UNIVERSITY OF CAMBRIDGE

Taut foliations from left orders, in Heegaard genus two

Sarah Dean Rasmussen

University of Cambridge

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Outline

I. Motivation

II. Left orders & Right orders

III. Taut foliations

IV. Heegaard foliations

I. Motivation

Why fundmental group left orders and taut foliations?

A. Big picture

B. Heegaard Floer homology

C. L-spaces

D. L-space conjecture

For duration of talk: M closed oriented 3-manifold.

Structures/Properties of M:

-interesting geodesics
-constrained 1-vertex triangulations
-taut foliations
-tight contact structures

Invariants of M:

-volume
-$H_1(M)$
-$\pi_1(M)$
-gauge/Floer-theoretic:
$HF/HM/ECH$, HI.

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Structures/Properties of M:

-interesting geodesics
-constrained 1-vertex triangulations
-taut foliations
-tight contact structures

-volume
-$H_1(M)$ (cycles / boundaries)
-$\pi_1(M)$ properties: geometric type,...
-????

Invariants of M:

-volume
-$H_1(M)$
-$\pi_1(M)$
-gauge/Floer-theoretic:
$HF/HM/ECH$, HI.

$M = U_\alpha \cup_\Sigma U_\beta$. Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

$HF(M) := HF_{\mathrm{Lag}}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$, $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \mathrm{Sym}^{g(\Sigma)}(\Sigma)$.

—$CF(M)$ generated by points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^{g(\Sigma)}(\Sigma)$.

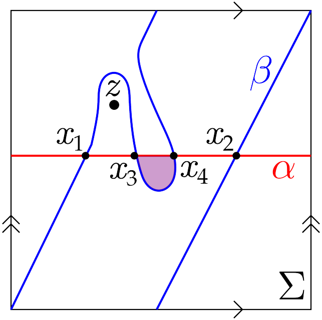
—Differentials: psuedoholomorphic Whitney disks.

Example:

$\widehat{CF}(M, \mathfrak{s}_1)$: $\langle x_1, x_4 \rangle \xrightarrow{x_4 \mapsto x_3} \langle x_3 \rangle$,

$\widehat{CF}(M, \mathfrak{s}_2)$: $\langle x_2 \rangle$,

$\implies \widehat{HF}(M, \mathfrak{s}_1) \simeq \widehat{HF}(M, \mathfrak{s}_2) \simeq \mathbb{Z}$.



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Example:

$\widehat{CF}(M, \mathfrak{s}_1)$: $\langle x_1 \rangle$,
$\widehat{CF}(M, \mathfrak{s}_2)$: $\langle x_2 \rangle$,
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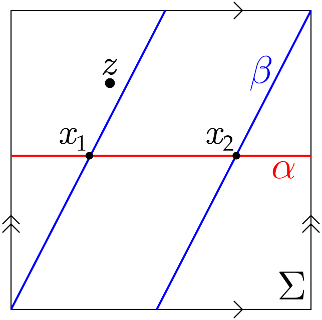


If M is a $\mathbb{Q}HS$ ($b_1(M) = 0$), then the smallest $\widehat{HF}(M)$ can be is

$$\widehat{HF}(M) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M)} \widehat{HF}(M, \mathfrak{s}) \simeq \bigoplus_{h \in H_1(M)} \mathbb{Z} \ \simeq \ \mathbb{Z}^{|H_1(M)|}.$$

Definition (L-space).
M is an *L-space* if $b_1(M) = 0$ and $\operatorname{rank} \widehat{HF}(M) = |H_1(M)|$,
or equivalently, if $HF_{\mathrm{red}}(M) = 0$.

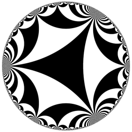
Example L-spaces:
-**L**ens spaces.
-Branched double covers of alternating knots.

L-space conjecture

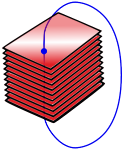
Conjecture. (Boyer-Gordon-Watson, Juhász, Ozsváth-Szabó, Némethi)

M is **not** an L-space $\iff$ ...

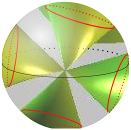
$\pi_1(M)$ has a left order (LO).
$g_1 > g_2 \iff hg_1 > hg_2$



M admits a
co-oriented taut foliation (CTF).



(if M a neg def graph manifold)
M links a nonrational singularity.



II. Left Orders and Right Orders

LO = Left order.   RO = Right order.

A. Definitions and positive cones

B. Real line actions.

G nontrivial group. LO = Left order. RO = Right order.

Definition (LOs & ROs).

LO $>_{\mathrm{R}}$ on G: $g_1 >_{\mathrm{R}} g_2 \iff hg_1 >_{\mathrm{R}} hg_2 \quad \forall g_1, g_2, h \in G$.

