7.3.1

$$\dot{x} = x - y - x(x^2+5y^2)$$
$$\dot{y} = x + y - y(x^2+y^2)$$

a) FIXED POINTS $\quad (x^*, y^*) = (0,0)$

STABILITY

$$\det \begin{bmatrix} 1-3x^2-5y^2-\lambda & -1-x\,10y \\ 1-2xy & 1-x^2-3y^2-\lambda \end{bmatrix}_{\substack{x=0\\y=0}} = 0$$

$$(1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0$$

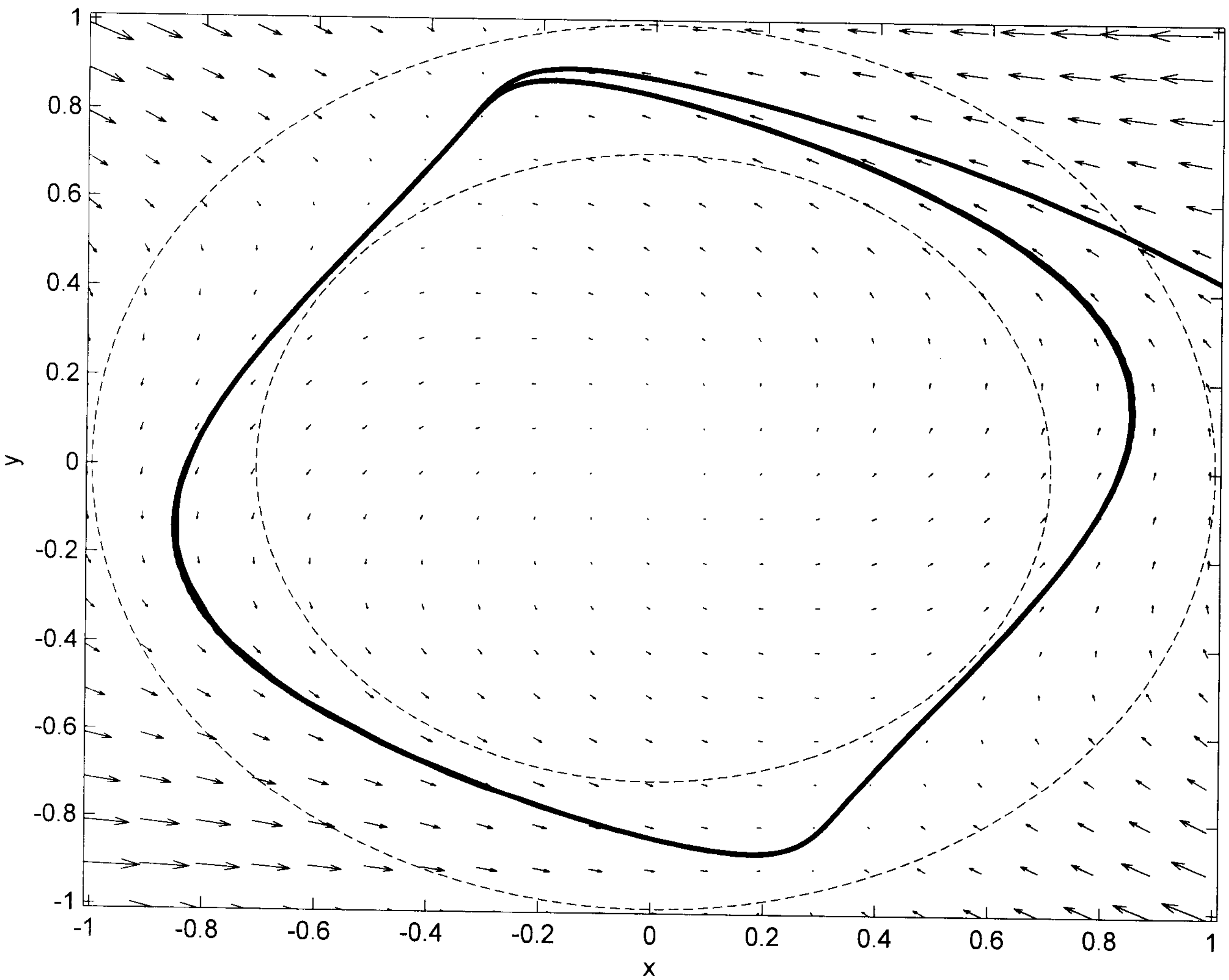
$$\lambda = 1 \pm i$$

$(0,0)$ IS AN UNSTABLE SPIRAL.

b) POLAR CO-ORDS

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$= x^2 - xy - x^2(x^2+5y^2)$$
$$+ xy + y^2 - y^2(x^2+y^2)$$
$$= r^2 - r^4 - 4r^4\sin^2\theta\cos^2\theta \qquad \left(\frac{\sin 2\theta}{2}\right)^2$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x}$$
$$= x^2 + \cancel{xy} - xy(x^2+y^2)$$
$$- \cancel{xy} + y^2 + xy(x^2+5y^2)$$
$$= r^2 + 4r^4\cos\theta\sin^3\theta$$



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x

$$\begin{cases} \dot{r} = r - r^3 - r^3 \sin^2 2\theta \\ \dot{\theta} = 1 + 2r^2 \sin 2\theta \sin^2 \theta \end{cases}$$

