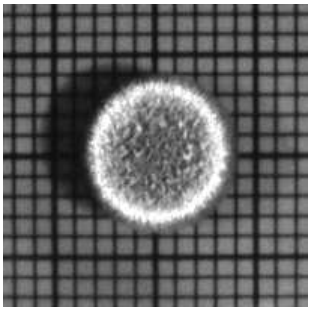
Chapter 1

Mathematical Models of Biological Processes

Where are we going?

Science involves observations, formulation of hypotheses, and testing of hypotheses. This book is directed to quantifiable observations about living systems and hypotheses about the processes of life that are formulated as mathematical models. Using three biologically important examples, growth of the bacterium *Vibrio natrigens*, depletion of light below the surface of a lake or ocean, and growth of a mold colony, we demonstrate how to formulate *mathematical models* that lead to *dynamic equations* descriptive of natural processes. You will see how to compute *solution equations* to the dynamic equations and to test them against experimental data.

Examine the picture of the mold colony (Day 6 of Figure 1.14) and answer the question, "Where is the growth?" You will find the answer to be a fundamental component of the process.



In our language, a **mathematical model** is a concise verbal description of the interactions and forces that cause change with time or position of a biological system (or physical or economic or other system). The modeling process begins with a clear verbal statement based on the scientist's understanding of the interactions and forces that govern change in the system. In order for mathematical techniques to assist in understanding the system, the verbal statement must be translated into an equation, called the **dynamic equation** of the model. Knowledge of the initial state of a system and the dynamic equation that describes the forces of change in the system is often sufficient to forecast an

observed pattern of the system. A **solution equation** may be derived from the dynamic equation and an initial state of the system and a graph or table of values of the solution equation may then be compared with the observed pattern of nature. The extent to which solution equation matches the pattern is a measure of the validity of the mathematical model.

Mathematical modeling is used to describe the underlying mechanisms of a large number of processes in the natural or physical or social sciences. The chart in Figure 1.1 outlines the steps followed in finding a mathematical model.

Initially a scientist examines the biology of a problem, formulates a concise description, writes equations capturing the essence of the description, solves the equations, and makes predictions about the biological process. This path is marked by the bold arrows in Figure 1.1. It is seldom so simple! Almost always experimental data stimulates exchanges back and forth between a biologist and a computational scientist (mathematician, statistician, computer scientist) before a model is obtained that explains some of the biology. This additional exchange is represented by lightly marked arrows in Figure 1.1.

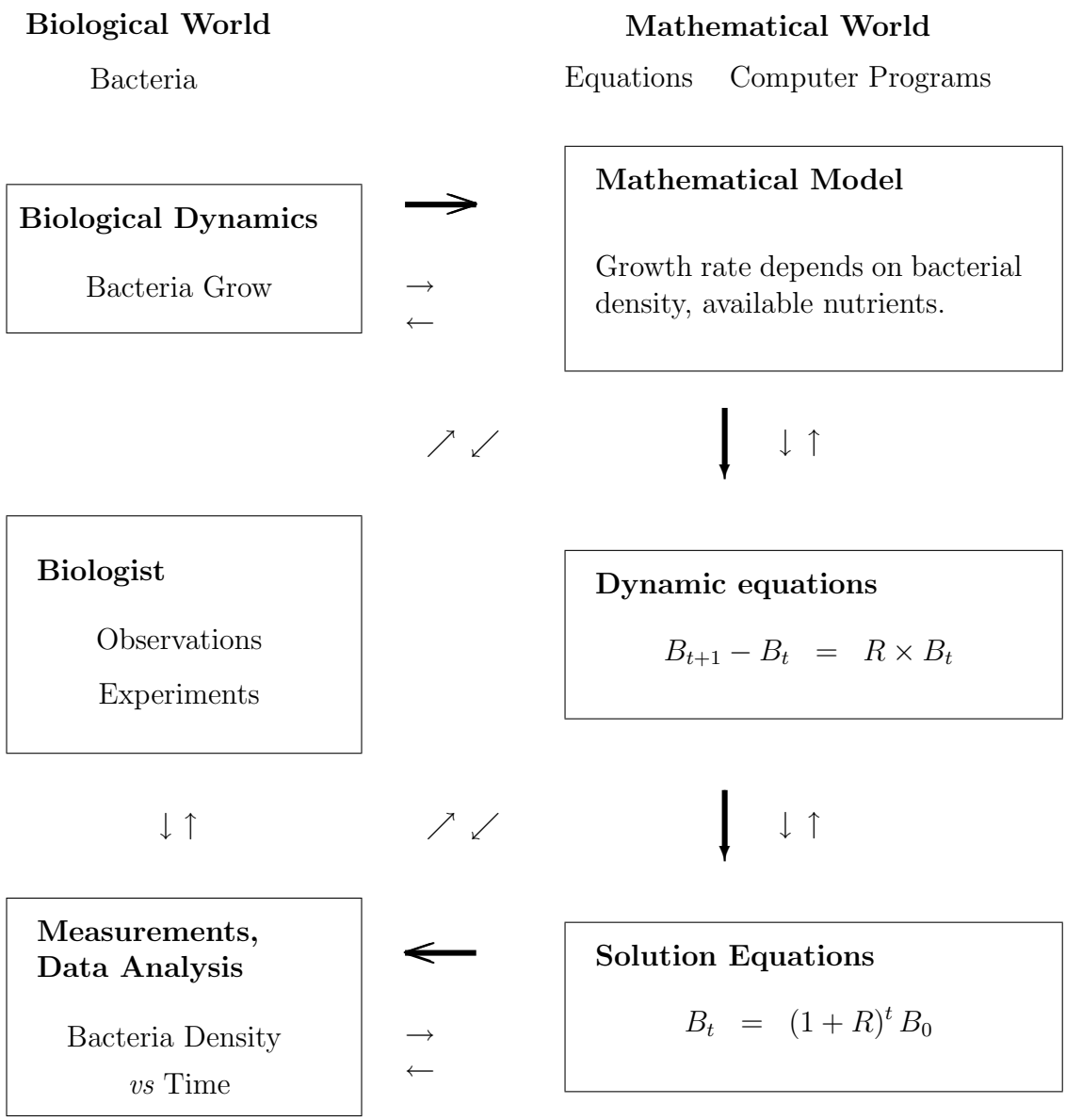


Figure 1.1: Biology - Mathematics: Information flow chart for bacteria growth.

An initial approach to modeling may follow the bold arrows in Figure 1.1. For *V. natrigens* growth the steps might be:

- The scientist 'knows' that 25% of the bacteria divide every 20 minutes.
- If so, then bacteria increase, $B_{t+1} - B_t$, should be $0.25B_t$ where t marks time in 20 minute intervals and B_t is the amount of bacteria at the end of the $t\underline{th}$ 20 minute interval.
- From $B_{t+1} - B_t = 0.25B_t$, the scientist may conclude that

$$B_t = 1.25^t B_0$$

(More about this conclusion later).

- The scientist uses this last equation to predict what bacteria density will be during an experiment in which *V. natrigens*, initially at 10^6 cells per milliliter, are grown in a flask for two hours.

- In the final stage, an experiment is carried out to grow the bacteria, their density is measured at selected times, and a comparison is made between observed densities and those predicted by equations.

  Reality usually strikes at this stage, for the observed densities may not match the predicted densities. If so, the additional network of lightly marked arrows of the chart is implemented.

  Adjustments to a model of *V. natrigen* growth that may be needed include:

  1. *Different growth rate:* A simple adjustment may be that only, say 20%, of the bacteria are dividing every 20 minutes and $B_t = 1.2^t B_0$ matches experimental observations.
  2. *Variable growth rate:* A more complex adjustment may be that initially 25% of the bacteria were dividing every 20 minutes but as the bacteria became more dense the growth rate fell to, say only 10% dividing during the last 20 minutes of the experiment.
  3. *Age dependent growth rate:* A different adjustment may be required if initially all of the bacteria were from newly divided cells so that, for example, most of them grow without division during the first two 20 minute periods, and then divide during the third 20 minute interval.
  4. *Synchronous growth:* A relatively easy adjustment may account for the fact that bacterial cell division is sometimes regulated by the photo period, causing all the bacteria to divide at a certain time of day. The green alga *Chlamydomonas moewussi*, for example, when grown in a laboratory with alternate 12 hour intervals of light and dark, accumulate nuclear subdivisions and each cell divides into eight cells at dawn of each day[1].

1.1 Experimental data, bacterial growth.

Population growth was historically one of the first concepts to be explored in mathematical biology and it continues to be of central importance. Thomas Malthus in 1821 asserted a theory

> "that human population tends to increase at a faster rate than its means of subsistence and that unless it is checked by moral restraint or disaster (as disease, famine, or war) widespread poverty and degradation inevitably result." [2]

In doing so he was following the bold arrows in Figure 1.1. World population has increased approximately six fold since Malthus' dire warning, and adjustments using the additional network of the chart are still being argued.

Bacterial growth data from a *V. natrigens* experiment are shown in Table 1.1. The population was grown in a commonly used nutrient growth medium, but the pH of the medium was adjusted to be pH 6.25 [3].

[1] Emil Bernstein, *Science* **131** (1960), 1528.

[2] Merriam Webster's Collegiate Dictionary, Tenth Edition.

[3] The experiment was a semester project of Deb Christensen in which several *V. natrigens* populations were grown in a range of pH values.

Table 1.1: Measurements of bacterial density.
The units of "Population Density" are those of absorbance as measured by the spectrophotometer.

pH 6.25			
Time (min)	Time Index t	Population Density B_t	Pop Change/ Unit Time $(B_{t+1} - B_t)/1$
0	0	0.022	0.014
16	1	0.036	0.024
32	2	0.060	0.041
48	3	0.101	0.068
64	4	0.169	0.097
80	5	0.266	

How do you measure bacteria density? Ideally you would place, say 1 microliter, of growth medium under a microscope slide and count the bacteria in it. This is difficult, so the procedure commonly used is to pass a beam of light through a sample of growth medium and measure the amount of light absorbed. The greater the bacterial density the more light that is absorbed and thus bacterial density is measured in terms of *absorbance* units. The instrument used to do this is called a *spectrophotometer*.

The spectrophotometer gives you a measure of light absorbance which is directly proportional to the bacterial density (that is, light absorbance is a constant times bacterial density). Absorbance is actually defined by

$$\text{Absorbance} \quad = \quad -\log_{10} \frac{I_t}{I_0}$$

where I_0 is the light intensity passing through the medium with no bacteria present and I_t is the light intensity passing through the medium with bacteria at time t.

It may seem more natural to use

$$\frac{I_0 - I_t}{I_0}$$

as a measure of absorbance. The reason for using $\log_{10} \frac{I_t}{I_0}$ is related to our next example for light absorbance below the surface of a lake, but is only easy to explain after continuous models are studied in Chapter 5 (Exercise 5.5.17).

1.1.1 Steps towards building a mathematical model.

This section illustrates one possible sequence of steps leading to a mathematical model of bacterial growth.

Step 1. Preliminary Mathematical Model: Description of bacterial growth. Bacterial populations increase rapidly when grown at low bacterial densities in abundant nutrient. The population

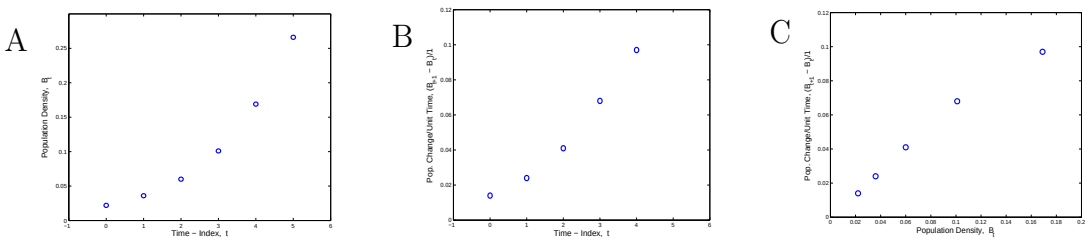


Figure 1.2: (A) Bacterial population density, B_t, *vs* time index, t. (B) Population density change per unit time, $B_{t+1} - B_t$, *vs* time index, t. (C) Population density change per unit time, $B_{t+1} - B_t$, *vs* population density, B_t.

increase is due to mitotic cell division – single cells divide asexually into two cells, subsequently the two cells divide to form four cells, and so on.

Step 2. Notation. The first step towards building equations for a mathematical model is an introduction of notation. In this case the data involve time and bacterial density, and it is easy to let t denote time and B_t denote bacterial density at time t. However, the data was read in multiples of 16 minutes, and it will help our notation to rescale time so that t is 0, 1, 2, 3, 4, or 5. Thus B_3 is the bacterial density at time $3 \times 16 = 48$ minutes. The rescaled time is shown under 'Time Index' in Table 1.1.

Step 3. Derive a dynamic equation. In some cases your mathematical model will be sufficiently explicit that you are able to write the dynamic equation directly from the model. For this development, we first look at supplemental computations and graphs of the data.

Step 3a. Computation of rates of change from the data. Table 1.1 contains a computed column, 'Population Change per Unit Time'. Change per unit time is often an important quantity in models.

Using our notation,

$$B_0 = 0.022, \qquad B_1 = 0.036, \qquad B_2 = 0.060, \qquad \text{etc.}$$

and

$$B_1 - B_0 = 0.014, \qquad B_2 - B_1 = 0.024, \qquad \text{etc.}$$

Step 3b. Graphs of the Data. Another important step in modeling is to obtain a visual image of the data. Shown in Figure 1.2 are three graphs that illustrate bacterial growth. Bacterial Growth A is a plot of column 3 *vs* column 2, B is a plot of column 4 *vs* column 2, and C is a plot of column 4 *vs* column 3 from the Table 1.1.

> Note: In plotting data, the expression 'plot B *vs* A' means that B is the vertical coordinate and A is the horizontal coordinate. Students sometimes reverse the axes, and disrupt a commonly used convention that began some 350 years ago. In plotting bacterial density *vs* time, students may put bacterial density on the horizontal axis and time on the vertical axis, contrary to widely used practice. Perhaps if the mathematician who introduced analytic geometry, Rene DesCartes of France and Belgium, had lived in China where documents are read from top to bottom by columns and from right to left, plotting of data would follow another convention.
>
> We suggest that you use the established convention.

The graph Bacterial Growth A is a classic picture of low density growth with the graph curving upward indicating an increasing growth rate. Observe from Bacterial Growth B that the *rate* of growth (column 4, vertical scale) is increasing with time.

The graph Bacterial Growth C is an important graph for us, for it relates the bacterial increase to bacterial density, and bacterial increase is based on the cell division described in our mathematical model. Because the points lie approximately on a straight line it is easy to get an equation descriptive of this relation.

Step 3c. An Equation Descriptive of the Data. Shown in Figure 1.3 is a reproduction of the graph Bacterial Growth C in which the point (0.15,0.1) is marked with an '+' and a line is drawn through (0,0) and (0.15,0.1). The slope of the line is 2/3. The line is 'fit by eye' to the first four points. The line can be 'fit' more quantitatively, but it is not necessary to do so at this stage.

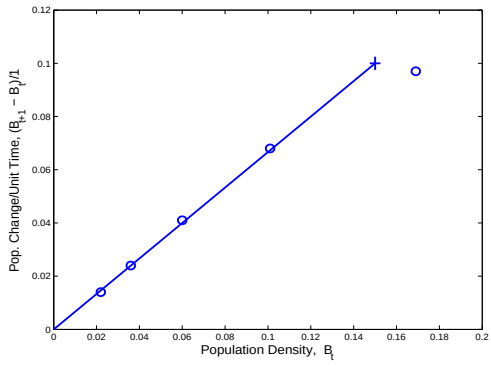


Figure 1.3: A line fit to the first four data points of bacterial growth. The line contains (0,0) and (0.15, 0.1) and has slope = 2/3.

The fifth point, which is $(B_4, B_5 - B_4) = (0.169, 0.097)$ lies below the line. Because the line is so close to the first four points, there is a suggestion that during the fourth time period, the growth, $B_5 - B_4 = 0.97$, is below expectation, or perhaps, $B_5 = 0.266$ is a measurement error and should be larger. These bacteria were actually grown and measured for 160 minutes and we will find in Section 11.6.1 that the measured value $B_5 = 0.266$ is consistent with the remaining data. The bacterial growth is slowing down after $t = 4$, or after 64 minutes.

The slope of the line in Bacterial Growth C is $\frac{0.1}{0.15} = \frac{2}{3}$ and the y-intercept is 0. Therefore an equation of the line is

$$y = \frac{2}{3}x$$

Step 3d. Convert data equation to a dynamic equation. The points in Bacterial Growth C were plotted by letting $x = B_t$ and $y = B_{t+1} - B_t$ for $t = 0$, 1, 2, 3, and 4. If we substitute for x and y into the data equation $y = \frac{2}{3}x$ we get

$$B_{t+1} - B_t = \frac{2}{3}B_t \tag{1.1}$$

Equation 1.1 is our first instance of a dynamic equation descriptive of a biological process.

Step 4. Enhance the preliminary mathematical model of Step 1. The preliminary mathematical model in Step 1 describes microscopic cell division and can be expanded to describe the macroscopic cell density that was observed in the experiment. One might extrapolate from the original statement, but the observed data guides the development.

In words Equation 1.1 says that the growth during the t-th time interval is $\frac{2}{3}$ times B_t, the bacteria present at time t, the beginning of the period. The number $\frac{2}{3}$ is called the *relative growth rate* - the growth per time interval is two-thirds of the current population size. More generally, one may say:

Mathematical Model 1.1.1 Bacterial Growth. A fixed fraction of cells divide every time period. (In this instance, two-thirds of the cells divide every 16 minutes.)

Step 5. Compute a solution to the dynamic equation. We first compute estimates of B_1 and B_2 predicted by the dynamic equation. The dynamic equation 1.1 specifies the change in bacterial density $(B_{t+1} - B_t)$ from t to time $t+1$. In order to be useful, an initial value of B_0 is required. We assume the original data point, $B_0 = 0.022$ as our reference point. It will be convenient to change $B_{t+1} - B_t = \frac{2}{3}B_t$ into what we call an **iteration** equation:

$$B_{t+1} - B_t = \frac{2}{3}B_t \qquad B_{t+1} = \frac{5}{3}B_t \tag{1.2}$$

Iteration Equation 1.2 is shorthand for at least five equations

$$B_1 = \frac{5}{3}B_0, \qquad B_2 = \frac{5}{3}B_1, \qquad B_3 = \frac{5}{3}B_2, \qquad B_4 = \frac{5}{3}B_3, \qquad \text{and} \qquad B_5 = \frac{5}{3}B_4.$$

Beginning with $B_0 = 0.022$ we can compute

$$\begin{aligned} B_1 &= \tfrac{5}{3}B_0 = \tfrac{5}{3}0.022 = 0.037 \\ B_2 &= \tfrac{5}{3}B_1 = \tfrac{5}{3}0.037 = 0.061 \end{aligned}$$

Explore 1.1.1 Use $B_0 = 0.022$ and $B_{t+1} = \frac{5}{3}B_t$ to compute B_1, B_2, B_3, B_4, and B_5. ■

There is also some important notation used to describe the values of B_t determined by the iteration $B_{t+1} = \frac{5}{3}B_t$. We can write

$$\begin{aligned} B_1 &= \frac{5}{3}B_0 \\ B_2 &= \frac{5}{3}B_1 \\ B_2 &= \frac{5}{3} \times \left(\frac{5}{3}B_0\right) = \frac{5}{3} \times \frac{5}{3}B_0 \\ B_3 &= \frac{5}{3}B_2 \\ B_3 &= \frac{5}{3} \times \left(\frac{5}{3} \times \frac{5}{3}B_0\right) = \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3}B_0 \end{aligned}$$

Explore 1.1.2 Write an equation for B_4 in terms of B_0, using the pattern of the last equations. ■

At time interval 5, we get

$$B_5 = \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} \times B_0$$

which is cumbersome and is usually written

$$B_5 = \left(\frac{5}{3}\right)^5 \times B_0.$$

The general form is

$$B_t = \left(\frac{5}{3}\right)^t B_0 = B_0 \left(\frac{5}{3}\right)^t \tag{1.3}$$

Using the starting population density, $B_0 = 0.022$, Equation 1.3 becomes

$$B_t = 0.022 \left(\frac{5}{3}\right)^t \tag{1.4}$$

and is the *solution* to the initial condition and dynamic equation 1.1

$$B_0 = 0.022, \qquad B_{t+1} - B_t = \frac{2}{3} B_t$$

Populations whose growth is described by an equation of the form

$$P_t = P_0 R^t \qquad \text{with} \qquad R > 1$$

are said to exhibit *exponential growth.*

Equation 1.4 is written in terms of the time index, t. In terms of time, T in minutes, $T = 16t$ and Equation 1.4 may be written

$$B_T = 0.022 \left(\frac{5}{3}\right)^{T/16} = 0.022 \times 1.032^T \tag{1.5}$$

Step 6. Compare predictions from the Mathematical Model with the original data. How well did we do? That is, how well do the computed values of bacterial density, B_t, match the observed values? The original and computed values are shown in Figure 1.4.

The computed values match the observed values closely except for the last measurement where the observed value is less than the value predicted from the mathematical model. The effect of cell crowding or environmental contamination or age of cells is beginning to appear after an hour of the experiment and the model does not take this into account. We will return to this population with data for the next 80 minutes of growth in Section 11.6.1 and will develop a new model that will account for decreasing rate of growth as the population size increases.

1.1.2 Concerning the validity of a model.

We have used the population model once and found that it matches the data rather well. The validity of a model, however, is only established after multiple uses in many laboratories and critical examination of the forces and interactions that lead to the model equations. Models evolve as knowledge accumulates. Mankind's model of the universe has evolved from the belief that Earth is the center of the universe, to

pH 6.25			
Time (min)	Time Index	Population Density	Computed Density
0	0	0.022	0.022
16	1	0.036	0.037
32	2	0.060	0.061
48	3	0.101	0.102
64	4	0.169	0.170
80	5	0.266	0.283

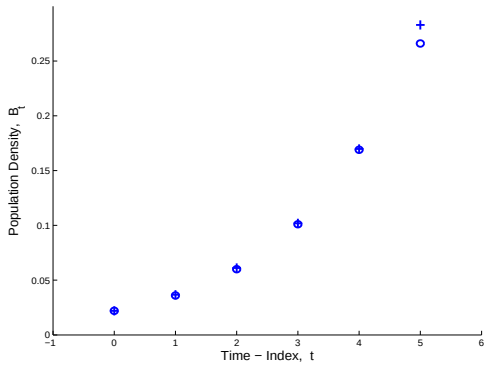


Figure 1.4: Tabular and graphical comparison of actual bacterial densities (o) with bacterial densities computed computed from Equation 1.3 (+).

the Copernican model that the sun is the center of the universe, to the realization that the sun is but a single star among some 200 billion in a galaxy, to the surprisingly recent realization (Hubble, 1923) that our Milky Way galaxy is but a single galaxy among an enormous universe of galaxies.

It is fortunate that our solution equation matched the data, but it must be acknowledged that two crucial parameters, P_0 and r, were computed from the data, so that a fit may not be a great surprise. Other equations also match the data. The parabola, $y = 0.0236 + .000186t + 0.00893t^2$, computed by least squares fit to the first five data points is shown in Figure 1.5, and it matches the data as well as does $P_t = 0.022(5/3)^t$. We prefer $P_t = 0.022(5/3)^t$ as an explanation of the data over the parabola obtained by the method of least squares because it is derived from an understanding of bacterial growth as described by the model whereas the parabolic equation is simply a match of equation to data.

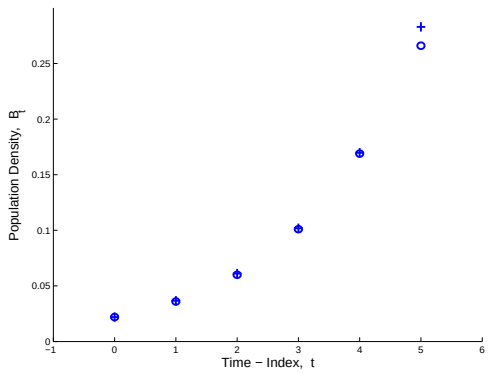


Figure 1.5: A graph of $y = 0.0236 + .000186t + 0.00893t^2$ (+) which is the quadratic fit by least squares to the first six data points (o) of *V. Natrigens* growth in Table 1.1

Exercises for Section 1.1, Experimental data, bacterial growth

Exercise 1.1.1 Compute B_1, B_2, and B_3 as in Step 5 of this section for

a.	B_0	$=$	4	$B_{t+1} - B_t$	$=$	$0.5 \times B_t$
b.	B_0	$=$	4	$B_{t+1} - B_t$	$=$	$0.1 \times B_t$
c.	B_0	$=$	0.2	$B_{t+1} - B_t$	$=$	$0.05 \times B_t$
d.	B_0	$=$	0.2	$B_{t+1} - B_t$	$=$	$1 \times B_t$
e.	B_0	$=$	100	$B_{t+1} - B_t$	$=$	$0.4 \times B_t$
f.	B_0	$=$	100	$B_{t+1} - B_t$	$=$	$0.01 \times B_t$

Exercise 1.1.2 Write a solution equation for the initial conditions and dynamic equations of Exercise 1.1.1 similar to the solution Equation 1.4, $B_t = (5/3)^t 0.022$ of the pair $B_0 = 0.022$, $B_{t+1} - B_t = (2/3)B_t$.

Exercise 1.1.3 Observe that the graph Bacterial Growth C is a plot of $B_{t+1} - B_t$ *vs* B_t. The points are $(B_0, B_1 - B_0)$, $(B_1, B_2 - B_1)$, etc. The second coordinate, $B_{t+1} - B_t$ is the population increase during time period t, given that the population at the beginning of the time period is B_t. Explain why the point (0,0) would be a point of this graph.

Exercise 1.1.4 In Table 1.2 are given four sets of data. For each data set, find a number r so that the values B_1, B_2, B_3, B_4, B_5 and B_6 computed from the difference equation

$$B_0 = \quad \text{as given in the table,} \qquad B_{t+1} - B_t = r \times B_t$$

are close to the corresponding numbers in the table. Compute the numbers, B_1 to B_6 using your value of r in the equation, $B_{t+1} = (1 + r)B_t$, and compare your computed numbers with the original data.

For each data set, follow steps 3, 5, and 6. The line you draw close to the data in step 3 should go through (0,0).

Table 1.2: Tables of data for Exercise 1.1.4

(a)

t	B_t
0	1.99
1	2.68
2	3.63
3	4.89
4	6.63
5	8.93
6	12.10

(b)

t	B_t
0	0.015
1	0.021
2	0.031
3	0.040
4	0.055
5	0.075
6	0.106

(c)

t	B_t
0	22.1
1	23.4
2	26.1
3	27.5
4	30.5
5	34.4
6	36.6

(d)

t	B_t
0	287
1	331
2	375
3	450
4	534
5	619
6	718

Exercise 1.1.5 The bacterium *V. natrigens* was also grown in a growth medium with pH of 7.85. Data for that experiment is shown in Table 1.3. Repeat the analysis in steps 1 - 9 of this section for this data. After completing the steps 1 - 9, compare your computed relative growth rate of *V. natrigens* at pH 7.85 with our computed relative growth rate of 2/3 at pH 6.25.

Table 1.3: Data for Exercise 1.1.5, *V. natrigens* growth in a medium with pH of 7.85.

pH 7.85	
Time (min)	Population Density
0	0.028
16	0.047
32	0.082
48	0.141
64	0.240
80	0.381

Exercise 1.1.6 What initial condition and dynamic equation would describe the growth of an *Escherichia coli* population in a nutrient medium that had 250,000 *E. coli* cells per milliliter at the start of an experiment and one-fourth of the cells divided every 30 minutes.

1.2 Solution to the dynamic equation $P_{t+1} - P_t = r\,P_t$.

The dynamic equation with initial condition, P_0,

$$P_{t+1} - P_t = r\,P_t, \qquad t = 0, 1, \cdots, \qquad P_0 \quad \text{a known value} \tag{1.6}$$

arises in many models of elementary biological processes. A *solution* to the dynamic equation 1.6 is a formula for computing P_t in terms of t and P_0.

Assume that $r \neq 0$ and $P_0 \neq 0$. The equation $P_{t+1} - P_t = r\,P_t$ can be changed to iteration form by

$$\begin{aligned} P_{t+1} - P_t &= r\,P_t \\ P_{t+1} &= (r+1)\,P_t \\ P_{t+1} &= R\,P_t \end{aligned} \tag{1.7}$$

where $R = r + 1$. The equation $P_{t+1} = R\,P_t$ is valid for $t = 0$, $t = 1$, $\cdots$ and represents a large number of equations, as in

$$\left.\begin{array}{rcll} P_1 &=& R\,P_0 & t = 0 \\ P_2 &=& R\,P_1 & t = 1 \\ \vdots &\vdots& \vdots & \vdots \\ P_{n-1} &=& R\,P_{n-2} & t = n-2 \\ P_n &=& R\,P_{n-1} & t = n-1 \end{array}\right\} \tag{1.8}$$

where n can be any stopping value.

Cascading equations. These equations (1.8) may be 'cascaded' as follows.

1. The product of all the numbers on the left sides of Equations 1.8 is equal to the product of all of the numbers on the right sides. Therefore

$$P_1 \times P_2 \times \cdots \times P_{n-1} \times P_n = R\,P_0 \times R\,P_1 \times \cdots \times R\,P_{n-2} \times R\,P_{n-1}$$

2. The previous equation may be rearranged to

$$P_1 \times P_2 \times \cdots \times P_{n-1} \times P_n = R^n \times P_0 \times P_1 \times \cdots \times P_{n-2} \times P_{n-1}.$$

3. P_1, P_2, $\cdots$ P_{n-1} are factors on both sides of the equation and (assuming no one of them is zero) may be divided from both sides of the equation, leaving

$$P_n = R^n \, P_0$$

Because n is arbitrary and the dynamic equation is written with t, we write

$$P_t = R^t \times P_0 = P_0 \times (1+r)^t \tag{1.9}$$

as the solution to the iteration

$$P_{t+1} = R \, P_t \qquad \text{with initial value, } P_0$$

and the solution to

$$P_{t+1} - P_t = r \, P_t \qquad \text{with initial value, } P_0.$$

Explore 1.2.1 a. Suppose $r = 0$ in $P_{t+1} - P_t = rP_t$ so that $P_{t+1} - P_t = 0$. What are P_1, P_2, $\cdots$?
b. Suppose $P_0 = 0$, and $P_{t+1} - P_t = rP_t$ for $t = 0, 1, 2, \cdots$. What are P_1, P_2, $\cdots$? ■

Example 1.2.1 Suppose a human population is growing at 1% per year and initially has 1,000,000 individuals. Let P_t denote the populations size t years after the initial population of $P_0 = 1,000,000$ individuals. If one asks what the population will be in 50 years there are two options.

Option 1. At 1% per year growth, the dynamic equation would be

$$P_{t+1} - P_t = 0.01 \, P_t$$

and the corresponding iteration equation is

$$P_{t+1} = 1.01 \, P_t$$

With $P_0 = 1,000,000$, $P_1 = 1.01 \times 1,000,000 = 1,010,000$, $P_2 = 1.01 \times 1,010,000 = 1,020,100$ and so on for 50 iterations.

Option 2. Alternatively, one may write the solution

$$P_t = 1.01^t \, (1,000,000)$$

so that

$$P_{50} = 1.01^{50} \, (1,000,000) = 1,644,631$$

The algebraic form of the solution, $P_t = R^t \, P_0$, with $r > 0$ and $R > 1$ is informative and gives rise to the common description of *exponential growth* attached to some populations. If r is negative and $R = 1 + r < 1$, the solution equation $P_t = R^t \, P_0$ exhibits *exponential decay.*

Exercises for Section 1.8, Solution to the dynamic equation, $P_{t+1} - P_t = rP_t + b$.

Exercise 1.2.1 Write a solution equation for the following initial conditions and difference equations or iteration equations. In each case, compute B_{100}.

$$\begin{array}{lrcl rcl}
\text{a.} & B_0 & = & 1,000 & B_{t+1} - B_t & = & 0.2 \times B_t \\
\text{b.} & B_0 & = & 138 & B_{t+1} - B_t & = & 0.05 \times B_t \\
\text{c.} & B_0 & = & 138 & B_{t+1} - B_t & = & 0.5 \times B_t \\
\text{d.} & B_0 & = & 1,000 & B_{t+1} - B_t & = & -0.2 \times B_t \\
\text{e.} & B_0 & = & 1,000 & B_{t+1} & = & 1.2 \times B_t \\
\text{f.} & B_0 & = & 1000 & B_{t+1} - B_t & = & -0.1 \times B_t \\
\text{e.} & B_0 & = & 1,000 & B_{t+1} & = & 0.9 \times B_t
\end{array}$$

Exercise 1.2.2 The equation, $B_t - B_{t-1} = rB_{t-1}$, carries the same information as $B_{t+1} - B_t = rB_t$. The index, t, is shifted by one unit.

a. Write the first four instances of $B_t - B_{t-1} = rB_{t-1}$ using $t = 1$, $t = 2$, $t = 3$, and $t = 4$.

b. Cascade these four equations to get an expression for B_4 in terms of r and B_0.

c. Write solutions to and compute B_{40} for

$$\begin{array}{lrcl rcl}
\text{(a.)} & B_0 & = & 50 & B_t - B_{t-1} & = & 0.2 \times B_{t-1} \\
\text{(b.)} & B_0 & = & 50 & B_t - B_{t-1} & = & 0.1 \times B_{t-1} \\
\text{(c.)} & B_0 & = & 50 & B_t - B_{t-1} & = & 0.05 \times B_{t-1} \\
\text{(d.)} & B_0 & = & 50 & B_t - B_{t-1} & = & -0.1 \times B_{t-1}
\end{array}$$

Exercise 1.2.3 Suppose a population is initially of size 1,000,000 and grows at the rate of 2% per year. What will be the size of the population after 50 years?

Exercise 1.2.4 The polymerase chain reaction is a means of making multiple copies of a DNA segment from only a minute amounts of original DNA. The procedure consists of a sequence of, say 30, cycles in which each segment present at the beginning of a cycle is duplicated once; at the end of the cycle that segment and one copy is present. Introduce notation and write a difference equation with initial condition from which the amount of DNA present at the end of each cycle can be computed. Suppose you begin with 1 picogram = 0.000000000001 g of DNA. How many grams of DNA would be present after 30 cycles.

Exercise 1.2.5 Write a solution to the dynamic equation you obtained for growth of *V. natrigens* in growth medium of pH 7.85 in Exercise 1.1.5. Use your solution to compute your estimate of B_4.

Exercise 1.2.6 There is a suggestion that the world human population is growing exponentially. Table 1.4 shows the human population in billions of people for the decades 1940 - 2000.

1. Test the equation
$$P_t = 2.2 \times 1.19^t$$
against the data where t is the time index in decades after 1940 and P_t is the human population in billions.

2. What percentage increase in human population each decade does the model for the equation assume?

3. What world human population does the equation predict for the year 2050?

Table 1.4: World Human Population.

Year	Index t	Population (billions)
1940	0	2.30
1950	1	2.52
1960	2	3.02
1970	3	3.70
1980	4	4.45
1990	5	5.30
2000	6	6.06

1.3 Experimental data: Sunlight depletion below the surface of a lake or ocean.

Light extinction with increasing depth of water determines underwater plant, algae, and phytoplankton growth and thus has important biological consequences. We develop and analyze a mathematical model of light extinction below the surface of the ocean.

Figure 1.6: There is less light in the water below the shark than there is in the water above the shark.

Sunlight is the energy source of almost all life on Earth and its penetration into oceans and lakes largely determines the depths at which plant, algae and phytoplankton life can persist. This life is important to us: some 85% of all oxygen production on earth is by the phytoplankta diatoms and dinoflagellates[4].

[4]PADI Diving Encyclopedia

Explore 1.3.1 *Preliminary analysis.* Even in a very clear ocean, light decreases with depth below the surface, as illustrated in Figure 1.6. Think about how light would decrease if you were to descend into the ocean. Light that reaches you passes through the water column above you.

a. Suppose the light intensity at the ocean surface is I_0 and at depth 10 meters the light intensity is $\frac{1}{2}I_0$. What light intensity would you expect at 20 meters?

b. Draw a candidate graph of light intensity versus depth.

We are asking you to essentially follow the bold arrows of the chart in Figure 1.7. ■

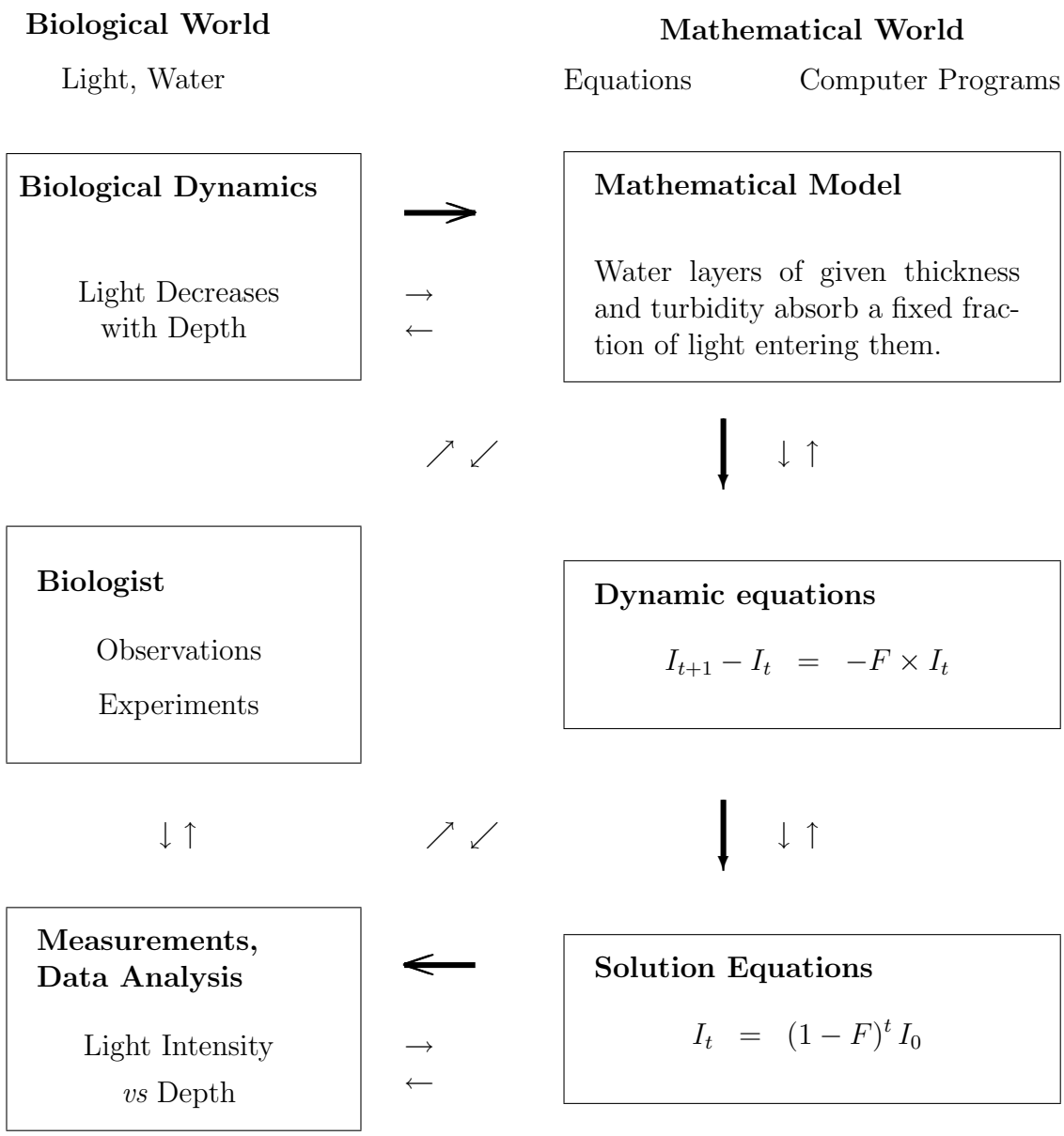


Figure 1.7: Biology - Mathematics: Information Flow Chart for the sunlight depletion example.

The steps in the analysis of light depletion are analogous to the steps for the analysis of bacterial growth in Section 1.1.

Step 1. Statement of a mathematical model. Think of the ocean as divided into layers, say one meter thick, as illustrated in Figure 1.8. As light travels downward from the surface, each layer will absorb some light. We will assume that the distribution of suspended particles in the water is uniform so that the light absorbing properties of each two layers are the same. We hypothesize that each two layers will absorb the same *fraction* of the light that enters it. The magnitude of the light absorbed will be greater in the top layers than in the lower layers simply because the intensity of the light entering the the top layers is greater than the intensity of light at the lower layers. We state the hypothesis as a mathematical model.

Mathematical Model 1.3.1 Light depletion with depth of water. Each layer absorbs a fraction of the light entering the layer from above. The *fraction of light absorbed*, f, is the same for all layers of a fixed thickness.

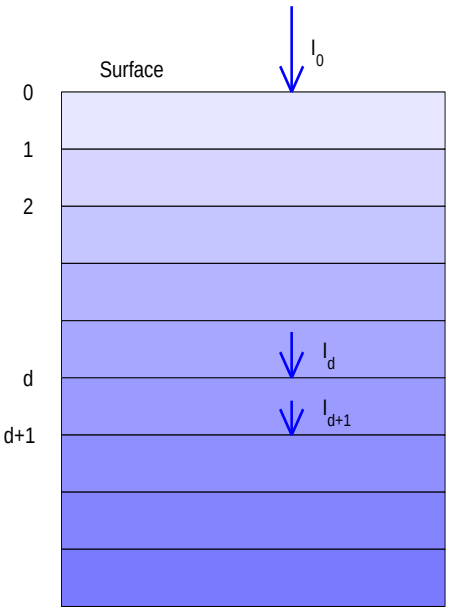


Figure 1.8: A model of the ocean partitioned into layers.

Step 2. Notation We will let d denote depth (an index measured in layers) and I_d denote light intensity at depth d. Sunlight is partly reflected by the surface, and I_0 (light intensity at depth 0) is to be the intensity of the light that penetrates the surface. I_1 is the light intensity at the bottom of the first layer, and at the top of the second layer.

The opacity of the water (Webster: opacity – 'the relative capacity of matter to obstruct the transmission of [light]') is due generally to suspended particles and is a measure of the *turbidity* of the water. In relatively clear ocean water, atomic interaction with light is largely responsible for light decay. In the bacterial experiments, the growth of the bacteria increases the turbidity of the growth serum, thus increasing the absorbance.

The fraction, f, of light absorbed by each layer is between 0 and 1 and although f is assumed to be the same for all layers, the value of f depends on the thickness of the layers and the distribution of suspended particles in the water and atomic interactions with light. Approximately,

$$f \quad = \quad \text{Layer thickness} \quad \times \quad \text{Opacity of the water.}$$

Layer thickness should be sufficiently thin that the preceding approximation yields a value of $f < 1$. Thus for high opacity of a muddy lake, layer thickness of 20 cm might be required, but for a sparkling ocean, layer thickness of 10 m might be acceptable.

Explore 1.3.2 Assume that at the surface light intensity I_0 is 400 watts/meter-squared and that each layer of thickness 2 meters absorbs 10% of the light that enters it. Calculate the light intensity at depths 2, 4, 6, $\cdots$, 20 meters. Plot your data and compare your graph with the graph you drew in Explore 1.3.1. ■

Step 3. Develop a dynamic equation representative of the model. Consider the layer between depths d and $d+1$ in Figure 1.8. The intensity of the light entering the layer is I_d.

> The light absorbed by the layer between depths d and $d+1$ is the difference between the light entering the layer at depth d and the light leaving the layer at depth $d+1$, which is $I_d - I_{d+1}$.
>
> The mathematical model asserts that
>
> $$I_d - I_{d+1} = f \times I_d \tag{1.10}$$

Note that I_d decreases as d increases so that both sides of equation 1.10 are positive. It is more common to write

$$I_{d+1} - I_d = -f \times I_d \tag{1.11}$$

and to put this equation in iteration form

$$I_{d+1} = (1-f)\, I_d \qquad\qquad I_{d+1} = F\, I_d \tag{1.12}$$

where $F = 1 - f$. Because $0 < f < 1$, also $0 < F < 1$.

Step 4. Enhance the mathematical model of Step 1. We are satisfied with the model of Step 1 (have not yet looked at any real data!), and do not need to make an adjustment.

Step 5. Solve the dynamical equation, $I_{d+1} - I_d = -f \times I_d$. The iteration form of the dynamical equation is $I_{d+1} = F\, I_d$ and is similar to the iteration equation $B_{t+1} = \frac{5}{3} B_t$ of bacterial growth for which the solution is $B_t = B_0 \left(\frac{5}{3}\right)^t$. We conclude that the solution of the light equation is

$$I_d = I_0 \times F^d \tag{1.13}$$

The solutions,

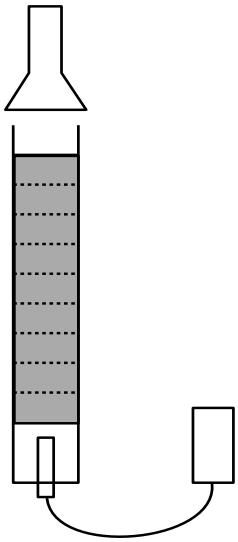
$$B_t = B_0 \left(\frac{5}{3}\right)^t \qquad \text{and} \qquad I_d = I_0 \times F^d \qquad F < 1$$

are quite different in character, however, because $\frac{5}{3} > 1$, $\left(\frac{5}{3}\right)^t$ increases with increasing t and for $F < 1$, F^d decreases with increasing d.

Step 6. Compare predictions from the model with experimental data. It is time we looked at some data. We present some data, estimate f of the model from the data, and compare values computed from Equation 1.13 with observed data.

In measuring light extinction, it is easier to bring the water into a laboratory than to make measurements in a lake, and our students have done that. Students collected water from a campus lake with substantial suspended particulate matter (that is, yucky). A vertically oriented 3 foot section of 1.5 inch diameter PVC pipe was blocked at the bottom end with a clear plastic plate and a flashlight was shined into the top of the tube (see Figure 1.9). A light detector was placed below the clear plastic at the bottom of the tube. Repeatedly, 30 cm^3 of lake water was inserted into the tube and the light intensity at the bottom of the tube was measured. One such data set is given in Figure 1.9[5].

[5]This laboratory is described in Brian A. Keller, Shedding light on the subject, *Mathematics Teacher* **91** (1998), 756-771.



Depth Layer	I_d d mW/cm^2	$I_{d+1} - I_d$ mW/cm^2
0	0.400	- 0.070
1	0.330	- 0.060
2	0.270	- 0.054
3	0.216	- 0.046
4	0.170	- 0.030
5	0.140	- 0.016
6	0.124	- 0.026
7	0.098	- 0.016
8	0.082	- 0.017
9	0.065	

Figure 1.9: Diagram of laboratory equipment and data obtained from a light decay experiment.

Explore 1.3.3 In Figure 1.10A is a graph of I_d *vs* d. Compare this graph with the two that you drew in Explore 1.3.1 and 1.3.2. ■

Our dynamical equation relates $I_{d+1} - I_d$ to I_d and we compute change in intensity, $I_{d+1} - I_d$, just as we computed changes in bacterial population. The change in intensity, $I_{d+1} - I_d$, is the amount of light absorbed by layer, d.

Graphs of the data. A graph of the original data, I_d *vs* d, is shown in Figure 1.10(A) and and a graph of $I_{d+1} - I_d$ *vs* I_d is shown in Figure 1.10(B).

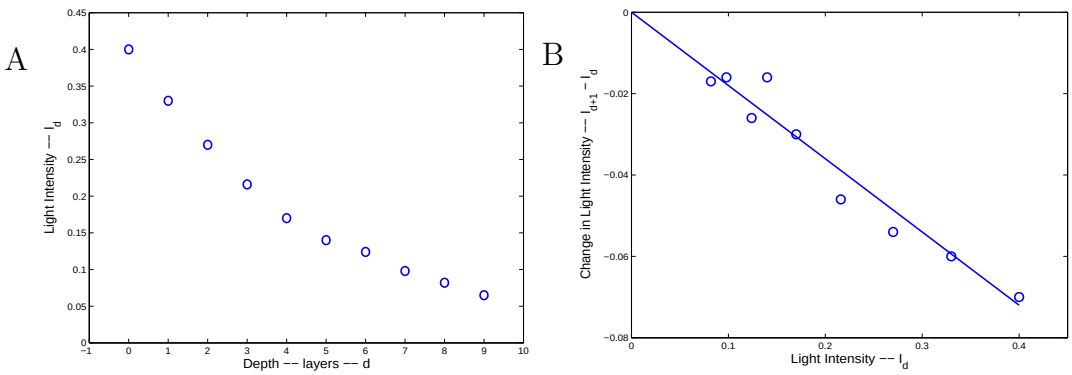


Figure 1.10: (A) The light decay curve for the data of Figure 1.9 and the graph of light absorbed by layers *vs* light entering the layer from Table 1.9.

In Figure 1.10(B) we have drawn a line close to the data that passes through (0,0). Our reason that the line should contain (0,0) is that if I_d, the amount of light entering layer d is small, then $I_{d+1} - I_d$, the amount of light absorbed by that layer is also small. Therefore, for additional layers, the data will cluster near (0,0).

The graph of $I_{d+1} - I_d$ *vs* I_d is a scattered in its upper portion, corresponding to low light intensities. There are two reasons for this.

1. Maintaining a constant light source during the experiment is difficult so that there is some error in the data.

2. *Subtraction of numbers that are almost equal emphasizes the error*, and in some cases the error can be as large as the difference you wish to measure. In the lower depths, the light values are all small and therefore nearly equal so that the error in the differences is a large percentage of the computed differences.

Use the data to develop a dynamic equation. The line goes through (0,0) and the point (0.4, -0.072). An equation of the line is

$$y = -0.18x$$

Remember that y is $I_{d+1} - I_d$ and x is I_d and substitute to get

$$I_{d+1} - I_d = -0.18\, I_d \tag{1.14}$$

This is the dynamic Equation 1.11 with $-f = -0.18$.

Solve the dynamic equation. (Step 5 for this data.) The iteration form of the dynamic equation is

$$I_{d+1} = I_d - 0.18 I_d \qquad\qquad I_{d+1} = 0.82\, I_d$$

and the solution is (with $I_0 = 0.400$)

$$I_d = 0.82^d\, I_0 = 0.400 \times 0.82^d \tag{1.15}$$

Compare predictions from the Mathematical Model with the original data. How well did we do? Again we use the equations of the model to compute values and compare them with the original data. Shown in Figure 1.11 are the original data and data computed with $\hat{I}_d = 0.400 \times 0.82^d$, and a graph comparing them.

Depth Layer	I_d Observed	$\hat{I}_d$ Computed
0	0.400	0.400
1	0.330	0.328
2	0.270	0.269
3	0.216	0.221
4	0.170	0.181
5	0.140	0.148
6	0.124	0.122
7	0.098	0.100
8	0.082	0.082
9	0.065	0.067

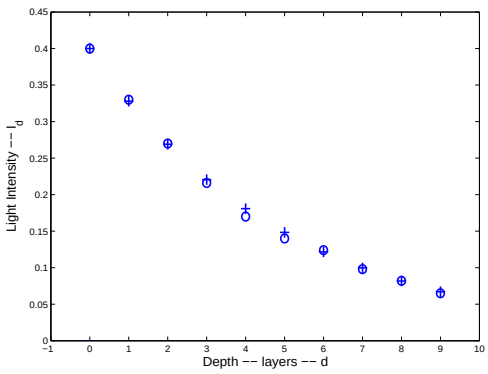


Figure 1.11: Comparison of original light intensities (o) with those computed from $\hat{I}_d = 0.82^d\, 0.400$ (+).

The computed data match the observed data quite well, despite the 'fuzziness' of the graph in Figure 1.10(b) of $I_{d+1} - I_d$ *vs* I_d from which the dynamic equation was obtained.

Exercises for Section 1.3, Experimental data: Sunlight depletion below the surface of a lake or ocean.

Exercise 1.3.1 In Table 1.5 are given four sets of data that mimic light decay with depth. For each data set, find a number r so that the values B_1, B_2, B_3, B_4, B_5 and B_6 computed from the difference equation

$$B_0 = \quad \text{as given in the table,} \qquad B_{t+1} - B_t = -r \times B_t$$

are close to the corresponding numbers in the table. Compute the numbers, B_1 to B_6 using your value of R in the equation and compare your computed numbers with the original data.

For each data set, follow steps 3, 5, and 6. The line you draw close to the data in step 3 should go through (0,0).

Table 1.5: Tables of data for Exercise 1.3.1

(a)

t	B_t
0	3.01
1	2.55
2	2.14
3	1.82
4	1.48
5	1.22
6	1.03

(b)

t	B_t
0	20.0
1	10.9
2	5.7
3	3.1
4	1.7
5	0.9
6	0.5

(c)

t	B_t
0	521
1	317
2	189
3	119
4	75
5	45
6	28

(d)

t	B_t
0	0.85
1	0.65
2	0.51
3	0.40
4	0.32
5	0.25
6	0.19

Exercise 1.3.2 Now it is your turn. Shown in the Exercise Table 1.3.2 are data from a light experiment using the laboratory procedure of this section. The only difference is the water that was used. Plot the data, compute differences and obtain a dynamic equation from the plot of differences *vs* intensities. Solve the dynamic equation and compute estimated values from the intensities and compare them with the observed light intensities. Finally, decide whether the water from your experiment is more clear or less clear than the water in Figure 1.9. (Note: Layer thickness was the same in both experiments.)

Table for Exercise 1.3.2 Data for Exercise 1.3.2.

Depth Layer	0	1	2	3	4	5	6	7	8
Light Intensity	0.842	0.639	0.459	0.348	0.263	0.202	.154	0.114	0.085

Exercise 1.3.3 One might reasonably conclude that the graph in Figure 1.10(a) looks like a parabola. Find an equation of a parabola close to the graph in Figure 1.10(a). You may wish to use P2Reg a TI-86 calculator. Compare the fit of the parabola to the data in Figure 1.10(a) with the fit of the graph of $I_d = 0.400 \times 0.82^d$ to the same data illustrated in Figure 1.11.

Why might you prefer $I_d = 0.400 \times 0.82^d$ over the equation of the parabola as an explanation of the data?

Exercise 1.3.4 A light meter suitable for underwater photography was used to measure light intensity in the ocean at Roatan, Honduras. The meter was pointed horizontally. Film speed was set at 400 ASA and time at 1/60 s. The recommended shutter apertures (*f* stop) at indicated depths are shown in Table Ex. 1.3.4. We show below that the light intensity is proportional to the square of the recommended shutter aperture. Do the data show exponential decay of light?

Notes. The quanta of light required to expose the 400 ASA film is a constant, C. The amount of light that strikes the film in one exposure is $A \times \Delta T \times I$, where A is area of shutter opening, ΔT is the time of exposure (set to be 1/60 s) and I is light intensity (quanta/(cm^2 -sec)). Therefore

$$C = A \times \Delta T \times I \qquad \text{or} \qquad I = C\frac{1}{T}\frac{1}{A} = 60C\frac{1}{A}$$

By definition of *f*-stop, for a lense of focal length, F, the diameter of the shutter opening is $F/(f\text{-stop})$ mm. Therefore

$$A = \pi\left(\frac{F}{f\text{-stop}}\right)^2$$

The last two equations yield

$$I = 60C\frac{1}{\pi}\frac{1}{F^2}(f\text{-stop})^2 = K(f\text{-stop})^2 \qquad \text{where} \qquad K = 60C\frac{1}{\pi}\frac{1}{F^2}.$$

Thus light intensity, I is proportional to $(f\text{-stop})^2$.

Table for Exercise 1.3.4 *f*-stop specifications for 400 ASA film at 1/60 s exposure at various depths below the surface of the ocean near Roatan, Honduras.

Depth m	*f*-stop
3	16
6	11
9	8
12	5.6
15	4

1.4 Doubling Time and Half-Life.

Populations whose growth can be described by an exponential function (such as $Pop = 0.022 \times 1.032^T$) have a characteristic *doubling time*, the time required for the population to double. The graph of the *V. natrigens* data and the graph of $P = 0.022 \times 1.032^T$ (Equation 1.5) that we derived from that data is shown in Figure 1.12.

Observe that at $T = 26$, $P = 0.05$ and at $T = 48$, $P = 0.1$; thus P doubled from 0.05 to 0.1 in the 22 minutes between $T = 26$ and $T = 48$. Also, at $T = 70$, $P = 0.2$ so the population also doubled from 0.1 to 0.2 between $T = 48$ and $T = 70$, which is also 22 minutes.

The doubling time, T_{Double}, can be computed as follows for exponential growth of the form

$$P = A \times B^t \qquad B > 1$$

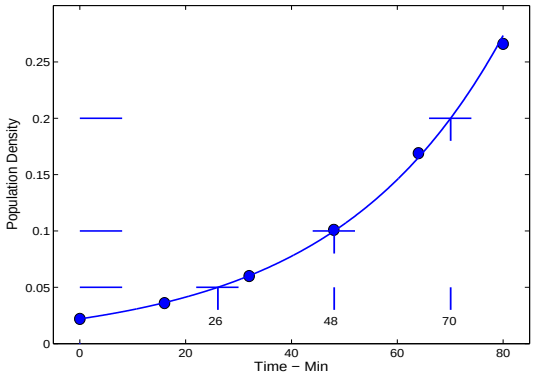


Figure 1.12: Data for *V. natrigens* growth and the graph of $P = 0.022 \times 1.032^T$ fit to the data.

Let P_1 be the population at any time T_1 and let T_2 be the time at which the population is twice P_1. The doubling time, T_{Double}, is by definition $T_2 - T_1$ and

$$P_1 = A \times B^{T_1} \qquad \text{and} \qquad P_{T_2} = 2 \times P_1 = A \times B^{T_2}$$

Therefore

$$\begin{aligned} 2 \times A \times B^{T_1} &= A \times B^{T_2} \\ 2 &= \frac{B^{T_2}}{B^{T_1}} \\ 2 &= B^{T_2 - T_1} \\ T_{\text{Double}} = T_2 - T_1 &= \frac{\log 2}{\log B} \quad \text{Doubling Time} \end{aligned} \tag{1.16}$$

Thus the doubling time for the equation $P = A\,B^T$ is $\log 2/\log B$. *The doubling time depends only on B and not on A nor on the base of the logarithm.*

For the equation, $Pop = 0.022 \times 1.032^T$, the doubling time is $\log 2/\log 1.032 = 22.0056$, as shown in the graph.

Half Life. For the exponential equation $y = A \times B^T$ with $B < 1$, y does not grow. Instead, y decreases. The half-life, T_{Half}, of y is the time it takes for y to decrease by one-half. The word, 'half-life' is also used in the context $y = A \times B^x$ where x denotes distance instead of time.

Data for light extinction below the surface of a lake from Section 1.3 on page 15 is shown in Figure 1.13 together with the graph of

$$I_d = 0.4 \times 0.82^d$$

Horizontal segments at $Light = 0.4$, $Light = 0.2$, and $Light = 0.1$ cross the curve in Figure 1.13 at $d = 0$, $d = 3.5$ and $d = 7$. The 'half-life' of the light is 3.5 layers of water.

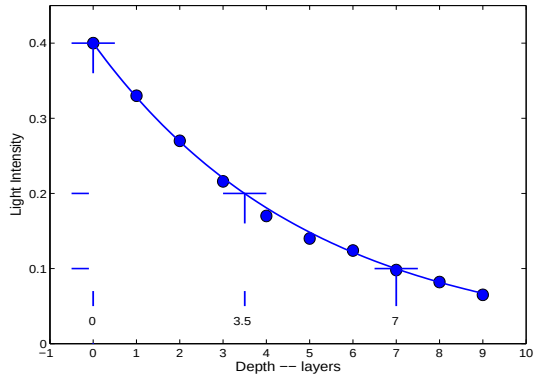


Figure 1.13: Graph of light depletion with depth and the curve $I_d = 0.4 \times 0.82^d$ fit to the data.

It might make more sense to you to call the number 3.5 the 'half-depth' of the light, but you will be understood by a wider audience if you call it 'half-life'.

In Exercise 1.4.9 you are asked to prove that the half-life, T_{Half}, of

$$y = A \times B^T, \quad \text{where} \quad B < 1 \qquad \text{is}$$

$$T_{\text{Half}} = \frac{\log \frac{1}{2}}{\log B} \qquad = \qquad \frac{-\log 2}{\log B} \tag{1.17}$$

Exercises for Section 1.4 Doubling Time and Half-Life.

Exercise 1.4.1 Determine the doubling times of the following exponential equations.

(a)	y	$=$	2^t	(b)	y	$=$	$2^{3\times t}$	(c)	y	$=$	$2^{0.1\times t}$
(d)	y	$=$	10^t	(e)	y	$=$	$10^{3\times t}$	(f)	y	$=$	$10^{0.1\times t}$

Exercise 1.4.2 Show that the doubling time of $y = A \times B^t$ is $1/\log_2 B$.

Exercise 1.4.3 Show that the doubling time of $y = A \times 2^{k\times t}$ is $1/k$.

Exercise 1.4.4 Determine the doubling times or half-lives of the following exponential equations.

(a)	y	$=$	0.5^t	(b)	y	$=$	$2^{3\times t}$	(c)	y	$=$	$0.1^{0.1\times t}$
(d)	y	$=$	100×0.8^t	(e)	y	$=$	$4 \times 5^{3\times t}$	(f)	y	$=$	$0.0001 \times 5^{0.1\times t}$
(g)	y	$=$	10×0.8^{2t}	(h)	y	$=$	$0.01^{3\times t}$	(i)	y	$=$	$0.01^{0.1\times t}$

Exercise 1.4.5 Find a formula for a population that grows exponentially and

a. Has an initial population of 50 and a doubling time of 10 years.

b. Has an initial population of 1000 and a doubling time of 50 years.

c. Has in intitial population of 1000 and a doubling time of 100 years.

Exercise 1.4.6 An investment of amount A_0 dollars that accumulates interest at a rate r compounded annually is worth

$$A_t = A_0(1+r)^t$$

dollars t years after the initial investment.

a. Find the value of A_{10} if $A_0 = 1$ and $r = 0.06$.

b. For what value of r will $A_8 = 2$ if $A_0 = 1$?

c. Investment advisers sometimes speak of the "Rule of 72", which asserts that an investment at R percent interest will double in $72/R$ years. Check the Rule of 72 for $R = 4$, $R = 6$, $R = 8$, $R = 9$ and $R = 12$..

Exercise 1.4.7 Light intensities, I_1 and I_2, are measured at depths d in meters in two lakes on two different days and found to be approximately

$$I_1 = 2 \times 2^{-0.1d} \qquad \text{and} \qquad I_2 = 4 \times 2^{-0.2d}.$$

a. What is the half-life of I_1?

b. What is the half-life of I_2?

c. Find a depth at which the two light intensities are the same.

d. Which of the two lakes is the muddiest?

Exercise 1.4.8 a. The mass of a single *V. natrigens* bacterial cell is approximately 2×10^{-11} grams. If at time 0 there are 10^8 *V. natrigens* cells in your culture, what is the mass of bacteria in your culture at time 0?

b. We found the doubling time for *V. natrigen* to be 22 minutes. Assume for simplicity that the doubling time is 30 minutes and that the bacteria continue to divide at the same rate. How may minutes will it take to have a mass of bacteria from Part a. equal one gram?

c. The earth weighs 6×10^{27} grams. How many minutes will it take to have a mass of bacteria equal to the mass of the Earth? How many hours is this? Why aren't we worried about this in the laboratory? Why hasn't this happened already in nature? Explain why it is not a good idea to extrapolate results far beyond the end-point of data gathering.

Exercise 1.4.9 Show that $y = A \times B^t$ with $B < 1$ has a half-life of

$$T_{\text{Half}} = \frac{\log \frac{1}{2}}{\log\ B} = \frac{-\log 2}{\log B}.$$

1.5 Quadratic Solution Equations: Mold growth

We examine mold growing on a solution of tea and sugar and find that models of this process lead to quadratic solution equations in contrast to the previous mathematical models which have exponential solution equations. Quadratic solution equations (equations of the form $y = at^2 + bt + c$) occur less frequently than do exponential solution equations in models of biological systems.

Shown in Figure 1.14 are some pictures taken of a mold colony growing on the surface of a mixture of tea and sugar. The pictures were taken at 10:00 each morning for 10 consecutive days. Assume that the area occupied by the mold is a reasonable measure of the size of the mold population. The grid lines are at 2mm intervals.

Explore 1.5.1 From the pictures, measure the areas of the mold for the days 2 and 6 and enter them into the table of Figure 1.15. The grid lines are at 2mm intervals, so that each square is 4mm^2. Check your additional data with points on the graph. ■

Explore 1.5.2 Do this. It is interesting and important to the remaining analysis. Find numbers A and B so that the graph of $y = A \times B^t$ approximates the mold growth data in Figure 1.14. Either chose two data points and insist that the points satisfy the equation or use a calculator or computer to compute the least squares approximation to the data. Similarly find a parabola $y = at^2 + bt + c$ that approximates the mold data. Draw graphs of the mold data, the exponential function, and the parabola that you found on a single set of axes. ■

Many people would expect the mold growth to be similar to bacterial growth and to have an *exponential* solution equation. We wish to explore the dynamics of mold growth.

