Math 145, Fall 2018, Midterm 2

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Please do not turn this page until you are told to start the exam. You are not allowed to use books, notes, calculators, or, in fact, anything except a pencil (or pen)(and your brain!). You should show all of your work. A correct answer with an incomplete or incorrect explanation will only get partial credit. An incorrect answer with a good explanation will get partial credit. MAKE SURE YOU HAVE ALL THE PAGES OF YOUR EXAM! There are 6 problems. Good Luck!
1 Problem 1

(20 points)

Prove:

\[ (1^3 + 2^3 + 3^3 + \ldots + n^3) = \frac{n^2(n + 1)^2}{4} \]

**Proof by induction on n:**

I. \( n = 1 \)

\[ 1^3 = \frac{1^2(2)^2}{4} = 1 \quad \checkmark \]

II. Assume

\[ (1^3 + 2^3 + \ldots + n^3) = \frac{n^2(n + 1)^2}{4} \]

III. Prove

\[ (1^2 + \ldots + n^3 + (n+1)^3) = \frac{(n+1)^2(n+2)^2}{4} \]

\[ (1^3 + \ldots + n^3 + (n+1)^3) = \]

\[ \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \]

\[ \frac{(n+1)^2 \left[ n^2 + 4(n+1) \right]}{4} = \]

\[ \frac{(n+1)^2(n+2)^2}{4} \]
2 Problem 2

(15 points)

Prove that a graph $G$ with 12 nodes and more than 55 edges must be connected.

If $G$ is not connected it must have at least two components $H$ with $h$ nodes and $K$ with $k$ nodes, with $h+k=12$, and no edges connecting a node in $H$ to a node in $K$.

Max # of edges in $H$ is \( \binom{h}{2} \).

Max # of edges in $K$ is \( \binom{k}{2} \).

Consider cases:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$k$</th>
<th>( \binom{h}{2} + \binom{k}{2} )</th>
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<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>0 + ( \binom{11}{2} ) = 55</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1 + ( \binom{10}{2} ) = 46</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>3 + ( \binom{9}{2} ) = 36</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>6 + ( \binom{8}{2} ) = 28</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>10 + ( \binom{7}{2} ) = 31</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>15 + ( \binom{6}{2} ) = 30</td>
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So if $G$ has more than 55 edges it cannot be an unconnected graph.
3 Problem 3

(15 points)

Suppose $S$ is a set of 41 distinct integers. Show there must exist distinct elements $a$ and $b$ in $S$ such that $a - b$ is divisible by 40.

Let $S = \{n_1, n_2, n_3, \ldots, n_{41}\}$

Let $R$ be the list of remainders of the $n_i$'s after dividing by 40.

By the P.H.P., at least two of the remainders are the same (there are 40 possible).

Therefore if $n_i, n_j < S$ such that

$n_i = k \cdot 40 + r$

$n_j = m \cdot 40 + r$

so $n_i - n_j = 40(k-m) + r-r = 40(k-m)$.

So $n_i - n_j$ is divisible by 40.
4 Problem 4

(20 points)

Find an explicit formula for the sequence $A_n$ if

\[ A_0 = 5 \]
\[ A_1 = 8 \]
\[ * \quad A_n = 5A_{n-1} + 14A_{n-2} \]

\[ X^n = 5X^{n-1} + 14X^{n-2} \quad X \neq 0 \]
\[ X^2 = 5X + 14 \]
\[ X^2 - 5X - 14 = 0 \]
\[ (x - 7)(x + 2) = 0 \]
\[ x = 7, -2 \]

So the two sequences
\[ 7^0, 7^1, 7^2, 7^3, \ldots = J_n \]
\[ (-2)^0, (-2)^1, (-2)^2, \ldots = I_n \]

Now find $c, d$ so that
\[ c \cdot 7^0 + d \cdot (-2)^0 = 5 \]
\[ c \cdot 7^1 + d \cdot (-2)^1 = 8 \]
\[ c + d = 5 \]
\[ 7c - 2d = 8 \]

\[ 9c = 18 \]
\[ c = 2 \]
\[ d = 3 \]

So $A_n = 2 \cdot 7^n + 3(-2)^n$
5 Problem 5

(15 points)

Does the graph drawn below have
a. an Eulerian walk?
b. a Hamiltonian cycle?

Justify your answers.

a. No, it has more than 2 vertices of odd degree.

b. Yes, shown on the graph.
6  Problem 6

(15 points)

Let $F_i$ be the $i$th Fibonacci number. Prove that $F_{mk}$ is a multiple of $F_k$ by induction on $m$. You may use the identity

$$F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b \quad \ast$$

I. True for $m = k$:

$$F_k \mid F_k$$

II. Assume $F_k \mid F_{(m-1)k}$

III. Prove $F_k \mid F_{mk}$

$$F_{mk} = F_{(m-1)k + k}$$

$$= F_{(m-1)k + (k-1) + 1}$$

$$\equiv F_{(m-1)k + 1} \cdot F_k + F_{(m-1)k} \cdot F_{k-1}$$

$$\leftarrow F_k \text{ divides this}$$

$$\Rightarrow F_k \mid F_{mk}.$$