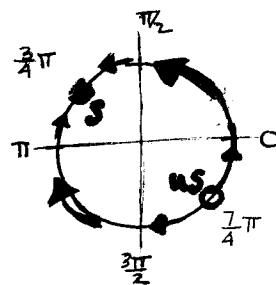
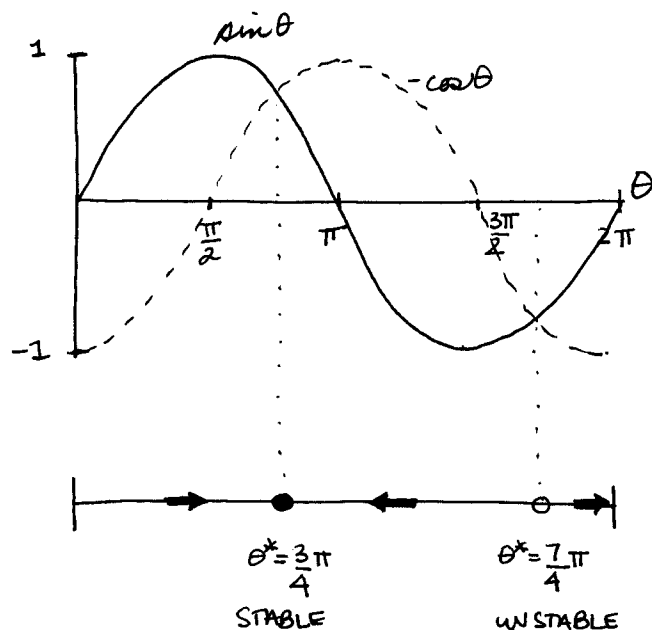


4.1.5 $\dot{\theta} = \sin \theta + \cos \theta = f(\theta)$
 $= \sin \theta - (-\cos \theta)$