Step 1. Mathematical model. Look carefully at the pictures in Figure 1.14. Observe that the features of the interior dark areas, once established, do not change. The growth is restricted to the perimeter of the colony[6]. On this basis we propose the model

Mathematical Model 1.5.1 Mold growth. Each day the increase in area of the colony is proportional to the length of the perimeter of the colony at the beginning of the day (when the photograph was taken).

Step 2. Notation. We will let t denote day of the experiment, A_t the area of the colony and C_t the length of the colony perimeter at the beginning of day t.

[6]The pictures show the mold colony from above and we are implicitly taking the area of the colony as a measure of the colony size. There could be some cell division on the underside of the colony that would not be accounted for by the area. Such was not apparent from visual inspection during growth as a clear gelatinous layer developed on the underside of the colony.

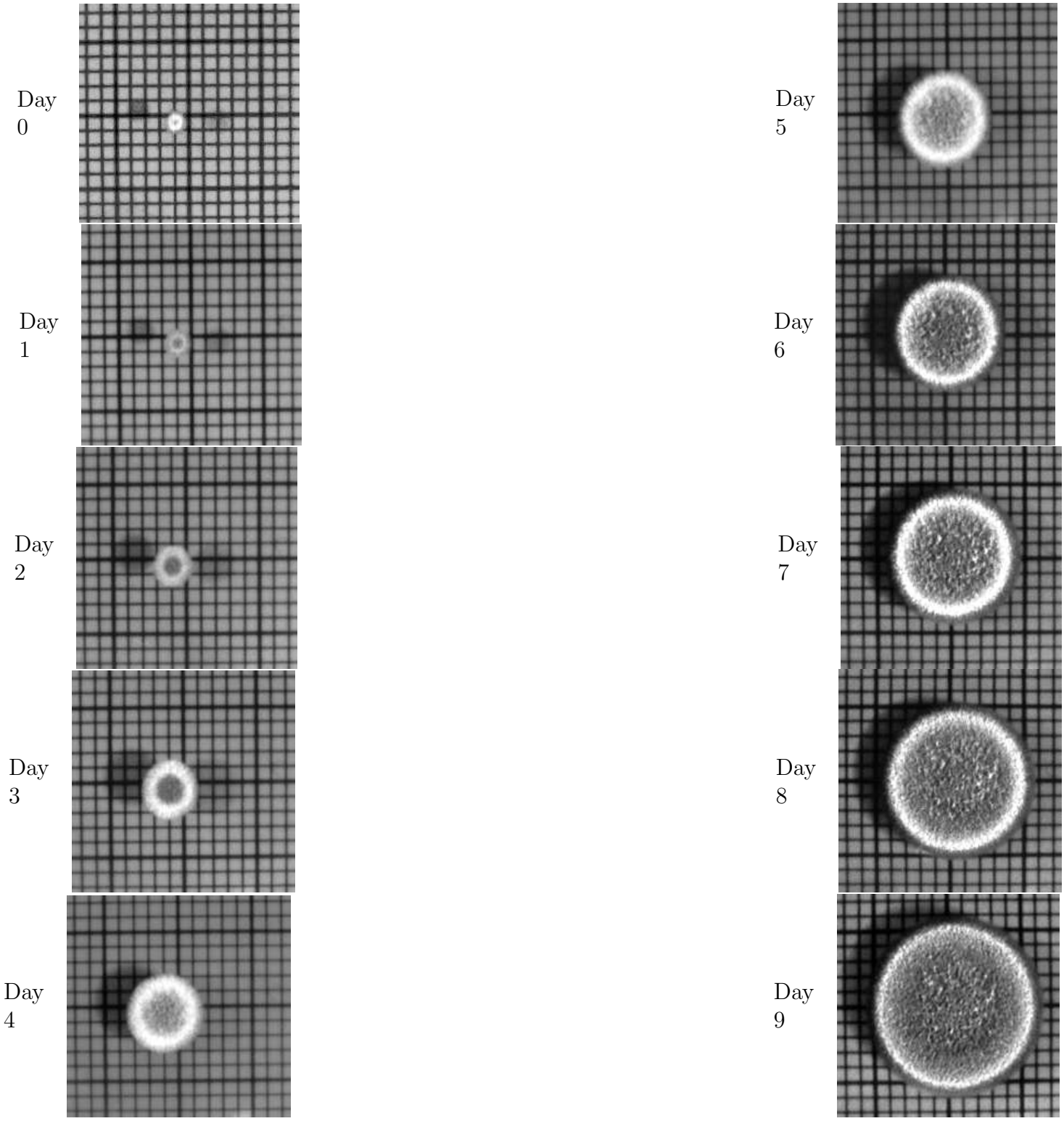


Figure 1.14: Pictures of a mold colony, taken on ten successive days. The grid lines are at 2mm intervals.

Day	Area	Day	Area
0	4	5	126
1	8	6	———
2	———	7	248
3	50	8	326
4	78	9	420

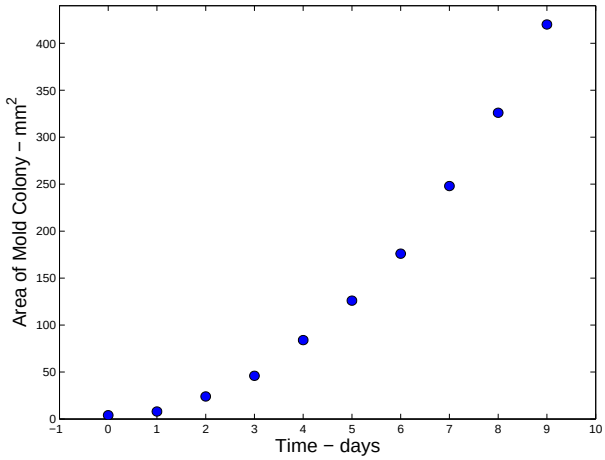


Figure 1.15: Areas of the mold colonies shown in Figure 1.14

Step 3. Dynamic equation. The statement that 'a variable A *is proportional to* a variable B' means that there is a constant, k, and

$$A = k \times B$$

Thus, 'the increase in area of the colony is proportional to the perimeter' means that there is a constant, k, such that

$$\text{increase in area} \quad = \quad k \times \text{perimeter}$$

The increase in the area of the colony on day t is $A_{t+1} - A_t$, the area at the beginning of day $t+1$ minus the area at the beginning of day t. We therefore write

$$A_{t+1} - A_t = k \times C_t$$

We now make the assumption that the mold colony is circular, so that[7]

$$C_t = 2\sqrt{\pi}\sqrt{A_t}$$

Therefore

$$A_{t+1} - A_t = k \times 2\sqrt{\pi}\sqrt{A_t} \tag{1.18}$$

Step 5. Solve the dynamic equation. The dynamic equation 1.18 is difficult to solve, and may not have a useful formula for its solution[8]. The formula, $A_t = 0$, defines a solution that is not useful. (Exercise 1.5.2).

A similar dynamic equation

$$A_{t+1} - A_{t-1} = K\sqrt{A_t} \tag{1.19}$$

has a solution

$$A_t = \frac{K^2}{16}t^2 \tag{1.20}$$

[7]For a circle of radius r,

$$\text{Area:} \quad A = \pi r^2, \quad r = \sqrt{A/\pi}, \qquad \text{Circumference:} \quad C = 2\pi r = 2\pi\sqrt{A/\pi} = 2\sqrt{\pi}\sqrt{A}$$

[8]We spare you the details, but it can be shown that there is neither a quadratic nor exponential solution to Equation 1.18

and we ask you to confirm this solution in Exercise 1.5.5. The solution $A_t = \frac{K^2}{16}t^2$ has $A_0 = 0$, which is not entirely satisfactory. (There was no mold on day 0.)

You are asked in Exercise 1.5.4 to estimate $k \times 2\sqrt{\pi}$ of Equation 1.18 and to iteratively compute approximations to $A_0, \cdots A_9$.

The difficulties at this stage lead us to reconsider the original problem.

Step 4. Reformulate the mathematical model. We have assumed the mold colony to be a circle expanding at its edges. We suggest that the radius is increasing at a constant rate, and write the following model.

Step 1. Mold growth, reformulated. Each day the radius of the colony increases by a constant amount.

Step 2. Notation, again. Let ρ_t be the radius[9] of the colony at the beginning of day t, and let Δ (Greek letter delta) denote constant daily increase in radius.

Step 3. Dynamic equation, again. The increase in radius on day t is $\rho_{t+1} - \rho_t$, the radius at the beginning of day $t+1$ minus the radius at the beginning of day t, and we write

$$\rho_{t+1} - \rho_t = \Delta \tag{1.21}$$

An important procedure in developing equations is to write a single thing two different ways, as we have just done for the increase in radius. Indeed, most equations do write a single thing two different ways.

Step 4. Solution equation, again. You will solve the equation $\rho_{t+1} - \rho_t = \Delta$ in Exercise 1.8.2 and find it to be

$$\rho_t = \rho_0 + t \times \Delta \tag{1.22}$$

Evaluate ρ_0 and Δ. From $A_t = \pi\rho_t^2$, $\rho_t = \sqrt{A_t/\pi}$, and

$$\begin{aligned} \rho_0 &= \sqrt{A_0/\pi} = \sqrt{4/\pi} = 1.13 \\ \rho_9 &= \sqrt{A_9/\pi} = \sqrt{420/\pi} = 11.56 \end{aligned}$$

There are 9 intervening days between measurements ρ_0 and ρ_9 so the average daily increase in radius is

$$\Delta = \frac{11.56 - 1.13}{9} = 1.16$$

We therefore write ($\rho_0 = 1.13$, $\Delta = 1.16$)

$$\rho_t = 1.13 + t \times 1.16$$

Now from $A_t = \pi\rho_t^2$ we write

$$A_t = \pi\left(1.13 + 1.16t\right)^2 \tag{1.23}$$

[9]We use the Greek letter ρ (rho) for the radius. We have already used R and r in another context.)

Step 6. Compare predictions from the model with observed data. A graph of $A_t = \pi\,(1.13 + 1.16t)^2$ and the original mold areas is shown in Figure 1.16 where we can see a reasonable fit, but there is a particularly close match at the two end points. It may be observed that only those two data points enter into calculations of the parameters 1.13 and 1.16 of the solution, which explains why the curve is close to them.

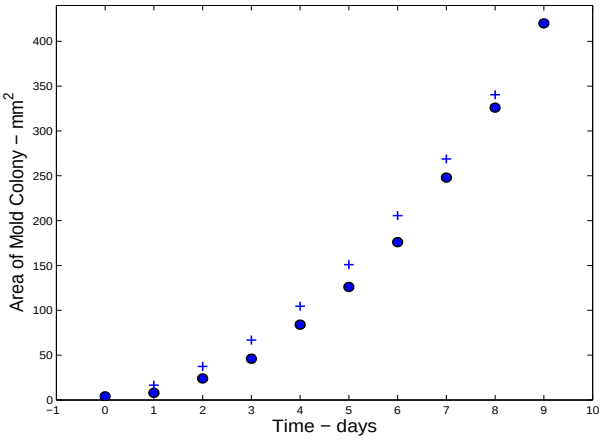


Figure 1.16: Comparison of the solution equation 1.23 with the actual mold growth data.

Exercises for Section 1.5, Quadratic solution equations: Mold growth.

Exercise 1.5.1 Do the exercise in Explore 1.5.2.

Exercise 1.5.2 Show that $A_t = 0$ for all t is a solution to Equation 1.18,

$$A_{t+1} - A_t = 2\pi k\sqrt{A_t}$$

Exercise 1.5.3 Use data at days 2 and 8, (2,24) and (8,326), to evaluate Δ and ρ_0 in Equation 1.22

$$\rho_t = \rho_0 + t \times \Delta.$$

See Step 4, Solution Equation, Again.

Use the new values of ρ_0 and Δ in $\rho_t = \rho_0 + t \times \Delta$ to compute estimates of ρ_0, ρ_1, $\cdots$, ρ_9 and A_0, A_1, $\cdots$, A_9. Plot the new estimates of A_t and the observed values of A_t and compare your graph with Figure 1.16.

Exercise 1.5.4 a. Compute $A_{t+1} - A_t$ and $\sqrt{A_t}$ for the mold data in Figure 1.15 and plot $A_{t+1} - A_t$ *vs* $\sqrt{A_t}$. You should find that the line $y = 5x$ lies close to the data. This suggests that 5 is the value of $k \times 2\sqrt{\pi}$ in Equation 1.18, $A_{t+1} - A_t = k \times 2\sqrt{\pi}\sqrt{A_t}$.
b. Use

$$A_0 = 4, \qquad A_{t+1} - A_t = 5\sqrt{A_t}$$

to compute estimates of $A_1, \cdots, A_9$. Compare the estimates with the observed data.

c. Find a value of K so that the estimates computed from

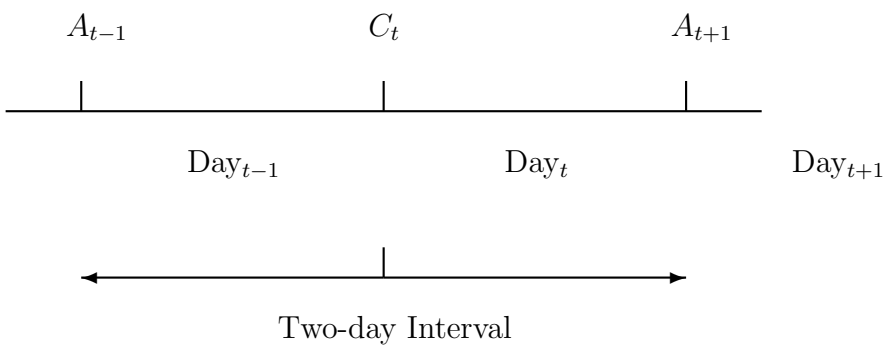
$$A_0 = 4, \qquad A_{t+1} - A_t = K\sqrt{A_t}$$

more closely approximates the observed data than do the previous approximations.

Exercise 1.5.5 Examine the following model of mold growth:

> **Model of mold growth, III.** The increase in area of the mold colony during any time interval is proportional to the length of the circumference of the colony at the midpoint of the time interval.

A schematic of a two-day time interval is



a. Explain the derivation of the dynamic equation

$$A_{t+1} - A_{t-1} = k \times C_t$$

from Model of mold growth, III.

With $C_t = 2\sqrt{\pi}\sqrt{A_t} = K\sqrt{A_t}$, the dynamic equation becomes

$$A_{t+1} - A_{t-1} = K \times \sqrt{A_t}$$

b. Show by substitution that

$$A_t = \frac{K^2}{16}t^2 \qquad \text{is a solution to} \qquad A_{t+1} - A_{t-1} = K\sqrt{A_t}$$

To do so, you will substitute $A_{t+1} = \frac{K^2}{16}(t+1)^2$, $A_{t-1} = \frac{K^2}{16}(t-1)^2$, and $A_t = \frac{K^2}{16}t^2$, and show that the left and right sides of the equation simplify to $\frac{K^2}{4}t$.

c. When you find a value for K, you will have a quadratic solution for mold growth.

In the Figure 1.17 are values of $A_{t+1} - A_{t-1}$ and $\sqrt{A_t}$ and a plot of $A_{t+1} - A_{t-1}$ *vs* $\sqrt{A_t}$, with one data point omitted. Compute the missing data and identify the corresponding point on the graph.

Time t Days	Area A_t mm^2	$\sqrt{A_t}$ mm	Change $A_{t+1} - A_{t-1}$ mm^2
0	4		
1	8	2.83	20
2	24	4.89	38
3	46	6.78	60
4	84	9.17	80
5	126		
6	176	12.00	122
7	248	13.27	150
8	326	18.06	172
9	420	20.49	

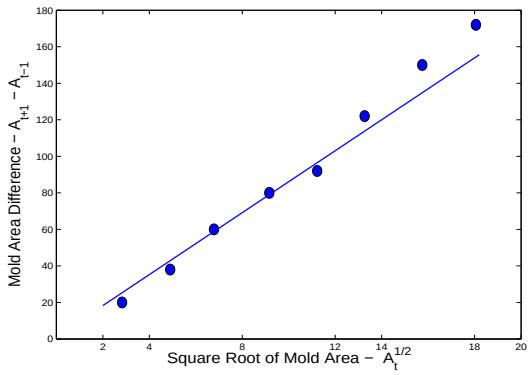


Figure 1.17: Data for Exercise 1.5.5, to estimate K in $A_{t+1} - A_{t-1} = K\sqrt{A_t}$.

d. Also shown on the graph of $A_{t+1} - A_{t-1}$ *vs* $\sqrt{A_t}$ is the graph of $y = 9.15x$ which is the line of the form $y = mx$ that is closest to the data. Let $K = 9.15$ in $A_t = \frac{K^2}{16}t^2$ and compute A_0, A_1, $\cdots$, A_9. Compare the computed values with the observed values of mold areas.

Exercise 1.5.6 Suppose a, b, and c are numbers and

$$P_t = at^2 + bt + c$$

where t is any number. Show that

$$m_t = P_{t+1} - P_t$$

is linearly related to t and that

$$a_t = m_{t+1} - m_t$$

is a constant.

1.6 Constructing a Mathematical Model of Penicillin Clearance.

In this section you will build a model of depletion of penicillin from plasma in a patient who has received a bolus injection of penicillin into the plasma.

When penicillin was first discovered, its usefulness was limited by the efficiency with which the kidney eliminates penicillin from the blood plasma passing through it. The modifications that have been made to penicillin (leading to amphicillin, moxicillin, mezlocillin, etc.) have enhanced its ability to cross membranes and reach targeted infections and reduced the rate at which the kidney clears the plasma of penicillin. Even with these improvements in penicillin, the kidneys remove 20 percent of the penicillin in the plasma passing through them. Furthermore, all of the blood plasma of a human passes through the kidneys in about 5 minutes. We therefore formulate

Mathematical Model 1.6.1 Renal clearance of penicillin. In each five minute interval following penicillin injection, the kidneys remove a fixed fraction of the penicillin that was in the plasma at the beginning of the five minute interval.

We believe the fraction to be about 0.2 so that 5 minutes after injection of penicillin into a vein, only 80% of the penicillin remains. This seems surprising and should cause you to question our assertion about renal clearance of penicillin[10].

Time min	Penicillin Concentration μg/ml
0	200
5	152
10	118
15	93
20	74

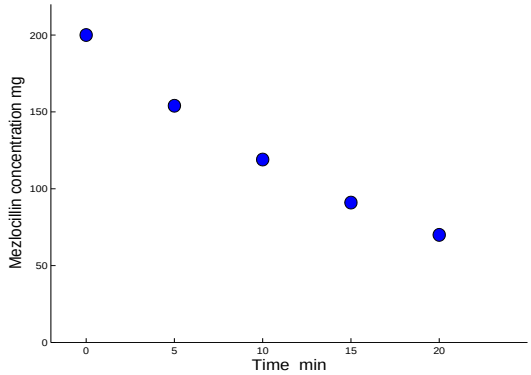


Figure 1.18: Data for serum penicillin concentrations at 5 minute intervals for the 20 minutes following a 'bolus' injection of 2 g into the serum of "five healthy young volunteers (read 'medical students')" taken from T. Bergans, Penicillins, in *Antibiotics and Chemotherapy*, Vol 25, H. Schøonfeld, Ed., S. Karger, Basel, New York, 1978. We are interpreting serum in this case to be plasma.

An amount of drug placed instantaneously (that is, over a short period of time) in a body compartment (plasma, stomach, muscle, etc.) is referred to as a *bolus injection*. An alternative procedure is *continuous infusion* as occurs with an intravenous application of a drug. Geologists refer to a meteor entering the atmosphere as a bolide. Data for penicillin concentration following a 2 g bolus injection are shown in Figure 1.18.

Explore 1.6.1 We think you will be able to develop this model without reading our development, and will find it much more interesting than reading a solution. You have a mathematical model and relevant data. You need to

Step 2. Introduce appropriate notation.

Step 3. Write a dynamic equation describing the change in penicillin concentration. You will need a crucial negative sign here, because penicillin is removed, and the change in penicillin concentration is negative. Evaluate a parameter of the dynamic equation.

[10]That all of the plasma passes through the kidney in 5 minutes is taken from Rodney A. Rhoades and George A. Tanner, Medical Physiology, Little, Brown and Company, Boston, 1995. "In resting, healthy, young adult men, renal blood flow averages about 1.2 L/min", page 426, and "The blood volume is normally 5-6 L in men and 4.5-5.5 L in women.", page 210. "Hematocrit values of the blood of health adults are $47 \pm 5\%$ for men and $42 \pm 5\%$ for women", page 210 suggests that the amount of plasma in a male is about 6 L $\times$ 0.53 = 3.18 L. J. A. Webber and W. J. Wheeler, Antimicrobial and pharmacokinetic properties, in Chemistry and Biology of β-Lactin Antibiotics, Vol. 1, Robert B. Morin and Marvin Gorman, Eds. Academic Press, New York, 1982, page 408 report plasma renal clearances of penicillin ranging from 79 to 273 ml/min. Plasma (blood minus blood cells) is approximately 53% of the blood so plasma flow through the kidney is about 6 liters $\times$ 0.53/5 min = 0.636 l/min. Clearance of 20% of the plasma yields plasma penicillin clearance of 0.636 = 0.2 = 0.127 l/min = 127 ml/min which is between 79 and 273 ml/min.

Step 4. Review the mathematical model (it will be excellent and this step can be skipped).

Step 5. Write a solution equation to the dynamic equation.

Step 6. Compare values computed with the solution equation with the observed data (you will find a very good fit).

You will find that from this data, 23 percent of the mezlocillin leaves the serum every five minutes.

■

Exercises for Section 1.6, Constructing a Mathematical Model of Penicillin Clearance.

Exercise 1.6.1 A one-liter flask contains one liter of distilled water and 2 g of salt. Repeatedly, 50 ml of solution are removed from the flask and discarded after which 50 ml of distilled water are added to the flask. Introduce notation and write a dynamic equation that will describe the change of salt in the beaker each cycle of removal and replacement. How much salt is in the beaker after 20 cycles of removal?

Exercise 1.6.2 A 500 milligram penicillin pill is swallowed and immediately enters the intestine. Every five minute period after ingestion of the pill

1. 10% of the penicillin in the intestine at the beginning of the period is absorbed into the plasma.
2. 15% of the penicillin in the plasma at the beginning of the period is removed by the kidney.

Let I_t be the amount of penicillin in the intestine and S_t be the amount of penicillin in the plasma at the end of the *tth* five minute period after ingestion of the pill. Complete the following equations, including + and - signs.

Initial conditions

$$I_0 = \underline{\qquad\qquad}$$

$$S_0 = \underline{\qquad\qquad}$$

Penicillin change per time period		Penicillin removed	Penicillin added
$I_{t+1} - I_t$	=	_____	_____
$S_{t+1} - S_t$	=	_____	_____

Exercise 1.6.3 Along with the data for 2 gm bolus injection of mezlocillin, T. Bergan reported serum mezlocillin concentrations following 1 g bolus injection in healthy volunteers and also data following 5 g injection in healthy volunteers. Data for the first twenty minutes of each experiment are shown in Table Ex. 1.6.3.

a. Analyze the data for 1 g injection as prescribed below.

b. Analyze the data for 5 g injection as prescribed below.

Analysis. Compute $P_{t+1} - P_t$ for $t = 0, 1, 2, 3$ and find a straight line passing through (0,0), ($y = mx$), close to the graph of $P_{t+1} - P_t$ *vs* P_t. Compute a solution to $P_{t+1} - P_t = mP_t$, and use your solution equation to compute estimated values of P_t. Prepare a table and graph to compare your computed solution with the observed data.

Table for Exercise 1.6.3 Plasma mezlocillin concentrations at five minute intervals following injection of either 1 g of mezlocillin or 5 g of mezlocillin into healthy volunteers.

1 g injection			5 g injection		
Time min	Time Index	Mezlocillin concentration μg/ml	Time min	Time Index	Mezlocillin concentration μg/ml
0	0	71	0	0	490
5	1	56	5	1	390
10	2	45	10	2	295
15	3	33	15	3	232
20	4	25	20	4	182

Exercise 1.6.4 Salt water with 15 grams of salt per liter is in a small container and separated from a large reservoir of fresh water by a membrane that is permeable to salt Na^+ and Cl^- ions. Because of osmosis, the salt ions move from the small container to the fresh water reservoir that is so large that effectively the reservoir concentration of salt will remain zero.

Write a mathematical model of the transfer of salt from the small container to the large container. Introduce notation and write an initial condition and a difference equation with that will describe the course of salt concentration in the small container.

Exercise 1.6.5 There is a standard osmosis experiment in biology laboratory as follows.

> Material: A thistle tube, a 1 liter flask, some 'salt water', and some pure water, a membrane that is impermeable to the salt and is permeable to the water.
>
> The bulb of the thistle tube is filled with salt water, the membrane is placed across the open part of the bulb, and the bulb is inverted in a flask of pure water so that the top of the pure water is at the juncture of the bulb with the stem.
>
> Because of osmotic pressure the pure water will cross the membrane pushing water up the stem of the thistle tube until the increase in pressure inside the bulb due to the water in the stem matches the osmotic pressure across the membrane.

Our problem is to describe the height of the water in the stem as a function of time. The following mathematical model would be appropriate.

Mathematical Model. The amount of water crossing the membrane during any minute is proportional to the osmotic pressure across the membrane minus the pressure due to the water in the stem at the beginning of the minute.

Assume that the volume of the bulb is much larger than the volume of the stem so that the concentration of 'salt' in the thistle tube may be assumed to be constant, thus making the osmotic pressure a constant, P_0 (This is fortunate!). Assume that the radius of the thistle tube stem is r cm. Then an amount, V cm^3, of additional water inside the thistle tube will cause the water to rise $V/(\pi r^2)$ cm. Assume the density of the salt water to be δ gm/cm.

Introduce notation and write a difference equation with initial condition that will describe the height of the water in the stem as a function of time.

1.7 Movement toward equilibrium.

Now we consider a new mathematical model that is used to describe the response of systems to constant infusion of material or energy. Examples include

- A pristine lake has a constant flow of fresh water into it and an equal flow of water out of the lake. A factory is built next to the lake and each day releases a fixed amount of chemical waste into the lake. The chemical waste will mix through out the lake and some will leave the lake in the water flowing out. The amount of chemical waste in the lake will increase until the amount of chemical leaving the lake each day is the same as the amount released by the factory each day.

- The nitrogen in the muscle of a scuba diver is initially at 0.8 atm, the partial pressure of N_2 in atmospheric air. She descends to 20 meters and breathes air with N_2 partial pressure 2.4 atm. Almost immediately her blood N_2 partial pressure is also 2.4 atm. Her muscle absorbs N_2 more slowly; each minute the amount of N_2 that flows into her muscle is proportional to the difference between the partial pressures of N_2 in her blood (2.4 atm) and in her muscle. Gradually her muscle N_2 partial pressure moves toward 2.4 atm.

- A hen leaves a nest and exposes her eggs to air at temperatures that are lower than the 37 °C of the eggs when she left. The temperature of the eggs will decrease toward the temperature of the air.

Example 1.7.1 *Chemical pollution in a lake.* A pristine lake of area 2 km^2 and average depth of 10 meters has a river flowing through it at a rate of 10,000 m^3 per day. A factory is built beside the river and releases 100 kg of chemical waste into the lake each day. What will be the amounts of chemical waste in the lake on succeeding days?

We propose the following mathematical model.

Step 1. Mathematical Model. The daily change in chemical waste in the lake is difference between the amount released each day by the factory and the amount that flows out of the lake down the exit river. The amount of chemical waste that leaves the lake each day is equal to the amount of water that leaves the lake that day times the concentration of chemical waste in that water. Assume that upon release from the factory, the chemical quickly mixes throughout the lake so that the chemical concentration in the lake is uniform.

Step 2. Notation. Let t be time measured in days after the factory opens and W_t be the chemical waste in kg and C_t the concentration of chemical waste in kg/m^3 in the lake on day t. Let V be the volume of the lake and F the flow of water through the lake each day.

Step 3. Equations. The lake volume its area times its depth; according to the given data,

$$V = 2\text{km}^2 \times 10\text{m} = 2 \times 10^7 \text{m}^3.$$

$$F = 10^4 \frac{\text{m}^3}{\text{day}}$$

The concentration of chemical in the lake is $\quad C_t = \dfrac{W_t}{V} = \dfrac{W_t}{2 \times 10^7} \dfrac{\text{kg}}{\text{m}^3}$

The change in the amount of chemical on day t is $W_{t+1} - W_t$ and

$$\begin{array}{rcccc} \text{Change per Day} & = & \text{Amount Added per Day} & - & \text{Amount Removed per Day} \\ W_{t+1} - W_t & = & 100 & - & F \times C_t \\ W_{t+1} - W_t & = & 100 & - & 10^4 \times \dfrac{W_t}{2 \times 10^7} \\ W_{t+1} - W_t & = & 100 & - & 5 \times 10^{-4} W_t \end{array}$$

The units in the equation are

$$\begin{array}{rcccc} W_{t+1} - W_t & = & 100 & - & 10^4 \dfrac{W_t}{2 \times 10^7} \\ \text{kg} - \text{kg} & = & \text{kg} & - & \text{m}^3 \dfrac{\text{kg}}{\text{m}^3} \end{array}$$

and they are consistent.

On day 0, the chemical content of the lake is 0. Thus we have

$$\begin{aligned} W_0 &= 0 \\ W_{t+1} - W_t &= 100 - 5 \times 10^{-4} W_t \end{aligned}$$

and we rewrite it as

$$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 100 + 0.9995 W_t \end{aligned}$$

Step 5. Solve the dynamic equation. We can compute the amounts of chemical waste in the lake on the first few days[11] and find 0, 100, 199.95, 299.85, 399.70 for the first five entries. We could iterate 365 times to find out what the chemical level will be at the end of one year (but would likely loose count).

Equilibrium State. The environmentalists want to know the 'eventual state' of the chemical waste in the lake. They would predict that the chemical in the lake will increase until there is no perceptual change on successive days. The *equilibrium state* is a number E such that if $W_t = E$, W_{t+1} is also E. From $W_{t+1} = 100 + 0.9995$ we write

$$E = 100 + 0.9995E, \qquad E = \frac{100}{1 - 0.9995} = 200000.$$

[11] On your calculator: 0 ENTER × 0.9995 + 100 ENTER ENTER ENTER ENTER

When the chemical in the lake reaches 200000 kg, the amount that flows out of the lake each day will equal the amount introduced from the factory each day.

The equilibrium E is also useful mathematically. Subtract the equations

$$\begin{aligned} W_{t+1} &= 100 + 0.9995 \\ E &= 100 + 0.9995E \end{aligned}$$

$$W_{t+1} - E = 0.9995(W_t - E)$$

With $D_t = W_t - E$, this equation is

$$D_t{+}1 = 0.9995D_t \qquad \text{which has the solution} \qquad D_t = D_0 \times 0.9995^t.$$

Then

$$W_t - E = (W_0 - E) \times 0.9995^t, \qquad W_t = 200000 - 200000 \times 0.9995^t$$

The graph of W_t is shown in Figure 1.19, and W_t is asymptotic to 200000.

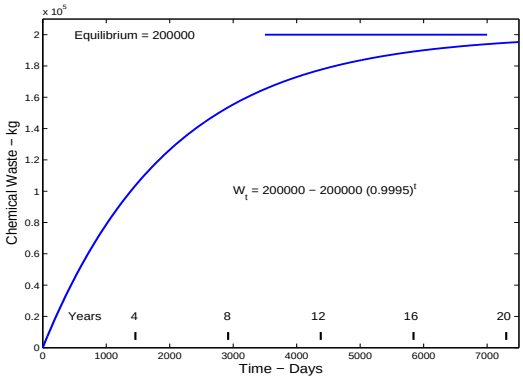


Figure 1.19: The amount of waste chemical in a lake.

You can read the chemical level of the lake at the end of one year from the graph or compute

$$W_{365} = 200000 - 200000 \times 0.9995^{365} \doteq 33,000kg$$

33,000 of the 36,500 kg of chemical released into the lake during the first year are still in the lake at the end of the year. Observe that even after 20 years the lake is not quite to equilibrium.

We can find out how long it takes for the lake to reach 98 percent of the equilibrium value by asking for what t is $W_t = 0.98 \times 200000$?. Thus,

$$\begin{aligned} W_t = 0.98 \times 200000 &= 200000 - 200000 \times 0.9995^t \\ 0.98 &= 1 - 0.9995^t \\ 0.9995^t &= 0.02 \\ \log 0.9995^t &= \log 0.02 \\ t \log 0.9995 &= \log 0.02 \\ t &= \frac{\log 0.02}{\log 0.9995} = 7822 \quad \text{days} = 21.4 \quad \text{years} \end{aligned}$$

Step 6. Compare the solution with data. Unfortunately we do not have data for this model. The volume and stream flow were selected to approximate Lake Erie, but the lake is much more complex than our simple model. However, a simulation of the process is simple.

Example 1.7.2 Simulation of chemical discharge into a lake. Begin with two one-liter beakers, a supply of distilled water and salt and a meter to measure conductivity in water. Place one liter of distilled water and 0.5 g of salt in beaker F (factory). Place one liter of distilled water in beaker L (lake). Repeatedly do

a. Measure and record the conductivity of the water beaker L.

b. Remove 100 ml of solution from beaker L and discard.

c. Transfer 100 ml of salt water from beaker F to beaker L.

The conductivity of the salt water in beaker F should be about 1000 microsiemens (μS). The conductivity of the water in beaker L should be initially 0 and increase as the concentration of salt in L increases. Data and a graph of the data are shown in Figure 1.20 and appears similar to the graph in Figure 1.19. In Exercise 1.7.4 you are asked to write and solve a mathematical model of this simulation and and compare the solution with the data. ■

Cycle Number	Conductivity μS
0	0
1	112
2	207
3	289
4	358
5	426
6	482
7	526
8	581

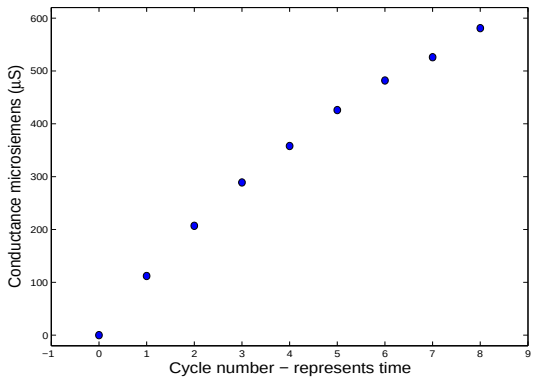


Figure 1.20: Data for Example 1.7.2, simulation of infusion of chemical waste product in a lake.

Exercises for Section 1.7 Movement toward equilibrium.

Exercise 1.7.1 For each of the following systems,

1. Compute W_0, W_1, W_2, W_3, and W_4.
2. Find the equilibrium value of W_t for the systems.
3. Write a solution equation for the system.
4. Compute W_{100}.
5. Compute the half-life, $T_{1/2} = -\log 2/\log B$ of the system.

a. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 1 + 0.2W_t \end{aligned}$$

b. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 + 0.2W_t \end{aligned}$$

c. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 100 + 0.2W_t \end{aligned}$$

d. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 + 0.1W_t \end{aligned}$$

e. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 + 0.05W_t \end{aligned}$$

f. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 + 0.01W_t \end{aligned}$$

Exercise 1.7.2 For each of the following systems,

1. Compute W_0, W_1, W_2, W_3, and W_4.
2. Find the equilibrium value of W_t for the systems.
3. Write a solution equation for the system.

4. Compute W_{100}.

a. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 1 - 0.2W_t \end{aligned}$$

b. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 - 0.2W_t \end{aligned}$$

c. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 100 - 0.2W_t \end{aligned}$$

d. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 - 0.1W_t \end{aligned}$$

e. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 - 0.05W_t \end{aligned}$$

f. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 10 - 0.01W_t \end{aligned}$$

Exercise 1.7.3 For each of the following systems,

a. Compute W_0, W_1, W_2, W_3, and W_4.

b. Describe the future terms, W_5, W_6, W_7, $\cdots$.

a. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 1 - W_t \end{aligned}$$

b. $$\begin{aligned} W_0 &= \tfrac{1}{2} \\ W_{t+1} &= 1 - W_t \end{aligned}$$

c. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 1 + W_t \end{aligned}$$

d. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 2 + W_t \end{aligned}$$

e. $$\begin{aligned} W_0 &= 0 \\ W_{t+1} &= 1 + 2W_t \end{aligned}$$

f. $$\begin{aligned} W_0 &= -1 \\ W_{t+1} &= 1 + 2W_t \end{aligned}$$

Exercise 1.7.4 Write equations and solve them to describe the amount of salt in the beakers at the beginning of each cycle for the simulation of chemical discharge into a lake of Example 1.7.2.

Exercise 1.7.5 For our model of lake pollution, we assume "that upon release from the factory, the chemical quickly mixes throughout the lake so that the chemical concentration in the lake is uniform." The time scale for 'quickly' is relative to the other parts of the model; in this case to the daily flow into and out of the lake. Suppose it takes 10 days for 100 kg of chemical released from the factory to mix uniformly throughout the lake. Write a mathematical model for this case. There are several reasonable models; your task to write one of them.

Exercise 1.7.6 An intravenous infusion of penicillin is initiated into the vascular pool of a patient at the rate of 10 mg penicillin every five minutes. The patients kidneys remove 20 percent of the penicillin in the vascular pool every five minutes.

a. Write a mathematical model of the change during each five minute period of penicillin in the patient.

b. Write a difference equation that describes the amount of penicillin in the patient during the five minute intervals.

c. What is the initial value of penicillin in the patient?

d. What will be the equilibrium amount of penicillin in the patient? (This is important to the nurse and the doctor!)

e. Write a solution to the difference equation.

f. At what time will the penicillin amount in the patient reach 90 percent of the equilibrium value? (The nurse and doctor also care about this. Why?)

g. Suppose the patients kidneys are weak and only remove 10 percent of the penicillin in the vascular pool every 5 minutes. What is the equilibrium amount of penicillin in the patient?

Exercise 1.7.7 The nitrogen partial pressure in a muscle of a scuba diver is initially 0.8 atm. She descends to 30 meters and immediately the N_2 partial pressure in her blood is 2.4 atm, and remains at 2.4 atm while she remains at 30 meters. Each minute the N_2 partial pressure in her muscle increases by an amount that is proportional to the difference in 2.4 and the partial pressure of nitrogen in her muscle at the beginning of that minute.

a. Write a dynamic equation with initial condition to describe the N_2 partial pressure in her muscle.

b. Your dynamic equation should have a proportionality constant. Assume that constant to be 0.07. Write a solution to your dynamic equation.

c. At what times will the N_2 partial pressure be 1.2 and 1.8?

d. What is the half-life of N_2 partial pressure in the muscle?

1.8 Solution to the dynamic equation $P_{t+1} - P_t = r\,P_t + b$.

The general form of the dynamic equation of the previous section is,

$$P_{t+1} - P_t = r\,P_t + b, \qquad t = 0, 1, \cdots, \qquad P_0 \quad \text{a known value.} \tag{1.24}$$

The constant b represents influence from outside the system, such as flow or immigration into ($b > 0$) or flow or emigration out of ($b < 0$) the system.

Suppose $r \neq 0$ and $b \neq 0$. We convert

$$P_{t+1} - P_t = r\,P_t + b$$

to iteration form

$$P_{t+1} = R\,P_t + b, \qquad \text{where} \qquad R = r + 1. \tag{1.25}$$

Next we look for an **equilibrium value** E satisfying

$$E = R \times E + b. \tag{1.26}$$

(Think, if $P_t = E$ then $P_{t+1} = E$.) We can solve for E

$$E = \frac{b}{1 - R} = -\frac{b}{r} \tag{1.27}$$

Now subtract Equation 1.26 from Equation 1.25.

$$\begin{aligned} P_{t+1} &= R\,P_t + b \\ E &= R\,E + b \end{aligned}$$

$$P_{t+1} - E = R\,(P_t - E). \tag{1.28}$$

Let $D_t = P_t - E$ and write

$$D_{t+1} = R\,D_t.$$

This is iteration Equation 1.7 with D_T instead of P_t and the solution is

$$D_t = R^t\,D_0.$$

Remembering that $D_t = P_t - E$ and $E = b/(1-R) = -b/r$ and $R = 1 + r$ we write

$$P_t - E = R^t\,(P_0 - E), \qquad P_t = -\frac{b}{r} + (P_0 + \frac{b}{r})(1+r)^t. \tag{1.29}$$

You are asked to show in Exercise 1.8.2 that the solution to

$$P_{t+1} - P_t = b \qquad \text{is} \qquad P_t = P_0 + t \times b \tag{1.30}$$

Exercises for Section 1.8, Solution to the dynamic equation, $P_{t+1} - P_t = rP_t + b$.

Exercise 1.8.1 Write a solution equation for the following initial conditions and difference equations or iteration equations. In each case, compute B_{100}.

a.	B_0	$=$	100	$B_{t+1} - B_t$	$=$	$0.2 \times B_t + 5$
b.	B_0	$=$	138	$B_{t+1} - B_t$	$=$	$0.05 \times B_t + 10$
c.	B_0	$=$	138	$B_{t+1} - B_t$	$=$	$0.5 \times B_t - 10$
d.	B_0	$=$	100	$B_{t+1} - B_t$	$=$	10
e.	B_0	$=$	100	B_{t+1}	$=$	$1.2 \times B_t - 5$
f.	B_0	$=$	100	$B_{t+1} - B_t$	$=$	$-0.1 \times B_t + 10$
e.	B_0	$=$	100	B_{t+1}	$=$	$0.9 \times B_t - 10$

Exercise 1.8.2 Equation 1.30

$$P_{t+1} - P_t = b$$

represents a large number of equations

$$\begin{aligned} P_1 - P_0 &= b \\ P_2 - P_1 &= b \\ P_3 - P_2 &= b \\ \vdots \quad \vdots \quad &\vdots \quad \vdots \\ P_{n-1} - P_{n-2} &= b \\ P_n - P_{n-1} &= b \end{aligned}$$

Add these equations to obtain

$$P_n = P_0 + n \times b$$

Substitute t for n to obtain

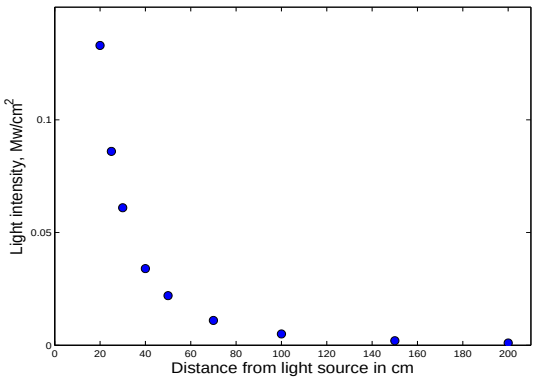
$$P_t = P_0 + t \times b$$

Exercise 1.8.3 Suppose a quail population would grow at 20% per year without hunting pressure, and 1000 birds per year are harvested. Describe the progress of the population over 5 years if initially there are
a. 5000 birds, b. 6000 birds, and c. 4000 birds.

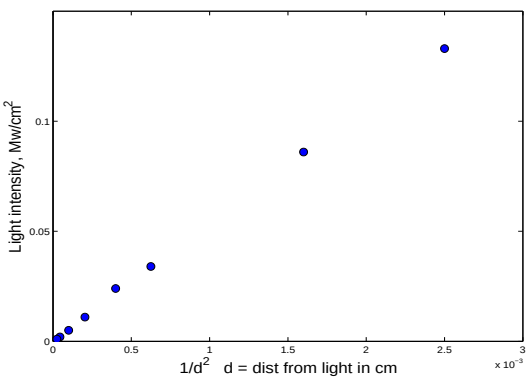
1.9 Light Decay with Distance.

Example 1.9.1 Some students measured the intensity of light from a 12 volt light bulb with 0.5 amp current at varying distances from the light bulb. The light decreases as the distance from the bulb increases, as shown in Figure 1.21A.

A



B



Distance cm	20	25	30	40	50	70	100	150	200
Light Intensity mW/cm^2	0.133	0.086	0.061	0.034	0.024	0.011	0.005	0.002	0.001

Figure 1.21: A. Light intensity *vs* distance from light source. B. Light intensity *vs* the reciprocal of the square of the distance from the light source. The light intensity is measured with a Texas Instruments Calculator Based Laboratory (CBL) which is calibrated in milliwatts/cm^2 = mW/cm^2.

If the data are transformed as shown in Figure 1.21B then they become very interesting. For each point, (d, I) of Figure 1.21A, the point $(1/d^2, I)$ is plotted in Figure 1.21B. *Light intensity vs* the *reciprocal of the square of the distance from the light source* is shown. It looks pretty linear.

Explore 1.9.1 a. One data point is omitted from Figure 1.21B. Compute the entry and plot the corresponding point on the graph. Note the factor $\times 10^{-3}$ on the horizontal scale. The point 2.5 on the horizontal scale is actually $2.5 \times 10^{-3} = 2.5 \times 0.001 = 0.0025$.

b. Explain why it would be reasonable for some points on the graph to be close to (0,0) in Figure 1.21B.

c. Is (0,0) a possible data point for Figure 1.21B?

■

It appears that we have found an interesting relation between light intensity and distance from a light source. In the rest of this section we will examine the geometry of light emission and see a fundamental reason suggesting that this relation would be expected. ■

In Section 1.3 we saw that sunlight decayed exponentially with depth in the ocean. In Example 1.9.1 we saw that light decreased proportional to the reciprocal of the square of the distance from the light source. These two cases are distinguished by the model of light decrease, and the crux of the problem is the definition of light intensity, and how the geometry of the two cases affects light transmission, and the medium through which the light travels.

> **Light Intensity.** Light intensity from a certain direction is defined to be the number of photons per second crossing a square meter region that is perpendicular to the chosen direction.

The numerical value of light intensity as just defined is generally large – outside our range of experience – and may be divided by a similarly large number to yield a measurement more practical to use. For example, some scientists divide by Avagadro's number, 6.023×10^{23}. A more standard way is to convert photons to energy and photons per second to watts and express light intensity in watts per meter squared.

Light rays emanating from a point source expand radially from the source. Because of the distance between the sun and Earth, sun light strikes the Earth in essentially parallel rays and light intensity remains constant along the light path, except for the interference from substance along the path.

When light is measured 'close to' a point source, however, the light rays expand and light intensity decreases as the observer moves away from the source. Consider a cone with vertex V at a point source of light. The number of photons per second traveling outward within the solid is cone constant. As can be seen in Figure 1.22, however, because the areas of surfaces stretching across the cone expand as the distance from the source increases, the density of photons striking the surfaces decreases.

Mathematical Model 1.9.1 Light Expansion: For light emanating uniformly from a point source and traveling in a solid cone, the number of photons per second crossing the surface of a sphere with center at the vertex of the cone is a constant, N, that is, independent of the radius of the sphere.

Because light intensity is 'number of photons per second per square meter of surface' we have

$$N = I_d \times A_d \tag{1.31}$$

where

N	is photons per second in a solid cone emanating from a light source at the vertex of the cone.
d	is distance from the light source.
I_d	is light intensity at distance d.
A_d	is the area of the intersection with the cone of the surface of a sphere of radius d and center at the vertex of the cone.

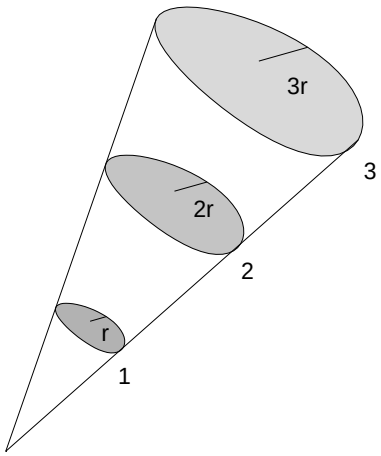


Figure 1.22: Light expansion in a cone.

Illustrated in Figure 1.23 is a cone with vertex V, a sphere with center V and radius d and the portion of the surface of the sphere that lies within the cone. If the vertex angle is α then

$$A_d = \quad \text{Area} = 2\pi\,(1 - \cos\alpha) \times d^2$$

Therefore in $N = I_d \times A_d$

$$N = I_d \times 2\pi\,(1 - \cos\alpha) \times d^2$$

or

$$I_d = K \times \frac{1}{d^2}$$

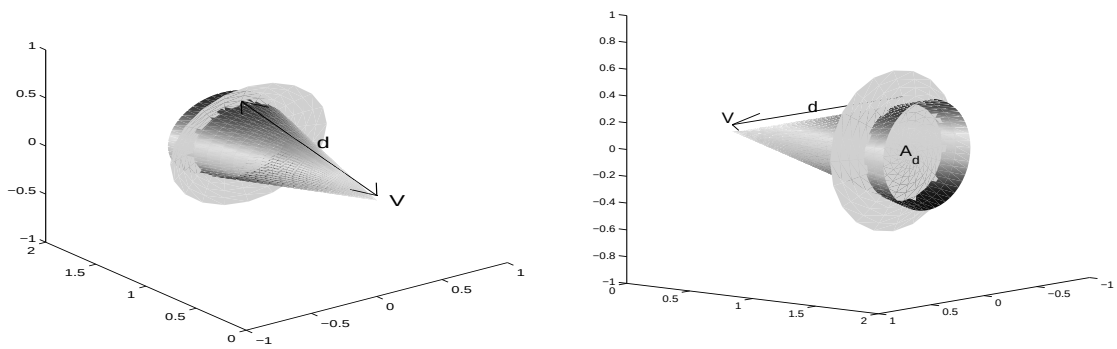


Figure 1.23: Two views of a cone and section of a sphere with center at the vertex of the cone.

Thus it is not a surprise that in Example 1.9.1, the intensity of light emanating from the 12 volt, 0.5 amp bulb was proportional to the reciprocal of the square of the light intensity.

For light emanating from a point source, the light intensity on the surface of a sphere with center at the point source and radius d is a constant, that is, the light intensity is the same at any two points of the sphere.

Exercise 1.9.1 Shown in Table 1.9.1 is data showing how light intensity from a linear light source decreases as distance from the light increases. Card board was taped on the only window in a room so that a one centimeter wide vertical slit of length 118 cm was left open. For this experiment, we considered 118 cm slit to be an infinitely long line of light. Other lights in the room were cut off. A light meter was pointed horizontally towards the center of the slit, 56 cm above the bottom of the slit, and light intensity was measured as the light meter was moved away from the window.

a. This experiment is easy to replicate. Try and use your own data for the next two parts. Best to have either a clear day or an overcast day; constant sunlight is needed.

b. Light emitted from a point surface is constant on surfaces of spheres with center at the light source. On which surfaces would you expect the intensity of light emitted from a linear source be constant?

c. (You may wish to do the next part before this one.) Explore the data and see whether you can find a relation between light intensity and some aspect of distance.

d. Formulate a model of how light decays with distance from a slit light source and relate your model to your observed relation.

Table for Exercise 1.9.1 Light intensity from a 1 cm strip of light measured at distances perpendicular to the strip.

Distance (cm)	3	4	5	6	7	8	9
Light Intensity mW/cm^2	0.474	0.369	0.295	0.241	0.208	0.180	0.159
Distance (cm)	10	11	12	13	14	15	16
Light Intensity mW/cm^2	0.146	0.132	0.120	0.111	0.100	0.092	0.086

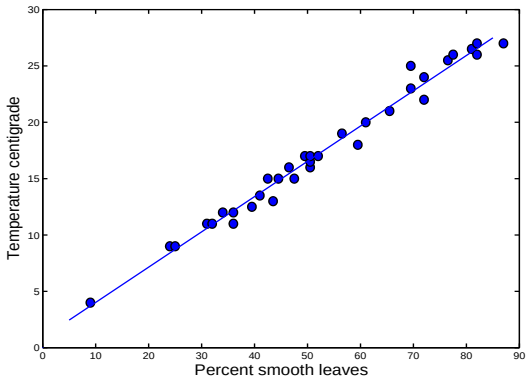
1.10 Data modeling *vs* mathematical models.

In Example 1.9.1 of light intensity from a 12 volt bulb we modeled data and only subsequently built a mathematical model of light decrease from a point source of light. There are other examples of modeling of data that are interesting but there is very little possibility of building a mathematical model to explain the process.

Example 1.10.1 Jack A. Wolfe[12] observed that leaves of trees growing in cold climates tend to be incised (have ragged edges) and leaves of trees growing in warm climates tend to have smooth edges (lacking lobes or teeth). He measured the percentages of species that have smooth margins among all species of the flora in many locations in eastern Asia. His data, as read from a graph in U. S. Geological Survey Professional Paper 1106, is presented in Figure 1.10.0.2.

[12] Jack A. Wolfe, A paleobotanical interpretation of tertiary climates in northern hemisphere, *American Scientist* **66** (1979), 694-703

Figure for Example 1.10.0.2 Average temperature C° *vs* percentage of tree species with smooth edge leaves in 33 forests in eastern Asia. The equation of the line is $y = -0.89 + 0.313x$.



The line, temp $= 0.89 + 0.313 \times \%$ smooth is shown in Figure 1.10.0.2 and is close to the data. The line was used by Wolfe to estimate temperatures over the last 65 million years based on observed fossil leaf composition (Exercise 1.10.1). The prospects of writing a mathematical model describing the relationship of smooth edge leaves to temperature are slim, however. ■

Example 1.10.1 On several nights during August and September in Ames, Iowa, some students listened to crickets chirping. They counted the number of chirps in a minute (chirp rate, R) and also recorded the air (ambient) temperature (T) in F° for the night. The data were collected between 9:30 and 10:00 pm each night, and are shown in the table and graph in Figure 1.24.

Temperature °F T	Chirps per Minute R
67	109
73	136
78	160
61	87
66	103
66	102
67	108
77	154
74	144
76	150

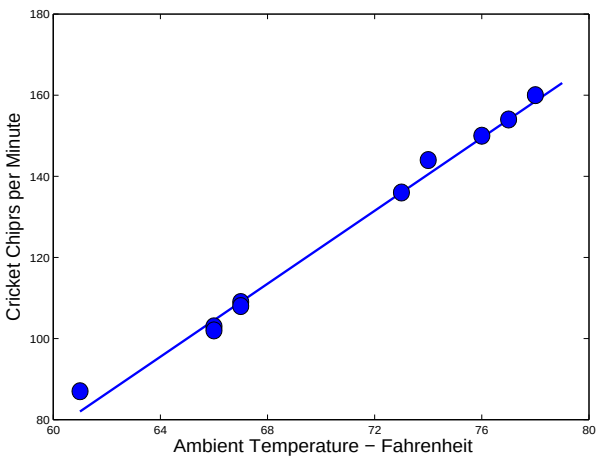


Figure 1.24: A table and graph of the frequency of cricket chirps *vs* temperature in degrees F°.

These data also appear linear and the line through (65,100) and (75,145),

$$\frac{R-100}{T-65} = \frac{145-100}{75-65}, \qquad R = 4.5T - 192.5 \tag{1.32}$$

lies close to the data. We can use the line to estimate temperature to be about 69.5 F° if cricket chirp rate is 120 chirps/minute.

Both of these examples are examples of data modeling. We fit a line to the data, but there is no underlying explanation of what mechanism is causing the relation.

Exercises for Section 1.10, Data modeling *vs* mathematical models.

Exercise 1.10.1 Jack Wolfe applied his data (Figure 1.10.0.2) to resolve a dispute about estimates of ambient temperatures during the last 65 million years. Fossil leaves from strata ranging in age back to 65 million years were examined for the percent of smooth-leafed species in each stratum. Under the hypothesis that the relation between percent of smooth-leafed species and temperature in modern species persisted over the last 65 million years, he was able to estimate the past temperatures. Your job is to replicate his work. Although Wolfe collected fossil leaf data from four locations ranging from southern Alaska to Mississippi, we show only his data from the Pacific Northwest, in Table Ex. 1.10.1, read from a graph in Wolfe's article in *American Scientist*.

Use the Percent Smooth data for the fossil leaves at previous times in Table Ex. 1.10.1, and the modern relation, temp $= 0.89 + 0.313 \times \%$ smooth, to draw a graph showing the history of ambient temperature for the land of the Pacific Northwest

a. Over the period from 50 to 40 million years ago.

b. Over the period from 35 to 26 million years ago.

c. Over the period from 26 to 16 million years ago.

d. Over the period from 16 to 6 million years ago.

Table for Exercise 1.10.1 Percentages of smooth-leafed species found in geological strata in the Pacific Northwest.

Age Myr	Percent Smooth	Age Myr	Percent Smooth	Age Myr	Percent Smooth	Age Myr	Percent Smooth
50	57	35	66	26	42	10	28
48	50	35	68	21	24	10	30
48	51	33	55	21	28	10	32
44	70	31	24	21	32	6	34
44	74	31	32	16	31	6	38
40	40	26	32	16	34		
40	44	26	40	16	38		

Exercise 1.10.2 a. Use the graph in Figure 1.24 to estimate the expected temperature if the cricket chirp rate were were 95 chirps per minute? Also use Equation 1.32 to compute the same temperature and compare your results.

b. On the basis of Equation 1.32, what cricket chirp frequencies might be expected for the temperatures 110 °F and 40 °F? Discuss your answers in terms of the data in Figure 1.24 and the interval on which the equation is valid.

c. A television report stated that in order to tell the temperature, count the number of cricket chirps in 13 seconds and add 40. The result will be the temperature in Fahrenheit degrees. Is that consistent with Equation 1.32? (Suggestion: Solve for Temperature °F in terms of Cricket Chirps per Minute.)

Exercise 1.10.3 Some students made a small hole near the bottom of the cylindrical section of a two-liter plastic beverage container, held a finger over the hole and filled the container with water to the top of the cylindrical section, at 15.3 cm. The small hole was uncovered and the students marked the height of the water remaining in the tube, reported here at eighty-second intervals, until the water ran out. They then measured the heights of the marks above the hole. The data gathered and a graph of the data are shown in Figure 1.25.

Time sec	Height cm
0	15.3
80	12.6
160	9.9
240	7.8
320	6.2
400	4.3
480	3.1
560	2.0

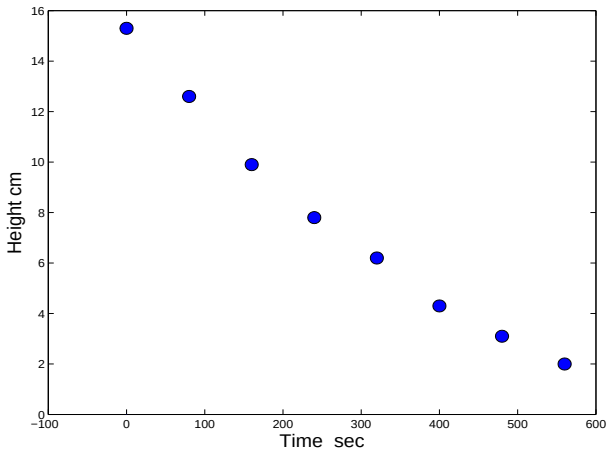


Figure 1.25: Height of water draining from a 2-liter beverage container measured at 40 second intervals.

a. Select a curve and fit an equation for the curve to the data in Figure 1.25.

b. Search your physics book for a mathematical model that explains the data. A fluid dynamicist named Evangelista Torricelli (1608-1649) provided such a model (and also invented the mercury thermometer).

c. The basic idea of Torricelli's formula is that the potential energy of a thin layer of liquid at the top of the column is converted to kinetic energy of fluid flowing out of the hole.

Let the time index, t_0, t_1, $\cdots$ measure minutes and h_k be the height of the water at time t_k. The layer between h_k and h_{k+1} has mass m_k and potential energy $m_k g h_k$. During the $k\underline{\text{th}}$ minute an equal mass and volume of water flows out of the hole with velocity, v_k, and kinetic energy $\frac{1}{2}m_k v_k^2$. Equating potential and kinetic energies,

$$m_k g h_k = \frac{1}{2} m_k v_k^2, \qquad g h_k = \frac{1}{2} v_k^2$$

Let A_{cylinder} and A_{hole} be the cross-sectional areas of the cylinder and hole, respectively.

1. Argue that
$$v_k \quad = \quad \frac{A_{\text{cylinder}}}{A_{\text{hole}}} \frac{h_k - h_{k+1}}{t_{k+1} - t_k} \quad = \quad \frac{A_{\text{cylinder}}}{A_{\text{hole}}} (h_k - h_{k+1})$$
2. Argue that
$$h_{k+1} \quad = \quad h_k - K\sqrt{h_k} \tag{1.33}$$
where K is a constant.
3. Compare Equation 1.33 with Equation 1.18

1.11 Summary

You have begun one of the most important activities in science, the writing of mathematical models. The best models are sufficiently detailed in their description that **dynamic equations** describing the progress of the underlying system can be written and lead to **solution equations** that describe the overall behavior. In our models there was a quantity (Q) that changed with either time or distance (v) and the dynamic equation specified how the change in Q, $Q_{v+1} - Q_v$, depended on Q or v. The solution equation explicitly expressed the relation of Q to v.

In some cases, such as the case of cricket chirp frequency dependence on ambient temperature in Example 1.10.1, the underlying mechanism is too complex to model it. The observation, however, stimulates considerable thought about why it should be. Presumably the metabolism of the cricket increases with temperature thus causing an increase in chirp frequency, but the phrase, 'metabolism of the cricket', masks a large complexity.

We have generally followed certain steps in developing our models. They are useful steps but by no means do they capture the way to model the biological universe. The modeling process is varied and has to be adapted to the questions at hand.

By the methods of this chapter you can solve every *first order linear finite difference equation with constant coefficients*:
$$y_0 \quad \text{given}, \qquad y_{t+1} - y_t = r \times y_t + b \tag{1.34}$$
The solution is
$$y_t = -\frac{b}{r} + (y_0 + \frac{b}{r}) \times (r+1)^t \qquad \text{if} \quad r \neq 0, \qquad \text{or} \qquad y_t = y_0 + t \times b \qquad \text{if} \quad r = 0.$$

Exercises for Chapter 1, Mathematical Models of Biological Processes.

Chapter Exercise 1.1 Two kilos of a fish poison, rotenone, are mixed into a lake which has a volume of $100 \times 20 \times 2 = 4000$ cubic meters. No water flows into or out of the lake. Fifteen percent of the rotenone decomposes each day.

a. Write a mathematical model that describes the daily change in the amount of rotenone in the lake.

b. Let R_0, P_1, R_2, $\cdots$ denote the amounts of rotenone in the lake, P_t being the amount of poison in the lake at the beginning of the $\underline{tth}$ day after the rotenone is administered. Write a dynamic equation representative of the mathematical model.

c. What is R_0? Compute R_1 from your dynamic equation. Compute R_2 from your dynamic equation.

d. Find a solution equation for your dynamic equation.

Note: Rotenone is extracted from the roots of tropical plants and in addition to its use in killing fish populations is used as a insecticide on such plants as tomatoes, pears, apples, roses, and african violets. Studies in which large amounts (2 to 3 mg/Kg body weight) of rotenone were injected into the jugular veins of laboratory rats produced symptoms of Parkinson's disease, including the reduction of dopamine producing cells in the brain. (R. Betarbet, et al. *Nature Neurosciences*, 3, 1301-1306.)

Chapter Exercise 1.2 Two kilos of a fish poison *that does not decompose* are mixed into a lake that has a volume of $100 \times 20 \times 2 = 4000$ cubic meters. A stream of clean water flows into the lake at a rate of 1000 cubic meters per day. Assume that it mixes immediately throughout the whole lake. Another stream flows out of the lake at a rate of 1000 cubic meters per day.

a. Write a mathematical model that describes the daily change in the amount of poison in the lake.

b. Let P_0, P_1, P_2, $\cdots$ denote the amounts of poison in the lake, P_t being the amount of poison in the lake at the beginning of the $t\underline{th}$ day after the poison is administered. Write a dynamic equation representative of the mathematical model.

c. What is P_0? Compute P_1 from your dynamic equation. Compute P_2 from your dynamic equation.

d. Find a solution equation for your dynamic equation.

Chapter Exercise 1.3 Two kilos of rotenone are mixed into a lake which has a volume of $100 \times 20 \times 2 = 4000$ cubic meters. A stream of clean water flows into the lake at a rate of 1000 cubic meters per day. Assume that it mixes immediately throughout the whole lake. Another stream flows out of the lake at a rate of 1000 cubic meters per day. Fifteen percent of the rotenone decompses every day.

a. Write a mathematical model that describes the daily change in the amount of rotenone in the lake.

b. Let R_0, R_1, R_2, $\cdots$ denote the amounts of rotenone in the lake, R_t being the amount of rotenone in the lake at the beginning of the $t\underline{th}$ day after the poison is administered. Write a dynamic equation representative of the mathematical model.

c. What is R_0? Compute R_1 from your dynamic equation. Compute P_2 from your dynamic equation.

d. Find a solution equation for your dynamic equation.

Chapter Exercise 1.4 Consider a chemical reaction

$$A + B \longrightarrow AB$$

in which a chemical, A, combines with a chemical, B, to form the compound, AB. Assume that the amount of B greatly exceeds the amount of A, and that in any second, the amount of AB that is formed is proportional to the amount of A present at the beginning of the second. Write a dynamic equation for this reaction, and write a solution equation to the dynamic equation.

Chapter Exercise 1.5 An egg is covered by a hen and is at 37° C. The hen leaves the nest and the egg is exposed to 17° C air. After 20 minutes the egg is at 34° C.

Draw a graph representative of the temperature of the egg t minutes after the hen leaves the nest.

> **Mathematical Model.** During any short time interval while the egg is uncovered, the decrease in egg temperature is proportional to the difference between the egg temperature and the air temperature.

a. Introduce notation and write a dynamic equation representative of the mathematical model.

b. Write a solution equation for your dynamic equation.

c. Your dynamic equation should have one parameter. Use the data of the problem to estimate the parameter.

