Geodesic currents and the smoothing property

Geometry and Topology seminar

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(joint work with Dylan Thurston)

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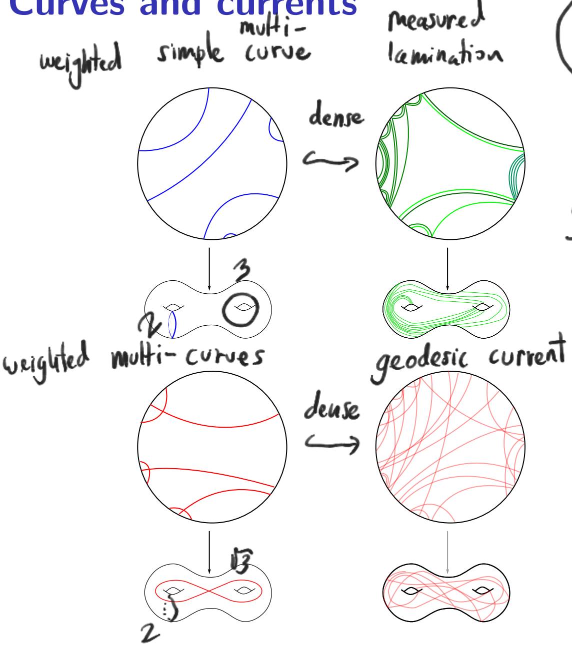
Curves and currents



closed
orientable
connected
genus $g \geq 2$

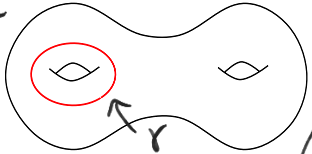
<u>curves</u> $C = [\gamma]$

free homotopy
classes of
closed paths

Definition of geodesic current

Fix Σ a hyperbolic metric on S, and let $C = [\gamma]$ be a curve

Σ



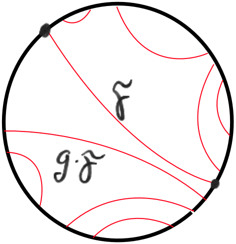
$\mathcal{G}(\widetilde{\Sigma})$ = space of geodesics of $\widetilde{\Sigma}$

locally finite

positive

Borel

$$\mu_\gamma := \sum_{g \in \pi_1(\Sigma)} \delta_{g\cdot\tilde{\gamma}}$$



$\widetilde{\Sigma}$

<u>geodesic current</u>

$\pi_1(\Sigma)$-inv locally finite Borel positive measure on $\widetilde{\Sigma}$

$\pi_1(\Sigma) \circlearrowright \widetilde{\Sigma}$

$\pi_1(\Sigma) \circlearrowright \partial_\infty \widetilde{\Sigma}$

$\pi_1(\Sigma) \circlearrowright \mathcal{G}(\widetilde{\Sigma})$

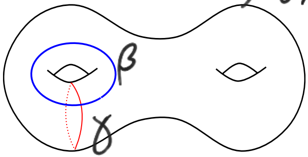
$h \in \pi_1(\Sigma)$, $h\cdot\mu_\gamma = \mu_\gamma$

Intersection number

Bonahon, 86 : there exist a bilinear continuous form

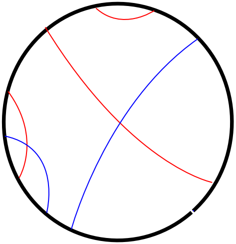
$$i : \mathcal{GC}(\Sigma) \times \mathcal{GC}(\Sigma) \to \mathbb{R} \qquad i(\mu_\gamma, \mu_\beta) = i(\gamma, \beta)$$

geometric intersection #

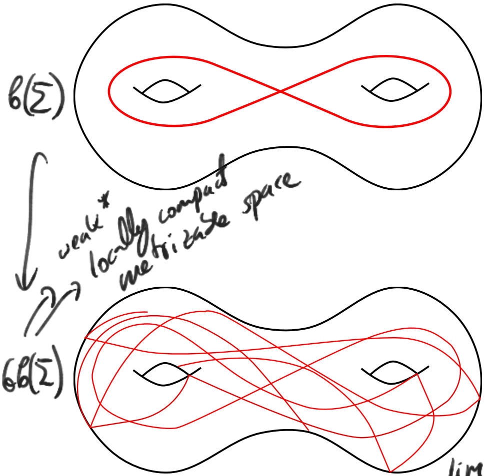


$\mathcal{GC}(\Sigma)$ = space of geodesic currents on Σ

weak* topology



Why care? Extensions of functions on curves



$f : \mathcal{C}(\Sigma) \longrightarrow \mathbb{R}$

extend?

