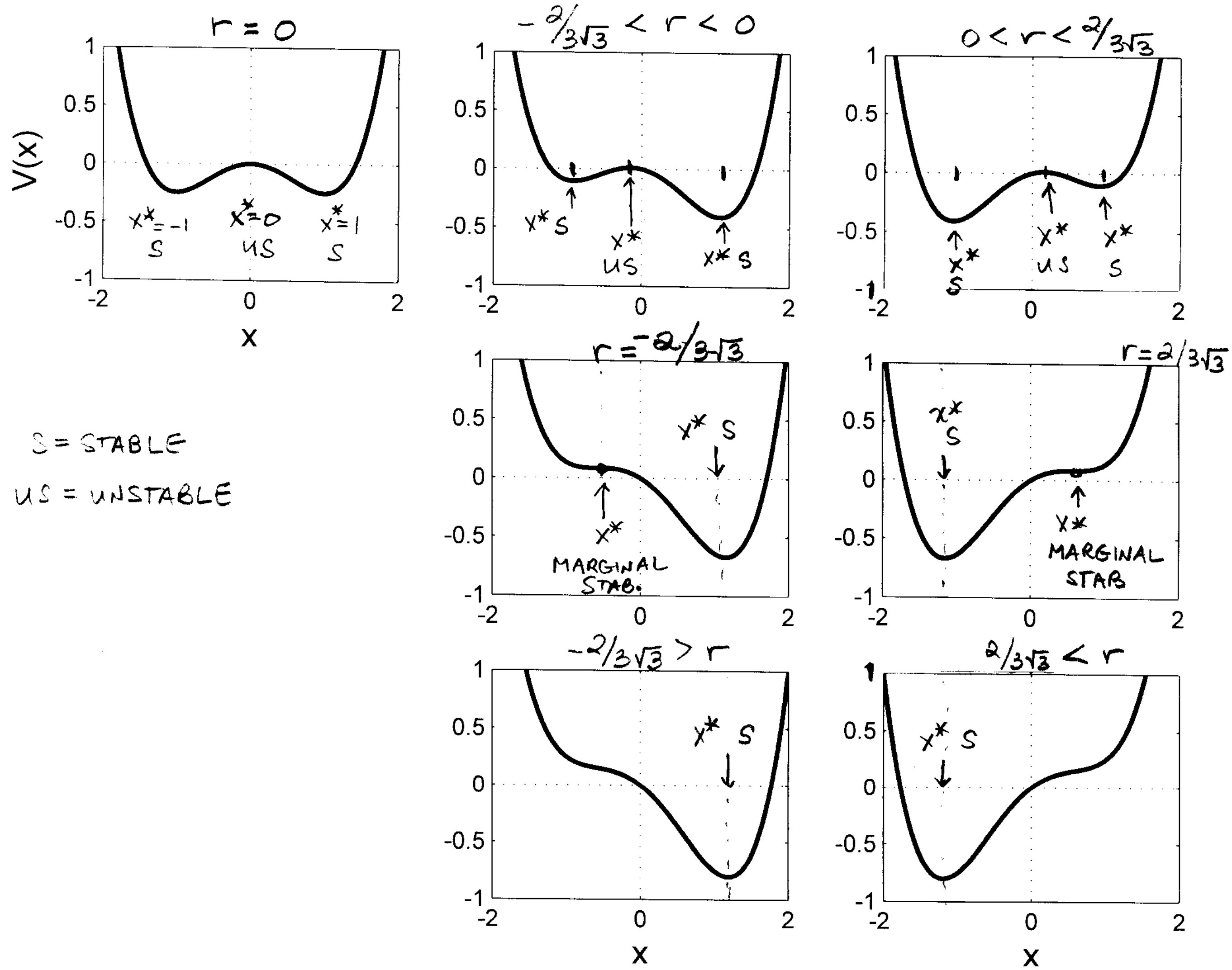
2.7.6

$$\dot{x} = r + x - x^3 = -\frac{dV}{dx} = f(x)$$

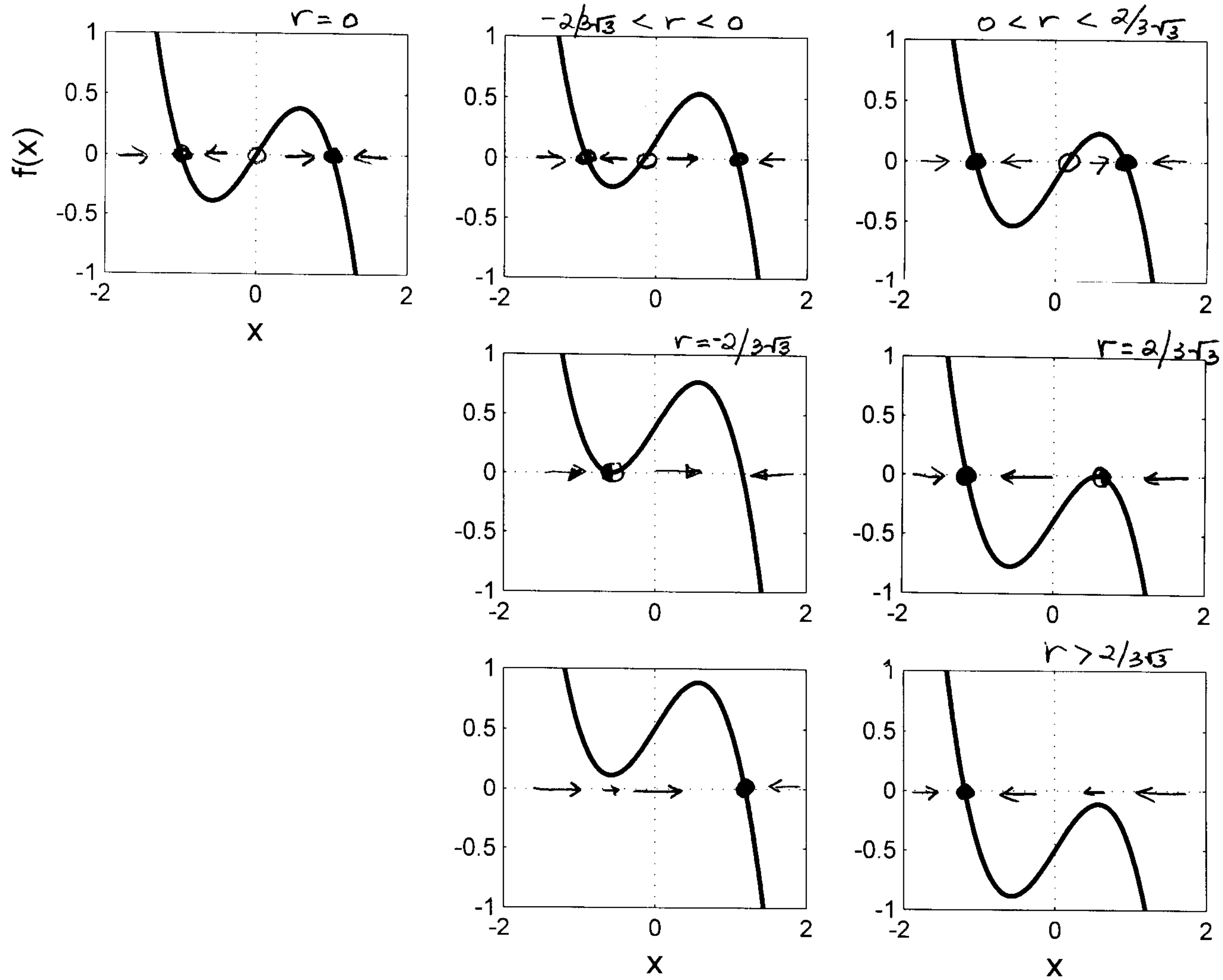
$$V(x) = -\int (r + x - x^3)\,dx = -\left(rx + \frac{x^2}{2} - \frac{x^4}{4}\right)$$

$r < 0$ $\qquad$ $r > 0$



- LOCAL MINIMA OF $V(x)$ CORRESPOND TO STABLE STEADY STATES
- LOCAL MAXIMA OF $V(x)$ CORRESPOND TO UNSTABLE STEADY STATES
- INFLECTION POINTS OF $V(x)$ CORRESPOND TO STEADY STATES WITH MARGINAL STABILITY.

