1 Problem 1

(15 points)

Find the coefficient of $x^3y^6$ in the expansion of $(2x - 2y)^9$.

$$(2x - 2y)^9 = \binom{9}{0}(2x)^9 + \binom{9}{1}(2x)^8(-2y) + \ldots + \binom{9}{9}(-2y)^9$$

Coefficient of $x^3y^6$ is

$$\binom{9}{6} \cdot 2^3 \cdot (-2)^6 = \binom{9}{6} \cdot 2^9$$
2 Problem 2

(20 points)

Prove by induction on \( n \) that the maximum number of pieces into which you can slice a pizza with \( n \) straight cuts is

\[
1 + \frac{n(n+1)}{2}
\]

I. \( n=1 \)

A single straight cut through the pizza cuts it into 2 pieces:

\[
2 = 1 + \frac{1 \cdot 2}{2} \quad \checkmark
\]

II. Assume \( n \) straight cuts can cut the pizza into \( 1 + \frac{n(n+1)}{2} \) pieces.

III. Show that the \( n+1 \) cut can cut the pizza into \( 1 + \frac{(n+1)(n+2)}{2} \) pieces, and show it cannot cut it into more. The \( n+1 \) cut line can be chosen to intersect each previous cut line exactly once (and not more). Therefore there are \( n \) points of intersection between the \( (n+1) \) line and all previous cut lines. Those \( n \) points divide the \( (n+1) \) cut line into \( n+1 \) segments. Each segment divides an initial slice into two pieces, so \( n+1 \) are
So the new number of pieces is

\[ 1 + \frac{n(n+1)}{2} + n + 1 \]

\[ = 1 + \frac{n(n+1)}{2} + \frac{2n+2}{2} \]

\[ = 1 + \frac{n^2 + 3n + 2}{2} \]

\[ = 1 + \frac{(n+1)(n+2)}{2} \]
3 Problem 3

(20 points)

How many ways are there to make up a box of a dozen doughnuts from a tray of 6 jelly, 16 sugar, and 14 chocolate and 10 glazed doughnuts?

If there were at least 12 of each kind, there would be \( \binom{12 + 4 - 2}{3} = \binom{15}{3} \) possibilities.

Of these possibilities, \( \frac{4}{3} \) use more than 10 glazed.

Of these possibilities, \( \binom{5 + 4 - 1 - \frac{8}{3}}{3} \) use more than 6 jelly [fill 7 spots with jelly, then fill the remaining 5 any way at all].

So the total is \( \binom{15}{3} - \binom{8}{3} - 3 \)
4 Problem 4

(10 points)

How many anagrams are there of the word "ANTIMACASSAR"?

\[
\begin{pmatrix} 12 \\ 3 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

\[= \frac{12!}{3!2!} \]

Place the "A"s

Place the "S"s

Place the rest

The explanation:
12 letters, one repeats 3 times, one repeats twice, no other repeats is fine.
5 Problem 5

(20 points)

By integrating the binomial expansion, prove that

\[
1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \ldots + \frac{1}{n+1} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1}
\]

\[
(1 + x)^n = \binom{n}{0} x^0 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n
\]

So

\[
\int_0^1 (1 + x)^n \, dx = \int_0^1 \left[ \binom{n}{0} x^0 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n \right] \, dx
\]

\[
= \left[ \binom{n}{0} x + \frac{\binom{n}{1} x^2}{2} + \frac{\binom{n}{2} x^3}{3} + \ldots + \frac{\binom{n}{n} x^{n+1}}{n+1} \right]_0^1
\]

\[
= \binom{n}{0} + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \ldots + \frac{1}{n+1} \binom{n}{n}
\]

But

\[
\int_0^1 (1 + x)^n \, dx = \frac{(1 + x)^{n+1}}{n+1} \bigg|_0^1 = \frac{2^{n+1} - 1}{n+1}
\]

\[
= \frac{2^{n+1} - 1}{n+1}
\]
6  Problem 6

(15 points)