* THERE'S SEVERAL WAYS TO EXPRESS THE TRIG FNS.

c) $\dot{r} = 0 \Rightarrow r = 0, \quad r = \dfrac{1}{\sqrt{1 + \sin^2 2\theta}}$

$\dot{r} > 0$ WHEN $0 < r < \dfrac{1}{\sqrt{2}} \leqslant \dfrac{1}{\sqrt{1+\sin^2 2\theta}}$

THEREFORE THE CIRCLE WITH MAXIMUM RADIUS CENTERED AT THE ORIGIN SUCH THAT ALL TRAJECTORIES HAVE A RADIALLY OUTWARD COMPONENT IS $r = r_1 = \frac{1}{\sqrt{2}} - \varepsilon$, WHERE $\varepsilon > 0$ IS ARBITRARILY SMALL.

d) $\dot{r} < 0$ WHEN $r > 1 \geqslant \dfrac{1}{\sqrt{1+\sin^2 2\theta}}$

THEREFORE THE CIRCLE WITH MINIMUM RADIUS CENTERED AT THE ORIGIN SUCH THAT ALL TRAJECTORIES HAVE A RADIALLY INWARD COMPONENT IS $r = r_2 = 1 + \varepsilon$, WHERE $\varepsilon > 0$ IS ARBITRARILY SMALL.

e) THE REGION D BETWEEN THE CIRCLES $r = r_1 = 1/\sqrt{2}$ AND $r = r_2 = 1$ IS A TRAPPING REGION WITH NO FIXED POINTS* IN IT. THEREFORE, BY THE POINCARÉ-BENDIXSON THM, ALL TRAJECTORIES IN D MUST APPROACH CLOSED PERIODIC ORBIT(S) AS $t \to \infty$.

* NO INTERSECTIONS OF $\dot{r} = 0$ AND $\dot{\theta} = 0$ IN D!

7.3.4 $\dot{x} = x(1-4x^2-y^2) - \frac{1}{2}(1+x)y$

$\dot{y} = y(1-4x^2-y^2) + 2x(1+x)$

a) FIXED POINT @ (0,0)

STABILITY

$$\det \begin{bmatrix} 1-12x^2-y^2-\frac{1}{2}y-\lambda & -2xy-\frac{1}{2}(x+1) \\ -8xy+2+4x & 1-4x^2-3y^2-\lambda \end{bmatrix}_{\substack{x=0\\y=0}} = 0$$

$$\det \begin{bmatrix} 1-\lambda & -\frac{1}{2} \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2 = 0$$

$\Rightarrow \lambda = 1 \pm i$

(0,0) IS AN UNSTABLE SPIRAL.

b) $V(x,y) = (1-4x^2-y^2)^2$

USE V AS A LYAPUNOV-LIKE FUNCTION

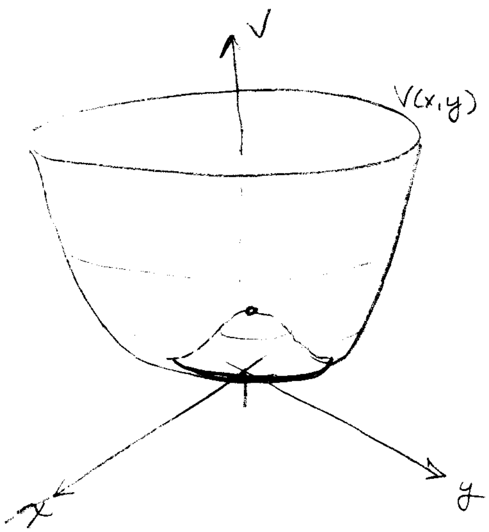
$$\frac{dV}{dt}(x(t), y(t)) = \nabla V \cdot (\dot{x}, \dot{y})$$
$$= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$\frac{dV}{dt} = -4(1-4x^2-y^2)\left[(4x,y)\cdot(\dot{x},\dot{y})\right]$$
$$= -4(1-4x^2-y^2)\left[4x^2(1-4x^2-y^2) - \cancel{2xy(1+x)} + y^2(1-4x^2-y^2) + \cancel{2xy(1+x)}\right]$$

$$\frac{dV}{dt}(x(t), y(t)) = -4\,(1-4x^2-y^2)^2\,(4x^2+y^2)$$

$$< 0 \quad \text{unless} \quad (x,y) = (0,0)$$
$$\text{or} \quad 4x^2+y^2 = 1.$$

- $(0,0)$ IS A LOCAL MAX OF $V(x,y)$
- $V(x,y)$ IS MINIMUM ON $4x^2+y^2=1$ WHERE $V(x,y) = 0$.



THEREFORE ALL TRAJECTORIES* MOVE DOWNHILL WITH RESPECT TO $V(x,y)$ AND APPROACH THE ELLIPSE $4x^2+y^2=1$, WHICH MINIMIZES $V(x,y)$ AND WHERE $\frac{dV}{dt}=0$.

* EXCEPT THE POINT $(0,0)$.