F.P. 2

$$f(\theta) = 0 \quad \sin \theta^* = -\cos \theta^*$$

$$\theta^* = \frac{3\pi}{4}, \frac{7\pi}{4}$$

STABILITY

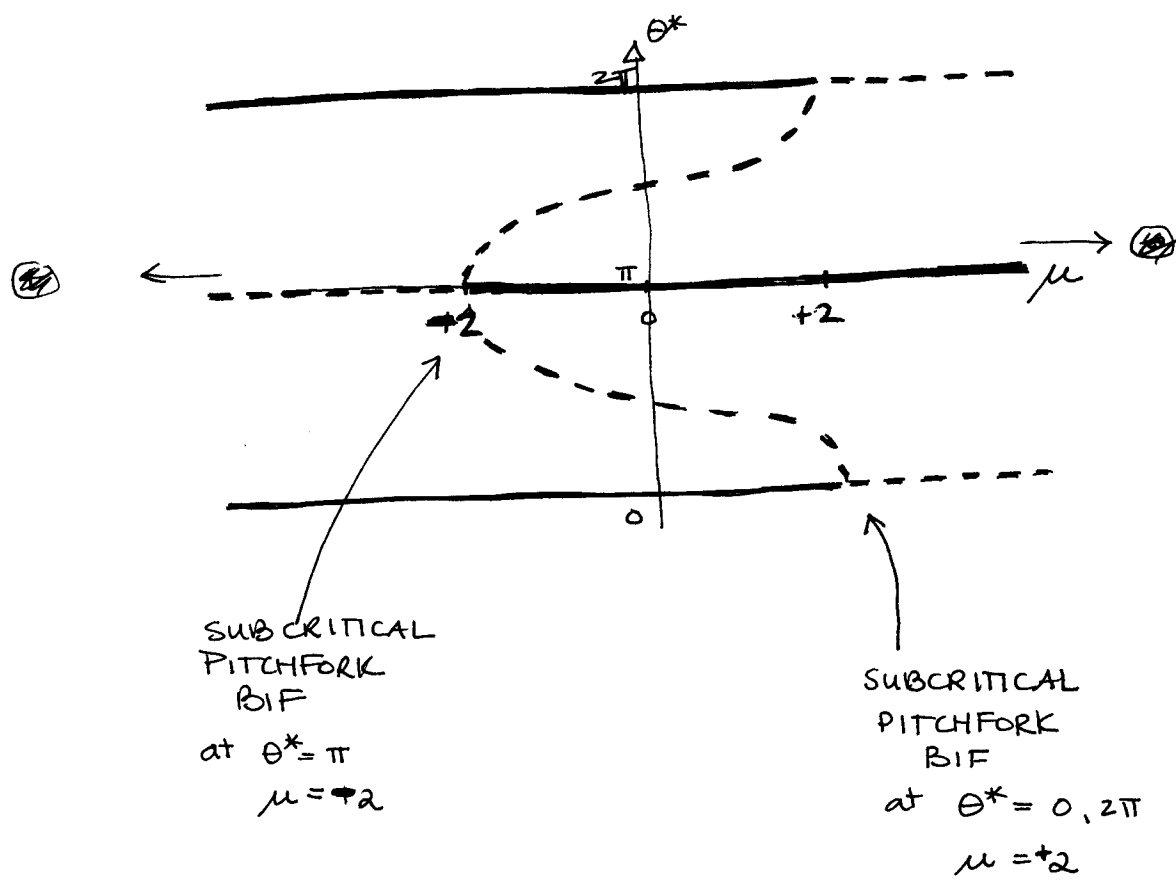
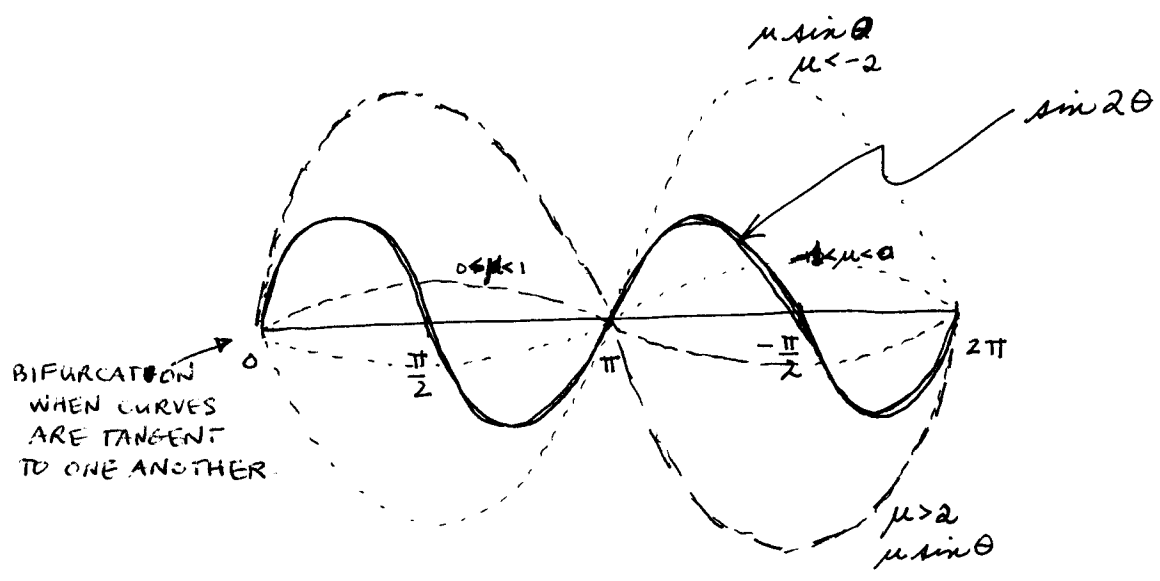
$$f'(\theta) = \cos \theta - \sin \theta$$

$$\theta^* = \frac{3\pi}{4} \quad f'(\frac{3\pi}{4}) = \cos \frac{3\pi}{4} - \sin \frac{3\pi}{4} < 0 \Rightarrow \theta^* = \frac{3\pi}{4} \text{ is STABLE.}$$

$$\theta^* = \frac{7\pi}{4} \quad f'(\frac{7\pi}{4}) = \cos \frac{7\pi}{4} - \sin \frac{7\pi}{4} > 0 \Rightarrow \theta^* = \frac{7\pi}{4} \text{ is UNSTABLE.}$$

4.3.3.

$$\dot{\theta} = \mu \sin \theta - \sin 2\theta = f(\theta; \mu)$$



ie. BIF. PTS WHEN $f'(\theta^*) = 0$

$f(\theta) = 0$ includes $\theta^* = 0, \pi$

$f'(\theta) = \mu \cos \theta - 2 \cos 2\theta = \mu - 2 = 0 \Rightarrow$ BIF OF $\theta^* = 0$ AT $\mu = 2$

$f'(\pi) = \mu \cos \pi - 2 \cos 2\pi = -\mu - 2 = 0 \Rightarrow$ BIF OF $\theta^* = \pi$ AT $\mu = -2$