Find the number of integers between 1 and 1000 which are NOT divisible by 6, 7 or 8.

\[ \text{Include } 1 \text{ and } 1000: \]
\[ A = \text{ those divisible by 6} \]
\[ B = \text{ those divisible by 7} \]
\[ C = \text{ those divisible by 8} \]

\[ |A| = 166 \]
\[ |B| = 142 \]
\[ |C| = 125 \]

\[ |A \cap B| = \left\lfloor \frac{1000}{42} \right\rfloor = 23 \]
\[ |A \cap C| = \left\lfloor \frac{1000}{24} \right\rfloor = 41 \]
\[ |B \cap C| = \left\lfloor \frac{1000}{56} \right\rfloor = 17 \]
\[ |A \cap B \cap C| = \left\lfloor \frac{1000}{168} \right\rfloor = 5 \]

\[ S = 166 + 142 + 125 - 23 - 41 - 17 + 5 \]
\[ \# \text{ that are divisible by 6, 7 or 8:} \]

\[ 1000 - S \]

\[ \# \text{ that are not divisible by 6, 7 or 8:} \]
1 Problem 1

(18 points)

Let $L_n$ be the sequence of numbers defined by the recursion relation

$$L_{n+1} = (-3/2)L_n + L_{n-1}$$

with initial values $L_0 = 5$, $L_1 = 0$.

a. Calculate: $L_2$ and $L_3$.

$$L_2 = -\frac{3}{2} \cdot 0 + 5 = 5$$
$$L_3 = -\frac{3}{2} \cdot 5 + 0 = -\frac{15}{2}$$

b. Find an explicit formula for $L_n$.

Solve: $q^{n+1} = -\frac{3}{2} q^n + q^{n-1}$

$$q^{n+1} + \frac{3}{2} q^n - q^{n-1} = 0$$

$$q^2 + \frac{3}{2} q - 1 = 0$$

$$2q^2 + 3q - 2 = 0$$

$$(2q-1)(q+2) = 0$$

$q = \frac{1}{2}$ or $q = -2$

$A + B = 5 \rightarrow A = 5 - B$

$$\frac{1}{2} A - 2B = 0 \rightarrow A = 4B$$

$\Rightarrow B = 1, A = 4$

$$L_n = 4 \left( \frac{1}{2} \right)^n + (-2)^n$$
2 Problem 2

(16 points)

Prove that a graph $G$ with 7 nodes and more than $\binom{7}{2}$ edges is always connected (assume there are no parallel edges or loops in the graph).

(Blunt force method - more elegant solutions welcome!)

Suppose $G$ is not connected.
Then $G$ has at least 2 components.
Let $H$ be a connected component of $G$ with $k$ nodes, and let $J$ be all other nodes and edges of $G$.
If $H$ has 1 node, $J$ has at most $\binom{6}{2}$ edges.
$\Rightarrow$ $G$ has at most $\binom{6}{2}$ edges.

$H$ has 2 nodes, $J$ has at most $\binom{5}{2} = 10$ edges.
$\Rightarrow$ $H$ has at most 11 edges.
$H$ has 3 nodes, $J$ has $\binom{4}{2} = 6$ edges;
$\Rightarrow$ $G$ has at most $3 + 6$ edges.
By exchanging $H$ + $J$, this covers all cases.
$\Rightarrow$ $G$ has $\binom{3}{2}$ edges. Contradiction.
$\Rightarrow$ $G$ is connected.
3 Problem 3

(16 points)

Decide which of the graphs below has an Eulerian walk. Decide which of the graphs below has a Hamiltonian cycle. **You do not have to justify your answers (which should be “YES” or “NO”) for this problem.**

<table>
<thead>
<tr>
<th></th>
<th>Eulerian Walk?</th>
<th>Hamiltonian cycle?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_6$ (complete graph on 6 vertices)</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>$K_7$ (complete graph on 7 vertices)</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

![Graph 1](image1)

![Graph 2](image2)

![Graph 3](image3)
4 Problem 4

(18 points)

Suppose that there are 13 people in a room. Show that there is either a
group of 3 people, none of whom know each other, or there is at (at least) 1
person who knows at least 6 others (or possibly both).

