Math 145, Winter 2013, Midterm 2

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<th>PROBLEM</th>
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Please do not turn this page until you are told to start the exam. You are not allowed to use books, notes, calculators, or, in fact, anything except a pencil (or pen) (and your brain!). You should show all of your work. A correct answer with an incomplete or incorrect explanation will only get partial credit. An incorrect answer with a good explanation will get partial credit. MAKE SURE YOU HAVE ALL THE PAGES OF YOUR EXAM! There are 6 problems. Good Luck!
1 Problem 1

(18 points)

Let $L_n$ be the sequence of numbers defined by the recursion relation

$$L_{n+1} = (-3/2)L_n + L_{n-1}$$

with initial values $L_0 = 5$, $L_1 = 0$.

a. Calculate: $L_2$ and $L_3$.

$$L_2 = -\frac{3}{2} \cdot 0 + 5 = 5$$

$$L_3 = -\frac{3}{2} \cdot 5 + 0 = -\frac{15}{2}$$

b. Find an explicit formula for $L_n$.

Solve: $q^{n+1} = -\frac{3}{2} q^n + q^{n-1}$

$q^{n+1} + \frac{3}{2} q^n - q^{n-1} = 0$

$q^2 + \frac{3}{2} q - 1 = 0$

$2q^2 + 3q - 2 = 0$

$(2q-1)(q+2) = 0$

$q = \frac{1}{2}$ or $q = -2$

Solve: $(\frac{1}{2})^n A + (-2)^n B = 5$

$(\frac{1}{2})^2 A + (-2)^1 B = 0$

$A + B = 5 \rightarrow A = 5 - B$

$\frac{1}{2} A - 2B = 0 \rightarrow A = 48$

$\Rightarrow B = 1, A = 4$

$L_n = 4 \left(\frac{1}{2}\right)^n + (-2)^n$
2 Problem 2

(16 points)

Prove that a graph $G$ with 7 nodes and more than $\binom{7}{2}$ edges is always connected (assume there are no parallel edges or loops in the graph).

(\textit{Brute force method - more elegant solutions welcome!})

Suppose $G$ is not connected.
Then $G$ has at least 2 components.
Let $H$ be a connected component of $G$ with $k$ nodes, and let $J$ be all other nodes and edges of $G$.
If $H$ has 1 node, $J$ has at most $\binom{6}{2}$ edges.

\[ \Rightarrow G \text{ has at most } \binom{6}{2} \text{ edges} \]

$H$ has 2 nodes, $J$ has at most $\binom{5}{2} = 10$ edges.

\[ \Rightarrow G \text{ has at most } 11 \text{ edges} \]

$H$ has 3 nodes, $J$ has $< \binom{4}{2} = 6$ edges.

\[ \Rightarrow G \text{ has at most } 3 + 6 \text{ edges} \]

By exchanging $H + J$, this covers all cases.

\[ \Rightarrow G \text{ has } < \binom{6}{2} \text{ edges} \text{. Contradiction,} \]

\[ \Rightarrow G \text{ is connected.} \]
3 Problem 3

(16 points)

Decide which of the graphs below has an Eulerian walk. Decide which of the graphs below has a Hamiltonian cycle. You do not have to justify your answers (which should be “YES” or “NO”) for this problem.

<table>
<thead>
<tr>
<th>Graph Description</th>
<th>Eulerian Walk?</th>
<th>Hamiltonian cycle?</th>
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</thead>
<tbody>
<tr>
<td>$K_6$ (complete graph on 6 vertices)</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>$K_7$ (complete graph on 7 vertices)</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>[Graph Image]</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>[Graph Image]</td>
<td>N</td>
<td>Y</td>
</tr>
</tbody>
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4 Problem 4

(18 points)

Suppose that there are 13 people in a room. Show that there is either a group of 3 people, none of whom know each other, or there is at (at least) 1 person who knows at least 6 others (or possibly both).

Let $p_1$ be one of the people.
Form a group $G_1$ consisting of $p_1$ and everyone whom $p_1$ knows.
If there are 7 or more people in this group, then $p_1$ knows at least 6 people and we are done.
So assume there are less than 7 in $G_1$.

Assume $p_2$, say, is not in $G_1$.
Form a group $G_2$ consisting of everyone $p_2$ knows who is not already in $G_1$. If $|G_2| > 7$, then $p_2$ knows at least 6 people, and we are done.
So assume $|G_2| < 7$. Then $|G_1| + |G_2| \leq 12$,
so there is yet a third person, $p_3$, who is not in $G_1$ or $G_2$,

Then $p_1, p_2, p_3$ are 3 people none of whom know each other.
5 Problem 5

(16 points)

Let $F_n$ be the Fibonacci sequence, so $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3,...,$

etc. Use induction to prove that

$$(F_1)^2 + (F_2)^2 + (F_3)^2 + (F_4)^2 + \ldots + (F_n)^2 = F_{n+1}F_n$$

D) Check for $n = 1$:

$$(F_1)^2 = 1 = 1 \cdot 1 = F_2 \cdot F_1 \checkmark$$

2) Assume for $n$, i.e., assume

$$(F_1)^2 + \ldots + (F_n)^2 = F_{n+1}F_n$$

3) Prove for $n+1$, i.e., prove

$$(F_1)^2 + \ldots + (F_n)^2 + (F_{n+1})^2 = F_{n+2}F_{n+1}$$

\[
(F_1)^2 + \ldots + (F_{n+1})^2 = F_{n+1}F_n + (F_{n+1})^2
\]

\[
= F_{n+1}(F_n + F_{n+1}) \quad (\text{def. of } F_n)
\]

\[
= F_{n+1}F_{n+2} \quad \checkmark
\]
6  Problem 6

(16 points)

**DEFINITION:** A graph $G$ is a tree if it is connected and contains no cycle as a subgraph.

*Using only the definition,* prove that if $G$ is a tree then $G$ is connected, but deleting any of its edges results in a disconnected graph.

$G$ is connected by definition.

Suppose $e$ is an edge of $G$, connecting $u$ to $v$.

Let $G' = G \setminus e$. Suppose $G'$ is connected. Let $P$ be a path in $G'$ from $u$ to $v$.

Then $P$ together with $e$ forms a cycle in $G$, so $G$ is not a tree.

Contradiction.

Hence $G'$ is not connected.