7.4.2

$$\ddot{x} + \underbrace{\mu(x^2-1)}_{f(x)}\dot{x} + \underbrace{\tanh x}_{g(x)} = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) - f(x)y \\ \quad = -\tanh x - \mu(x^2-1)y \end{cases}$$

a) $\underline{\mu > 0}$

(i) $f(x)$ and $g(x)$ are continuously differentiable

(ii) $g(x) = \tanh(x)$ is ODD

i.e. $g(-x) = -g(x)$ for all x

(iii) $\tanh(x) = g(x) > 0$ for all $x > 0$

(iv) $f(x) = \mu(x^2-1)$ is EVEN

i.e. $f(-x) = +f(x)$ for all x

(v) $F(x) = \int_0^x f(\tilde{x})d\tilde{x} = \int_0^x \mu(\tilde{x}^2-1)d\tilde{x} = \mu\left(\frac{x^3}{3} - x\right)$

NOTE: $F(x)$ IS ODD.

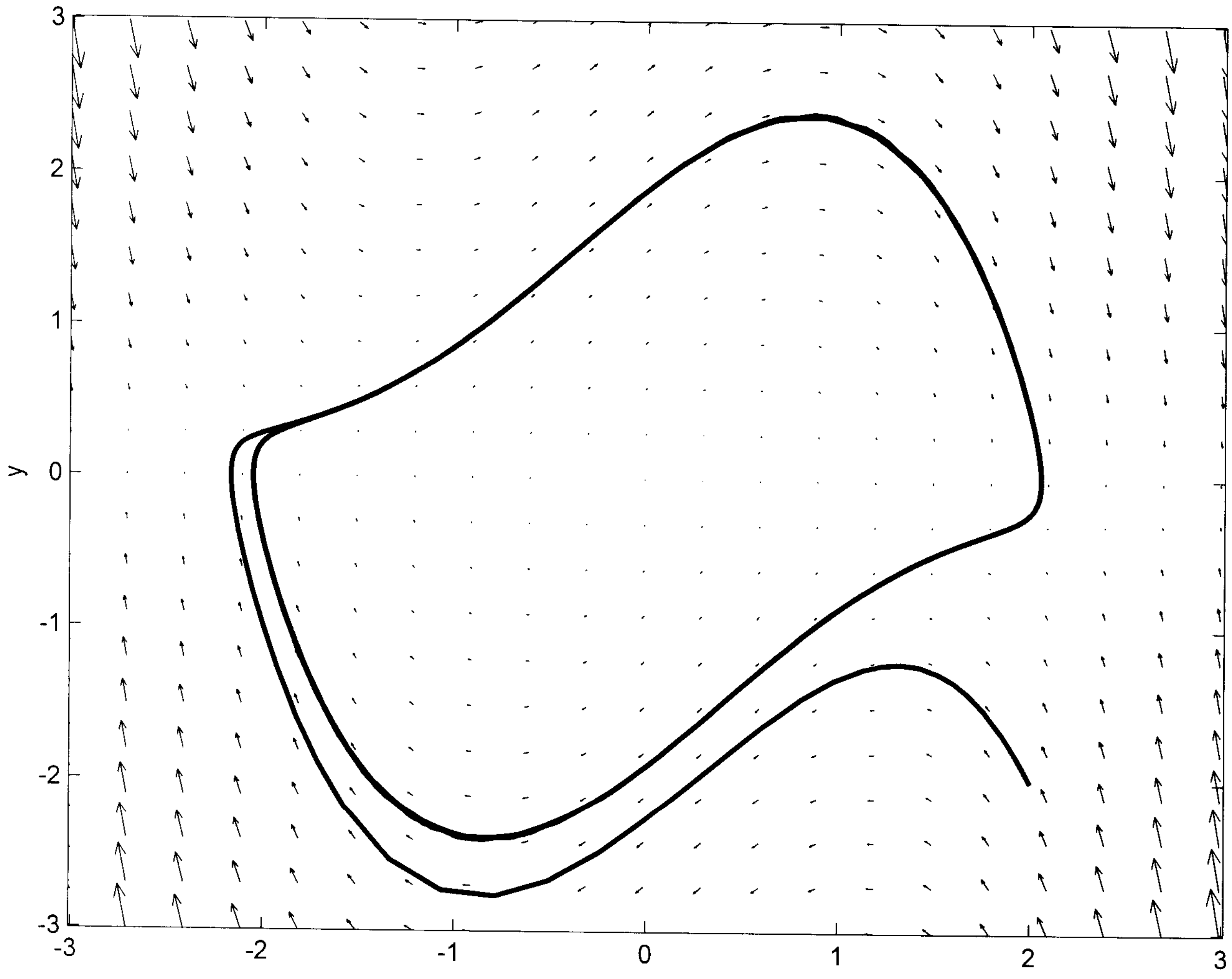
$F(x) = 0$ at $x = 0$ and $x = \pm\sqrt{3}$

$F(x) < 0$ for $0 < x < \sqrt{3}$

$F(x) > 0$ and $F'(x) \geq 0$ for $x > \sqrt{3}$

and $F(x) \to \infty$ as $x \to \infty$

i.e. ALL CONDITIONS FOR LIÉNARD'S THM HOLD, AND THEREFORE THE SYSTEM HAS AN UNIQUE <u>STABLE</u> LIMIT CYCLE IN THE PHASE PLANE FOR $\mu > 0$.



