1 Problem 1

(15 points)

Find the coefficient of x^3y^6 in the expansion of $(2x-2y)^9$.

$$(2x-2y)^9 = \binom{9}{0}(2x)^9 + \binom{9}{1}(2x)^8(-2y)^1 + \cdots + \binom{9}{9}(-2y)^9$$

coefficient of x^3y^6 is

$$\binom{9}{6}\cdot 2^3 \cdot (-2)^6 = \binom{9}{6}\cdot 2^9$$

2 Problem 2

(20 points)

Prove by induction on n that the maximum number of pieces into which you can slice a pizza with n straight cuts is

$$1 + \frac{n(n+1)}{2}$$

I. $n=1$

A single straight cut through the pizza cuts it into 2 pieces;

$$2 = 1 + \frac{1 \cdot 2}{2} \quad \checkmark$$

II. Assume n straight cuts can cut the pizza into $1 + \frac{n(n+1)}{2}$ pieces.

III. Show that the $n+1$ cut can cut the pizza into $1 + \frac{(n+1)(n+2)}{2}$ pieces, and show it cannot cut it into more.

The $n+1$ cut line can be chosen to intersect each previous cut line exactly once (and not more). Therefore there are n points of intersection between the $(n+1)$ line and all previous cut lines. Those n point divide the $(n+1)$ cut line into $n+1$ segments. Each segment divides an

$n+1$ new pieces (at most).

So the new number of pieces is

$$1+\frac{n(n+1)}{2}+n+1$$

$$=1+\frac{n(n+1)}{2}+\frac{2n+2}{2}$$

$$=1+\frac{n^2+3n+2}{2}$$

$$=1+\frac{(n+1)(n+2)}{n}$$

3 Problem 3

(20 points)

How many ways are there to make up a box of a dozen doughnuts from a tray of 6 jelly, 16 sugar, and 14 chocolate and 10 glazed doughnuts?

If there were at least 12 of each kind, there would be $\binom{12+4-1}{3} = \binom{15}{3}$ possibilities.

Of these possibilities, 4 use more than 10 glazed.

Of these possibilities, $\binom{5+4-1}{3} = \binom{8}{3}$ use more than 6 jelly [fill 7 spots with jelly, then fill the remaining 5 any way at all].

So the total is $\binom{15}{3} - \binom{8}{3} - 3$

4 Problem 4

(10 points)

How many anagrams are there of the word "ANTIMAÇASSAR."?

$$\binom{12}{3}\binom{9}{2}\underbrace{\binom{7}{1}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}}_{\text{Place the rest}}$$

Place the "A"s

Place the "S"s

$$\boxed{= \frac{12!}{3!\,2!}}$$

the explanation "12 letters, one repeats 3 times, one repeats twice, no othe repeats" is fine.

5 Problem 5

(20 points)

By **integrating** the binomial expansion, from 0 to 1, prove that

$$1 + 1/2\binom{n}{1} + 1/3\binom{n}{2} + ... + 1/(n+1)\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

$$\text{So } \int_0^1 (1+x)^n dx = \int_0^1 \left[\binom{n}{0}x^0 + \cdots + \binom{n}{n}x^n\right] dx$$

$$= \left[\binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n}$$

$$\text{But } \int_0^1 (1+x)^n dx \text{ also } = \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^1 = \frac{2^{n+1}}{n+1} - \frac{1^{n+1}}{n+1}$$

$$= \frac{2^{n+1}-1}{n+1}$$

6 Problem 6

(15 points)

Find the number of integers between 1 and 1000 which are **NOT** divisible by 6, 7 or 8.

<u>Include</u> 1 and 1000:

$A = \{\text{those divisible by } 6\}$

$B = \{\text{those divisible by } 7\}$

$C = \{\text{those divisible by } 8\}$

$|A| = 166$

$|B| = 142$

$|C| = 125$

$|A \cap B| = \left\lfloor \frac{1000}{42} \right\rfloor = 23$

$|A \cap C| = \left\lfloor \frac{1000}{24} \right\rfloor = 41$

$|B \cap C| = \left\lfloor \frac{1000}{56} \right\rfloor = 17$

$|A \cap B \cap C| = \left\lfloor \frac{1000}{168} \right\rfloor = 5$

that <u>are</u> divisible by 6, 7 or 8:

$S = 166 + 142 + 125 - 23 - 41 - 17 + 5$

that are <u>not</u> divisible by 6, 7 or 8:

$1000 - S.$

1 Problem 1

(18 points)

Let L_n be the sequence of numbers defined by the recursion relation

$$L_{n+1} = (-3/2)L_n + L_{n-1}$$

with initial values $L_0 = 5$, $L_1 = 0$.

a. Calculate: L_2 and L_3.

$$L_2 = -3/2 \cdot 0 + 5 = 5$$
$$L_3 = -3/2 \cdot 5 + 0 = -15/2$$

b. Find an explicit formula for L_n.

