21B, Winter 2017, Midterm 2

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Please circle the time your section meets and your TA’s name:
6-7 PM, 7-8 PM, 8-9 PM, don’t know.
Norman Sheu, Armaun Emami, Subhadip Dey, Matthew Lin, don’t know.

Please do not turn this page until you are instructed to start the exam. You are not allowed to use books, notes, calculators, or, in fact, anything except a pencil (or pen). You should show all of your work. MAKE SURE YOU HAVE ALL THE PAGES OF YOUR EXAM! There are 8 problems.
1 Problem 1

(15 points)

Use the shell method to find the volume of the solid obtained by rotating the region bounded by $y = \ln x$, $y = 0$, $x = 2$ about the $y$-axis.

Area of shell of $y = 2\pi x \ln y$

Volume $= \int_0^\infty 2\pi x \ln y \, dy$

$x \ln x \, dx = \frac{1}{2} x \ln x - \frac{1}{2} x \bigg|_1^2 = \frac{1}{2} \ln 2 - \frac{1}{2} \ln 2 + \frac{1}{2} = \frac{1}{2}$

So volume $= 2\pi \left[ \frac{1}{2} \ln 2 - \frac{1}{2} \right]^2 = 2\pi \left[ 2 \ln 2 - 1 - (0 - \frac{1}{4}) \right] = 2\pi \left[ 2 \ln 2 - \frac{3}{4} \right]$
2 Problem 2

(10 points)

A population of bacteria is grown under ideal conditions so that the population increases exponentially with time. Initially there are 1000 bacteria, and at the end of three hours there are 10,000. How many are there after five hours? You do not have to simplify your answer.

\[ P(t) = P(0)e^{kt} \]

\[ P(0) = 1000, \quad P(3) = 10,000 \]

\[ 1000e^{3k} = 10,000 \]

\[ e^{3k} = 10 \]

\[ 3k = \ln 10 \]

\[ k = \frac{\ln 10}{3} \]

\[ P(t) = 1000 e^{\frac{\ln 10}{3} t} \]

\[ P(5) = 1000 e^{\frac{5}{3} \ln 10} = \# \text{ of bacteria after five hours} \]
3 Problem 3

(10 points)

Solve the differential equations:

A. 
\[ (\sec x) \frac{dy}{dx} = e^{y+\sin x} \]
\[ \int 2 \ dy = \int \left( \cos y \right) e^{\sin x} \ dx \quad (\text{let } u = \sin x) \]
\[ 2y = x + C \]

B. 
\[ \frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y} \]
\[ \int \frac{\sec^2 \sqrt{y}}{\sqrt{y}} \ dy = \int \sec^2 u \ du \quad (\text{let } u = \sqrt{y}) \]
\[ 2 \tan \sqrt{y} = x + C \]
4  Problem 4

(15 points)

Find the center of mass of a thin plate of constant density $\delta$ covering the region cut from the first quadrant by the circle $x^2 + y^2 = 9$. (Hint: the mass of the plate is the density times its area.)

\[ \text{Area} = \frac{9\pi}{4} \]
\[ \text{Mass} = \frac{9\pi}{4} \cdot \delta \]

The center of mass must lie on the line $y = x$.

Find $M_y$:

Find $M_x$:

Strip at $x$: height: $\sqrt{9-x^2}$

width: $dx$

center: $(x, \frac{1}{2}\sqrt{9-x^2})$

\[ M_x = \delta \int_0^3 \left( \frac{1}{2} \sqrt{9-x^2} \cdot \sqrt{9-x^2} \right) dx \]
\[ = \delta \left[ \frac{9-x^3}{3} \right]_0^3 = \frac{\delta}{2} \left[ 27 - 9 \right] = 9 \delta \]
\[ M_y = \frac{9\delta}{4\pi} \cdot 8 = \frac{4\delta}{\pi} \]

C.O.M. of plate: \( \left( \frac{1}{M} \frac{1}{M} \right) \)
5 Problem 5

(15 points)

Evaluate the integrals

\[
\int x \tan^2 x \, dx = \int x(1 - \sec^2 x) \, dx = \frac{x^2}{2} - \int x \sec^2 x \, dx
\]

\[
\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx
\]

\[
u = x \quad dv = \sec^2 x \, dx
\]
\[
 du = dx \quad v = \tan x
\]

\[
\int \tan x \, dx = \int \frac{\cos x}{\cos x} \, dx = \ln |\cos x|
\]

\[
\int e^x \ln x \, dx =
\]

\[
u = \ln x \quad dv = x^3 \, dx
\]
\[
 du = \frac{1}{x} \, dx \quad v = \frac{x^4}{4}
\]

\[
\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16}
\]

\[
= \frac{e^4}{4} - \frac{1}{16} + \frac{1}{16}
\]
6 Problem 6

(15 points)

A. Define $\ln x, x > 0$.

$$\ln x = \int_{1}^{x} \frac{1}{t} dt \quad \text{for } x > 0$$

B. Using the definition, approximate $\ln 5$ by partitioning the appropriate interval into four equal pieces and using the right hand endpoints as sample points. You don’t have to simplify your answer.

$$\ln 5 = \int_{1}^{5} \frac{1}{t} dt$$

\( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \)
7 Problem 7

(10 points)

A force of 40 N is required to hold a spring that has been stretched 1 meter from its natural length. Assume the spring obeys Hooke's Law.

A. How much force is required to hold the spring stretched 1.5 meters from its natural length?

\[ F(x) = kx \]

Given: \( F(1) = 40 \), \( k = 40 \)

\[ F(1.5) = 60 \text{ (N)} \]

B. How much work is done to stretch it from 1 meter to 1.5 meters?

\[
\text{Work} = \int_{1}^{1.5} 40x \, dx = 20x^2 \Bigg|_{1}^{1.5} = 20 \left[ (1.5)^2 - 1 \right]
\]
8 Problem 8

(10 points)

Set up (but do not evaluate) an integral for the surface area of the surface generated by rotating the curve

\[ y = \frac{e^x + e^{-x}}{2} \]

\[ 0 \leq x \leq \ln 2 \]

about the x-axis.

\[
\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}
\]

\[
\left(\frac{dy}{dx}\right)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2
\]

Surface area:

\[
2\pi \int_{0}^{\ln 2} \left(\frac{e^x + e^{-x}}{2}\right) \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} \, dx
\]