- at $\mu = 2, \theta^* = 0$
 $= \mu_{\text{CRIT}}$

$$f(\theta^*=0, \mu=2) = 0$$

$$\frac{\partial f}{\partial \theta}(\theta^*=0, \mu=2) = 0$$

$$\frac{1}{2} \frac{\partial^2 f}{\partial \theta^2}(\theta^*=0, \mu=2) = 0$$

$$\frac{1}{6} \frac{\partial^3 f}{\partial \theta^3}(\theta^*=0, \mu=2) = -\frac{\mu+8}{6} \Big|_{\mu=2} = \frac{6}{6} = 1$$

$$\frac{\partial f}{\partial \mu}(\theta^*=0, \mu=2) = 0$$

$$\frac{\partial^2 f}{\partial \mu \partial \theta}(\theta^*=0, \mu=2) = 1$$

$$\Rightarrow \dot{\theta} \simeq (\mu-2)\theta + \theta^3 = r\theta + \theta^3$$

which is the normal form for a SUBCRITICAL
 PITCHFORK BIFURCATION (at $\mu=2, \theta^*=0$).
 $r=0$

- similarly for $\mu = -2, \theta^* = \pi$

$$\Rightarrow \dot{\theta} \simeq -(\mu+2)(\theta - \pi) + (\theta - \pi)^3$$

$$y = \theta - \pi \quad \Rightarrow \quad \dot{y} = r y + y^3$$

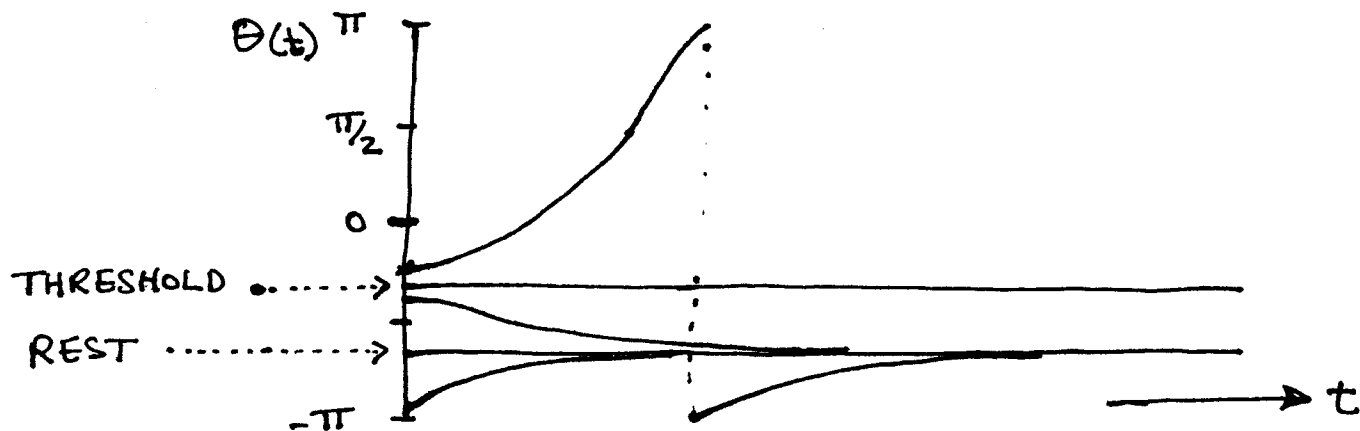
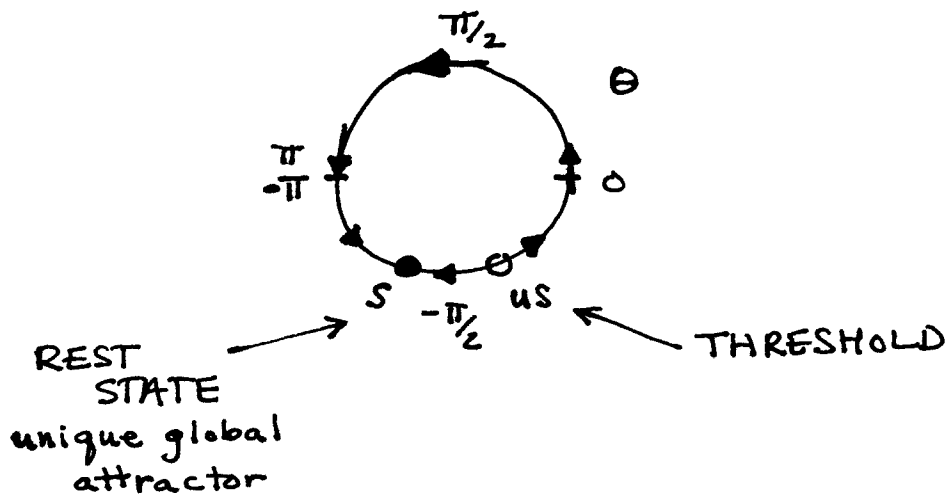
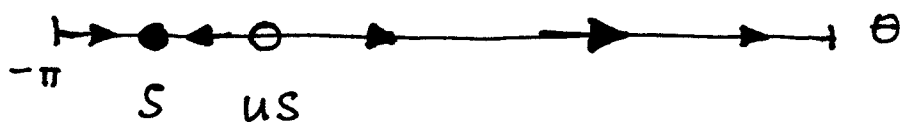
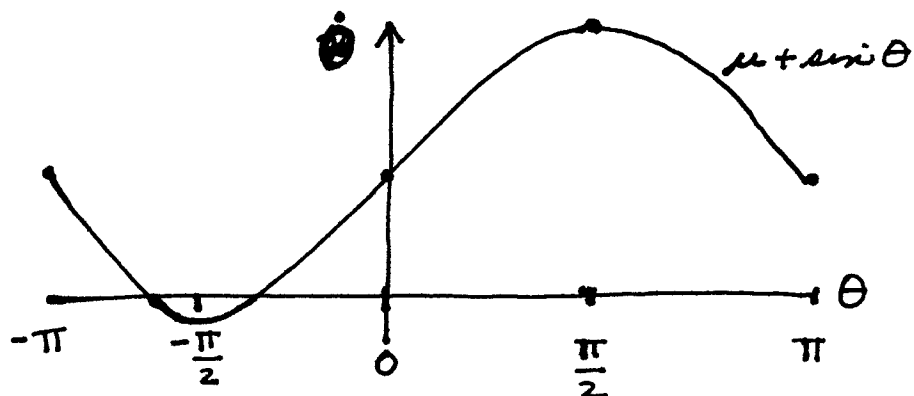
$r = \mu + 2$