FIXED POINTS AND THEIR STABILITY CAN ALSO BE EASILY DETERMINED BY DIRECTLY EXAMINING $f(x)$.

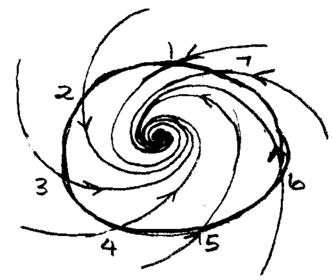


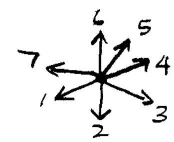
COMPARE TO PREVIOUS FIGURE (i.e., $V(x)$)

6.8.1

(a) • STABLE SPIRAL

PHASE - PORTRAIT

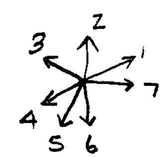




$I_c = 1$

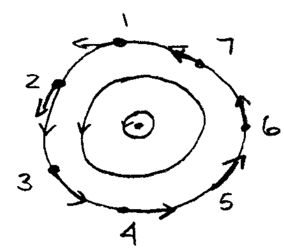
(b) • UNSTABLE SPIRAL

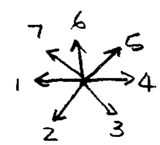
same picture as above but reverse direction of flow