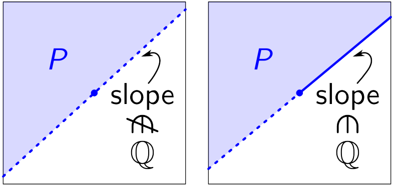
RO $>_{\mathrm{L}}$ on G: $g_1 >_{\mathrm{L}} g_2 \iff g_1 h >_{\mathrm{L}} g_2 h \quad \forall g_1, g_2, h \in G$.

Definition (positive cone P).

$P \subset G$ is a *positive cone* if

(i) $P \cdot P \subset P$

(ii) $G = P \amalg \{\mathrm{id}\} \amalg P^{-1}$



$G = \mathbb{Z} \oplus \mathbb{Z}$

Proposition (Alternative Definition).

G is LO $\iff$ G is RO $\iff$ G admits a positive cone P.

$g >_{\mathrm{L}} h \iff g^{-1} >_{\mathrm{R}} h^{-1} \iff g^{-1}h \in P$.

Theorem (classical). G countable nontrivial group.
G is LO $\iff$ G admits faithful $\mathbb{R}$-action, $\rho : G \to \mathrm{Homeo}_+ \mathbb{R}$.

$(\Rightarrow)$: *Dynamically realized action* ρ.
Choose $\rho(\cdot)(0) : G \hookrightarrow \mathbb{R}$ dense and order-preserving:

$$\rho(g)(0) < \rho(h)(0) \iff g <_{\mathrm{L}} h.$$

Set $\rho(g)(\rho(h)(0)) := \rho(gh)(0) \quad \forall g, h \in G$.
Extend by limit points.

$(\Leftarrow)$: *Lexicographical ordering.*
Choose ordering on $\mathbb{Q} \subset \mathbb{R}$: $\mathbb{Q} = \{q_1, q_2, \ldots, \}$.
For $g \neq h$, to see if $g <_{\mathrm{L}} h$, ask "is $\rho(g)(q_1) < \rho(h)(q_1)$?"
If $\rho(g)(q_i) = \rho(g)(q_i)\ \forall i \leq k$, ask "is $\rho(g)(q_{k+1}) < \rho(h)(q_{k+1})$?," $\ldots$.

Theorem (Boyer-Rolfsen-Wiest). If M is prime, closed, oriented, then $\pi_1(M)$ is LO if $\pi_1(M)$ admits *any* nontrivial $\mathbb{R}$-action.

III. Taut foliations (CTFs)

CTF = Cooriented taut foliation.

A. Foliations

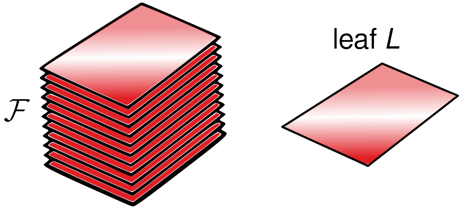
B. Taut foliation definition

C. $\pi_1(M)$ LOs from CTFs on M?

D. Known constructions of taut foliations

E. Transversely foliated bundles + holonomy reps

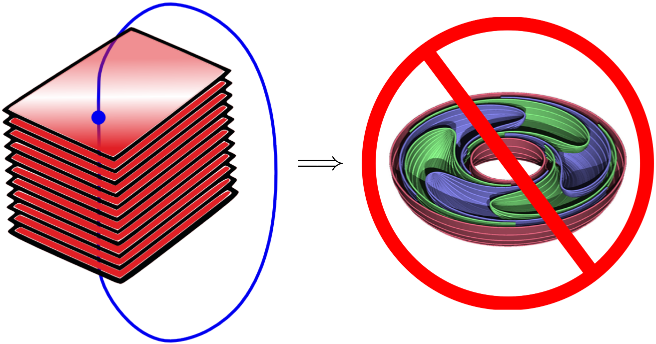
Definition (product foliation). A *codim-k product foliation* $\mathcal{F}$ on X is a decomposition $\mathcal{F} = \coprod_{b\in B} \pi^{-1}(b)$ of X into fibers $\pi^{-1}(b) \cong L$ of a trivial fibration $\pi : X \to B$ over a k-dim base B. ($X \cong L \times B$)
The fibers $\pi^{-1}(b)$, for $b \in B$, are called the *leaves* of $\mathcal{F}$.



Definition (foliation). A *codimension-k foliation* $\mathcal{F}$ on X^n is a globally compatible decomposition of X into leaves that looks locally like the product foliation associated to the trivial fibration $\mathbb{R}^n \to \mathbb{R}^k$.

Coorientation on $\mathcal{F}$ $\leftrightarrow$ globally compatible coorientations on $\mathbb{R}^k$s.

Definition (taut foliation). A codimension-1 foliation $\mathcal{F}$ on a closed oriented 3-manifold M is called *taut* if for every $x \in M$, there is a closed *transversal* containing x, i.e. a closed curve transverse to $\mathcal{F}$.



Convention. All foliations cooriented unless otherwise specified: CTF.

III. CTFs. $\pi_1(M)$ LOs from CTFs on M?

Given a CTF $\mathcal{F}$ on M ...