which is the normal form for a
 SUBCRITICAL PITCHFORK BIF.

at $y=0, r=0$ i.e. $\mu = -2, \theta^* = \pi$

4.5.3 $\dot{\theta} = \mu + \sin \theta$, $\theta \in (-\pi, \pi]$

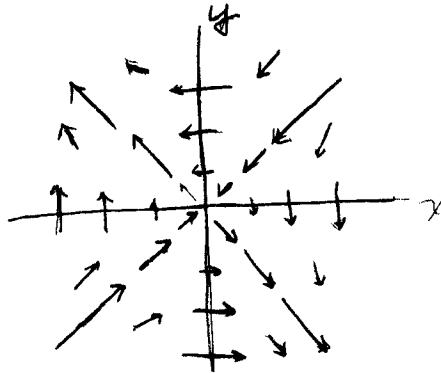
μ slightly less than 1



5.1.9

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

A)



B) note that $x\dot{x} - y\dot{y} = 0$
 i.e. $x(-y) - y(-x) = 0$.

$$x \frac{dx}{dt} = y \frac{dy}{dt}$$

$$\int_0^t \tilde{x} \frac{d\tilde{x}}{dt} dt = \int_0^t \tilde{y} \frac{d\tilde{y}}{dt} dt$$

$$\int_{x(0)}^{x(t)} \tilde{x} d\tilde{x} = \int_{y(0)}^{y(t)} \tilde{y} d\tilde{y}$$

$$\frac{x(t)^2}{2} - \frac{x(0)^2}{2} = \frac{y(t)^2}{2} - \frac{y(0)^2}{2}$$

$$\Rightarrow x(t)^2 - y(t)^2 = C$$

where $C = x(0)^2 - y(0)^2$

(6)

Find eigenvalues

$$\det \begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda_+ = 1, \lambda_- = -1$$

 $\Rightarrow (0,0)$ is A
SADDLE POINT.

$$\lambda_+ = 1 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ v_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{aligned} -1 - v_+ &= 0 \\ v_+ &= -1 \end{aligned}$$

$$\lambda_- = -1 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ v_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{aligned} 1 - v_- &= 0 \\ v_- &= 1 \end{aligned}$$

$$\Rightarrow \text{eigenpairs} \quad \begin{aligned} \lambda_+ &= 1, v_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \lambda_- &= -1, v_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

• UNSTABLE MANIFOLD OF SADDLE PT IS ALONG $\bar{v}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\text{i.e. } y = -x$$

• STABLE MANIFOLD OF SADDLE PT IS ALONG $\bar{v}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\text{i.e. } y = x$$

(7)

$$\begin{aligned} \text{let } u &= x+y \\ v &= x-y \end{aligned} \Rightarrow \begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \end{aligned}$$

$$\frac{du}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -y - x = -u$$

$$\frac{dv}{dt} = \frac{dx}{dt} - \frac{dy}{dt} = -y + x = v$$

$$\Rightarrow \begin{cases} \frac{du}{dt} = -u \\ \frac{dv}{dt} = v \end{cases} \longrightarrow \begin{aligned} u(t) &= u_0 e^{-t} \\ v(t) &= v_0 e^t \end{aligned}$$

where $u(0) = u_0$
 $v(0) = v_0$

E)