$I_c = 1$

(c) • CENTER

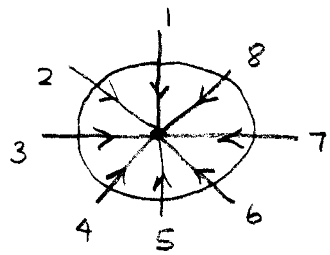


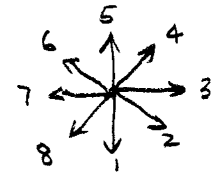


$I_c = 1$

IF FLOW IS CIRCULATING IN THE OPPOSITE DIRECTION, ARROW ARE FLIPPED BUT STILL $I_c = 1$

(d) (STABLE) STAR NODE

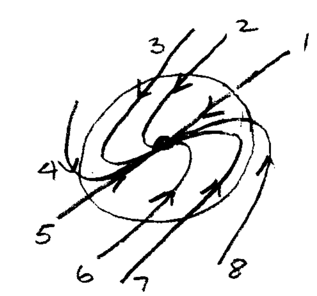


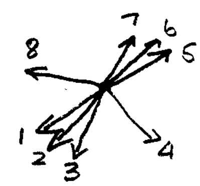


$I_c = 1$

FOR UNSTABLE STAR NODE, FLOW IS IN OPPOSITE DIRECTION AND THEREFORE ARROWS ARE REVERSED, BUT STILL $I_c = 1$

(e) STABLE DEGENERATE NODE





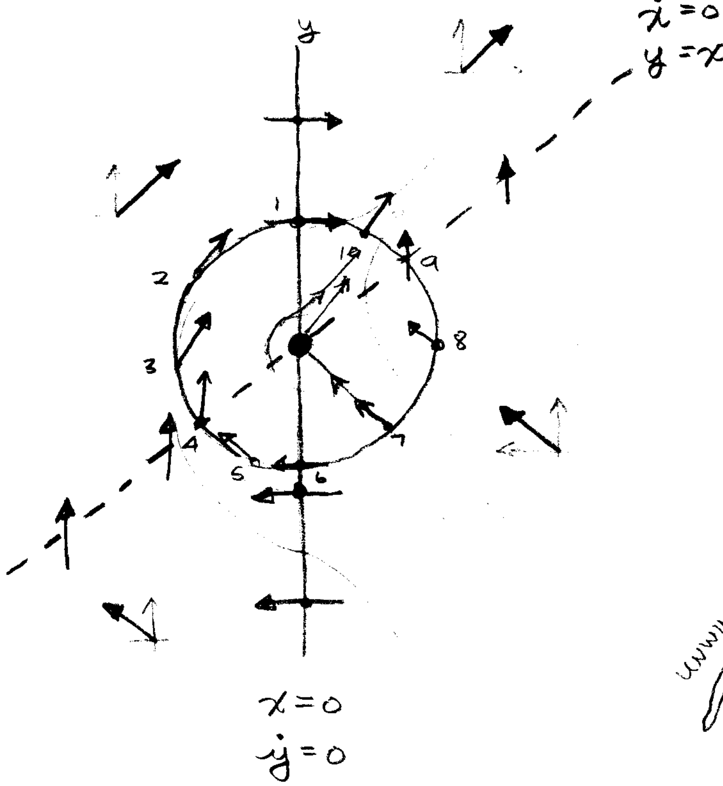
$I_c = 1$

UNSTABLE DEGENERATE NODE $I_c = 1$

(SAME ARGUMENT AS ABOVE).

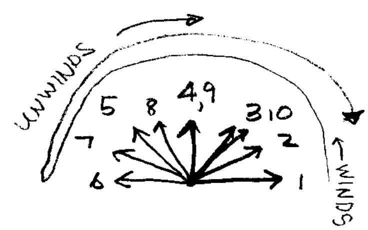
6.8.3

$$\begin{cases} \dot{x} = y - x \\ \dot{y} = x^2 \end{cases}$$



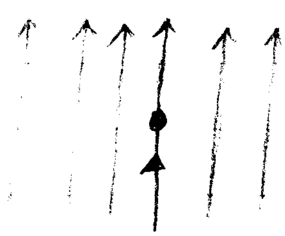
FIXED PT
(0, 0)

INDEX CALC.