$\tilde{f} : \mathcal{GC}(\Sigma) \to \mathbb{R}$

$\tilde{f}|_{\mathcal{C}(\Sigma)} = f.$

α is a filling geodesic

$f : \mathcal{GC}(\Sigma) \to \mathbb{R}$ continuous + homogeneous

$$\lim_{L \to \infty} \frac{\#\{\phi \in MCG \mid f(\phi(\alpha)) \leq L\}}{L^{6g-6}} \text{ exist}$$

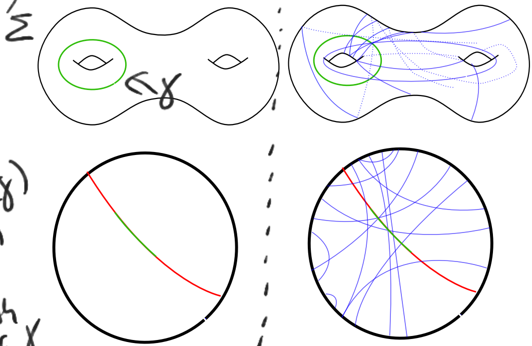
e.g. $f = \ell_\Sigma$

- Goldman-Labourie-Margulis 2009: proper affine actions
- Rafi-Souto 2017: curve counting problems

Why care? Intersection numbers

$\mu \in \mathcal{GB}(\Sigma)$, $i(\mu, \cdot) : \mathcal{G}(\Sigma) \to \mathbb{R}$ extends to currents

Bonahon, 86



$\ell_\Sigma(\gamma)$ = hyp length of γ

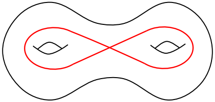
There exist $\mu_\Sigma \in \mathcal{GB}(\Sigma)$

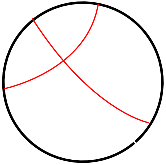
$i(\mu_\Sigma, \mu_\gamma) = \ell_\Sigma(\gamma)$

- Martone-Zhang 2019: entropy-systole bounds for Anosov lengths (systole: length of shortest curve)
- Burger-Iozzi-Parreau-Pozzetti 2019: domains of discontinuity for mapping class group action on the boundary of higher rank Teichmüller spaces

Definition of essential crossing

Let $C = [\gamma]$ be a curve, $p \in \gamma$ is a crossing, we say p is <u>essential</u> if persists under homotopy.

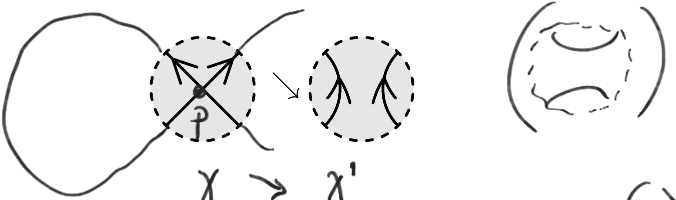




Definition of smoothing and smoothing property

<u>smoothing</u> $C=[\gamma]$ a curve, $p \in \gamma$ essential crossing,

a smoothing is a cut and reglue operation on p:



$\gamma \to \gamma'$

<u>Fact</u> smoothing is well-def for homotopy classes, so $C \searrow C'$ makes sure

<u>Smoothing property</u> $f: \mathcal{C}(S) \to \mathbb{R}$, if $f(C) \geq f(C')$ whenever $C \searrow C'$

Examples of functionals on curves

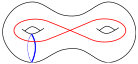
smoothing

- Intersection numbers: (Bonahon, 86)
- Negatively curved lengths: (Bonahon, 88), (Otal, 90)

hyp length

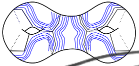
variable curvature

non-pos



quadratic differential

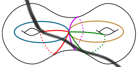
- Flat lengths: (Hersonsky-Paulin, 97), (Duchin-Leininger-Rafi, 10)



quasi-smoothing

- Word length: (Erlandsson, 16), (Erlandsson-Parlier-Souto, 16)

some gen sets



stable length wrt any gen set of π_1

- Anosov 'lengths': (Martone-Zhang, 16), (Bridgeman-Canary-Labourie-Sambarino, 17)

smoothing

not smoothing

- Extremal length

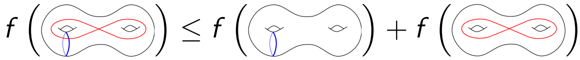
Extension result

"From curves to currents"

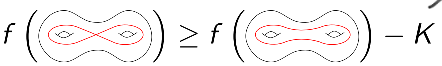
$f : \mathcal{C}(S) \to \mathbb{R}$

Theorem (MG–Thurston, 20)

1. (convex union) $f(C_1 \cup C_2) \leq f(C_1) + f(C_2)$.