3
2
1
0
-1
-2
-3
y
-3
-2
-1
0
1
2
3
x

c) $\underline{\mu < 0}$

LET $t = -\tau$ AND $\gamma = -\mu > 0$

$$\Rightarrow \quad \frac{d^2x}{d\tau^2} + \underbrace{\gamma(x^2-1)}_{f(x)} \frac{dx}{d\tau} + \underbrace{\tanh x}_{g(x).} = 0 \quad \ldots(*)$$

AS BEFORE, ALL CONDITIONS FOR LIÉNARD'S THM HOLD, AND THEREFORE SYSTEM (*) HAS AN UNIQUE STABLE LIMIT CYCLE FOR $\gamma > 0$.

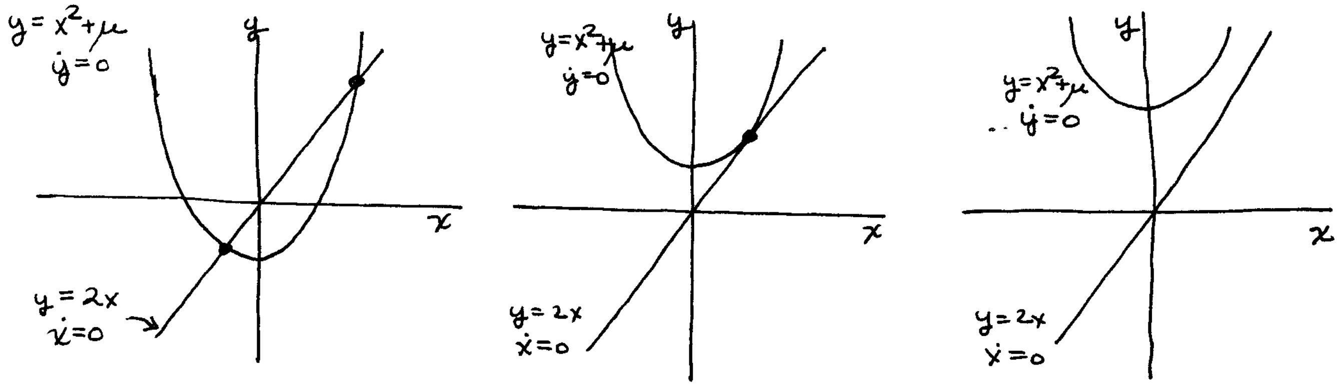
THIS IMPLIES THAT, WHEN THE SYSTEM IS REVERTED BACK TO ORIGINAL VARIABLE t AND PARAMETER μ, THE ORIGINAL SYSTEM HAS AN UNSTABLE LIMIT CYCLE FOR $\mu < 0$.

