MIDTERM 119A

① $\dot{x} = -x^2 + 2\alpha x = f(x;\alpha)$

a) FIXED PTS $\dot{x} = 0$

$$-x^{*2} + 2\alpha x^* = 0$$

$$\boxed{x^* = 0, \quad x^* = 2\alpha}$$

b) STABILITY OF x^*

$$\frac{\partial f}{\partial x}(x^*;\alpha) = -2x^* + 2\alpha \begin{cases} < 0, & x^* \text{ STABLE} \\ > 0, & x^* \text{ UNSTABLE} \\ = 0, & \text{INCONCLUSIVE} \end{cases}$$

- $x^* = 0$ $\quad \frac{\partial f}{\partial x}(0;\alpha) = 2\alpha$

$$\Rightarrow \quad \begin{array}{ll} \alpha > 0, & x^* = 0 \text{ IS UNSTABLE} \\ \alpha < 0, & x^* = 0 \text{ IS STABLE} \end{array}$$

- $x^* = 2\alpha$ $\quad \frac{\partial f}{\partial x}(2\alpha;\alpha) = -2\alpha$

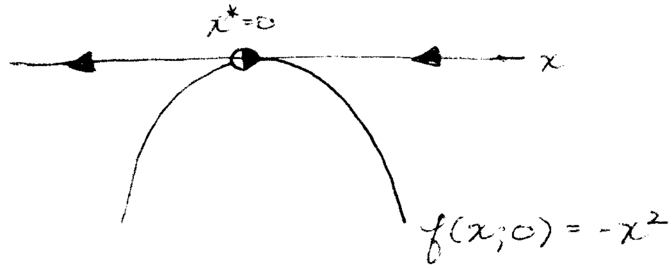
$$\Rightarrow \quad \begin{array}{ll} \alpha > 0, & x^* = 2\alpha \text{ IS STABLE} \\ \alpha < 0, & x^* = 2\alpha \text{ IS UNSTABLE} \end{array}$$

① cont'd.

- if $\alpha = 0$, then $x^* = 0 = 2\alpha$
  (i.e. there is only one fixed pt and linear stability analysis is inconclusive.)

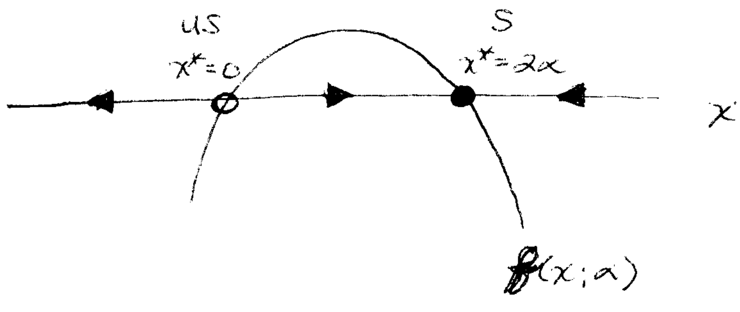
GRAPHICALLY

$\alpha = 0$



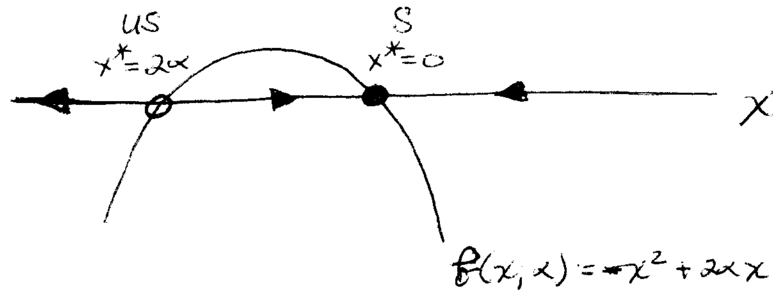
when $\alpha = 0$, $x^* = 0$ is MARGINALLY STABLE.

