MAT119A Midterm Exam
February 11, 2004

Write solutions in the exam booklet provided. Start each problem on a new page. Be sure to label diagrams carefully and justify all of your answers. Remember to print your name on the exam booklet. Good Luck!

1. Consider the equation

$$\frac{dx}{dt} = -x^2 + 2\alpha x = f(x;\alpha), \tag{1}$$

with parameter α.

(a) Calculate the location of ALL fixed points x^* of equation 1 as a function of the parameter α.

(b) Determine the stability of these fixed points analytically.

(c) Plot the phase portraits corresponding to equation 1 for $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$. Indicate the direction of the flow and the stability of the fixed points.

(d) Plot the bifurcation diagram for the system using α as the control parameter. Indicate the stability of the fixed points on the diagram. At what values of x and α does a bifurcation occur? What type of bifurcation is it?

2. Consider the equation

$$\frac{dx}{dt} = -x^2 + 2\alpha x - \beta = f(x;\alpha,\beta), \tag{2}$$

with parameters α and β.

(a) Calculate the location of ALL fixed points x^* for equation 2 in terms of the parameters α and β. For what values of α and β do fixed points exist (i.e. when is x^* real). In the β vs α plane, plot the region in which fixed points exist. On the plot, indicate the regions (i.e. the values of α and β) where the equation has one fixed point, two fixed points or no fixed points.

(b) In problem 1, you determined the location and the type of bifurcation that occurs when $\beta = 0$ and α is varied. What kind of bifurcations occur when $\beta > 0$ is fixed and α is varied? (Base your answer on the change in the number of fixed points). What happens when $\beta < 0$ is fixed and α is varied? Label the types of bifurcations on your plot in (a).

(c) To confirm that bifurcations occur on the boundary between the region in the β vs α plane where fixed points exist and the region where they do not exist, find the relationship between α and β for which $f'(x^*) = 0$.

3. Consider the nonlinear system of ODEs

$$\begin{aligned}\frac{dx}{dt} &= -x + y - 1 \\ \frac{dy}{dt} &= e^x - y\end{aligned} \tag{3}$$

(a) Use linear stabilty analysis to assess the stability of the fixed point at (0,1). What conclusions can be made about the flow around the fixed point in the nonlinear system?

(b) Plot the nullclines of the system. Indicate the direction of the flow on the nullclines and in the different regions of phase space.

(c) On a new plot, re-sketch the nullclines and sketch trajectories in the phase plane.

(d) What is the stability of the fixed point at (0,1)? Can the fixed point be classified as one of the types of fixed points found in linear systems? If so, which one? If not, can you suggest a possible name for the fixed point?