$I_c = 0$

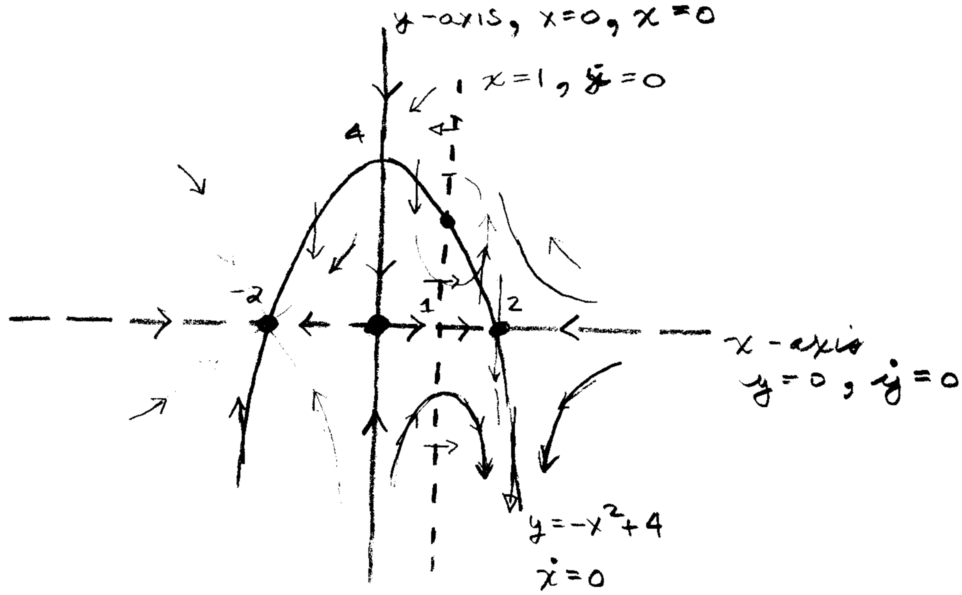
TOPOLOGY OF
FLOW



* USUAL FIXED PT!

6.8.7

$$\begin{cases} \dot{x} = x(4 - y - x^2) \\ \dot{y} = y(x-1) \end{cases}$$



$\dot{x} = 0 \Rightarrow x = 0, \; y = -x^2 + 4$

$\dot{y} = 0 \Rightarrow y = 0, \; x = 1$

FIXED PTS AT $(0,0)$, $(-2,0)$, $(2,0)$, $(1,3)$

- $(0,0)$: SADDLE PT, $I_c = -1$
- $(-2,0)$: STABLE NODE, $I_c = +1$
- $(2,0)$: SADDLE PT, $I_c = -1$
- $(1,3)$: STABLE SPIRAL (BY LINEAR STABILITY ANALYSIS), $I_c = 1$

- A CLOSED ORBIT MUST ENCIRCLE AT LEAST ONE FIXED POINT WITH INDEX 1.
- A CLOSED ORBIT CANNOT CROSS THE x-AXIS OR THE y-AXIS, BECAUSE STRAIGHT LINE TRAJECTORIES THAT FLOW INTO FIXED POINTS LIE ON THESE AXES.
- THEREFORE, THE ONLY POSSIBILITY LEFT IS THAT THE CLOSED ORBIT LIES IN $x > 0$, $y > 0$ AND CIRCUMNAVIGATES THE STABLE SPIRAL AT $(1,3)$.

IN THE "ANSWERS TO SELECTED EXERCISES" ON p. 459, STROGATZ SAYS THAT THE SPIRAL IS JOINED TO THE SADDLE POINT AT $(2,0)$ BY A BRANCH OF THE UNSTABLE MANIFOLD, THUS A CLOSED ORBIT CANNOT SURROUND $(1,3)$ BECAUSE IT CANNOT CROSS THIS TRAJECTORY...

HOWEVER THIS ARGUMENT PRESUPPOSES THAT YOU KNOW THAT THE UNSTABLE MANIFOLD OF $(2,0)$ [FOR $y>0$] WINDS INTO $(1,3)$. WHY COULDN'T THIS PORTION OF THE UNSTABLE MANIFOLD OF $(2,0)$ NOT WIND INTO A CLOSED ORBIT INSTEAD ?!

6.8.9

A SMOOTH VECTOR FIELD ON THE PHASE PLANE IS KNOWN TO HAVE EXACTLY TWO CLOSED TRAJECTORIES, ONE OF WHICH LIES INSIDE THE OTHER. THE INNER CIRCLE RUNS CLOCKWISE AND THE OUTER CIRCLE RUNS COUNTER-CLOCKWISE.

CLAIM: THERE MUST BE AT LEAST ONE FIXED POINT IN THE REGION BETWEEN THE CYCLE.