UNSTABLE MANIFOLD

$$x=0$$

$$\Rightarrow u=0$$

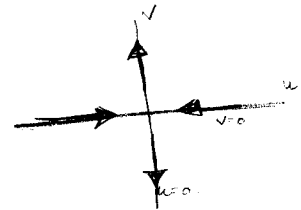
$$(0, y) \rightarrow (0, -y)$$

STABLE MANIFOLD

$$y=0$$

$$\Rightarrow v=0$$

$$(x, 0) \rightarrow (-x, 0)$$



$$\begin{aligned} F) \quad x(t) &= \frac{u(t) + v(t)}{2} = \frac{u_0 e^{-t} + v_0 e^t}{2} \\ &= \frac{x_0 + y_0}{2} e^{-t} + \frac{x_0 - y_0}{2} e^t \\ y(t) &= \frac{u(t) - v(t)}{2} = \frac{x_0 + y_0}{2} e^{-t} - \frac{x_0 - y_0}{2} e^t \end{aligned}$$

5.2.1

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \bar{x}' = A\bar{x}$$

a) eigenvalues and eigenvectors

$$\det \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$$

$$\lambda_+ = 3, \lambda_- = 2$$

$$\lambda_+ = 3 \Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ v_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v_+ = 1$$

$$\lambda_- = 2 \Rightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ v_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v_- = 2$$

$$\Rightarrow \left\{ \lambda_+ = 3, \bar{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \lambda_- = 2, \bar{v}_- = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

b) general sol'n of $\bar{x}' = A\bar{x}$:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c) $\lambda_+ = 3, \lambda_- = 2 \Rightarrow (0,0)$ IS AN UNSTABLE NODE.

$$d) \quad \bar{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$c_2 = 1, \quad c_1 = 2$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

5.2.2 $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$

a) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\bar{x}' = A \bar{x}$$

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 + 1$$

$$= \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

$$\begin{bmatrix} 1 - (1 \pm i) & -1 \\ 1 & 1 - (1 \pm i) \end{bmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = 0$$

$$\begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = 0 \Rightarrow v = \mp i$$

eigenpairs

$$\lambda_1 = 1 + i, \bar{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \leftarrow \text{or } \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 - i, \bar{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \leftarrow \text{or } \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

(11)

$$\bar{x}(t) = c_1 e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(1-i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\text{where } c_1, c_2 \in \mathbb{C}$$

$$\bar{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\Rightarrow c_1 + c_2 = x_0 \Rightarrow \operatorname{Im}(c_1) + \operatorname{Im}(c_2) = 0$$

$$\operatorname{Re}(c_1) + \operatorname{Re}(c_2) = x_0$$

$$-ic_1 + ic_2 = y_0 \Rightarrow -\operatorname{Re}(c_1) + \operatorname{Re}(c_2) = 0$$

$$\operatorname{Im}(c_1) - \operatorname{Im}(c_2) = y_0$$

$$c_1 = \frac{x_0 + i y_0}{2}, \quad c_2 = \frac{x_0 - i y_0}{2}$$

$$\bar{x}(t) = \frac{e^t}{2} (\cos t + i \sin t) \begin{pmatrix} x_0 + i y_0 \\ y_0 - i x_0 \end{pmatrix}$$

$$+ \frac{e^t}{2} (\cos t - i \sin t) \begin{pmatrix} x_0 - i y_0 \\ y_0 + i x_0 \end{pmatrix}$$

$$= \frac{e^t}{2} \begin{pmatrix} x_0 \cos t - y_0 \sin t + i x_0 \sin t + i y_0 \cos t \\ y_0 \cos t + x_0 \sin t - i x_0 \cos t + i y_0 \sin t \end{pmatrix}$$

$$+ \frac{e^t}{2} \begin{pmatrix} x_0 \cos t - y_0 \sin t - i x_0 \sin t - i y_0 \cos t \\ y_0 \cos t + x_0 \sin t + i x_0 \cos t - i y_0 \sin t \end{pmatrix}$$

$$\bar{x}(t) = e^t \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ y_0 \cos t + x_0 \sin t \end{pmatrix}$$

