1 Problem 1

(15 points)

Let \( L_n \) be the sequence of numbers defined by the recursion relation

\[
L_{n+1} = -2L_n + 15L_{n-1}, \quad (n \geq 1)
\]

with initial values \( L_0 = -1, \ L_1 = 0 \).

a. Calculate: \( L_2 \) and \( L_3 \).

\[
L_2 = -15
\]
\[
L_3 = 30
\]

b. Find an explicit formula for \( L_n \).

\( \text{Solution:} \quad q^2 = -2q + 15 \)

\[
q^2 + 2q - 15 = 0
\]

\[
(q + 5)(q - 3) = 0
\]

\[
q = -5 \quad \text{or} \quad q = 3
\]

\( \text{Find} \quad a, b \neq 0 \)

\[
2a(-5)^n + 5b(3)^n = -1
\]

\[
a(-5) + 6(3) = 0
\]

\( \text{Solution:} \quad a = -2/3, \quad b = 5/3 \)

\[
L_n = (-2/3)(5)^n + (5/3)(3)^n
\]
2 Problem 2

(20 points)

Consider a $1 \times n$ chessboard. Let $h_n$ be the number of ways to color the board using red, white and blue so that no two squares colored red are adjacent.

Examples:
Acceptable board:

\[
\begin{array}{cccccc}
R & W & W & B & R & W \\
\end{array}
\]

$1 \times 6$

Not Acceptable Board:

\[
\begin{array}{cccccc}
W & B & R & R & B & R & W \\
\end{array}
\]

$1 \times 7$

a. Calculate: $h_1$, $h_2$ and $h_3$.

\[
h_1 = 3
\]
\[
h_2 = 3 \times 3 - 1 = 8
\]
\[
h_3 = 3 \times 3 \times 3 - 1 - 2 - 2 = 22
\]

b. Find a recursion formula for $h_n$.

In a $1 \times n$ board, the right-most square is adjacent square is Blue $\rightarrow$ any acceptable $n-2$ board can fill

Case 1: Red $\rightarrow$ adjacent square is White $\rightarrow$ any acceptable $n-2$ board can fill

Case 2: Blue $\rightarrow$ any acceptable $1 \times (n-1)$ board can fill squares

Case 3: White $\rightarrow$ any acceptable $1 \times (n-1)$ board can fill squares

$\Rightarrow \ \ h_n = 2h_{n-1} + 2h_{n-2}$
2 Problem 2

(16 points)

Prove that a graph $G$ with 7 nodes and more than $\binom{7}{2}$ edges is always connected (assume there are no parallel edges or loops in the graph).

(\textit{Brute force method - more elegant solutions welcome!})

Suppose $G$ is not connected.

Then $G$ has at least 2 components, $H + J$.

Let $H$ be a connected component of $G$ with $k$ nodes, and let $J$ be all other nodes and edges of $G$.

If $H$ has 1 node, $J$ has at most $\binom{6}{2} = 15$

edges

$\Rightarrow G$ has at most $\binom{6}{2}$ edges

$H$ has 2 nodes, $J$ has at most $\binom{5}{2} = 10$

edges

$\Rightarrow G$ has at most $11$ edges

$H$ has 3 nodes, $J$ has $< \binom{4}{2} = 6$ edges

$\Rightarrow G$ has at most $3 + 6$ edges

By exchanging $H + J$, this covers all cases.

$\Rightarrow G$ has $< \binom{6}{2}$ edges. Contradiction.

$\Rightarrow G$ is connected.
3 Problem 3

(15 points)

Decide which of the graphs below has an Eulerian walk. Decide which of the graphs below has a Hamiltonian cycle. You do not have to justify your answers (which should be “YES” or “NO”) for this problem.

<table>
<thead>
<tr>
<th></th>
<th>Eulerian Walk?</th>
<th>Hamiltonian cycle?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_6$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td><img src="image1" alt="Graph" /></td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td><img src="image2" alt="Graph" /></td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>
4 Problem 4

(17 points)

Prove that if $n$ is a multiple of 6 then $F_n$ is divisible by 8. You may use the identity:

$$F_6 = 8$$

Prove $F_6$ divides $F_{6m}$.

$$F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$$

Proof by induction on $m$.

1. $m = 1$; $F_6 | F_6$.
2. Assume $F_6 | F_{6(m-1)}$.
3. Prove $F_6 | F_{6m}$.

$$F_{6m} = F_{6(m-1)} + 6 = F_{6(m-1) + 5} + 1$$

$$= F_{6(m-1)} + F_6 + F_{6(m-1)}F_5$$

$$= F_{6(m-1) + 1} F_6 + F_{6(m-1)} F_5$$

A | B since (I.H.) $F_6 | F_{6(m-1)}$.

$F_6 | (A + B) \Rightarrow F_6 | F_{6m}$. 

6
6 Problem 6

(15 points)

There is no partial credit on this problem; your answer for each question should be "YES" or "NO".

a. Is there a simple graph on 5 nodes with degrees $4, 4, 3, 2, 2$? \[ \text{NO} \]

b. Is there a simple graph on 5 nodes with degrees $3, 3, 3, 2, 2$? \[ \text{NO} \]

c. Is there a simple graph on 5 nodes with degrees $4, 3, 3, 2, 2$? \[ \text{YES} \]

d. Is there a simple graph on 5 nodes with degrees $3, 3, 2, 3, 2$? \[ \text{NO} \]

e. Is there a simple graph on 5 nodes with degrees $3, 3, 3, 3, 2$? \[ \text{YES} \]