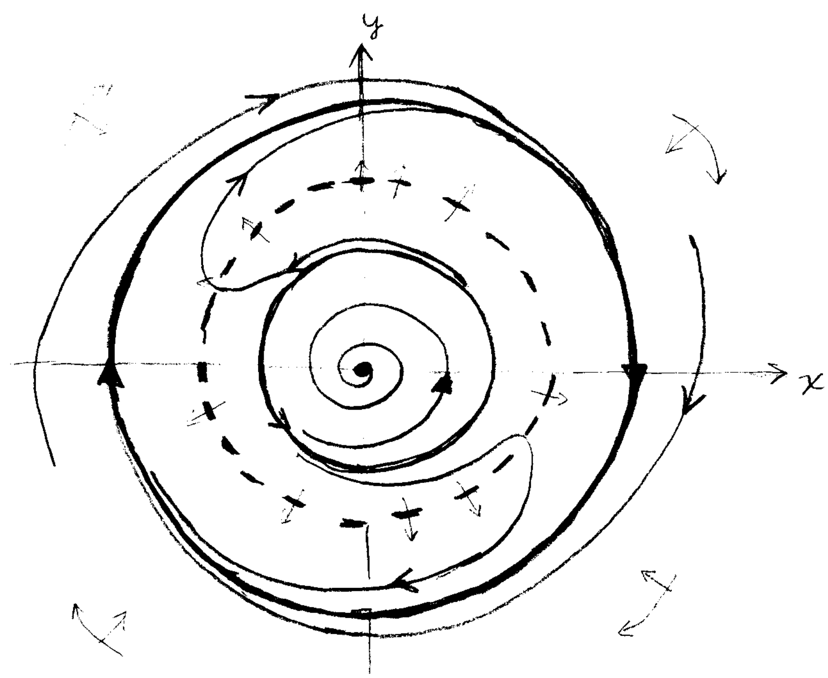
... TRUE OR FALSE?

FALSE, SEE PROBLEM 7.1.3 FOR A COUNTER EXAMPLE (JUST REVERSE TIME).

7.1.3 $\begin{cases} \dot{r} = r(1-r^2)(4-r^2) \\ \dot{\theta} = 2 - r^2 \end{cases}$

nullclines for r : $\dot{r} = 0$, $r = 0$, $r = 1$, $r = 2$

nullclines for θ : $\dot{\theta} = 0$, $r = \sqrt{2}$



FIXED PT AT $r = 0$, STABLE SPIRAL.

$\begin{cases} \dot{\theta} > 0 & \text{for } r < \sqrt{2} \\ \dot{\theta} < 0 & \text{for } r > \sqrt{2} \\ \dot{\theta} = 0 & \text{for } r = \sqrt{2} \end{cases}$

i.e. FLOW REVERSE WINDING DIRECTION AT $r = \sqrt{2}$

STABLE LIMIT CYCLE AT $r = 2$.

UNSTABLE LIMIT CYCLE AT $r = 1$.

7.1.8 $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, $a > 0$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ay(x^2+y^2-1) - x \end{cases}$$

a) FIXED PT $y^* = 0$, $x^* = 0$ i.e. $(0,0)$

STABILITY

$$\det \begin{bmatrix} 0 - \lambda & 1 \\ -ay^* 2x^* - 1 & -ax^{*2} - a3y^{*2} + a - \lambda \end{bmatrix}_{(0,0)}$$

$$= \det \begin{bmatrix} -\lambda & 1 \\ -1 & a - \lambda \end{bmatrix}$$

$$= \lambda^2 - a\lambda + 1 = 0$$

$$\lambda = +\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - 1}$$

eigenpairs $\lambda_1 = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - 1}$, $\bar{V}_1 = \begin{pmatrix} 1 \\ \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - 1} \end{pmatrix}$

$\lambda_2 = \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - 1}$, $\bar{V}_2 = \begin{pmatrix} 1 \\ \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - 1} \end{pmatrix}$

$\Rightarrow$ (0,0) IS AN UNSTABLE NODE IF $a > 2$.

(0,0) IS AN UNSTABLE SPIRAL IF $2 > a > 0$.

(0,0) IS AN UNSTABLE DEGENERATE NODE IF $a = 2$.

b) change to polar coordinates

$$x = r\cos\theta \ , \quad y = r\sin\theta$$

note that we can do this as done in class or by using the identities

$$r\dot{r} = x\dot{x} + y\dot{y} \ , \quad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

(see problem 7.3.1 for example)

$$\Rightarrow \begin{cases} \dot{r} = -a\sin^2\theta \, r(r^2-1) \\ \dot{\theta} = -a\cos\theta\sin\theta\,(r^2-1) - 1 \end{cases}$$

NOTE THAT $r = 1 \Rightarrow \dot{r} = 0$,

AND $\dot{\theta} = -1$ when $r = 1$.

THIS IMPLIES THAT THERE IS A CLOSED ORBIT (PERIODIC ORBIT) AT $r = 1$. FURTHERMORE, BECAUSE $\dot{r} \neq 0$ IN A NEIGHBORHOOD AROUND $r = 1$, THE ORBIT IS A LIMIT CYCLE, I.E. AN ISOLATED PERIODIC ORBIT.