8.1.6

$$\dot{x} = y - 2x$$
$$\dot{y} = \mu + x^2 - y$$

a) NULLCLINES

$$\dot{x} = 0 \Rightarrow y = 2x$$
$$\dot{y} = 0 \Rightarrow y = x^2 + \mu$$



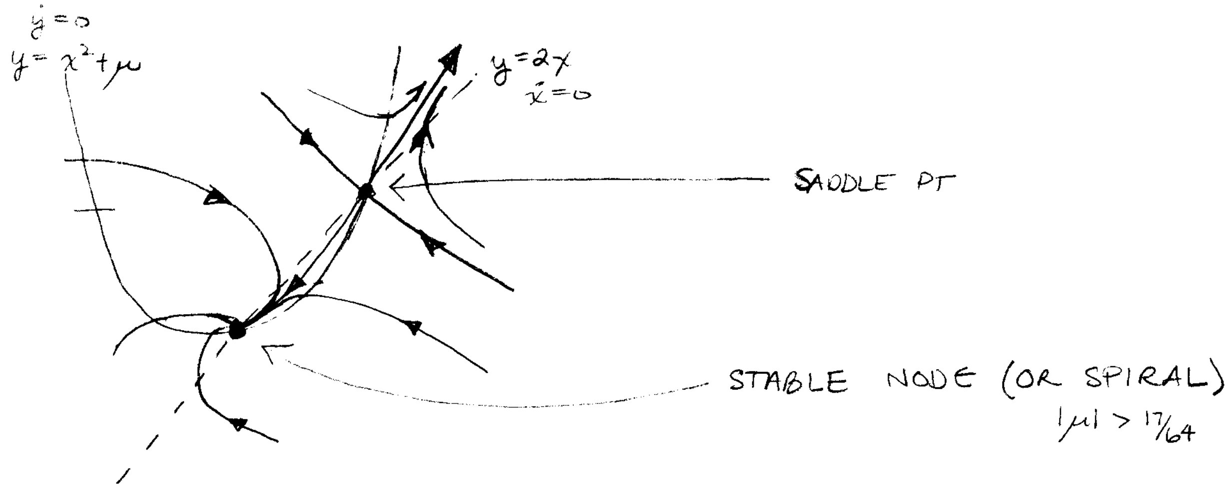
$\mu < 1$ $\mu = 1$ $\mu > 1$

b) FIXED PTS $\dot{x} = 0$, $\dot{y} = 0$

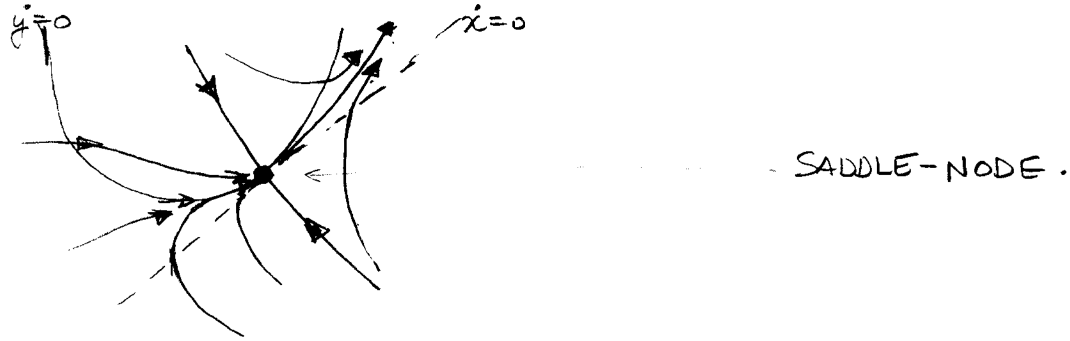
$$y^* = 2x^* = x^{*2} + \mu$$
$$x^{*2} - 2x^* + \mu = 0$$
$$x^* = 1 \pm \sqrt{1 - \mu}$$

$\Rightarrow$
- ONE FIXED PT AT $\mu = 1$ $(x^* = 1, y^* = 2)$.
- TWO FIXED PTS FOR $\mu < 1$.
- NO FIXED PTS FOR $\mu > 1$.

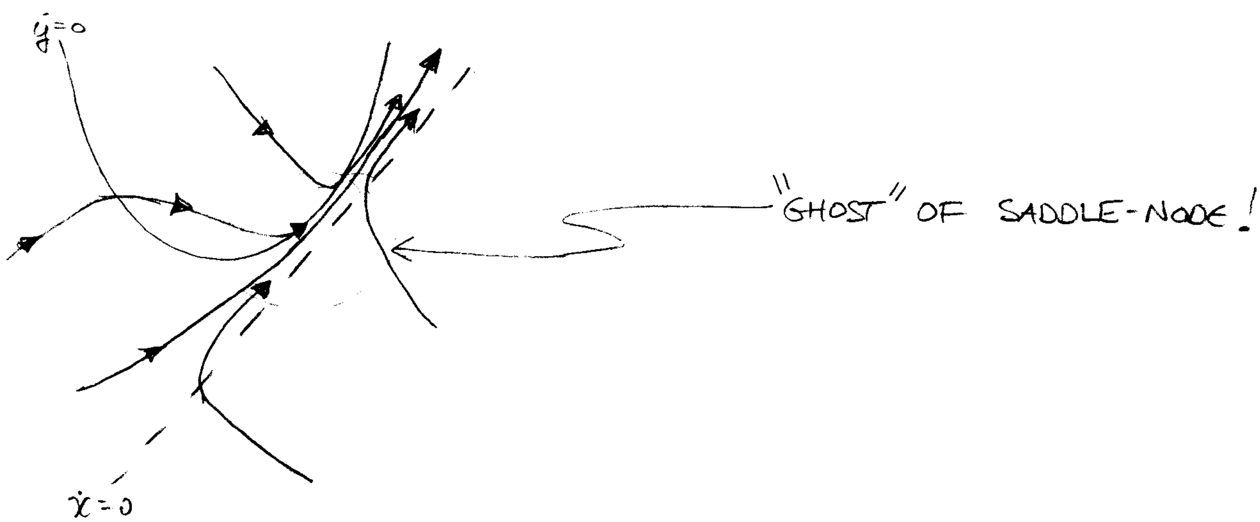
THERE IS A SADDLE-NODE BIFURCATION AT $\mu_{CRIT} = 1$, $(x^*, y^*)_{CRIT} = (1, 2)$.



$\mu < 1$



$\mu = 1$ (BIF. PT)



8.1.9 $\ddot{x} + b\dot{x} - kx + x^3 = 0$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -by - x^3 + kx \end{cases}$$

- FIXED PTS. $\dot{x} = 0, \dot{y} = 0$

$$\Rightarrow (x^*, y^*) = (0,0), \underbrace{(\pm\sqrt{k}, 0)}_{\text{ONLY EXIST IF } k>0.}$$

* PITCHFORK BIF OCCURS AT $k=0$, $(0,0)$.

- STABILITY OF $(0,0)$, $(\pm\sqrt{k}, 0)$

$$\det \begin{bmatrix} 0-\lambda & 1 \\ -3x^2+k & -b-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + b\lambda - (k - 3x^2) = 0$$

$$\lambda = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + k - 3x^2}$$

$(0,0) \Rightarrow \lambda = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + k}$

- IF $k > 0$, THEN $\lambda_1 < 0 < \lambda_2$ AND $(0,0)$ IS A SADDLE PT.
- IF $k < 0$, THEN
  (i) IF $|k| < \left(\frac{b}{2}\right)^2$ THEN λ REAL. $(0,0)$ IS A STABLE NODE IF $b>0$, AND AN UNSTABLE NODE IF $b<0$.