c) $\alpha > 0$



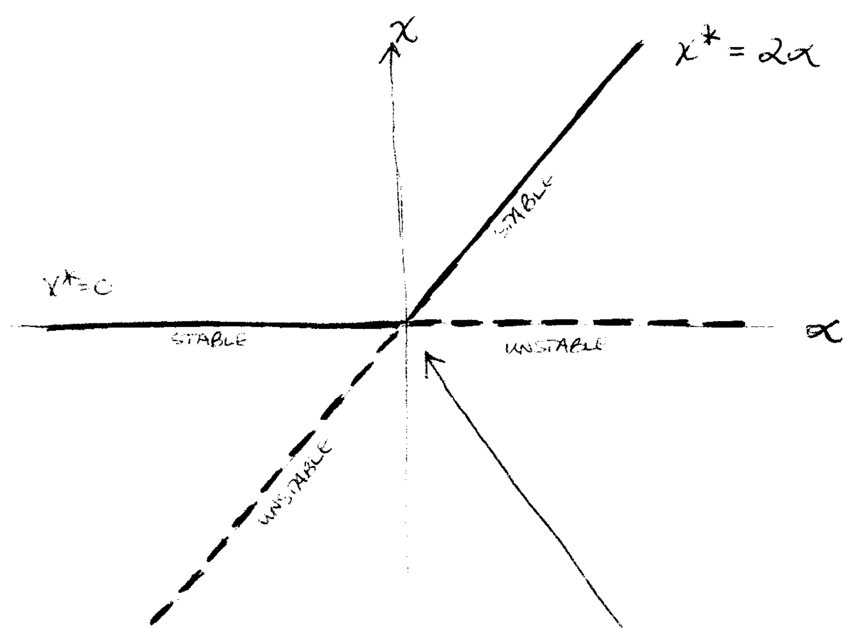
c) cont'd

$\alpha < 0$



($\alpha = 0$, see (b)).

d)



TRANSCRITICAL BIFURCATION OCCURS AT ($x^* = 0$, $\alpha = 0$)

MIDTERM 119A

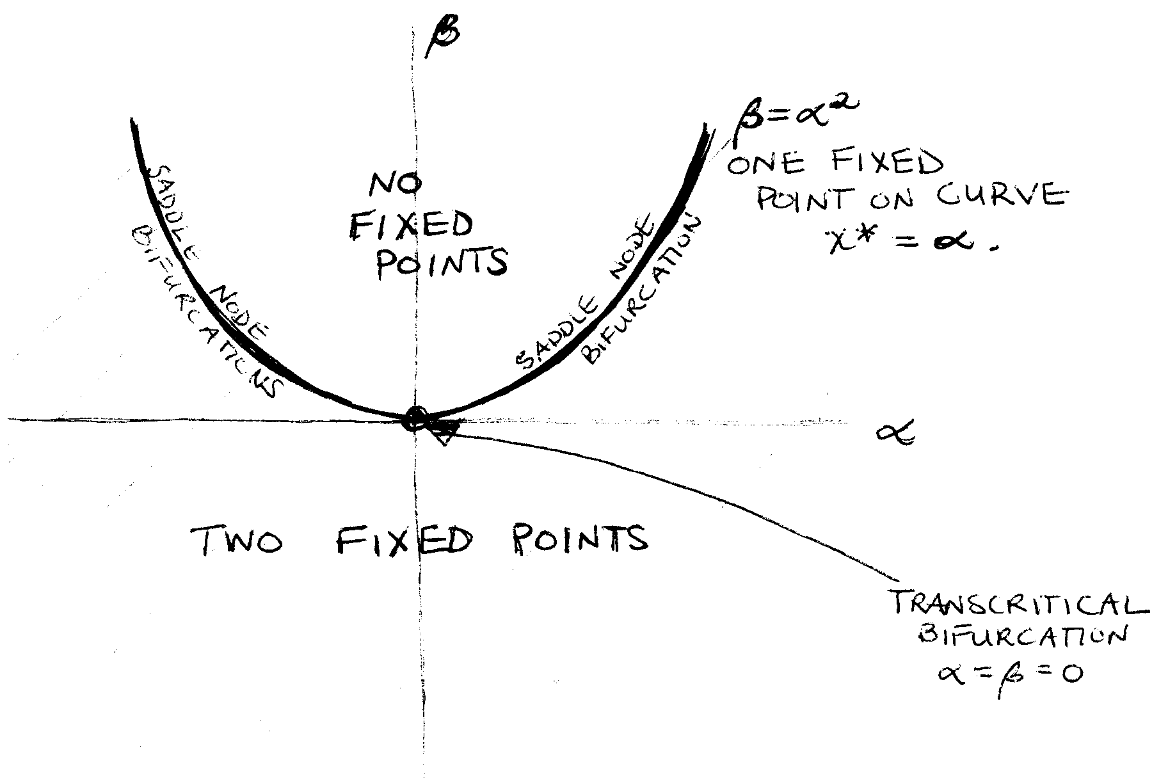
(2) $\dot{x} = -x^2 + 2\alpha x - \beta = f(x;\alpha,\beta)$