THE PERIOD OF THE LIMIT CYCLE IS 2π.

$$\left| \int_0^{2\pi} \frac{d\theta}{dt}\, dt \right| = \left| \int_0^{2\pi} (-1)\, dt \right| = 2\pi$$

THE AMPLITUDE OF THE LIMIT CYCLE IS 1.
I.E. $r = 1$.

NOTE THAT $r = 1$, $\dot{\theta} = -1$

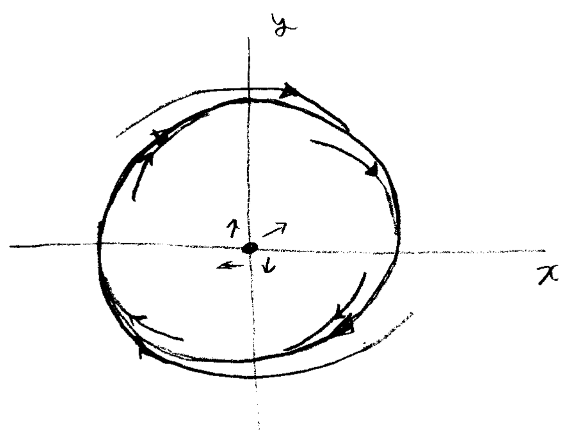
$\theta(t) = -t + t_0$

$$\Rightarrow \begin{cases} x(t) = \cos(-t+t_0) = \cos(t-t_0) \\ y(t) = \sin(-t+t_0) = -\sin(t-t_0) \end{cases}$$

ON LIMIT CYCLE.

c) STABILITY OF THE LIMIT CYCLE CAN BE DETERMINED BY THE VECTOR FLD:

$\dot{\theta} \simeq -1$
FOR $r \simeq 1$



$\dot{r} > 0$ FOR $0 < r < 1$
$\dot{r} < 0$ FOR $r > 1$

$\Rightarrow$ LIMIT CYCLE $r=1$ IS STABLE.

NOTE: $r=0$ CORRESPONDS TO THE UNSTABLE NODE OR SPIRAL AT $(0,0)$.

ALTERNATE STABILITY ANALYSIS

let $r = 1 + \underbrace{\varepsilon R}$ where $0 < \varepsilon \ll 1$

perturbation to LIMIT CYCLE.

$$\varepsilon \dot{R} = -a \sin^2\theta \, (1 + \varepsilon R)\left((1+\varepsilon R)^2 - 1\right)$$
$$= -a \sin^2\theta \, (1 + \varepsilon R)\left(1 + 2\varepsilon R + \varepsilon^2 R^2 - 1\right)$$
$$\dot{R} = -2a \sin^2\theta \, R + O(\varepsilon)$$

$$\dot{\theta} = -a \sin\theta \cos\theta \left((1+\varepsilon R)^2 - 1\right) - 1$$
$$= -1 + O(\varepsilon)$$

$$\Rightarrow \begin{cases} \dot{R} \simeq (-2a \sin^2\theta)\, R \\ \dot{\theta} \simeq 1 \end{cases}$$

perturbations to limit cycle die out. ie. $R \to 0$.

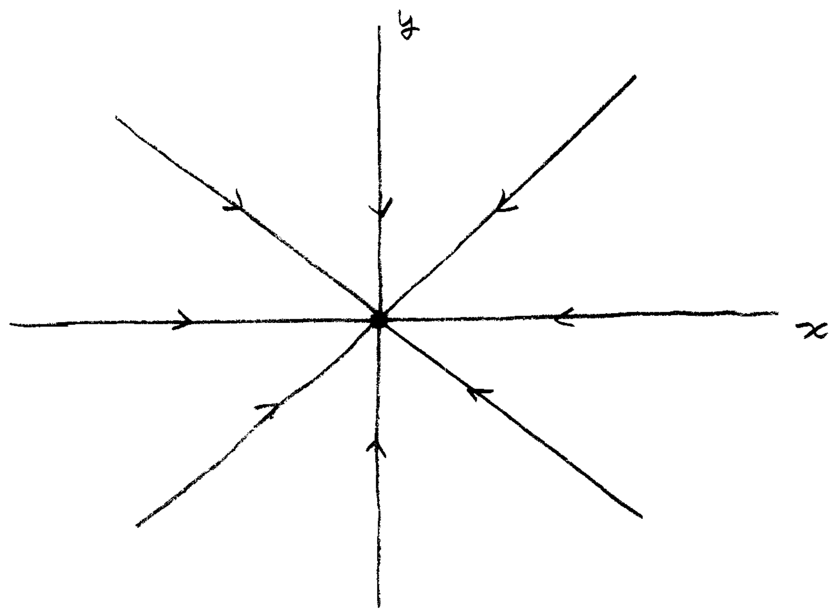
$\therefore$ $r = 1$ is a STABLE LIMIT CYCLE.