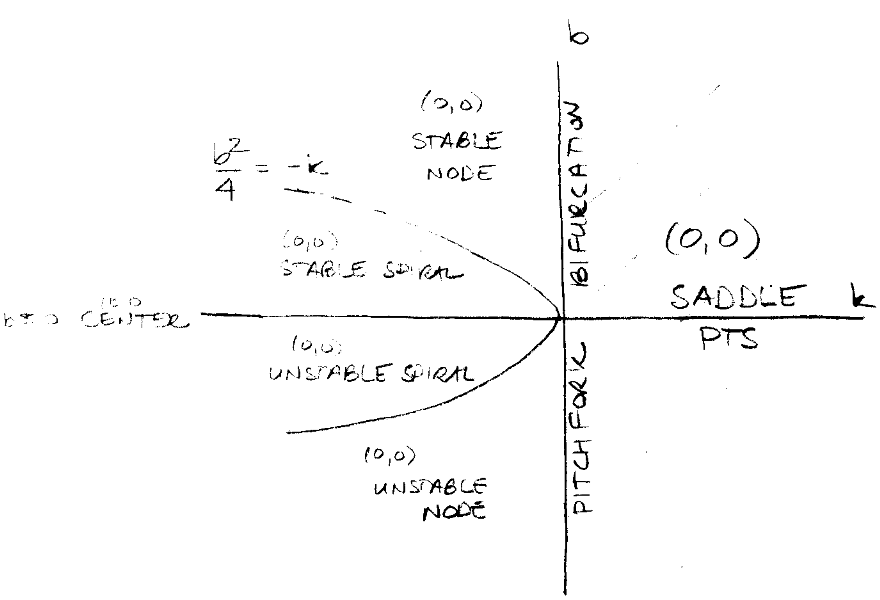
(ii) IF $|k| > (\frac{b}{2})^2$, THEN $\lambda \in \mathbb{C}$ w. $Im(\lambda) \neq 0$ AND (0,0) IS A STABLE ($b>0$) OR UNSTABLE ($b<0$) SPIRAL.

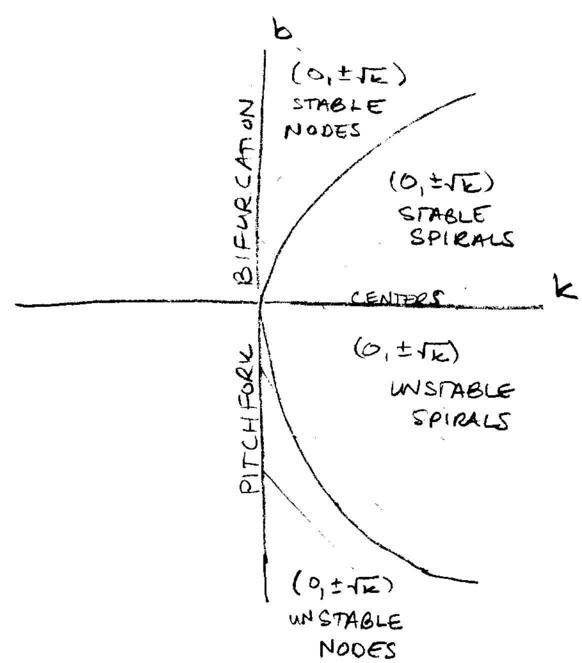
(iii) IF $b = 0$, THEN (0,0) IS A (NON-LINEAR) CENTER.

$$(0, \pm\sqrt{k}) \Rightarrow \lambda = -\frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - 2k}$$

$k>0$, (i) $(\frac{b}{2})^2 > 2k$, $(0, \pm\sqrt{k})$ ARE STABLE NODES IF $b>0$ OR UNSTABLE NODES IF $b<0$.

(ii) $(\frac{b}{2})^2 < 2k$, $(0, \pm\sqrt{k})$ ARE STABLE SPIRALS IF $b>0$ OR UNSTABLE SPIRALS IF $b<0$.





8.2.1 $\ddot{x} + \mu(x^2-1)\dot{x} + x = a$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\mu(x^2-1)y - x + a \end{cases}$$

- UNIQUE FIXED PT $y^* = 0, \; x^* = a.$

- STABILITY

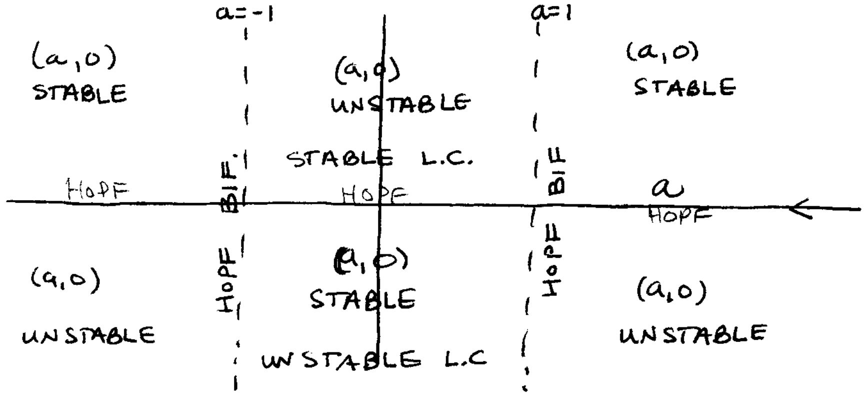
$$\det \begin{bmatrix} -\lambda & 1 \\ -\mu\, 2x^*y^* - 1 & -\mu(x^{*2}-1) - \lambda \end{bmatrix}_{\substack{x^*=a \\ y^*=0}} = 0$$