a) FIXED PTS $\dot{x} = 0$

$$-x^{*2} + 2\alpha x^* - \beta = 0$$

$$\Rightarrow \quad x^* = \alpha \pm \sqrt{\alpha^2 - \beta}$$

x^* exist when $\alpha^2 \geq \beta$.



2 cont'd

b) $\beta > 0$.

as x increases (from below $-\sqrt{\beta}$),
the number of fixed points are

2 for $x < -\sqrt{\beta}$
1 for $x = -\sqrt{\beta}$
0 for $-\sqrt{\beta} < x < \sqrt{\beta}$
1 for $\alpha = \sqrt{\beta}$
2 for $\alpha > -\sqrt{\beta}$

This implies that there as SADDLE NODE BIFURCATIONS at $\alpha = -\sqrt{\beta}$ and $\alpha = +\sqrt{\beta}$.

$\beta < 0$.

there are always two fixed points (independent of the value of α); there are no bifurcations.

c)

$$
\begin{aligned}
f'(x^*) &= -2x^* + 2\alpha \\
&= -2\left(\alpha \pm \sqrt{\alpha^2 - \beta}\right) + 2\alpha \\
&= \mp 2\sqrt{\alpha^2 - \beta} \\
&= 0
\end{aligned}
$$

$\Rightarrow$ BIFURCATIONS OCCUR WHEN $\alpha^2 = \beta$.

MIDTERM 119A

(3)
$$\begin{cases} \dot{x} = -x + y - 1 = f(x,y) \\ \dot{y} = e^x - y = g(x,y) \end{cases}$$

a) FIXED POINT AT $(x^*, y^*) = (0, 1)$

LINEAR STABILITY ANALYSIS

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ e^x & -1 \end{bmatrix}$$

$$J\Big|_{(x^*,y^*)=(0,1)} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = A$$

$$\det(A - I\lambda) = \det \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = 0$$

$$= (\lambda+1)^2 - 1 = 0$$

$$= \lambda^2 + 2\lambda = 0$$

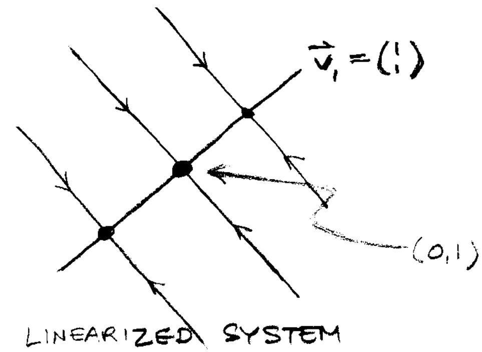
$$\Rightarrow \lambda_1 = 0, \; \lambda_2 = -2 \quad \text{EIGENVALUES}$$