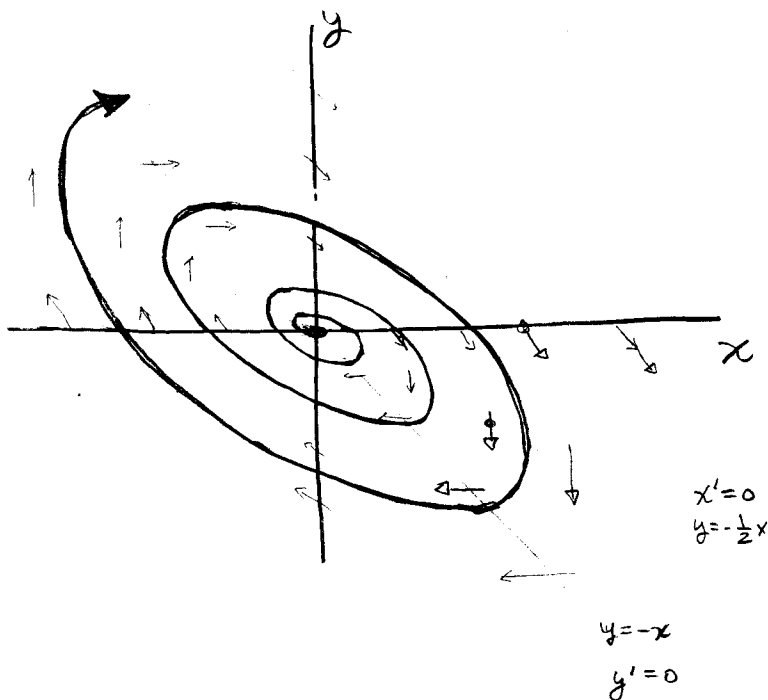
5.2.4 $\begin{cases} x' = 5x + 10y \\ y' = -x - y \end{cases}$

• eigenvalues: $\det \begin{bmatrix} 5-\lambda & 10 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 5 = 0$

$$\lambda = 2 \pm i$$

$\Rightarrow (0,0)$ is an UNSTABLE SPIRAL.

* note: eigenvectors will be complex.



5.2.8
$$\begin{cases} x' = -3x + 4y \\ y' = -2x + 3y \end{cases}$$

eigenvalues:

$$\det \begin{bmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 1 = 0$$

$$\lambda_{1,2} = \pm 1$$

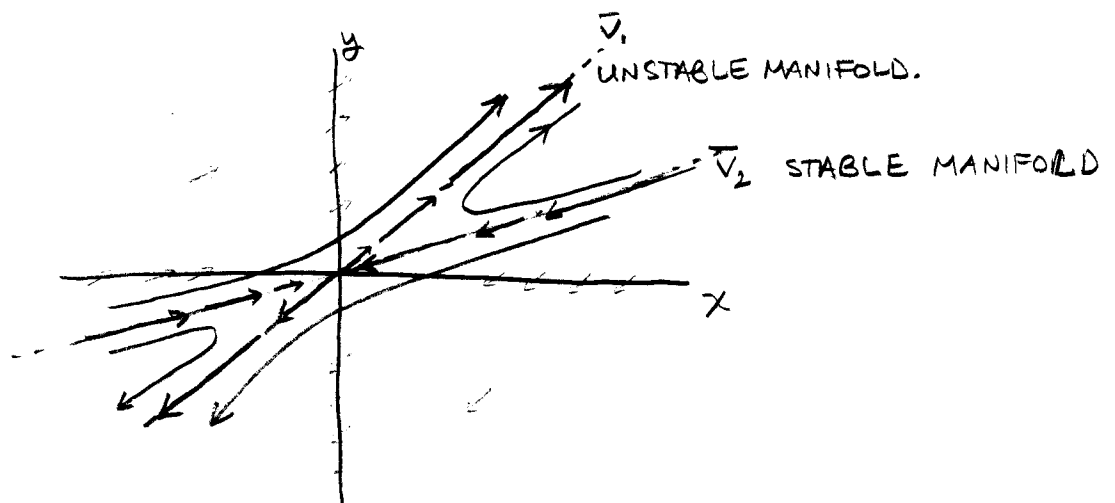
eigenvector:

$$\lambda_1 = 1 \quad \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ v_1 \end{pmatrix} = \vec{0} \Rightarrow v_1 = 1$$

$$\lambda_2 = -1 \quad \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} = \vec{0} \Rightarrow v_2 = \frac{1}{2}$$

$$\Rightarrow \left\{ \lambda_1 = 1, \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\}$$

$(0,0)$ is a SADDLE POINT



5.2.13

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$b > 0, m, k > 0$$

(a) $\dot{x} = v$

$$\dot{v} = -\frac{k}{m}x - \frac{b}{m}v$$

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \Rightarrow \bar{x}' = A\bar{x}$$

