3t} \), now notice that

\( x + 1 = Ae^{3t} \) and so \( \frac{dx}{dt} = 3(x + 1) \) as desired.

(b) \( \frac{dy}{dt} = ky^2 \)

Solution: Again separating variables and integrating we have

\[
\int \frac{dy}{y^2} = \int k dt
\]

This gives

\[
-y^{-1} = kt + C \Rightarrow y^{-1} = -C - kt \Rightarrow y = \frac{1}{A - kt}.
\]

Checking this we have

\[
\frac{dy}{dt} = -(A - kt)^{-2}(-k) = k \left( \frac{1}{A - kt} \right)^2 = ky^2.
\]
(c) \( \frac{dA}{dt} + 3t^2A = 4t^2 \)

Solution: Use the integrating factor method to solve this problem. First find \( u(t) = e^{\int P(t)dt} \) where here \( P(t) = 3t^2 \) so \( \int P(t)dt = t^3 \) and therefore \( u(t) = e^{t^3} \). We also need to compute

\[
\int u(t)Q(t)dt = \int e^{t^3}4t^2dt.
\]

Using substitution (let \( z = t^3 \) so \( dz = 3t^2dt \)) we have

\[
\int 4e^{t^3}\frac{dz}{3} = \frac{4}{3}e^{t^3} + C = \frac{4}{3}e^{t^3} + C.
\]

Finally, the answer

\[
A(t) = \frac{1}{u(t)}\int u(t)Q(t)dt = e^{-t^3}\left[\frac{4}{3} + C\right] = \frac{4}{3} + Ce^{-t^3}.
\]

Checking this, we have \( A'(t) = -3t^2Ce^{-t^3} \) so plugging this in

\[
A'(t) + 3t^2A = -3t^2Ce^{-t^3} + 3t^2\left[\frac{4}{3} + Ce^{-t^3}\right] = 4t^2
\]

4. Newton’s law of cooling states that the rate of change in the temperature \( T \) of an object is proportional to the difference between the temperature of the object (\( T \)) and the surrounding temperature \( T_0 \). This can be expressed by the differential equation:

\[
\frac{dT}{dt} = k(T - T_0).
\]

A room is kept at a constant temperature of 75\(^\circ\)F. A cake is taken out of a 375\(^\circ\) oven and placed on the counter. If the cake has reduced in temperature to 275\(^\circ\) in 1 hour, when will the cake reach 100\(^\circ\)? (Hint: First find the particular solution, and then solve the problem given.)

Solution:

We have to solve \( T'(t) = k(T - 75) \) which we solve using separation of variables

\[
\int \frac{dT}{T - 75} = \int k \, dt
\]
which leads to

\[ \ln|T - 75| = kt + C \Rightarrow T(t) = Ae^{kt} + 75 \]

To find \( A \) we use the initial condition

\[ 375 = T(0) = 75 + A \Rightarrow A = 300. \]

To find \( k \) we use the fact that \( T(1) = 275 \)

\[ 275 = T(1) = 75 + 300e^k \Rightarrow \frac{2}{3} = e^k \text{ or } k = \ln(2/3) \approx -0.4 \]

Now we solve

\[ 100 = T(t) = 75 + 300e^{-0.4t} \Rightarrow \frac{25}{300} = e^{-0.4t} \]

\[ \Rightarrow \ln\left(\frac{25}{300}\right) = -0.04t \Rightarrow t \approx 6.25 \]

5. Suppose that you know a population grows according to the logistic growth model, in other words the population is bounded above by a maximum \( L \) and the population grows proportional to the size of the population times the difference between the population and the maximum. What model below describes this type of growth? Circle the correct answer, no justification is necessary.

(a) \( \frac{dA}{dt} = A(L - A) \)

(b) \( \frac{dA}{dt} = kA(L - A) \)

(c) \( \frac{dA}{dt} = k(L - A)^2 \)

(d) \( \frac{dA}{dt} = kLA^2 \)

(e) None of the above

Solution:

The answer is (b). (a) is similar but the constant of proportionality is \( k = 1 \) which is not general enough for the model. (c) has the population growing proportional to the square of the difference between the population and the maximum and (d) has the population growing proportional to the population size.
6. For the following, match each of the contour maps below (a,b,c,d) to a possible corresponding quadric surface.

i) \(x^2 + (y/2)^2 = z^2\)  ___b___

ii) \(x^2 - \frac{1}{4}y^2 = z\)  ___c___

iii) \(x^2 + y^2 - z^2 = 1\)  ___a___

iv) \(z = x^2 + y^2\)  ___d___
7. Find the domain and range of the following functions. Also evaluate the functions at the specified point.

(a) \( f(x, y) = \frac{1}{x + 4y}, \quad (x, y) = (1, 2) \)

Solution: The domain is all \( x, y \) such that \( x + 4y \neq 0 \), the range is all real numbers except 0 and \( f(1, 2) = \frac{1}{1+4(2)} = 1/9 \).

(b) \( f(x, y, z) = \ln(|x - y + 2z|), \quad (x, y, z) = (1, 4, -1) \)

Solution: The domain is all \( x, y, z \) such that \( x - y + 2z \neq 0 \) the range is all real numbers and \( f(1, 4, -1) = \ln(|1 - 4 + 2(-1)|) = \ln(5) \).

(c) \( f(x, y) = \sqrt{9 - x^2 - y^2}, \quad (x, y) = (1, 1) \)

Solution: The domain is all \( x, y \) such that \( 9 - x^2 - y^2 \geq 0 \) the range is all real numbers between 0 and 3 and \( f(1, 1) = \sqrt{7} \).