$$\lambda^2 + \mu(a^2-1)\lambda + 1 = 0$$

$$\lambda = -\frac{\mu(a^2-1)}{2} \pm \sqrt{\frac{\mu^2(a^2-1)^2}{4} - 1}$$

HOPF BIFURCATION WHEN $Re(\lambda) = 0$

ie. $-\mu(a^2-1) = 0$ WHERE $\lambda = \pm i$.



$\mu > 0$
$a = \pm 1$, $(a,0)$
SUPERCRITICAL HOPF BIFURCATION PTS

WHEN $\mu = 0$, $(a,0)$ IS A ~~[illegible]~~ CENTER.

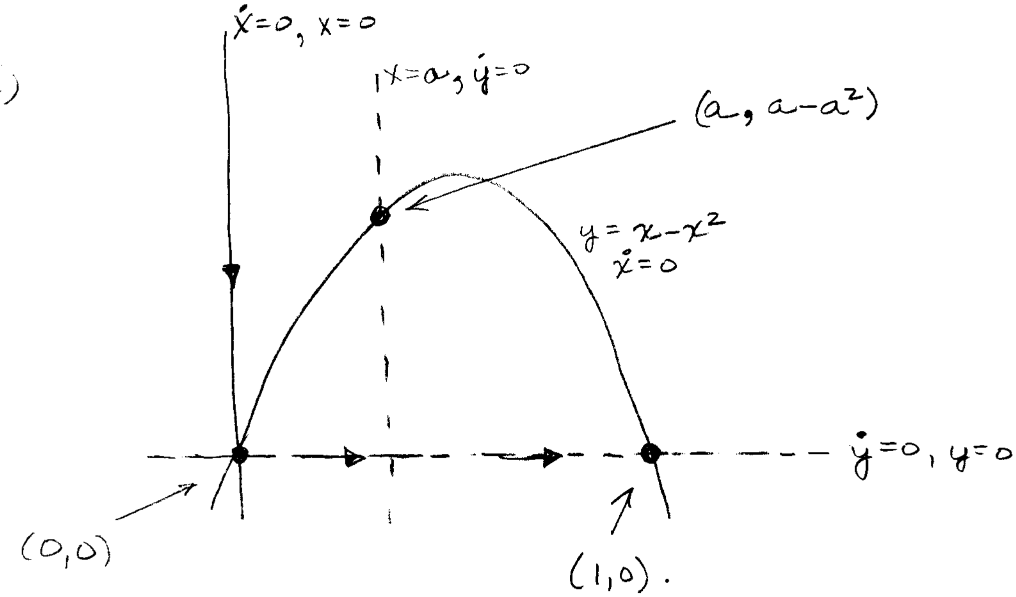
$\mu < 0$
$a = \pm 1$, $(a,0)$
SUBCRITICAL HOPF BIFURCATION PTS.

8.2.8

$$\dot{x} = x\,[x(1-x) - y] = x^2 - x^3 - xy$$
$$\dot{y} = y\,(x-a) = yx - ya$$

$$x \geq 0,\ y \geq 0\ ,\quad a > 0.$$

a)



c) $$\det \begin{bmatrix} 2x^* - 3x^{*2} - y^* - \lambda & -x^* \\ y^* & (x^*-a) - \lambda \end{bmatrix} = 0$$

(0,0) : $\lambda^2 + \lambda a = 0 \qquad \lambda = 0, -a$

by examining phase plane, <u>SADDLE-NODE</u>

(1,0) : $\lambda = -1$, $\lambda = 1-a$

FOR $0 < a < 1$, SADDLE POINT

FOR $a > 1$, STABLE NODE

FOR $a = 1$, SADDLE-NODE

$(a, a-a^2)$: $\lambda^2 - (a - 2a^2)\lambda + a^2(1-a) = 0$

$$\lambda = +\frac{a(1-2a)}{2} \pm \sqrt{\left(\frac{a(1-2a)}{2}\right)^2 - a^2(1-a)}$$

$$\lambda = a\left[\frac{(1-2a)}{2} \pm \frac{\sqrt{-3+4a^2}}{2}\right]$$

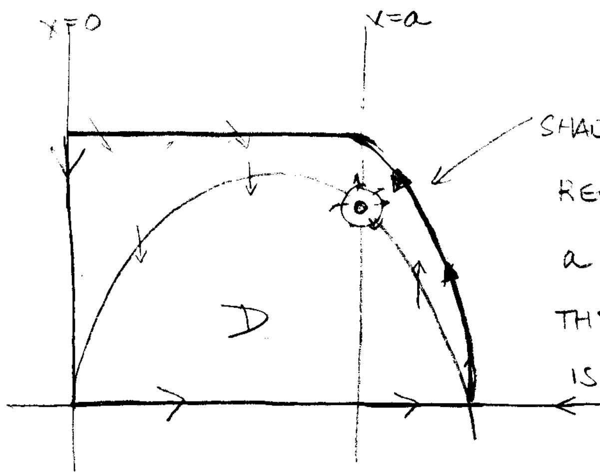
FOR $0 < a < \frac{1}{2}$, STABLE SPIRAL

FOR $\frac{1}{2} < a < \sqrt{3/4}$, UNSTABLE SPIRAL.