1. If $e(\mathcal{F}) = 0$, then $\pi_1(M)$ is LO. (Calegari-Dunfield)

$\mathcal{F}$ CTF $\rightsquigarrow$ faithful "universal S^1 action": $\rho_{\mathcal{F}}^{S_1} : \pi_1(M) \to \mathrm{Homeo}_+ S^1$.

$e(\rho_{\mathcal{F}}^{S_1}) = e(\mathcal{F}) = 0 \implies \rho_{\mathcal{F}}^{S_1}$ lifts to $\mathbb{R}$-action, $\pi_1(M) \to \mathrm{Homeo}_+ \mathbb{R}$.

2. If $\mathcal{F}$ is $\mathbb{R}$-covered, then $\pi_1(M)$ is LO.

Leafspace $\Lambda_{\mathcal{F}}$ of CTF $\mathcal{F}$ given by $\widetilde{M} \xrightarrow{\text{leaf} \mapsto \text{point}} \Lambda_{\mathcal{F}}$.

$\mathcal{F}$ $\mathbb{R}$-covered means leafspace $\Lambda_{\mathcal{F}} \cong \mathbb{R}$.

$\pi_1(M)$ acts on $\widetilde{M} \implies \pi_1(M)$ acts on $\Lambda_{\mathcal{F}} \cong \mathbb{R}$.

1. Dunfield: $e(\mathcal{F})$ has approx uniform random distribution in $H^2(M)$.
2. $\mathbb{R}$-covered foliations mostly only known for Seifert-fibered manifolds.

$\star$ LOs $\to$ CTFs: ???? (previously unknown)

Thurston: Slitherings around S^1.

Gabai: Intersections with $\mathbb{R}$-bundles over M?

Danny Calegari: Generalising Ziggurats (Jankins-Neumann-Naimi).

III. CTFs. Known constructions of taut foliations

Only 2 known types of strategies for constructing CTFs on M prime.

1. M arbitrary: branched surfaces.

-Sutured hierarchy (but requires $b_1(M) > 0$) (Gabai),

-Knot exteriors (Roberts et al),

-Foliar orientations on one-vertex triangulations (Dunfield).

2. M Seifert fibered: fiber-transverse foliations.

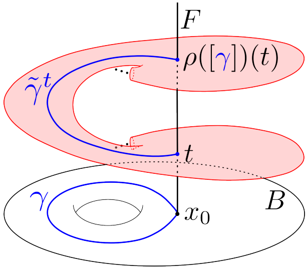
-Fiber-transverse foliation on S^1-fibration over orbifold.

-For appropriate graph manfiolds, such foliations can be glued together.

Fiber-transverse foliation analog for arbitrary M?

Definition (complete transversely foliated bundle).
An F-bundle $\pi : E \to B$ with foliation $\mathcal{F}$
is a *complete transversely foliated bundle* if for each leaf $L \subset E$ of $\mathcal{F}$,
(i) (transversality) L is transverse to each fiber $\pi^{-1}(b) \cong F$ of E, and
(ii) (completeness) π restricts on L to a covering map $\pi|_L : L \to B$.

Definition (holonomy representation).
For a basepoint $x_0 \in B$ and base-fiber embedding $F \xrightarrow{\sim} \pi^{-1}(x_0) \subset E$,
$\mathcal{F}$ has *holonomy representation* $\mathrm{Hol}\,\mathcal{F} = \rho : \pi_1(B, x_0) \to \mathrm{Homeo}_+ F$,
$\rho([\gamma]) : t \mapsto \tilde{\gamma}^t(1)$, $\tilde{\gamma}^t : I \to E$ lifts $\gamma : (I, \partial I) \to (B, x_0)$ with $\tilde{\gamma}^t(0) = t$.



Proposition (classical).
Given an oriented manifold F, a closed oriented based manifold (B, x_0),
and a representation $\rho : \pi_1(B, x_0) \to \mathrm{Homeo}_+ F$,
one can construct
the *complete transversely foliated F-bundle E_ρ*
with *transverse foliation $\mathcal{F}_\rho$* of *holonomy representation ρ*, by setting

$$E_\rho := (\widetilde{B} \times F)/(x, t) \sim (x \cdot g, \rho(g^{-1})(t)), \text{ for all } g \in \pi_1(B, x_0),$$

$$\pi : E_\rho \to B, \quad [(x, t)] \mapsto [x] \text{ for } (x, t) \in \widetilde{B} \times F.$$

$$\mathcal{F}_\rho := \coprod_{t \in F} \widetilde{B} \times \{t\} / \sim,$$

for $\widetilde{B}$ the universal cover of B.

$\sim$ identifies each orbit of the diagonal action of $\pi_1(B)$
by deck transformations on $\widetilde{B}$ and by ρ^{-1} on F.