Chapter Exercise 1.6 The length of an burr oak leaf was measured on successive days in May. The data are shown in Table 1.6. Select an appropriate equation to approximate the data and compute the coefficients of your equation. Do you have a mathematical model of leaf growth?

Table 1.6: Length of a burr oak leaf.

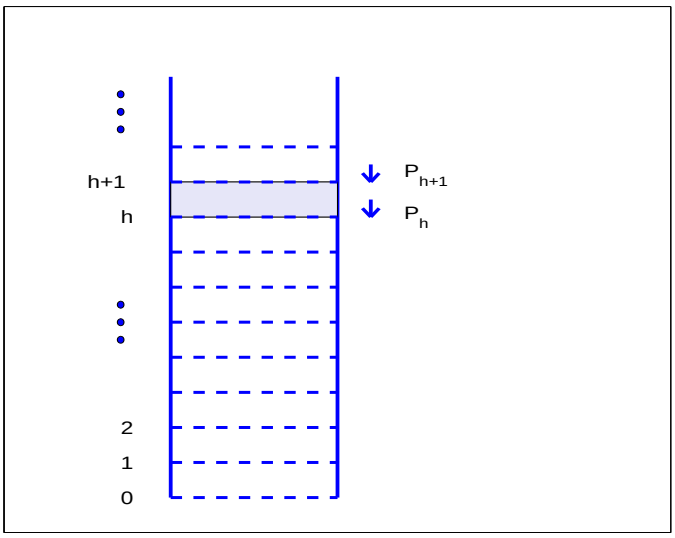
Day	May 7	May 8	May 9	May 10	May 11
Length mm	67	75	85	98	113

Chapter Exercise 1.7 Atmospheric pressure decreases with increasing altitude. Derive a dynamic equation from the following mathematical model, solve the dynamic equation, and use the data to evaluate the parameters of the solution equation.

Mathematical Model 1.11.1 Mathematical Model of Atmospheric Pressure. Consider a vertical column of air based at sea level divided at intervals of 10 meters and assume that the temperature of the air within the column is constant, say 20°C. The pressure at any height is the weight of air in the column above that height divided by the cross sectional area of the column. In a 10-meter section of the column, by the ideal gas law the the mass of air within the section is proportional to the product of the volume of the section and the pressure within the section (which may be considered constant and equal to the pressure at the bottom of the section). The weight of the air above the lower height is the weight of air in the section plus the weight of air above the upper height.

Sea-level atmospheric pressure is 1 atm and the pressure at 18,000 feet is one-half that at sea level (an easy to remember datum from NASA).

Figure for Exercise 1.11.0 Figure for Exercise 1.7.



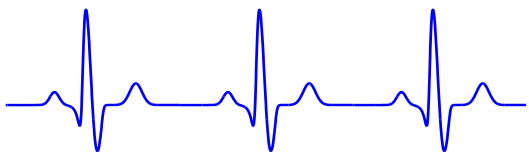
Chapter 2

Functions as Descriptions of Biological Patterns.

Where are we going?

Relations between quantities are so central to science and mathematics that the concept has been carefully defined and is named *function*. You will see how tables of data, graphs, and equations are unified into a single concept, function. This chapter will present examples of some unusual relations between quantities that suggest the usefulness of the general concept, and it will introduce three definitions of **function** with increasing levels of precision.

Electrocardiograms are examples of relations between voltage and time that are functions. Equations to describe the relation are difficult to find, and once found, are not informative. It is better to think of each trace as simply a function.



Many biological relations may be described by equations — linear, quadratic, hyperbolic, exponential, and logarithmic. These together with the trigonometric equations cover most of the commonly encountered equations. Scientific calculators readily compute these values that in the past were laboriously computed by hand and recorded in lengthy tables. There are some relations between measured quantities that are not so easily described by equations, and in this chapter we see how we can extend the use of equations to describe relations to a general concept referred to as a **function**. Formal definitions of *function* and ways of combining functions are presented.

2.1 Environmental Sex Determination in Turtles

It is a well known, but little understood fact that the sex of some reptiles depends on the environment in which the egg is incubated (temperature is important), and is not purely a genetic determination. A graph of some data on a species of fresh water turtles appears in Figure 2.1[1].

Explore 2.1.1 Examine the graph for *Chrysemys picta* and write a brief verbal description of the dependence of the % Female on incubation temperature for a clutch of *Chrysemys picta* eggs.

■

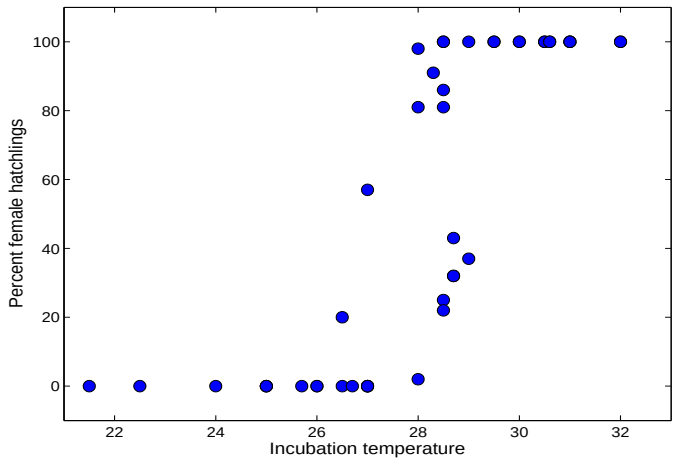


Figure 2.1: Percent female hatchlings from incubation of *Chrysemys picta* eggs at various temperatures.

Your experience probably does not include a single analytic formula that describes the dependence of the % Female on incubation temperature for a clutch of turtle eggs. It is easiest to use a *piecewise* definition; use one formula for a range of temperatures and other formulas for other ranges of temperatures. Your verbal description in the case of *Chrysemys picta* could say that if the temperature is less than 28 °C, the percent of female is zero, and if the temperature is greater than or equal to 29 °C, the percent of female is 100. One could describe what happens between 28 °C and 29 °C, but the accuracy of the data probably does not warrant such a refinement. This piecewise procedure of describing data occurs often enough to have a special notation.

$$\text{Percent female} = \begin{cases} 0 & \text{if Temp} < 28\ ^\circ\text{C} \\ \text{Uncertain} & \text{if } 28\ ^\circ\text{C} \leq \text{Temp} < 29\ ^\circ\text{C} \\ 100 & \text{if } 29\ ^\circ\text{C} \leq \text{Temp} \end{cases} \tag{2.1}$$

The designation Uncertain for 28 °C $\leq$ Temp $<$ 29 °C is a bit unsatisfactory but we hope the example sufficiently illustrates piecewise definition of dependence.

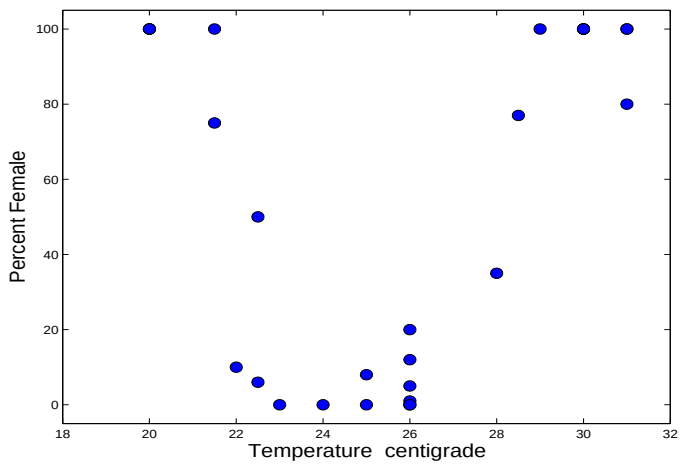
[1]This data was collected by Ralph Ackerman from a number of publications, most of which are referenced in, Fredric J. Janzen and Gary L. Paukstis, Environmental sex determination in reptiles: Ecology, evolution, and experimental design, *The Quarterly Review of Biology* **66** (1991) 149-179.

Exercise 2.1.1 Data on the temperature determination of sex for the snapping turtle, *Chelydra serpentina* appears in Figure Ex. 2.1.1.

a. Write a verbal description of the dependence of the % Female on incubation temperature for a clutch of *Chelydra serpentina* eggs.

b. Write formulas similar to Formulas 2.1 to describe the dependence of the % Female on incubation temperature for a clutch of *Chelydra serpentina* eggs.

Figure for Exercise 2.1.1 Percent females from clutches of *Chelydra serpentina* (snapping turtle) eggs incubated at various temperatures.



2.2 Functions and Simple Graphs.

A large part of science may be described as the study of the dependence of one measured quantity on another measured quantity. The word *function* is used in this context in a special way. In previous examples, the word *function* may have been used as follows:

- The density of *V. natrigens* is a *function* of time.
- Light intensity is a *function* of depth below the surface in an ocean.
- Light intensity is a *function* of distance from the light source.
- The frequency of cricket chirps is a *function* of the ambient temperature.
- The % female turtles from a clutch of eggs is a *function* of the incubation temperature.

2.2.1 Three definitions of "Function".

Because of its prevalence and importance in science and mathematics, the word *function* has been defined several ways over the past three hundred years, and now is usually given a very precise formal meaning. More intuitive meanings are also helpful and we give three definitions of function, all of which will be useful to us.

The word *variable* means a symbol that represents any member of a given set, most often a set of numbers, and usually denotes a value of a measured quantity. Thus density of *V. natrigens*, time, % female, incubation temperature, light intensity, depth and distance are all variables.

The terms *dependent* variable and *independent* variable are useful in the description of an experiment and the resulting functional relationship. The density of *V. natrigens* (dependent variable) is a function of time (independent variable). The % female turtles from a clutch of eggs (dependent variable) is a *function* of the incubation temperature (independent variable).

Using the notion of variable, a function may be defined:

Definition 2.2.1 Function I *Given two variables, x and y, a* function *is a rule that assigns to each value of x a unique value of y.*

In this context, x is the *independent variable*, and y is the *dependent variable.* In some cases there is an equation that nicely describes the 'rule'; in the % female in a clutch of turtle eggs examples of the preceding section, there was not a simple equation that described the rule, but the rule met the definition of function, nevertheless.

The use of the words dependent and independent in describing variables may change with the context of the experiment and resulting function. For example, the data on the incubation of turtles implicitly assumed that the temperature was held constant during incubation. For turtles in the wild, however, temperature is not held constant and one might measure the temperature of a clutch of eggs as a function of time. Then, temperature becomes the dependent variable and time is the independent variable.

In Definition 2.2.1, the word 'variable' is a bit vague, and 'a function is a rule' leaves a question as to 'What is a rule?'. A 'set of objects' or, equivalently, a 'collection of objects', is considered to be easier to understand than 'variable' and has broader concurrence as to its meaning. Your previous experience with the word *function* may have been that

Definition 2.2.2 Function II *A function is a rule that assigns to each number in a set called the* domain *a unique number in a set called the* range *of the function.*

Definition 2.2.2 is similar to Definition 2.2.1, except that 'a number in a set called the domain' has given meaning to *independent variable* and 'a unique number in a set called the range' has given meaning to *dependent variable.*

The word 'rule' is at the core of both definitions 2.2.1 **Function I.** and 2.2.2 **Function II.** and is still a bit vague. The definition of function currently considered to be the most concise is:

Definition 2.2.3 Function III *A* function *is a collection of ordered number pairs no two of which have the same first number.*

A little reflection will reveal that 'a table of data' is the motivation for Definition 2.2.3. A data point is actually a number pair. Consider the tables of data shown in Table 2.1 from *V. natrigens* growth and human population records. (16,0.036) is a data point. (64,0.169) is a data point. (1950,2.52) and (1980,4.45) are data points. These are basic bits of information for the functions. On the other hand, examine the data for cricket chirps in the same table, from Chapter 1. That also is a collection of ordered pairs, but the collection does not satisfy Definition 2.2.3. There are two ordered pairs in the table with the same first term – (66,102) and (66,103). Therefore the collection does not constitute a function.

In a function that is a collection of ordered number pairs, the first number of a number pair is always a value of the independent variable and a member of the domain and the second number is always

Table 2.1: Examples of tables of data

V. natrigens Growth pH 6.25	
Time (min)	Population Density
0	0.022
16	0.036
32	0.060
48	0.101
64	0.169
80	0.266

World Population	
Year	Population (billions)
1940	2.30
1950	2.52
1960	3.02
1970	3.70
1980	4.45
1990	5.30
2000	6.06

Cricket Chirps	
Temperature °F	Chirps per Minute
67	109
73	136
78	160
61	87
66	103
66	102
67	108
77	154
74	144
76	150

a value of the dependent variable and a member of the range. Almost always in recording the results of an experiment, the numbers in the domain are listed in the column on the left and the numbers in the range are listed in the column on the right. Formally,

Definition 2.2.4 Domain and Range *For Definition 2.2.3 of function, the* **domain** *is defined as the set of all numbers that occur as the first number in an ordered pair of the function and the* **range** *of the function is the set of all numbers that occur as a second number in an ordered pair of the function.*

Example 2.2.1 Data for the percentage of U.S. population in 1955 that had antibodies to the polio virus as a function of age is shown in Table 2.2.1.1. The domain of the function is the set of all ages of people, although only 14 such ages are listed. The range is contained in the interval of percentages, [0,100], but may not actually exceed 90, say. The data show an interesting fact that a high percentage of the population in 1955 had been infected with polio. A much smaller percentage were crippled or killed by the disease. ■

Table for Example 2.2.1.1 Data for percentage of the U.S. population in 1955 at various ages that had antibodies to the polio virus. Data read from Anderson and May, Vaccination and herd immunity to infectious diseases, *Nature* **318** 1985, pp 323-9, Figure 2f. Permission kindly provided without charge by Nature Publishing Group.

Age	0.8	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	12.5	17.5	22.5	27.5
%	3	6	13	19	27	35	40	43	46	49	64	72	78	87

Remember that only a very few data 'points' were listed from the large number of possible points in each of the experiments we considered. The function is the set of all possible ordered pairs associated with the experiment.

Because many biological quantities change with time, the domain of a function of interest is often an interval of time. In some cases a biological reaction depends on temperature (%Females in a clutch of turtle eggs for example) so that the domain of a function may be an interval of temperatures. In cases of spatial distribution of a disease or light intensity below the surface of a lake, the domain may be an interval of distances.

It is implicit in the bacteria growth data that at any specific time, there is only one value of the absorbance associated with that time. It may be incorrectly or inaccurately read, but a fundamental assumption is that there is only one correct absorbance for that specific time. The condition that no two of the ordered number pairs have the same first term is a way of saying that each number in the domain has a unique number in the range associated with it.

All three of the definitions of *function* are helpful, as are brief verbal descriptions, and we will rely on all of them. Our basic definition, however, is the ordered pair definition, Definition 2.2.3.

2.2.2 Simple graphs.

Coordinate geometry associates ordered number pairs with points of the plane so that by Definition 2.2.3 a function is automatically identified with a point set in the plane called a *simple graph*:

Definition 2.2.5 Simple Graph *A simple graph is a point set, G, in the plane such that no vertical line contains two points of G.*

The **domain** *of G is the set of x-coordinates of points of G and the* **range** *of G is the set of all y-coordinates of points of G.*

Note: For use in this book, every set contains at least one element.

The domain of a simple graph G is sometimes called the x-projection of G, meaning the vertical projection of G onto the x-axis and the range of G is sometimes called the y-projection of G meaning the horizontal projection of G onto the y-axis.

A review of the graphs of incubation temperature - % Female turtles in Figure 2.1 and Exercise Fig. 2.1.1 will show that in each graph at least one vertical line contains two points of the graph. Neither of these graphs is a simple graph, but the graphs convey useful information.

A circle is not a simple graph. As shown in Figure 2.2A there is a vertical line that contains two points of it. There are a lot of such vertical lines. The circle does contain a simple graph, and contains one that is 'as large as possible'. The upper semicircle shown in Figure 2.2B is a simple graph. The points (-1,0) and (1,0) are filled to show that they belong to it. It is impossible to add any other points of the circle to this simple graph and still have a simple graph — thus it is 'as large as possible'. An equation of the upper semicircle is

$$y = \sqrt{1 - x^2}\,, \qquad -1 \le x \le 1$$

The domain of this simple graph is [-1,1], and the range is [0,1]. Obviously the lower semicircle is a

maximal simple graph also, and it has the equation

$$y = -\sqrt{1-x^2}\,, \qquad -1 \le x \le 1$$

The domain is again [-1,1], and the range is [-1,0].

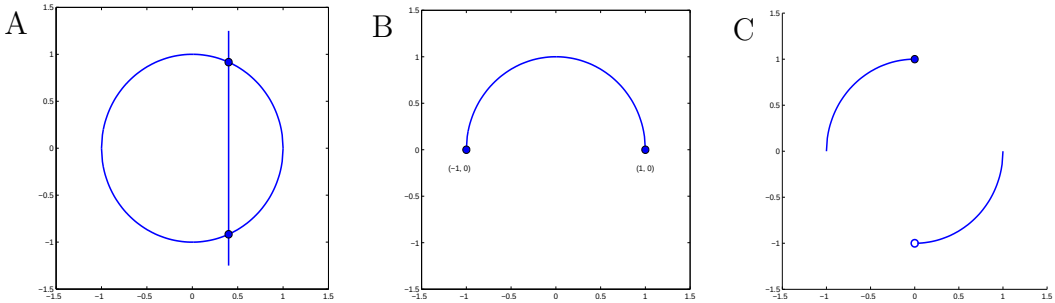


Figure 2.2: A. A circle; a vertical line contains two point of the circle so that it is not a simple graph. B. A subset of the circle that is a simple graph. C. Another subset of the circle that is a simple graph. The simple graphs in (b) and (c) are maximal in the sense that any point from the circle added to the graphs would create a set that is not a simple graph — the vertical line containing that point would also contain a point from the original simple graph.

There is yet a third simple graph contained in the circle, shown in Figure 2.2C, and it is 'as large as possible'. An equation for that simple graph is

$$y = \begin{cases} \sqrt{1-x^2} & \text{if} \quad -1 \le x \le 0 \\ -\sqrt{1-x^2} & \text{if} \quad 0 < x \le 1 \end{cases}$$

The domain is [-1,1] and range is [-1,1] Because of the intuitive advantage of geometry, it is often useful to use simple graphs instead of equations or tables to describe functions, but again, we will use any of these as needed.

Exercises for Section 2.2 Functions and Simple Graphs.

Exercise 2.2.1 Which of the tables shown in Table Ex. 2.2.1 reported as data describing the growth of *V. natrigens* are functions?

Table for Exercise 2.2.1 Hypothetical data for *V. natrigens* growing in Nutrient Broth.

Time	Abs
0	0.018
12	0.023
24	0.030
36	0.039
48	0.049
48	0.065
60	0.085
78	0.120
96	0.145
110	0.195
120	0.240

Time	Abs
0	0.018
12	0.023
24	0.030
36	0.039
48	0.049
60	0.065
72	0.065
87	0.065
96	0.080
110	0.095
120	0.120

Time	Abs
0	0.018
12	0.023
24	0.030
48	0.049
48	0.049
48	0.049
72	0.065
87	0.065
96	0.080
110	0.095
120	0.120

Exercise 2.2.2 For the following experiments, determine the independent variable and the dependent variable, and draw a simple graph or give a brief verbal description (your best guess) of the function relating the two.

a. A rabbit population size is a function of the number of coyotes in the region.

b. An agronomist, interested in the most economical rate of nitrogen application to corn, measures the corn yield in test plots using eight different levels of nitrogen application.

c. An enzyme, E, catalyzes a reaction converting a substrate, S, to a product P according to

$$\mathrm{E} + \mathrm{S} \rightleftharpoons \mathrm{ES} \rightleftharpoons \mathrm{E} + \mathrm{P}$$

Assume enzyme concentration, [E], is fixed. A scientist measures the rate at which the product P accumulates at different concentrations, [S], of substrate.

d. A scientist titrates a 0.1 M solution of HCl into 5 ml of an unknown basic solution containing litmus (litmus causes the color of the solution to change as the pH changes).

Exercise 2.2.3 A table for bacterial density for growth of *V. natrigens* is repeated in Exercise Table 2.2.3. There are two functions that relate population density to time in this table, one that relates population density to time and another that relates population to time index.

a. Identify an ordered pair that belongs to both functions.

b. One of the functions is implicitly only a partial list of the order pairs that belong to it. You may be of the opinion that both functions have that property, but some people may think one is more obviously only a sample of the data. Which one?

c. What is the domain of the other function?

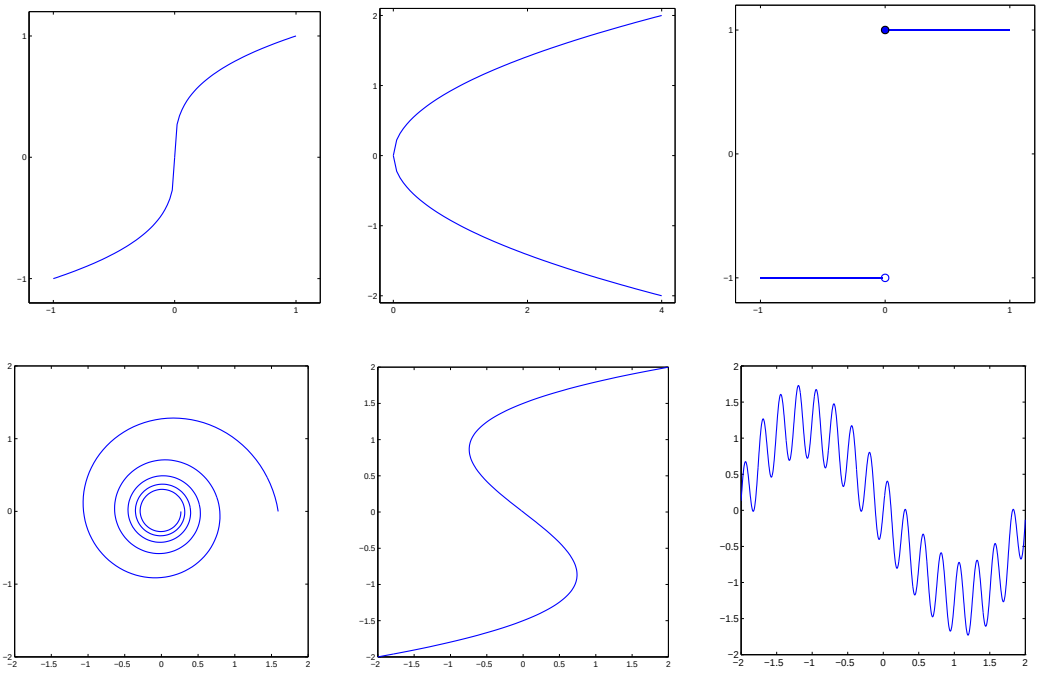
Table for Exercise 2.2.3 Data for *V. natrigens* growing in pH 6.25 nutrient broth.

pH 6.25		
Time (min)	Time Index t	Population Density B_t
0	0	0.022
16	1	0.036
32	2	0.060
48	3	0.101
64	4	0.169
80	5	0.266

Exercise 2.2.4 Refer to the graphs in Figure Ex. 2.2.4.

1. Which of the graphs are simple graphs?
2. For those that are not simple graphs,
   (a) Draw, using only the points of the graph, a simple graph that is 'as large as possible', meaning that no other points can be added and still have a simple graph.
   (b) Draw a second such simple graph.
   (c) Identify the domains and ranges of the two simple graphs you have drawn.
   (d) How many such simple graphs may be drawn?

Figure for Exercise 2.2.4 Graphs for Exercise 2.2.4. Some are simple graphs; some are not simple graphs.

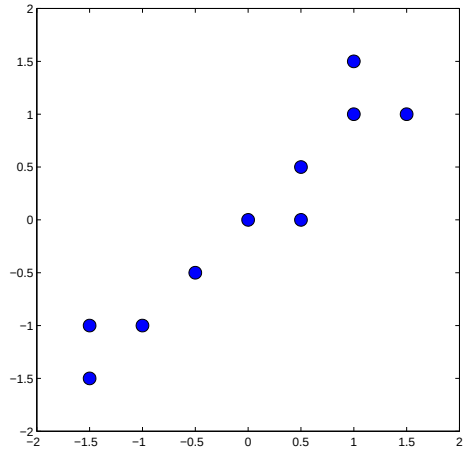


Exercise 2.2.5 Make a table showing the ordered pairs of a simple graph contained in the graph in Figure Ex. 2.2.5 and that has domain

$$\{-1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5\}$$

How many such simple graphs are contained in the graph of Figure Ex. 2.2.5 and that have this domain?

Figure for Exercise 2.2.5 Graph for Exercise 2.2.5.



Exercise 2.2.6 a. Does every subset of the plane contain a simple graph?

b. Does every subset of the plane contain two simple graphs?

c. Is there a subset of the plane that contains two and only two simple graphs?

d. Is there a line in the plane that is not the graph of a function?

e. Is there a function whose graph is a circle?

f. Is there a simple graph in the plane whose domain is the interval [0,1] (including 0 and 1) and whose range is the interval [0,3]?

g. Is there a simple graph in the plane whose domain is [0,1] and whose range is the y-axis?

Exercise 2.2.7 A bit of a difficult exercise. For any location, λ on Earth, let Annual Daytime at λ, $AD(\lambda)$, be the sum of the lengths of time between sunrise and sunset at λ for all of the days of the year. Find a reasonable formula for $AD(\lambda)$.

You may guess or find data to suggest a reasonable formula, but we found proof of the validity of our formula a bit arduous. As often happens in mathematics, instead of solving the actual problem posed, we found it best to solve a 'nearby' problem that was more tractable. The 365.24... days in a year is a distraction, the eliptical orbit of Earth is a downright hinderance, and the wobble of Earth on its axis can be overlooked. Specifically, we find it helpful to assume that there are precisely 366 days in the year (after all this was true about 7 or 8 million years ago), the Earth's orbit about the sun is a circle, the Earth's axis makes a constant angle with the plane of the orbit, and that the rays from the sun to Earth are parallel. We hope you enjoy the question.

2.2.3 Functions in other settings.

There are extensions of the function concept to settings where the ordered pairs are not ordered *number* pairs. A prime example of this is the genetic code shown in Figure 2.3. The relation is a true function (no two ordered pairs have the same first term), and during the translation of proteins, the ribosome and the transfer RNA's use this function reliably.

1st position (5' end) ↓	2nd position U	C	A	G	3rd position (3' end) ↓
U	Phe	Ser	Tyr	Cys	U
	Phe	Ser	Tyr	Cys	C
	Leu	Ser	STOP	STOP	A
	Leu	Ser	STOP	Trp	G
C	Leu	Pro	His	Arg	U
	Leu	Pro	His	Arg	C
	Leu	Pro	Gln	Arg	A
	Leu	Pro	Gln	Arg	G
A	Ile	Thr	Asn	Ser	U
	Ile	Thr	Asn	Ser	C
	Ile	Thr	Lys	Arg	A
	Met*	Thr	Lys	Arg	G
G	Val	Ala	Asp	Gly	U
	Val	Ala	Asp	Gly	C
	Val	Ala	Glu	Gly	A
	Val	Ala	Glu	Gly	G

Figure 2.3: The genetic code (for human nuclear RNA). Sets of three nucleotides in RNA (codons) are translated into amino acids in the course of proteins synthesis. CAA codes for Gln (glutamine). *AUG codes for Met (methionine) and is also the START codon.

Explore 2.2.1 List three ordered pairs of the genetic code. What is the domain of the genetic code? What is the range of the genetic code? ■

The ordered pair concept is retained in the preceding example; the only change has been in the types of objects that are in the domain and range. When the objects get too far afield from simple numbers, the word *transformation* is sometimes used in place of function. The genetic code is a transformation of the codons into amino acids and start and stop signals.

Another commonly encountered extension of the kinds of objects in the domain of a function occurs when one physical or biological quantity is dependent on two others. For example, the widely known Charles' Law in Chemistry can be stated as

$$P = \frac{nRT}{V}$$

where P = pressure in atmospheres, n = number of molecular weights of the gas, $R = 0.0820$ Atmospheres/degree Kelvin-mol = 8.3 /degree Kelvin-mol (the gas constant), T = temperature in kelvins, and V = volume in liters. For a fixed sample of gas, the pressure is dependent on two quantities, temperature and volume. The domain is the set of all feasible temperature-volume pairs, the range is the set of all feasible pressures. The function in this case is said to be a function of two variables. The ordered pairs in the function are of the form

$$((x, y), z)\,, \quad \text{or} \quad (\,(\text{temperature}, \text{volume})\,,\ \text{pressure})$$

There may also be multivalued transformations. For example, doctors prescribe antibiotics. For each bacterial infection, there may be more than one antibiotic effective against that bacterium; there may be a list of such antibiotics. The domain would be a set of bacteria, and the range would be a set of lists of antibiotics.

Exercises for Section 2.2.3 Functions in other settings.

Exercise 2.2.8 Describe the domain and range for each of the following transformations.

a. Bird identification guide book.

b. A judge sentences defendants to jail terms.

c. The time between sunrise and sunset.

d. Antibiotic side effects.

2.3 Function notation.

It is common to let a letter denote a set of ordered pairs that is a function.

> If F denotes a function, then for any ordered pair, (x, y) in F, y is denoted by $F(x)$.

Thus the ordered pair $(x, F(x))$ is an ordered pair of F. The notation makes it very easy to describe a function by an equation. Instead of

> 'Let F be the collection of ordered number pairs to which an ordered pair (x, y) belongs if and only if x is a number and $y = x^2 + x$.'

one may write

> 'Let F be the function such that for all numbers x, $F(x) = x^2 + x$.'

Operationally, you will find that often you can simply replace y in an equation by $F(x)$ and define a function. Furthermore, because you are used to using y in an equation, you can often replace $F(x)$ in the definition of a function by y and use the resulting equation which is more familiar.

The following expressions all may be used in the definition of a function.

$$\begin{array}{llllllll} \text{(a)} & F(x) & = & x^2 & \text{(b)} & F(x) & = & x+5 \\ \text{(c)} & F(x) & = & x+\frac{1}{x} \quad x \neq 0 & \text{(d)} & F(x) & = & e^x \\ \text{(e)} & F(x) & = & \sqrt{x} \quad x \geq 0 & \text{(f)} & F(x) & = & \log_{10} x \quad x > 0 \end{array}$$

Observe that some conditions, $x \neq 0$ and $x \geq 0$ and $x > 0$, are included for some of the expressions. Those conditions describe the domain of the function. For example, the domain of the function f defined by $F(x) = \log_{10} x$ is the set of positive numbers.

You may also see something like

$$\begin{array}{lllllll} F(x) & = & \sqrt{1+x} & & F(x) & = & \frac{1-x}{1+x} \\ F(x) & = & \sqrt{1-x^2} & & F(x) & = & \log_{10}(x^2-x) \end{array}$$

The intention is that the domain is the set of all values of x for which the expressions can be computed even though no restrictions are written. Often the restrictions are based on these rules:

1. Avoid dividing by zero.
2. Avoid computing the square root of negative numbers.
3. Avoid computing the logarithm of 0 and negative numbers.

Assuming we use only real numbers and not complex numbers, complete descriptions of the previous functions would be

$$\begin{array}{llllllll} f(x) & = & \sqrt{1+x}\,, & x \geq -1 & F(x) & = & \frac{1-x}{1+x}\,, & x \neq -1 \\ F(x) & = & \sqrt{1-x^2}\,, & -1 \leq x \leq 1 & F(x) & = & \log_{10}(x^2-x)\,, & x < 0 \quad \text{or} \quad 1 < x \end{array}$$

Use of parentheses. The use of parentheses in the function notation is special to functions and *does not mean multiplication.* The symbol inside the parentheses is always the independent variable, a member of the domain, and $F(x)$ is a value of the dependent variable, a member of the range. It is particularly tricky in that we will often need to use the symbol $F(x+h)$, and students confuse this with a multiplication and replace it with $F(x) + F(h)$. Seldom is this correct.

Example 2.3.1 For the function, R, defined by

$$R(x) = x + \frac{1}{x} \qquad x \neq 0$$

$$\begin{aligned} R(1+3) &= R(4) = 4 + \frac{1}{4} = 4.25 \\ R(1) &= 1 + \frac{1}{1} = 2.0 \qquad \text{and} \\ R(3) &= 3 + \frac{1}{3} = 3.3333\cdots \\ R(1)+R(3) &= 2 + 3.3333\cdots = 5.3333\cdots \neq 4.25 = R(4) \end{aligned}$$

In this case

$$R(1+3) \neq R(1) + R(3)$$

■

Exercises for Section 2.3 Function Notation.

Exercise 2.3.1 Let F be the collection of ordered number pairs to which an ordered pair (x, y) belongs if and only if x is a number and $y = x^2 + x$.

a. Which of the ordered number pairs belong to F? (0,1), (0,0), (1,1), (1,3), (1,-1), (-1,1), (-1,0), (-1,-1).

b. Is there any uncertainty as to the members of F?

c. What is the domain of F?

d. What is the range of F?

Exercise 2.3.2 For the function, F, defined by $F(x) = x^2$,

1. Compute $F(1+2)$, and $F(1) + F(2)$. Is $F(1+2) = F(1) + F(2)$?
2. Compute $F(3+5)$, and $F(3) + F(5)$. Is $F(3+5) = F(3) + F(5)$?
3. Compute $F(0+4)$, and $F(0) + F(4)$. Is $F(0+4) = F(0) + F(4)$?

Exercise 2.3.3 Find a function, L, defined for all numbers (domain is all numbers) such that for all numbers a and b, L(a + b) = L(a) + L(b). Is there another such function?

Exercise 2.3.4 For the function, $F(x) = x^2 + x$, compute the following

(a) $\frac{F(5)-F(3)}{5-3}$ (b) $\frac{F(3+2)-F(3)}{2}$

(c) $\frac{F(b)-F(a)}{b-a}$ (d) $\frac{F(a+h)-F(a)}{h}$

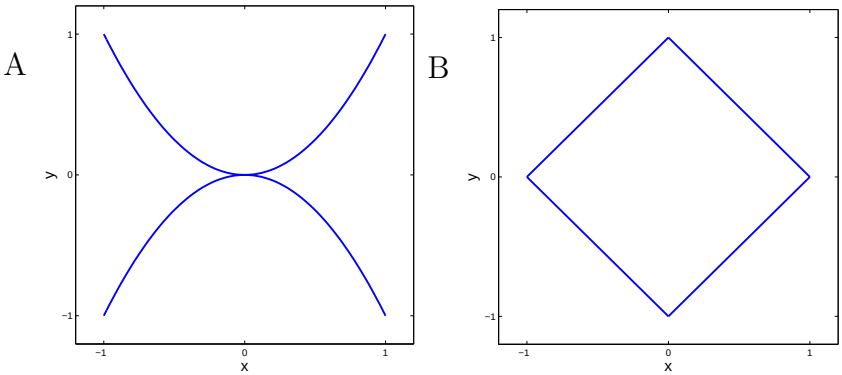
Exercise 2.3.5 Repeat steps (a) - (d) of Exercise 2.3.4 for the functions

$$\text{(a)} \quad F(x) = 3x \qquad \text{(b)} \quad F(x) = x^3$$

$$\text{(c)} \quad F(x) = 2^x \qquad \text{(d)} \quad F(x) = \sin x$$

Exercise 2.3.6 a. In Figure 2.3.6A is the graph of $y^4 = x^2$ for $-2 \le x \le 2$. Write equations that define five different maximal simple subgraphs.
b. In Figure 2.3.6B is the graph of $|x| + |y| = 1$ for $-1 \le x \le 1$. Write equations that define five different maximal simple subgraphs.

Figure for Exercise 2.3.6 Graph of A $y^4 = x^2$, and B $|x| + |y| = 1$, for Exercise 2.3.6.



Exercise 2.3.7 What are the implied domains of the functions

$$F(x) = \sqrt{x-1} \qquad F(x) = \tfrac{1+x^2}{1-x^2}$$

$$F(x) = \sqrt{4-x^2} \qquad F(x) = \log_{10}(x^2)$$

2.4 Polynomial functions.

Data from experiments (and their related functions) are often described as being linear, parabolic, hyperbolic, polynomial, harmonic (sines and cosines), exponential, or logarithmic either because their graphs have some resemblance to the corresponding geometric object or because equations describing their related functions use the corresponding expressions. In this section we extend linear and quadratic equations to more general polynomial functions.

Definition 2.4.1 Polynomial *For n a positive integer or zero, a* **polynomial of degree n** *is a function, P defined by an equation of the form*

$$P(x) = a_0 + a_1 \times x + a_2 \times x^2 + a_3 \times x^3 + \cdots + a_n \times x^n$$

where a_0, a_1, a_2, a_3, $\cdots$, a_n are numbers, independent of x, called the **coefficients of p**, *and if $n > 0$ $a_n \neq 0$.*

Functions of the form

$$P(x) = C \qquad \text{where } C \text{ is a number}$$

are said to be constant functions and also polynomials of degree zero. Functions defined by equations

$$P(x) = a + bx \qquad \text{and} \qquad P(x) = a + bx + cx^2$$

are linear and quadratic polynomials, respectively, and are polynomials of degree one and degree two. The equation

$$P(x) = 3 + 5 \times x + 2 \times x^2 + (-4) \times x^3$$

is a polynomial of degree 3 with coefficients 3, 5, 2, and -4 and is said to be a cubic polynomial.

Polynomials are important for four reasons:

1. Polynomials can be computed using only the arithmetic operations of addition, subtraction, and multiplication.

2. Most functions used in science have polynomials "close" to them over finite intervals.

3. The sum of two polynomials is a polynomial, the product of two polynomials is a polynomial, and the composition of two polynomials is a polynomial.

4. Polynomials are 'linear' in their coefficients a fact which makes them suitable for least squares 'fit' to data.

Reason 1 is obvious, but even the meaning of Reason 2 is opaque. An illustration of Reason 2 follows, and we return to the question in Chapter 9. Sum, product, and composition of two functions are described in Section 2.6. Reason 4 is illustrated in Section 2.5.

A tangent to a graph is a polynomial of degree one close to the graph. The graph of $F(x) = 2^x$ and the tangent

$$P_1(x) = 1 + 0.69315x \qquad -1 \le x \le 2$$

at (0,1) are shown in Figure 2.4(a). The graph of the cubic polynomial

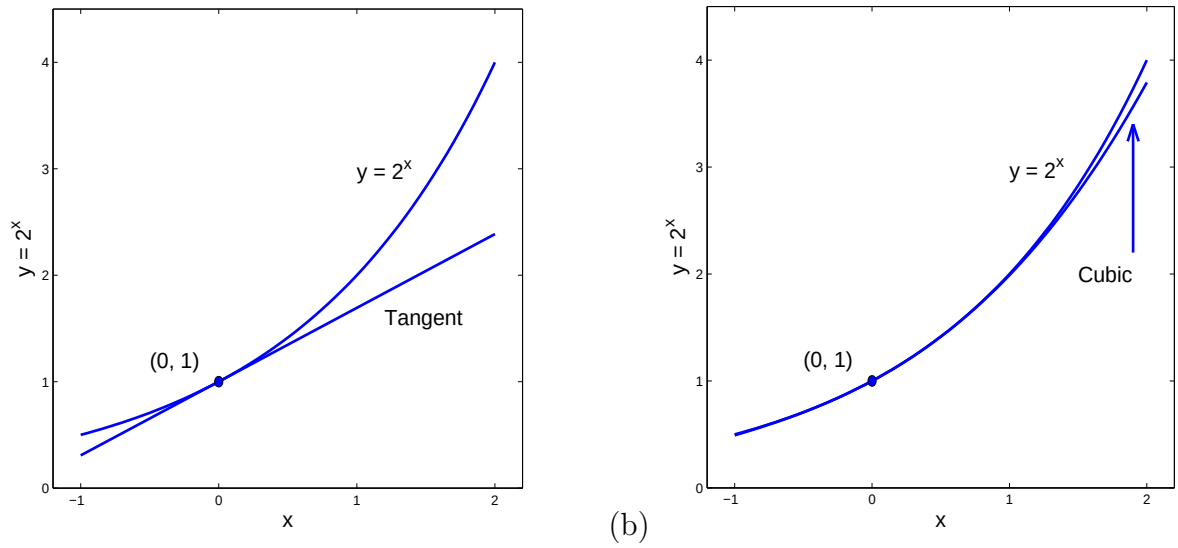
$$P_3(x) \;=\; 1 + 0.69315x + 0.24023x^2 + 0.05550x^3 \qquad -1 \le x \le 2$$

shown in Figure 2.4(b) is even closer to the graph of $F(x) = 2^x$. These graphs are hardly distinguishable on $-1 \le x \le 1$ and only clearly separate at about $x = 1.5$. The function $F(x) = 2^x$ is difficult to evaluate (without a calculator) except at integer values of x, but $P_3(x)$ can be evaluated with just multiplication and addition. For $x = 0.5$, $F(0.5) = 2^{0.5} = \sqrt{2} = 1.41421$ and $P_3(0.5) = 1.41375$. The relative error in using $P_3(0.5)$ as an approximation to $\sqrt{2}$ is

$$\text{Relative Error} \qquad = \left| \frac{P_3(0.5) - F(0.5)}{F(0.5)} \right| = \left| \frac{1.41375 - 1.41421}{1.41421} \right| = 0.00033$$

The relative error is less than 0.04 percent.

The tangent $P_1(x)$ is a good approximation to $F(x) = 2^x$ near the point of tangency (0,1) and the cubic polynomial $P_3(x)$ is an even better approximation. The coefficients 0.69135, 0.24023, and 0.05550 are presented here as Lightning Bolts Out of the Blue. A well defined procedure for selecting the coefficients is defined in Chapter 9.



(a) (b)

Figure 2.4: (a) Graph of $y = 2^x$ and its tangent at (0,1), $P_1(x) = 1 + 0.69315x$. (b) Graph of $y = 2^x$ and the cubic, $P_3(x) = 1 + 0.69315x + 0.24023x^2 + 0.05550x^3$.

Explore 2.4.1 Find the relative error in using the tangent approximation, $P_1(0.5) = 1 + 0.69315 \times 0.5$ as an approximation to $F(0.5) = 2^{0.5}$. ■

Exercises for Section 2.4 Polynomial functions.

Exercise 2.4.1 Let $F(x) = \sqrt{x}$.

a. Use a graphing calculator and draw the graphs of F and

$$P_2(x) = \frac{3}{4} + \frac{3}{8}x - \frac{1}{64}x^2$$

on the range $1 \le x \le 8$.

b. Compute the relative error in $P_2(2)$ as an approximation to $F(2) = \sqrt{2}$.

c. Use a graphing calculator and draw the graphs of F and

$$P_3(x) = \frac{5}{8} + \frac{15}{32}x - \frac{5}{128}x^2 + \frac{1}{512}x^3$$

on the range $1 \le x \le 8$.

d. Compute the relative error in $P_3(2)$ as an approximation to $F(2) = \sqrt{2}$.

Exercise 2.4.2 Let $F(x) = \sqrt[3]{x}$.

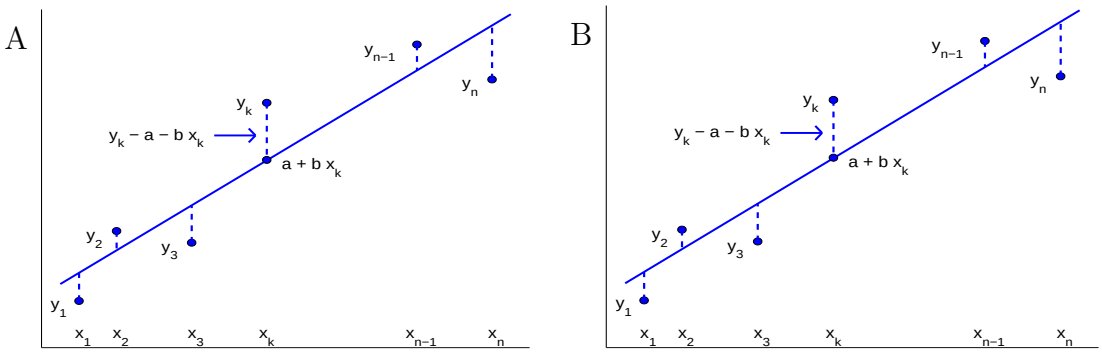


Figure 2.5: A. Least squares fit of a line to data. B. Least squares fit of a parabola to data.

a. Use a graphing calculator and draw the graphs of F and

$$P_2(x) = \frac{5}{9} + \frac{5}{9}x - \frac{1}{9}x^2$$

on the range $0 \le x \le 3$.

b. Compute the relative error in $P_2(2)$ as an approximation to $F(2) = \sqrt[3]{2}$.

c. Use a graphing calculator and draw the graphs of F and

$$P_3(x) = P_2(x) + \frac{5}{81}(x-1)^2$$

on the range $1 \le x \le 3$.

d. Compute the relative error in $P_3(2)$ as an approximation to $F(2) = \sqrt[3]{2}$.

2.5 Least squares fit of polynomials to data.

Polynomials and especially linear functions are often 'fit' to data as a means of obtaining a brief and concise description of the data. The most common and widely used method is the method of *least squares*. To fit a line to a data set, (x_1, y_1), (x_2, y_2), $\cdots$ (x_n, y_n), one selects a and b that minimizes

$$\sum_{k=1}^{n} (y_k - a - b\,x_k)^2 \tag{2.2}$$

The geometry of this equation is illustrated in Figure 2.5. The goal is to select a and b so that the sum of the squares of the lengths of the dashed lines is as small as possible.

We show in Example 10.2.2 that the optimum values of a and b satisfy

$$\begin{aligned} an \quad + \quad b\textstyle\sum_{k=1}^{n} x_k \quad &= \quad \textstyle\sum_{k=1}^{n} y_k \\ a\textstyle\sum_{k=1}^{n} x_k \quad + \quad b\textstyle\sum_{k=1}^{n} x_k^2 \quad &= \quad \textstyle\sum_{k=1}^{n} x_k y_k \end{aligned} \tag{2.3}$$

Table 2.2: Relation between temperature and frequency of cricket chirps.

Temperature °F	67	73	78	61	66	66	67	77	74	76
Chirps per Minute	109	136	160	87	103	102	108	154	144	150

The solution to these equations is

$$
\begin{aligned}
a_0 &= \frac{\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k y_k\right)}{\Delta} \\
b_0 &= \frac{n \sum_{k=1}^{n} x_k\, y_k - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right)}{\Delta} \\
\Delta &= n \sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2
\end{aligned}
\tag{2.4}
$$

Example 2.5.1 If we use these equations to fit a line to the cricket data of Example 1.10.1 showing a relation between temperature and cricket chirp frequency, we get

$$y = 4.5008x - 192.008, \qquad \text{close to the line} \qquad y = 4.5x - 192$$

that we 'fit by eye' using the two points, (65,100) and (75,145).

■

Explore 2.5.1 Your TI-86 calculator will hide all of the arithmetic of Equations 2.4 and give you the answer. Enter the data from Table 2.2 in xStat and yStat and compute LinR(xStat,yStat). ■

To fit a parabola to a data set, (x_1, y_1), (x_2, y_2), $\cdots$ (x_n, y_n), one selects a, b and c that minimizes

$$\sum_{k=1}^{n} (y_k - a - b\, x_k - c x_k^2)^2 \tag{2.5}$$

The geometry of this equation is illustrated in Figure 2.5B. The goal is to select a, b and c so that the sum of the squares of the lengths of the dashed lines is as small as possible.

The optimum values of a, b, and c satisfy (Exercise 10.2.4)

$$
\begin{aligned}
a\, n + b \sum x_k + c \sum x_k^2 &= \sum y_k \\
a \sum x_k + b \sum x_k^2 + c \sum x_k^3 &= \sum x_k y_k \\
a \sum x_k^2 + b \sum x_k^3 + c \sum x_k^4 &= \sum x_k^2 y_k
\end{aligned}
\tag{2.6}
$$

A methodical procedure for solving three linear equations in three variables is given in the Appendix to this section. For now it is best to rely on your calculator.

Table 2.3: A tube is filled with water and a hole is opened at the bottom of the tube. Relation between height of water remaining in the tube and time.

Time sec	0	5	10	15	20	25	30	35	40	45	50	55	60
Height cm	85	73	63	54	45	36	29	22	17	12	7	4	2

Explore 2.5.2 Use 'P2Reg' on your calculator to fit a parabola to the water draining from a tube data of Figure 1.25 reproduced in Table 2.3. The 'Reg' in 'P2Reg' stands for 'regression.' Statisticians 'regress' parabolas on or to the data. ■

Example 2.5.2 A graph of the polio data from Example 2.2.1 showing the percent of U.S. population that had antibodies to the polio virus in 1955 is shown in Figure 2.6. Also shown is a graph of the fourth degree polynomial

$$P_4(x) = -4.13 + 7.57x - 0.136x^2 - 0.00621x^3 + 0.000201x^4$$

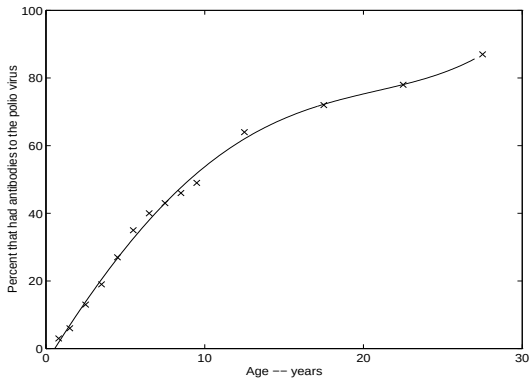


Figure 2.6: A fourth-degree polynomial fit to data for percent of people in 1955 who had antibodies to the polio virus as a function of age. Data read from Anderson and May, Vaccination and herd immunity to infectious diseases, *Nature* **318** 1985, pp 323-9, Figure 2f. Permission kindly provided without charge by Nature Publishing Group.

The polynomial that 'fit' the polio data above was obtained using P4Reg on a TI-86 calculator. The calculator selects the coefficients, -4.14, 7.57, $\cdots$ so that the sum of the squares of the distances from the polynomial to the data is as small as possible. ■

Exercises for Section 2.5, Least squares fit of polynomials to data.

Exercise 2.5.1 Use Equations 2.3 to find the linear function that is the least squares fit to the data:

$$(-2, 5) \qquad (3, 12)$$

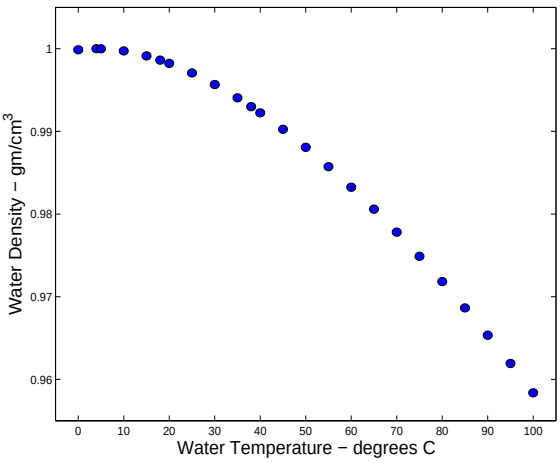
Exercise 2.5.2 Use Equations 2.6 to find the quadratic function that is the least squares fit to the data:

$$(-2, 5) \qquad (3, 12) \qquad (10, 0)$$

Exercise 2.5.3 Shown in the Table 2.4 are the densities of water at temperatures from 0 to 100 °C Use your calculator to fit a cubic polynomial to the data. Compare the graphs of the data and of the cubic.

Table 2.4: The density of water at various temperatures Source: Robert C. Weast, Melvin J. Astle, and William H. Beyer, *CRC Handbook of Chemistry and Physics*, 68th Edition, 1988, CRC Press, Boca Raton, FL, p F-10.

Temp °C	Density g/cm^3	Temp °C	Density g/cm^3
0	0.99987	45	0.99025
3.98	1.00000	50	0.98807
5	0.99999	55	0.98573
10	0.99973	60	0.98324
15	0.99913	65	0.98059
18	0.99862	70	0.97781
20	0.99823	75	0.97489
25	0.99707	80	0.97183
30	0.99567	85	0.96865
35	0.99406	90	0.96534
38	0.99299	95	0.96192
40	0.99224	100	0.95838



2.6 New functions from old.

It is often important to recognize that a function of interest is made up of component parts — other functions that are combined to make up the function of central interest. Researchers monitoring natural populations (deer, for example) partition the dynamics into the algebraic *sum* of births, deaths, and harvest. Researchers monitoring annual grain production in the United States decompose the production into the *product* of the number of acres planted and yield per acre.

$$\text{Total Corn Production} \quad = \quad \text{Acres Planted to Corn} \quad \times \quad \text{Yield per Acre}$$

$$P(t) \quad = \quad A(t) \times Y(t)$$

Factors that influence $A(t)$, the number of acres planted (government programs, projected corn price, alternate cropping opportunities, for example) are qualitatively different from the factors that influence $Y(t)$, yield per acre (corn genetics, tillage practices, and weather).

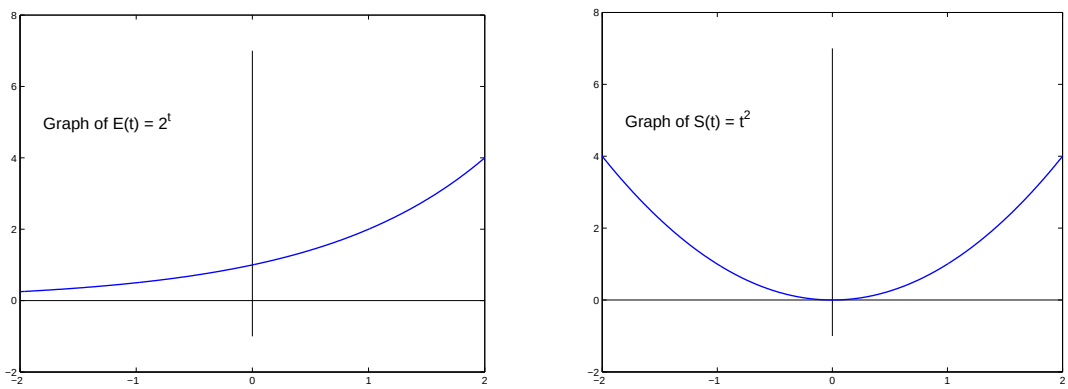
2.6.1 Arithmetic combinations of functions.

A common mathematical strategy is "divide and conquer" — partition your problem into smaller problems, each of which you can solve. Accordingly it is helpful to recognize that a function is composed of component parts. Recognizing that a function is the sum, difference, product, or quotient of two functions is relatively simple.

Definition 2.6.1 Arithmetic Combinations of Functions. *If F and G are two functions with common domain, D, The sum, difference, product, and quotient of F and G are functions, F+G, F-G, F×G, and F ÷ G, respectively defined by*

$$\begin{aligned}
(F+G)(x) &= F(x)+G(x) && \textit{for all } x \textit{ in } D \\
(F-G)(x) &= F(x)-G(x) && \textit{for all } x \textit{ in } D \\
(F\times G)(x) &= F(x)\times G(x) && \textit{for all } x \textit{ in } D \\
(F\div G)(x) &= F(x)\div G(x) && \textit{for all } x \textit{ in } D \textit{ with } \quad G(x)\neq 0
\end{aligned}$$

For example $F(t) = 2^t + t^2$ is the sum of an exponential function, $E(t) = 2^t$ and a quadratic function, $S(t) = t^2$. Which of the two functions dominate? Which contributes most to the value of F? Shown in Figure 2.7 are the graphs of E and S.



(a) (b)

Figure 2.7: Graphs of $E(t) = 2^t$ and $S(t) = t^2$.

Next in Figure 2.8 are the graphs of

$$E+S, \qquad E-S, \qquad E\times S, \qquad \text{and} \qquad \frac{E}{S}$$

but not necessarily in that order. Which graph depicts which of the combinations of S and E?

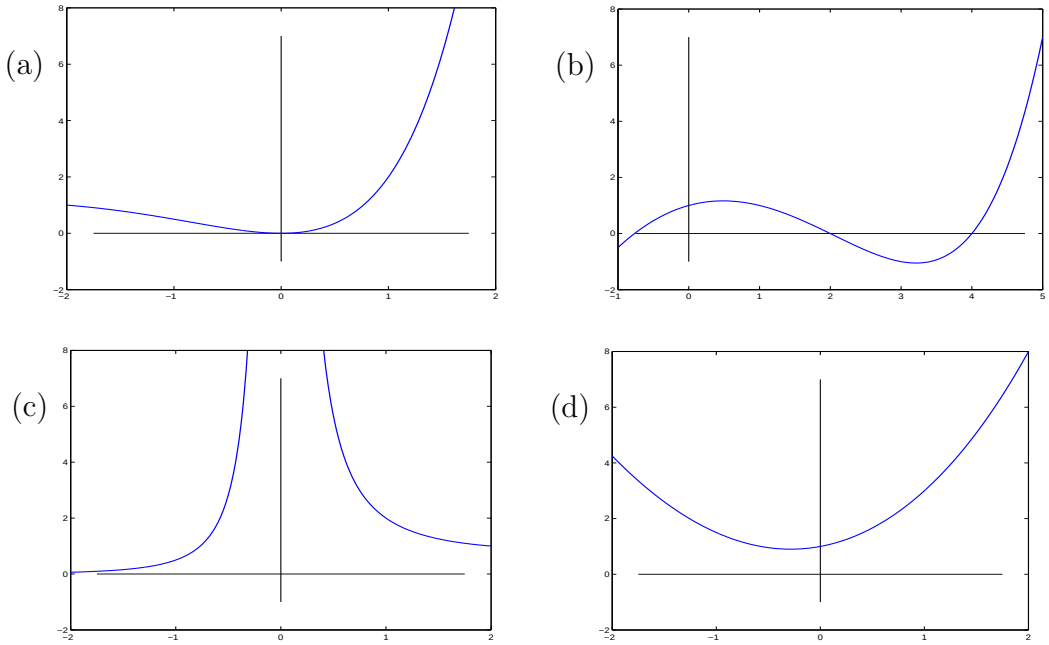


Figure 2.8: Graphs of the sum, difference, product, and quotient of $E(t) = 2^t$ and $S(t) = t^2$.

Note first that the domains of E and S are all numbers, so that the domains of $E + S$, $E - S$, and $E \times S$ are also all numbers. However, the domain of E/S excludes 0 because $S(0) = 0$ and $E(0)/S(0) = 1/0$ is meaningless.

The graph in Figure 2.8(c) appears to not have a point on the y-axis, and that is a good candidate for E/S. $E(t)$ and $S(t)$ are never negative, and the sum, product, and quotient of non-negative numbers are all non-negative. However, the graph in Figure 2.8(b) has some points below the x-axis, and that is a good candidate for $E - S$.

The product, $E \times S$ is interesting for $t < 0$. The graph of $E = 2^t$ is asymptotic to the negative t-axis; as t progresses from -1 to -2 to -3 to $\cdots$, $E(t)$ is $2^{-1} = 0.5$, $2^{-2} = 0.25$, $2^{-3} = 0.125$, $\cdots$ and gets close to zero. But $S(t) = t^2$ is $(-1)^2 = 1$, $(-2)^2 = 3$, $(-3)^2 = 9$, $\cdots$ gets very large. What does the product do?

2.6.2 The inverse of a function.

Suppose you have a travel itinerary as shown in Table 2.5. If your traveling companion asks, "What day

Table 2.5: Itinerary for (brief) European trip.

Day	June 1	June 2	June 3	June 4	June 5	June 6	June 7
City	London	London	London	Brussels	Paris	Paris	Paris

were we in Brussels?", you may read the itinerary 'backward' and respond that you were in Brussels on June 4. On the other hand, if your companion asks, "I cashed a check in Paris, what day was it?", you may have difficulty in giving an answer.

An itinerary is a function that specifies that on day, x, you will be in location, y. You have inverted the itinerary and reasoned that the for the city Brussels, the day was June 4. Because you were in Paris

June 5 - 7, you can not specify the day that the check was cashed.

Charles Darwin exercised an inverse in an astounding way. In his book, *On the Various Contrivances by which British and Foreign Orchids are Fertilised by Insects* he stated that the angraecoids were pollinated by specific insects. He noted that *A. sesquipedale* in Madagascar had nectaries eleven and a half inches long with only the lower one and one-half inch filled with nectar. He suggested the existence of a 'huge moth, with a wonderfully long proboscis' and noted that if the moth 'were to become extinct in Madagascar, assuredly the *Angraecum* would become extinct.' Forty one years later *Xanthopan morgani praedicta* was found in tropical Africa with a proboscis of ten inches.

Such inverted reasoning occurs often.

Explore 2.6.1 Answer each of the following by examining the inverse of the function described.

a. Rate of heart beat increases with level of exertion; heart is beating at 175 beats per minute; is the level of exertion high or low?

b. Resting blood pressure goes up with artery blockage; resting blood pressure is 110 (systolic) 'over' 70 (diastolic); is the level of artery blockage high or low?

c. Diseases have symptoms; a child is observed with a rash over her body. Is the disease chicken pox?

■

The child with a rash in Example c. illustrates again an ambiguity often encountered with inversion of a function; the child may in fact have measles and not chicken pox. The inverse information may be multivalued and therefore not a function. Nevertheless, the doctor may make a diagnosis as the most probable disease, given the observed symptoms. She may be influenced by facts such as

- Blood analysis has demonstrated that five other children in her clinic have had chicken pox that week and,
- Because of measles immunization, measles is very rare.

It may be that she can actually distinguish chicken pox rash from measles rash, in which case the ambiguity disappears.

The definition of the inverse of a function is most easily made in terms of the ordered pair definition of a function. Recall that a function is a collection of ordered number pairs, no two of which have the same first term.

Definition 2.6.2 Inverse of a Function. *A function F is invertible if no two ordered pairs of F have the same* second number*. The inverse of an invertible function, F, is the function G to which the ordered pair (x,y) belongs if and only if (y,x) is an ordered pair of F. The function, G, is often denoted by F^{-1}.*

Explore 2.6.2 Do This. Suppose G is the inverse of an invertible function F. What is the inverse of G? ■

Table 2.6: Of the two itineraries shown below, the one on the left is invertible.

Day	Location	Day	Location
June 4	London	June 8	Vienna
June 5	Amsterdam	June 9	Zurich
June 6	Paris	June 10	Venice
June 7	Berlin	June 11	Rome

Day	Location	Day	Location
June 4	London	June 8	Zurich
June 5	Paris	June 9	Zurich
June 6	Paris	June 10	Rome
June 7	Paris	June 11	Rome

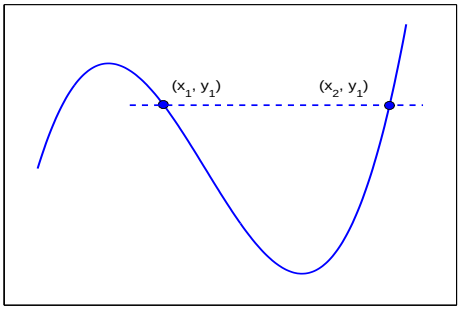
The notation F^{-1} for the inverse of a function F is distinct from the use of $^{-1}$ as an exponent meaning division, as in $2^{-1} = \frac{1}{2}$. In this context, F^{-1} does not mean $\frac{1}{F}$, even though students have good reason to think so from previous use of the symbol, $^{-1}$. The TI-86 calculator (and others) have keys marked $\sin^{-1}$ and x^{-1}. In the first case, the $^{-1}$ signals the inverse function, in the second case the $^{-1}$ signals reciprocal. Given our desire for uniqueness of definition and notation, the ambiguity is unfortunate and a bit ironic. There is some recovery, however. You will see later that the composition of functions has an algebra somewhat like 'multiplication' and that 'multiplying' by an inverse of a function F has some similarity to 'dividing' by F. At this stage, however, the only advice we have is to interpret h^{-1} as 'divide by h' if h is a number and as 'inverse of h' if h is a function or a graph.

The graph of a function easily reveals whether it is invertible. Remember that the graph of a function is a *simple graph*, meaning that no vertical line contains two points of the graph.

> A simple graph G is the graph of an invertible function if no *horizontal* line contains two points of G.

The simple graph G in Figure 2.9(a) has two points on the same horizontal line. The points have the same y-coordinate, y_1, and thus the function defining G has two ordered pairs, (x_1, y_1) and (x_2, y_1) with the same second term. The function is not invertible. The same simple graph G does contain a simple graph that is invertible, as shown as the solid curve in Figure 2.9(b)

a.



b.

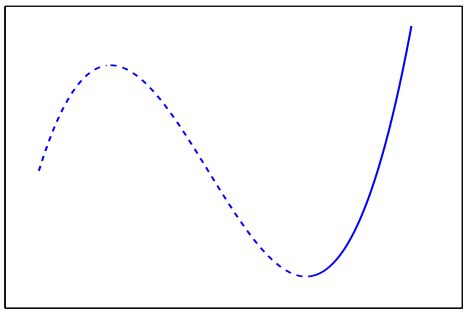
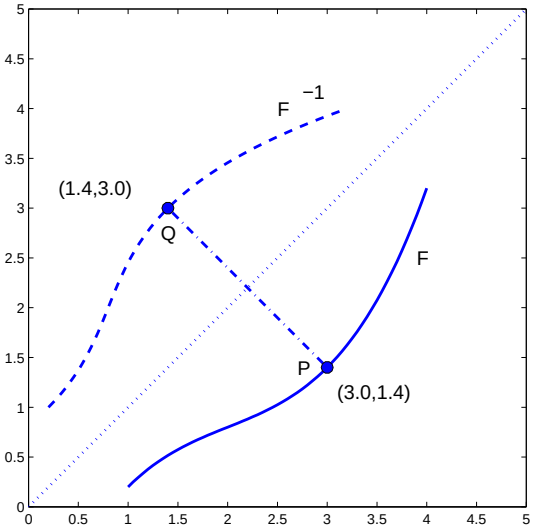


Figure 2.9: Graph of a function that is not invertible.