Let $p_1$ be one of the people.
Form a group $G_1$ consisting of $p_1$
and everyone whom $p_1$ knows.
If there are 7 or more people in
this group, then $p_1$ knows at least 6
people and we are done.
So assume there are less than 7 in $G_1$.
Assume $p_2$, say, is not in $G_1$.
Form a group $G_2$ consisting of
everyone $p_2$ knows who is not already in
$G_1$. If $|G_2| > 7$, then $p_2$ knows at
least 6 people, and we are done.
So assume $|G_2| < 7$. Then $|G_1| + |G_2| \leq 12$.
so there is yet a third person, $p_3$,
who is not in $G_1$ or $G_2$.

Then $p_1, p_2, p_3$ are 3 people
none of whom know each other.
5 Problem 5

(16 points)

Let $F_n$ be the Fibonacci sequence, so $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \ldots$, etc. Use induction to prove that

$$(F_1)^2 + (F_2)^2 + (F_3)^2 + (F_4)^2 + \ldots + (F_n)^2 = F_{n+1}F_n$$

1) Check for $n = 1$:

$$(F_1)^2 = 1 = 1 \cdot 1 = F_2 \cdot F_1$$

2) Assume for $n$, i.e., assume

$$(F_1)^2 + \ldots + (F_n)^2 = F_{n+1}F_n$$

3) Prove for $n+1$, i.e., prove

$$(F_1)^2 + \ldots + (F_{n+1})^2 = F_{n+2}F_{n+1}$$

$$(F_1)^2 + \ldots + (F_n)^2 = F_{n+1}F_n + (F_{n+1})^2$$

$$(F_1)^2 + \ldots + (F_{n+1})^2 = F_{n+1}F_n + F_{n+1}(F_n + F_{n+1}) \quad \text{(def. of $F_n$)}$$

$$(F_1)^2 + \ldots + (F_{n+1})^2 = F_{n+1}F_{n+2} \quad \checkmark$$
6 Problem 6

(16 points)

**DEFINITION:** A graph $G$ is a **tree** if it is connected and contains no cycle as a subgraph.

*Using only the definition*, prove that if $G$ is a tree then $G$ is connected, but deleting any of its edges results in a disconnected graph.

$G$ is connected by definition.

Suppose $e$ is an edge of $G$, connecting $u$ to $v$.

Let $G' = G - e$. Suppose $G'$ is connected. Let $P$ be a path in $G'$ from $u$ to $v$.

Then $P$ together with $e$ forms a cycle in $G$, so $G$ is not a tree.

Contradiction.

Hence $G'$ is not connected.
6 Problem 6

(14 points)

Let $F_n$ be the Fibonacci sequence, so $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, and $F_{n+1} = F_n + F_{n-1}$. Use induction to prove that

\[ F_n \leq \left(\frac{5}{3}\right)^n \]

1. Check $F_1 \leq \frac{5}{3}$ (1 ≤ 5/3) √
   
   $F_2 \leq \left(\frac{5}{3}\right)^2$ (1 ≤ 25/9) √

2. Assume $F_k \leq \left(\frac{5}{3}\right)^k$ for all $k \leq n$.

3. Prove $F_{n+1} \leq \left(\frac{5}{3}\right)^{n+1}$.

   $F_{n+1} = F_n + F_{n-1}$ (by definition)

   $F_{n+1} \leq \left(\frac{5}{3}\right)^n + \left(\frac{5}{3}\right)^{n-1}$ (I. H.)