Theorem (classical).
Complete transversely-foliated F-bundles over (B, x_0) are classified by their holonomy representation, up to isomorphism of foliated based F-bundles,

In other words, there is a bijection,

$$\left\{\begin{array}{c}\text{complete transversely-foliated}\\ F\text{-bundles over } (B, x_0)\end{array}\right\} \Big/ \begin{array}{c}\text{isomorphism of}\\ \text{foliated based bundles}\end{array}$$

$$\updownarrow (\mathcal{F} \mapsto \mathrm{Hol}\,\mathcal{F})$$

$$\left\{\begin{array}{c}\text{representations}\\ \pi_1(B, x_0) \to \mathrm{Homeo}_+ F\end{array}\right\}.$$

(For a Seifert fibered space M, this gives a correspondence between CTFs on M and $\mathbb{R}$-actions of $\pi_1(M)$, up to suitable equivalence.)

Classification of Seifert Fibered Spaces with CTFs:
genus > 0 case: Eisenbud-Hirsch-Neumann.
genus 0 case: Jankins-Neumann, Naimi; Calegari-Walker.

Theorem (J-N & N / C-W)

If $M = M(\frac{\beta_0}{\alpha_0}; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ is Seifert fibered over S^2, then M admits a CTF $\iff$ $\pi_1(M)$ admits an LO $\iff$

$$\min_{k>0} -\frac{1}{k}\left(-1 + \sum \left\lceil \frac{\beta_i}{\alpha_i} k \right\rceil\right) < 0 < \max_{k>0} -\frac{1}{k}\left(1 + \sum \left\lfloor \frac{\beta_i}{\alpha_i} k \right\rfloor\right).$$

Theorem (–R)

An analogous classification result holds for graph manifolds.

IV. Heegaard foliations

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Definition (efficient Heegaard diagram).
A Heegaard diagram $\mathcal{H}$ for M is *efficient* if in its associated presentation for $\pi_1(M)$, no proper nontrivial subword of a relator is trivial in $\pi_1(M)$.

Theorem (—R)

Suppose M is a prime closed oriented 3-manifold with an efficient Heegaard diagram of genus ≤ 2. Then for any left order $>_{\mathrm{L}}$ on $\pi_1(M)$, one can use $\mathcal{H}$ and $>_{\mathrm{L}}$ to build a cooriented taut foliation on M called a Heegaard foliation.

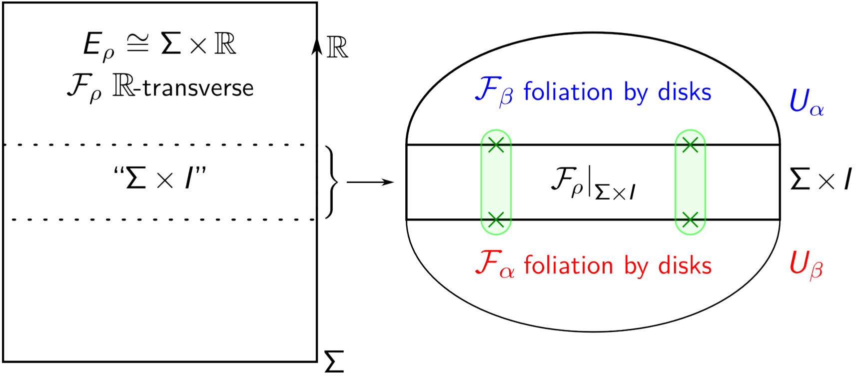
Corollary

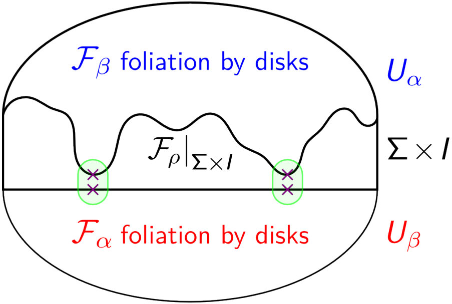
Suppose M is a prime closed oriented 3-manifold with an efficient Heegaard diagram of genus ≤ 2. If $\pi_1(M)$ is left-orderable, then M is not an L-space.

$\rho' : \pi_1(M) \to \mathrm{Homeo}_+ \mathbb{R}, \qquad \rho(g)(0) < \rho(h)(0) \iff g <_{\mathrm{L}} h.$

$\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ efficient Heegaard diagram for M,
$\iota : \Sigma \hookrightarrow M = U_\alpha \cup_\Sigma U_\beta$.

$$\rho := \rho' \circ \iota_* : \pi_1(\Sigma) \to \mathrm{Homeo}_+ \mathbb{R}$$
$$\rightsquigarrow \quad E_\rho \cong \Sigma \times \mathbb{R},\ \mathcal{F}_\rho \text{ with } \mathrm{Hol}\, \mathcal{F}_\rho = \rho.$$