(b) calculate eigenvalues of A

$$\det \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{bmatrix} = \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

because $b, k, m > 0$, $\operatorname{Re}(\lambda_{1,2}) < 0$

ie. $(0,0)$ is always STABLE

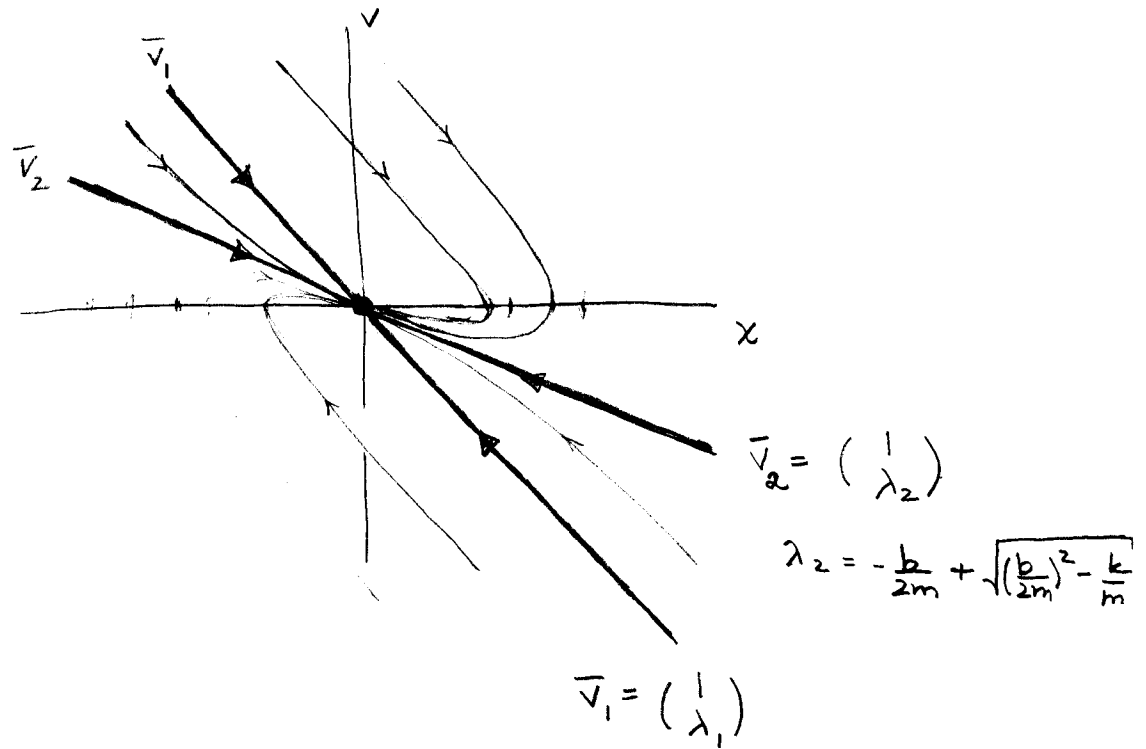
(i) if $\left(\frac{b}{2m}\right)^2 > \frac{k}{m}$, then

$$\lambda_1 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} < \lambda_2 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} < 0$$

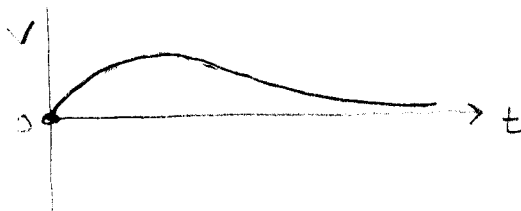
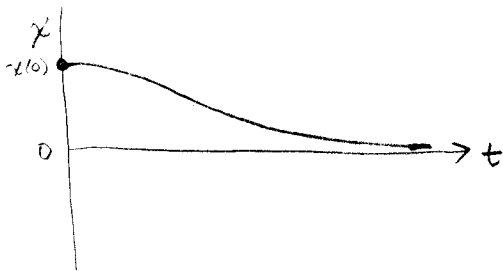
$\Rightarrow (0,0)$ is a STABLE NODE.

note that the eigenvectors of A are

$$\bar{V}_{1,2} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \end{pmatrix}$$



assuming $v(0) = 0$
 $x(0) > 0$



$$\lambda_1 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

eigenvalue with
 larger magnitude
 i.e. faster decay.