(3) cont'd

EIGENVALUES & EIGENVECTORS

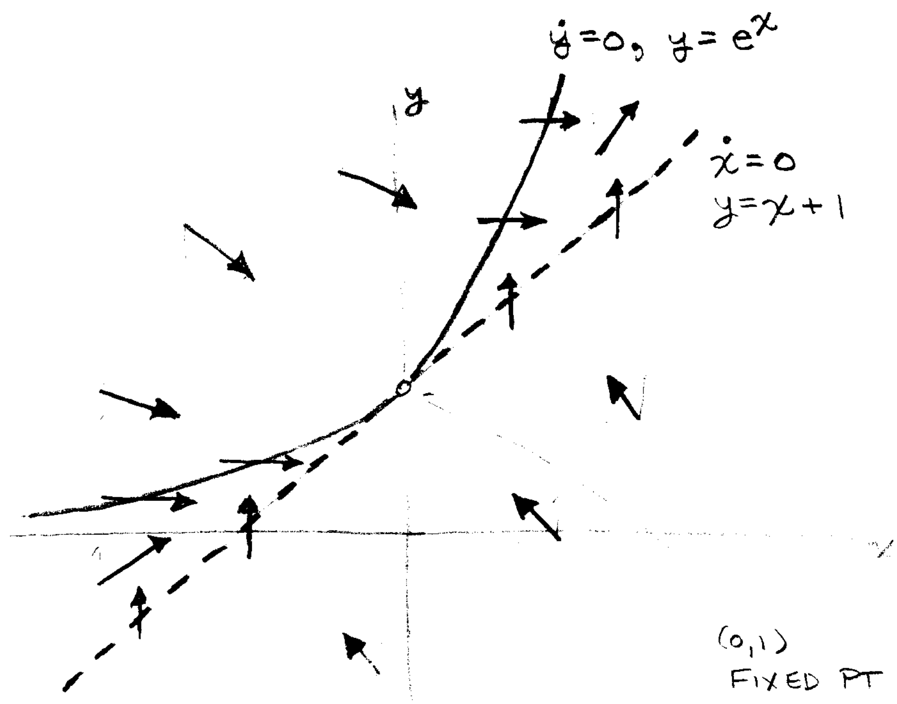
$\lambda_1 = 0$, $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$, $\bar{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



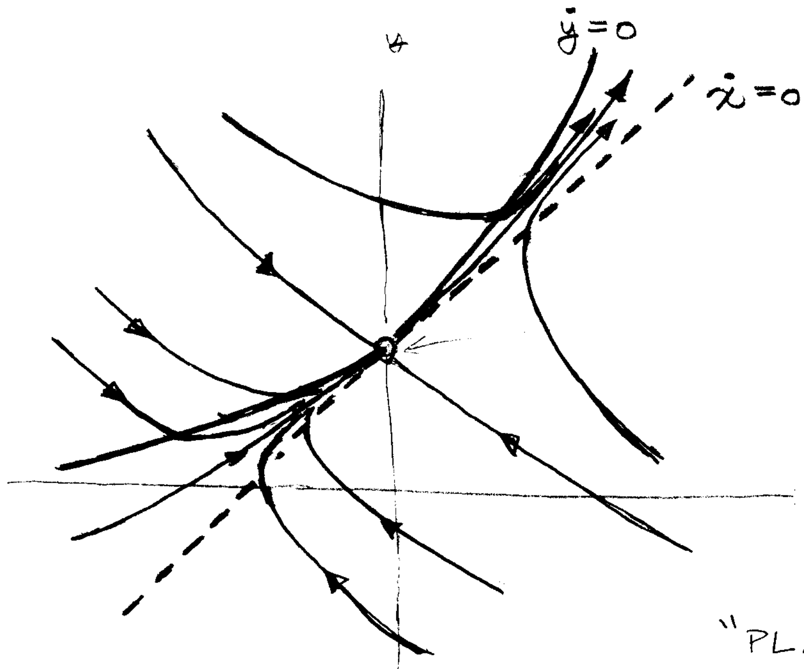
$\Rightarrow$ LINEAR STABILITY ANALYSIS SUGGESTS THAT THERE IS A LINE OF FIXED POINTS (A LINE ATTRACTOR) ALONG $y = x$ i.e $\bar{v}_1$, HOWEVER BECAUSE $\lambda_1 = 0$, RESULT IS INCONCLUSIVE.

b)



③ cont'd

c)



"PLAUSIBLE"
PHASE-PORTRAIT.

d) • THE FIXED POINT AT (0,1) IS UNSTABLE. IT LOOKS LIKE A STABLE NODE FROM THE FLOW IN THE LOWER LEFT PORTION OF THE PHASE PLANE, BUT IT LOOKS LIKE A SADDLE POINT IN THE UPPER RIGHT PORTION OF THE PHASE PLANE.

• IT CANNOT BE CLASSIFIED AS ONE OF THE FIXED POINTS FOUND IN LINEAR SYSTEMS.

• IT IS A HYBRID STABLE NODE - SADDLE POINT AND IS CALLED A SADDLE-NODE.