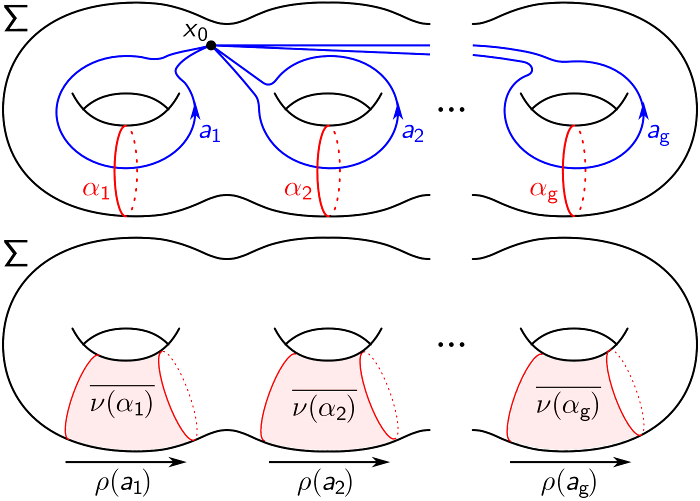




Subtleties:

1. The $\mathbb{R}$-transverse foliation $\mathcal{F}_\rho$ must admit sections $\mathcal{F}_{0,\alpha}$ and $\mathcal{F}_{0,\beta}$ that respectively extend to $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$. $\implies$ *Foliation Templates*.

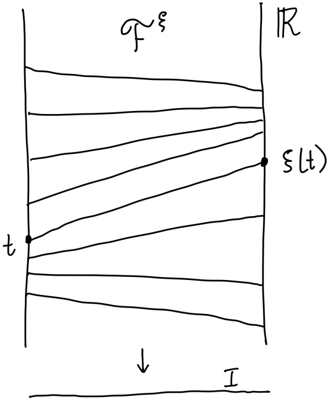
2. Singularities must be contained in special neighborhoods conducive to cancellation. $\implies$ *Extremal regions*.



$\alpha_1, \ldots, \alpha_g$ freely homotopic to

$\hat{\alpha}_1, \ldots, \hat{\alpha}_g \in \ker \rho = \ker [\iota_* : \pi_1(\Sigma) \to \pi_1(M)]$

Definition. To any $\xi \in \mathrm{Homeo}_+ \mathbb{R}$, we associate the codim-1, 2-dim *suspension foliation* $\mathcal{F}^\xi$ on $I \times \mathbb{R}$, rel boundary.
$I \times R$ regarded as mapping cylinder for ξ.
$\{\text{Leaves of } \mathcal{F}^\xi\} = \{\text{orbits of points under } \xi\}$.



$$\left[\mathcal{F}^\xi / (t,0) \sim (t,1) \right]$$
$$=$$
$$\left[\begin{array}{l} \mathbb{R}\text{-transverse foliation on} \\ \mathbb{R}\text{-bundle over } S^1 = I/\{0\}\sim\{1\}, \\ \text{w/ } \left(\mathrm{Hol}\, \mathcal{F}^\xi/\sim\right) : \pi_1(S^1) \to \mathrm{Homeo}_+ \mathbb{R}, \\ \qquad [I \mapsto S^1] \mapsto \xi. \end{array} \right]$$

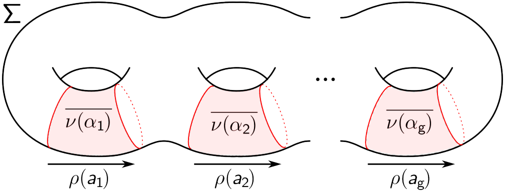
Definition. A *foliation template* $T = (\boldsymbol{\varphi}, \boldsymbol{\xi})$ *of length* n *on* Σ is an ordered pair of ordered n-tuples with respective i^{th} entries

(*i*) *template charts* $\varphi_i : S^1 \times [-\frac{1}{2}, +\frac{1}{2}] \to A_i \subset \Sigma$

determining the i^{th} *template triple* (A_i, μ_i, η_i):

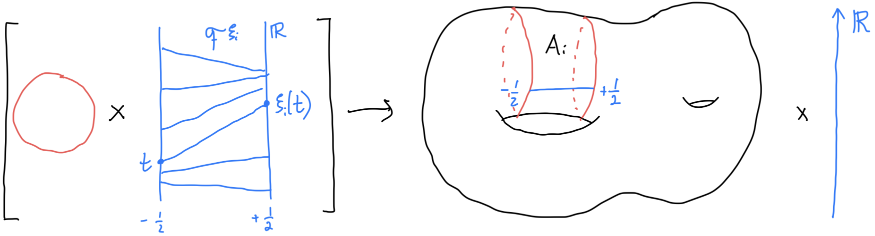
- *template pinched annulus* $A_i \subset \Sigma$ (pairwise disjoint interiors),
- *template curve* $\mu_i = \text{core}(\mathring{A}_i)$,
- *local coorientation* $\eta_i : I \to \Sigma$, coorientation for μ_i;

(*ii*) *local holonomy* $\xi_i \in \mathrm{Homeo}_+ \mathbb{R}$.



$$\text{im}(\varphi_i) = A_i = \overline{\nu(\alpha_i)}, \quad \mu_i = \alpha_i, \quad \eta_i \sim a_i; \qquad \xi_i = \rho(a_i)$$

Definition. Given $T = (\varphi, \xi)$ with triple $(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\eta})$, (recall $A_i = \overline{\nu}(\mu_i)$), define the *ith suspension foliation of* T, $\mathcal{F}^i_T$, on $A_i \times \mathbb{R}$ by associating the foliation $S^1 \times \mathcal{F}^{\xi_i}$ to $A_i \times \mathbb{R}$ via $\varphi_i : S^1 \times [-\frac{1}{2}, +\frac{1}{2}] \to A_i \subset \Sigma$.