   $\Rightarrow \quad \left(\frac{5}{3}\right)^n \cdot \left(\frac{5}{3} + 1\right)

   \quad \Rightarrow \quad \left(\frac{5}{3}\right)^n \cdot \left(\frac{8}{3}\right)

   \quad \Rightarrow \quad \left(\frac{5}{3}\right)^{n-1} \cdot \left(\frac{5}{3}\right)^2 = \left(\frac{5}{3}\right)^{n+1}$
3 Problem 3

(14 points)

Let \( G \) be a connected graph with no loops and no parallel edges. Show it is not possible for \( G \) to have exactly eight vertices with degrees 1, 1, 1, 2, 3, 4, 5, 7.

Suppose there is such a graph \( G \).

Label the vertices \( v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \) with degrees, respectively, 1, 1, 1, 2, 3, 4, 5, 7.

Since \( v_7 \) has degree 7, it is connected to every other vertex.

Since \( v_1, v_2, v_3 \) each have degree 1, the only vertex each is connected to is \( v_8 \).

Hence \( v_7 \) is at most connected to \( v_4, v_5, v_6 \) and \( v_8 \).

Hence degree \( (v_7) \leq 4 \).

Contradiction.

Hence no such \( G \) exists.
1 Problem 1

(15 points)

How many sets of three numbers can be formed from \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\} if no two consecutive numbers are to be in the set?

\[
\binom{20}{3} - 18 - 17 \times 2 - 16 \times 17
\]

\[
\text{all sets of 3}
\]

\[
\# \text{with 3 consecutive}
\]

\[
\# \text{with 21, 23 or 19, 20}
\]

\[
\# \text{all others with 2 consecutive}
\]
4 Problem 4

(14 points)

For each of the two graphs $G, H$ drawn below, answer the questions below. You do not have to justify your answers for this problem, and there is no partial credit.

For each of the two graphs $G, H$, answer the following questions:

b. Does $G$ have a Hamiltonian cycle? Yes

Does $H$ have a Hamiltonian cycle? No

c. Does $G$ have an Eulerian walk? No

Does $H$ have an Eulerian walk? No
6 Problem 6

(14 points)

Determine the number of solutions of the equation

\[ x + y + z + w = 14 \]

if \( x, y, z, w \) are all non-negative integers less than or equal to eight. (Note that \( x = 6, y = 8, z = 0, w = 0 \) and \( x = 8, y = 6, z = 0, w = 0 \), for example, are different solutions.)

\[ \binom{17}{3} - 4 \binom{8}{3} = 456 \]

\# of non-neg. solutions with no restrictions

4 choices of variable \( x \)

\# of ways to fill 5 spots (after having filled first nine with one variable)
5 Problem 5
(14 points)

Let \( a_n \) be the sequence defined as follows: \( a_0 = 1, a_1 = 3 \), and if \( n > 0 \),
\[
a_{n+2} = 2a_{n+1} + 2a_n
\]
a. Find \( a_2 \) and \( a_3 \).
\[
a_2 = 8
\]
\[
a_3 = 22
\]
b. Find an explicit formula for \( a_n \).

Solve
\[
q^{n+2} = 2q^{n+1} + 2q^n \quad \text{or} \quad q = 0
\]
\[
q^2 - 2q - 2 = 0
\]
\[
q = 1 \pm \sqrt{3}
\]
\[
u = 1 + \sqrt{3}, \quad v = 1 - \sqrt{3}
\]
\[
a_n = A(u)^n + B(v)^n
\]
\[
1 = a_0 = A + B
\]
\[
3 = a_1 = A(1 + \sqrt{3}) + B(1 - \sqrt{3})
\]
\[
3 = A(1 + \sqrt{3}) + (1 - A)(1 - \sqrt{3}) \quad \text{and} \quad A = \frac{2 + \sqrt{3}}{2\sqrt{3}}
\]
\[
B = \frac{\sqrt{3} - 2}{2\sqrt{3}}
\]

\[
\therefore a_n = \left(\frac{2 + \sqrt{3}}{2\sqrt{3}}\right)(1 + \sqrt{3})^n + \left(\frac{\sqrt{3} - 2}{2\sqrt{3}}\right)(1 - \sqrt{3})^n
\]