Explore 2.6.3 Find another simple graph contained in the graph G of Figure 2.9(a) that is invertible. ■

Example 2.6.1 Shown in Figure 2.10 is the graph of an invertible function, F, as a solid line and the graph of F^{-1} as a dashed line. Tables of seven ordered pairs of F and seven ordered pairs of F^{-1} are given. Corresponding to the point, P (3.0,1.4) of F is the point, Q (1.4,3.0) of F^{-1}. The line $y = x$ is the perpendicular bisector of the interval, $\overline{PQ}$. Observe that the domain of F^{-1} is the range of F, and the range of F^{-1} is the domain of F. ■ ■



F	
1.0	0.200
1.5	0.575
2.0	0.800
2.5	1.025
3.0	1.400
3.5	2.075
4.0	3.200

F^{-1}	
0.200	1.0
0.575	1.5
0.800	2.00
1.025	2.5
1.400	3.0
2.075	3.5
3.200	4.0

Figure 2.10: Graph of a function F (solid line) and its inverse F^{-1} (dashed line) and tables of data for F and F^{-1}.

The preceding example lends support for the following observation.

> The graph of the inverse of F is the reflection of the graph of F about the diagonal line, $y = x$.
>
> The *reflection of G with respect to the diagonal line, $y = x$* consists of the points Q such that either Q is a point of G on the diagonal line, or there is a point P of G such that the diagonal line is the perpendicular bisector of the interval $\overline{PQ}$.

The concept of the inverse of a function makes it easier to understand logarithms. Shown in Table 2.7 are some ordered pairs of the exponential function, $F(x) = 10^x$ and some ordered pairs of the logarithm function $G(x) = \log_{10}(x)$. The logarithm function is simply the inverse of the exponential function.

The function $F(x) = x^2$ is not invertible. In Figure 2.11 both (-2,4) and (2,4) are points of the graph of F so that the horizontal line $y = 4$ contains two points of the graph of F. However, the function S defined by

$$S(x) = x^2 \qquad \text{for } x \geq 0 \tag{2.7}$$

is invertible and its inverse is

$$S^{-1}(x) = \sqrt{x}.$$

Table 2.7: Partial data for the function $F(x) = 10^x$ and its inverse $G(x) = \log_{10}(x)$.

x	10^x
-2	0.01
-1	0.1
0	1
1	10
2	100
3	1000

x	$\log_{10} x$
0.01	-2
0.1	-1
1	0
10	1
100	2
1000	3

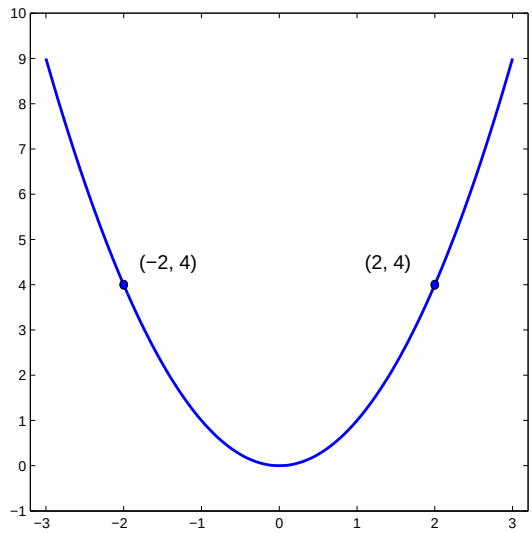


Figure 2.11: Graph of $F(x) = x^2$; (-2,4) and (2,4) are both points of the graph indicating that F is not invertible.

2.6.3 Finding the equation of the inverse of a function.

There is a straightforward means of computing the equation of the inverse of a function from the equation of the function. The inverse function reverses the role of the independent and dependent variables. The independent variable for the function is the dependent variable for the inverse function. To compute the equation for the inverse function, it is common to interchange the symbols for the dependent and independent variables.

Example 2.6.2 To compute the equation for the inverse of the function S,

$$S(x) = x^2 \qquad \text{for} \qquad x \geq 0$$

let the equation for S be written as

$$y = x^2 \qquad \text{for} \qquad x \geq 0$$

Then interchange y and x to obtain

$$x = y^2 \qquad \text{for} \qquad y \geq 0$$

and solve for y.

$$\begin{aligned} x &= y^2 \qquad \text{for} \qquad y \geq 0 \\ y^2 &= x \qquad \text{for} \qquad y \geq 0 \\ \left(y^2\right)^{\frac{1}{2}} &= x^{\frac{1}{2}} \\ y^{2\times\frac{1}{2}} &= x^{\frac{1}{2}} \\ y &= x^{\frac{1}{2}} \end{aligned}$$

Therefore

$$S^{-1}(x) = \sqrt{x} \qquad \text{for} \quad x \geq 0 \qquad \blacksquare$$

$\blacksquare$

Example 2.6.3 The graph of function $F(x) = 2x^2 - 6x + 3/2$ shown in Figure 2.12 has two invertible portions, the left branch and the right branch. We compute the inverse of each of them.

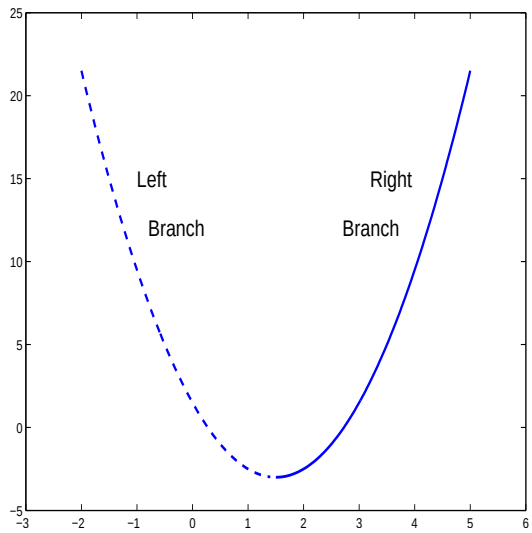


Figure 2.12: Graph of $F(x) = 2x^2 - 6x + 3/2$ showing the left branch as dashed line and right branch as solid line.

Let $y = 2x^2 - 6x + 5/2$, exchange symbols $x = 2y^2 - 6y + 5/2$, and solve for y. We use the steps of

'completing the square' that are used to obtain the quadratic formula.

$$\begin{aligned}
x &= 2y^2 - 6y + 5/2 \\
&= 2(y^2 - 3x + 9/4) - 9/2 + 5/2 \\
&= 2(y - 3/2)^2 - 2 \\
(y - 3/2)^2 &= \frac{x+2}{2} \\
y - 3/2 &= \sqrt{\frac{x+2}{2}} \qquad \text{or} \qquad -\sqrt{\frac{x+2}{2}} \\
y &= \tfrac{3}{2} + \sqrt{\frac{x+2}{2}} && \text{Right branch inverse.} \\
y &= \tfrac{3}{2} - \sqrt{\frac{x+2}{2}} && \text{Left branch inverse.}
\end{aligned}$$

■

Exercises for Section 2.6 New functions from old.

Exercise 2.6.1 In Figure 2.8 identify the graphs of $E + S$ and $E \times S$.

Exercise 2.6.2 Three examples of biological functions and questions of inverse are described in Explore 2.6.1. Identify two more functions and related inverse questions.

Exercise 2.6.3 The genetic code appears in Figure 2.3. It occasionally occurs that the amino acid sequence of a protein is known, and one wishes to know the DNA sequence that coded for it. That the genetic code is not invertible is illustrated by the following problem.

Find two DNA sequences that code for the amino acid sequence KYLEF. (Note: K = Lys = lysine, Y = Tyr = tyrosine, L = Leu = leucine, E = Glu = glutamic acid, F = Phe = phenylalanine. The sequence KYLEF occurs in sperm whale myoglobin.)

Exercise 2.6.4 Which of the following functions are invertible?

a. The distance a DNA molecule will migrate during agarose gel electrophoresis as a function of the molecular weight of the molecule, for domain: 1kb $\leq$ Number of bases $\leq$ 20kb.

b. The density of water as a function of temperature.

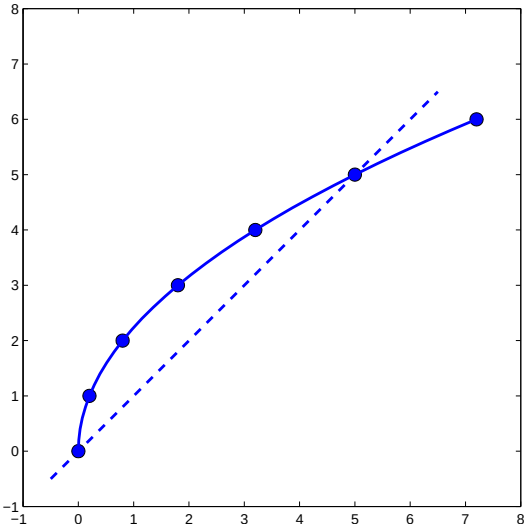
c. Day length as a function of elevation of the sun above the horizon (at, say, 40 degrees North latitude).

d. Day length as a function of day of the year.

Exercise 2.6.5 Shown in Figure Ex. 2.6.5 is the graph of a function, F. Some ordered pairs of the function are listed in the table and plotted as filled circles. What are the corresponding ordered pairs of F^{-1}? Plot those points and draw the graph of F^{-1}.

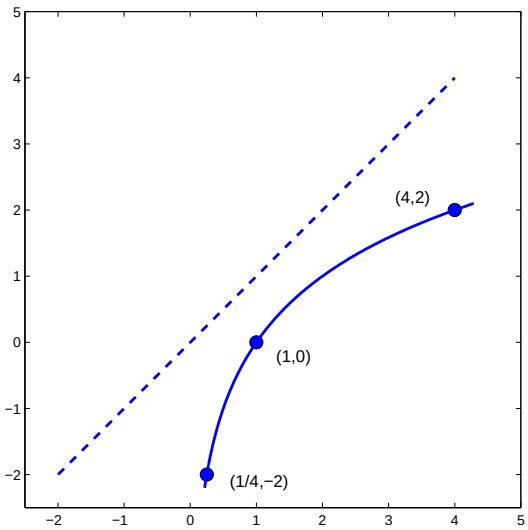
Figure for Exercise 2.6.5 Partial data and a graph of an invertible function, F, and the diagonal, $y = x$. See Exercise 2.6.5.

x	$F(x)$
0	0
1	0.2
2	0.8
3	1.8
4	3.2
5	1.0
6	7.2



Exercise 2.6.6 Shown in Figure Ex. 2.6.6 is the graph of $F(x) = 2^x$. (-2,1/4), (0,1), and (2,4) are ordered pairs of F. What are the corresponding ordered pairs of F^{-1}? Plot those points and draw the graph of F^{-1}.

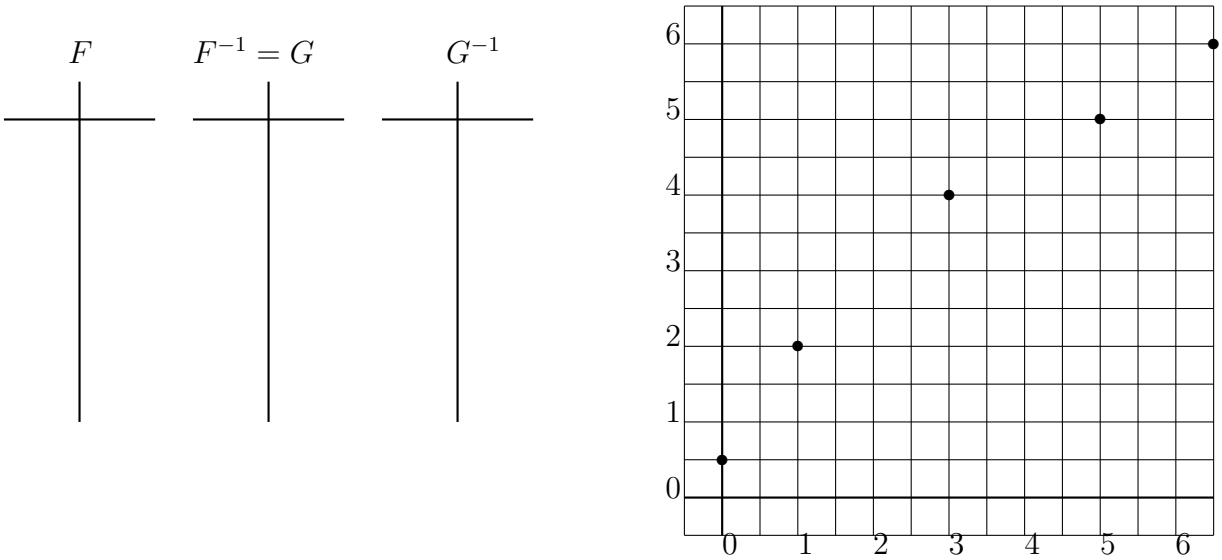
Figure for Exercise 2.6.6 Graph of $F(x) = 2^x$ and the diagonal, $y = x$. See Exercise 2.6.6



Exercise 2.6.7 In Subsection 2.2.2 Simple Graphs and Figure 2.2 it is observed that the circle is not a simple graph but contained several simple graphs that were 'as large as possible', meaning that if another point of the circle were added to them they would not be simple graphs. Neither of the examples in Figure 2.2 is invertible. Does the circle contain a simple graph that is as large as possible and that is an invertible simple graph?

Exercise 2.6.8 Shown in Figure Ex. 2.6.8 is a graph of a function, F. Make a table of F and F^{-1} and plot the points of the inverse. Let G be F^{-1}. Make a table of G^{-1} and plot the points of G^{-1}.

Figure for Exercise 2.6.8 Graph of a function F. See Exercise 2.6.8.



Exercise 2.6.9 Incredible! Find the inverse of the function, F, defined by

$$F(x) = x^{-1}$$

Hint: Look at its graph.

Exercise 2.6.10 Answer the question in Explore 2.6.2, Suppose G is the inverse of an invertible function F. What is the inverse of G?

Exercise 2.6.11 Find equations for the inverses of the functions defined by

(a)	$F_1(x)$	$=$	$\frac{1}{x+1}$	(e)	$F_5(x)$	$=$	10^{-x^2} for x ≥ 0
(b)	$F_2(x)$	$=$	$\frac{x}{x+1}$	(f)	$F_6(z)$	$=$	$\frac{z+\frac{1}{z}}{2}$ for z ≥ 1
(c)	$F_3(x)$	$=$	$1 + 2^x$	(g)	$F_7(x)$	$=$	$\frac{2^x - 2^{-x}}{2}$
(d)	$F_4(x)$	$=$	$\log_2 x - \log_2(x+1)$				

Hint for (g): Let $y = \frac{2^x - 2^{-x}}{2}$, interchange x and y so that $x = \frac{2^y - 2^{-y}}{2}$, then substitute $z = 2^y$ and solve for z in terms of x. Then insert $2^y = z$ and solve for y.

Exercise 2.6.12 Mutations in mitochondrial DNA occur at the rate of 15 per 10^2 base pairs per million years. Therefore, the number of differences, D, expected between two present mitochondrial DNA sequences of length L would be

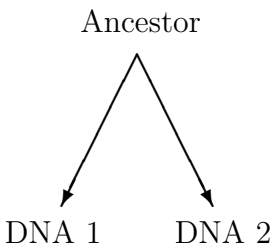
$$D = 2 \times \frac{15}{100} \times L \times \frac{T}{1000000} \tag{2.8}$$

where T is number of years since the most recent ancestor of the mitochondrial sequences.

1. Explain the factor of 2 in Equation 2.8. (Hint: consider the phylogenetic tree shown in Figure Ex. 2.6.12)

2. The African pygmy and the Papua-New Guinea aborigine mitochondrial DNA differ by 2.9%. How many years ago did their ancestral populations diverge?

Figure for Exercise 2.6.12 Phylogenetic tree showing divergence from an ancestor.



2.7 Composition of functions

Another important combination of functions is illustrated by the following examples. The general picture is that A depends on B, B depends on C, so that A depends on C.

1. **The coyote population is affected by a rabbit virus, *Myxomatosis cuniculi.*** The size of a coyote population depends on the number of rabbits in the system; the rabbits are affected by the virus *Myxomatosis cuniculi*; the size of the coyote population is a function of the prevalence of *Myxomatosis cuniculi* in rabbits.

2. **Heart attack incidence is decreased by low fat diets.** Heart attacks are initiated by atherosclerosis, a buildup of deposits in the arteries; in people with certain genetic makeups[2], the deposits are decreased with a low fat diet. The risk of heart attacks in some individuals is decreased by low fat diets.

3. **You shiver in a cold environment.** You step into a cold environment and cold receptors (temperature sensitive nerves with peak response at 30°C) send signals to your hypothalamus; the hypothalamus causes signals to be sent to muscles, increasing their tone; once the tone reaches a threshold, rhythmic muscle contractions begin. See Figure 2.13.

[2]see the Web page, http://www.heartdisease.org/Traits.html

4. **Severity of allergenic diseases is increasing.** Childhood respiratory infections such as measles, whooping cough, and tuberculosis stimulate a helper T-cell, T_H1 activity. Increased T_H1 activity inhibits T_H2 (another helper T-cell) activity. Absence of childhood respiratory diseases thus releases T_H2 activity. But T_H2 activity increases immunoglobulin E which is a component of allergenic diseases of asthma, hay fever, and eczema. Thus reducing childhood respiratory infections may partially account for the observed recent increase in severity of allergenic diseases [3].

Definition 2.7.1 Composition of Functions. *If F and G are two functions and the domain of F contains part of the range of G, then the composition of F with G is the function, H, defined by*

$$H(x) = F(G(x)) \qquad \textit{for all } x \textit{ for which } G(x) \textit{ is in the domain of } F$$

The composition, H, is denoted by $F \circ G$.

The notation $F \circ G$ for the composition of F with G means that

$$(F \circ G)(x) = F(G(x))$$

The parentheses enclosing $F \circ G$ insures that $F \circ G$ is thought of as a single object (function). The parentheses usually are omitted and one sees

$$F \circ G(x) = F(G(x))$$

Without the parentheses, the novice reader may not know which of the following two ways to group

$$(F \circ G)(x) \qquad \text{or} \qquad F \circ (G(x))$$

The experienced reader knows the right hand way does not have meaning, so the left hand way must be correct.

In the "shivering example" above, the nerve cells that perceive the low temperature are the function G and the hypothalamus that sends signals to the muscle is the function F. The net result is that the cold signal increases the muscle tone. This relation may be diagrammed as in Figure 2.13. The arrows show the direction of information flow.

Formulas for function composition.

It is helpful to recognize that a complex equation defining a function is a composition of simple parts. For example

$$H(x) = \sqrt{1 - x^2}$$

is the composition of

$$F(z) = \sqrt{z} \qquad \text{and} \qquad G(x) = 1 - x^2 \qquad\qquad F(G(x)) = \sqrt{1 - x^2}$$

[3] Shirakawa, T. et al, *Science* **275** 1997, 77-79.

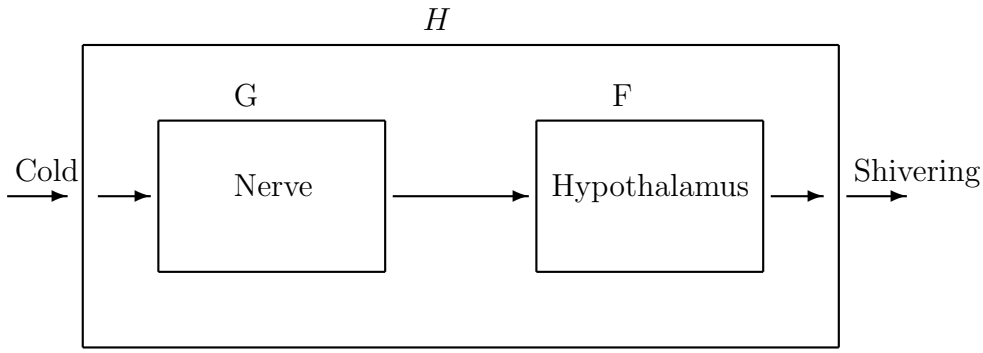


Figure 2.13: Diagram of the composition of F – increase of muscle tone by the hypothalamus – with G – stimulation from nerve cells by cold.

The domain of G is all numbers but the domain of F is only $z \geq 0$ and the domain of $F \circ G$ is only $-1 \leq x \leq 1$.

The order of function composition is very important. For

$$F(z) = \sqrt{z} \qquad \text{and} \qquad G(x) = 1 - x^2,$$

the composition, $G \circ F$ is quite different from $F \circ G$.

$$F \circ G(x) = \sqrt{1 - x^2}$$

and its graph is a semicircle.

$$G \circ F(z) = G(F(z)) = 1 - (F(z))^2 = 1 - \left(\sqrt{z}\right)^2 = 1 - z,$$

and its graph is part of a line (defined for $z \geq 0$).

Occasionally it is useful to recognize that a function is the composition of three functions, as in

$$K(x) = \log(\sin(x^2))$$

K is the composition, $F \circ G \circ H$ where

$$F(u) = \log(u) \qquad G(v) = \sin(v) \qquad H(x) = x^2$$

The composition of F and $\mathbf{F^{-1}}$. The composition of a function with its inverse is special. The case of $F(x) = x^2, \quad x \geq 0$ with $F^{-1}(x) = \sqrt{x}$ is illustrative.

$$(F \circ F^{-1})(x) = F(F^{-1}(x)) = F(\sqrt{x}) = \left(\sqrt{x}\right)^2 = x \qquad \text{for} \quad x \geq 0$$

Also

$$(F^{-1} \circ F)(x) = F^{-1}(F(x)) = F^{-1}(x^2) = \sqrt{x^2} = x \qquad \text{for} \quad x \geq 0$$

The *identity function* I is defined by

$$I(x) = x \qquad \text{for } x \text{ in a domain } D \tag{2.9}$$

where the domain D is adaptable to the problem at hand.

For $F(x) = x^2$ and $F^{-1}(x) = \sqrt{x}$, $F \circ F^{-1}(x) = F^{-1} \circ F(x) = x = I(x)$, where D should be $x \geq 0$. In the next paragraph we show that

$$F \circ F^{-1} = I \qquad \text{and} \qquad F^{-1} \circ F = I \tag{2.10}$$

for all invertible functions F.

The ordered pair (a, b) belongs to F^{-1} if and only if (b, a) belongs to F. Then

$$(F \circ F^{-1})(a) = F(F^{-1}(a)) = F(b) = a \qquad \text{and} \qquad (F^{-1} \circ F)(b) = F^{-1}(F(b)) = F^{-1}(a) = b$$

Always, $F \circ F^{-1} = I$ and with an appropriate domain D for I. Also $F^{-1} \circ F = I$ with possibly a different domain D for I.

Example 2.7.1 Two properties of the logarithm and exponential functions are

$$\text{(a)} \;\; \log_b b^\lambda = \lambda \qquad \text{and} \qquad \text{(b)} \;\; u = b^{\log_b u}$$

The logarithm function, $L(x) = \log_b(x)$ is the inverse of the exponential function, $E(x) = b^x$, and the properties simply state that

$$L \circ E = I \qquad \text{and } E \circ L = I \qquad \blacksquare$$

$\blacksquare$

The identity function composes in a special way with other functions.

$$(F \circ I)(x) = F(I(x)) = F(x) \qquad \text{and} \qquad (I \circ F)(x) = I(F(x)) = F(x)$$

Thus

$$F \circ I = F \qquad \text{and} \qquad I \circ F = F$$

Because 1 in the numbers has the property that

$$x \times 1 = x \qquad \text{and} \qquad 1 \times x = x$$

the number 1 is the identity for multiplication. Sometimes $F \circ G$ is thought of as multiplication also and I has the property analogous to 1 of the real numbers. Finally the analogy of

$$x \times x^{-1} = x \times \frac{1}{x} = 1 \qquad \text{with} \qquad F \circ F^{-1} = I$$

suggests a rationale for the symbol F^{-1} for the inverse of F.

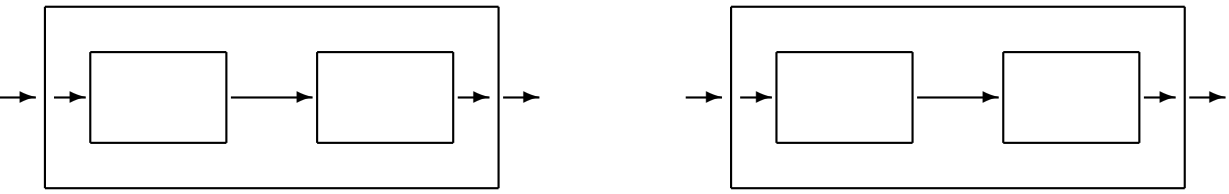
With respect to function composition $F \circ G$ as multiplication, recall the example of $F(x) = \sqrt{x}$ and $G(x) = 1 - x^2$ in which $F \circ G$ and $G \circ F$ were two different functions. Composition of functions is not *commutative*, a property of real number multiplication that does not extend to function composition.

Exercises for Section 2.7 Composition of functions

Exercise 2.7.1 Four examples of composition of two biological processes (of two functions) were described at the beginning of this section on page 87. Write another example of the composition of two biological processes.

Exercise 2.7.2 Put labels on the diagrams in Figure Ex. 2.7.2 to illustrate the dependence of coyote numbers on rabbit *Myxomatosis cuniculi* and the dependence of the frequency of heart attacks on diet of a population.

Figure for Exercise 2.7.2 Diagrams for Exercise 2.7.2



Exercise 2.7.3 a. In Explore 1.5.1 of Section 1.5 on page 27 you measured the area of the mold colony as a function of day. Using the same pictures in Figure 1.14, measure the *diameters* of the mold colony as a function of day and record them in Exercise Table 2.7.3. Remember that grid lines are separated by 2mm. Then use the formula, $A = \pi r^2$, for the area of a circle to compute the third column showing area as a function of day.

b. Determine the dependence of the colony diameter on time.

c. Use the composition of the relation between the area and diameter of a circle ($A = \pi r^2$) with the dependence of the colony diameter on time to describe the dependence of colony area on time.

Table for Exercise 2.7.3 Table for Exercise 2.7.3

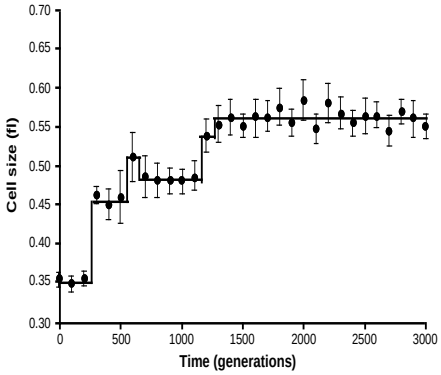
Mold colony, page 28		
Day	Diameter	Area
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		

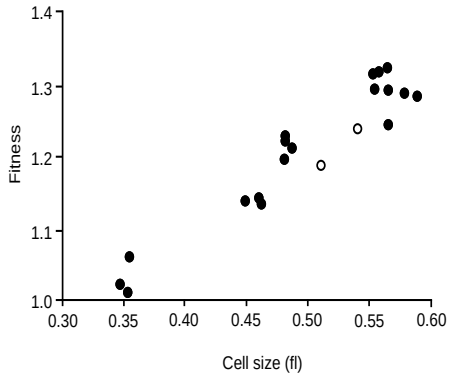
Exercise 2.7.4 S. F. Elena, V. S. Cooper, and R. E. Lenski have grown an *V. natrigens* population for 3000 generations in a constant, nutrient limited environment. They have measured cell size and fitness of cell size and report (*Science* **272**, 1996, 1802-1804) data shown in the graphs below (permission to use these copyrighted figures purchased June 2010). The thrust of their report is the observed abrupt changes in fitness, supporting the hypothesis of "punctuated evolution."

a. Make a table showing cell size as a function of time for generations 0, 100, 200, 300, 400 and 500.

b. Make a table showing fitness as a function of time for generations 0, 100, 200, 300, 400 and 500.

c. Are the data consistent with the hypothesis of 'punctuated equilibrium'?





Exercise 2.7.5 *Find functions, $F(z)$ and $G(x)$ so that the following functions, H, may be written as $F(G(x))$.*

a. $H(x) = (1+x^2)^3$	b. $H(x) = 10^{\sqrt{x}}$	c. $H(x) = \log(2x^2+1)$
d. $H(x) = \sqrt{x^3+1}$	e. $H(x) = \dfrac{1-x^2}{1+x^2}$	f. $H(x) = \log_2(2^x)$

Exercise 2.7.6 *Find functions, $F(u)$ and $G(v)$ and $H(x)$ so that the following functions, K, may be written as $F(G(H(x)))$.*

a. $K(x) = \sqrt{1-\sqrt{x}}$	b. $K(x) = (1+2^x)^3$	c. $K(x) = \log(2x^2+1)$
d. $K(x) = \sqrt{x^3+1}$	e. $K(x) = (1-2^x)^3$	f. $K(x) = \log_2(1+2^x)$

Exercise 2.7.7 Compute the compositions, $f(g(x))$, of the following pairs of functions. In each case specify the domain and range of the composite function, and sketch the graph. Your calculator may assist you. For example, the graph of part A can be drawn on the TI-86 calculator with GRAPH , y(x) = , y1 = x^2 , y2 = 1/(1+y1) You may wish to suppress the display of y1 with SELCT in the y(x)= menu.

a. $f(z) = \frac{1}{1+z}$, $g(x) = x^2$	b. $f(z) = \frac{z}{1+z}$, $g(x) = \frac{x}{1-x}$	c. $f(z) = 5^z$, $g(x) = \log x$
d. $f(z) = \frac{1}{z}$, $g(x) = 1+x^2$	e. $f(z) = \frac{z}{1-z}$, $g(x) = \frac{x}{1+x}$	f. $f(z) = \log z$, $g(x) = 5^x$
g. $f(z) = 2^z$, $g(x) = -x^2$	h. $f(z) = 2^z$, $g(x) = -1/x^2$	i. $f(z) = \log z$, $g(x) = 1-x^2$

Exercise 2.7.8 For each part, find two pairs, F and G, so that $F \circ G$ is H.

a	$H(x) = \sqrt{1-\sqrt{x}}$		b	$H(x) = \dfrac{1}{1-\sqrt{x}}$		c	$H(x) = (1+x^2)^3$	
d	$H(x) = \left(x^{(x^2)}\right)^3$		e	$H(x) = 2^{(x^2)}$		f	$H(x) = (2^x)\,2$	

Exercise 2.7.9 Air is flowing into a spherical balloon at the rate of 10 cm^3/s. What volume of air is in the balloon t seconds after there was no air in the balloon? The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. What will be the radius of the balloon t seconds after there is no air in the balloon?

Exercise 2.7.10 Why are all the points of the graph of $y = \log_{10}(\sin(x))$ on or below the X-axis? Why are there no points of the graph with x-coordinates between π and 2π?

Exercise 2.7.11 Use your calculator to draw the graph of the composition of $F(x) = 10^x$ with $G(x) = \log_{10} x$. Now draw the graph of the composition of G with F. Explain the difference between the two graphs. Use xMin = -2, xMax = 2, yMin = -2, yMax = 2.

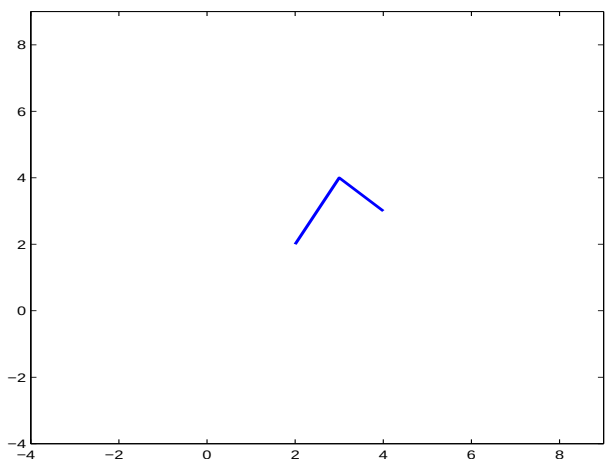
Exercise 2.7.12 Let $P(x) = 2x^3 - 7x^2 + 5$ and $Q(x) = x^2 - x$. Use algebra to compute $Q(P(x))$. You may conclude (correctly) from this exercise that the composition of two polynomials is always a polynomial.

Exercise 2.7.13 Shown in Figure 2.7.13 is the graph of a function, G.

Sketch the graphs of

a.	$G_a(x)$	$=$	$-3 + G(x)$	b.	$G_b(x)$	$=$	$G((x-3))$
c.	$G_c(x)$	$=$	$2 \times G(x)$	d.	$G_d(x)$	$=$	$G(2 \times x)$
e.	$G_e(x)$	$=$	$5 - 2 \times G(x)$	f.	$G_f(x)$	$=$	$G(2 \times (x-3))$
g.	$G_g(x)$	$=$	$3 + G(2 \times (x-3)$	h.	$G_h(x)$	$=$	$4 + G(x+4)$

Figure for Exercise 2.7.13 Graph of G for Exercise 2.7.13.



2.8 Periodic functions and oscillations.

There are many periodic phenomena in the biological sciences. Examples include wingbeat of insects and of birds, nerve action potentials, heart beat, breathing, rapid eye movement sleep, circadian rhythms (sleep-wake cycles), women's menstrual cycle, bird migrations, measles incidence, locust emergence. All of these examples are periodic repetition with time, the variable usually associated with periodicity. The examples are listed in order of increasing period of repetition in Table 2.8.

Table 2.8: Characteristic periods of time-periodic biological processes.

Biological process	Period	Biological process	Period
Insect wingbeat	0.02 sec	Circadian cycle (sleep-wake)	1 day
Nerve action potential	0.2 sec	Menstrual cycle	28 days
Heart beat	1 sec	Bird migrations	1 year
Breathing (rest)	5 sec	Measles	2 years
REM sleep	$\approx$ 90 min	Locust	17 years
Tides	6 hours		

A measurement usually quantifies the state of a process periodic with time and defines a function characteristic of the process. Some examples are shown in Figure 2.14. The measurement may be of physical character as in electrocardiograms, categorical as in stages of sleep, or biological as in measurement of hormonal level.

There are also periodic variations with space that result in the color patterns on animals – stripes on zebras and tigers, spots on leopards – , regular spacing of nesting sites, muscle striations, segments in a segmented worm, branching in nerve fibers, and the five fold symmetry of echinoderms. Some spatially periodic structures are driven by time-periodic phenomena – branches in a tree, annular tree rings, chambers in a nautilus, ornamentation on a snail shell. The pictures in Figure 2.8 illustrate some periodic functions that vary with linear space. The brittle star shown in Figure 2.16 varies periodically with angular change in space.

In all instances there is an independent variable, generally time or space, and a dependent variable that is said to be periodic.

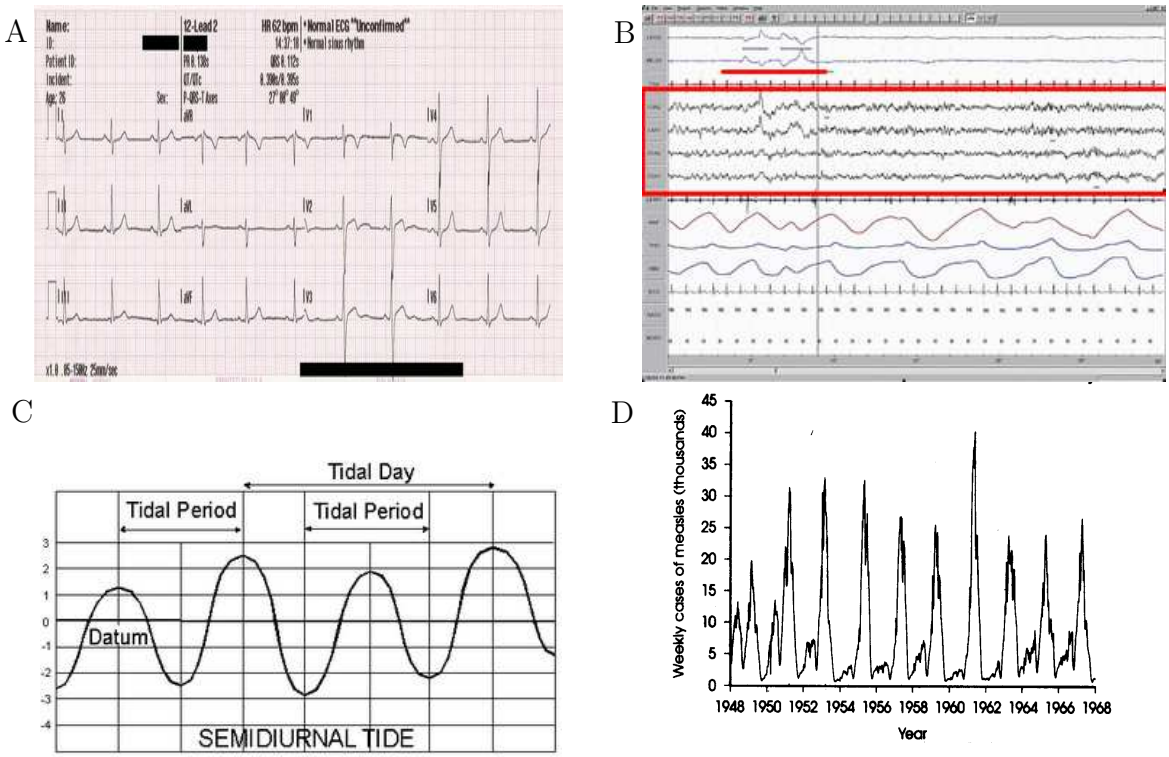


Figure 2.14: Figures demonstrating periodic repetition with time. (a) Electorcardiogram. http://en.wikipedia.org/wiki/Electrocardiography uploaded by MoodyGroove. (b) This is a screenshot of a polysomnographic record (30 seconds) representing Rapid Eye Movement Sleep. EEG highlighted by red box. Eye movements highlighted by red line. http://en.wikipedia.org/wiki/File:REM.png uploaded by Mr. Sandman. (c) Tidal movement (≈12 hr) (http://en.wikipedia.org/wiki/Tide), uploaded from NOAA, http://co-ops.nos.noaa.gov/images/restfig6.gif. (d) Recurrent epidemics of measles (≈2 yr), Anderson and May, Vaccination and herd immunity to infectious diseases, *Nature* **318** 1985, pp 323-9, Figure 1a. Permission kindly provided without charge by Nature Publishing Group.

Definition 2.8.1 Periodic Function. *A function, F, is said to be periodic if there is a positive number, p, such that for every number x in the domain of F, $x + p$ is also in the domain of F and*

$$F(x + p) = F(x). \tag{2.11}$$

and for each number q where $0 < q < p$ there is some x in the domain of F for which

$$F(x + q) \neq F(x)$$

The period of F is p.

The **amplitude** *of a periodic function F is one-half the difference between the largest and least values of $F(t)$, when these values exist.*

Figure 2.15: Figures demonstrating periodic repetition with space. A. Muscle striations, Kansas University Medical School, http://www.kumc.edu/instruction/medicine/anatomy/histoweb/muscular/muscle02.htm. B. Bristle worm, http://www.photolib.noaa.gov/htmls/reef1016.htmImage ID: reef1016, Dr. Anthony R. Picciolo. C. Annular rings, uploaded by Arnoldius to commons.wikimedia.org/wiki D. A common genet (*Genetta genetta*), http://en.wikipedia.org/wiki/File:Genetta_genetta_felina_(Wroclaw_zoo).JPG Uploaded by Guérin Nicolas. Note a periodic distribution of spots on the body and stripes on the tail.

The condition that 'for every number x in the domain of F, $x + p$ is also in the domain of F' implies that the domain of F is infinite in extent — it has no upper bound. Obviously, all of the examples that we experience are finite in extent and do not satisfy Definition 2.8.1. We will use 'periodic' even though we do not meet this requirement.

The condition 'when these numbers exist' in the definition of amplitude is technical and illustrated in Figure 2.17 by the graph of

$$F(t) = t - [t]$$

where $[t]$ denotes the integer part of t. ($[\pi] = 3$, $[\sqrt{30}] = 5$). There is no largest value of $F(t)$.

$$F(n) = n - [n] = 0 \quad \text{for integer} \quad n$$

If t is such that $F(t)$ is the largest value of F then t is not an integer and is between an integer n and $n + 1$. The midpoint s of $[t, n + 1]$ has the property that $F(t) < F(s)$, so that $F(t)$ is not the largest value of F.

A  B 

Figure 2.16: A. Periodic distribution of Gannet nests in New Zealand. B. A star fish demonstrating angular periodicity, NOAA's Coral Kingdom Collection, Dr. James P. McVey, http://www.photolib.noaa.gov/htmls/reef0296.htm.

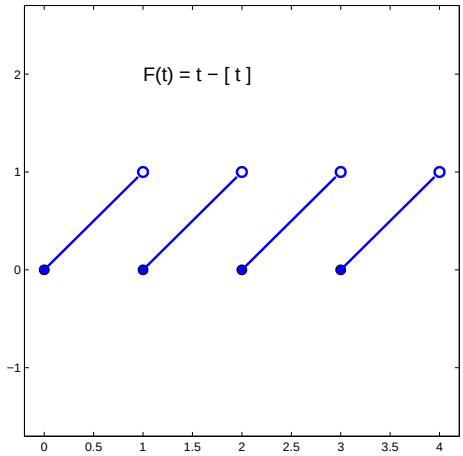


Figure 2.17: The graph of $F(t) = t - [t]$ has no highest point.

The function, $F(t) = t - [t]$ is pretty clearly periodic of period 1, and we might say that its amplitude is 0.5 even though it does not satisfy the definition for amplitude. Another periodic function that has no amplitude is the tangent function from trigonometry.

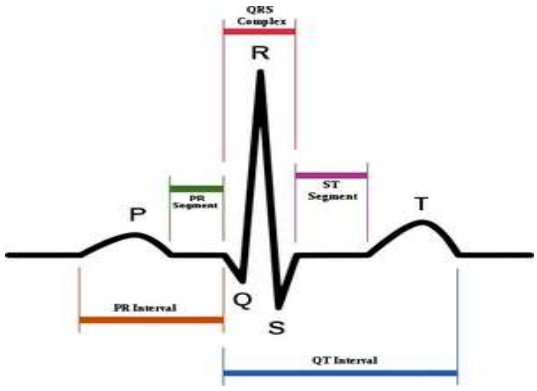
Periodic Extension Periodic functions in nature do not strictly satisfy Definition 2.8.1, but can be approximated with strictly periodic functions over a finite interval of their domain. Periodic functions in nature also seldom have simple equation descriptions. However, one can sometimes describe the function over one period and then assert that the function is periodic – thus describing the entire function.

Example 2.8.1 Electrocardiograms have a very characteristic periodic signal as shown in Figure 2.14.

A picture of a typical signal from 'channel I' is shown in Figure 2.18; the regions of the signal, P-R, QRS, etc. correspond to electrical events in the heart that cause contractions of specific muscles. Is there an equation for such a signal? Yes, a very messy one!

The graph of the following equation is shown in Figure 2.18(b). It is similar to the typical

(a)



(b)

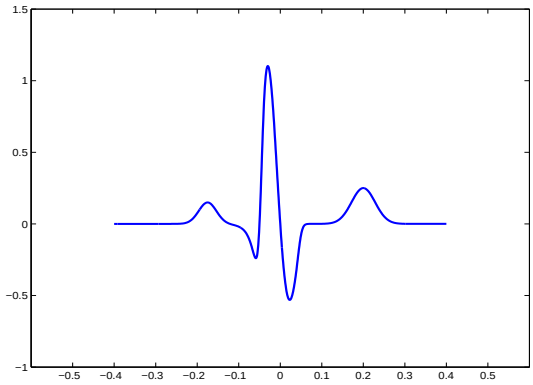


Figure 2.18: (a) Typical signal from an electrocardiogram, created by Agateller (Anthony Atkielski) http://en.wikipedia.org/wiki/Electrocardiography, . (b) Graph of Equation 2.12

electrocardiogram in Figure 2.18(a).

$$H(t) = 25000\frac{(t+0.05)\,t\,(t-0.07)}{\left(1+(20t)^{10}\right)\times 2^{\left(2^{(40t)}\right)}} + 0.15\times 2^{-1600(t+0.175)^2} + 0.25\times 2^{-900(t-0.2)^2} \tag{2.12}$$

$$-0.4 \le t \le 0.4$$

Explore 2.8.1 Draw the graph of the heart beat Equation 2.12 on your calculator. You will find it useful to break the function into parts. On the TI-86 it was useful to define

$$\begin{aligned} y1 &= 25000\times\frac{(x+0.05)\,x\,(x-0.07)}{\left(1+(20x)^{10}\right)} \\ y2 &= (y1)/2^{\left(2^{(40x)}\right)} \end{aligned}$$

and $y3$ and $y4$ for the other two terms, then combine $y2$, $y3$ and $y4$ into $y5$ and select only $y5$ to graph. ■

The heart beat function H in Equation 2.12 is not a periodic function and is defined only for $-0.4 \le t \le 0.4$. Outside that interval, the expression defining $H(t)$ is essentially zero. However, we can simultaneously extend the definition of H and make it periodic by

$$H(t) = H(t-0.8) \qquad \text{for all } t$$

What is the impact of this? $H(0.6)$, say is now defined, to be $H(-0.2)$. Immediately, H has meaning for $0.4 \le t \le 1.2$, and the graph is shown in Figure 2.19(a). Now because $H(t)$ has meaning on $0.4 \le t \le 1.2$, H also has meaning on $0.4 \le t \le 2.0$ and the graph is shown in Figure 2.19(b). The extension continues indefinitely. ■ ■

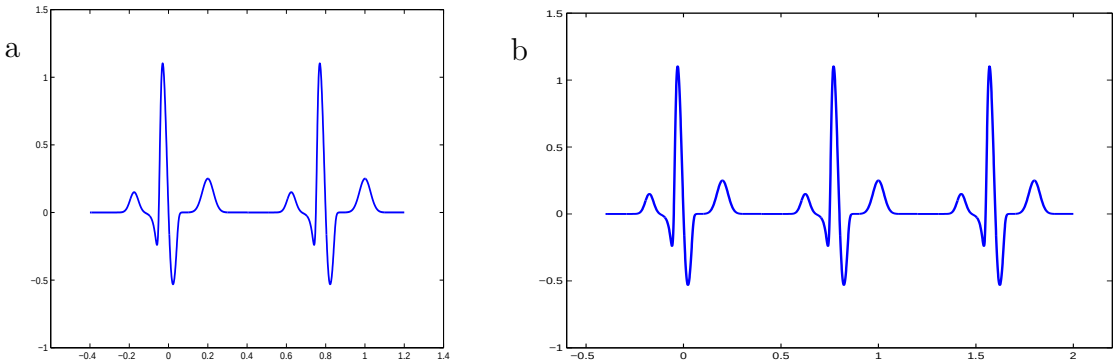


Figure 2.19: (a) Periodic extension of the heart beat function H of Equation 2.12 by one period. (a) Periodic extension of the heart beat function H to two periods.

Definition 2.8.2 *Periodic extension of a function. If f is a function defined on an interval $[a, b)$ and $p = b - a$, the periodic extension of F of f of period p is defined by*

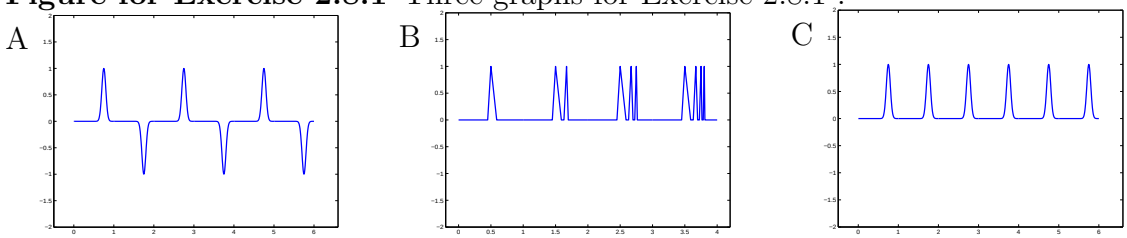
$$\begin{array}{rcl} F(t) & = & f(t) \quad \textit{for} \quad a \leq t < b, \\ F(t+p) & = & F(t) \quad \textit{for} \quad -\infty < t < \infty \end{array} \tag{2.13}$$

Equation 2.13 is used recursively. For t in $[a\, b)$, $t + p$ is in $[b, b + p)$ and Equation 2.13 defines F on $[b, b + p)$. Then, for t in $[b, b + p)$, $t + p$ is in $[b + p, b + 2p)$ and Equation 2.13 defines F on $[b + p, b + 2p)$. Continue this for all values of $t > b$. If t is in $[a - p, a)$ then $t + p$ is in $[a, b)$ and $F(t) = F(t + p)$. Continuing in this way, F is defined for all t less than a.

Exercises for Section 2.8, Periodic functions and oscillations.

Exercise 2.8.1 Which of the three graphs in Figure Ex. 2.8.1 are periodic. For any that is periodic, find the period and the amplitude.

Figure for Exercise 2.8.1 Three graphs for Exercise 2.8.1 .



Exercise 2.8.2 Shown in Figure Ex. 2.8.2 is the graph of a function, f defined on the interval [1,6]. Let F be the periodic extension of f of period 5.

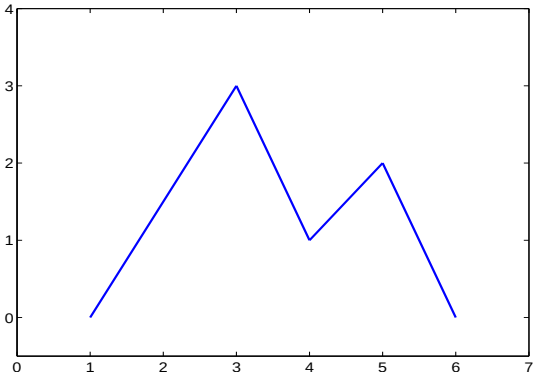
a. What is the period of F?

b. Draw a graph of F over three periods.

c. Evaluate $F(1)$, $F(3)$, $F(8)$, $F(23)$, $F(31)$ and $F(1004)$.

d. Find the amplitude of F.

Figure for Exercise 2.8.2 Graph of a function f for Exercise 2.8.2 .



Exercise 2.8.3 Suppose your are traveling an interstate highway and that every 10 miles there is an emergency telephone. Let D be the function defined by

$$D(x) \qquad \text{is the distance to the nearest emergency telephone}$$

where x is the mileage position on the highway.

a. Draw a graph of D.

b. Find the period and amplitude of D.

Exercise 2.8.4 Your 26 inch diameter bicycle wheel has a patch on it. Let P be the function defined by

$$P(x) \qquad \text{is the distance from patch to the ground}$$

where x is the distance you have traveled on a bicycle trail.

a. Draw a graph of P (approximate is acceptable).

b. Find the period and amplitude of P.

Exercise 2.8.5 Let F be the function defined for all numbers, x, by

$$F(x) = \qquad \text{the distance from } x \text{ to the even integer nearest } x.$$

a. Draw a graph of F.

b. Find the period and amplitude of F.

Exercise 2.8.6 Let f be the function defined by

$$f(x) = 1 - x^2 \qquad -1 \leq x \leq 1$$

Let F be the extension of f with period 2.

a. Draw a graph of f.

b. Draw a graph of F.

c. Evaluate $F(1)$, $F(2)$, $F(3)$, $F(12)$, $F(31)$ and $F(1002)$.

d. Find the amplitude of F.

2.8.1 Trigonometric Functions.

The trigonometric functions are perhaps the most familiar periodic functions and often are used to describe periodic behavior. However, not many of the periodic functions in biology are as simple as the trigonometric functions, even over restricted domains.

Amplitude and period and frequency of a Cosine Function. The function

$$H(t) = A\cos\left(\frac{2\pi}{P}t + \phi\right) \qquad A > 0 \qquad P > 0$$

and ϕ any angle has amplitude A, period P, and frequency $1/P$.

Graphs of rescaled cosine functions shown in Figure 2.20 demonstrate the effects of A and P.

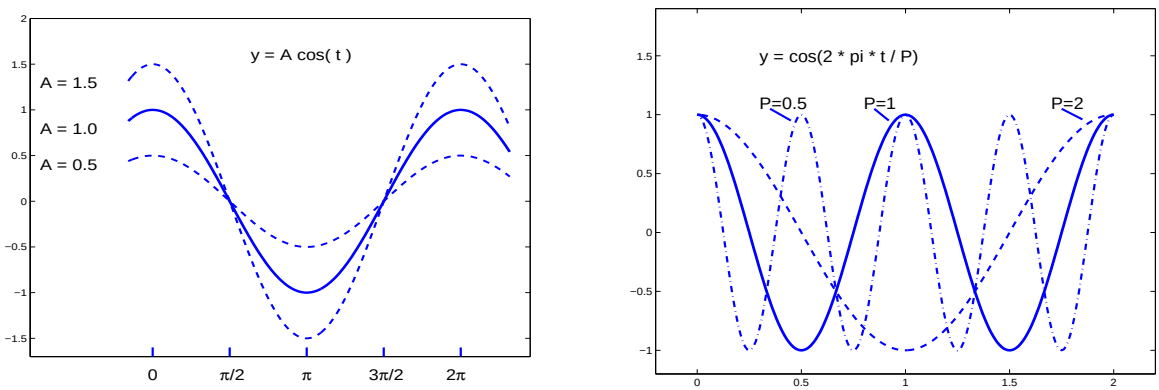


Figure 2.20: (a) Graphs of the cosine function for amplitudes 0,5 1.5 and 1.5. (b) Graphs of the cosine function for periods 0.5, 1.0 and 2.0.

Example 2.8.2 *Problem.* Find the period, frequency, and amplitude of

$$H(t) = 3\sin(5t + \pi/3)$$

Solution. Write $H(t) = 3\sin(5t + \pi/3)$ as

$$H(t) = 3\sin\left(\frac{2\pi}{2\pi/5}t + \pi/3\right).$$

Then the amplitude of P is 3, and the period is $2\pi/5$ and the frequency is $5/(2\pi)$. ■

Motion of a Spring-Mass System A mass suspended from a spring, when vertically displaced from equilibrium a small amount will oscillate above and below the equilibrium position. A graph of displacement from equilibrium *vs* time is shown in in Figure 2.21A for a certain system. Critical points of the graph are

$$(2.62\text{ s}, 72.82\text{ cm}), \qquad (3.18\text{ s}, 46.24\text{ cm}), \qquad \text{and} \qquad (3.79\text{ s}, 72.80\text{ cm})$$

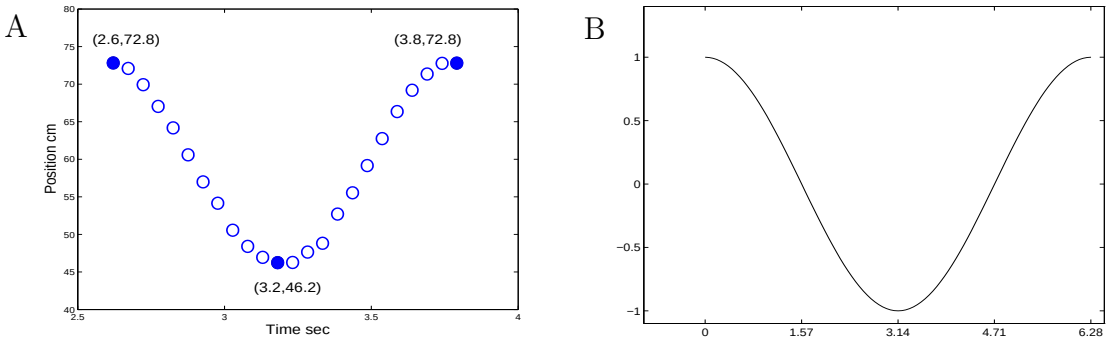


Figure 2.21: Graphs of A. the motion of a spring-mass system and B. $H(x) = \cos t$.

Also shown is the graph of the cosine function, $H(t) = \cos t$.

It is clear that the data and the cosine in Figure 2.21 have similar shapes, but examine the axes labels and see that their periods and amplitudes are different and the graphs lie in different regions of the plane. We wish to obtain a variation of the cosine function that will match the data.

The period of the harmonic motion is 3.79 - 2.62 = 1.17 seconds, the time of the second peak minus the time of the first peak. The amplitude of the harmonic motion is $0.5(72.82 - 46.24) = 13.29$ cm, one-half the difference of the heights of the highest and lowest points.

Now we expect a function of the form

$$H_0(t) = 13.3\cos(\frac{2\pi}{1.17}t)$$

to have the shape of the data, but we need to translate vertically and horizontally to match the data. The graphs of H_0 and the data are shown in Figure 2.22A, and the shapes are similar. We need to match the origin (0,0) with the corresponding point (2.62, 59.5) of the data. We write

$$H(t) = 59.5 + 13.3\ \cos\left(\frac{2\pi}{1.17}(t - 2.62)\right)$$

The graphs of H and the data are shown in Figure 2.22B and there is a good match.

Polynomial approximations to the sine and cosine functions. Shown in Figure 2.23 are the graphs of $F(x) = \sin x$ and the graph of (dashed curve)

$$P_3(x) \quad = \quad x - \frac{x^3}{120}$$

on $[0, \pi/2]$.

The graph of P_3 is hardly distinguishable from the graph of F on the interval $[0, \pi/4]$, although they do separate near $x = \pi/2$. $F(x) = \sin x$ is difficult to evaluate (without a calculator) except for special values such as $F(0) = \sin 0 = 0$, $F(\pi/3) = \sin \pi/3 = 0.5$ and $F(\pi/2) = \sin \pi/2 = 1.0$. However, $P_3(x)$ can be calculated using only multiplication, division and subtraction. The maximum difference between P_3 and F on $[0, \pi/4]$ occurs at $\pi/4$ and $F(\pi/4) = \sqrt{2}/2 \doteq 0.70711$ and $P_3(\pi/4) = 0.70465$. The relative error in using $P_3(\pi/4)$ as an approximation to $F(\pi/4)$ is

$$\text{Relative Error} \qquad = \frac{|P_3(\pi/4) - F(\pi/4)|}{F(\pi/4)} \doteq \frac{|0.70465 - 0.70711|}{0.70711} = 0.0036$$

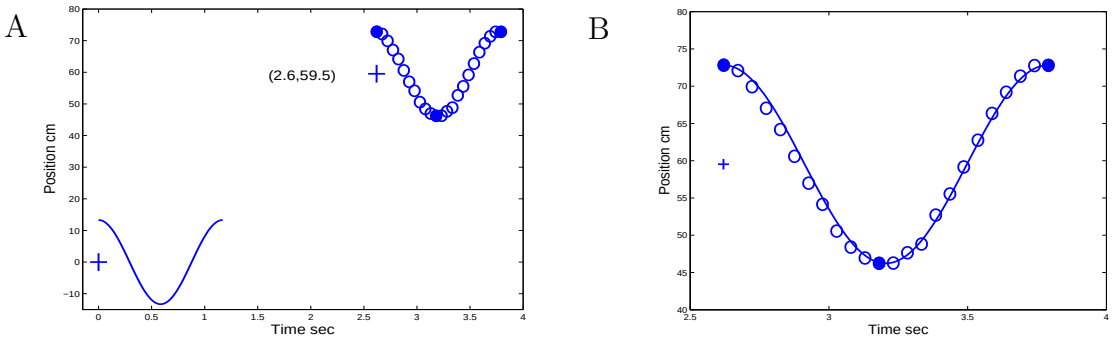


Figure 2.22: On the left is the graph of $H_0(t) = 13.3\cos\left(\frac{2\pi}{1.17}t\right)$ and the harmonic oscillation data. The two are similar in form. The graph to the right shows the translation, H, of H_0, $H(t) = 59.5 + 13.3\cos\left(\frac{2\pi}{1.17}(t-2.62)\right)$ and its approximation to the harmonic oscillation data.

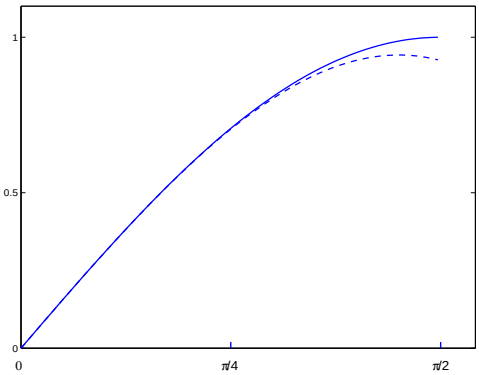


Figure 2.23: The graphs of $F(x) = \sin x$ and $P_3(x) = x - \frac{x^3}{120}$ on $[0, \pi/2]$.

thus less than 0.5% error is made in using the rather simple $P_3(x) = x - (x^3)/120$ in place of $F(x) = \sin x$ on $[0, \pi/4]$.

Exercises for Section 2.8.1, Trigonometric functions.

Exercise 2.8.7 Find the periods of the following functions.

a.	$P(t)$	$=$	$\sin(\frac{\pi}{3}t)$	d.	$P(t)$	$=$	$\sin(t) + \cos(t)$
b.	$P(t)$	$=$	$\sin(t)$	e.	$P(t)$	$=$	$\sin(\frac{2\pi}{2}t) + \sin(\frac{2\pi}{3}t)$
c.	$P(t)$	$=$	$5 - 2\sin(t)$	f.	$P(t)$	$=$	$\tan 2t$

Exercise 2.8.8 Sketch the graphs and label the axes for

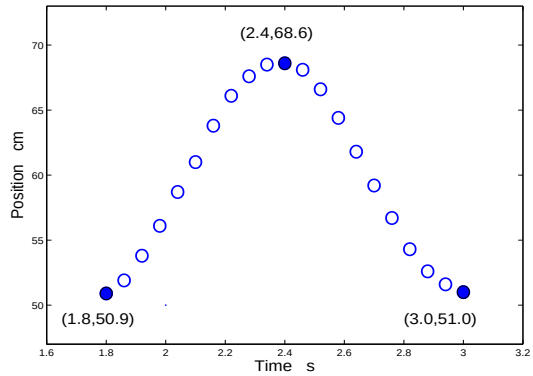
$$\text{(a)} \qquad \mathrm{y} = 0.2\cos\left(\frac{2\pi}{0.8}\mathrm{t}\right) \qquad \text{and} \qquad \text{(b)} \qquad \mathrm{y} = 5\cos\left(\frac{1}{8}\mathrm{t} + \pi/6\right)$$

Exercise 2.8.9 Describe how the harmonic data of Figure 2.22A would be translated so that the graph of the new data would match that of H_0.

Exercise 2.8.10 Use the identity, $\cos t = \sin(t + \frac{\pi}{2})$, to write a sine function that approximates the harmonic oscillation data.

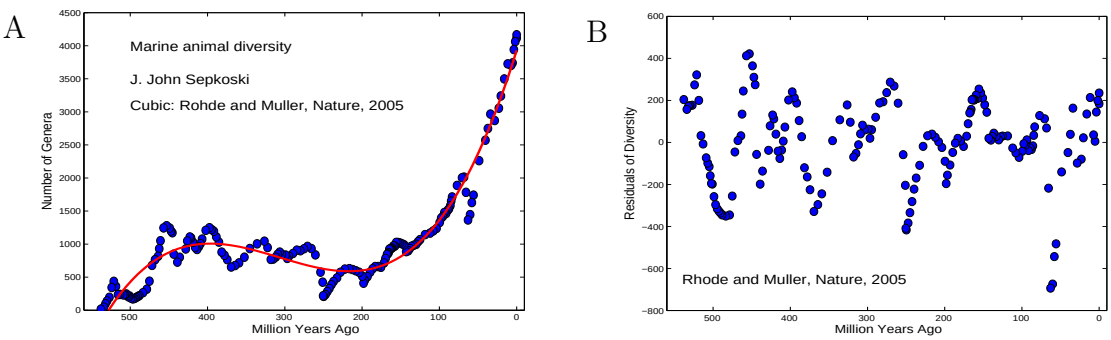
Exercise 2.8.11 Fit a cosine function to the spring-mass oscillation shown in Exercise Figure 2.8.11.

Figure for Exercise 2.8.11 Graph of a spring-mass oscillation for Exercise 2.8.11.



Exercise 2.8.12 A graph of total marine animal diversity over the period from 543 million years ago until today is shown in Exercise Figure 2.8.12A. The data appeared in a paper by Robert A. Rohde and Richard A. Muller[4], and are based on work by J. John Sepkoski[5]. Also shown is a cubic polynomial fit to the data by Robert Rohde and Richard Muller who were interested in the the difference between the data and the cubic shown in Exercise Figure 2.8.12B. They found that a sine function of period 62 million years fit the residuals rather well. Find an equation of such a sine function.

Figure for Exercise 2.8.12 A. Marine animal diversity and a cubic polynomial fit to the data. B. The residuals of the cubic fit. Figures adapted by permission without charge from Nature Publishing Group, Ltd. Robert A. Rohde and Richard A. Muller, Cycles in fossil diversity, *Nature* 434, 208-210, Copyright 2005, http://www.nature.com



Exercise 2.8.13 Use a graphing calculator draw the graphs of $F(x) = \sin x$ and

$$P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

on the range $0 \le x \le \pi$. Compute the relative error in $P_5(\pi/4)$ as an approximation to $F(\pi/4)$ and in $P_5(\pi/2)$ as an approximation to $F(\pi/2)$.

[4]Robert A. Rohde and Richard A. Muller, Cycles in fossil diversity, *Nature* 434, 208-210.

[5]J. John Sepkoski, A compendium of Fossil Marine Animal Genera, Eds David Jablonski and Michael Foote, *Bulletins of AMerican Paleontology*, 363, 2002.