Solve: $q^{n+1} = -3/2\, q^n + q^{n-1}$

$q^{n+1} + 3/2\, q^n - q^{n-1} = 0$ $\qquad q = 0$ or

$q^2 + 3/2\, q - 1 = 0$

$2q^2 + 3q - 2 = 0$

$(2q-1)(q+2) = 0$

$q = 1/2$ or $q = -2$

Solve: $(1/2)^0 A + (-2)^0 B = 5$

$(1/2)^1 A + (-2)^1 B = 0$

$A + B = 5 \rightarrow A = 5 - B$

$\frac{1}{2}A - 2B = 0 \rightarrow A = 4B$

$\Rightarrow B = 1,\ A = 4$

$L_n = 4(1/2)^n + (-2)^n$

2 Problem 2

(16 points)

Prove that a graph G with 7 nodes and more than $\binom{6}{2}$ edges is always connected (assume there are no parallel edges or loops in the graph).

(Brute force method – more elegant solutions welcome!)

Suppose G is not connected.

Then G has at least 2 components.

Let H be a connected component of G with k nodes, and let J be all other nodes and edges of G.

If H has 1 node, J has at most $\binom{6}{2}$ edges

$\Rightarrow G$ has at most $\binom{6}{2}$ edges

H has 2 nodes, J has at most $\binom{5}{2}=10$ edges

$\rightarrow G$ has at most 11 edges

H has 3 nodes, J has $< \binom{4}{2} = 6$ edges

$\Rightarrow G$ has at most $3+6$ edges

By exchanging H + J, this covers all cases.

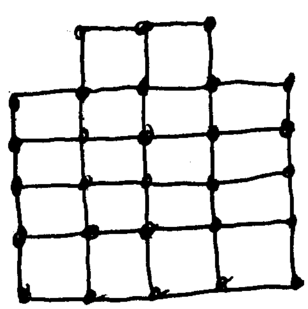
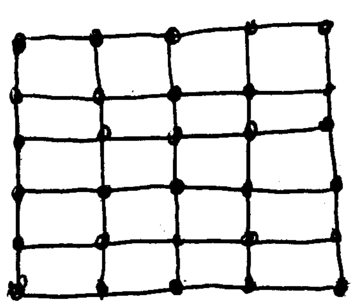
$\Rightarrow G$ has $< \binom{6}{2}$ edges. Contradiction.

$\Rightarrow G$ is connected.

3 Problem 3

(16 points)

Decide which of the graphs below has an Eulerian walk. Decide which of the graphs below has a Hamiltonian cycle. **You do not have to justify your answers (which should be "YES" or "NO") for this problem.**

	Eulerian Walk?	Hamiltonian cycle?
K_6 (complete graph on 6 vertices)	N	Y
K_7 (complete graph on 7 vertices)	Y	Y
	N	N
	N	Y

4 Problem 4

(18 points)

Suppose that there are 13 people in a room. Show that there is either a group of 3 people, none of whom know each other, or there is at (at least) 1 person who knows at least 6 others (or possibly both).

Let p_1 be one of the people.
Form a group G_1 consisting of p_1 and everyone whom p_1 knows.
If there are 7 or more people in this group, then p_1 knows at least 6 people and we are done.
So assume there are less than 7 in G_1.

Assume p_2, say, is not in G_1.
Form a group G_2 consisting of everyone p_2 knows who is not already in G_1. If $|G_2| \geq 7$, then p_2 knows at least 6 people, and we are done.
So assume $|G_2| < 7$. Then $|G_1| + |G_2| \leq 12$, so there is yet a third person, p_3, who is not in G_1 or G_2.

Then p_1, p_2, p_3 are 3 people none of whom know each other.

5 Problem 5

(16 points)

Let F_n be the Fibonacci sequence, so $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3....,$ etc. Use induction to prove that

$$(F_1)^2 + (F_2)^2 + (F_3)^2 + (F_4)^2 + ... + (F_n)^2 = F_{n+1}F_n$$

1) Check for $n = 1$:

$$(F_1)^2 = 1 = 1 \cdot 1 = F_2 \cdot F_1 \quad \checkmark$$

2) Assume for n, i.e., assume

$$(F_1)^2 + \dots + (F_n)^2 = F_{n+1} F_n$$

3) Prove for $n+1$, i.e., prove

$$(F_1)^2 + \dots + (F_{n+1})^2 = F_{n+2} F_{n+1}$$

$$(F_1)^2 + \dots + (F_{n+1})^2 \overset{\text{I.H.}}{=} F_{n+1} F_n + (F_{n+1})^2$$

$$= F_{n+1} (F_n + F_{n+1}) \quad (\text{def. of } F_n)$$

$$= F_{n+1} F_{n+2} \quad \checkmark$$

6 Problem 6

(16 points)

DEFINITION: A graph G is a **tree** if it is connected and contains no cycle as a subgraph.

Using only the definition, prove that if G is a tree then G is connected, but deleting any of its edges results in a disconnected graph.

G is connected by definition.

Suppose e is an edge of G, connecting u to v.

Let $G' = G \setminus e$. Suppose G' is connected. Let P be a path in G' from u to v.