FOR $\sqrt{3/4} < a < 1$, UNSTABLE NODE

d) HOPF BIF OCCUR AT $(a, a-a^2)$ WHEN $a = 1/2$ i.e. WHEN $Re(\lambda) = 0$, $Im(\lambda) \neq 0$.

ie $a_{CRIT,HB} = \frac{1}{2}$, $(x^*, y^*) = (\frac{1}{2}, \frac{1}{4})$.



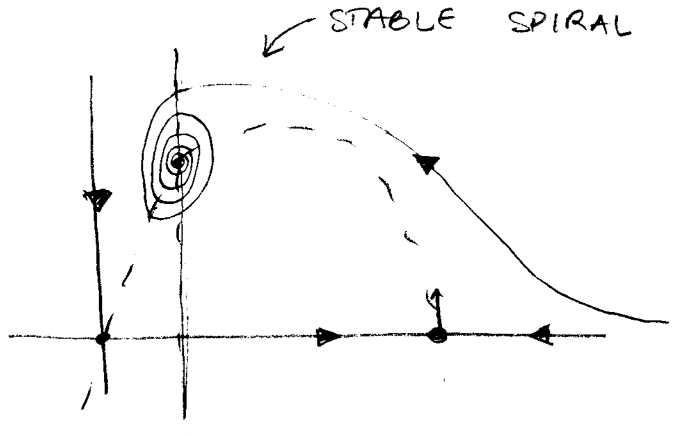
SHADED AREA IS A TRAPPING REGION FOR ALL VALUES OF a $(0<a<1)$.

THEREFORE BY PB THM THERE IS A LIMIT CYCLE IN D WHEN FIXED PT $(a, a-a^2)$ IS UNSTABLE.

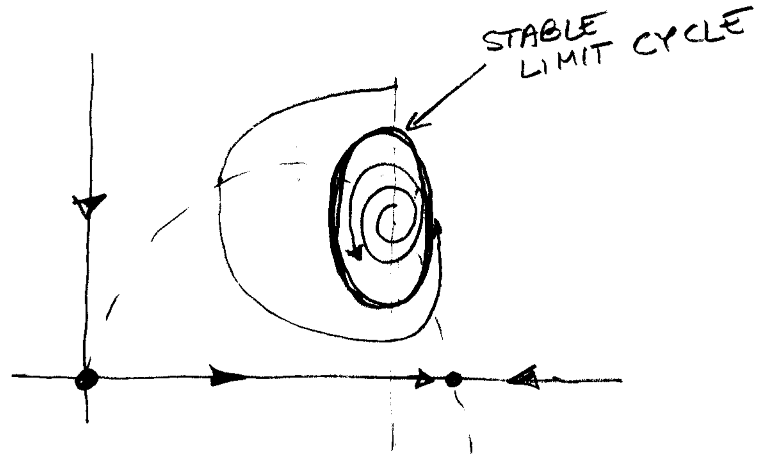
THIS SUGGEST A SUPERCRITICAL HOPF BIF. OCCURS AT $a_{CRIT} = 1/2$.

e) freq. of LIMIT CYCLE CLOSED TO BIF PT IS $f \cong \frac{2\pi}{|Im(\lambda(a=\frac{1}{2}))|} = \frac{2\pi}{1/2\sqrt{2}} = \sqrt{2}\,\pi$

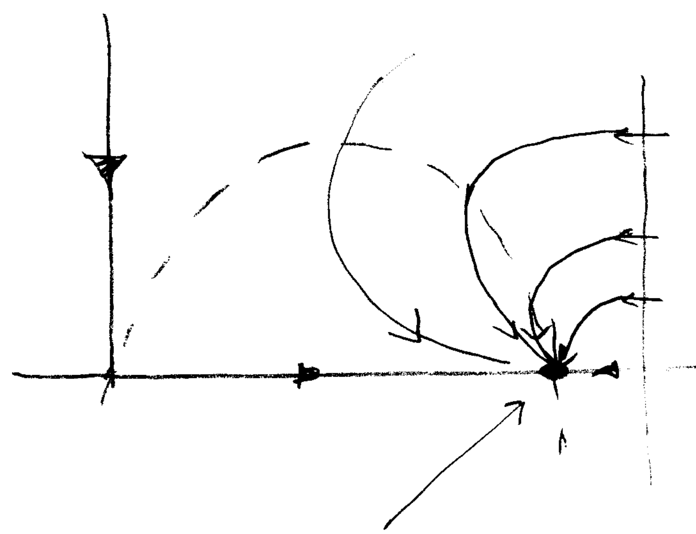
f) $0 < a < 1/2$



$\frac{1}{2} < a < 1$



$a > 1$



$(1,0)$ IS A STABLE NODE AND IS GLOBALLY ATTRACTING IN $x, y > 0$

ie. PREDITORS ALWAY DIE OUT!