Exercise 2.8.14 Polynomial approximations to the cosine function.

a. Use a graphing calculator draw the graphs of $F(x) = \cos x$ and

$$P_2(x) = 1 - \frac{x^2}{2}$$

on the range $0 \leq x \leq \pi$.

b. Compute the relative error in $P_2(\pi/4)$ as an approximation to $F(\pi/4)$ and in $P_2(\pi/2)$ as an approximation to $F(\pi/2)$.

c. Use a graphing calculator draw the graphs of $F(x) = \cos x$ and

$$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

on the range $0 \leq x \leq \pi$.

d. Compute the relative error in $P_4(\pi/4)$ as an approximation to $F(\pi/4)$ and the absolute error in $P_4(\pi/2)$ as an approximation to $F(\pi/2)$.

Chapter 3

The Derivative

Where are we going?

In this chapter, you will learn about the rate of change of a function, a concept at the heart of calculus.

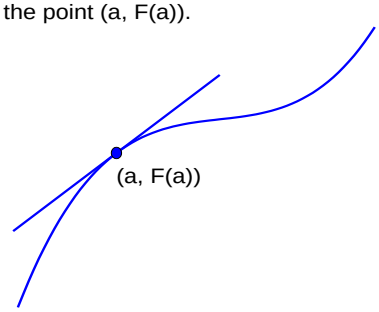
Suppose F is a function.

The *rate of change of* F at a point a in its domain is the slope of the tangent to the graph of F at $(a, F(a))$, if such a tangent exists.

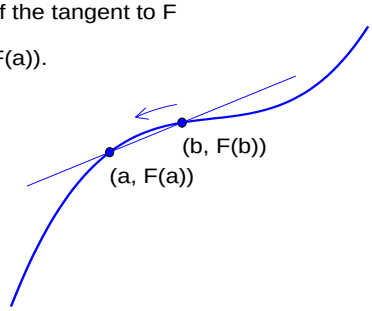
The slope of the tangent is approximated by slopes of lines through $(a, F(a))$ and points $(b, F(b))$ when b is close to a.

We initiate the use of rate of change to form models of biological systems.

The *rate of change* of a function F at a number a is the *slope of the tangent* to the graph of F at the point (a, F(a)).



As (b, f(b)) approaches (a, f(a)) the *slope of the secant* through (a, f(a)) and (b, f(b)) approaches the slope of the tangent to F at (a, F(a)).



Calculus is the study of change and rates of change. It has two primitive concepts, the *derivative* and the *integral*. Given a function relating a dependent variable to an independent variable, the derivative is the *rate of change* of the dependent variable as the independent variable changes. We determine the derivative of a function when we answer questions such as

1. Given population size as a function of time, at what rate is the population growing?
2. Given the position of a particle as a function of time, what is its velocity?
3. At what rate does air density decrease with increasing altitude?
4. At what rate does the pressure of one mole of O_2 at 300°K change as the volume changes if the temperature is constant?

On the other hand, given the rate of change of a dependent variable as an independent variable changes, the integral is the function that relates the dependent variable to the independent variable.

1. Given the growth rate of a population at all times in a time interval, how much did the population size change during that time interval?
2. Given the rate of renal clearance of penicillin during the four hours following an initial injection, what will be the plasma penicillin level at the end of that four hour interval?
3. Given that a car left Chicago at 1:00 pm traveling west on I80 and given the velocity of the car between 1 and 5 pm, where was the car at 5 pm?

The derivative is the subject of this chapter; the integral is addressed in Chapter 13. The derivative and integral are independently defined. Chapter 13 can be studied before this one and without reference to this one. However, the two concepts are closely related, and the relation between them is The Fundamental Theorem of Calculus, developed in Chapter 14.

Explore 3.0.2 You use both the derivative and integral concepts of calculus when you cross a busy street. You observe the nearest oncoming car and subconsciously estimate its distance from you and its speed (use of the derivative). You decide whether you have time to cross the street before the car arrives at your position (a simple use of the integral). Might there be a car different from the nearest car that will affect your estimate of the time available to cross the street? You may even observe that the car is slowing down and you may estimate whether it will stop before it gets to your crossing point (involves the integral).

Give an example in which estimates of distances and speeds and times are important for successful performance in a sport. ■

3.1 Tangent to the graph of a function

At what rate was the world human population increasing in 1980? Shown in Figure 3.1 are data for the twentieth century and a graph of an approximating function, F. A tangent to the graph of F at $(1980, F(1980))$ is drawn and has a slope of $0.0781 \times 10^9 = 78,100,000$. Now,

$$\text{slope is} \quad \frac{\text{rise}}{\text{run}} = \frac{\text{change in population}}{\text{change in years}} \approx \frac{\text{people}}{\text{year}}.$$

The units of slope, then, are people/year. Therefore,

$$\text{slope} = 78,100,000 \ \frac{\text{people}}{\text{year}}.$$

The world population was increasing approximately 78,100,000 people per year in 1980.

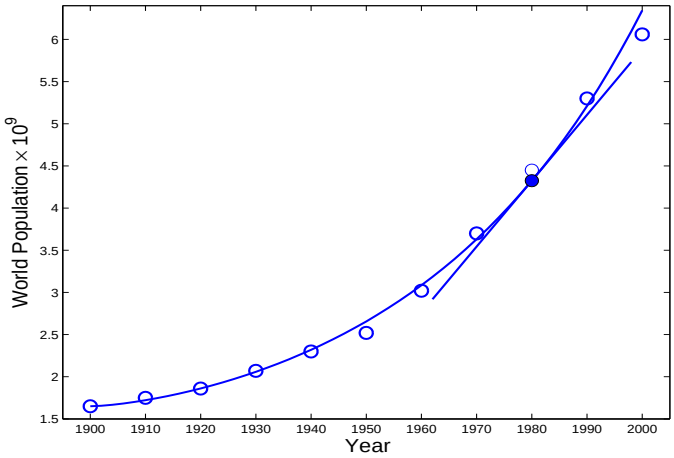


Figure 3.1: Graph of United Nations estimates of world human population for the twentieth century, an approximating curve, and a tangent to the curve. The slope of the tangent is $0.0781 \times 10^9 = 78,100,000$.

Explore 3.1.1 At approximately what rate was the world human population increasing in 1920? ■

Definition 3.1.1 Rate of change of a function at a point. The rate of change of a function, F, at a number a in its domain is the slope of the line that is tangent to the graph of F at the point $(a, F(a))$, if there is a tangent.

To be of use, Definition 3.1.1 requires a definition of tangent to a graph which is given below in Definition 3.1.2. In some cases there will be no tangent. In each graph shown in Figure 3.2 there is no tangent at the point (3,2) of the graph. Students usually agree that there is no tangent in graphs A and B, but sometimes argue about case C.

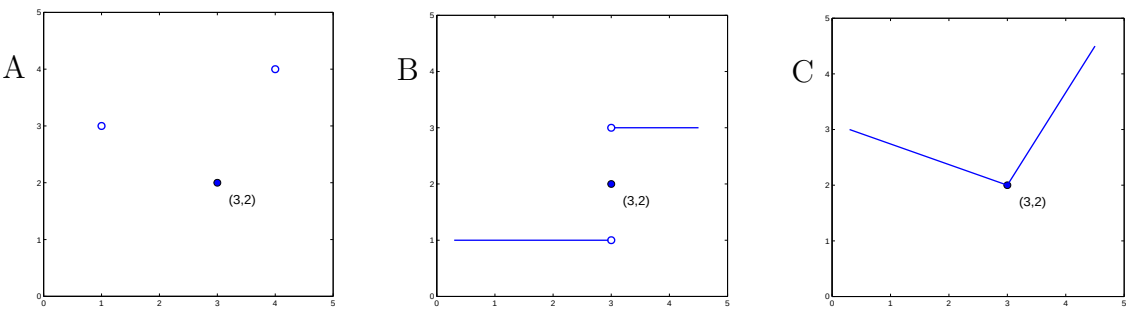


Figure 3.2: In neither of these graphs will we accept a line as tangent to the graph at the point (2,3)

Explore 3.1.2 Do you agree that there is no line tangent to any of the graphs in Figure 3.2 at the point (3,2)? ■

Examples of tangents to graphs are shown in Figure 3.3; all the graphs have tangents at the point $(2, 4)$. In Figure 3.3C, however, the line shown is not the tangent line. The graphs in B and C are the same and the tangent is the line drawn in B.

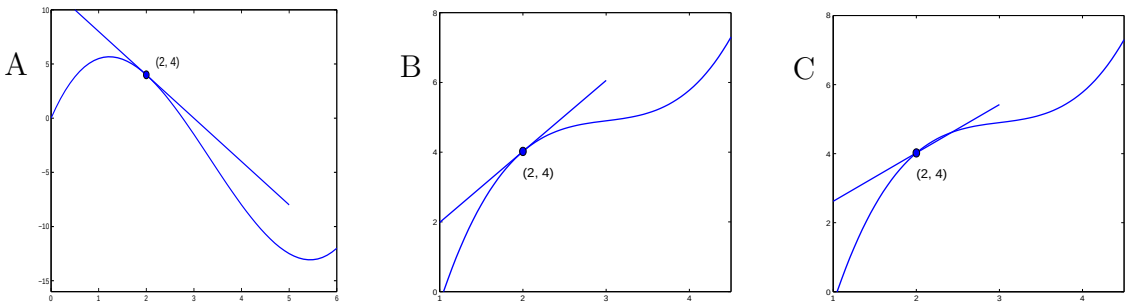


Figure 3.3: All of these graphs have a tangent to the graph at the point (2,4). However, the line drawn in C is not the tangent.

A line tangent to the graph of F at a point $(a, F(a))$ contains $(a, F(a))$ so in order to find the tangent we only need to find the slope of the tangent, which we denote by m_a. To find m_a we consider points b in the domain of F that are different from a and compute the slopes,

$$\frac{F(b) - F(a)}{b - a},$$

of the lines that contain $(a, F(a))$ and $(b, F(b))$. The line containing $(a, F(a))$ and $(b, F(b))$ is called a *secant* of the graph of F. The slope, $\dfrac{F(b) - F(a)}{b - a}$, of the secant is a 'good' approximation to the slope of the tangent when b is 'close to' a. A graph, a tangent to the graph, and a secant to the graph are shown in Figure 3.4. If we could animate that figure, we would slide the point $(b, F(b))$ along the curve towards $(a, F(a))$ and show the secant moving toward the tangent.

A substitute for this animation is shown in Figure 3.5. Three points are shown, B_1, B_2, and B_3 with $B_1 = (\, b_1, F(b_1)\,)$, $B_2 = (\, b_2, F(b_2)\,)$, and $B_3 = (\, b_3, F(b_3)\,)$. The numbers b_1, b_2, and b_3 are progressively closer to a, and the slopes of the dashed lines from B_1, B_2, and B_3 to (a,F(a)) are progressively closer to the slope of the tangent to the graph of F at $(a, F(a))$. Next look at the magnification of F in Figure 3.5B. The progression toward a continues with b_3, b_4, and b_5, and the slopes from B_3, B_4, and B_5 to $(a, F(a))$ move even closer to the slope of the tangent.

Explore 3.1.3 It is important in Figure 3.5 that as the x-coordinates b_1, b_2, $\cdots$ approach a the points $B_1 = (\, b_1, F(b_1)\,)$, $B_2 = (\, b_2, F(b_2)\,)$, $\cdots$ *on the curve approach* $(a, f(a))$. Would this be true in Figure 3.2B? ■

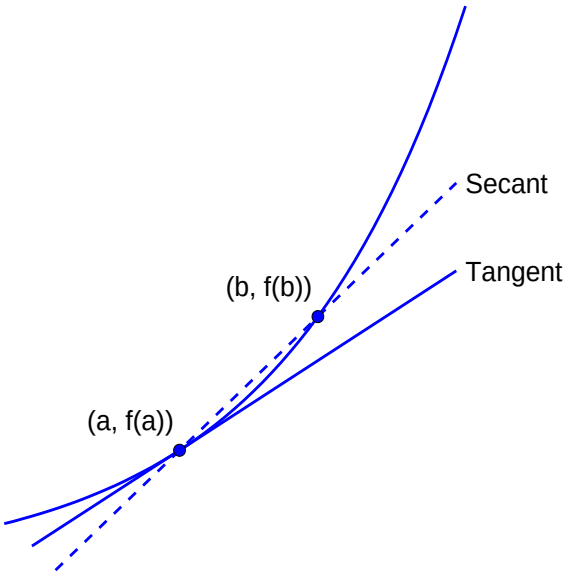


Figure 3.4: A graph, a tangent to the graph, and a secant to the graph.

Definition 3.1.2 Tangent to a graph. Suppose the domain of a function F contains an open interval that contains a number a. Suppose further that there is a number m_a such that that for points b in the interval different from a,

$$\text{as } b \text{ approaches } a \qquad \frac{F(b)-F(a)}{b-a} \qquad \text{approaches} \quad m_a.$$

Then m_a is the slope of the tangent to F at $(a, F(a))$. The graph of $y = F(a) + m_a(x-a)$ is the tangent to the graph of F at $(a, F(a))$.

We are making progress. We now have a definition of tangent to a graph and therefore have given meaning to rate of change of a function. However, we must make sense of the phrase

$$\text{as } b \text{ approaches } a \qquad \frac{F(b)-F(a)}{b-a} \qquad \text{approaches} \quad m_a.$$

This phrase is a bridge between geometry and analytical computation and is formally defined in Definition 3.2.1. We first use it on an intuitive basis. Some students prefer an alternate, similarly intuitive statement:

$$\text{if } b \text{ is close to } a \qquad \frac{F(b)-F(a)}{b-a} \qquad \text{is close to} \quad m_a.$$

Both phrases are helpful.

Consider the parabola, shown in Figure 3.6,

$$F(t) = t^2 \qquad \text{for all } t \text{ and a point } (a, a^2) \text{ of } F.$$

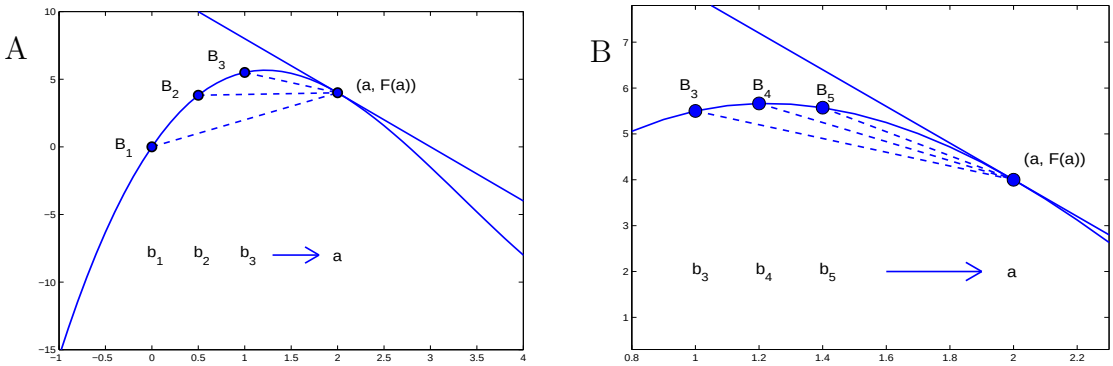


Figure 3.5: A. A graph and tangent to the graph at $(a, F(a))$. Slopes of the secant lines from B_1, B_2, and B_3 to $(a, F(a))$ progressively move toward the slope of the tangent. B. A magnification of A with the progression continued.

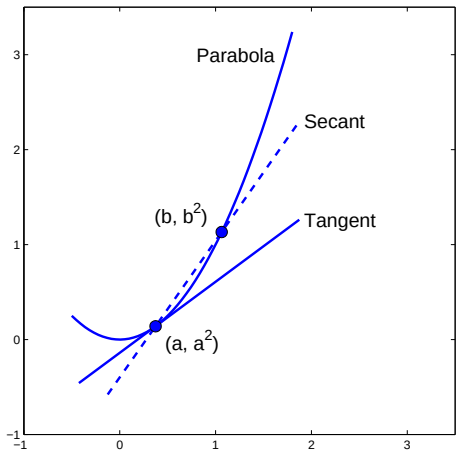


Figure 3.6: The parabola, $F(t) = t^2$, a tangent to the parabola at (a, a^2), and a secant through (a, a^2) and (b, b^2).

The slope of the secant is

$$\frac{F(b) - F(a)}{b - a} = \frac{b^2 - a^2}{b - a} = \frac{(b - a)(b + a)}{b - a} = b + a.$$

Although 'approaches' has not been carefully defined, it should not surprise you if we conclude that

$$\text{as } b \text{ approaches } a \qquad \frac{F(b) - F(a)}{b - a} = \frac{b^2 - a^2}{b - a} = b + a \qquad \text{approaches} \quad a + a = 2a.$$

Alternatively, we might conclude that

$$\text{if } b \text{ is close to } a \qquad \frac{F(b) - F(a)}{b - a} = \frac{b^2 - a^2}{b - a} = b + a \qquad \text{is close to} \quad a + a = 2a.$$

We make either conclusion, and along with it conclude that the slope of the tangent to the parabola $y = x^2$ at the point (a, a^2) is $2a$. Furthermore, the rate of change of $F(t) = t^2$ at a is $2a$. This is the first of many examples.

Explore 3.1.4 Do this. Use your intuition to answer the following questions. You will not answer g. or h. easily, if at all, but think about it.

a. As b approaches 4,	what number does $3b$ approach?
b. As b approaches 2,	what number does b^2 approach?
c. If b is close to 5,	what number is $3b + b^3$ close to?
d. As b approaches 0,	what number does $\frac{b^2}{b}$ approach?
e. If b is close to 0,	what number is 2^b close to?
f. As b approaches 0,	does $\frac{2^b}{b}$ approach a number?
g. As b approaches 0,	what number does $\frac{\sin b}{b}$ approach? Use radian measure of angles.
g*. If b is close to 0,	what number is $\frac{\sin b}{b}$ close to? Use radian measure of angles.
h. As b approaches 0,	what number does $\frac{2^b-1}{b}$ approach?
h*. If b is close to 0,	what number is $\frac{2^b-1}{b}$ close to?

■

One may look for an answer to c., for example, by choosing a number, b, close to 5 and computing $3b + b^3$. Consider 4.99 which some would consider close to 5. Then $3 \times 4.99 + 4.99^3$ is 139.22. 4.99999 is even closer to 5 and $3 \times 4.99999 + 4.99999^3$ is 139.99922. One may guess that $3b + b^3$ is close to 140 if b is close to 5. Of course in this case $3b + b^3$ can be evaluated for $b = 5$ and is 140. The approximations seem superfluous.

Explore 3.1.5 Item g*. is more interesting than item c because $\frac{\sin b}{b}$ is meaningless for $b = 0$. Compute $\frac{\sin b}{b}$ for $b = 0.1$, $b = 0.01$, and $b = 0.001$ **(put your calculator in radian mode)** and answer the question of g*. ■

Item h is more interesting than g*. Look at the following computations.

$$b = 0.1 \qquad \frac{2^{0.1} - 1}{0.1} = 0.717734625$$

$$b = 0.01 \qquad \frac{2^{0.01} - 1}{0.01} = 0.69555006$$

$$b = 0.001 \qquad \frac{2^{0.001} - 1}{0.001} = 0.69338746$$

$$b = 0.0001 \qquad \frac{2^{0.000001} - 1}{0.000001} = 0.6931474$$

It is not clear what the numbers on the right are approaching, and, furthermore, the number of digits reported are decreasing. This will be explained when we compute the derivative of the exponential functions in Chapter 5

Explore 3.1.6 Set your calculator to display the maximum number of digits that it will display. Calculate $2^{0.00001}$ and explain why the number of reported digits is decreasing in the previous computations.

Your calculator probably has a button marked 'LN' or 'Ln' or 'ln'. Use that button to compute $\ln 2$ and compare $\ln 2$ with the previous calculations. ■

In the next examples, you will find it useful to recall that for numbers a and b and n a positive integer,

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + b^2a^{n-3} + ba^{n-2} + a^{n-1}). \quad (3.1)$$

Problem. Find the rate of change of

$$F(t) = 2t^4 - 3t \qquad \text{at} \qquad t = 2.$$

Equivalently, find the slope of tangent to the graph of F at the point (2,26).
Solution. For b a number different from 2,

$$\begin{aligned} \frac{F(b)-F(2)}{b-2} &= \frac{(2b^4-3b)-(2\times 2^4 - 3\times 2)}{b-2} \\ &= 2\frac{b^4-2^4}{b-2} - 3\frac{b-2}{b-2} \\ &= 2\left(b^3 + b^2\times 2 + b\times 2^2 + 2^3\right) - 3 \end{aligned}$$

We claim that

$$\text{As } b \text{ approaches 2,} \qquad \frac{F(b)-F(2)}{b-2} = 2\left(b^3 + b^2\times 2 + b\times 2^2 + 2^3\right) - 3 \qquad \text{approaches 61.}$$

Therefore, the slope of the tangent to the graph of F at (2,26) is 61, and the rate of change of $F(t) = 2t^4 - 3t$ at $t = 2$ is 61. An equation of the tangent to the graph of F at (2,26) is

$$\frac{y-26}{t-2} = 61, \qquad y = 61t - 96$$

Graphs of F and $y = 61t - 96$ are shown in Figure 3.7.
Problem. Find an equation of the line tangent to the graph of

$$F(t) = \frac{1}{t^2} \qquad \text{at the point} \quad (2, 1/4)$$

Solution. Graphs of $F(t) = 1/t^2$ and a secant to the graph through the points $(b, 1/b^2)$ and $(2, 1/4)$ for a number b not equal to 2 are shown in Figure 3.8A.

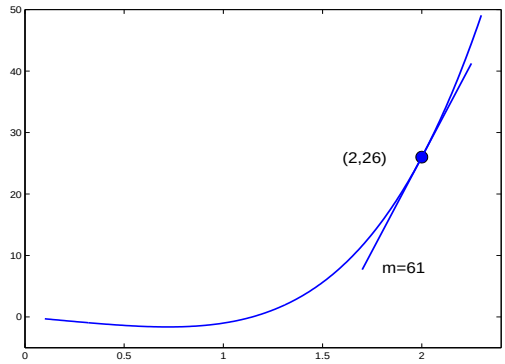


Figure 3.7: Graphs of $F(t) = 2t^4 - 3t$ and the line $y = 61t - 96$.

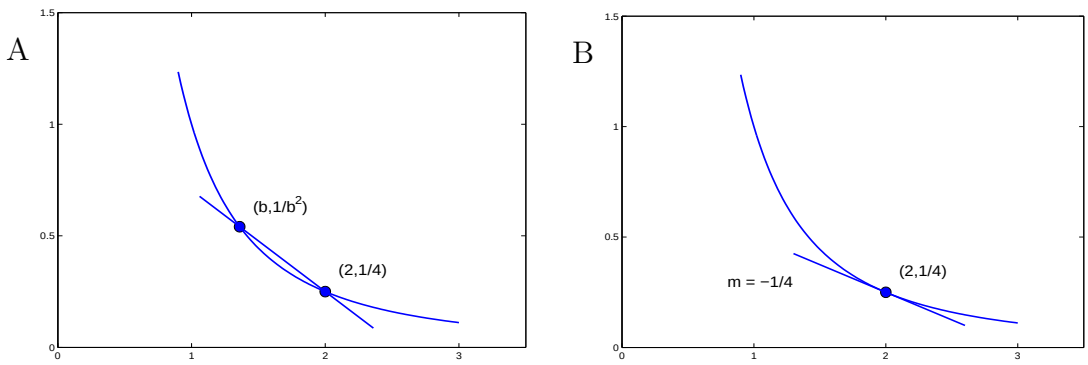


Figure 3.8: A. Graphs of $F(t) = 1/t^2$ and a secant to the graph through $(b, 1/b^2)$ and $(2, 1/4)$. B. Graphs of $F(t) = 1/t^2$ and the line $y = -1/4t + 3/4$.

The slope of the secant is

$$\begin{aligned} \frac{F(b) - F(2)}{b-2} &= \frac{\frac{1}{b^2} - \frac{1}{2^2}}{b-2} \\ &= \frac{2^2 - b^2}{b^2 \times 2^2} \frac{1}{b-2} \\ &= -\frac{b+2}{2^2 b^2} \end{aligned}$$

We claim that

$$\text{As } b \text{ approaches 2,} \qquad \frac{F(b) - F(2)}{b-2} = -\frac{b+2}{2^2 b^2} \qquad \text{approaches} \qquad -\frac{1}{4}$$

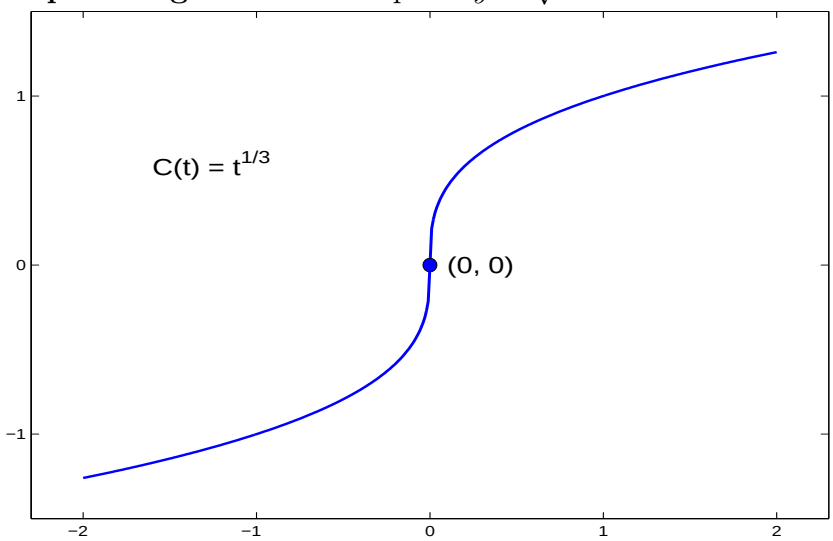
An equation of the line containing (2,1/4) with slope -1/4 is

$$\frac{y - 1/4}{t-2} = -1/4, \qquad y = -\frac{1}{4}t + \frac{3}{4}$$

This is an equation of the line tangent to the graph of $F(t) = 1/t^2$ at the point (2,1/4). Graphs of F and $y = -1/4t + 3/4$ are shown in Figure 3.8B.

Explore 3.1.7 In Explore Figure 3.1.7 is the graph of $y = \sqrt[3]{x}$. Does the graph have a tangent at (0,0)? **Your vote counts.** ■

Explore Figure 3.1.7 Graph of $y = \sqrt[3]{x}$.



Problem. At what rate is the function $F(t) = \sqrt[3]{t}$ increasing at $t = 8$?
Solution. For a number b not equal to 8,

$$\begin{aligned}\frac{F(b) - F(8)}{b - 8} &= \frac{\sqrt[3]{b} - \sqrt[3]{8}}{b - 8} \\ &= \frac{\sqrt[3]{b} - 2}{\left(\sqrt[3]{b}\right)^3 - 2^3} \qquad \textbf{Lightning Bolt!} \\ &= \frac{1}{(\sqrt[3]{b})^2 + \sqrt[3]{b} \times 2 + 2^2}\end{aligned}$$

Now we claim that

$$\text{as } b \text{ approaches } 8 \qquad \frac{F(b) - F(8)}{b - 8} = \frac{1}{(\sqrt[3]{b})^2 + \sqrt[3]{b} \times 2 + 2^2} \qquad \text{approaches} \quad \frac{1}{12}.$$

Therefore, the rate of increase of $F(t) = \sqrt[3]{t}$ at $t = 8$ is 1/12. A graph of $F(t) = \sqrt[3]{t}$ and $y = t/12 + 4/3$ is shown in Figure 3.9.
Pattern. In each of the computations that we have shown, we began with an expression for

$$\frac{F(b) - F(a)}{b - a}$$

that was meaningless for $b = a$ because of $b - a$ in the denominator. We made some algebraic rearrangement that neutralized the factor $b - a$ in the denominator and obtained an expression $E(b)$ such that

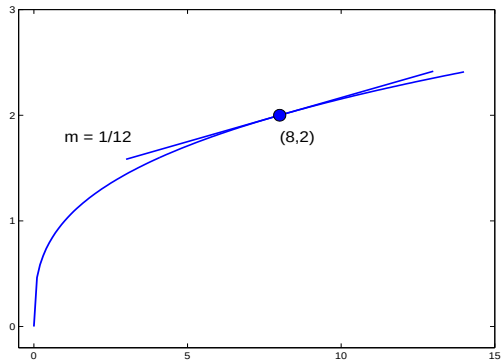


Figure 3.9: Graphs of $F(t) = \sqrt[3]{t}$ and the line $y = t/12 + 4/3$.

1. $\dfrac{F(b) - F(a)}{b - a} = E(b) \qquad$ for $b \neq a$, and
2. $E(a)$ is defined, and
3. As b approaches a, $E(b)$ approaches $E(a)$.

We then claimed that

$$\text{as } b \text{ approaches } a \qquad \frac{F(b) - F(a)}{b - a} \qquad \text{approaches} \qquad E(a).$$

This pattern will serve you well until we consider exponential, logarithmic and trigonometric functions where more than algebraic rearrangement is required to neutralize the factor $b - a$ in the denominator. Item 3 of this list is often given scant attention, but deserves your consideration.

Exercises for Section 3.1, Tangent to the graph of a function.

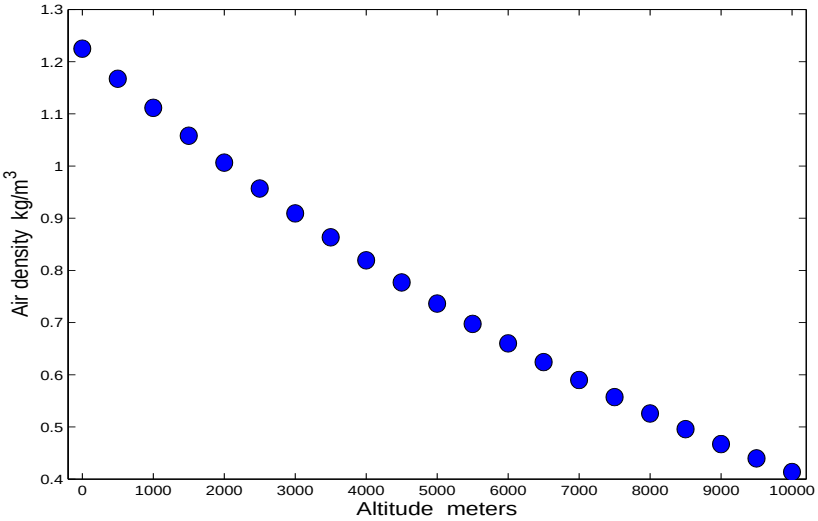
Exercise 3.1.1 Shown in Exercise Figure 3.1.1 is a graph of air densities in Kg/m^3 as a function of altitude in meters (U.S. Standard Atmospheres 1976, National Oceanic and Atmospheric Administration, NASA, U.S. Air Force, Washington, D.C. October 1976). You will find the rate of change of density with altitude. Because the independent variable is altitude, a distance, the rate of change is commonly called the *gradient*.

a. At what rate is air density changing with increase of altitude at altitude = 2000 meters? Alternatively, what is the gradient of air density at 2000 meters?

b. What is the gradient of air density at altitude = 5000 meters?

c. What is the gradient of air density at altitude = 8000 meters?

Figure for Exercise 3.1.1 Graph of air density (Kg/m^3) *vs* altitude (m).



Exercise 3.1.2 An African honey bee *Apis mellifera scutellata* was introduced into Brazil in 1956 by geneticists who hoped to increase honey production with a cross between the African bee which was native to the tropics and the European species commonly used by bee keepers in South America and in the United States. Twenty six African queens escaped into the wild in 1957 and the subsequent feral population has been very aggressive and has disrupted or eliminated commercial honey production in areas where they have spread.

Shown in Exercise Figure 3.1.2 is a map [1] that shows the regions occupied by the African bee in the years 1957 to 1983, and projections of regions that would be occupied by the bees during 1985-1995.

a. At what rate did the bees advance during 1957 to 1966?

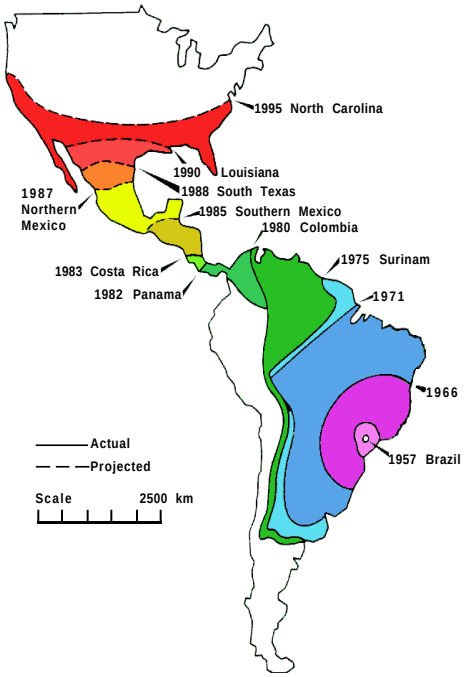
b. At what rate did the bees advance during 1971 to 1975?

c. At what rate did the bees advance during 1980 to 1982?

d. At what rate was it assumed the bees would advance during 1983 - 1987?

[1] Orley R. Taylor, African bees: potential impact in the United States, *Bull Ent Soc of America*, Winter, 1985, 15-24. Copyright,1985, The Entomological Society of America.

Figure for Exercise 3.1.2 The spread of the African bee from Brazil towards North America. The solid curves with dates represent observed spread. The dashed curves and dates are projections of spread.



Exercise 3.1.3 If b approaches 3

a. b approaches _____.

b. $2 \div b$ approaches _____. Note: Neither 0.6666 nor 0.6667 is the answer.

c. π approaches _____.

d. $\frac{2}{\sqrt{b}+\sqrt{3}}$ approaches _____. Note: Neither 0.577 nor 0.57735026919 is the answer.

e. $b^3 + b^2 + b$ approaches _____.

f. $\frac{b}{1+b}$ approaches _____.

g. 2^b approaches _____.

h. $\log_3 b$ approaches _____.

Exercise 3.1.4 In a classic study[2], David Ho and colleagues treated HIV-1 infected patients with ABT-538, an inhibitor of HIV-1 protease. HIV-1 protease is an enzyme required for viral replication so that the inhibitor disrupted HIV viral reproduction. Viral RNA is a measure of the amount of virus in the serum. Data showing the amount of viral RNA present in serum during two weeks following drug

[2]David D. Ho, Avidan U Neumann, Alan S. Perlson, Wen Chen, John M. Leonard, & Martin Markowitz, Rapid turnover of plasma virions and CD4 lymphocytes in HIV-1 infection, *Nature* 373, 123-126, (1995)

administration are shown in Table Ex. 3.1.4 for one of the patients. Before treatment the patients serum viral RNA was roughly constant at 180,000 copies/ml. On day 1 of the treatment, viral production was effectively eliminated and no new virus was produced for about 21 days after which a viral mutant that was resistant to ABT-538 arose. The rate at which viral RNA decreased on day 1 of the treatment is a measure of how rapidly the patient's immune system eliminated the virus before treatment.

a. Compute an estimate of the rate at which viral RNA decreased on day 1 of the treatment.

b. Assume that the patient's immune system cleared virus at that same rate before treatment. What percent of the virus present in the patient was destroyed by the patient's immune system each day before treatment?

c. At what rate did the virus reproduce in the absence of ABT-538.

Table for Exercise 3.1.4 RNA copies/ml in a patient during treatment with an inhibitor of HIV-1 protease.

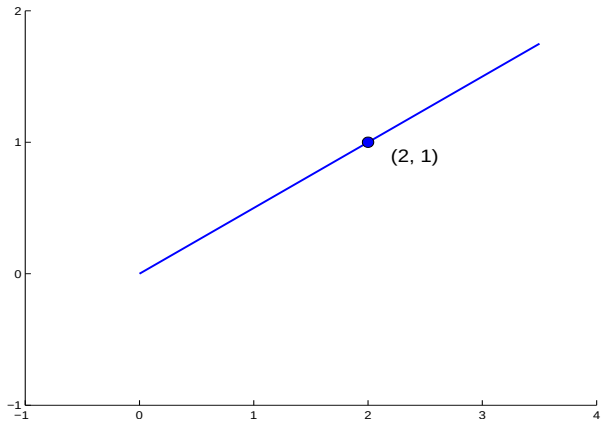
Observed	
Time Days	RNA copies/ml Thousands
1	80
4	50
8	18
11	9.5
15	5

Exercise 3.1.5 Shown in Figure Ex. 3.1.5 is the graph of the line, $y = 0.5x$, and the point (2,1).

a. Is there a tangent to the graph of $y = 0.5x$ at the point (2,1)?

b. Suppose $H(t) = 0.5t$ is the height in feet of water above flood stage in a river t hours after midnight. At what rate is the water rising at time $t = 2$ am?

Figure for Exercise 3.1.5 Graph of the line $y = 0.5x$. See Exercise 3.1.5.

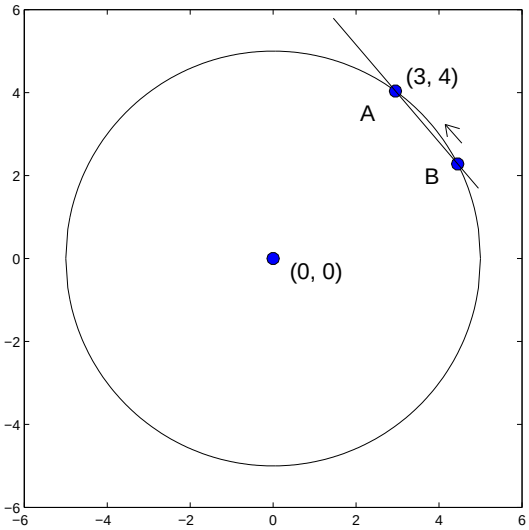


Exercise 3.1.6 See Figure Ex. 3.1.6. Let A be the point (3,4) of the circle

$$x^2 + y^2 = 25$$

Let B be a point of the circle different from A. What number does the slope of the line containing A and B approach as B approaches A?

Figure for Exercise 3.1.6 Graph of the circle $x^2 + y^2 = 25$ and a secant through (3,4) and a point B. See Exercise 3.1.6.



Exercise 3.1.7 Suppose plasma penicillin concentration in a patient following injection of 1 gram of penicillin is observed to be

$$P(t) = 200 \times 2^{-0.03t},$$

where t is time in minutes and $P(t)$ is μg/ml of penicillin. Use the following steps to approximate the rate at which the penicillin level is changing at time $t = 5$ minutes and at $t = 0$ minutes.

a. **t = 5 minutes.** Use your graphing calculator to draw the graph of $P(t)$ vs t for $4.9 \le t \le 5.1$. (The graph should appear to be a straight line on this short interval.)

b. Complete the table on the left, computing the average rates of change of penicillin level.

b	$\frac{P(b)-P(5)}{b-5}$
4.9	-3.7521
4.95	
4.99	
4.995	
5.005	
5.01	
5.05	
5.1	-3.744

b	$\frac{P(b)-P(0)}{b-0}$
-0.1	OMIT
-0.05	OMIT
-0.01	OMIT
-0.005	OMIT
0.005	
0.01	
0.05	
0.1	-4.155

c. What is your best estimate of the rate of change of penicillin level at the time $t = 5$ minutes? Include units in your answer.

d. **t = 0 minutes.** Complete the second table above. The OMIT entries in the second table refer to the fact that the level of penicillin, $P(t)$, may not be given by the formula for negative values of time, t. What is your best estimate of the rate of change of penicillin level at the time $t = 0$ minutes?

Exercise 3.1.8 The patient in Exercise 3.1.7 had penicillin level 200 μg/ml at time $t = 0$ following a 1 gram injection. What is the approximate volume of the patient's vascular pool? If you wished to maintain the patient's penicillin level at 200 μg/ml, at what rate would you continuously infuse the patient with penicillin?

Exercise 3.1.9 Find equations of the lines tangent to the graphs of the function F at the indicated points.

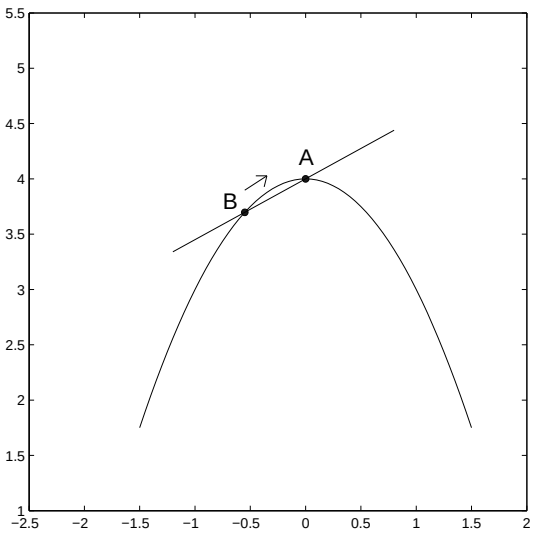
a.	$F(t)$	$=$	t^2	at	$(2,4)$	b.	$F(t)$	$=$	t^2+2	at	$(2,6)$
c.	$F(t)$	$=$	$t+1$	at	$(2,3)$	d.	$F(t)$	$=$	$3t^3-4t^2$	at	$(1,-1)$
e.	$F(t)$	$=$	$1/(t+1)$	at	$(1,1/2)$	f.	$F(t)$	$=$	$\sqrt[4]{t}$	at	$(1,1)$

Exercise 3.1.10 Find the rates of change of the function F at the indicated points.

a.	$F(t)$	$=$	t^3	at	$(2,8)$	b.	$F(t)$	$=$	$1/t$	at	$(2,1/2)$
c.	$F(t)$	$=$	$2t+t^2$	at	$(2,8)$	d.	$F(t)$	$=$	$\sqrt{t}$	at	$(4,2)$
e.	$F(t)$	$=$	$t/(t+1)$	at	$(1,1/2)$	f.	$F(t)$	$=$	$(t+1)/t$	at	$(1,2)$

Exercise 3.1.11 In Figure Ex. 3.1.11 is the graph of a parabola with vertex at A and a secant to the parabola through A and a point B What number does the slope of the line containing A and B approach as B approaches A?

Figure for Exercise 3.1.11 A parabola with vertex at A and a secant through A and a point B. See Exercise 3.1.11.



3.2 Limit and rate of change as a limit

The phrase

$$\text{as } b \text{ approaches } a \qquad \frac{F(b)-F(a)}{b-a} \qquad \text{approaches} \quad m_a \tag{3.2}$$

introduced in the previous section to define slope of a tangent to the graph of F at a point $(a, F(a))$ can be made more explicit. Almost 200 years after Isaac Newton and Gottfried Leibniz introduced calculus (about 1665), Karl Weierstrass introduced (about 1850) a concise statement of **limit** that clarifies the phrase. Phrase 3.2 would be rewritten

$$\text{The limit as } b \text{ approaches } a \text{ of} \quad \frac{F(b)-F(a)}{b-a} \quad \text{is} \quad m_a. \tag{3.3}$$

We introduce notation separate from that of the phrase 3.2 because 'limit' has application well beyond its use in defining tangents.

Definition 3.2.1 Definition of limit. Suppose G is a function, a is a number, (p, q) is an open interval that contains a, and the domain of G contains every point of (p, q) except a. Also suppose L is a number. The statement that

$$\text{as } x \text{ approaches } a \qquad \text{G(x) approaches } L$$

means that if ϵ is a positive number, there is a positive number δ such that if x is in the domain of G and $0 < |x-a| < \delta$ then $|G(x) - L| < \epsilon$. This relation is denoted by

$$\lim_{x \to a} G(x) = L \tag{3.4}$$

and is read, 'The limit as x approaches a of $G(x)$ is L.'

The statement that '$\lim_{x \to a} G(x)$ exists' means that G is a function, an open interval (p, q) contains a number a, the domain of G contains all of (p, q) except a and for some number L, $\lim_{x \to a} G(x) = L$.

The use of the word 'limit' in Definition 3.2.1 is different from its usual use as a bound, as in 'speed limit' or in "1. the point, line or edge where something ends or must end; $\cdots$" (Webster's new College Dictionary, Fourth Edition). Perhaps 'goal' (something aimed at or striven for) would be a better word, but we and you have no choice: assume a new and well defined meaning for 'limit.'

Important. In Definition 3.2.1, the question of whether $G(a)$ is defined is irrelevant, and if $G(a)$ is defined, the value of $G(a)$ is irrelevant. The inequality $0 < |x-a| < \delta$ specifically excludes consideration of $x = a$. Two illustrative examples appear in Figure 3.10. They are the graphs of F_1 and F_2, defined by

$$F_1(x) = \begin{cases} \frac{1}{2} * x & \text{for} \quad x \neq 1 \\ \text{not defined} & \text{for} \quad x = 1 \end{cases} \qquad F_2(x) = \begin{cases} \frac{1}{2}x & \text{for} \quad x \neq 1 \\ 1 & \text{for} \quad x = 1 \end{cases} \tag{3.5}$$

Intuitively: If x is close to 1 *and different from 1*, $F_1(x) = \frac{1}{2}x$ is close to 0.5 and $F_2(x)$ is close to 0.5.

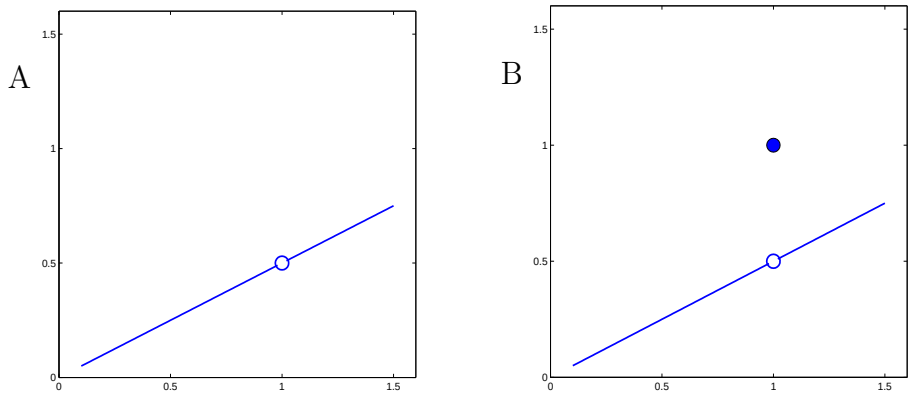


Figure 3.10: A. Graphs of F_1 and F_2 as defined in Equation 3.5.

$$\lim_{x\to 1} F_1(x) = 0.5, \qquad \text{and} \qquad \lim_{x\to 1} F_2(x) = 0.5$$

By Definition 3.2.1: If ϵ is a positive number, choose $\delta = \epsilon$.

If x is in the domain of F_1 and $\quad 0 < |x-1| < \delta$,

then $\quad \left|F_1(x) - \frac{1}{2}\right| = \left|\frac{1}{2}x - \frac{1}{2}\right| = \frac{1}{2}|x-1| < \frac{1}{2}\delta < \epsilon.$

The same argument applies to F_2. That $F_2(1) = 1$ does not change the argument because $0 < |x-1| < \delta$ excludes $x = 1$.

Example 3.2.1 Practice with ϵ's and δ's.

1. Suppose administration of $a = 3.5$ mg of growth hormone produces the optimum serum hormone level $L = 8.1$ μg in a 24 kg boy. Suppose further that an amount x mg of growth hormone produces serum hormone level $G(x)$ μg. You may wish to require $\epsilon = 1.5$ μg accuracy in serum hormone levels, and need to know what specification δ mg accuracy to require in preparation of the growth hormone to be administered. That is, if the actual amount administered is between $3.5 - \delta$ mg and $3.5 + \delta$ mg then the resulting serum hormone level will be between $8.1 - 1.5$ μg and $8.1 + 1.5$ μg. If instead, your tolerance is $\epsilon = 0.5$ μg, the specified δ mg would be smaller.

2. Show that if a is a positive number, then

$$\lim_{x\to a} \sqrt{x} = \sqrt{a} \tag{3.6}$$

Suppose $\epsilon > 0$. Let $\delta = \epsilon \times \sqrt{a}$. Suppose x is in the domain of $\sqrt{x}$ and $0 < |x-a| < \delta$. Then

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \le \frac{|x-a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} = \frac{\epsilon \times \sqrt{a}}{\sqrt{a}} = \epsilon.$$

3. Find a number $\delta > 0$ so that

$$\text{if} \qquad 0 < |x-1| < \delta \qquad \text{then} \qquad \left|\frac{1}{1+x^2} - 0.5\right| < 0.01.$$

A graph of $y = 1/(1+x^2)$ is shown in Figure 3.11 along with lines a distance 0.01 above and below $y = 0.5$. We solve (The ISECT feature on your TI-86 is handy here.)

$$\frac{1}{1+x^2} = 0.49, \qquad x = 1.0202; \qquad \text{and} \qquad \frac{1}{1+x^2} = 0.51, \qquad x = 0.98019.$$

From the graph it is clear that

$$\text{if} \qquad 0.981 < x < 1.02 \qquad \text{then} \qquad 0.49 < \frac{1}{1+x^2} < 0.51.$$

We choose $\delta = 0.019$, the smaller of $1.0 - 0.981$ and $1.02 - 1.0$.

$$\text{Then if} \qquad 0 < |x-1| < \delta, \quad \left|\frac{1}{1+x^2} - 0.5\right| < 0.01.$$

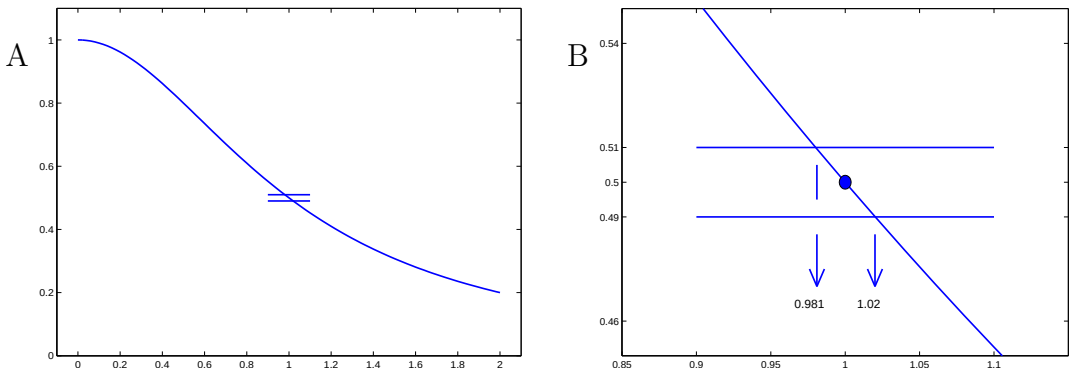


Figure 3.11: A. Graph of $F(x) = 1/(1+x^2)$, and B a magnification of A. In both graphs, segments of the lines $y = 0.49$ and $y = 0.51$ are drawn.

4. Find a number $\delta > 0$ so that

$$\text{if} \qquad 0 < |x-4| < \delta \qquad \text{then} \qquad \left|\frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4} + \frac{1}{16}\right| < 0.01.$$

This algebraic challenge springs from the problem of showing that the slope of

$$f(x) = \frac{1}{\sqrt{x}} \quad \text{at the point} \quad (4, \tfrac{1}{2}) \quad \text{is} \quad \frac{-1}{16}.$$

We first do some algebra.

$$\frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4} + \frac{1}{16} = \frac{\frac{2-\sqrt{x}}{2\sqrt{x}}}{(\sqrt{x}-2)(\sqrt{x}+2)} + \frac{1}{16} = \frac{-1}{2\sqrt{x}(\sqrt{x}+2)} + \frac{1}{16} = \frac{-8+\sqrt{x}(\sqrt{x}+2)}{16\sqrt{x}(\sqrt{x}+2)}$$

It is sufficient to find $\delta > 0$ so that

$$\text{if} \quad 0 < |x-4| < \delta \quad \text{then} \quad \left| \frac{-8+\sqrt{x}(\sqrt{x}+2)}{16\sqrt{x}(\sqrt{x}+2)} \right| < 0.01 \quad \text{or} \quad \left| \frac{-8+\sqrt{x}(\sqrt{x}+2)}{\sqrt{x}(\sqrt{x}+2)} \right| < 0.16$$

Assume that x is 'close to' 4 and certainly bigger than 1 so that the denominator, $\sqrt{x}(\sqrt{x}+2)$ is greater than 1. Then it is sufficient to find $\delta > 0$ so that

$$\left| -8+\sqrt{x}(\sqrt{x}+2) \right| = \left| -8+x+2\sqrt{x} \right| < 0.16.$$

Let $z = \sqrt{x}$ so that $z^2 = x$ and solve

$$\begin{aligned} -8+z^2+2z &= -0.16 \quad \text{and} \quad & -8+z^2+2z &= 0.16 \\ z_1 &= 1.97321 & z_2 &= 2.026551.97321 \\ x_1 &= 3.89356 & x_2 &= 4.10690 \end{aligned}$$

We will choose $\delta = 0.1$ so that

$$\text{if} \quad 0 < |x-4| < 0.1 \quad \text{then } x \text{ is between } x_1 \text{ and } x_2 \text{ and} \quad \left| \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4} + \frac{1}{16} \right| < 0.01.$$

5. Suppose you try to 'square the circle.' That is, suppose you try to construct a square the area of which is exactly the area, π, of a circle of radius 1. A famous problem from antiquity is whether one can construct such a square using only a straight edge and a compass. The answer is no. But can you get close?

Suppose you will be satisfied if the area of your square is within 0.01 of π (think, $\epsilon = 0.01$). It helps to know that

$$\pi = 3.14159265\cdots, \qquad \text{and} \qquad 1.77^2 = 3.1329,$$

and that every interval of rational length (for example, 1.77) can be constructed with straight edge and compass. Because $1.77^2 = 3.1329$ is within 0.01 of π, you can 'almost' square the circle with straight edge and compass.

Explore 3.2.1 Suppose you are only satisfied if the area of your square is within 0.001 of π. What length should the edges of your square be in order to achieve that accuracy? ■

Now suppose ϵ is a positive number and ask, how close (think, δ) must your edge be to $\sqrt{\pi}$ in order to insure that

$$|\,\text{the area of your square} - \pi\,| \quad < \quad \epsilon \quad ?$$

Let x be the length of the side of your square. Your problem is to find δ so that

$$\text{if} \quad |x-\sqrt{\pi}| < \delta, \qquad \text{then} \quad |x^2-\pi| < \epsilon.$$

Danger: This may fry your brain. Look at the target $|x^2 - \pi| < \epsilon$ and write

$$\begin{aligned} |x^2 - \pi| &< \epsilon \\ |x - \sqrt{\pi}| \times |x + \sqrt{\pi}| &< \epsilon \end{aligned}$$

Lets require first of all that $0 < x < 5$. (We get to write the specifications for x.) Then $x + \sqrt{\pi} < 10$. (Actually, $\sqrt{\pi} \doteq 1.7725$ so that $x + \sqrt{\pi}$ is less than 7, but we can be generous.) Now look at our target,

$$|x^2 - \pi| \quad = \quad |x - \sqrt{\pi}| \times |x + \sqrt{\pi}| \quad < \quad \epsilon$$

If we insist that $|x - \sqrt{\pi}| < \delta = \text{Minimum}(\epsilon/10, 1)$, then $0 < x < 5$, and

$$|x^2 - \pi| \quad = \quad |x - \sqrt{\pi}| \times |x + \sqrt{\pi}| \quad < \quad \frac{\epsilon}{10} \times 10 \quad = \quad \epsilon.$$

If $|x - \sqrt{\pi}| < \delta = \text{Minimum}(\epsilon/10, 1)$, then $|x^2 - \pi| < \epsilon$.

Explore 3.2.2 We required that $0 < x < 5$. (We get to write the specifications for x.) Would it work to require that $0 < x < 1$? ■

Using the definition of limit we rewrite the definitions of tangent to the graph of a function and the rate of change of a function.

Definition 3.2.2 Definition of tangent and rate of change, revisited. Suppose a function F is defined on an open interval that contains a point a and m_a is a number. The statement that the slope of the tangent to the graph of F at $(a, F(a))$ is m_a and that the rate of change of F at a is m_a means that

$$\lim_{b \to a} \frac{F(b) - F(a)}{b - a} = m_a. \tag{3.7}$$

In the previous section we found that the slope of the tangent to the graph of $F(t) = t^2$ at a point (a, a^2) was $2a$. Using the limit notation we would write that development as

$$\begin{aligned} \lim_{b \to a} \frac{F(b) - F(a)}{b - a} &= \lim_{b \to a} \frac{b^2 - a^2}{b - a} \\ &= \lim_{b \to a} \frac{(b - a)(b + a)}{b - a} \\ &= \lim_{b \to a} (b + a) \\ &= a + a = 2a \end{aligned}$$

There is some algebra of the limit symbol, $\lim\limits_{x\to a}$, that is important. Suppose each of F_1 and F_2 is a function and a is a number and $\lim\limits_{x\to a} F_1(x)$ and $\lim\limits_{x\to a} F_2(x)$ both exist. Suppose further that C is a number. Then

$$\lim_{x\to a} C = C \tag{3.8}$$

$$\lim_{x\to a} x = a \tag{3.9}$$

$$\text{If } a \neq 0, \text{ then} \qquad \lim_{x\to a} \frac{1}{x} = \frac{1}{a} \tag{3.10}$$

$$\lim_{x\to a} C\, F_1(x) = C \lim_{x\to a} F_1(x) \tag{3.11}$$

$$\lim_{x\to a} (\, F_1(x) + F_2(x)) = \lim_{x\to a} F_1(x) + \lim_{x\to a} F_2(x) \tag{3.12}$$

$$\lim_{x\to a} (F_1(x) \times F_2(x)) = \left(\lim_{x\to a} F_1(x)\right) \times \left(\lim_{x\to a} F_2(x)\right) \tag{3.13}$$

$$\tag{3.14}$$

Probably all of these equations are sufficiently intuitive that proofs of them seem superfluous. Equations 3.8 and 3.9 could be more accurately expressed. For Equation 3.8 one might say,

$$\text{"If } F_1(x) = C \text{ for all } x \neq a \text{ then} \qquad \lim_{x\to a} F_1(x) = C\text{."} \tag{3.15}$$

Explore 3.2.3 Suppose F_1 is defined by

$$F_1(x) = C \qquad \text{if} \qquad x \neq a$$

$$F_1(a) = C + 1$$

Is it true that

$$\lim_{x\to a} F_1(x) = C \quad ?$$

■

Explore 3.2.4 Write a more accurate expression for Equation 3.9 similar to that of 3.15. ■

Proof of Equations 3.10, 3.11 and 3.12 are rather easy and are included to illustrate the process.
Proof of Equation 3.10. Suppose $a \neq 0$ and $\epsilon > 0$. Let

$$\delta = \text{Minimum}(|a|/2, \epsilon \times a^2/2). \qquad \text{Suppose} \qquad 0 < |x - a| < \delta.$$

Note: Because $|x - a| < \delta \leq |a|/2$, $|x| > |a|/2$ and $|x \times a| > |a^2|/2$.

$$\text{Then} \qquad \left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{x-a}{x \times a}\right|$$
$$< \frac{\delta}{|x \times a|}$$
$$< \frac{\delta}{a^2/2}$$
$$\leq \frac{\epsilon \times a^2/2}{a^2/2} \qquad = \qquad \epsilon$$

Proof of Equation 3.11. Let $\lim_{x \to a} F_1(x) = L_1$. Suppose ϵ is a positive number.
Case $C = 0$. Let $\delta = 1$. Then if x is in the domain of F_1 and $0 < |x - a| < \delta$,

$$|C\, F_1(x) - C\, L_1| = |0 - 0| = 0 < \epsilon.$$

Case $C \neq 0$. Let

$$\epsilon_0 = \frac{\epsilon}{|C|}$$

Because $\lim_{x \to a} F_1(x) = L_1$, there is $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|F_1(x) - L_1| < \epsilon_0$. Then if x is in the domain of F and $0 < |x - a| < \delta$,

$$|C\, F_1(x) - C\, L_1| = |C|\, |F_1(x) - L_1| < |C| \times \epsilon_0 = |C| \times \frac{\epsilon}{|C|} = \epsilon.$$

Thus

$$\lim_{x \to a} C\, F_1(x) = C \times L_1 = C \lim_{x \to a} F_1(x)$$

Proof of Equation 3.12. Suppose $\lim_{x \to a} F_1(x) = L_1$ and $\lim_{x \to a} F_2(x) = L_2$. Suppose $\epsilon < 0$. There are numbers δ_1 and δ_2 such that

$$\text{if } 0 < |x - a| < \delta_1 \text{ then} |F_1(x) - L_1| < \frac{\epsilon}{2} \qquad \text{and} \qquad \text{if } 0 < |x - a| < \delta_2 \text{ then} |F_2(x) - L_2| < \frac{\epsilon}{2}$$

Let $\delta = \text{Minimum}(\delta_1, \delta_2)$. Suppose $0 < |x - a| < \delta$. Then

$$|F_1(x) + F_2(x) - (L_1 + L_2)| \leq |F_1(x) - L_1| + |F_2(x) - L_2|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Honest Exposition. Almost always in evaluating

$$\lim_{x \to a} F(x),$$

if F is a familiar function and $F(a)$ is defined then

$$\lim_{x \to a} F(x) = F(a).$$

If this happens, F is said to be *continuous* at a (more about continuity later). It follows from Equations 3.8 through 3.13, for example, that

$$\text{if } P(x) \text{ is any polynomial, then} \qquad \lim_{x\to a} P(x) = P(a). \qquad \text{See Exercise 3.2.7.}$$

Moreover, even if $F(a)$ is not meaningful but there is an expression $E(x)$ for which

$$F(x) = E(x) \quad \text{for} \quad x \neq a, \quad \text{and} \quad E(a) \quad \text{is defined,}$$

then almost always

$$\lim_{x\to a} F(x) = E(a).$$

For example, for

$$F(x) = \frac{x^4 - a^4}{x - a}, \qquad F(a) \quad \text{is meaningless.}$$

But,

$$\begin{aligned} \text{for } x \neq a, \qquad F(x) &= \frac{(x-a)(x^3 + ax^2 + a^2x + a^3)}{x - a} \\ &= x^3 + x^2\,a + x\,a^2 + a^3 \\ &= E(x). \end{aligned}$$

$$\text{and} \qquad E(a) = 4a^3, \qquad \text{and} \qquad \lim_{x\to a} F(x) = 4a^3.$$

The simple procedure just discussed fails when considering more complex limits such as

$$\lim_{x\to 0} \frac{\sin x}{x}, \qquad \lim_{x\to 1} \frac{\log x}{x-1}, \qquad \text{and} \qquad \lim_{x\to 1} \frac{2^x - 2}{x - 1}.$$

Limit of the composition of two functions. A formula for the limit of the composition of two functions has very broad application.

Suppose u is a function, L is a number, a is in the domain of u, and there is an open interval (p, q) in the domain of u that contains a and if x is in (p, q) and is not a then $u(x) \neq L$. Suppose further that

$$\lim_{x\to a} u(x) = L$$

Suppose F is a function and for x in (p, q), $u(x)$ is in the domain of F, and

$$\lim_{s\to L} F(s) = \lambda.$$

Then

$$\lim_{x\to a} F(u(x)) = \lambda \tag{3.16}$$

Intuitively, if x is close to but distinct from a, then $u(x)$ is close to but distinct from L and $F(u(x))$ is close to λ. The formal argument is perhaps easier than the statement of the property, but is omitted.

Some formulas that follow from Equation 3.16 and previous formulas include

$$\text{If} \quad \lim_{x\to a} F(x) = L > 0, \qquad \text{then} \qquad \lim_{x\to a} \sqrt{F(x)} = \sqrt{L}.$$

$$\text{If} \quad \lim_{x\to a} F(x) = L \neq 0, \qquad \text{then} \qquad \lim_{x\to a} \frac{1}{F(x)} = \frac{1}{L}.$$

$$\text{If} \quad \lim_{x\to a} F_2(x) \neq 0, \qquad \text{then} \qquad \lim_{x\to a} \frac{F_1(x)}{F_2(x)} = \frac{\lim_{x\to a} F_1(x)}{\lim_{x\to a} F_2(x)}. \tag{3.17}$$

Exercises for Section 3.2, Limit and rate of change as a limit.

Exercise 3.2.1

a. Find a number, $\delta > 0$, so that if x is a number and $0 < |x-2| < \delta$ then $|2x-4| < 0.01$.

b. Find a number, δ, so that if x is a number and $0 < |x-2| < \delta$ then $|x^2-4| < 0.01$.

c. Find a number, δ, so that if x is a number and $0 < |x-2| < \delta$ then $\left|\frac{1}{x} - \frac{1}{2}\right| < 0.01$.

d. Find a number, δ, so that if x is a number and $0 < |x-2| < \delta$ then $|x^3-8| < 0.01$.

e. Find a number, δ, so that if x is a number and $0 < |x-2| < \delta$ then $\left|\frac{x}{x+1} - \frac{2}{3}\right| < 0.01$.

f. Find a number, $\delta > 0$, so that if x is a number and $0 < |x-9| < \delta$ then $|\sqrt{x}-3| < 0.01$.

g. Find a number, δ, so that if x is a number and $0 < |x-8| < \delta$ then $|\sqrt[3]{x}-2| < 0.01$.

h. Find a number, $\delta > 0$, so that if x is a number and $0 < |x-2| < \delta$ then $|x^4-3x-13| < 0.01$.