Then P together with e forms a cycle in G, so G is not a tree.

Contradiction.

Hence G' is not connected.

6 Problem 6

(14 points)

Let F_n be the Fibonacci sequence, so $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, and $F_{n+1} = F_n + F_{n-1}$. Use induction to prove that

$$F_n \leq (\frac{5}{3})^n$$

① Check $F_1 \leq \frac{5}{3}$ $(1 \leq \frac{5}{3})$ ✓

$F_2 \leq \left(\frac{5}{3}\right)^2$ $\left(1 \leq \frac{25}{9}\right)$ ✓

② Assume $F_K \leq \left(\frac{5}{3}\right)^k$ for all $k \leq n$.

③ Prove $F_{n+1} \leq \left(\frac{5}{3}\right)^{n+1}$.

$F_{n+1} = F_n + F_{n-1}$ (by definition)

$F_{n+1} \leq \left(\frac{5}{3}\right)^n + \left(\frac{5}{3}\right)^{n-1}$ (I.H.)

$$= \left(\frac{5}{3}\right)^{n-1}\left(\frac{5}{3}+1\right)$$

$$= \left(\frac{5}{3}\right)^{n-1}\left(\frac{8}{3}\right)$$

$$\leq \left(\frac{5}{3}\right)^{n-1}\left(\frac{5}{3}\right)^2 = \left(\frac{5}{3}\right)^{n+1}$$

3 Problem 3

(14 points)

Let G be a connected graph with no loops and no parallel edges. Show it is not possible for G to have exactly eight vertices with degrees $1, 1, 1, 2, 3, 4, 5, 7$.

Suppose there is such a graph G.

Label the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ with degrees, respectively, $1, 1, 1, 2, 3, 4, 5, 7$.

Since v_8 has degree 7, it is connected to every other vertex.

Since v_1, v_2, v_3 each have degree 1, the only vertex each is connected to is v_8.

Hence v_7 is at most connected to v_4, v_5, v_6 and v_8.

Hence $\text{degree}(v_7) \leq 4$.

Contradiction.

Hence no such G exists.

1 Problem 1

(15 points)

How many sets of three numbers can be formed from $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ if no two consecutive numbers are to be in the set?

$$\underbrace{\binom{20}{3}}_{\text{all sets of 3}} - \underbrace{18}_{\#\text{ with 3 consecutive}} - \underbrace{17 \times 2}_{\#\text{ with } \{1,2\} \text{ or } \{19,20\}} - \underbrace{16 \times 17}_{\#\text{ all others with 2 consecutive}}$$

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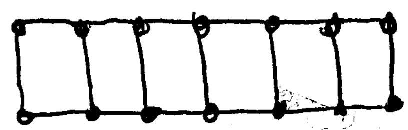
4 Problem 4

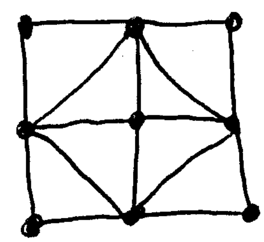
(14 points)

For each of the two graphs G, H drawn below, answer the questions below. **You do not have to justify your answers for this problem, and there is no partial credit.**

G



H



b. Does G have a Hamiltonian cycle? Yes

Does H have a Hamiltonian cycle? No

c. Does G have an Eulerian walk? No

Does H have an Eulerian walk? No

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6 Problem 6

(14 points)

Determine the number of solutions of the equation

$$x + y + z + w = 14$$

if x, y, z, w are all non-negative integers less than or equal to eight. (Note that $x = 6, y = 8, z = 0, w = 0$ and $x = 8, y = 6, z = 0, w = 0$, for example, are *different* solutions.)

$$\binom{17}{3} - 4\binom{8}{3} = 456$$

# of non-neg. solutions with no restrictions

4 choices of variable × # of ways to fill 5 spots (after having filled first nine with one variable)

5 Problem 5

(14 points)

Let a_n be the sequence defined as follows: $a_0 = 1, a_1 = 3$, and if $n > 0$,

$$a_{n+2} = 2a_{n+1} + 2a_n$$

a. Find a_2 and a_3.

$a_2 = 8$

$a_3 = 22$

b. Find an explicit formula for a_n.

Solve $q^{n+2} = 2q^{n+1} + 2q^n$; $q = 0$ or

$q^2 - 2q - 2 = 0$

$q = 1 \pm \sqrt{3}$

$u = 1+\sqrt{3}$, $v = 1-\sqrt{3}$

$a_n = A(u)^n + B(v)^n$

$1 = a_0 = A + B$

$3 = a_1 = A(1+\sqrt{3}) + B(1-\sqrt{3})$

$3 = A(1+\sqrt{3}) + (1-A)(1-\sqrt{3})$; $A = \frac{2+\sqrt{3}}{2\sqrt{3}}$

$B = \frac{\sqrt{3}-2}{2\sqrt{3}}$

So $a_n = \left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)(1+\sqrt{3})^n + \left(\frac{\sqrt{3}-2}{2\sqrt{3}}\right)(1-\sqrt{3})^n$