7.2.1

$$V(x,y) = x^2 + y^2$$

$$\dot{\bar{x}} = -\nabla V(\bar{x})$$

$$\begin{cases} \dfrac{dx}{dt} = -2x = -\dfrac{\partial V}{\partial x} \\[2ex] \dfrac{dy}{dt} = -2y = -\dfrac{\partial V}{\partial y} \end{cases}$$



LINEAR (SEPERATED!) SYSTEM
(0, 0) IS A STABLE STAR NODE
WITH EIGENVALUE $\lambda = -2$.

7.2.9

a) $\dot{x} = y + x^2 y = f(x,y)$

$\dot{y} = -x + 2xy = g(x,y)$

$$\frac{\partial f}{\partial y} = 1 + x^2 \neq \frac{\partial g}{\partial x} = -1 + 2y$$

SYSTEM IS NOT A GRADIENT SYSTEM.
(BY THM DISCUSSED IN CLASS & IN EXER. 7.2.5).

b) $\dot{x} = 2x = f(x,y) = -\frac{\partial V}{\partial x}$

$\dot{y} = 8y = g(x,y) = -\frac{\partial V}{\partial y}$

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial x} = 0 = -\frac{\partial^2 V}{\partial x \partial y}$$

SYSTEM IS A GRADIENT SYSTEM

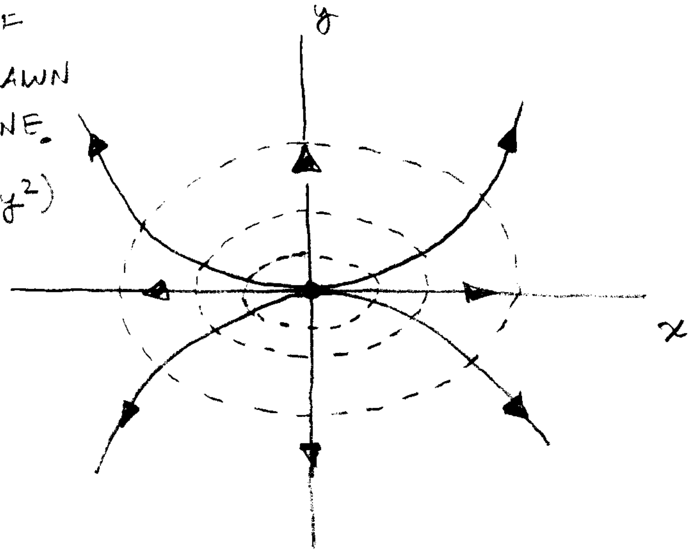
$$-\frac{\partial V}{\partial x} = 2x \Rightarrow V(x,y) = -x^2 + C_1(y)$$

$$-\frac{\partial V}{\partial y} = 8y \Rightarrow V(x,y) = -4y^2 + C_1(x)$$

$$\Rightarrow V(x,y) = -x^2 - 4y^2$$

LEVEL SETS OF $V(x,y)$ ARE DRAWN AS DASHED LINE.

$V(x,y) = -(x^2+4y^2)$



(SEPARATED) LINEAR SYSTEM WITH (0,0) UNSTABLE NODE $\lambda_1 = 2,\ \lambda_2 = 8$.

c) $$\dot{x} = -2x e^{x^2+y^2} = f(x,y) = -\frac{\partial V}{\partial x}$$
$$\dot{y} = -2y e^{x^2+y^2} = g(x,y) = -\frac{\partial V}{\partial y}$$

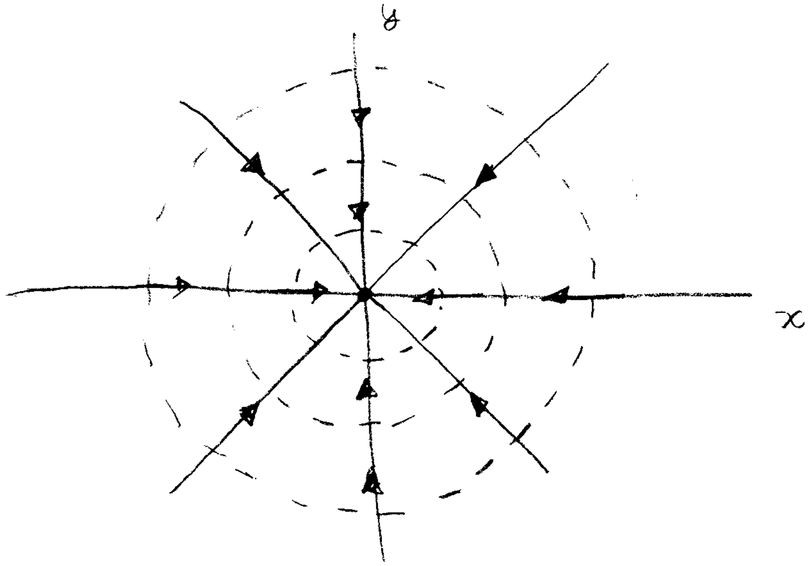
$$\frac{\partial f}{\partial y} = -4xy\, e^{x^2+y^2} = \frac{\partial g}{\partial x} = -4xy\, e^{x^2+y^2} = \frac{-\partial^2 V}{\partial x \partial y}$$