Exercise 3.2.2 a. Find a number, $\delta > 0$, so that

$$\text{if } x \text{ is a number and} \qquad |x-3| < \delta \qquad \text{then} \qquad \left|\frac{x^2-9}{x-3} - 6\right| < 0.01.$$

b. Find a number, $\delta > 0$, so that

$$\text{if } x \text{ is a number and} \qquad |x-4| < \delta \qquad \text{then} \qquad \left|\frac{\sqrt{x}-2}{x-4} - \frac{1}{4}\right| < 0.01.$$

c. Find a number, $\delta > 0$, so that

$$\text{if } x \text{ is a number and} \qquad |x-2| < \delta \qquad \text{then} \qquad \left|\frac{\frac{1}{x}-\frac{1}{2}}{x-2} + \frac{1}{4}\right| < 0.01.$$

d. Find a number, $\delta > 0$, so that

$$\text{if } x \text{ is a number and} \qquad |x-1| < \delta \qquad \text{then} \qquad \left|\frac{x^2+x-2}{x-1} - 3\right| < 0.01.$$

Exercise 3.2.3 a. Use Equations 3.9 and 3.13,

$$\lim_{x\to a} x = a \qquad \text{and} \qquad \lim_{x\to a} F_1(x) \times F_2(x) = \left(\lim_{x\to a} F_1(x)\right) \times \left(\lim_{x\to a} F_2(x)\right),$$

to show that

$$\lim_{x\to a} x^2 = a^2.$$

b. Show that

$$\lim_{x\to a} x^3 = a^3.$$

c. Show by induction that if n is a positive integer,

$$\lim_{x\to a} x^n = a^n. \tag{3.18}$$

Exercise 3.2.4 Prove the following theorem.

Theorem 3.2.1 : Limit is Unique Theorem. Suppose G is a function and

$$\lim_{x\to a} G(x) = L_1 \qquad \text{and} \qquad \lim_{x\to a} G(x) = L_2.$$

Then $L_1 = L_2$.

Begin your proof with,

1. Suppose that $L_1 < L_2$.
2. Let $\epsilon = (L_2 - L_1)/2$.

Exercise 3.2.5 Evaluate the limits.

a. $\lim_{x\to 5} 3x^2 - 15x$	b. $\lim_{x\to 0} 3x$	c. $\lim_{x\to 50} \pi$
d. $\lim_{x\to -2} 3x^2 - 15x$	e. $\lim_{x\to -\pi} x$	f. $\lim_{x\to \pi} 50$
g. $\lim_{x\to -5} \dfrac{x-1}{x-1}$	h. $\lim_{x\to 1} \dfrac{x^2-2x+1}{x-1}$	i. $\lim_{x\to 1} \dfrac{x^4-1}{x-1}$
j. $\lim_{x\to 4} \dfrac{x}{\sqrt{x+2x^2}}$	k. $\lim_{x\to 1} \sqrt{\dfrac{x-2}{x^3+1}}$	l. $\lim_{x\to 1} \sqrt{\dfrac{x-3}{x^3-27}}$

Exercise 3.2.6 Sketch the graph of $y = \sqrt[3]{x}$ for $-1 \le x \le 1$. Does the graph have a tangent at (0,0)? Remember, **Your vote counts.**

Exercise 3.2.7 Suppose $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ is a polynomial and a is a number. For each step a through g, identify the equation of Equations 3.8 - 3.12 and 3.18 that justify the step.

$$\begin{aligned}
\lim_{x\to a} P(x) &= \lim_{x\to a}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0\right) \\
&= \lim_{x\to a}\left(a_n x^n\right) + \lim_{x\to a}\left(a_{n-1}x^{n-1} + \cdots a_1 x + a_0\right) && \text{a.} \\
&= \lim_{x\to a}\left(a_n x^n\right) + \lim_{x\to a}\left(a_{n-1}x^{n-1}\right) + \lim_{x\to a}\left(a_{n-2}x^{n-2} \cdots a_1 x + a_0\right) && \text{b.} \\
\vdots \quad & \vdots \\
&= \lim_{x\to a}\left(a_n x^n\right) + \cdots + \lim_{x\to a}\left(a_2 x^2\right) + \lim_{x\to a}\left(a_1 x + a_0\right) \\
&= \lim_{x\to a}\left(a_n x^n\right) + \cdots + \lim_{x\to a}\left(a_2 x^2\right) + \lim_{x\to a}\left(a_1 x\right) + \lim_{x\to a}\left(a_0\right) && \text{c.} \\
&= a_n \lim_{x\to a}\left(x^n\right) + \cdots + a_2 \lim_{x\to a}\left(x^2\right) + a_1 \lim_{x\to a}\left(x\right) + \lim_{x\to a}\left(a_0\right) && \text{d.} \\
&= a_n\, a^n + \cdots + a_2\, a^2 + a_1 \lim_{x\to a}\left(x\right) + \lim_{x\to a}\left(a_0\right) && \text{e.} \\
&= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + \lim_{x\to a}\left(a_0\right) && \text{f.} \\
&= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0 && \text{g.} \\
&= P(a)
\end{aligned}$$

Exercise 3.2.8 . Show that Equation 3.11,

$$\lim_{x\to a} C \times F_1(x) = C \times \left(\lim_{x\to a} F_1(x)\right)$$

follows from Equations 3.8 and 3.13,

$$\lim_{x\to a} C = C \qquad \text{and} \qquad \lim_{x\to a}\left(F_1(x) \times F_2(x)\right) = \left(\lim_{x\to a} F_1(x)\right) \times \left(\lim_{x\to a} F_2(x)\right)$$

Exercise 3.2.9 Some of the following statements are true and some are false. For those that are true, provide proofs using Equations 3.8 - 3.13. For those that are false, provide functions F_1 and F_2 to show that they are false.

Suppose F_1 and F_2 are functions defined for all numbers, x.

a. If $\lim_{x\to a} (F_1(x) - F_2(x)) = 0$, then $\lim_{x\to a} F_1(x) = \lim_{x\to a} F_2(x)$.

b. If $\lim_{x\to a} (F_1(x) \times F_2(x)) = 0$, then $\lim_{x\to a} F_1(x) = \lim_{x\to a} F_2(x) = 0$.

c. If $\lim_{x\to a} \dfrac{F_1(x)}{F_2(x)} = 0$, then $\lim_{x\to a} F_1(x) = 0$.

d. If $\lim_{x\to a} F_1(x) = 0$, then $\lim_{x\to a} \dfrac{F_1(x)}{F_2(x)} = 0$.

c. If $\lim_{x\to a} (F_1(x) \times F_2(x)) = 0$, then $\lim_{x\to a} F_1(x) = 0$ or $\lim_{x\to a} F_2(x) = 0$.

Exercise 3.2.10

Evaluate $\lim_{x\to a} \dfrac{F(x) - F(a)}{x - a}$ for

a.	$F(x)$	$=$	x^2	a	$=$	-2	b.	$F(x)$	$=$	17	$a = 0$
c.	$F(x)$	$=$	$2x^3$	a	$=$	2	d.	$F(x)$	$=$	$x^2 + 2x$	$a = 1$
e.	$F(x)$	$=$	$\frac{1}{x}$	a	$=$	$\frac{1}{2}$	f.	$F(x)$	$=$	$3x^2 - 5x$	$a = 7$
g.	$F(x)$	$=$	$3\sqrt{x}$	a	$=$	4	h.	$F(x)$	$=$	$x^2 + 2x + 1$	$a = -1$
i.	$F(x)$	$=$	$\frac{4}{x} + 5$	a	$=$	2	j.	$F(x)$	$=$	x^6	$a = 2$
k.	$F(x)$	$=$	$\frac{1}{x^3}$	a	$=$	2	l.	$F(x)$	$=$	x^{10}	$a = 2$
m.	$F(x)$	$=$	$\frac{4}{x^5}$	a	$=$	2	n.	$F(x)$	$=$	x^{67}	$a = 1$

Exercise 3.2.11 Suppose $F(x)$ is a function and

$$\lim_{b\to 2} \frac{F(b) - F(2)}{b - 2} \quad \text{exists.}$$

What is $\lim_{b\to 2} F(b)$?

3.3 The derivative function, F′

.

Definition 3.3.1 The function, F′. Suppose F is a function and for some number, x, in its domain the rate of change of F exists at x. Then the function F' (read 'F prime') is defined by

$$F'(x) = \text{the rate of change of } F \text{ at } x \tag{3.19}$$

for all numbers x in the domain of F for which the rate of change of F at x exists.
Alternatively, we may write

$$F'(x) = \lim_{b \to x} \frac{F(b) - F(x)}{b - x}. \tag{3.20}$$

The function F' is called the *derivative*[3] of F (the function derived from F). When the independent variable of F is expressed as in $F(x)$, $F'(x)$ is the derivative of $F(x)$ with respect to x.

The derivative is one-half of calculus, perhaps 4 percent of your university education, and requires your attention. The next 200? pages of this text present biological and physical interpretations of the derivative, formulas for computing the derivatives of commonly encountered functions, and uses of derivatives in writing equations for mathematical models of biological and physical systems.

Example 3.3.1 You found in Chapter 1, Problem 1.9.1 on page 48 that light intensity, L, at a distance, d, from a linear slit of light was

$$L(d) = \frac{1.45}{d} \qquad 3 \le d \le 16$$

where L was measured in mW/cm^2 and d was measured in cm.

At what rate is the light decreasing when $d = 5$ cm? We find the derivative of L. For any value of d between 3 and 16 cm,

$$\begin{aligned} L'(d) &= \lim_{b \to d} \frac{L(b) - L(d)}{b - d} \frac{\text{mW/cm}^2}{\text{cm}} \\ &= \lim_{b \to d} \frac{\frac{1.45}{b} - \frac{1.45}{d}}{b - d} \frac{\text{mW/cm}^2}{\text{cm}} \\ &= \lim_{d \to 5} \left(-1.45 \frac{1}{d \times b} \right) \frac{\text{mW/cm}^2}{\text{cm}} \end{aligned}$$

[3]Joseph-Louis Lagrange (a French mathematician of Italian descent, 1736-1813) used the word 'derivative', and may have been the first to do so (H. L. Vacher, Computational geology 5 – If geology, then calculus, *J. Geosci. Educ.*, 1999, **47** 186-195.

$$= \frac{-1.45}{d^2} \quad \frac{\text{mW/cm}^2}{\text{cm}}$$

$$\text{For } d = 5, \qquad L'(5) = \left.\frac{-1.45}{d^2}\right|_{d=5} = \frac{-1.45}{5^2} = -0.058 \quad \frac{\text{mW/cm}^2}{\text{cm}}.$$

We have just used a helpful notation: $|_{x=a}$

$$\text{For any function, } G, \qquad G(x)\,|_{x=a} = G(a).$$

It is useful to compare the algebraic forms of L and L',

$$L(d) = \frac{k}{d} \qquad \text{and} \qquad L'(d) = -k\frac{1}{d^2}.$$

Explore 3.3.1 For one mole of oxygen at 300° Kelvin the pressure, P in atmospheres, and volume, V in liters, are related by

$$P = \frac{1 \times 0.0820 \times 300}{V}$$

a. Find the derivative of P with respect to V.

b. What is $P'(1)$?

■

The question of units on the rate of change is important. We have used the following.

Units on F′

$$\text{If} \quad \lim_{x \to a} G(x) = L, \qquad \text{then the units on } L \text{ are the units on } G.$$

It follows that the units on

$$F'(x) = \lim_{b \to x} \frac{F(b) - F(x)}{b - x}$$

are the units on F divided by the the units on the domain of F (the units on x).

- If $F(t)$ is distance in meters and t is time in seconds, then $F'(t)$ is meters per second, or velocity.

- If $F(x)$ is pressure in atmospheres and x is altitude in km, then $F'(x)$ is commonly called the pressure *gradient* and is measured in atm/km.
- If $F(t)$ is population size in individuals and t is time in years, then $F'(t)$ is population growth rate (which might be negative) in individuals per year.

It is useful to see the rate of change of F over the whole domain of F. In Figure 3.12A is the graph of a function, called a logistic function, that is typical of the size of population that starts at low density and grows in a limited environment. At time $t = 0$ the population size is $P(0) = 1$ and the maximum supportable population is $M = 10$. Its derivative is shown in 3.12B and illustrates population growth rate. The maximum of the derivative occurs at $t = 3.15$ and this marks the steepest part of the population graph. The growth rate initially is low, rises to a maximum at $t = 3.15$, and decreases again as population density nears its maximum.

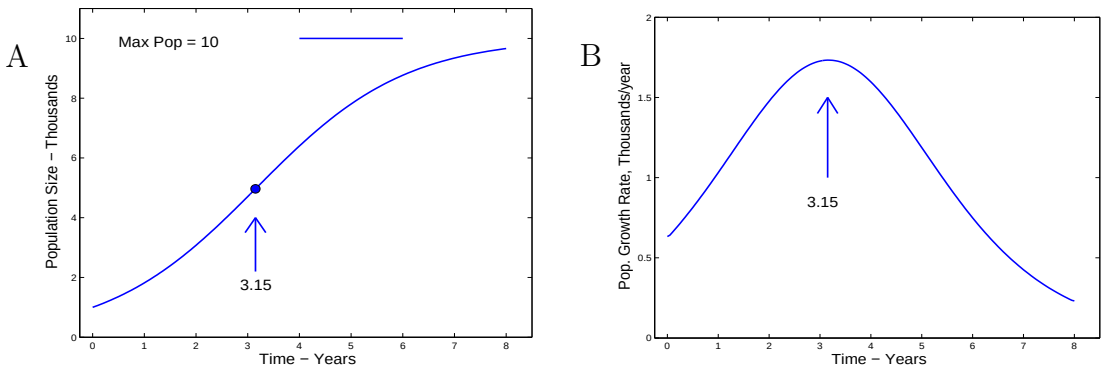


Figure 3.12: A. Graph of a classical logistic curve, L, describing population size as a function of time. B. A graph of L', the population growth rate.

Example 3.3.2 *Problem.* The graphs of a function P and its derivative P' are shown inFigure 3.13. Which graph is the graph of P?

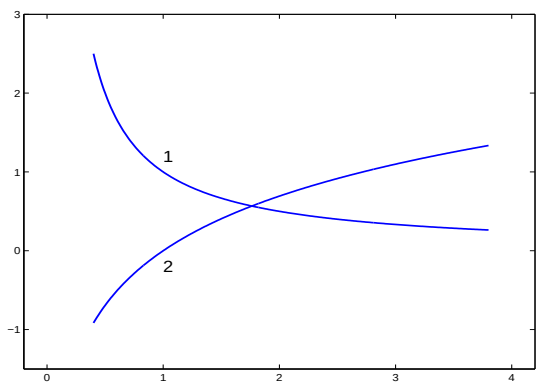


Figure 3.13: The graph of a function P and its derivative P'.

Solution. We claim that graph 1 is not P, because every tangent to graph 1 has negative slope and some y-coordinates of graph 2 are positive. Therefore graph 2 must be the graph of P.

Alternate notation. Calculus originated in England with Sir Isaac Newton (1642-1727) and in Germany with Gottfried Wilhelm Leibniz (1646-1716), and indeed some elements of it were anticipated by the Greek mathematicians. Given the multiple origins and a 300 year history, it is not surprising that there are several notations for derivative. Newton used y' for the 'fluxion' of y (rate of change of y). But when the independent variable was time, Newton used the symbol $\dot{y}$ for the rate of change of y and $\ddot{y}$ for the second derivative of y (the derivative of the derivative of y). Therefore if y denotes distance, $\dot{y}$ denotes velocity and $\ddot{y}$ denotes acceleration. The most common notation, $\frac{dF}{dt}$, for the derivative was introduced and used by Leibnitz. When discussing the rate of change of F at a number a one may see

$$F'(a) \qquad \text{or} \qquad \left.\frac{dF}{dx}\right|_{x=a} \qquad \text{or} \qquad \frac{dF}{dx} \qquad \text{or} \qquad \dot{F}$$

A function that has a derivative is called a *differentiable* function. To *differentiate* F means to compute the derivative, F'. "Differentiable" comes from the concept of a 'differential' which is a cousin of an elusive concept called 'infinitesimal'. An infinitesimal is a positive number that is less than all other positive numbers, which is possible only in an extension of the number system that we commonly use. An infinitesimal change, dx, in x causes an infinitesimal change, dF, in $F(x)$ and the derivative is the ratio $\frac{dF}{dx}$. The concept of a limit is considered to be less mysterious than is infinitesimal, and can easily be put on a quite sound footing, whereas it is difficult to define 'infinitesimal' clearly[4].

It is sometimes preferable to substitute h for $b - x$ so that $b = x + h$, and write

$$F'(x) = \lim_{b \to x} \frac{F(b) - F(x)}{b - x} \qquad \text{as} \qquad F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}. \tag{3.21}$$

Exercises for Section 3.3, The derivative function, F′.

Exercise 3.3.1 Use Equation 3.20,

$$F'(x) = \lim_{b \to x} \frac{F(b) - F(x)}{b - x},$$

to compute $F'(x)$ for

a.	$F(x)$	$=$	x^2	b.	$F(x)$	$=$	$2x^2$	c.	$F(x)$	$=$	x^2+1
d.	$F(x)$	$=$	x^3	e.	$F(x)$	$=$	$4x^3$	f.	$F(x)$	$=$	x^3-1
g.	$F(x)$	$=$	x^2+x	h.	$F(x)$	$=$	x^2+x^3	i.	$F(x)$	$=$	$3x+1$
j.	$F(x)$	$=$	$\sqrt{x}$	k.	$F(x)$	$=$	$4\sqrt{x}$	l.	$F(x)$	$=$	$4+\sqrt{x}$
m.	$F(x)$	$=$	5	n.	$F(x)$	$=$	$\frac{1}{x}$	o.	$F(x)$	$=$	$5+\frac{1}{x}$
p.	$F(x)$	$=$	$\frac{1}{x^2}$	q.	$F(x)$	$=$	$\frac{5}{x^2}$	r.	$F(x)$	$=$	$5+\frac{1}{x^2}$

[4] Abraham Robinson, *Nonstandard Analysis*, Princeton University Press, 1996 is a recent book defining calculus based on infinitesimals.

Exercise 3.3.2 Use the equation,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h},$$

to comupute $F'(x)$ for:

a.	$F(x)$	$=$	x^2	b.	$F(x)$	$=$	$3x^2$	c. $F(x) = x^2 + 5$
d.	$F(x)$	$=$	x^{-1}	e.	$F(x)$	$=$	$2x^{-1}$	f. $F(x) = x^{-1} - 7$

Exercise 3.3.3 In Figure Ex. 3.3.3A and Figure Ex. 3.3.3B are four pairs of graphs of a function P and and its derivative, P'. For each pair, which is the graph of P? Explain your choice.

Figure for Exercise 3.3.3 A. Graphs of a function and its derivative for Exercise 3.3.3.

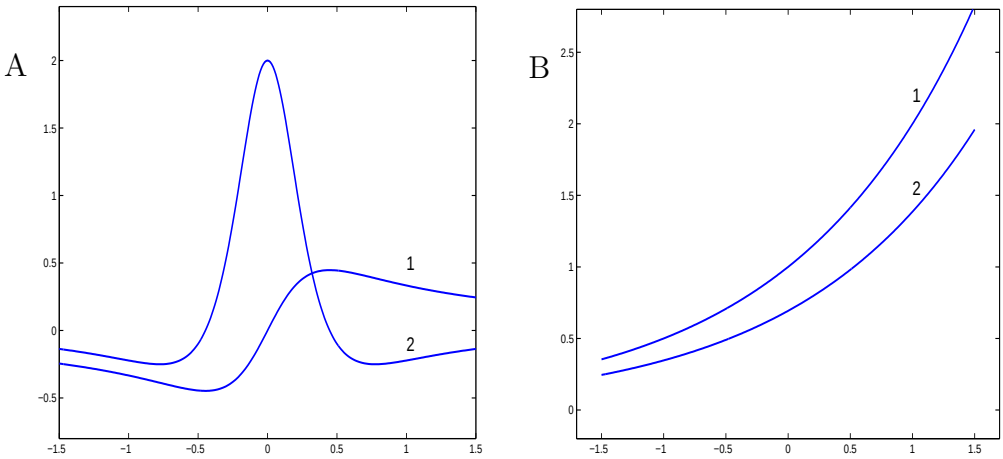
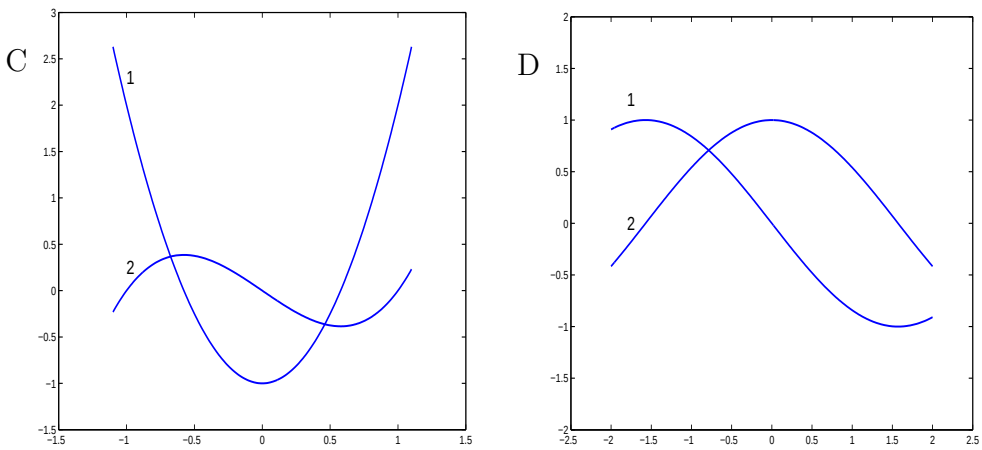
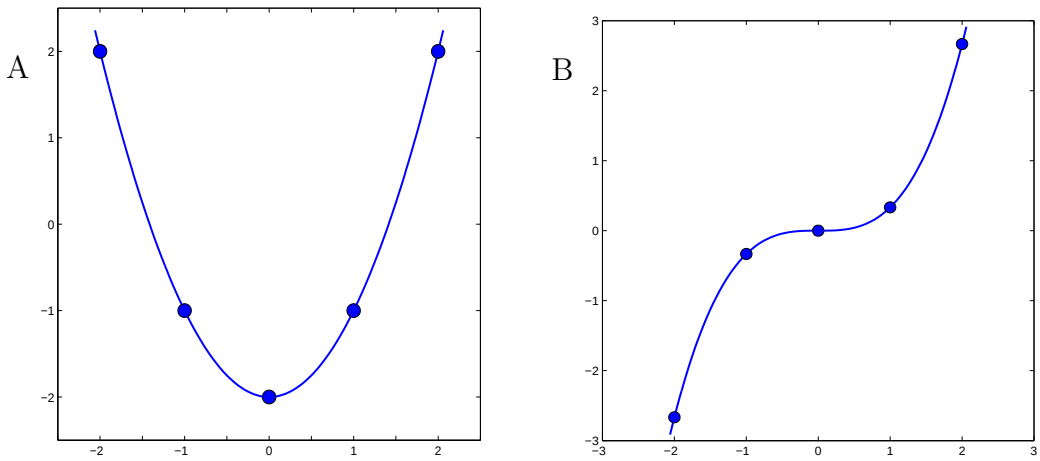


Figure for Exercise 3.3.3 B. Graphs of a function and its derivative for Exercise 3.3.3.



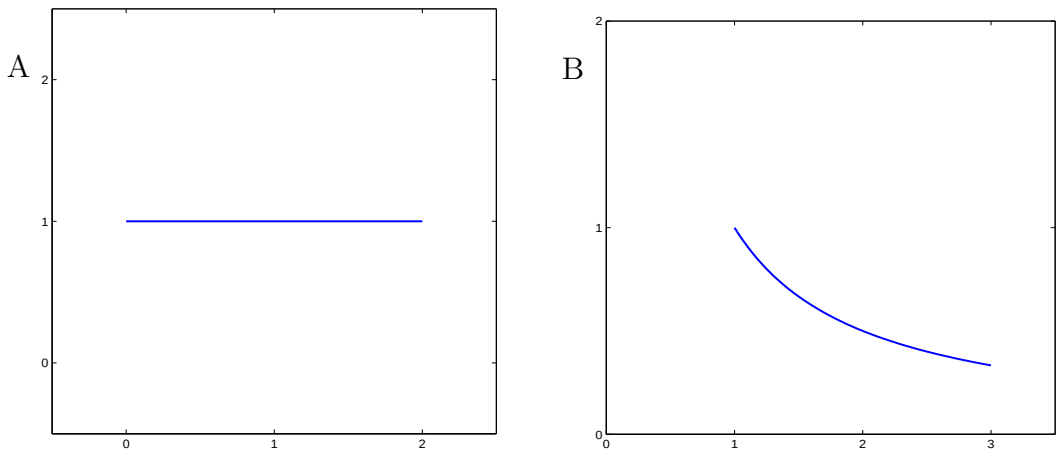
Exercise 3.3.4 a. In Exercise Figure 3.3.4A is the graph of a function, F. Estimate the slopes of the tangents to that graph at the points marked on the graph. Plot a new graph of slopes *vs* the dependent variable for F. Sketch a graph of F'.
b. Repeat the steps of the part a. for the graph of a function, G, in Exercise Figure 3.3.4B.

Figure for Exercise 3.3.4 A. Graph of a function F for Exercise 3.3.4a. B. Graph of a function G for Exercise 3.3.4b.



Exercise 3.3.5 Danger: Obnubilation Zone.
a. Shown in Exercise Figure 3.3.5A is a graph of the derivative, F', of a function, F. One point of the graph of F is (0,2) (that is, $F(0) = 2$). Your job, should you accept it, is to sketch a reasonable graph of F.
b. Repeat the steps of a. to sketch a graph of a function G knowing that $G(1) = 0$ and the graph of $G'(x)$ is shown in Exercise Figure 3.3.5B. (Consider: What is the slope of G at $x = 1$? What is the slope of G at $x = 2$? What is the slope of G at $x = 3$?)

Figure for Exercise 3.3.5 A. Graph of F' for a function F for Exercise 3.3.5a. B. Graph of G' for a function G for Exercise 3.3.5b.



Exercise 3.3.6 In Figure 3.12A it appears that at $t = 6$ years the population was about 8.75 thousand, or 8,750 individuals and that the growth rate was about 0.75 thousand per year, or 750 individuals per year. Suppose this is a deer population and you wished to allow hunters to harvest some deer each year. How many deer could be harvested each year and the population size remain at 8,750 individuals?

Exercise 3.3.7 The function

$$P(t) = \frac{10 \times 2^t}{9 + 2^t} = \frac{10}{(9 \times 2^{-t} + 1)}$$

is an example of a logistic function, a type of function that often is used to describe the growth of populations. Use your calculator to plot the graph of this function for $0 \le t \le 10$, $0 \le y \le 20$. Find how to plot $P'(t)$ on your calculator. On the TI-86, P and P' are drawn by

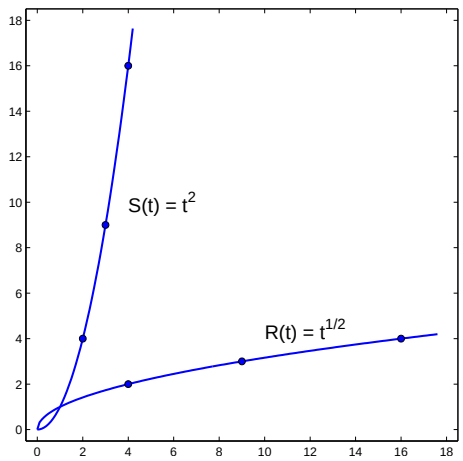
y1 = 10/(9*2^-x+1) y2 = der1(y1,x)

Note: The - is negative and not subtraction. Trace the graph of P' and find at what time it is a maximum. Identify the point on the graph of P corresponding to that time.

At what time and population size is the population growing the fastest?

Exercise 3.3.8 The square function, $S(t) = t^2$, $t \ge 0$, and the square root function, $R(t) = \sqrt{t}$, $t \ge 0$, are each inverses of the other. See Figure Ex. 3.3.8. Compare the slopes of the tangents to S at the points (2,4), (3,9), and (4,16) with the slopes of R at the respectively corresponding points, (4,2), (9,3), and (16,4) of R. Compare the slope of the graph of S at the point (a, a^2), $a > 0$ with the slope of the graph of R at the corresponding point (a^2, a).

Figure for Exercise 3.3.8 Graphs of $S(t) = t^2$ and $R(t) = \sqrt{t}$. See Exercise 3.3.8.



The next two exercises explore the reflective property of a parabola which asserts that light rays originating from the focal point of a parabola will strike the parabola and be reflected in a direction parallel to the axis of the parabola. We choose the parabola that is the graph of $y = 2\sqrt{t}$ which has (1,0) as its focal point and $x = -1$ as its directrix.

Exercise 3.3.9 Shown in Figure Ex. 3.3.9 is the graph of $y = 2\sqrt{t}$ and a ray emanating vertically from the focal point at (1,0) and reflected (apparently horizontally) by the tangent to the parabola at (1,2). The angles A (of incidence) and B (of reflection) are equal.

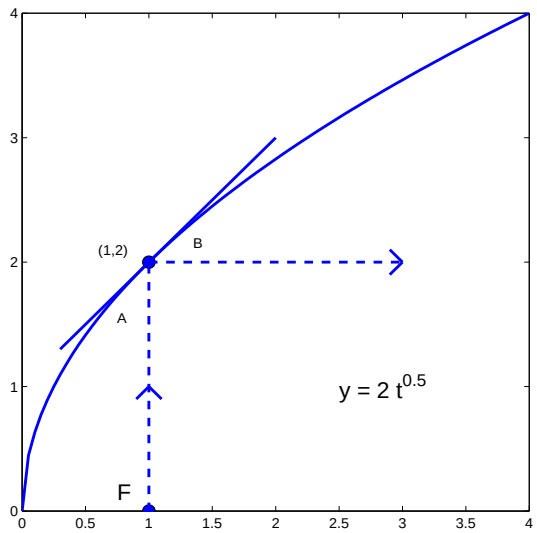
a. Compute the slope of the tangent to $y = 2\sqrt{t}$ at the point (1,2).

b. Argue that the angle A in Figure Ex. 3.3.9 is 45°.

c. Argue that the angle B in Figure Ex. 3.3.9 is 45°.

d. Argue that the reflected ray from (1,2) is horizontal.

Figure for Exercise 3.3.9 Graph of the parabola $y = 2\sqrt{t}$ and a ray (dashed line) emanating vertically from the focal point (1,0) and reflected at (1,2). See Exercise 3.3.9.



Exercise 3.3.10 Shown in Figure Ex. 3.3.10 is the graph of $y = 2\sqrt{t}$ a ray (dashed line) from the focal point, (1,0), to the point $(a,\ 2\sqrt{a})$, and a tangent, T, to the parabola at $(a,\ 2\sqrt{a})$. Our goal is to show that the reflected ray (dashed line with arrow head) is horizontal (but it has not been drawn that way). The two angles marked β are equal because they are vertical angles of intersecting lines.

It will be sufficient to show that the angle of reflection, B, is also β, the angle of inclination of the tangent T. Because B and β are acute, it will be sufficient to show that $\tan B = \tan\beta$.

a. Argue that $C = A + \beta$, so that $A = C - \beta$.

b. Argue that $B = A$, so that $B = C - \beta$.

c. Compute the slope of the tangent T to the graph of $y = 2\sqrt{t}$ at $(a, 2\sqrt{a})$. By definition, this number is also $\tan\beta$.

d. Compute $\tan C$.

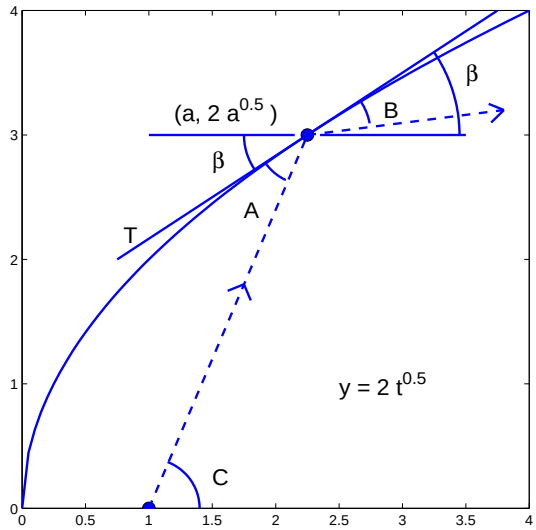
e. Use the trigonometric identity

$$\tan(C - \beta) = \frac{\tan C - \tan\beta}{1 + \tan C\ \tan\beta}$$

to show that $\tan B = \tan(C - \beta) = \frac{1}{\sqrt{a}}$.

Because $\tan B = \tan \beta$, $B = \beta$ and the reflected ray is horizontal.

Figure for Exercise 3.3.10 Graph of the parabola $y = 2\sqrt{t}$ and a ray (dashed line) emanating vertically from the focal point (1,0) and reflected at a point $(a, 2\sqrt{a})$. See Exercise 3.3.10.



3.4 Mathematical models using the derivative.

The rate of change of a function provides a powerful new way of thinking about models of biological processes.

The changes in biological and physical properties that were measured in discrete packages ($P_{t+1} - P_t$, $A_{t+1} - A_t$, $I_{t+1} - I_t$) in Chapter 1 can be more accurately represented as instantaneous rates of change ($P'(t)$, $A'(t)$, $I'(t)$) using the derivative. The *V. natrigens* populations were measured at 16 minute intervals but were growing continuously. The kidneys filter continuously, not in 5-minute spurts. Here we begin a process of using the derivative to interpret mathematical models that continues through out the book.

3.4.1 Mold growth.

We wrote in Section 1.5 that the *daily* increase in the area of a mold colony is proportional to the circumference of the colony at the beginning of the day. Alternatively, we might say:

Mathematical Model 3.4.1 Mold growth. The rate of increase in the area of the mold colony at time t is proportional to the circumference of the colony at time t.

Letting $A(t)$ be area and $C(t)$ be circumference of the mold colony at time t, we would write

$$A'(t) = k \times C(t)$$

Because $C(t) = 2\sqrt{\pi}\sqrt{A(t)}$ (assuming the colony is circular)

$$A'(t) = k \times 2\sqrt{\pi}\sqrt{A(t)} = K\sqrt{A(t)} \tag{3.22}$$

where $K = k \times 2\sqrt{\pi}$.

Equation 3.22, $A'(t) = K\sqrt{A(t)}$, is a statement about the function A. We recall also that the area of the colony on day 0 was 4 mm^2. Now we search for a function A such that

$$A(0) = 4 \qquad A'(t) = K\sqrt{A(t)} \qquad t \geq 0 \tag{3.23}$$

Warning: Incoming Lightning Bolt. Methodical ways to search for functions satisfying conditions such as Equations 3.23 are described in Chapter 16. At this stage we only write that the function

$$A(t) = \left(\frac{K}{2}t + 2\right)^2 \qquad t \geq 0 \qquad \textbf{Bolt Out of Chapter 16.} \tag{3.24}$$

satisfies Equations 3.23 and is the only such function. In Exercise 3.6.6 you are asked to confirm that $A(t)$ of Equation 3.24 is a solution to Equation 3.23.

Finally, we use an additional data point, $A(8) = 266$ to find an estimate of K in $A(t) = \left(\frac{K}{2}t + 2\right)^2$.

$$\begin{aligned} A(8) &= \left(\frac{K}{2}8 + 2\right)^2 \\ 266 &= (4K + 2)^2 \\ K &\doteq 3.58 \end{aligned}$$

Therefore, $A(t) = (1.79t + 2)^2$ describes the area of the mold colony for times $0 \leq t \leq 9$. Furthermore, A has a quadratic expression as suggested in Section 1.5 based on a discrete model. A graph of the original data and A appears in Figure 3.14.

3.4.2 Chemical kintetics.

Chemists use the *rate of change in the amount of product* from a chemical reaction as a measure of the *reaction rate*. You will compute some rates of chemical reactions from discrete data of chemical concentration *vs* time.

Chemical reactions in which one combination of chemicals changes to another are fundamental to the study of chemistry. It is important to know how rapidly the reactions occur, and to know what factors affect the rate of reaction. A reaction may occur rapidly as in an explosive mixture of chemicals

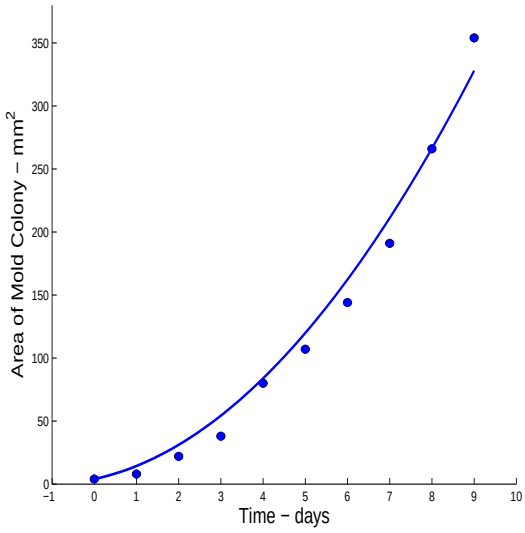


Figure 3.14: Graph of $A(t) = (\,1.79t + 2\,)^2$ and the area of a mold colony from Table 1.5

or slowly, as when iron oxidizes on cars in a junk yard. Temperature and concentration of reactants often affect the reaction rate; other chemicals called catalysts may increase reaction rates; in many biological processes, there are enzymes that regulate the rate of a reaction. Consider

$$\mathrm{A} \longrightarrow \mathrm{B}$$

to represent (part of) a reaction in which a reactant, A, changes to a product, B. The rate of the reaction may be measured as the rate of disappearance of A or the rate of appearance of B.

Butyl chloride.

When butyl chloride, C_4H_9Cl, is placed in water, the products are butyl alcohol and hydrochloric acid. The reaction is

$$\mathrm{C_4H_9Cl + H_2O \longrightarrow C_4H_9OH + H^+ + Cl^-}$$

As it takes one molecule of C_4H_9Cl to produce one atom of Cl^-, the rate at which butyl chloride disappears is the same as the rate at which hydrochloric acids appears. The presence of Cl^- may be measured by the conductivity of the solution. Two students measured the conductivity of a solution after butyl chloride was added to water, and obtained the results shown in Figure 3.15. The conductivity probe was calibrated with 8.56 mmol NaCl, and conductivity in the butyl chloride experiment was converted to mmol Cl^-. The experiment began with butyl chloride being added to water to yield 9.6 mmol butyl chloride.

The average rate of change over the time interval [30,40]

$$m_{30,40} = \frac{1.089 - 0.762}{40 - 30} = 0.0327,$$

and the average rate of change over the time interval [40,50]

$$m_{40,50} = \frac{1.413 - 1.089}{50 - 40} = 0.0324.$$

Time sec	Cl^- mmol	Time sec	Cl^- mmol
0	0.014	80	2.267
10	0.188	90	2.486
20	0.456	100	2.683
30	0.762	110	2.861
40	1.089	120	3.010
50	1.413	130	3.144
60	1.728	140	3.255
70	2.015	150	3.355

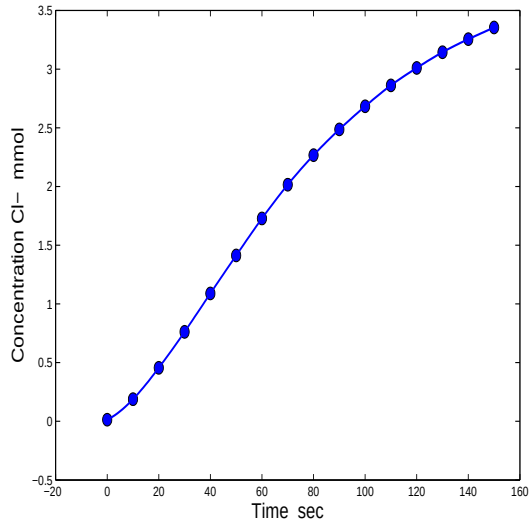


Figure 3.15: Conductivity of a water and butyl chloride solution at times after butyl chloride was added to water.

both approximate the reaction rate. A better estimate is the average of these two numbers, $\frac{0.0327+0.0324}{2} = 0.03255$. We only use the average when the backward and forward time increments, -10 seconds and +10 seconds, are of the same magnitude. The average can be computed without computing either of the backward or forward average rates, as

$$m_{30,50} = \frac{1.413 - 0.762}{50 - 30} = 0.03255$$

In the case of $t = 150$, knowledge of the forward time increment is not available, and we use the backward time increment only.

$$m_{140,150} = \frac{3.355 - 3.255}{150 - 140} = 0.01$$

Explore 3.4.1 Estimate the reaction rate at time $t = 80$ seconds. ■

Example 3.4.1 It is useful to plot the reaction rate *vs* the concentration of Cl^- as shown in Figure 3.16. The computed reaction rates for times $t = 0, 10, \cdots 40$ are less than we expected. At these times, the butyl chloride concentrations are highest and we expect the reaction rates to also be highest. Indeed, we expect the rate of the reaction to be proportional to the butyl chloride concentration. If so then the relation between reaction rate and Cl^- concentration should be linear, as in the parts corresponding to times $t = 60, 70, \cdots 160$ s. The line in Figure 3.16 has equation $y = 0.0524 - 0.0127x$. We can not explain the low rate of appearance of Cl^- at this time.

The reaction is not quite so simple as represented, for if butyl alcohol is placed in hydrochloric acid, butylchloride and water are produced. You may see from the data that the molarity of Cl^- is tapering off and indeed later measurements showed a maximum Cl^- concentration of 3.9 mmol. If all of the butyl

Time sec	Cl^- mmol	Reaction Rate mmol/sec	Time sec	Cl^- mmol	Reaction Rate mmol/sec
0	0.014	0.0175	80	2.267	0.0235
10	0.188	0.0221	90	2.486	0.0208
20	0.456	0.0287	100	2.683	0.0188
30	0.762	0.0317	110	2.861	0.0164
40	1.089	0.0326	120	3.010	0.0141
50	1.413	0.0319	130	3.144	0.0122
60	1.728	0.0301	140	3.255	0.0105
70	2.015	0.0270	150	3.355	0.010

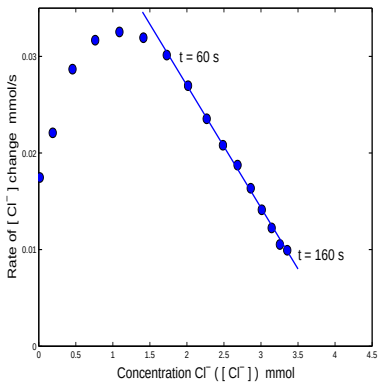


Figure 3.16: Rate at which CL^- accumulates as a function of CL^- concentration after butyl chloride is added to water.

chloride decomposed, the maximum Cl^- concentration would be 9.6, the same as the initial concentration of butyl chloride. There is a reverse reaction and the total reaction may be represented

$$C_4H_9Cl(aq) + H_2O(l) \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C_4H_9OH(aq) + H^+ + Cl^-(aq)$$

The numbers, k_1 and k_2, are called rate constants of the reaction. The number k_1 is the negative of the slope of the line computed in Example 3.4.1 (that is, 0.0127).

Exercises for Section 3.4 Mathematical models using the derivative.

Exercise 3.4.1 Write a derivative equation that describes the following model of mold growth.

> **Mathematical Model.** A mold colony is growing in a circular pattern. The radius of the colony is increasing at a constant rate.

Exercise 3.4.2 Write a derivative equation that describes the following model of light depletion below the surface of a lake.

> **Mathematical Model.** The rate at which light intensity decreases at any depth is proportional to the intensity at that depth.

Exercise 3.4.3 Write a derivative equation that describes the following model of penicillin clearance.

> **Mathematical Model.** The rate at which the kidneys remove penicillin is proportional to the concentration of penicillin.

Exercise 3.4.4 Data from Michael Blaber of Florida State University College of Medicine[5] for the butyl chloride experiment

[5] http://wine1.sb.fsu.edu/chem1046/notes/Kinetics/Rxnrates.htm

$$C_4H_9Cl(aq) + H_2O(l) \rightarrow C_4H_9OH(aq) + HCl(aq)$$

are shown in Table 3.4.4. These are more nearly what one would expect from this experiment.

a. Graph the data.

b. Estimate the rate of change of the concentration of C_4H_9Cl for each of the times shown.

c. Draw a graph of the rate of reaction versus concentration of C_4H_9Cl.

Table for Exercise 3.4.4 Data for Ex. 3.4.4.

Time (sec)	0	50	100	150	200	300	400	500	800
$C[_4H_9Cl]$ (M)	0.1	0.0905	0.0820	0.0741	0.0671	0.0549	0.0448	0.0368	0.0200

Exercise 3.4.5 Data from Purdue University[6] for the decrease of the titration marker phenolphthalein (Hln) in the presence of excess base are shown in Table 3.4.5. The data show the concentration of phenolphthalein that was initially at 0.005 M in a solution with 0.61 M OH^- ion.

a. Graph the data.

b. Estimate the rate of change of the concentration of phenolphthalein (Hln) for each of the times shown.

c. Draw a graph of the rate of reaction versus concentration of phenolphthalein.

Table for Exercise 3.4.5 Data for Ex. 3.4.5.

Time (sec)	0	10.5	22.3	35.7	51.1	69.3	91.6	120.4	160.9
Hln (M)	0.005	0.0045	0.0040	0.0035	0.0030	0.0025	0.0020	0.0015	0.0010

Exercise 3.4.6 Data from Michael J. Mombourquette, Queens University, Kingston, Ontario, Canada[7] for the decrease of CO in the reaction

$$CO(g) + NO_2(g) \rightarrow CO_2(g) + NO\ (g)$$

are shown in Table 3.4.6A. and the decrease of N_2O_5 in the reaction

$$2N_2O_5(g) \rightarrow 4NO_2(g) + O_2$$

are shown in Table 3.4.6B. Initially, 0.1 g/l of CO was mixed with 0.1 g/l of NO_2.

For each table,

a. Graph the data.

[6] http://chemed.chem.purdue.edu/genchem/topicreview/bp/ch22/rate.html

[7] http://www.chem.queensu.ca/people/faculty/mombourquette/FirstYrChem/kinetics/index.htm

b. Estimate the rate of change of the concentration of CO or of N_2O_5 for each of the times shown.

c. Draw a graph of the rate of reaction versus concentration of reactant.

Table for Exercise 3.4.6 Data for Ex. 3.4.6.

A.

Time (sec)	[CO] g/l
0	0.100
10	0.067
20	0.050
30	0.040

B.

Time (sec)	$[N_2O_5]$
0	0.0172
10	0.0113
20	0.0084
30	0.0062
40	0.0046
50	0.0035
60	0.0026

3.5 Derivatives of Polynomials, Sum and Constant Factor Rules

We begin a common strategy to find derivatives of functions:

- Use Definition 3.2.2 to find the derivative of a few elementary functions (such as $F(t) = C$, $F(t) = t$, and $F(t) = t^n$). We call these formulas *primary formulas*.
- Use Definition 3.2.2 to prove some rules about derivatives of combinations of functions.
- Use the combination rules and primary formulas to compute derivatives of more complex functions.

We are using the symbol t for the independent variable. We could as well use x as in previous sections, but time is a common independent variable in biological models and we wish to use both symbols.

The following are derivative formulas for this section.

Primary formulas: First Set

If C is a number and n is a positive integer and u, v, and F are functions with a common domain, D, then

If for all t,	$F(t) = C$	then	$F'(t) = 0$	**Constant Rule**
If for all t,	$F(t) = t$	then	$F'(t) = 1$	**t Rule**
If for all t,	$F(t) = t^n$	then	$F'(t) = nt^{n-1}$	**$\mathbf{t}^n$ Rule**

Combination Rules: First Set

If for all t in D	$F(t) = u(t) + v(t)$	then	$F'(t) = u'(t) + v'(t)$	**Sum Rule**
If for all t in D	$F(t) = C\,u(t)$	then	$F'(t) = C\,u'(t)$	**Constant Factor Rule**

We use a notation that simplifies the statements of the rules. An example is

$$\text{If} \quad F(t) = t^4 \quad \text{then} \quad F'(t) = 4t^3 \qquad \text{is shortened to} \qquad \left[\, t^4 \,\right]' = 4t^3.$$

More generally

$$\text{If} \quad F(t) = E(t) \quad \text{then} \quad F'(t) = H(t) \qquad \text{is shortened to} \qquad [\, E(t) \,]' = H(t).$$

Alternatively, we may use Leibnitz' notation and write

$$\frac{d}{dt}\,[\, E(t) \,] = H(t).$$

The first set of rules may be written,

Derivative Rules: First Set

$$[\,C\,]' = 0 \qquad \textbf{Constant Rule} \tag{3.25}$$

$$[\,t\,]' = 1 \qquad \textbf{t Rule} \tag{3.26}$$

$$[\,t^n\,]' = n\,t^{n-1} \qquad \textbf{t}^n \textbf{ Rule} \tag{3.27}$$

$$[\,u(t) + v(t)\,]' = u'(t) + v'(t) \qquad \textbf{Sum Rule} \tag{3.28}$$

$$[\,C\,F(t)]' = C\,F'(t) \qquad \textbf{Constant Factor Rule} \tag{3.29}$$

The short forms depend on your recognizing that C is a constant, t is an independent variable, n is a positive integer, and that u and v are functions with a common domain, and that F is a function.

The distinction between $[C]' = 0$ and $[t]' = 1$ is that the symbol $'$ means rate of change **as t changes**, sometimes said to be **derivative with respect to t**. C denotes a number that does not change with t and therefore its rate of change as t changes is zero. The Leibnitz notation, $\frac{d}{dt}$, is read, 'derivative with respect to t', and gives a better distinction between these two formulas.

Explore 3.5.1 Write short forms of the first set of derivative rules using the Leibnitz notation, $\frac{d}{dt}$. ■

Proofs of the Derivative Rules: First Set. You have sufficient experience that you can prove each of these rules using the Definition of Derivative3.20. We prove the t^n Rule and the Sum Rule and leave the others for you in Exercise 3.5.4.

Explore 3.5.2 The steps of the arguments for the Sum Rule and the Constant Factor Rule will be made apparent if you use Definition of Derivative 3.20 to compute the derivatives of

$$P(t) = t^2 + t^3 \qquad \text{and} \qquad P(t) = 4t^2$$

■

t^n Rule. Suppose n is a positive integer and a function F is defined by

$$F(t) = t^n \qquad \text{for all numbers } t. \qquad \text{Then} \qquad F'(t) = n\,t^{n-1}$$

Proof.

$$F'(t) = \lim_{b \to t} \frac{F(b) - F(t)}{b - t}$$

$$
\begin{aligned}
&= \lim_{b\to t} \frac{b^n - t^n}{b-t} \\
&= \lim_{b\to t} \frac{(b-t)(b^{n-1} + b^{n-2}t + \cdots + bt^{n-2} + t^{n-1})}{b-t} \\
&= \lim_{b\to t} \left(b^{n-1} + b^{n-2}t + \cdots + bt^{n-2} + t^{n-1}\right) \\
&= \underbrace{t^{n-1} + t^{n-1} + \cdots + t^{n-1} + t^{n-1}}_{n \text{ terms}} \\
&= n\,t^{n-1}
\end{aligned}
$$

We have stated and proved the t^n Rule assuming that n is a positive integer. It is also true for n a negative integer when $t \neq 0$ (Exercise 3.5.5). When t is restricted to being greater than zero, the t^n rule is valid for *every* number n (integer, rational, and irrational). You will prove it for *rational* numbers $\frac{p}{q}$, p and q integers in the Chapter 4, Exercise 4.3.4. Meanwhile, we encourage you to use formulas such as

$$\left[t^{-5}\right]' = -5t^{-5-1} = -5t^{-6}, \qquad \left[t^{5/3}\right]' = \frac{5}{3}t^{(5/3)-1} = \frac{5}{3}t^{2/3}, \qquad \text{and even} \qquad [\,t^{\pi}\,]' = \pi t^{\pi-1}.$$

Sum Rule. Suppose you and your lab partner are growing two populations of *Vibrio natrigens*, one growing at the rate of 4.32×10^5 cells per minute and the other growing at the rate of 2.19×10^5 cells per minute. It is a fairly easy conclusion that the growth rate of all *V. natrigens* in the two populations is 4.32×10^5 plus $2.19 \times 10^5 = 6.51 \times 10^5$ cells per minute.

It is your experience that rates are additive, and we will include a proof. The proof is needed to distinguish the intuitive for addition from the non-intuitive for multiplication. Many students 'intuitively think' that rates are multiplicative, and they are not!

Proof. Suppose u and v are functions with a common domain, D, and for all t in D, $F(t) = u(t) + v(t)$.

$$
\begin{aligned}
F'(t) &= \lim_{b\to t} \frac{F(b) - F(t)}{b-t} \\
&= \lim_{b\to t} \frac{u(b) + v(b) - (u(t) + v(t))}{b-t} \\
&= \lim_{b\to t} \frac{u(b) - u(t) + v(b) - v(t)}{b-t} \\
&= \lim_{b\to t} \frac{u(b) - u(t)}{b-t} + \frac{v(b) - v(t)}{b-t} \\
&= u'(t) + v'(t)
\end{aligned}
$$

The sum rule has a companion, the difference rule,

$$[\,u(t) - v(t)\,]' = u'(t) - v'(t)$$

that we also call the sum rule. Furthermore,

$$[\,u(t)+v(t)+w(t)\,]' = [\,u(t)+v(t)\,]' + w'(t) = u'(t)+v'(t)+w'(t).$$

It can be shown by induction that

$$[\,u_1(t)+u_2(t)+\cdots+u_n(t)\,]' = u_1'(t)+u_2'(t)+\cdots+u_n'(t)$$

'Sum Rule' encompasses all of these possibilities.

Use of the derivative rules. Using the first set of derivative rules, we can compute the derivative of any polynomial *without explicit reference* to the Definition of Derivative 3.20, as the following example illustrates.

Example 3.5.1 Let $P(t) = 3t^4 - 5t^2 + +2t + 7$. Then

$$\begin{aligned}
P'(t) &= [\,P(t)\,]' && \text{Notation shift}\\
&= [3t^4 - 5t^2 + 2t + 7]' && \text{Definition of } P\\
&= [3t^4]' - [5t^2]' + [2t]'\,[7]' && \text{Sum Rule}\\
&= 3\,[t^4]' - 5\,[t^2]' + 2\,[t]'\,[7]' && \text{Constant Factor Rule}\\
&= 3\times 4t^3 - 5\times 2t^1 + 2\,[t]'\,[7]' && t^n \text{ Rule}\\
&= 12t^3 - 10t + 2\times 1 + [7]' && \text{t Rule}\\
&= 12t^3 - 10t + 2 + 0 && \text{Constant Rule}
\end{aligned}$$

Example 3.5.2 It is useful to write some fractions in forms so that the Constant Factor Rule obviously applies. For example

$$P(t) = \frac{t^3}{7} = \frac{1}{7}t^3$$

and to compute P' you may write

$$P'(t) = \left[\frac{t^3}{7}\right]' = \left[\frac{1}{7}t^3\right]' = \frac{1}{7}\left[t^3\right]' = \frac{1}{7}3t^2 = \frac{3}{7}t^2$$

Similarly

$$P(t) = \frac{8}{3t} = \frac{8}{3}\frac{1}{t}$$

and to compute P' you may write

$$P'(t) = \left[\frac{8}{3t}\right]' = \left[\frac{8}{3}\frac{1}{t}\right]' = \frac{8}{3}\left[\frac{1}{t}\right]' = \frac{8}{3}\left[t^{-1}\right]' = \frac{8}{3}\left((-1)t^{-1-1}\right) = -\frac{8}{3}\left(\frac{1}{t^2}\right)$$

Explore 3.5.3 Let $P(t) = 1 + 2\,t^3/5$. Cite the formulas that justify the steps a - d.

$$\begin{aligned}
P'(t) &= \left[1 + 2\,\frac{t^3}{5}\right]' \\
&= [1]' + \left[2\,\frac{t^3}{5}\right]' && \text{a.} \\
&= 0 + \left[2\,\frac{t^3}{5}\right]' && \text{b.} \\
&= 0 + \left[\tfrac{2}{5}\,t^3\right]' \\
&= 0 + \tfrac{2}{5} \times [t^3]' && \text{c.} \\
&= 0 + \tfrac{2}{5} \times 3\,t^2 && \text{d.} \\
&= \tfrac{6}{5}\,t^2 && \blacksquare
\end{aligned}$$

$\blacksquare$

Example 3.5.3 The derivative of a quadratic function is a linear function.

Solution. Suppose $P(t) = at^2 + bt + c$ where a, b, and c are constants. Then

$$\left.\begin{aligned}
P'(t) &= [\,at^2 + bt + c\,]' && \text{a.} \\
&= [\,at^2]' + [bt]' + [c\,]' && \text{b.} \\
&= a\,[\,t^2]' + b\,[t]' + [c\,]' && \text{c.} \\
&= a \times 2t + b \times 1 + [c\,]' && \text{d.} \\
&= 2at + b + 0 && \text{e.}
\end{aligned}\right\} \tag{3.30}$$

We see that $P'(t) = 2at + b$ which is a linear function. $\blacksquare$

3.5.1 Velocity as a derivative.

If $P(t)$ denotes the position of a particle along an axis at time t, then for any time interval $[a, b]$,

$$\frac{P(b) - P(a)}{b - a}$$

is the **average velocity** of the particle during the time interval $[a, b]$. The rate of change of P at $t = a$,

$$\lim_{b \to a} \frac{P(b) - P(a)}{b - a},$$

is the **velocity** of the particle at time $t = a$.

Example 3.5.4 In baseball, a 'pop fly' is hit and the ball leaves the bat traveling vertically at 30 meters per second. How high will the ball go and how much time does the catcher have to get in position to catch it?

Solution. Using a formula from Section 3.6.1, the ball will be at a height $s(t) = -4.9t^2 + 30t$ meters t seconds after it is released, where $s(t)$ is the height above the point of impact with the bat. The velocity, $v(t) = s'(t)$ is

$$[\,s(t)\,]' = \left[-4.9t^2 + 30t\right]' = -4.9 \times 2t + 30 \times 1 = -9.8t + 30$$

The ball will be at its highest position when the velocity $v(t) = s'(t) = 0$ (which implies that the ball is not moving and identifies the time at which the ball is at its highest point, is not going up and is not going down).

$$s'(t) = 0 \qquad \text{implies} \qquad -9.8t + 30 = 0, \qquad \text{or} \qquad t = \frac{30}{9.8} \approx 3.1 \quad \text{seconds.}$$

The height of the ball at $t = 3.1$ seconds is

$$s\,(3.1) = -4.9\,(3.1)^2 + 30 \times 3.1 \approx 45.9 \quad \text{meters}$$

Thus in about 3.1 seconds the ball reaches a height of about 45.9 meters. The catcher will have about 6 seconds to position to catch the ball. Furthermore, at time $t = 6.2$

$$s'(6.2) = -9.8 \times 6.2 + 30 = -30.76 \quad \text{meters/second.}$$

The velocity of the falling ball is ≈ -30.76 m/s when the catcher catches it. Its magnitude will be exactly 30 m/s, the speed at which it left the bat. ■

3.5.2 Local Maxima and Local Minima.

Example 3.5.4 illustrates a useful technique that will be expanded in the next chapter: if one is seeking the high point of a graph, it is useful to examine the points at which the derivative of the related function is zero and the tangent is horizontal. However, tangents at local minima are also horizontal, so that knowing the location of a horizontal tangent does not insure that the location is a local maximum – it might be a local minimum or it may be neither a local minimum nor a local maximum!

Shown in Figure 3.17A is the graph of a function and two horizontal tangents to the graph. The horizontal tangent at A marks a local maximum of the graph, the tangent at B marks a local minimum of the graph. A is a 'local' maximum because in an interval surrounding A, A is the highest point, but note that there are points of the graph above A. In Figure 3.17B is the graph of $P(t) = t^3$ which has a horizontal tangent at (0,0). The horizontal tangent at $(0, 0)$, however, signals neither a local maximum nor a local minimum.

Example 3.5.5 A farmer's barn is 60 feet long on one side. He has 100 feet of fence and wishes to build a rectangular pen along that side of his barn. What should be the dimensions of the pen to maximize the area?

A diagram of a barn and fence with some important labels, L and W, is shown in Figure 3.18. Because there are 100 feet of fence,

$$2 * W + L = 100$$

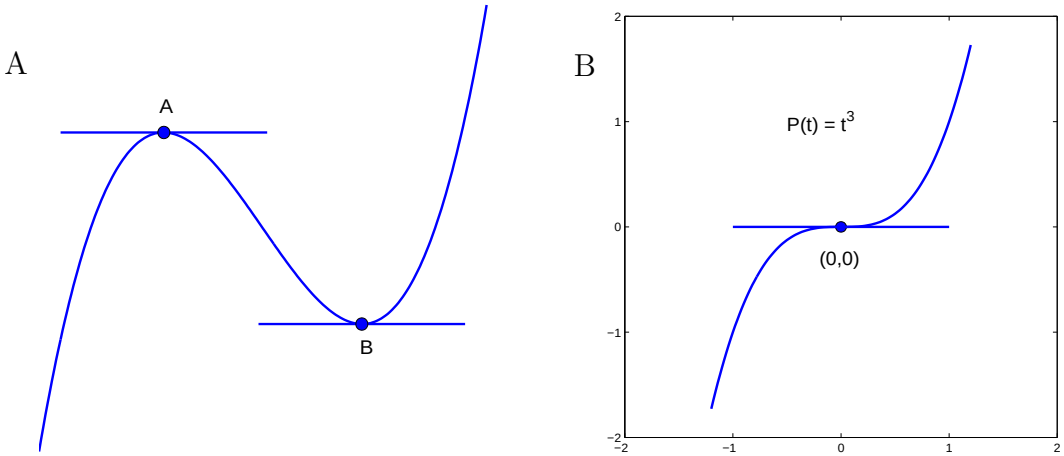


Figure 3.17: A. Graph with a local maximum at A and a local minimum at B. B. Graph of $P(t) = t^3$ that has a horizontal tangent at (0,0); $(0, 0)$ is neither a local maximum nor a local minimum of P.

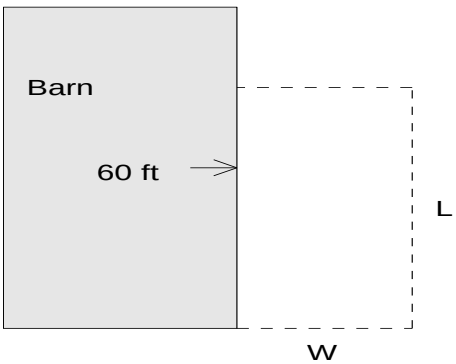


Figure 3.18: Diagram of a barn with adjacent pen bounded by a fence of length 100 feet.

The area, A, is

$$A = L \times W$$

Because $2W + L = 100$,

$$L = 100 - 2W$$

and

$$A = L \times W = (100 - 2W) \times W$$

or

$$A = 100W - 2W^2$$

The question becomes now, for what value of W will A be the largest. The graph of A *vs* W is a parabola with its highest point at the vertex. The tangent to the parabola at the vertex is horizontal,

and we find a value of W for which $A'(W) = 0$.

$$\left.\begin{aligned} A'(W) &= [100W - 2W^2]' && \text{a.} \\ &= [100W]' - [2W^2]' && \text{b.} \\ &= 100\,[W]' - 2\,[W^2]' && \text{c.} \\ &= 100 \times 1 - 2 \times 2W && \text{d.} \end{aligned}\right\} \tag{3.31}$$

The optimum dimensions, W and L, are found by setting $A'(W) = 0$, so that

$$\begin{aligned} A'(W) &= 100 - 4W = 0 \\ W &= 25 \\ \\ L &= 100 - 2W = 50 \end{aligned}$$

Thus the farmer should build a 25 by 50 foot pen.

Example 3.5.6 This problem is written on the assumption, to our knowledge untested, that spider webs have an optimum size. Seldom are they so small as 1 cm in diameter and seldom are they so large as 2 m in diameter. If they are one cm in diameter, there is a low probability of catching a flying insect; if they are 2 m in diameter they require extra strength to withstand wind and rain. We will examine circular webs, for convenience, and determine the optimum diameter for a web so that it will catch enough insects and not fall down.

Solution. Assume a circular spider web of diameter, d. It is reasonable to assume that the amount of food gathered by the web is proportional to the area, A, of the web. Because $A = \pi d^2/4$, the amount of food gather is proportional to d^2. We also *assume* that the energy required to build and maintain a web of area A is proportional to d^3. (The basic assumption is that the work to build a square centimeter of web increases as the total web area increases because of the need to have stronger fibers. If, for example, the area of the fiber cross-section increases linearly with A and the mesh of the web is constant, the mass of the web increases as d^3.)

With these assumptions, the net energy, E, available to the spider is of the form

$$\begin{aligned} E &= \text{Energy from insects caught} - \text{Energy expended building the web} \\ &= k_1 d^2 - k_2 d^3 \end{aligned}$$

where d is measured in centimeters and k_1 and k_2 are proportionality constants.

For illustration we will assume that $k_1 = 0.01$ and $k_2 = 0.0001$. A graph of $E(d) = 0.01d^2 - 0.0001d^3$ is shown in Figure 3.19 where it can be seen that there are two points, A and B, at which the graph has horizontal tangents. We find where the derivative is zero to locate A and B.