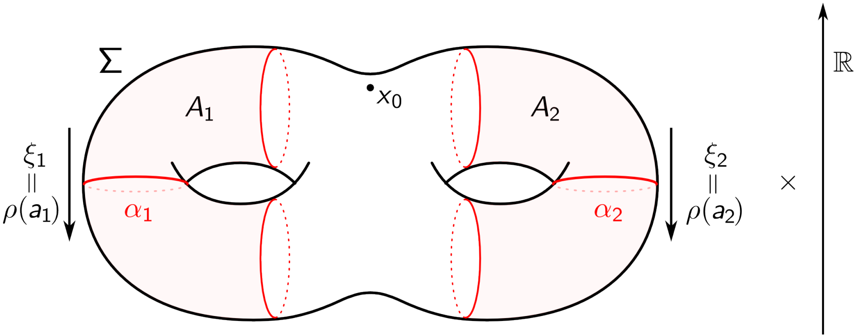


Definition. The global *T-foliation* $\mathcal{F}_T$ is then given by

$$\mathcal{F}_T := \left(\coprod_{i=1}^n \mathcal{F}^i_T\right) \cup \mathcal{F}^{\mathrm{prod}}_{\widehat{\Sigma}\times\mathbb{R}} \quad \text{on} \quad \Sigma \times \mathbb{R},$$

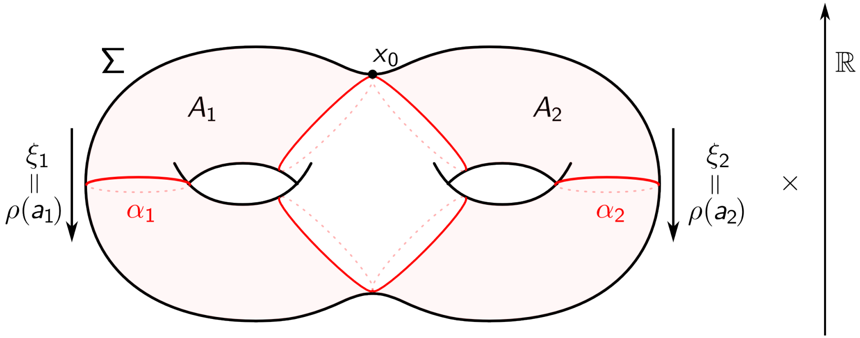
for $\widehat{\Sigma} := \Sigma \setminus \coprod_{i=1}^n \mathring{A}_i$
and $\mathcal{F}^{\mathrm{prod}}_{\widehat{\Sigma}\times\mathbb{R}}$ the product foliation on $\widehat{\Sigma} \times \mathbb{R}$ by $\widehat{\Sigma} \times \{\mathrm{pt}\}$.

$T_\alpha := (\boldsymbol{\varphi}, \boldsymbol{\xi})$ with triple $(\mathbf{A}, \boldsymbol{\alpha}, \boldsymbol{\eta})$. (so $A_i = \overline{\nu}(\alpha_i)$).



$$\langle a_1, a_2\rangle \,/\, \ker\rho = \pi_1(\Sigma)/\ker\rho \implies \mathcal{F}_{T_\alpha} = \mathcal{F}_\rho.$$

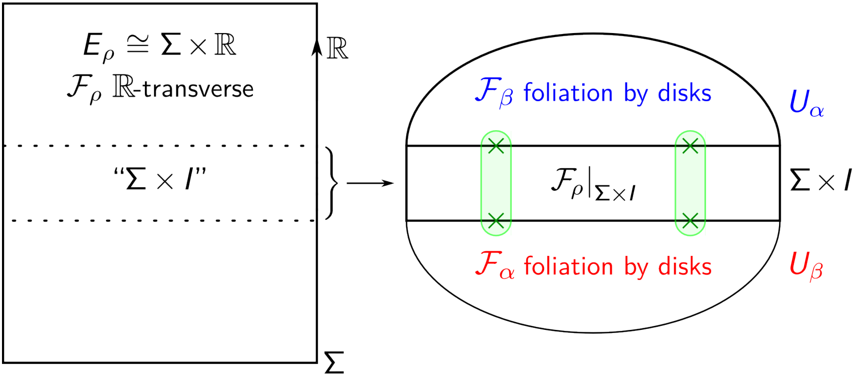
$T_\alpha := (\varphi, \boldsymbol{\xi})$ with triple $(\mathbf{A}, \boldsymbol{\alpha}, \boldsymbol{\eta})$. (so $A_i = \overline{\nu}(\alpha_i)$).