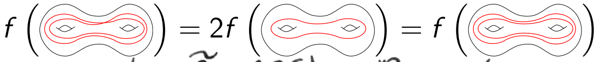


2. (quasi-smoothing)



uniform constant

3. (homogeneity) $f(nC) = nf(C)$.
4. (stability) $f(C^n) = f(nC)$



Then there exist $\tilde{f} : \mathcal{GC}(S) \to \mathbb{R}$, continuous and $\tilde{f}|_{\mathcal{C}(S)} = f$.

Question

- Given $\rho : \pi_1(S) \to SL(n, \mathbb{R})$ a 1-Anosov representation, does the function on oriented curves $\ell_1 := \log \frac{\lambda_1}{\lambda_2}(\rho(\cdot))$ satisfy quasi-smoothing?

Question

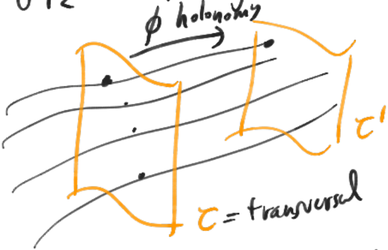
- Given f satisfying the smoothing and convex union property, what convexity properties does its restriction to measured laminations have?

Proof of Extension

Coming up next:
a quick sketch of the proof of the extension theorem.

Proof of Extension. Step 1: another definition of currents

$UT\Sigma$ = unit tangent bundle



$\phi : \tau \to \tau'$

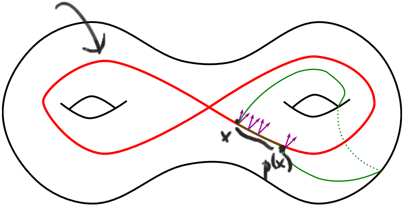
$(\mu_\tau)_\tau$ is invariant under flow : $\mu_{\tau'} = \phi_* \mu_\tau$

Proof of Extension. Step 2: cross-section, return map and homotopy return map

Build a transversal τ. Fix a geodesic δ,

$\tau = \{ (x,v) \in UT\Sigma \mid x \in \delta$, v almost perp to$_\epsilon$ $\delta \}$

δ



first return map $p : \tau \to \tau$

$m : \tau \to \mathcal{B}(S)$ homotopy return

↳ iterate $m^n : \tau \to \mathcal{B}(S)$
nth return map

Proof of Extension. Step 3: defining the extension

$$f_\tau^n(\mu) := f\left(\int m^n \, d\mu_\tau\right)$$

μ geodesic current

weighted curve

$$f_\tau(\mu) := \lim_{n\to\infty} \frac{f_\tau^n(\mu)}{n}.$$

Fix Σ hyp metric, let $\rho\cdot\Sigma$ metric obtained by scaling Σ by $\rho : \Sigma \to \mathbb{R}_{\geq 0}$

For a $C = [\gamma]$,

$$EL_\Sigma(C) := \sup_\rho \frac{\ell^2_{\rho\Sigma}(C)}{\mathrm{Area}(\rho\Sigma)}$$

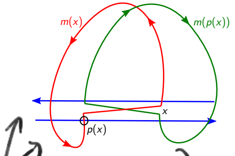
$0 < \mathrm{Area}(\rho\Sigma) < \infty.$

Proof of Extension. Step 4: relating iterates of homotopy return map

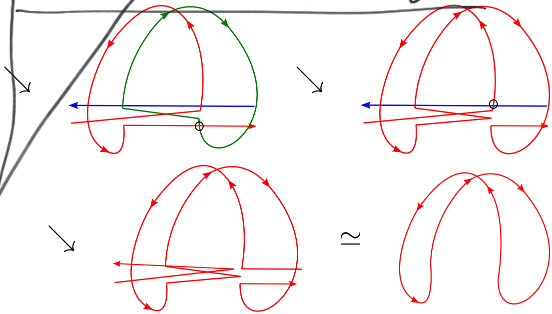
$m(x) \cup m(p(x)) \cup 2\delta \to \cdots \to m^2(x)$

$\text{const} + f^n + f^m \geq f^{n+m}$

smoothing



δ (two copies)



subadditive → Fekete's lemma

limit exist.