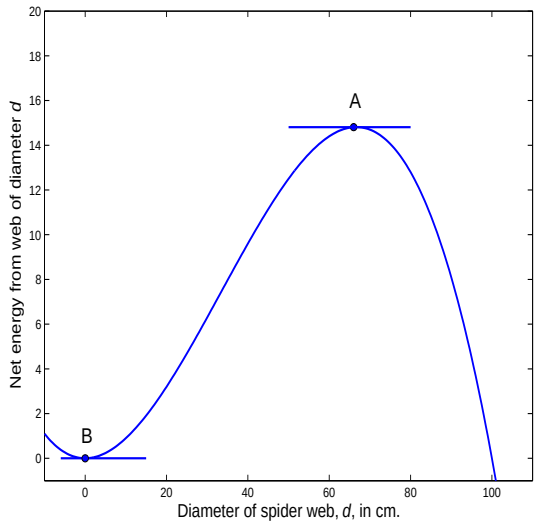


Figure 3.19: Graph of $E(d) = 0.01d^2 - 0.0001d^3$ representative of the energy gain from a spider web of diameter d cm. There is a local maximum at A and a local minimum at B.

$$\left.\begin{aligned} E'(d) &= [\,0.01d^2 - 0.0001d^3\,]' && \text{a.} \\ &= [\,0.01d^2]' - [0.0001d^3\,]' && \text{b.} \\ &= 0.01\,[\,d^2]' - 0.0001\,[d^3\,]' && \text{c.} \\ &= 0.01 \times 2d - 0.0001 \times 3d^2 && \text{d.} \\ &= 0.02d - 0.0003d^2 \end{aligned}\right\} \tag{3.32}$$

Then $E'(d) = 0$ yields

$$\begin{aligned} 0.02d - 0.0003d^2 &= 0 \\ d \times (0.02 - 0.0003d) &= 0 \\ d &= 0 \qquad \text{or} \qquad d = 0.02/0.0003 = 66.7 \quad \text{cm} \end{aligned}$$

The value $d = 0$ locates the local minimum at B and has an obvious interpretation: if there is no web there is no energy gain. At $d = 66.7$ cm, $E(d) = 14.8$ (units unspecified) suggests that a positive net energy will accrue with a web of diameter 66.7 cm and that weaving a web of 66.7cm diameter is the optimum strategy for the spider. Note that if our model and its parameters are correct we have determined in a rather short time what it took spiders many generations to work out. At the very least we could have moved the spiders ahead several generations with our model. ▬

Exercises for Section 3.5, Derivatives of Polynomials, Sum and Constant Factor Rules.

Exercise 3.5.1 Use Definition of the Derivative 3.20 to compute $P'(t)$ for

(a)	$P(t) = 1 + t^2$		(b)	$P(t) = t - t^2$		(c)	$P(t) = t^2 - t$	
(d)	$P(t) = 5\,t^2$		(e)	$P(t) = 5 \times 3$		(f)	$P(t) = 1 + 5t^2$	
(g)	$P(t) = 1 + t + t^2$		(h)	$P(t) = 2t + 3t^2$		(i)	$P(t) = 5 + 3t - 2t^2$	

Exercise 3.5.2 Suppose u and v are functions with a common domain and $P = u - v$. Write $P(t) = u(t) + (-1) \times v(t)$ and use the Sum and Constant Factor rules to show that $P'(t) = u'(t) - v'(t)$.

Exercise 3.5.3 Provide reasons for the steps a - e in Equations 3.30 use to show that the derivatives of quadratic functions are linear functions.

Exercise 3.5.4 a. Prove Equation 3.25, $[\ C\ ]' = 0$.

b. Prove Equation 3.26, $[\ t\ ]' = 1$.

c. Prove Equation 3.29, $[\ C\,F(t)\ ]' = C\ \ [\ F(t)\ ]'$.

Exercise 3.5.5 Suppose m is a positive integer and $u(t) = t^{-m} = 1/t^m$ for $t \neq 0$. Show that $u'(t) = -mt^{-m-1}$, thus proving the t^n rule for negative integers. Begin your argument with

$$\begin{aligned} u'(t) &= \lim_{b \to t} \frac{\dfrac{1}{b^m} - \dfrac{1}{t^m}}{b - t} \\ &= \lim_{b \to t} \frac{t^m - b^m}{b^m \times t^m} \times \frac{1}{b - t} \end{aligned}$$

Exercise 3.5.6 Provide reasons for the steps a - d in Equations 3.31 used to find the optimum dimensions of a lot adjacent to a barn.

Exercise 3.5.7 A farmer's barn is 60 feet long on one side. He has 120 feet of fence and wishes to build a rectangular pen along that side of his barn. What should be the dimensions of the pen to maximize the the area of the pen?

Exercise 3.5.8 A farmer's barn is 60 feet long on one side. He has 150 feet of fence and wishes to build two adjacent rectangular pens of equal area along that side of his barn. What should be the arrangement and dimensions of the pen to maximize the sum of the areas of the two pens?

Exercise 3.5.9 A farmer's barn is 60 feet long on one side. He has 150 feet of fence and wishes to build two adjacent rectangular pens of equal area along that side of his barn. What should be the arrangement and dimensions of the pen to maximize the sum of the areas of the two pens?

Exercise 3.5.10 A farmer's barn is 60 feet long on one side. He has 280 feet of fence and wishes to build two adjacent rectangular pens of equal area along that side of his barn. What should be the arrangement and dimensions of the pen to maximize the sum of the areas of the two pens?

Exercise 3.5.11 A farmer's barn is 60 feet long on one side. He wishes to build a rectangular pen of area 800 square feet along that side of his barn. What should be the dimension of the pen to minimize the amount of fence used?

Exercise 3.5.12 Show that the derivatives of cubic functions are quadratic functions.

Exercise 3.5.13 Probably baseball statistics should be discussed in British units rather than metric units. Professional pitchers throw fast balls in the range of 90+ miles per hour. Suppose the pop fly ball leaves the bat traveling 60 miles per hour (88 feet/sec), in which case the height of the ball in feet will be $s(t) = -16t^2 + 88t$ t seconds after the batter hits the ball. How high will the ball go, and how long will the catcher have to position to catch it? How fast is the ball falling when the catcher catches it?

Exercise 3.5.14 A squirrel falls from a tree from a height of 10 meters above the ground. At time t seconds after it slips from the tree, the squirrel is a distance $s(t) = 10 - 4.9t^2$ meters above the ground. How fast is the squirrel falling when it hits the ground?

Exercise 3.5.15 What is the optimum radius of the trachea when coughing? The objective is for the flow of air to create a strong force outward in the throat to clear it.

For this problem you should perform the following experiment.

Hold your hand about 10 cm from your mouth and blow on it (a) with your lips compressed almost closed but with a small stream of air escaping, (b) with your mouth wide open, and (c) with your lips adjusted to create the largest force on your hand. With (a) your lips almost closed there is a high pressure causing rapid air flow but a small stream of air and little force. With (b) your mouth wide open there is a large stream of air but with little pressure so that air flow is slow. The largest force (c) is created with an intermediate opening of your lips where there is a notable pressure and rapid flow of substantial volume of air.

Let R be the normal radius of the trachea and $r < R$ be the tracheal radius when coughing. Assume that the velocity of air flow through the trachea is proportional to pressure difference across the trachea and that the pressure difference is proportional to $R - r$, the constriction of the trachea. Assume that the mass of air flow is proportional to the area of the trachea (πr^2). Finally, the momentum, M, of the air flow is mass times velocity:

$$M = k \times r^2(R - r)$$

What value of r, $0 < r < R$, maximizes M?

Exercise 3.5.16 Cite formulas that justify the steps a - d in Equations 3.32 for the analysis of the spider web.

Exercise 3.5.17 Let $P(t) = -3 + 5\,t - 2\,t^2$. Cite the formulas that justify steps a - f below:

$$\begin{aligned}
P'(t) &= [-3 + 5 \times t + (-2) \times t^2]' && \text{a.}\\
&= [-3]' + [5 \times t]' + [(-2) \times t^2]' && \text{b.}\\
&= 0 + [5 \times t]' + [(-2) \times t^2]' && \text{c.}\\
&= 0 + 5 \times [t]' + (-2) \times [t^2]' && \text{d.}\\
&= 0 + 5 \times 1 + (-2) \times [t^2]' && \text{e.}\\
&= 0 + 5 \times 1 + (-2) \times 2\,t && \text{f.}\\
&= 5 - 4\,t
\end{aligned}$$

Exercise 3.5.18 Compute the derivatives of the following polynomials as in the previous exercises. Use only one rule for each step written, and write the name of the rule used for each step.

a. $P(t) = 15t^2 - 32t^6$		c. $P(t) = \frac{t^4}{4} + \frac{t^3}{3}$	
b. $P(t) = 1 + t + t^2 + t^3$		d. $P(t) = (1 + t^2)^2$	
e. $P(t) = 31t^{52} - 82t^{241} + \pi t^{314}$		f. $P(t) = 2^5 + 17t^5$	
g. $P(t) = \sqrt{2} - \frac{t^7}{427} + 18t^{35}$		h. $P(t) = 17^3 - \frac{t^{23}}{690} + 5t^{705}$	

Exercise 3.5.19 Find values of t for which $P'(t) = 0$ for:

a. $P(t) = t^2 - 10t + 35$		c. $P(t) = 5t^2 - t + 1$	
b. $P(t) = t^3 - 3t + 8$		d. $P(t) = t^3 - 6t^2 + 9t + 7$	
e. $P(t) = 7t^4 - 56t^2 + 8$		f. $P(t) = t + \frac{1}{t}$	$t > 0$
g. $P(t) = \frac{t}{2} + \frac{2}{t}$		h. $P(t) = \frac{t^3}{3} - t^2 + t$	

Exercise 3.5.20 Suppose that in the problem of Example 3.5.6, the work of building a web is proportional to d^4, the fourth power of the diameter, d. Then the energy available to the spider is

$$E = k_1 d^2 - k_2 d^4$$

Assume that $k_1 = 0.01$ and $k_2 = 0.000001$.

a. Draw a graph of $E = 0.01d^2 - 0.000001d^4$ for $-10 \le d \le 110$.

b. Find $E'(d)$ for $E(d) = 0.01d^2 - 0.000001d^4$.

c. Find two numbers, d, for which $E'(d) = 0$.

d. Find the highest point of the graph between $d = 0$ and $d = 100$.

Exercise 3.5.21 Consider a territorial bird that harvests only in its defended territory (assumed to be circular in shape). The amount of food available can be assumed to be proportional to the area of the territory and therefore proportional to d^2, the square of the diameter of the territory. Assume that the food gathered is proportional to the amount of food available times the time spent gathering food. Let the unit of time be one day, and suppose the amount of time spent defending the territory is proportional to the length of the territory boundary and therefore equal to $k \times d$ for some constant, k. Then $1 - k\,d$ is the amount of time available to gather food, and the amount, F of food gathered will be

$$F = k_2 \times d^2 \times (1 - k\,d)$$

Find the value of d that will maximize the amount of food gathered.

3.6 The second derivative and higher order derivatives.

You may read or hear statements such as "the rate of population growth is decreasing", or "the rate of inflation is increasing", or the velocity of the particle is increasing." In each case the underlying quantity is a rate and *its* rate of change is important.

Definition 3.6.1 The second derivative. *The second derivative of a function, P, is the derivative of the derivative of P, or the derivative of P'.*

The second derivative of P may be denoted by

$$P'', \quad P''(t), \quad \frac{d^2P}{dt^2}, \quad \ddot{P}, \quad \ddot{P}(t), \quad P^{(2)}, \quad P^{(2)}(t), \quad \text{or} \quad D_t^2 P(t)$$

Geometry of the first and second derivatives. That a function, P is *increasing* on an interval $[a, b]$ means that

$$\text{if } s \text{ and } t \text{ are in } [a, b] \text{ and} \qquad s < t \qquad \text{then} \qquad P(s) < P(t)$$

It should be fairly intuitive that if the first derivative of a function, P, is positive throughout $[a, b]$, then P is increasing on $[a, b]$. Both graphs in Figure 3.20 have positive first derivatives and are increasing. The graphs also illustrate the geometry of the second derivative. In Figure 3.20A P' is increasing ($P'(s) < P'(t)$). P has a *positive second derivative* ($P'' > 0$) and the graph of P is *concave upward*. In Figure 3.20B, P' is decreasing ($P'(s) > P'(t)$). P has a *negative second derivative* ($P'' < 0$) and the graph of P is concave downward. We will expand on this interpretation in Chapters 8 and 9.

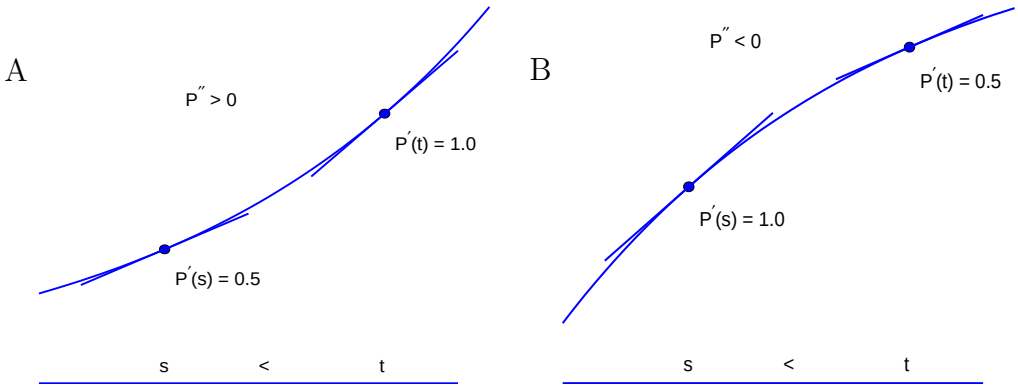


Figure 3.20: A. Graph of a function with a positive second derivative; it is concave up. B. Graph of a function with a negative second derivative; it is concave down.

The higher order derivatives are a natural extension of the transition from the derivative to the second derivative. The third derivative is the derivative of the second derivative; the fourth derivative is the derivative of the third derivative; and the process continues. In this language, the derivative of P is called the first derivative of P (and sometimes P itself is called the zero-order derivative of P).

Definition 3.6.2 Inductive definition of higher order derivatives. *The derivative of a function, P, is the order 1 derivative of P. For n an integer greater than 1, the order n derivative of P is the derivative of the order $n-1$ derivative of P.*

The third order derivative of P may be denoted by

$$P''', \quad P'''(t), \quad \frac{d^3P}{dt^3}, \quad P^{(3)}, \quad P^{(3)}(t), \quad \text{or} \quad D_t^3 P(t)$$

For $n > 3$ the $n\underline{\text{th}}$ order derivative of P may be denoted by

$$P^{(n)}, \quad P^{(n)}(t), \quad \frac{d^nP}{dt^n}, \quad \text{or} \quad D_t^n P(t)$$

If $P(t) = \mu t + \beta$ is a linear function, then $p'(t) = \mu$, is a constant function, and $P''(t) = [\,\mu\,]' = 0$ is the zero function. A similar pattern occurs with quadratic polynomials. Suppose $P(t) = at^2 + bt + c$ is a

quadratic polynomial. Then

$$\begin{aligned} P'(t) &= [\,at^2 + bt + c\,]' \\ &= 2at + b && P'(t) \text{ is a linear function.} \\ P''(t) &= [\,2at + b\,]' \\ &= 2a && P''(t) \text{ is a constant function.} \\ P'''(t) &= [\,2a\,]' \\ &= 0 && P'''(t) \text{ is the zero function.} \end{aligned} \tag{3.33}$$

Explore 3.6.1 Suppose $P(t) = a + bt + ct^2 + dt^3$ is a cubic polynomial. Show that P' is a quadratic polynomial, P'' is a linear function, P''' is a constant function, and $P^{(4)}$ is the zero function.

If $S(t)$ is the position of a particle along an axis at time t, then $s'(t)$ is the velocity of the particle and the rate of change of velocity, $s''(t)$, is called the *acceleration* of the particle. Sometimes when $s''(t)$ is negative the word *deceleration* is used to describe the motion of the particle. The word acceleration is used to describe second derivatives in other contexts. An accelerating economy is one in which the rate of increase of the gross national product is increasing.

Example 3.6.1 Shown in Figure 3.21A is the graph of the logistic function, $L(t)$, first shown in Figure 3.12. Some tangents are drawn on the graph of L. The slope at b is greater than the slope at a; the slope, $L'(t)$, is increasing on the interval to the left of the point marked, I=Inflection Point.

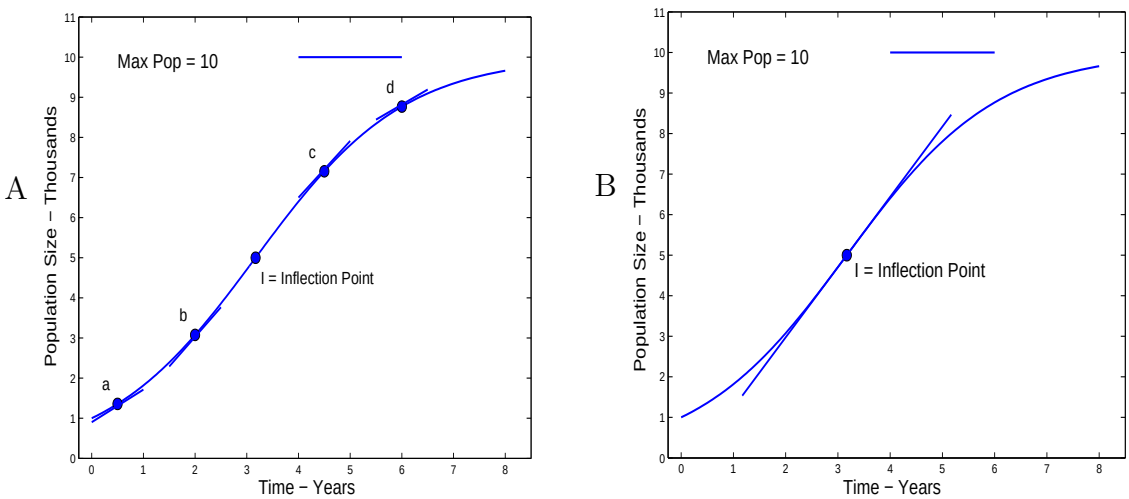


Figure 3.21: A. The graph of a logistic equation and an inflection point I. B. The same graph with tangent segments. The slope of the segment at a is less than the slope at b. The slope at c is greater than the slope at d.

Explore 3.6.2 Do this.

a. The slopes of L at b and d are approximately 1.5 and 0.75. Estimate the slopes at a and c. Confirm that the slope at a is less than the slope at b and that the slope at c is greater than the slope at d.

b. Let t_I be the time of the inflection point I. Argue that $L''(t) > 0$ on $[0, t_I]$.

c. Argue that $L''(t) < 0$ for $t > t_I$.

d. Argue that $L''(t_I) = 0$.

e. On what interval is the graph of L concave downward?■

The tangent at the inflection point is shown in Figure 3.21B, and it is interesting that the tangent crosses the curve at I. The slope of that tangent $\doteq 1.733$, the largest of the slopes of all of the tangents. The maximum population growth rate occurs at t_I and is approximately 1733 individuals per year.

Explore 3.6.3 Danger: Obnubilation Zone. You have to think about this. Suppose the population represented by the logistic curve in Figure 3.21 is a natural population such as deer, fish, geese or shrimp, and suppose you are responsible for setting the size of harvest each year. What is your strategy?

Argue with a friend about this. You should observe that the population size at the inflection point, I, is 5 which is one-half the maximum supportable population of 10. You should discuss the fact that variable weather, disease and other factors may disrupt the ideal of logistic growth. You should discuss how much harvest you could have if you maintained the population at points a, b, c or d in Figure 3.21A. ■

■

3.6.1 Falling objects.

We describe the position, $s(t)$, of an object falling in the earth's gravity field near the Earth's surface t seconds after release. We assume that the velocity of the object at time of release is v_0 and the height of the object above the earth (or some reference point) at time of release is s_0.

Gravity near the Earth's surface is constant, equal to g = -980 cm/sec^2. We write the

Mathematical Model 3.6.1 Free falling object. The acceleration of a free falling object near the Earth's surface is the acceleration of gravity, g.

Because $s''(t)$ is the acceleration of the falling object, we write

$$s''(t) = g.$$

Now, s'' is a constant and is the derivative of s'. The derivative of a linear function, $P(t) = at + b$, is a constant ($P'(t) = a$). We invoke some **advertising logic**[8] and guess that $s'(t)$ is a linear function (proved in Chapter 14).

$$s'(t) = g\,t + b \qquad \textbf{Bolt out of Chapter 14.}$$

[8] A tall, muscular, rugged man drives a Dodge Ram. If you buy a Dodge Ram, you will be a tall muscular, rugged man.

Because $s'(0) = v_0$ is assumed known, and $s'(0) = g\,0 + b = b$, we write

$$s'(t) = g\,t + s_0$$

Using equally compelling logic, because the derivative of a quadratic function, $P(t) = a\,t^2 + b\,t + c$, is a linear function ($P'(t) = 2a\,t + b$), and because $s(0) = s_0$, we write

$$s(t) = \frac{g}{2}t^2 + v_0 t + s_0 \tag{3.34}$$

.

Example 3.6.2 Students measured height *vs* time of a falling bean bag using a Texas Instruments Calculator Based Laboratory Motion Detector, and the data are shown in Figure 3.22A. Average velocities were computed between data points and plotted against the midpoints of the data intervals in Figure 3.22B. Mid-time is (Time$_{i+1}$ + Time$_i$)/2 and Ave. Vel. is (Height$_{i+1}$ - Height$_i$)/(Time$_{i+1}$ - Time$_i$).

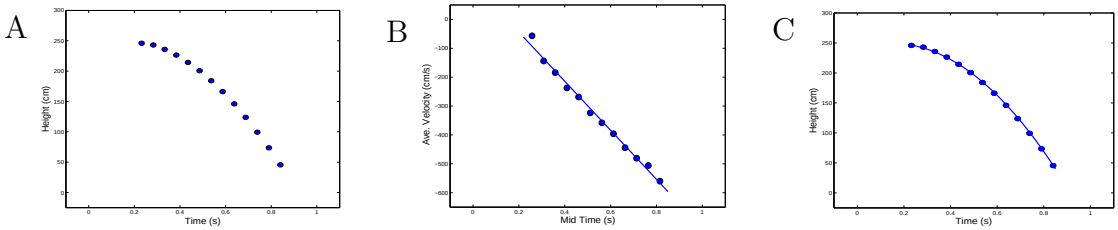


Figure 3.22: Graph of height *vs* time and velocity *vs* time of a falling bean bag.

An equation of the line fit by least squares to the graph of Average Velocity *vs* Midpoint of time interval is

$$v_{\text{ave}} = -849 t_{\text{mid}} + 126 \qquad \text{cm/s.}$$

If we assume a continuous model based on this data, we have

$$s'(t) = -849t + 126, \qquad s(t) = \frac{-849}{2}t^2 + 126t + s_0 \quad \text{cm}$$

From Figure 3.22, the height of the first point is about 240. We write

$$s(t) = \frac{-849}{2}t^2 + 126t + 240 \quad \text{cm}$$

The graph of s along with the original data is shown in Figure 3.22C. The match is good. Instead of g -980 cm/s^2 that applies to objects falling in a vacuum we have acceleration of the bean bag in air to be -849cm/sec^2. ∎

Exercises for Section 3.6, The second derivative and higher order derivatives.

Exercise 3.6.1 Compute P', P'' and P''' for the following functions.

a. $P(t) = 17$	b. $P(t) = t$	c. $P(t) = t^2$
d. $P(t) = t^3$	e. $P(t) = t^{1/2}$	f. $P(t) = t^{-1}$
g. $P(t) = t^8$	h. $P(t) = t^{125}$	i. $P(t) = t^{5/2}$

Exercise 3.6.2 Find the acceleration of a particle at time t whose position, $P(t)$, on an axis is described by

a. $P(t) = 15$	b. $P(t) = 5t + 7$
c. $P(t) = -4.9t^2 + 22t + 5$	d. $P(t) = t - \frac{t^3}{6} + \frac{t^5}{120}$

Exercise 3.6.3 For each figure in Exercise Figure 3.6.3, state whether:

a. P is increasing or decreasing?

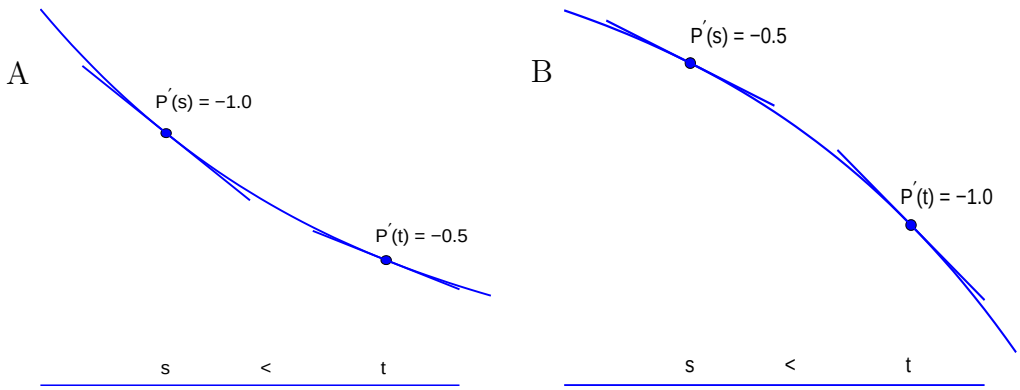
b. P' is positive or negative?

c. P' is increasing or decreasing?

d. $P''(a)$ positive or negative?

e. The graph of P is concave up or concave down

Figure for Exercise 3.6.3 Graphs for exercise 3.6.3



Exercise 3.6.4 The function, P, graphed in Figure Ex. 3.6.4 has a local minimum at the point $(a, P(a))$.

a. What is $P'(a)$?

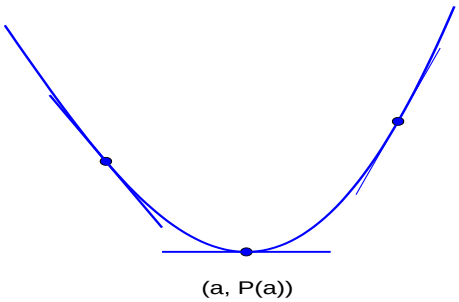
b. For $t < a$, $P'(a)$ is (positive or negative)?

c. For $a < t$, $P'(a)$ is (positive or negative)?

d. $P'(t)$ is (increasing or decreasing)?

e. $P''(a)$ is (positive or negative)?

Figure for Exercise 3.6.4 Graph of a function P with a minimum at $(a, P(a))$.
See Exercise 3.6.4.



Exercise 3.6.5 The function, P, graphed in Figure Ex. 3.6.5 has a local maximum at the point $(a, P(a))$.

a. What is $P'(a)$?

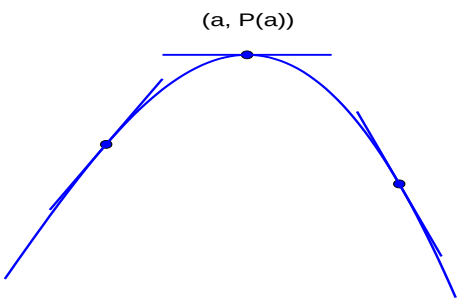
b. For $t < a$, $P'(a)$ is (positive or negative)?

c. For $a < t$, $P'(a)$ is (positive or negative)?

d. $P'(t)$ is (increasing or decreasing)?

e. $P''(a)$ is (positive or negative)?

Figure for Exercise 3.6.5 Graph of a function P with a maximum at $(a, P(a))$.
See Exercise 3.6.5.



Exercise 3.6.6 Show that $A(t)$ of Equation 3.24,

$$A(t) = \left(\frac{K}{2}\,t + 2\right)^2 \qquad t \geq 0,$$

satisfies Equation 3.23,

$$A(0) = 4 \qquad A'(t) = K\sqrt{A(t)} \qquad t \geq 0$$

You will need to compute $A'(t)$ and to do so expand

$$A(t) = \left(\frac{K}{2}\,t + 2\right)^2 \quad \text{to} \quad A(t) = \frac{K^2}{4}\,t^2 + Kt + 4$$

and show that

$$A'(t) = K\left(\frac{K}{2}\,t + 2\right) = K\sqrt{A(t)}.$$

Exercise 3.6.7 Show that for $s(t) = \frac{g}{2}t^2 + v_0 t + \gamma$, $s'(t) = gt + v_0$.

Exercise 3.6.8 Evaluate γ if

a.	$s(t)$	$=$	$5t^2 + \gamma$	and	$s(0)$	$=$	15
b.	$s(t)$	$=$	$-8t^2 + 12t + \gamma$	and	$s(0)$	$=$	11
c.	$s(t)$	$=$	$-8t^2 + 12t + \gamma$	and	$s(1)$	$=$	11
d.	$s(t)$	$=$	$-\frac{849}{2}t^2 + 126t + \gamma$	and	$s(0.232)$	$=$	245.9

Exercise 3.6.9 Show that

a.	$P(t)$	$=$	$5t+3$	satisfies	$P(0)$	$=$	3	and	$P'(t)$	$=$	5
b.	$P(t)$	$=$	$8t+2$	satisfies	$P(0)$	$=$	2	and	$P'(t)$	$=$	8
c.	$P(t)$	$=$	t^2+3t+7	satisfies	$P(0)$	$=$	7	and	$P'(t)$	$=$	$2t+3$
d.	$P(t)$	$=$	$-2t^2+5t+8$	satisfies	$P(0)$	$=$	8	and	$P'(t)$	$=$	$-4t+5$
e.	$P(t)$	$=$	$(3t+4)^2$	satisfies	$P(0)$	$=$	16	and	$P'(t)$	$=$	$6\sqrt{P(t)}$
f.	$P(t)$	$=$	$(5t+1)^2$	satisfies	$P(0)$	$=$	1	and	$P'(t)$	$=$	$10\sqrt{P(t)}$
g.	$P(t)$	$=$	$(1-2t)^{-1}$	satisfies	$P(0)$	$=$	1	and	$P'(t)$	$=$	$2\,(\,P(t)\,)^2$
h.	$P(t)$	$=$	$\dfrac{5}{1-15t}$	satisfies	$P(0)$	$=$	5	and	$P'(t)$	$=$	$3\,(\,P(t)\,)^2$
i.	$P(t)$	$=$	$(6t+9)^{1/2}$	satisfies	$P(0)$	$=$	3	and	$P'(t)$	$=$	$3/P(t)$
j.	$P(t)$	$=$	$(4t+4)^{1/2}$	satisfies	$P(0)$	$=$	2	and	$P'(t)$	$=$	$2/P(t)$
k.	$P(t)$	$=$	$(4t+4)^{3/2}$	satisfies	$P(0)$	$=$	8	and	$P'(t)$	$=$	$6\sqrt[3]{P(t)}$

For parts g - k, use the Definition of Derivative 3.2.2 to compute P'.

Exercise 3.6.10 Add the equations,

$$\begin{aligned} H_2 - H_1 &= \frac{-849}{2}\left(t_2^2 - t_1^2\right) + 126\,(\,t_2 - t_1\,) \\ H_3 - H_2 &= \frac{-849}{2}\left(t_3^2 - t_2^2\right) + 126\,(\,t_3 - t_2\,) \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ H_n - H_{n-1} &= \frac{-849}{2}\left(t_n^2 - t_{n-1}^2\right) + 126\,(\,t_n - t_{n-1}\,)\,. \end{aligned}$$

to obtain

$$H_n - H_1 = \frac{-849}{2}\left(t_n^2 - t_1^2\right) + 126\,(\,t_n - t_1\,)\,.$$

Exercise 3.6.11 In a chemical reaction of the form

$$2\mathrm{A} + \mathrm{B} \longrightarrow \mathrm{A_2B},$$

where the reaction does not involve intermediate compounds, the reaction rate is proportional to $[\mathrm{A}]^2 \times [\mathrm{B}]$ where [A] and [B] denote, respectively, the concentrations of the components A and B. Let a and b denote [A] and [B], respectively, and assume that [B] is much greater than [A] ($b(t) \gg a(t)$. The rate at which a changes may be written

$$a'(t) = -k(\,a(t)\,)^2 b(t) = -K\,(\,a(t)\,)^2$$

We have assumed that $b(t)$ is (almost) constant because [A] is the limiting concentration of the reaction.

Let

$$a(t) = \frac{a_0}{1 - a_0 K t}.$$

Show that $\quad a(0) = a_0$.

Use the Definition of Derivative 3.20,

$$a'(t) = \lim_{b \to t} \frac{a(b) - a(t)}{b - t}$$

to compute $a'(t)$. Then compute $(\,a(t)\,)^2$ and show that

$$a'(t) = -K\,(\,a(t)\,)^2$$

Exercise 3.6.12 Show that for any quadratic function, $Q(t) = a + bt + ct^2$ (a, b and c are constants), and any interval, $[u, v]$, the average rate of change of Q on $[u, v]$ is equal to the rate of change of Q at the midpoint, $(u + v)/2$, of $[u, v]$.

3.7 Left and right limits and derivatives; limits involving infinity.

Suppose F is a function defined for all x. We give meaning to the following symbols.

$$\lim_{x \to a^-} F(x) \qquad \text{and} \qquad F'^{-}(a).$$

Having done so, we ask you to define

$$\lim_{x \to a^+} F(x), \qquad \text{and} \qquad F'^{+}(a).$$

Next we will define limits involving infinity,

$$\lim_{x \to \infty} F(x) \qquad \text{and} \qquad \lim_{x \to a} F(x) = \infty,$$

and ask you to define

$$\lim_{x \to -\infty} F(x) \qquad \text{and} \qquad \lim_{x \to a^+} F(x) = \infty.$$

In reading the following two definitions, it will be helpful to study the graphs in Figure 3.23 and assume $a = 3$.

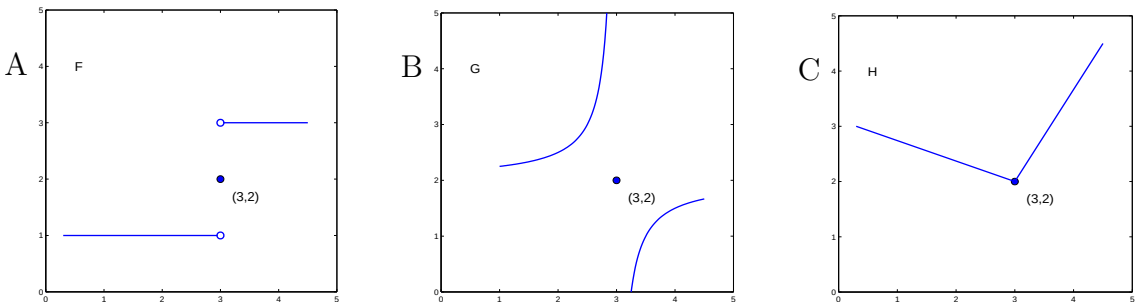


Figure 3.23: Graphs of functions F, G, and H that illustrate Definitions 3.7.1 and 3.7.2.

Definition 3.7.1 *Left hand limit and derivative.* Suppose F is a function defined for all numbers x except perhaps for a number a and L is a number.

The statement that $\lim_{x \to a^-} F(x) = L$ means that

if ϵ is a positive number there is a positive number δ such that if x is in the domain of F and $0 < a - x < \delta$ then $|F(x) - L| < \epsilon$.

If F is defined at a, then $F'^{-}(a)$ is defined by

$$F'^{-}(a) = \lim_{x \to a^-} \frac{F(x) - F(a)}{x - a}. \tag{3.35}$$

The condition $0 < a - x < \delta$ restricts x to being 'to the left' of a ($x < a$) and within δ of a. In Figure 3.23A, $\lim_{x \to 3^-} F(x) = 1$. In Figure 3.23C, $H'^-(3) = -1/3$. This may help resolve some dispute as to whether H has a tangent at (3,2) that was mentioned early in this chapter, Explore 3.1.2.

Definition 3.7.2 *Limits involving infinity.* Suppose F is a function defined for all numbers x except perhaps for a number a and L is a number.

The statement that $\lim_{x \to \infty} F(x) = L$ means that

if ϵ is a positive number there is a number M so that if $M < x$, $|F(x) - L| < \epsilon$.

The half-line $y = L$ for $x > 0$ is said to be a *horizontal asymptote* of F.

The statement that $\lim_{x \to a} F(x) = \infty$ means that

if M is a number there is a positive number δ so that if $0 < |a - x| < \delta$, $F(x) > M$.

In Figure 3.23B, $\lim_{x \to \infty} G(x) = 2$, and $\lim_{x \to 3} G(x)$ does not exist. However, $\lim_{x \to 3^-} G(x) = \infty$

Explore 3.7.1 Refer to Figure 3.23. For each part below, either evaluate the expression or explain why it is not defined.

a. $\lim_{x \to 3+} F(x)$	b. $\lim_{x \to 3} F(x)$	c. $\lim_{x \to 3+} G(x)$
d. $F'^-(3)$	e. $H'^+(3)$	f. $H'(3)$
g. $\lim_{x \to \infty} F(x)$	h. $\lim_{x \to -\infty} F(x)$	i. $\lim_{x \to -\infty} H(x)$

■

Explore 3.7.2 Write definitions for:

a. $\lim_{x \to a+} F(x) = L$ b. $F'^+(a)$ c. $\lim_{x \to -\infty} F(x) = L$
d. $\lim_{x \to a+} F(x) = -\infty$

■

Explore 3.7.3 Attention: Solving this problem may require a significant amount of thought.
We say that $\lim_{x \to a^-} F(x)$ exists if either

$$\lim_{x \to a^-} F(x) = -\infty, \qquad \lim_{x \to a^-} F(x) = \infty, \qquad \text{or for some number } L \qquad \lim_{x \to a^-} F(x) = L.$$

Is there a function, F, defined for all numbers x such that

$$\lim_{x \to 1^-} F(x) \qquad \text{does } \textbf{not} \text{ exist.}$$

■

Proof of the following theorem is only technical and is omitted.

Theorem 3.7.1 Suppose (p, q) an open interval containing a number a and F is a function defined on (p, q) excepts perhaps at a and L is a number.

$$\lim_{x \to a} F(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} F(x) = L \quad \text{and} \quad \lim_{x \to a^+} F(x) = L. \tag{3.36}$$

Furthermore, if F is defined at a

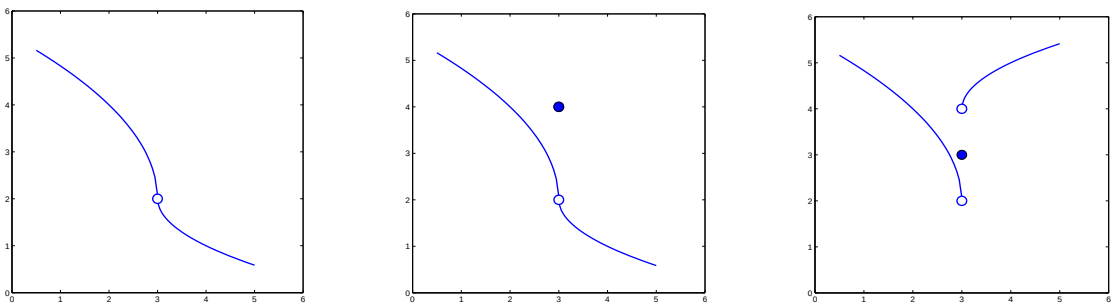
$$F'(a) \quad \text{exists if and only if} \quad F'^-(a) = F'^+(a), \quad \text{in which case} \quad F'(a) = F'^-(a). \tag{3.37}$$

Exercises for Section 3.7 Left and right limits and derivatives; limits involving infinity.

Exercise 3.7.1 For each of the graphs in Figure 3.7.1 of a function, F, answer the questions or assert that there is no answer available.

1. What does $F(t)$ approach as t approaches 3?
2. What does $F(t)$ approach as t approaches 3^-?
3. What does $F(t)$ approach as t approaches 3^+?
4. What is $F(3)$?
5. Use the limit symbol to express answers to a. - c.

Figure for Exercise 3.7.1 Graphs of three functions for Exercise 3.7.1.



Exercise 3.7.2 Let functions D, E, F, G, and H be defined by

$$\begin{aligned} D(x) &= |x| \quad \text{for all } x \\ E(x) &= \begin{cases} -1 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad 0 < x \end{cases} \\ F(x) &= \begin{cases} x^2 & \text{for} \quad x < 0 \\ \sqrt{x} & \text{for} \quad x \geq 0 \end{cases} \\ G(x) &= \begin{cases} 1 & \text{for} \quad x \neq 0 \\ 2 & \text{for} \quad x = 0 \end{cases} \\ H(x) &= \begin{cases} x & \text{for} \quad x < 0 \\ x^2 & \text{for} \quad x \geq 0 \end{cases} \end{aligned}$$

A. Sketch the graphs of D, E, F, G, and H.
B. Let K be either of D, E, F, G, or H. Evaluate the limits or show that they do not exist.

a. $\lim_{x \to 0-} K(x)$ b. $\lim_{x \to 0+} K(x)$ c. $\lim_{x \to 0} K(x)$

d. $K'^{-}(0)$ e. $K'^{+}(0)$ f. $K'(0)$

Exercise 3.7.3 Define the term 'tangent from the left'. (We assume you would have a similar definition for 'tangent from the right'; no need to write it.) What is the relation between tangent from the left, tangent from the right, and tangent?

3.8 Summary of Chapter 3

The Derivative.

You now have an introduction to the concept of rate of change and to the derivative of a function. The derivative and its companion, the integral that is studied in Chapter 13, enabled [9] an explosion in science and mathematics beginning in the late seventeenth century, and remain at the core of science and mathematics today. Briefly, for a suitable function, P, the derivative of P is the function P' defined by

$$P'(t) = \lim_{h \to 0} \frac{P(t+h) - P(t)}{h} \tag{3.38}$$

We also wrote $P'(t)$ as $[\,P(t)\,]'$. A helpful interpretation of $P'(a)$ is that it is the slope of the tangent to the graph of P at the point $(a, P(a))$.

The derivatives of three functions were computed (for C a number and n a positive integer) and we wrote what we call *primary* formulas:

$$\left.\begin{aligned} P(t) &= C \implies P'(t) = 0 \quad \text{or} \quad [\,C\,]' = 0 \\ P(t) &= t \implies P'(t) = 1 \quad \text{or} \quad [\,t\,]' = 1 \\ P(t) &= t^n \implies P'(t) = nt^{n-1} \quad \text{or} \quad [\,t^n\,]' = nt^{n-1} \end{aligned}\right\} \tag{3.39}$$

[9] It might also be argued that the explosion in science enabled or caused the creation of the derivative and integral. The two are inseparable.

Two *combination* formulas were developed:

$$\left.\begin{array}{rclcrcl} P(t) & = & C \times u(t) & \Longrightarrow & P'(t) & = & C \times u'(t) \\ P(t) & = & u(t) + v(t) & \Longrightarrow & P'(t) & = & u'(t) + v'(t) \end{array}\right\} \tag{3.40}$$

They can also be written

$$\begin{aligned} [\,Cu(t)\,]' &= C\,[\,u(t)\,]' \\ [\,u(t) + v(t)\,]' &= [\,u(t)\,]' + [\,v(t)\,]' \end{aligned}$$

We will expand both the list of primary formulas and the list of combination formulas in future chapters and thus expand the array of derivatives that you can compute without explicit reference to the Definition of Derivative Equation 3.38.

We saw that derivatives describe rates of chemical reactions, and used the derivative function to find optimum values of spider web design and the height of a pop fly in baseball. We examined two cases of dynamic systems, mold growth and falling objects, using rates of change rather than the average rates of change used in Chapter 1. A vast array of dynamical systems and optimization problems have been solved since the introduction of calculus. We will see some of them in future chapters.

Finally we defined and computed the second derivative and higher order derivatives and gave some geometric interpretations (concave up and concave down) and some physical interpretation (acceleration).

Exercises for Chapter 3, The Derivative.

Chapter Exercise 3.1 Use Definition of the Derivative 3.20, $\lim_{b \to t} \dfrac{P(b) - P(t)}{b - t}$, to compute the rates of change of the following functions, P.

a.	$P(t) = t^3$		b.	$P(t) = 5t^2$		c.	$P(t) = \dfrac{t^3}{4}$	
d.	$P(t) = t^2 + t^3$		e.	$P(t) = 2\sqrt{t}$		f.	$P(t) = \sqrt{2t}$	
g.	$P(t) = 7$		h.	$P(t) = 5 - 2t$		i.	$P(t) = \dfrac{1}{1+t}$	
j.	$P(t) = \dfrac{1}{3t}$		k.	$P(t) = 5t^7$		l.	$P(t) = \dfrac{1}{t^2}$	
m.	$P(t) = \dfrac{1}{(3t+1)^2}$		n.	$P(t) = \sqrt{t+3}$		o.	$P(t) = \dfrac{1}{\sqrt{t+1}}$	

Chapter Exercise 3.2 Data from David Dice of Carlton Comprehensive High School in Canada[10] for the decrease in mass of a solution of 1 M HCl containing chips of $CaCO_3$ is shown in Table 3.8.0. The reaction is

[10] http://www.carlton.paschools.ps.sk.ca/chemical/chem

$$CaCO_3(s) + 2HCl(aq) \rightarrow CO_2(g) + H_2O(l) + CaCl_2(aq).$$

The reduction in mass reflects the release of CO_2.

a. Graph the data.

b. Estimate the rate of change of the mass at each of the times shown.

c. Draw a graph of the rate of change of mass versus the mass.

Table for Exercise 3.8.0 Data for Ex. 3.2.

Time (min)	Mass (g)	
0.25	248.46	
0.50	247.95	
1.00	246.83	
1.50	245.95	
2.00	245.22	
2.50	244.67	
3.00	244.27	
3.50	243.95	
4.00	243.72	
4.50	243.52	
5.00	243.37	

Chapter Exercise 3.3 Use derivative formulas 3.39 and 3.40 to compute the derivative of P. Use Primary formulas only in the last step. Assume the t^n rule $[\,t^n\,]' = nt^{n-1}$. to be valid for all numbers n, integer, rational, irrational, positive, and negative. In some cases, algebraic simplification will be required before using a derivative formula.

a. $P(t) = 3t^2 - 2t + 7$ b. $P(t) = t + \frac{2}{t}$ c. $P(t) = \sqrt{2t}$

d. $P(t) = (\,t^2 + 1\,)^3$ e. $P(t) = 1/\sqrt{2t}$ f. $P(t) = 2t^{-3} - 3t^{-2}$

g. $P(t) = \frac{5+t}{t}$ h. $P(t) = 1 + \sqrt{t}$ i. $P(t) = (\,1 + 2t\,)^2$

j. $P(t) = (\,1 + 3t\,)^3$ k. $P(t) = \frac{1+\sqrt[3]{t}}{\sqrt{t}}$ l. $P(t) = \sqrt[3]{2t}$

Chapter Exercise 3.4 Find an equation of the tangent to the graph of P at the indicated points. Draw the graph P and the tangent.

a. $P(t) = t^4$ at $(1,1)$ b. $P(t) = t^{12}$ at $(1,1)$

c. $P(t) = t^{1/2}$ at $(4,2)$ d. $P(t) = \frac{5}{2}$ at $(5,1)$

e. $P(t) = \sqrt{1+t}$ at $(8,3)$ f. $P(t) = \frac{1}{2t}$ at $(\frac{1}{2},1)$

Chapter 4

Continuity and the Power Chain Rule

4.1 Continuity.

We require the concept of *continuity* of a function. Most, but not all, of the functions in previous sections and that you will encounter in biology are continuous. Four equivalent definitions of continuity follow, with differing levels of intuition and formality.

Definition 4.1.1 Continuity of a function at a number in its domain.

Intuitive: *A function, f, is continuous at a number a in its domain means that if x is in the domain of f and x is close to a, then $f(x)$ is close to $f(a)$.*

Symbolic: *A function, f, is continuous at a number a in its domain means that either*

$$\lim_{x \to a} f(x) = f(a)$$

or there is an open interval containing a and no other point of the domain of f.

Formal: *A function, f, is continuous at a number a in its domain means that for every positive number, ϵ, there is a positive number δ such that if x is in the domain of f and $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.*

Geometrical *A simple graph G is continuous at a point P of G means that if α and β are horizontal lines with P between them there are vertical lines h and k with P between them such that every point of of G between h and k is between α and β.*

Definition 4.1.2 Discontinuous. *If a is a number in the domain of a function f at which f is not continuous then f is said to be discontinuous at a.*

Definition 4.1.3 Continuous function. *A function, f, is continuous means that f is continuous at every number, a, in its domain.*

Almost all of the functions that you have experienced are continuous, and to many students it is intuitive obvious from the notation that $u(b)$ approaches $u(a)$ as b approaches a; but in some cases $u(b)$ does not approach $u(a)$ as b approaches a. Some examples demonstrating the exception follow.

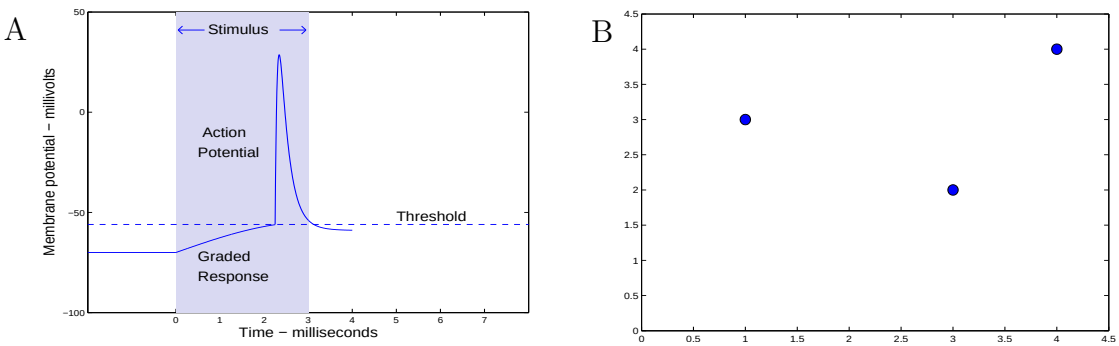
Example 4.1.1 1. You showed in Exercise 3.2.7 that every polynomial is continuous.

2. The function describing serum insulin concentration as a function of time is a continuous function even though it may change quickly with serious consequences.

3. The word 'threshold' suggests a discontinuous change in one parameter as a related parameter crosses the threshold. Stimulus to neurons causes ion gates to open. When an Na^+ ion gate is opened, Na^+ flows into the cell gradually increasing the membrane potential, called a 'graded response' (continuous response), up to a certain threshold at which an action potential is triggered (a rapid increase in membrane potential) that appears to be a discontinuous response. As in almost all biological examples, however, the function is actually continuous. See Example Figure 4.1.1.1A.

4. **Surprise.** The graph in Figure 4.1.1.1B is continuous. There are only three points of the graph. The graph is continuous at, for example, the point (3,2). There is an open interval containing 3, $(2.5, 3.5)$, for example, that contains 3 and no other point of the domain.

Figure for Example 4.1.1.1 A. Events leading to a nerve action potential. B. A discrete graph is continuous.



5. The function *approximating* % Female hatched from a clutch of turtle eggs

$$\text{Percent female} = \begin{cases} 0 & \text{if} \quad Temp < 28 \\ 50 & \text{if} \quad Temp = 28 \\ 100 & \text{if} \quad 28 < Temp \end{cases} \tag{4.1}$$

is not continuous. The function is discontinuous at $t = 28$. If the temperature, T, is close to 28 and less than 28, then % Female(T) is 0, which in usual measures is not 'close to' 50 = % Female(28).

6. As you move up a mountain side, the flora is usually described as being a discontinuous function of altitude. There is a 'tree line', below which the dominant plant species are pine and spruce and above which the dominant plant species are low growing brushes and grasses, as illustrated in Figure 4.1.1.1C[1]Such a region of apparent discontinuity is termed an 'ecotone' by ecologists.

Figure for Example 4.1.1.1 (Continued.) C. A tree line.

[1]Ke4roh picture of Berthoud pass, Colorado http://en.wikipedia.org/wiki/Tree_line

C 

7. The ecotone illustrated by the tree line in Figure 4.1.1.1C appears sharp. Sharpness, however, may just be a matter of scale: if you walk through the tree line, you will a gradual decrease in tree density and pockets of trees above the tree line.

   It is important to the concept of discontinuity that there be an abrupt change in the dependent variable with only a gradual change in the independent variable. Charles Darwin expressed it:

   > Charles Darwin, *Origin of Species.* Chap. VI, Difficulties of the Theory. "We see the same fact in ascending mountains, and sometimes it is quite remarkable how abruptly, as Alph. de Candolle has observed, a common alpine species disappears. The same fact has been noticed by E. Forbes in sounding the depths of the sea with the dredge. To those who look at climate and the physical conditions of life as the all-important elements of distribution, these facts ought *to cause surprise, as climate and height or depth graduate away insensibly* [our emphasis]."

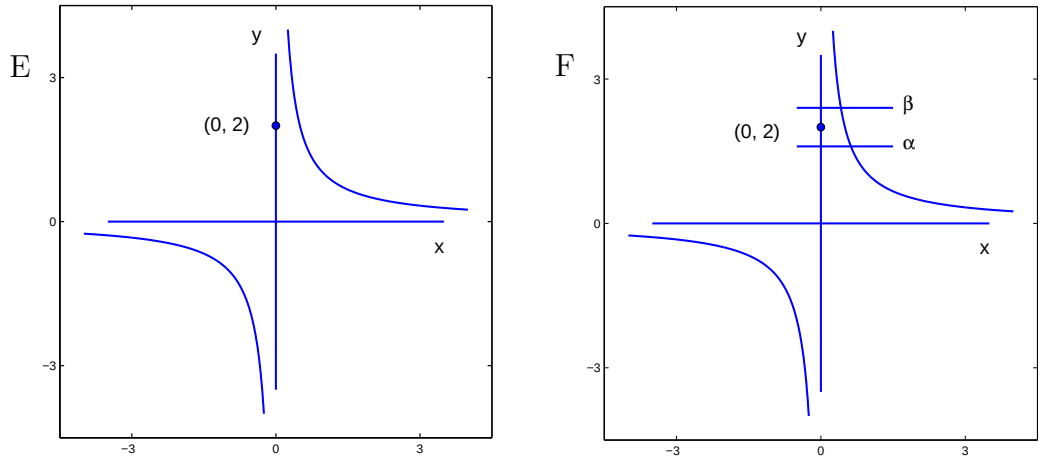
8. From Equation 3.10, $\lim_{x \to a} \dfrac{1}{x} = \dfrac{1}{a}$, the function, $f(x) = \frac{1}{x}$ is continuous. The graph of $f(x)$ certainly changes rapidly near $x = 0$, and one may think that f is not continuous at $x = 0$. However, *0 is not in the domain of* f, so that the function is neither continuous nor discontinuous at $x = 0$.

9. Let the function g be defined by

$$g(x) = \frac{1}{x} \qquad \text{for} \quad x \neq 0, \qquad g(0) = 2$$

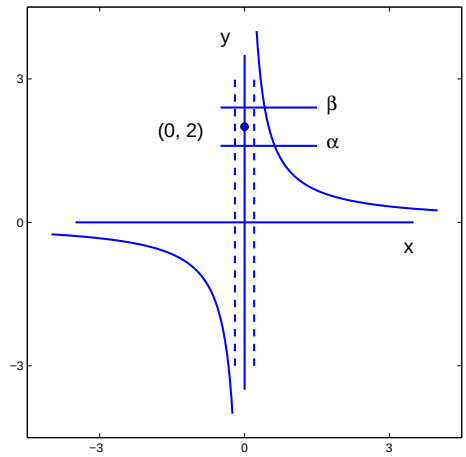
   A graph of g is shown in Example Figure 4.1.1.1E. The function g is not continuous at 0 and the graph of g is not continuous at $(0, 2)$. Two horizontal lines above and below $(0, 2)$ are drawn in Figure 4.1.1.1F. For every pair of vertical lines h and k with $(0, 2)$ between them there are points of the graph of g between h and k that are not between α and β.

   Figure for Example 4.1.1.1 (Continued) E. The graph of $g(x) = 1/x$ for $x \neq 0$, $g(0) = 2$. F. The point $(0, 2)$ and horizontal lines above and below $(0, 2)$.



Explore 4.1.1 In Figure 4.1.1 there are two vertical dashed lines with (0,2) between them. It appears that every point of the graph of g between the vertical dashed lines is between the two horizontal lines. Does this contradict the claim made in Item 9? ■

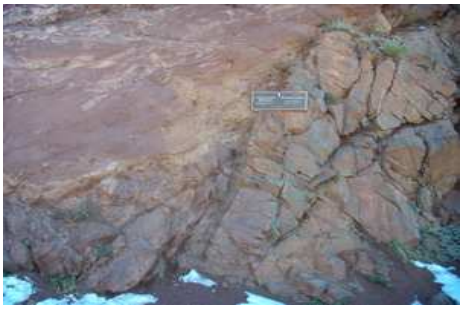
Explore Figure 4.1.1 The graph of $g(x) = 1/x$ for $x \neq 0$, $g(0) = 2$, horizontal lines above and below $(0, 2)$, and vertical lines (dashed) with (0,2) between them.



10. The geological age of soil is not a continuous function of depth below the surface. Older soils are at a greater depth, so that the age of soils is (almost always) an increasing function of depth. However, in many locations, soils of some ages are missing: soils of age 400 million years may rest directly on top of soils of age 1.7 billion years as shown in Figure 4.1.1.1. Either soils of the intervening ages were not deposited in that location or they were deposited and subsequently eroded. Geologist speak of an "unconformity" occurring at that location and depth.

Figure for Example 4.1.1.1 (Continued) G. Picture of an unconformity at Red Rocks Park and Amphitheatre near Denver, Colorado. Red 300 million year-old sedimentary rocks rest on gray 1.7 billion year-old metamorphic rocks. (Better picture in "Messages in Stone: Colorado's Colorful Geology" Vincent Matthews, Katie KellerLynn, and Betty Fox, Colorado Geological Survey, Denver, Colorado, 2003) H. Snow line figure for Exercise 4.1.4.

G

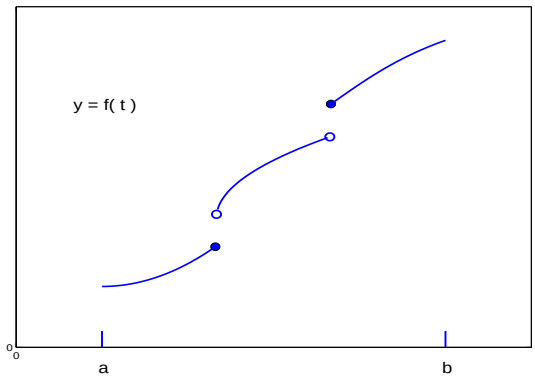


H

11. Every increasing function defined on an interval that is discontinuous at some point has a vertical gap in the graph at that point. Every increasing function with no gap is continuous.

Figure for Example 4.1.1.1 (Continued) H. An increasing function. There are two gaps and two points of discontinuity.

H



Combinations of continuous functions. We showed in Chapter 3 that

$$\lim_{x\to a} \left(F_1(x) + F_2(x)\right) = \lim_{x\to a} F_1(x) + \lim_{x\to a} F_2(x) \qquad \text{Equation 3.12}$$

$$\lim_{x\to a} \left(F_1(x) \times F_2(x)\right) = \left(\lim_{x\to a} F_1(x) \right) \times \left(\lim_{x\to a} F_2(x)\right) \qquad \text{Equation 3.13}$$

$$\text{If} \quad \lim_{x\to a} F_2(x) \neq 0, \qquad \text{then} \qquad \lim_{x\to a} \frac{F_1(x)}{F_2(x)} = \frac{\lim\limits_{x\to a} F_1(x)}{\lim\limits_{x\to a} F_2(x)}. \qquad \text{Equation 3.17}$$

From these results it follows that if u and v are continuous functions with common domain, D, then

$$u + v \qquad \text{and} \qquad u \times v \qquad \text{are continuous,} \tag{4.2}$$

and if $v(t)$ is not zero for any t in D, then

$$\frac{u}{v} \qquad \text{is continuous.} \tag{4.3}$$

Of particular interest is the equation on the limit of composition of two functions,

$$\text{If} \quad \lim_{x\to a} u(x) = L \quad \text{and} \quad \lim_{s\to L} F(s) = \lambda, \quad \text{then} \quad \lim_{x\to a} F(u(x)) = \lambda \qquad \text{Equation 3.16}$$

From this it follows that if F and u are continuous and the domain of F contains the range of u then

$$F \circ u, \qquad \text{the composition of } F \text{ with } u\text{, is continuous.} \tag{4.4}$$

Exercises for Section 4.1, Continuity.

Exercise 4.1.1 1. Find an example of a plant ecotone distinct from the treeline and moss on pine tree examples.

2. Find an example of a discontinuity of animal type.

Exercise 4.1.2 For $f(x) = 1/x$,

a. How close must x be to 0.5 in order that $f(x)$ is within 0.01 of 2?

b. How close must x be to 3 in order to insure that $\frac{1}{x}$ be within 0.01 of $\frac{1}{3}$

c. How close must x be to 0.01 in order to insure that $\frac{1}{x}$ be within 0.1 of 10

Exercise 4.1.3 Find the value for $u(2)$ that will make u continuous if

a. $u(t) = 2t + 5$ for $t \neq 2$ b. $u(t) = \dfrac{t^2 - 4}{t - 2}$ for $t \neq 2$

c. $u(t) = \dfrac{(t-2)^2}{|t-2|}$ for $t \neq 2$ d. $u(t) = \dfrac{t+1}{t-3}$ for $t \neq 2$

e. $u(t) = \dfrac{t-2}{t^3 - 8}$ for $t \neq 2$ f. $u(t) = \dfrac{\frac{1}{t} - \frac{1}{2}}{t - 2}$ for $t \neq 2$

Exercise 4.1.4 In Example Figure 4.1.1.1 H is a picture of snow that fell on the side of a mountain the night before the picture was taken. There is a 'snow line', a horizontal separation of the snow from terrain free of snow below the line. 'Snow' is a discontinuous function of altitude. Explain the source of the discontinuity.

Exercise 4.1.5 The TI-86 calculator has a 'TEST' (2nd 2) key that causes a menu to appear that can be used to define some interesting functions.

a. In GRAPH, first select the Dot format for graphs (press MORE FORMT DrawDot). Now define a function (press MORE MORE y(x)=) and

y1= $(x < 2) * x^2 + (2 \leq x) * (3 - x)$

Set the range $0 \leq x \leq 4$, $-1 \leq y \leq 5$, and GRAPH.

b. The function so defined is not continuous. Find a number, A, so that

y1= $(x < 2) * x^2 + (2 \leq x) * (A - x)$

is continuous.

Exercise 4.1.6 Is the temperature of the water in a lake a continuous function of depth? Write a paragraph discussing water temperature as a function of depth in a lake and how water temperature affects the location of fish.

Exercise 4.1.7 To reduce inflammation in a shoulder, a doctor prescribes that twice daily one Voltaren tablet (25 mg) to be taken with food. Draw a graph representative of the amount of Voltaren in the body as a function of time for a one week period. Is your graph continuous?

Exercise 4.1.8 The function, $f(x) = \sqrt[3]{x}$ is continuous.

1. How close must x be to 1 in order to insure that $f(x)$ is within 0.1 of $f(1) = 1$ (that is, to insure that $0.9 < f(x) < 1.1$)?
2. How close must x be to 1/8 in order to insure that $f(x)$ is within 0.0001 of $f(1/8) = 1/2$ (that is, to insure that $0.4999 < f(x) < 0.5001$)?
3. How close must x be to 0 in order to insure that $f(x)$ is within 0.1 of $f(0) = 0$?

Exercise 4.1.9 a. Draw the graph of a function, f, defined on the interval $[1, 3]$ such that $f(1) = -2$ and $f(3) = 4$.

b. Does your graph intersect the X-axis?

c. Draw a graph of of a function, f, defined on the interval $[1, 3]$ such that $f(1) = -2$ and $f(3) = 4$ that does not intersect the X-axis. Be sure that its X-projection is all of $[1, 3]$.

d. Write equations to define a function, f, on the interval $[1, 3]$ such that $f(1) = -2$ and $f(3) = 4$ and the graph of f does not intersect the X-axis.

e. There is a theorem that asserts that the function you just defined must be discontinuous at some number in [1,3]. Identify such a number for your example.

The preceding problem illustrates a general property of continuous functions called the *intermediate value property.* Briefly it says that a continuous function defined on an interval that has both positive and negative values on the interval, must also be zero somewhere on the interval. In language of graphs, the graph of a continuous function defined on an interval that has a point above the X-axis and a point below the X-axis must intersect the X-axis. The proof of this property requires more than the familiar properties of addition, multiplication, and order of the real numbers. It requires the *completion* property of the real numbers.

Exercise 4.1.10 A nutritionist studying plasma epinephrine (EPI) kinetics with tritium labeled epinephrine, [^{3}H]EPI, observes that after a bolus injection of [^{3}H]EPI into plasma, the time-dependence of [^{3}H]EPI level is well approximated by $L(t) = 4e^{-2t} + 3e^{-t}$ where $L(t)$ is the level of [^{3}H]EPI t hours after infusion. Sketch the graph of L. Observe that $L(0) = 7$ and $L(2) = 0.479268$. The intermediate value property asserts that at some time between 0 and 2 hours the level of [^{3}H]EPI will be 1.0. At what time will $L(t) = 1.0$? (Let $A = e^{-t}$ and observe that $A^2 = e^{-2t}$.)

Exercise 4.1.11 For the function, $f(x) = 10 - x^2$, find an an open interval, $(3 - \delta, 3 + \delta)$ so that $f(x) > 0$ for x in $(3 - \delta, 3 + \delta)$.

Exercise 4.1.12 For the function, $f(x) = \sin(x)$, find an an open interval, $(3 - \delta, 3 + \delta)$ so that $f(x) > 0$ for x in $(3 - \delta, 3 + \delta)$.