$$\langle a_1, a_2 \rangle / \ker \rho = \pi_1(\Sigma) / \ker \rho \implies \mathcal{F}_{T_\alpha} = \mathcal{F}_\rho.$$

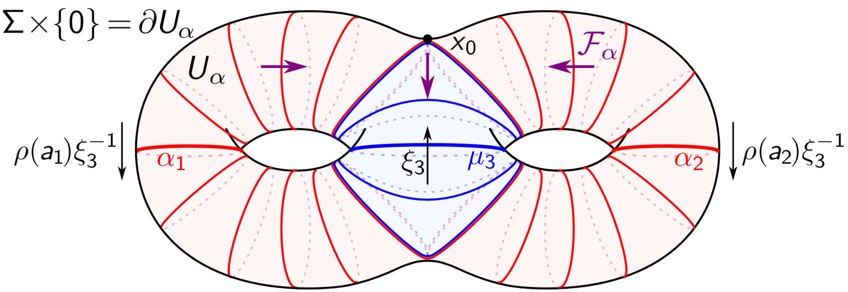
Recall:

$T_\alpha := (\boldsymbol{\varphi}, \boldsymbol{\xi})$ with triple $(\mathbf{A}, \boldsymbol{\alpha}, \boldsymbol{\eta})$. (so $A_i = \overline{\nu}(\alpha_i)$).



1. The $\mathbb{R}$-transverse foliation $\mathcal{F}_\rho$ must admit sections $\mathcal{F}_{0,\alpha}$ and $\mathcal{F}_{0,\beta}$ that respectively extend to $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$. $\Longrightarrow$ *Foliation Templates*.

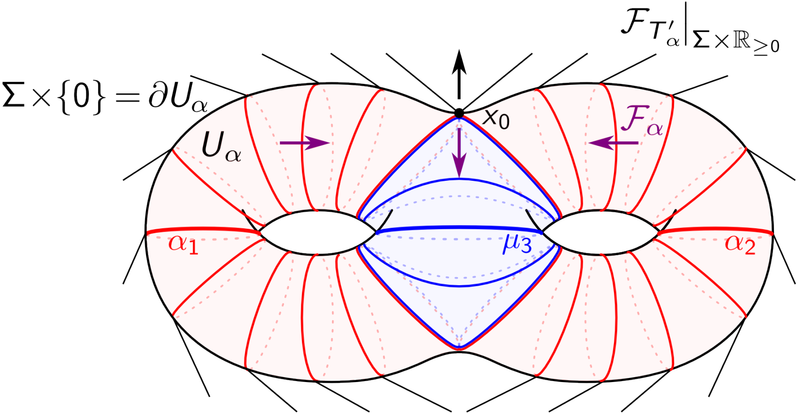
$$\mathcal{F}_\alpha|_{\partial U_\alpha} := \mathcal{F}_{T'_\alpha}|_{\Sigma\times\{0\}} = \text{“}\mathcal{F}_{\alpha,0}\text{”}.$$



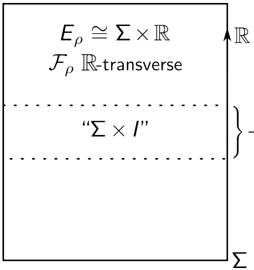
$a_1, a_2 >_{\mathrm{L}} 1 \implies \rho(a_1)(0),\ \rho(a_2)(0) > 0,$

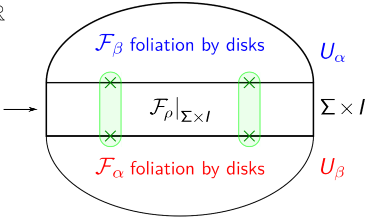
$\xi_3 : t \mapsto t+\varepsilon \implies \xi_3(0) > 0,$

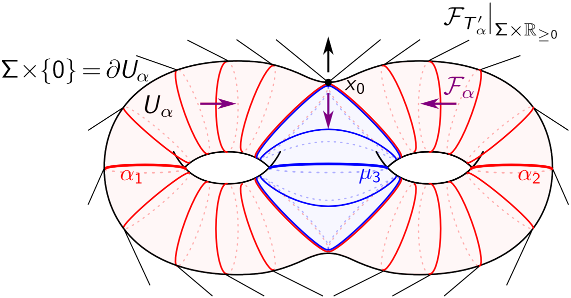
Coorientation of $\mathcal{F}_\alpha = \eta_i^{-1} = -$(Coorientation of μ_i).