SYSTEM IS A GRADIENT SYSTEM

$$\frac{\partial V}{\partial x} = -2x\, e^{x^2+y^2} \Rightarrow V(x,y) = \int e^{x^2+y^2} 2x\, dx + C_1(y)$$
$$= e^{x^2+y^2} + C_1(y)$$

$$-\frac{\partial V}{\partial y} = -2y\, e^{x^2+y^2} \Rightarrow V(x,y) = \int e^{x^2+y^2} 2y\, dy + C_2(x)$$
$$= e^{x^2+y^2} + C_2(x)$$

$$\Rightarrow \quad V(x,y) = e^{x^2+y^2}$$



- NOTE FLOW VECTORS ARE $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \underbrace{-2e^{x^2+y^2}}_{\text{negative.}}(x, y)$

- LEVEL SET OF $V(x,y) = e^{x^2+y^2}$ ARE

$V(x,y) = e^{x^2+y^2} = c$ OR $x^2+y^2 = \ln c$, $c \geq 1$

i.e. CIRCLES.

DRAWN ABOVE AS DASHED LINES.

IN BOTH (b) AND (c) ...

** NOTE THAT, AS FOR ALL GRADIENT SYSTEMS, TRAJECTORIES ARE ORTHOGONAL TO THE LEVEL SETS OF $V(x,y)$.

$\underline{7.2.12}$

$$\begin{cases} \dot{x} = -x + 2y^3 - 2y^4 \\ \dot{y} = -x - y + xy \end{cases}$$

LYAPUNOV FUNCTION $\quad V(x,y) = x^m + a y^n$

$*$ need $a > 0$, m, n even.

$$\frac{dV}{dt} = \nabla V \cdot (\dot{x}, \dot{y})$$

$$= (m x^{m-1}, a n y^{n-1}) \cdot (-x + 2y^3 - 2y^4, -x - y + xy)$$

TRY TO ELIMINATE NON-NEGATIVE TERMS.

$$= -m x^m + 2y^3 m x^{m-1} - 2y^4 m x^{m-1}$$
$$- a n y^{n-1} x - a n y^n + a n y^n x$$

CHOOSE $m = 2$

$$= -2x^2 + 4y^3 x - 4y^4 x$$
$$- a n y^{n-1} x - a n y^n + a n y^n x$$

CHOOSE $n = 4$

$$= -2x^2 + 4y^3 x - 4y^4 x$$
$$- a 4 y^3 x - a 4 y^4 + a 4 y^4 x$$

CHOOSE $a = 1$

$$= -2x^2 + \cancel{4y^3 x} - \cancel{4y^4 x}$$
$$- 4y^3 x - 4y^4 + \cancel{4y^4 x}$$

$$= -(x^2 + 2y^2) < 0$$ EXCEPT AT $(0,0)$ WHERE $\frac{dV}{dt} = 0$.

$\Rightarrow$ THE LYAPUNOV FUNCTION

$$V(x,y) = x^2 + y^4$$

SHOWS THAT $(0,0)$ IS ASYMPTOTICALLY STABLE.

$b_1, b_2, r_1, r_2, K_1, K_2 > 0$

<u>7.2.13</u>

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$

let $\rho(N_1, N_2) = \dfrac{1}{N_1 N_2}$.

then $\nabla \cdot (\rho (\dot{N}_1, \dot{N}_2))$

$$= \nabla \cdot \left(\frac{r_1 (1 - N_1/K_1)}{N_2} - b_1 , \frac{r_2 (1 - N_2/K_2)}{N_1} - b_2 \right)$$

$$= -\frac{r_1}{N_2} \frac{1}{K_1} - \frac{r_2}{N_1} \frac{1}{K_2}$$

$\Rightarrow$ IN THE FIRST QUADRANT . $N_1 > 0$, $N_2 > 0$,

$$\nabla \cdot (\rho (\dot{N}_1, \dot{N}_2)) < 0 .$$

THEREFORE BY DULAC'S CRITERION, THE SYSTEM HAS NO PERIODIC ORBITS IN THE FIRST QUADRANT.