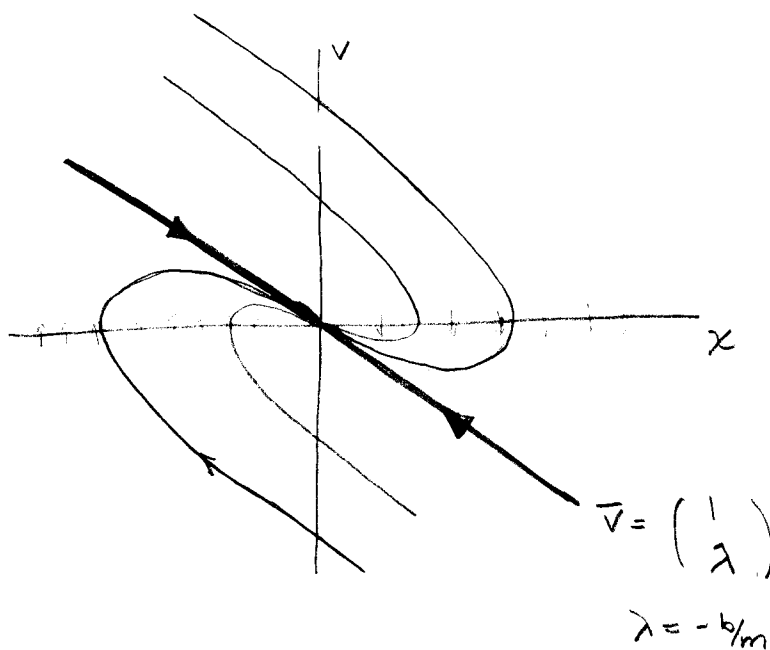
"OVERDAMPED"

iii. if $\left(\frac{b}{2m}\right)^2 = \frac{k}{m}$, then $\lambda_1 = \lambda_2 = \lambda = -\frac{b}{m} < 0$

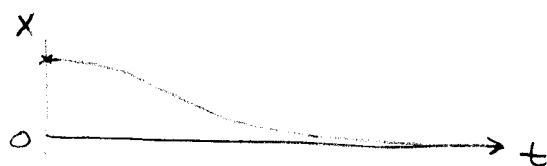
with only one eigenvector $\bar{v} = \begin{pmatrix} 1 \\ -\frac{b}{m} \end{pmatrix}$

$\Rightarrow (0,0)$ is a STABLE DEGENERATE NODE.

aside { note that as $\left(\frac{b}{2m}\right)^2 - \frac{k}{m}$ approaches 0 from above (going from case (i) to case (iii)), the eigenvectors $\bar{v}_1$ and $\bar{v}_2$ are squeezed together until they coalesce when $\left(\frac{b}{2m}\right)^2 - \frac{k}{m} = 0$.



$x(0) > 0$
 $v(0) = 0$



"CRITICALLY DAMPED"

(iii) if $\left(\frac{b}{2m}\right)^2 < \frac{k}{m}$, then $\lambda_{1,2} \in \mathbb{C}$

$$\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

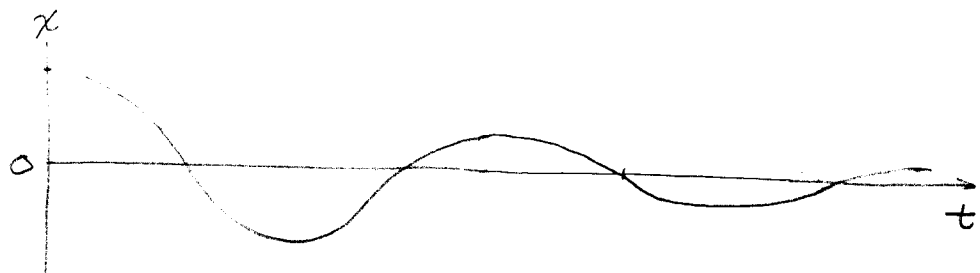
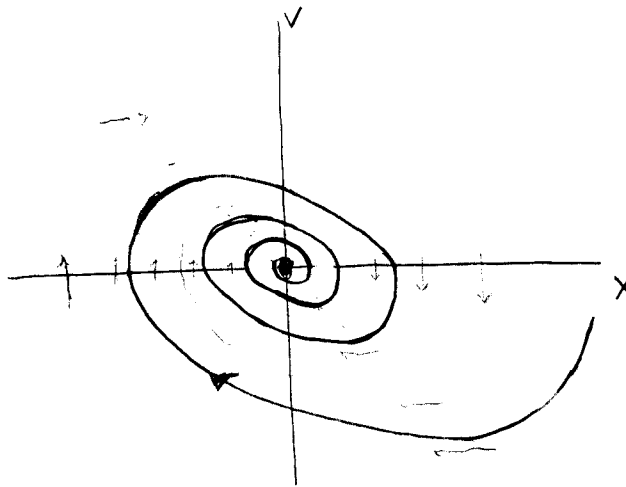
$$= \alpha \pm i\omega$$

$(0,0)$ is a

STABLE FOCUS.

$$\alpha = \operatorname{Re}(\lambda_{1,2}) = -\frac{b}{2m}$$

$$\omega = \operatorname{Im}(\lambda_{1,2}) = \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$



damped oscillations

"UNDER DAMPED"