$\mathcal{F}_{T'_\alpha}\big|_{\Sigma\times\mathbb{R}_{\geq 0}}$
$\Sigma\times\{0\}=\partial U_\alpha$
U_α
x_0
$\mathcal{F}_\alpha$
α_1
μ_3
α_2

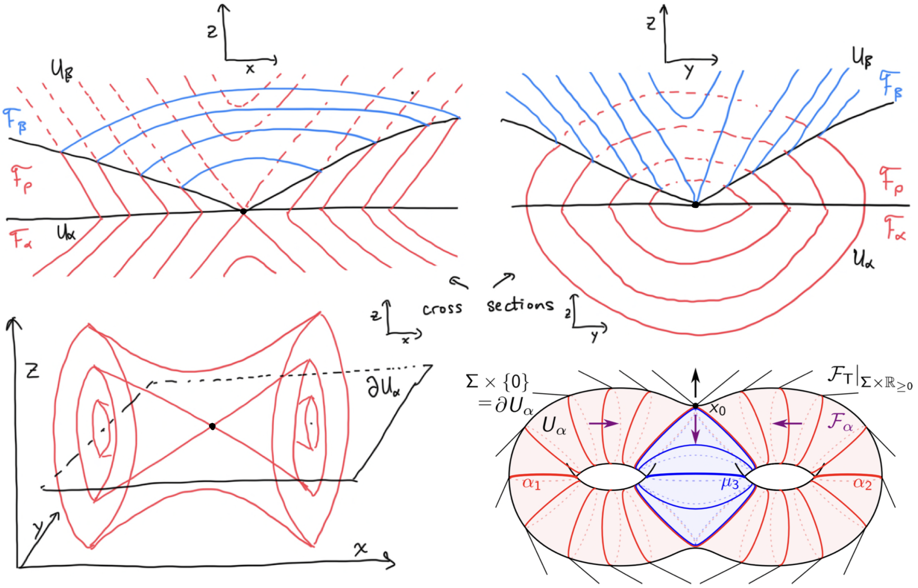
IV. Heegaard foliations. Singularities



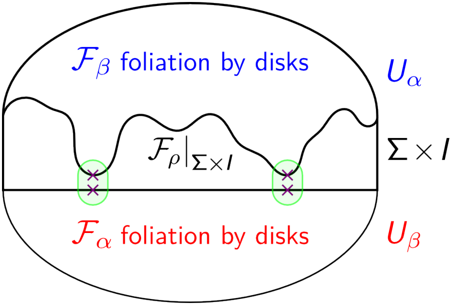




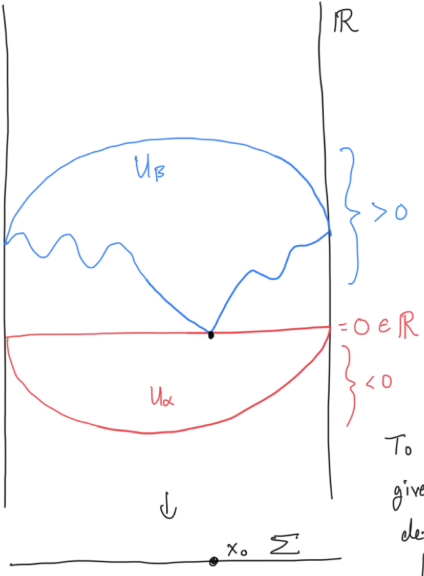
IV. Heegaard foliations. Singularity cancellation



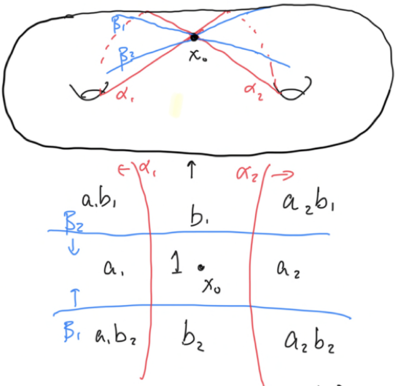
Recall:



2. Singularities must be contained in special neighborhoods conducive to cancellation. $\implies$ *Extremal regions*.



$$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, x_0)$$



To each region r, assign $g(r) \in \pi_1(M)$, given by the homotopy class of the knot determined by x_0 + a pt in r. Then

$$\left[\begin{array}{l}\text{height of center of } r \\ \text{in } \partial U_\beta\end{array}\right] = \rho(g(r)).$$

IV. Heegaard foliations

Definition (Heegaard foliation)

We call the cooriented taut foliation we have just now constructed a *Heegaard foliation*.

Theorem (—R)

Suppose M is a prime closed oriented 3-manifold with an efficient Heegaard diagram of genus ≤ 2. Then for any left order $>_{\mathrm{L}}$ on $\pi_1(M)$, one can use $\mathcal{H}$ and $>_{\mathrm{L}}$ to build a Heegaard foliation on M.

Theorem (—R)

Suppose M is a prime closed oriented 3-manifold with an efficient Heegaard diagram of genus ≤ 2. Then for any left order $>_{\mathrm{L}}$ on $\pi_1(M)$,

Thanks!