Exercise 4.1.13 The previous two problems illustrate a property of continuous functions formulated in the Locally Positive Theorem:

Theorem 4.1.1 Locally Positive Theorem. *If a function, f, is continuous at a number a in its domain and $f(a)$ is positive, then there is a positive number, δ, such that $f(x)$ is positive for every number x in $(a - \delta, a + \delta)$ and in the domain of f.*

Prove the Locally Positive Theorem. Your proof may begin:

1. Suppose the hypothesis of the Locally Positive Theorem.
2. Let $\epsilon = f(a)$.
3. Use the hypothesis that $\lim_{x \to a} f(x) = f(a)$.

Exercise 4.1.14 Is it true that if a function, f, is positive at a number a in its domain, then there is a positive number, δ, such that if x is in $(a - \delta, a + \delta)$ and in the domain of f then $f(x) > 0$?

4.2 The Derivative Requires Continuity.

Suppose u is a function.

$$\text{If} \qquad \lim_{b \to 3} \frac{u(b) - 5}{b - 3} = 4, \qquad \text{what is} \qquad \lim_{b \to 3} u(b)\,?$$

The answer is that

$$\lim_{b \to 3} u(b) = 5$$

We reason that

$$\text{for } b \text{ close to 3 the numerator of} \qquad \frac{u(b) - 5}{b - 3} \qquad \text{is close to 4 times the denominator.}$$

$$\text{That is,} \qquad u(b) - 5 \qquad \text{is close to} \qquad 4 \times (b - 3).$$

But $4 \times (b - 3)$ is also close to zero. Therefore

If b is close to 3, $u(b) - 5$ is close to zero and $u(b)$ is close to 5.

The general question we address is:

Theorem 4.2.1 The Derivative Requires Continuity. If u is a function and $u'(t)$ exists at $t = a$ then u is continuous at $t = a$.

Proof. In Exercise 4.2.2 you are asked to give reasons for the following steps, a - e.
Suppose the hypothesis of Theorem 4.2.1.

$$\begin{aligned}
\left(\lim_{b\to a} u(b)\right) - u(a) &= \lim_{b\to a} (u(b) - u(a)) && \text{a.}\\
&= \lim_{b\to a} \frac{u(b) - u(a)}{b - a} \times (b - a) \\
&= \lim_{b\to a} \frac{u(b) - u(a)}{b - a} \times \lim_{b\to a} (b - a) && \text{b.}\\
&= u'(a) \lim_{b\to a} (b - a) && \text{c.}\\
&= 0 && \text{d.}\\
\left(\lim_{b\to a} u(b)\right) &= u(a) && \text{e.}
\end{aligned} \tag{4.5}$$

End of proof.

A graph of a function u defined by

$$u(t) = \begin{cases} 0 & \text{for} \quad 20 \le t < 28 \\ 50 & \text{for} \quad t = 28 \\ 100 & \text{for} \quad 28 < t \le 30 \end{cases} \tag{4.6}$$

is shown in Figure 4.1A. We observed in Section 4.1 that u is not continuous at $t = 28$. ($\lim_{t\to 28^-} u(t) = 0 \neq 50 = u(28)$.) Furthermore, $u'(28)$ does not exist. A secant to the graph through $(b, 0)$ and $(28, 50)$ with $b < 28$ is drawn in Figure 4.1B, and

$$\text{for } b < 28, \qquad \frac{u(b) - u(28)}{b - 28} = \frac{0 - 50}{b - 28} = \frac{50}{28 - b}$$

The slope of the secant gets greater and greater as b gets close to 28.

Explore 4.2.1 Is there a line tangent to the graph of u shown in Figure 4.1 at the point (28,50) of the graph? ■

Explore 4.2.2 In Explore Figure 4.2.2 is the graph of $y = \sqrt{|x|}$. Does the graph have a tangent at (0,0)? **Your vote counts.** ■

A B

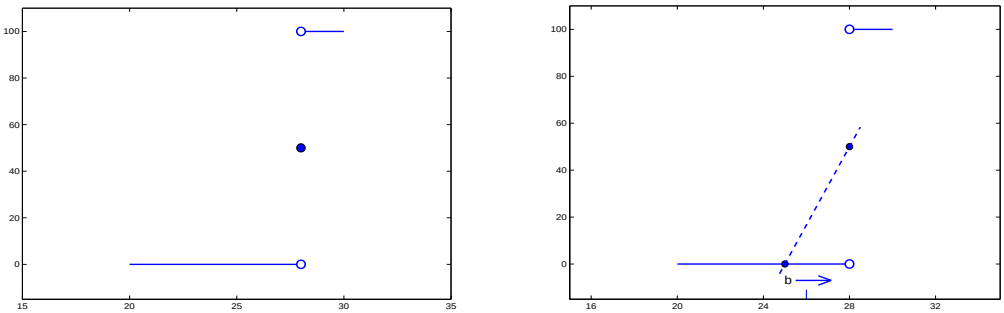
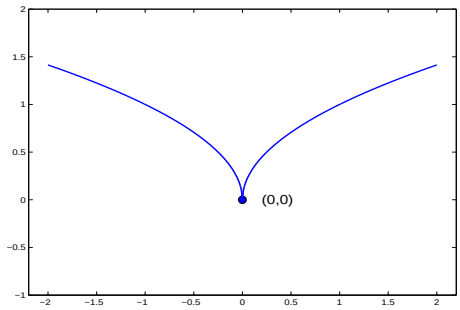


Figure 4.1: A. Graph of The function u defined in Equation 4.6. B. Graph of u and a secant to the graph through $(b, 0)$ and $(28, 50)$.

Explore Figure 4.2.2 Graph of $y = \sqrt{|x|}$.



The graph of $P(t) = |t|$ for all t is shown in Figure 4.2A. P is continuous, but $P'(0)$ does not exist.

$$\frac{P(b) - P(0)}{b - 0} = \frac{|b| - 0}{b - 0} = \frac{|b|}{b} = \begin{cases} -1 & \text{for} \quad b < 0 \\ 1 & \text{for} \quad b > 0 \end{cases}$$

A graph of $\dfrac{P(b) - P(0)}{b - 0}$ is shown in Figure 4.2B. It should be clear that

$$\lim_{b \to 0} \frac{P(b) - P(0)}{b - 0} \qquad \text{does not exist, so that} \qquad P'(0) \qquad \text{does not exist.}$$

Therefore, the converse of Theorem 4.2.1 is not true. Continuity does not imply that the derivative exists.

Exercises for Section 4.2, The Derivative Requires Continuity.

Exercise 4.2.1 Shown in Figure 4.2.1 is the graph of $C(t) = \sqrt[3]{t}$.

a. Use Definition of Derivative Equation 3.20 to show that $C'(0)$ does not exist.

b. Is $C(t)$ continuous?

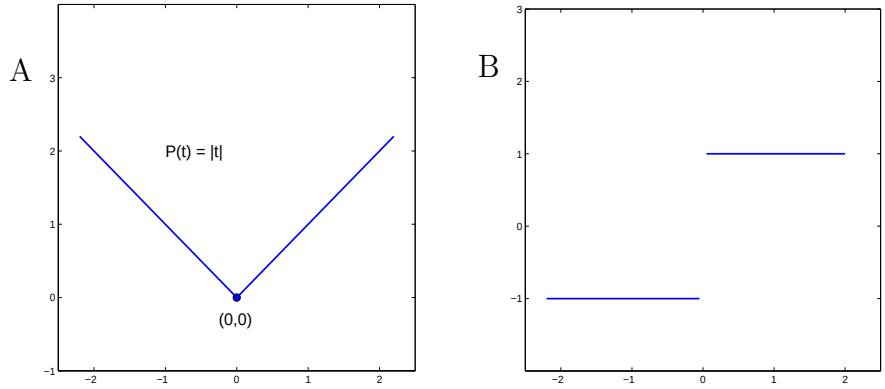
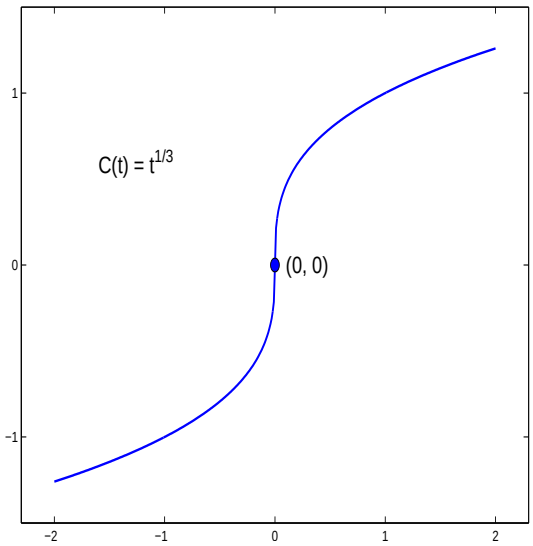


Figure 4.2: a. Graph of $P(t) = |t|$ for all t. B. Graph of $(P(t) - P(0))/(t - 0)$.

c. Is there a line tangent to the graph of C at (0,0)?

Figure for Exercise 4.2.1 Graph of $C(t) = \sqrt[3]{t}$ for Exercise 4.2.1.



Exercise 4.2.2 Justify the steps a - e Equations 4.5.

Exercise 4.2.3 For for the function $P(t) = |t|$ for all t compute $P'^{-}(0)$ and $P'^{+}(0)$.

Explore 4.2.3 This problem may require extensive thought. Is there a function defined for all numbers t and continuous at every number t and for which $f'^{-}(1)$ does not exist? ■

4.3 The generalized power rule.

In Section 3.5 we proved the Power Rule: for all positive integers, n,

$$[\,t^n\,]'\,(t) = nt^{n-1}$$

We show here the **generalized power rule**.

Suppose n is a positive integer and $u(t)$ is a function that has a derivative for all t. We use the notation

$$(\,u(t))^n \;=\; u^n(t). \qquad \text{Then} \qquad [\,u^n(t)\,]' = n\,u^{n-1}(t) \times u'(t). \tag{4.7}$$

The generalized power rule is used in the following setting. Suppose

$$P(t) = \left(1 + t^2\right)^3$$

There are two options for computing $P'(t)$.

Option A. Expand the binomial:

$$\begin{aligned} P(t) &= \left(1 + t^2\right)^3 \\ &= 1 + 3t^2 + 3t^4 + t^6 \end{aligned}$$

Then use the Sum, Constant, Constant Factor, and Power Rules to show that

$$P'(t) \;=\; 0 + 6t + 12t^3 + 6t^5$$

Option B. Use the Generalized Power Rule with $u(t) = 1 + t^2$. Then

$$\begin{aligned} P(t) &= (\,1 + t^2\,)^3 & &= u^3(t) \\ P'(t) &= 3\,(\,1 + t^2\,)^2 \times [1 + t^2]' & &= 3u^2(t) \times u'(t) \\ &= 3\,(\,1 + t^2\,)^2 \times 2t \end{aligned}$$

The answers are the same, for

$$3\left(1 + t^2\right)^2 \times 2t = 3\left(1 + 2t^2 + t^4\right) \times 2t = 6t + 12t^3 + 6t^5$$

Option A (expand the binomial) may appear easier than Option B (use the generalized power rule), but the generalized power rule is clearly easier for a problem like

$$\text{Compute} \qquad P'(t) \qquad \text{for} \qquad P(t) = \left(1 + t^2\right)^{10}$$

Expanding $(\,1 + t^2\,)^{10}$ into polynomial form is tedious (if you try it you may conclude that it is a more than tedious). On the other hand, using the generalize power rule

$$P'(t) = 10\left(1 + t^2\right)^9 \times \left[1 + t^2\right]' = 10\left(1 + t^2\right)^9 \times 2t$$

A special strength of the generalized power rule is that when u is positive, Equation 4.7 is valid for all numbers n (integer, rational, irrational, positive, negative). Thus for $P(t) = \sqrt{1+t^2}$,

$$\begin{aligned}
P'(t) &= \left[\sqrt{1+t^2}\,\right]' \\
&= \left[\,(1+t^2)^{\frac{1}{2}}\,\right]' && \text{Change to fractional exponent.} \\
&= \tfrac{1}{2}\,(1+t^2)^{\frac{1}{2}-1} \times [\,1+t^2\,]' && \text{Generalized Power Rule.} \\
&= \tfrac{1}{2}\,(1+t^2)^{-\frac{1}{2}} \times 2t && \text{Sum, Constant, Power Rules}
\end{aligned}$$

You will prove that when u is positive Equation 4.7, $[\,u^n(t)\,]' = n\,u^{n-1} \times u'(t)$, is valid for n a negative integer (Exercise 4.3.3) and for n a rational number (Exercise 4.3.4).

Proof of the Generalized Power Rule. Assume that n is a positive integer, $u(t)$ is a function and $u'(t)$ exists. Then

$$\text{Equations to prove the GPR. See Exercise 4.3.2.} \tag{4.8}$$

$$
\begin{aligned}
[\,u^n(t)\,]' &\overset{a}{=} \lim_{b\to t} \frac{u^n(b)-u^n(t)}{b-t} \\
&\overset{b}{=} \lim_{b\to t} \frac{\Big(u^{n-1}(b)+u^{n-2}(b)\,u(t)+\cdots+u(b)\,u^{n-2}(t)+u^{n-1}(t)\Big)\times(u(b)-u(t))}{b-t} \\
&\overset{c}{=} \lim_{b\to t} \left(\Big(u^{n-1}(b)+u^{n-2}(b)\,u(t)+\cdots+u(b)\,u^{n-2}(t)+u^{n-1}(t)\Big)\times\frac{(u(b)-u(t)}{b-t}\right) \\
&\overset{d}{=} \lim_{b\to t} \Big(u^{n-1}(b)+u^{n-2}(b)\,u(t)+\cdots+u(b)\,u^{n-2}(t)+u^{n-1}(t)\Big)\times\lim_{b\to t}\frac{(u(b)-u(t))}{b-t} \\
&\overset{e}{=} \lim_{b\to t} \Big(u^{n-1}(b)+u^{n-2}(b)\,u(t)+\cdots+u(b)\,u^{n-2}(t)+u^{n-1}(t)\Big)\times u'(t) \\
&\overset{f}{=} \left(\lim_{b\to t} u^{n-1}(b)+\lim_{b\to t} u^{n-2}(b)\,u(t)+\cdots+\lim_{b\to t} u(b)\,u^{n-2}(t)+\lim_{b\to t} u^{n-1}(t)\right)\times u'(t) \\
&\overset{g}{=} \left(\lim_{b\to t} u^{n-1}(b)+u(t)\lim_{b\to t} u^{n-2}(b)\;+\cdots+u^{n-2}(t)\lim_{b\to t} u(b)+\lim_{b\to t} u^{n-1}(t)\right)\times u'(t) \\
&\overset{h}{=} \left(\lim_{b\to t} u^{n-1}(b)+u(t)\lim_{b\to t} u^{n-2}(b)\;+\cdots+u^{n-2}(t)\lim_{b\to t} u(b)+\;u^{n-1}(t)\right)\times u'(t) \\
&\overset{i}{=} \left(\lim_{b\to t} u^{n-1}(b)+u(t)\lim_{b\to t} u^{n-2}(b)\;+\cdots+u^{n-2}(t)\lim_{b\to t}\times u(t)+\;u^{n-1}(t)\right)\times u'(t) \\
&\overset{j}{=} \left(\underbrace{u^{n-1}(t)+u^{n-1}(t)+\cdots+u^{n-1}(t)+u^{n-1}(t)}_{n\text{ terms}}\right)\times u'(t) \\
&= nu^{n-1}(t)\times u'(t)
\end{aligned}
$$

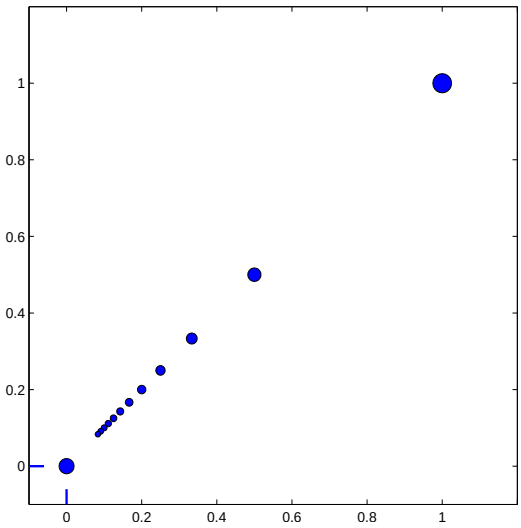
End of Proof.

Explore 4.3.1 In Explore Figure 4.3.1 is a graph of F defined by

$$F(x)=x \qquad \text{for } x=0 \text{ or } x \text{ is the reciprocal of a positive integer.}$$

Only 13 of the infinitely many points of the graph of F are plotted. What is the graph of F'? **Your vote counts.** ■

Explore Figure 4.3.1 Thirteen of the infinitely many points of the graph of $y = x$ for $x = 0$ or x is the reciprocal of a positive integer.



To demonstrate use of the generalized power rule, we announce a *Primary Formula* that is proved in Chapter 7.

$$[\sin t]' = \cos t, \tag{4.9}$$

The derivative of the sine function is the cosine function.

Then, for $(\sin t)^2 = \sin^2 t$,

$$\begin{aligned} \left[\sin^2 t\right]' &= 2\sin^{2-1} t \times [\sin t]' && \text{Generalized Power Rule} \\ &= 2\sin t \times \cos t && \text{Equation 4.9} \end{aligned} \tag{4.10}$$

Now consider that $\cos t = \sqrt{1-\sin^2 t}$ for $0 \le t \le \pi/2$.

$$\begin{aligned} [\cos t]' &= \left[\left(1-\sin^2 t\right)^{\frac{1}{2}}\right]' && \text{Definition of } P \\ &= \tfrac{1}{2}\left(1-\sin^2 t\right)^{\frac{1}{2}-1} \times \left[1-\sin^2 t\right]' && \text{Generalized Power Rule} \\ &= \tfrac{1}{2}\left(1-\sin^2 t\right)^{\frac{1}{2}-1} \times \left([1]' - \left[\sin^2 t\right]'\right) && \text{Sum Rule for Derivatives} \\ &= \tfrac{1}{2}\frac{1}{\sqrt{1-\sin^2 t}} \times \left(0 - 2\sin t\cos t\right) && [C]' = 0 \text{ and Eq 4.10} \\ &= -\sin t && \text{Trigonometric simplification.} \end{aligned}$$

One might then guess (correctly) that

$$[\cos t]' = -\sin t \qquad \text{for all} \quad t.$$

Observe the exaggerated $\Big(\quad\Big)$'s in the step marked 'Sum Rule for Derivatives.' Students tend to omit writing those parentheses. They may carry them mentally or may loose them. The ( )'s are necessary. Without them, the steps would lead to

$$\begin{aligned}
P' &= \tfrac{1}{2}\left(1-\sin^2 t\right)^{\frac{1}{2}-1} \times \left[1 - \sin^2 t\right]' && \text{Generalized Power Rule}\\
&= \tfrac{1}{2}\left(1-\sin^2 t\right)^{\frac{1}{2}-1} \times [\,1\,]' - \left[\sin^2 t\right]' && \text{Sum Rule for Derivatives}\\
&= \tfrac{1}{2}\left(1-\sin^2 t\right)^{\frac{1}{2}-1} \times 0 - 2\sin t\cos t && [\,C\,]' = 0 \text{ and Eq 4.10}\\
&= -2\sin t\cos t && \text{Trigonometric simplification.}
\end{aligned}$$

Unfortunately, the answer is incorrect. Always:

First Notice.

Use parentheses, ( )′s, they are cheap.

4.3.1 The Power Chain Rule.

The Generalized Power Rule is one of a collection of rules called **chain rules** and henceforth we will refer to it as the Power Chain Rule. The reason for the word, 'chain' is that the rule is often a 'link' in a 'chain' of steps leading to a derivative. Because of its form

$$[\,u(t)^n]' = nu(t)^{n-1} \times [\,u(t)\,]',$$

when the Power Chain Rule is used, there always remains a derivative, $[\,u(t)\,]'$, to compute after the Power Chain Rule is used. **The power chain rule is never the final step – the final step is always one or more of the *Primary Formulas.***

For example, compute the derivative of

$$y = \frac{1}{1+\sqrt{x+1}} = \left(1+(x+1)^{1/2}\right)^{-1}$$

$$
\begin{aligned}
y' &= \left[\left(1+(x+1)^{1/2}\right)^{-1}\right]' && \text{Logical Identity}\\
&= (-1)\left(1+(x+1)^{1/2}\right)^{-2} \times \left[1+(x+1)^{1/2}\right]' && \text{Power Chain Rule}\\
&= (-1)\left(1+(x+1)^{1/2}\right)^{-2} \left(0+\left[(x+1)^{1/2}\right]'\right) && \text{Sum and Constant Rules}\\
&= (-1)\left(1+(x+1)^{1/2}\right)^{-2} (1/2)(x+1)^{-1/2}\,[x+1]' && \text{Power Chain Rule}\\
&= (-1)\left(1+(x+1)^{1/2}\right)^{-2} (1/2)(x+1)^{-1/2}(1+0) && \text{Power and Constant Rules}
\end{aligned}
$$

Exercises for Section 4.3, The generalized power rule.

Exercise 4.3.1 Compute $P'(t)$ for

a.	$P(t)$	$=$	$(2+t^2)^4$	b.	$P(t)$	$=$	$(1+\sin t)^3$
c.	$P(t)$	$=$	$(t^4+5)^2$	d.	$P(t)$	$=$	$(6t^7+5^4)^9$
e.	$P(t)$	$=$	$\left(\frac{t}{8}+t^2\right)^2$	f.	$P(t)$	$=$	$(t^2+\sin t)^{13}$
g.	$P(t)$	$=$	$\left(\frac{1}{t}+t\right)^2$	h.	$P(t)$	$=$	$\left(\frac{5}{t}+\frac{t}{3}\right)^2$
i.	$P(t)$	$=$	$\frac{1}{t+5}$	j.	$P(t)$	$=$	$\frac{2}{(1+t)^2}$

Exercise 4.3.2 Give reasons to support each of the equality signs labeled a - j in Equations 4.8 to prove the Generalized Power Rule. Each equality can be justified by reference to algebra, to one of the limit formulas Equations 3.8 through 3.13, or Theorem 4.2.1, The Derivative Requires Continuity, or the definition of the derivative, Equation 3.20. Cryptic versions appear below (or inside cover of the book).

Eq 3.8 $\lim_{x\to a} C = C$ Eq 3.9 $\lim_{x\to a} x = a$

Eq 3.10 $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ Eq 3.11 $\lim_{x\to a} C\,F(x) = C \lim_{x\to a} F(x)$

Eq 3.12 $\lim_{x\to a} (F_1(x) + F_2(x)) = \lim_{x\to a} F_1(x) + \lim_{x\to a} F_2(x)$

Eq 3.13 $\lim_{x\to a} (F_1(x) \times F_2(x)) = \left(\lim_{x\to a} F_1(x)\right) \times \left(\lim_{x\to a} F_2(x)\right)$

Eq 3.20 $F'(x) = \lim_{b\to x} \frac{F(b)-F(x)}{b-x}$

Exercise 4.3.3 Suppose m is a positive integer and a function u has a derivative at t and that $u(t) \neq 0$. Give reasons for the equalities a - g below that show

$$\left[u^{-m}(t)\right]' = (-m) \times u^{-m-1}(t) \times u'(t)$$

$$\begin{aligned}
[\,u^{-m}(t)\,]' &\overset{a}{=} \lim_{b\to 0} \frac{u^{-m}(b) - u^{-m}(t)}{b-t} \\
&= \lim_{b\to t} \frac{\dfrac{1}{u^m(b)} - \dfrac{1}{u^m(t)}}{b-t} \\
&= \lim_{b\to t} \frac{u^m(t) - u^m(b)}{u^m(b) \times u^m(t)} \times \frac{1}{b-t} \\
&\overset{b}{=} -\lim_{b\to t} \frac{u^{m-1}(t) + u^{m-2}(t)u(b) + \cdots + u(t)u^{m-2}(b) + u^{m-1}(b)}{u^m(b) \times u^m(t)} \times \frac{u(b) - u(t)}{b-t} \\
&\overset{c}{=} -\frac{\lim\limits_{b\to t}\left(u^{m-1}(t) + u^{m-2}(t)u(b) + \cdots u(t)u^{m-2}(b) + u^{m-1}(b)\right)}{\lim\limits_{b\to t} u^m(b) \times u^m(t)} \times \lim_{b\to t} \frac{u(b) - u(t)}{b-t} \\
&\overset{d}{=} -\frac{\lim\limits_{b\to t}\left(u^{m-1}(t) + u^{m-2}(t)u(b) + \cdots u(t)u^{m-2}(b) + u^{m-1}(b)\right)}{u^m(t) \times u^m(t)} \times \lim_{b\to t} \frac{u(b) - u(t)}{b-t} \\
&\overset{e}{=} -\frac{\lim\limits_{b\to t}\left(u^{m-1}(t) + u^{m-2}(t)u(b) + \cdots u(t)u^{m-2}(b) + u^{m-1}(b)\right)}{u^m(t) \times u^m(t)} \times u'(t) \\
&\overset{f}{=} -\frac{u^{m-1}(t) + u^{m-1}(t) + \cdots u^{m-1}(t) + u^{m-1}(t)}{u^m(t) \times u^m(t)} \times u'(t) \\
&\overset{g}{=} (-m) \times u^{-m-1}(t) \times u'(t)
\end{aligned}$$

Exercise 4.3.4 Suppose p and q are integers and u is a positive function that has a derivative at all numbers t. Assume that $\left[u^{\frac{p}{q}}(t)\right]'$ exists. Give reasons for the steps a - d below that show

$$\left[u^{\frac{p}{q}}(t)\right]' = \frac{p}{q}\, u^{\frac{p}{q}-1}(t) \times u'(t).$$

Let

$$v(t) = u^{\frac{p}{q}}(t).$$

Then

$$
\begin{aligned}
v^q(t) &= u^p(t) && \text{a.}\\
[\,v^q(t)\,]' &= [\,u^p(t)\,]' \\
q\,v^{q-1}(t)\times v'(t) &= p\,u^{p-1}(t)\times u'(t) && \text{b.}\\
v'(t) &= \tfrac{p}{q}\,\frac{u^{p-1}(t)}{\left(u^{\frac{p}{q}}\right)^{q-1}}\times u'(t) && \text{c.}\\
\left[u^{\frac{p}{q}}(t)\right]' &= \tfrac{p}{q}\,u^{\frac{p}{q}-1}(t)\times u'(t) && \text{d.}
\end{aligned}
$$

4.4 Applications of the Power Chain Rule.

The Power Chain Rule

$$\text{PCR:}\qquad [\,u^n(t)\,]' = n\,u^{n-1}(t)\times u'(t) \tag{4.11}$$

greatly expands the diversity and interest of problems that we can analyze. Some introductory examples follow.

Example 4.4.1 A. Find the slope of the tangent to the circle, $x^2+y^2=13$, at the point (2,3). See Figure 4.3.
B. Also find the slope of the tangent to the circle, $x^2+y^2=13$, at the point (3,-2).

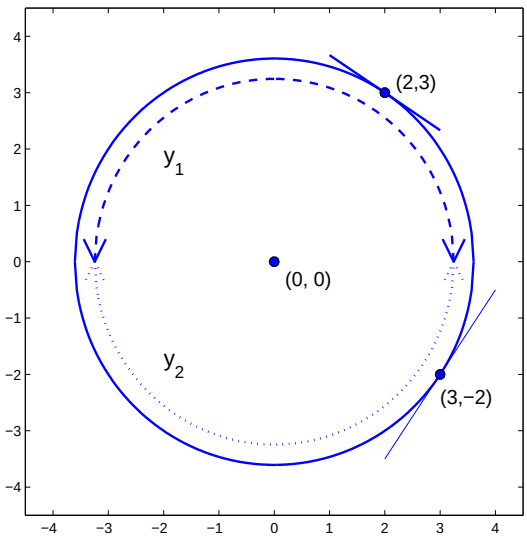


Figure 4.3: Graph of the circle $x^2+y^2=13$ and tangents drawn at the points (2,3) and (3,-2) of the circle.

Solution First check that $2^2+3^2=4+9=13$, so that (2,3) is indeed a point of the circle. Then solve for y in $x^2+y^2=13$ to get

$$y_1=\sqrt{13-x^2}$$

Then the Power Chain Rule with $n = \frac{1}{2}$ yields

$$\begin{aligned}
y_1' &= \left[\sqrt{13 - x^2}\right]' = \left[\,(\,13 - x^2\,)^{\frac{1}{2}}\,\right]' && \text{a. Symbolic identity} \\
&= \tfrac{1}{2}\,(\,13 - x^2\,)^{\frac{1}{2}-1}\,[\,13 - x^2]' && \text{b. PCR, } n = 1/2 \ u = 1 - x^2 \\
&= \tfrac{1}{2}\,(\,13 - x^2\,)^{-\frac{1}{2}} \times \Big(\,[13]' - [x^2]'\,\Big) && \text{c. Sum Rule} \\
&= \tfrac{1}{2}\,(\,13 - x^2\,)^{-\frac{1}{2}} \times \Big(\,0 - 2x\,\Big) && \text{d. Constant and Power Rules} \\
&= -x\,(\,13 - x^2\,)^{-\frac{1}{2}}
\end{aligned}$$

Observe the exaggerated $\Big(\quad\Big)$'s in steps c and d. Students tend to omit writing them, but they are necessary. Without the ( )'s, steps c and d would lead to

$$\begin{aligned}
y_1' &= \tfrac{1}{2}\,(\,13 - x^2\,)^{-\frac{1}{2}} \times [13]' - [x^2]' && \text{c. Sum Rule} \\
&= \tfrac{1}{2}\,(\,13 - x^2\,)^{-\frac{1}{2}} \times\ 0 - 2x && \text{d. Constant and Power Rules} \\
&= -2x
\end{aligned}$$

a notably simpler answer, but unfortunately incorrect. Always:

Second Notice.
Use parentheses, ( )'s, they are cheap.

To finish the computation, we compute y_1' at $x = 2$ and get

$$y_1'(2) = \frac{1}{2}\Big(\,13 - 2^2\,\Big)^{-\frac{1}{2}} \times (-2 \times 2) = -\frac{2}{3}$$

and the slope of the tangent to $x^2 + y^2 = 13$ at (2,3) is -2/3. An equation of the tangent is

$$\frac{y-3}{x-2} = -\frac{2}{3} \qquad \text{or} \qquad y = -\frac{2}{3}x + 5$$

B. Now we find the tangent to the circle $x^2 + y^2 = 13$ at the point (2,-3). Observe that $2^2 + (-3)^2 = 4 + 9 = 13$ so that (2,-3) is a point of $x^2 + y^2 = 13$, but (2,-3) **does not** satisfy

$$y_1 = \Big(\,13 - x^2\,\Big)^{\frac{1}{2}}$$

because

$$-3 \neq \left(13 - 2^2\right)^{\frac{1}{2}} = 3$$

For (2,-3) we must use the lower semicircle and

$$y_2 = -\left(13 - x^2\right)^{\frac{1}{2}}$$

Then

$$\begin{aligned} y_2' &= -\left[\left(13 - x^2\right)^{\frac{1}{2}}\right]' \\ &= -\frac{1}{2}\left(13 - x^2\right)^{\frac{1}{2}-1}\left[1 - x^2\right]' \\ &= -\frac{1}{2}\left(13 - x^2\right)^{-\frac{1}{2}}(-2x) \end{aligned}$$

At $x = 3$

$$y_2'(3) = -\frac{1}{2}\left(13 - 3^2\right)^{-\frac{1}{2}}(-2 \times 3) = \frac{3}{2}$$

and the slope of the line drawn is 3/2. ▬

It is often helpful to put the denominator of a fraction into the numerator with a negative exponent. For example:
Problem. Compute $P'(t)$ for $P(t) = \dfrac{5}{(1+t)^2}$. *Solution*

$$\begin{aligned} P'(t) &= \left[\frac{5}{(1+t)^2}\right]' \\ &= \left[5(1+t)^{-2}\right]' \\ &= 5\left[(1+t)^{-2}\right]' \\ &= 5 \times (-2) \times (1+t)^{-3}\left[(1+t)\right]' \\ &= 5 \times (-2) \times (1+t)^{-3} \times 1 = -10 \times (1+t)^{-3} \end{aligned}$$

We will find additional important uses of the power rules in the next section.

Exercises for Section 4.4, Applications of the Power Chain Rule.

Exercise 4.4.1 Compute $y'(x)$ for

a.	$y = 2x^3 - 5$		b.	$y = \dfrac{2}{x^2}$		c.	$y = \dfrac{5}{(x+1)^2}$	
d.	$y = (1+x^2)^{0.5}$		e.	$y = \sqrt{1-x^2}$		f.	$y = (1-x^2)^{-0.5}$	
g.	$y = (2-x)^4$		h.	$y = (3-x^2)^4$		i.	$y = \dfrac{1}{\sqrt{7-x^2}}$	
j.	$y = (1+(x-2)^2)^2$		k.	$y = (1+3x)^{1.5}$		l.	$y = \dfrac{1}{\sqrt{16-x^2}}$	
m.	$y = \sqrt{9-(x-4)^2}$		n.	$y = (9-x^2)^{1.5}$		o.	$y = \sqrt[3]{1-3x}$	

Exercise 4.4.2 Shown in Figure Ex. 4.4.2 is the ellipse,

$$\frac{x^2}{18} + \frac{y^2}{8} = 1$$

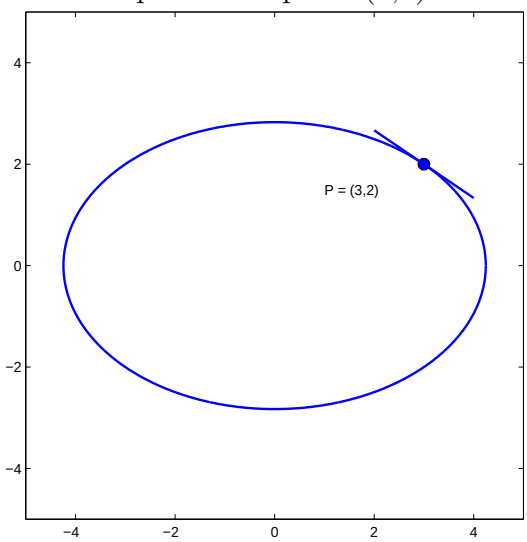
and a tangent to the ellipse at (3,2).

a. Find the slope of the tangent.

b. Find an equation of the tangent.

c. Find the x- and y-intercepts of the tangent.

Figure for Exercise 4.4.2 Graph of the ellipse $x^2/18 + y^2/8 = 1$ and a tangent to the ellipse at the point (3,2).



Exercise 4.4.3 Shown Figure Ex. 4.4.3 is the ellipse,

$$\frac{2x^2}{35} + \frac{3y^2}{35} = 1$$

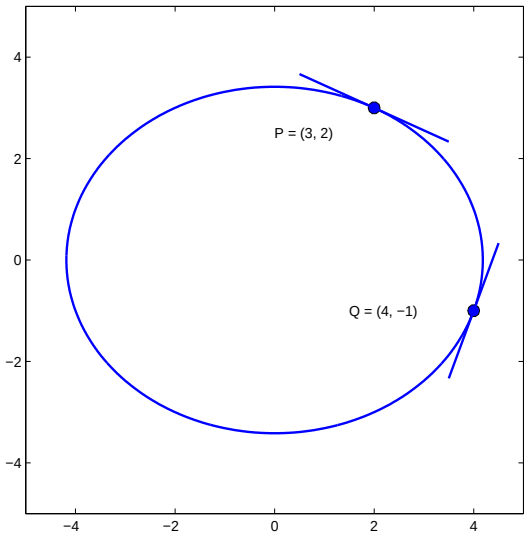
and tangents to the ellipse at (2, 3) and at (4, -1).

a. Find the slopes of the tangents.

b. Find equations of the tangents.

c. Find the point of intersection of the tangents.

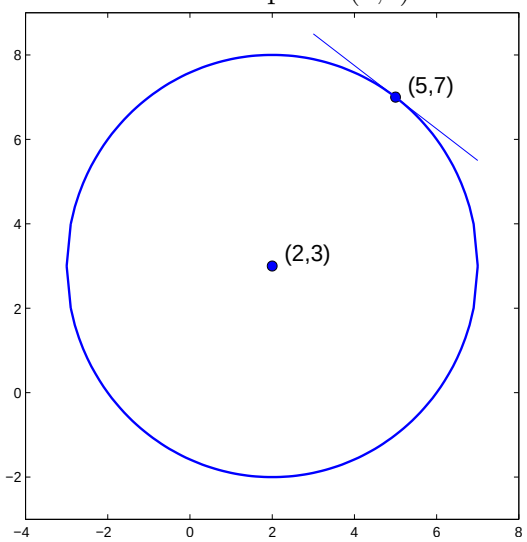
Figure for Exercise 4.4.3 Graph of the ellipse $2x^2/35 + 3y^2/35 = 1$ and tangents to the ellipse at the points (2,3) and (4,-1).



Exercise 4.4.4 Shown in Figure Ex. 4.4.4 is the circle with center at (2,3) and radius 5. find the slope of the tangent to this circle at the point (5,7). An equation of the circle is

$$(x-2)^2 + (y-3)^2 = 5^2$$

Figure for Exercise 4.4.4 Graph of the circle $(x-2)^2 + (y-3)^2 = 5^2$ and tangent to the circle at the point (5,7).



4.5 Some optimization problems.

In Section 3.5.2 we found that local maxima and minima are often points at which the derivative is zero. The algebraic functions for which we can now compute derivatives have only a finite number of points at which the derivative is zero or does not exist and it is usually a simple matter to search among them for the highest or lowest points of their graphs. Such a process has long been used to find optimum parameter values and a few of the traditional problems that can be solved using the derivative rules of this chapter are included here. More optimization problems appear in Chapter 8 Applications of the Derivative.

Assume for this section only that all local maxima and local minima of a function, F, are found by computing F' and solving for x in $F'(x) = 0$.

Example 4.5.1 A forester needs to get from point A on a road to point B in a forest (see diagram in Figure 4.4). She can travel 5 km/hr on the road and 3 km/hr in the forest. At what point, P, should she leave the road and enter the forest in order to minimize the time required to travel from A to B?

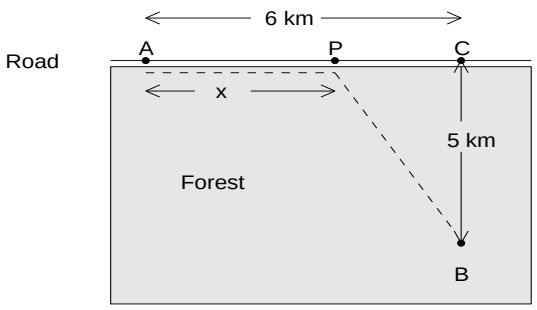


Figure 4.4: Diagram of a forest and adjacent road for Example 4.5.1

Solution. She might go directly from A to B through the forest; she might travel from A to C and then to B; or she might, as illustrated by the dashed line, travel from A to a point, P, along the road and then from P to B.

Assume that the road is straight, the distance from B to the road is 5 km and the distance from A to the projection of B onto the road (point Q) is 6 km. The point, P, is where the forester leaves the road; let x be the distance from A to P. The basic relation between distance, speed, and time is that

$$\text{Distance (km)} = \text{Speed (km/hr)} \times \text{Time (hr)}$$

so that

$$\text{Time} \quad = \quad \frac{\text{Distance}}{\text{Speed}}$$

The distance traveled and time required are

	Along the road	In the forest
Distance	x	$\sqrt{(6-x)^2+5^2}$
Time	$\frac{x}{5}$	$\frac{\sqrt{(6-x)^2+5^2}}{3}$

The total trip time, T, is written as

$$T = \frac{x}{5} + \frac{\sqrt{(6-x)^2+5^2}}{3} \tag{4.12}$$

A graph of T *vs* x is shown in Figure 4.5. It appears that the lowest point on the curve occurs at about $x = 2.5km$ and $T = 2.5$ hours. ■

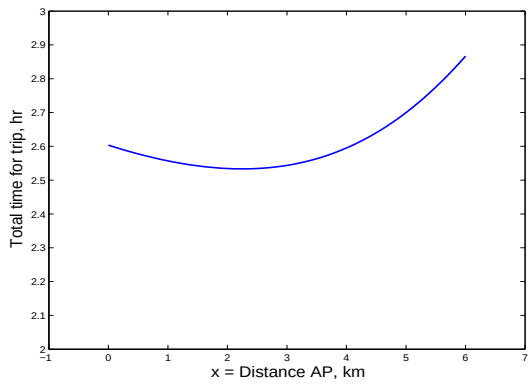


Figure 4.5: Graph of Equation 4.12; total trip time, T, *vs* distance traveled along the road, x before entering the forest.

Explore 4.5.1 It appears that to minimize the time of the trip, the forester should travel about 2.5 km along the road from A to a point P and enter the forest to travel to B. Observe that the tangent to the graph at the lowest point is horizontal, and that no other point of the graph has a horizontal tangent.

Compute the derivative of $T(x)$ for

$$T(x) = \frac{x}{5} + \frac{\sqrt{(6-x)^2+5^2}}{3}$$

Note: The constant denominators may be factored out, as in

$$\left[\frac{x}{5}\right]' = \left[\frac{1}{5}x\right]' = \frac{1}{5}[x]' = \frac{1}{5} \times 1 = \frac{1}{5}$$

You should get

$$T'(x) = \frac{1}{5} + \frac{1}{3}\frac{1}{2}\left((6-x)^2 + 5^2\right)^{-1/2} \times (6-x) \times (-1)$$

Find the value of x for which $T'(x) = 0$.

Your conclusion should be that the forester should travel 2.25 km from A to P and that the time for the trip, $T(2.25) = 2.53\overline{3}$ hours. ■

Exercises for Section 4.5, Some optimization problems.

Exercise 4.5.1 In Example 4.5.1, what should be the path of the forester is she can travel 10km/hr on the road and 4km/hr in the forest?

Exercise 4.5.2 The air temperature is -10°F and Linda has a ten mile bicycle ride from the university to her home. There is no wind blowing, but riding her bicycle increases the effects of the cold, according to the wind chill chart in Figure 4.5.2 provided by the Centers for Disease Control. The formula for computing windchill is

Windchill (F) = 35.74 + 0.6215T - 35.75$V^{0.16}$ + 0.4275T$V^{0.16}$,

where: T = Air Temperature (F) and V = Wind Speed (mph).

Assume that if she travels at a speed, s, then she looses body heat a rate proportional to the difference between her body temperature and wind chill temperature for speed s.

a. At what speed should she travel in order to minimize the amount of body heat that she looses during the 10 mile bicycle ride?

b. Frostbite is skin tissue damage caused by prolonged skin tissue temperature of 23°F. The time for frostbite to occur is also shown in Figure 4.5.2. What is her optimum speed if she wishes to avoid frostbite.

c. Discuss her options if the ambient air temperature is -20°F.

Figure for Exercise 4.5.2 Table of windchill temperatures for values of ambient air temperatures and wind speeds provided by the Center for Disease Control at http://emergency.cdc.gov/disasters/winter/pdf/cold_guide.pdf. It was adapted from a more detailed

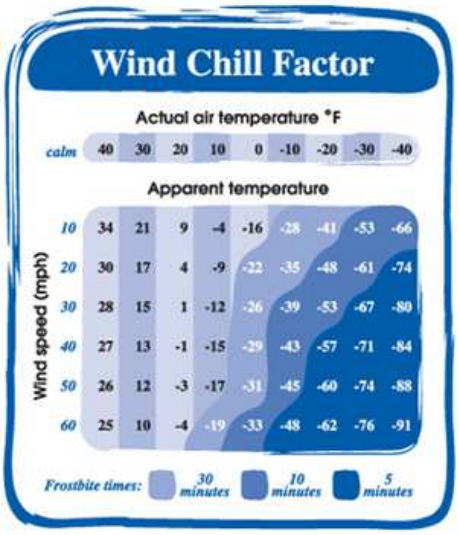


chart at http://www.nws.noaa.gov/om/windchill.

Exercise 4.5.3 If x pounds per acre of nitrogen fertilizer are spread on a corn field, the yield is

$$200 - \frac{4000}{x+25}$$

bushels per acre. Corn is worth \$4.50 per bushel and nitrogen costs \$0.50 per pound. All other costs of growing and harvesting the crop amount to \$600 per acre, and are independent of the amount of nitrogen fertilizer applied. How much nitrogen per acre should be used to maximize the net dollar return per acre?

Exercise 4.5.4 Optimum cross section of your femur. R. M. Alexander[2] has an interesting analysis of the cross section of mammal femurs. Femurs are hollow tubes filled with marrow. They should resist forces that tend to bend them, but not be so massive as to impair movement. An optimum femur will be the lightest bone that is strong enough to resist the maximum bending moment, M, that will be applied to it during the life of the animal.

A hollow tube of mass m kg/m is may be stronger than a solid rod of the same weight, depending on two parameters of the tube, the outside radius, R, and the inside radius, $x \times R$ $(0 \le x < 1)$, see Figure 4.6. For a given moment, M, the relation between R and x is

$$R = \left[\frac{M}{K(1-x^4)}\right]^{\frac{1}{3}} = \left(\frac{M}{K}\right)^{\frac{1}{3}} (1-x^4)^{-\frac{1}{3}} \tag{4.13}$$

The constant K describes the strength of the material.

Let ρ be bone density and assume marrow density is $\frac{1}{2}\rho$. Then the mass per unit length of bone, m_b, is

$$\begin{aligned} m_b &= \rho \times (\pi R^2 - \pi(R \times x)^2) \\ &= \rho \times \pi \times (1-x^2) \times R^2 \\ &= \rho\pi \left(\frac{M}{K}\right)^{\frac{2}{3}} \times (1-x^2) \times (1-x^4)^{-\frac{2}{3}} \end{aligned} \tag{4.14}$$

a. Write an equation for the mass per unit length of the bone marrow similar to Equation 4.14.

[2] R. McNeil Alexander, Optima for Animals, Princeton University Press, Princeton, NJ, 1996, Section 2.1, pp 17-22.

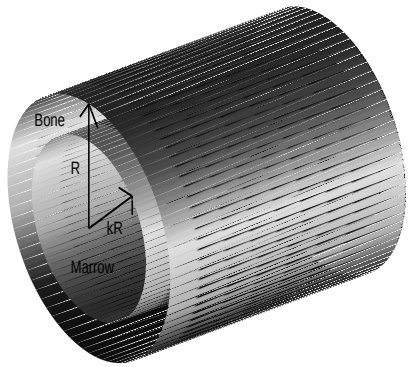


Figure 4.6: Consider a femur to be a tube of radius R with solid bone between kR and R and marrow inside the tube of radius kR.

b. We would like to know the derivative of m with respect to x for

$$m = C\left((1-\frac{1}{2}x^2)\times(1-x^4)^{-\frac{2}{3}}\right). \qquad C = \rho\pi\left(\frac{M}{K}\right)^{\frac{2}{3}} \tag{4.15}$$

You will see in Chapter 6 that

$$[\,m\,]' = C\times\left(\left[1-\frac{1}{2}x^2\right]'\times(1-x^4)^{-\frac{2}{3}} + (1-\frac{1}{2}x^2)\times\left[(1-x^4)^{-\frac{2}{3}}\right]'\right)$$

Finish the computation of $[\,m\,]'$.

c. Find a value of x for which

$$-x\left(1-x^4\right)^{-\frac{2}{3}} + \frac{8}{3}x^3\left(1-\frac{1}{2}x^2\right)\times\left(1-x^4\right)^{-\frac{5}{3}} = 0$$

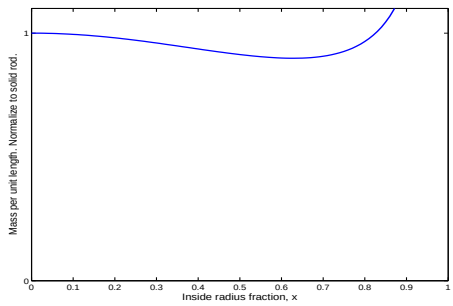
d. The value $\overline{x}$ computed in Part c. is the x-coordinate of the lowest point of the graph of m shown in Exercise Figure 4.5.4. Alexander shows the values for x for five mammalian species; for the humerus they range from 0.42 to 0.66 and for the femur they range from 0.54 to 0.63. Compare $\overline{x}$ with these values.

e. Exercise Figure 4.5.4B is a cross section of the human leg at mid-thigh. Estimate x for the femur.

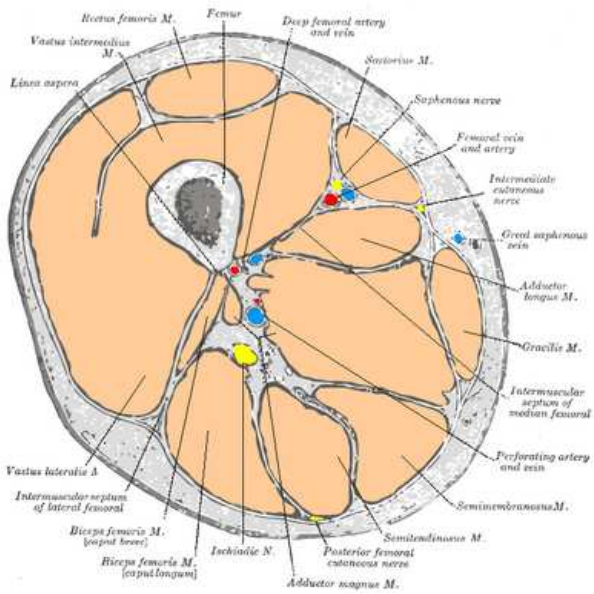
Alexander modifies this result, noting that Equation 4.13 is the breaking moment, and a bone with walls this thin would buckle before it broke, and noting that bones are tapered rather than of uniform width.

Figure for Exercise 4.5.4
A. Graph of Equation 4.15, the mass of bone plus marrow as a function of the ratio, x. Mass per unit length of a solid bone has been arbitrarily set equal to one. B. Cross section of a human leg at mid thigh. http://en.wikipedia.org/wiki/Fascial_compartments_of_thigh. Lithograph plate from Gray's Anatomy, not copyrightable

A.



B



4.6 Implicit differentiation.

In Section 4.4, Applications of the Power Chain Rule, we found slopes of circles and ellipses. There is a procedure for finding these slopes that requires less algebra, but more mathematical sophistication. In each of the equations,

$$\begin{aligned} x^2 + y^2 &= 13 \\ \frac{x^2}{18} + \frac{y^2}{8} &= 1 \\ \frac{2x^2}{35} + \frac{3y^2}{35} &= 1 \end{aligned}$$

we might solve for y in terms of x, being careful to choose the correct square root to match the point of tangency, and then compute $y'(x)$.

In $x^2 + y^2 = 13$, we may chose $y(x) = \sqrt{13 - x^2}$. Note that

$$x^2 + y^2 = x^2 + (\sqrt{13 - x^2})^2 = x^2 + 13 - x^2 = 13$$

so that $y(x) = \sqrt{13 - x^2}$ is a function that 'satisfies' and is said to be *implicitly* defined by the equation.

Definition 4.6.1 Implicit Function. *Suppose we are given an equation*

$$E(x, y) = 0$$

and point (a, b) *for which*

$$E(a, b) = 0$$

A function, f*, defined on an interval* $(a - h, a + h)$ *surrounding* a *and satisfying*

$$E(x, f(x)) = 0 \qquad \textit{and} \qquad f(a) = b$$

is said to be implicitly defined *by* E*. There may be no such function,* f*, one such function, or many such functions.*

Now we assume without solving for $y(x)$ that there is a function $y(x)$ for which

$$x^2 + (\, y(x)\,)^2 = 13 \qquad \text{and} \qquad y(2) = 3$$

and use the power rule and power chain rule to differentiate the terms in the equation, as follows.

$$\begin{aligned} x^2 + (\, y(x)\,)^2 &= 13 \\ \left[x^2\right]' + \left[(\, y(x)\,)^2\right]' &= [13]' \\ 2x + 2y(x)y'(x) &= 0 \end{aligned}$$

The **power rule** is used for

$$\left[x^2\right]' = 2x$$

The **power chain rule** is used for

$$\left[(\, y(x)\,)^2\right]' = 2y(x)y'(x)$$

We use $x = 2$ and $y(2) = 3$ in the last equation to get

$$2 \times 2 + 2 \times 3 \times y'(2) = 0$$

and solve for $y'(2)$ to get

$$y'(2) = -\frac{2}{3}$$

as was found in Example 4.4.1 to be the slope of the tangent to $x^2 + y^2 = 13$ at (2,3).

It is important to remember in the above steps the $[\quad]'$ means derivative with respect to the independent variable, x. The Leibnitz notation, $\frac{d}{dx}$, explicitly shows this and may be easier to use. We repeat this problem with Leibnitz notation.

$$\begin{aligned} x^2 + (\,y(x)\,)^2 &= 13 \\ \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\,y(x)\,)^2 &= \frac{d}{dx}(13) \\ 2x + 2y(x)\frac{d}{dx}y(x) &= 0 \end{aligned}$$

The **power rule** is used for	The **power chain rule** is used for
$\frac{d}{dx}\left(x^2\right) = 2x$	$\frac{d}{dx}(\,y(x)\,)^2 = 2y(x)\frac{d}{dx}y(x)$

Example 4.6.1 We consider another example of implicit differentiation. Find the slope of the graph of

$$\sqrt{x} + \sqrt{5 - y^2} = 5 \qquad \text{at (9,1) and at (4,2)}$$

First we check to see that (9,1) satisfies the equation:

$$\sqrt{9} + \sqrt{5 - 1^2} = \sqrt{9} + \sqrt{4} = 3 + 2 = 5. \qquad \text{It checks.}$$

Then we assume there is a function $y(x)$ such that

$$\sqrt{x} + \sqrt{5 - (y(x))^2} = 5 \qquad \text{and that} \qquad y(9) = 1.$$

We convert the square root symbols to fractional exponents and differentiate using Leibnitz notation.

$$\begin{aligned} x^{\frac{1}{2}} + (5 - (y(x))^2)^{\frac{1}{2}} &= 5 \\ \tfrac{d}{dx}\left(x^{\frac{1}{2}} + (5 - (y(x))^2)^{\frac{1}{2}}\right) &= \tfrac{d}{dx}5 \\ \tfrac{d}{dx}x^{\frac{1}{2}} + \tfrac{d}{dx}(5 - (y(x))^2)^{\frac{1}{2}} &= 0 && \text{Sum and Constant Rules} \\ \tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}(5 - y^2)^{-\frac{1}{2}}\tfrac{d}{dx}(5 - (y(x))^2) &= 0 && \text{Power and Power Chain Rules} \\ \tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}(5 - y^2)^{-\frac{1}{2}}\left(\tfrac{d}{dx}5 - \tfrac{d}{dx}(y(x))^2\right) &= 0 && \text{Sum Rule} \\ \tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}(5 - y^2)^{-\frac{1}{2}}\left(0 - 2y\tfrac{d}{dx}y(x)\right) &= 0 && \text{Constant and Power Chain Rules} \end{aligned}$$

Next we solve for $\frac{d}{dx}y(x)$ and get

$$\frac{d}{dx}y(x) = \frac{\sqrt{5 - y^2}}{2yx^{\frac{1}{2}}} \qquad \text{and evaluate at (9,1)} \qquad \left.\frac{\sqrt{5 - y^2}}{2yx^{\frac{1}{2}}}\right|_{(x,y)=(9,1)} = \frac{1}{3}$$

So the slope of the graph at (9,1) is 1/3. You may notice that we have selectively used $y(x)$ and y; often only y is used to simplify notation.

An equation of the tangent to the graph of $\sqrt{x}+\sqrt{5-y^2}=5$ at the point (9,1) is $y-1=\frac{1}{3}(x-9)$.

Now for the point (4,2), the differentiation is exactly the same and we might (**alert!**) evaluate

$$\frac{d}{dx}y(x)=\frac{\sqrt{5-y^2}}{2yx^{\frac{1}{2}}} \qquad \text{at } (4,2) \qquad \left.\frac{\sqrt{5-y^2}}{2yx^{\frac{1}{2}}}\right|_{(x,y)=(4,2)}=\frac{1}{8}$$

However, the point (4,2) **does not satisfy the original equation** and is not a point of its graph. Finding the slope at that point is meaningless, so we punt.

All of this solution is algebraic. The graph of the equation shown in Figure 4.7 is of considerable help. ■

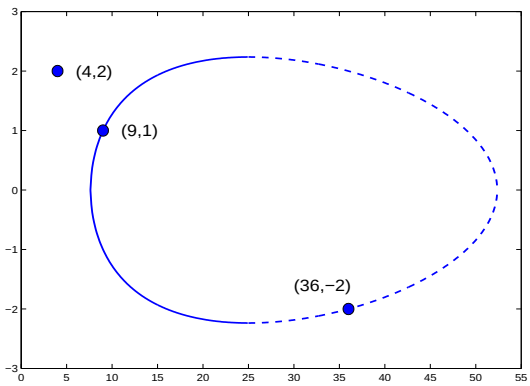


Figure 4.7: Solid curve: Graph of $\sqrt{x}+\sqrt{5-y^2}=5$ and the points (9,1) and (4,2) for Example 4.6.1. Dashed curve: Graph of $\sqrt{x}-\sqrt{5-y^2}=5$ and the point (36,-2) for Exercise 4.6.2.

Exercises for Section 4.6 Implicit Differentiation.

Exercise 4.6.1 Find the slopes of the tangents to the graph of

a. $\dfrac{x^2}{18}+\dfrac{y^2}{8} = 1$ at the points $(3,2)$ and $(-3,2)$

b. $\dfrac{2x^2}{35}+\dfrac{3y^2}{35} = 1$ at the points $(4,1)$, $(-3,-2)$, and $(4,-1)$

Exercise 4.6.2 Find the slope of the graph of $\sqrt{x}-\sqrt{5-y^2}=5$ at the point (36,-2). Is there a slope to the graph at the point (46,1)?

Exercise 4.6.3 A Finnish landscape architect laid out gardens in the shape of the pseudo ellipsoid

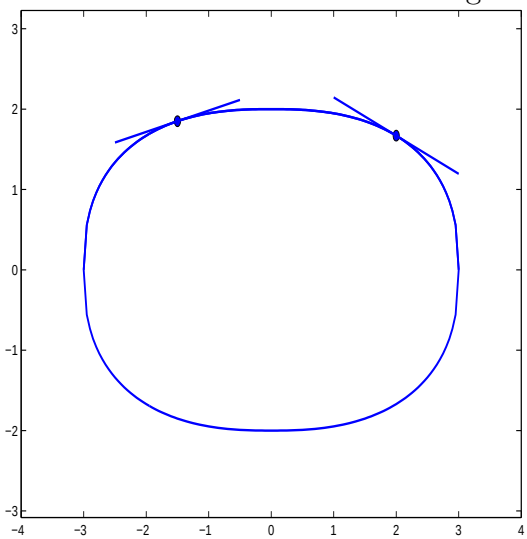
$$\frac{|x|^{2.5}}{a^{2.5}}+\frac{|y|^{2.5}}{b^{2.5}}=1$$

a shape that became commonly used in design of Scandinavian furniture and table ware. In Figure Ex. 4.6.3 is the graph of,

$$\frac{|x|^{2.5}}{3^{2.5}} + \frac{|y|^{2.5}}{2^{2.5}} = 1$$

and tangents drawn at (2, 1.67) and (-1.5, 1.85). Find the slopes of the tangents.

Figure for Exercise 4.6.3 Graph of the equation $|x|^{2.5}/3^{2.5} + |y|^{2.5}/2^{2.5} = 1$ representative of a Scandinavian design.



Exercise 4.6.4 Draw the graph and find the slopes of the tangents to the graph of

a.	$x^2 - 2y^2$	$=$	1	at the points	$(3,2)$ and $(-3,-2)$
b.	$2x^4 + 3y^4$	$=$	35	at the points	$(2,1)$, $(1,-2)$, and $(2,-1)$
c.	$\sqrt{\lvert x\rvert} + \sqrt{\lvert y\rvert}$	$=$	5	at the points	$(9,4)$ and $(1,-16)$
d.	$\sqrt{x} + \sqrt[3]{y}$	$=$	9	at the points	$(64,1)$, $(36,27)$, and $(16,125)$
e.	$x^2 + y^2$	$=$	$(x+y)^2$	at the points	$(1,0)$ and $(0,1)$
f.	$(x^2+4)y$	$=$	24	at the points	$(2,3)$ and $(0,6)$
g.	$x^3 + y^3$	$=$	$(x+y)^3$	at the points	$(1,-1)$ and $(-2,0)$
h.	$x^4 + x^2y^2$	$=$	$20y^2$	at the points	$(2,1)$ and $(2,-1)$

Exercise 4.6.5 This is an interesting and challenging problem. The goal is to explain, among other things, the historical instance of John Adams overhearing the plans of his opponents in Statuary Hall just outside the U.S. congressional chamber.

Ellipses have an interesting reflective property explained by tangents to an ellipse (see Figure 4.8A). Light or sound originating at one focal point of an ellipse is reflected by the ellipse to the other focal point. Statuary Hall is in the shape of an ellipse. John Adams opponents had a desk at one of the focal points and Adams arranged to stand at the other focal point. This property also is a factor in the acoustics of the Mormon Tabernacle in Salt Lake City, Utah and the Smith Civil War Memorial in Philadelphia, Pennsylvania.

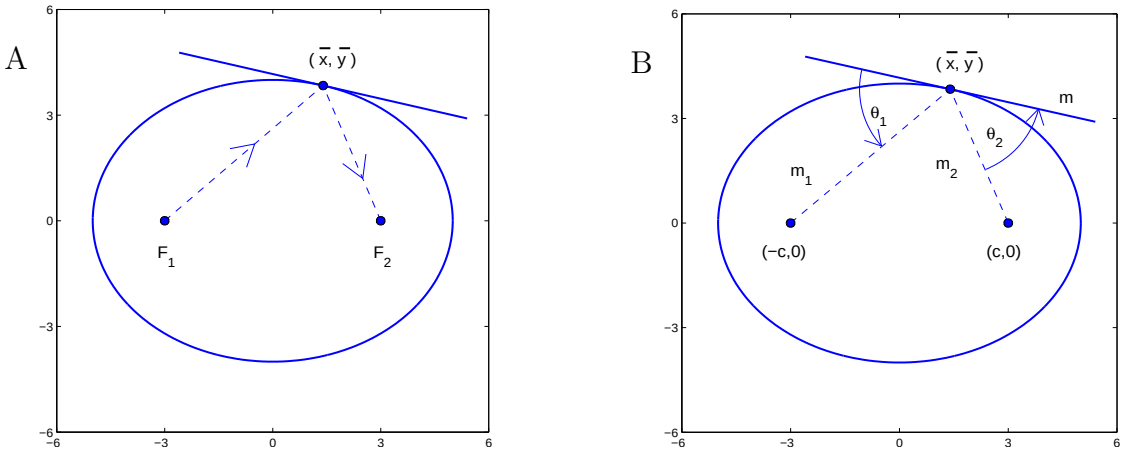


Figure 4.8: A. An ellipse. Light or sound originating at focal point F_1 and striking the ellipse at $(\overline{x}, \overline{y})$ is reflected to F_2. B. The angle of incidence, θ_1, is equal to the angle of reflection, θ_2.

In case you have forgotten: For two intersecting lines with inclinations α_1 and α_2 and $\alpha_1 > \alpha_2$ and slopes $m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$, one of the angles between the two lines is $\theta = \alpha_1 - \alpha_2$ (Figure 4.9). If neither line is vertical and the lines are not perpendicular,

$$\tan\theta = \tan(\alpha_1 - \alpha_2) = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{4.16}$$

Refer to Figure 4.8B. Assume an equation of the ellipse is $b^2x^2 + a^2y^2 = a^2\,b^2$, $a > b > 0$. Then the focal points will be at $(-c, 0)$ and $(c, 0)$ where $c = \sqrt{a^2 - b^2}$. Let $(\overline{x}, \overline{y})$ denote a point of the ellipse. In order to establish the reflective property of ellipses, it is sufficient to show that the angle of incidence, θ_1 is equal to the angle of reflection, θ_2.

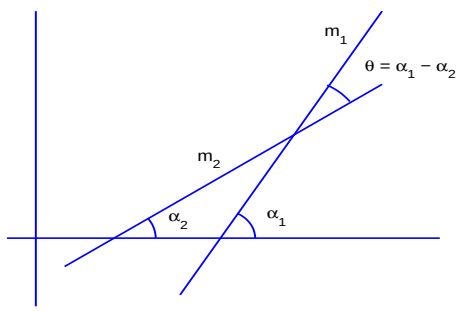


Figure 4.9: Two lines with inclinations α_1 and $\alpha_2 < \alpha_1$ and slopes m_1 and m_2. An angle of intersection is $\theta = \alpha_1 - \alpha_2$.

a. Find the slope, m, of the tangent at $(\overline{x}, \overline{y})$.

b. Show that

$$\tan\theta_1 = \frac{\dfrac{\overline{y}}{\overline{x}+c} - \left(-\dfrac{b^2\,\overline{x}}{a^2\,\overline{y}}\right)}{1 + \dfrac{\overline{y}}{\overline{x}+c} \times \left(-\dfrac{b^2\,\overline{x}}{a^2\,\overline{y}}\right)}$$

c. Write a similar expression for $\tan\theta_2$.

d. **For the algebraically bold.** Show that $\tan\theta_1 = \tan\theta_2$.

e. Both tangents are positive, both angles are acute, and the angles are equal.

4.7 Summary of Chapter 4

We have defined *continuity* of a function, shown that if a function F has a derivative at a point x in its domain then F is continuous at x, and used this property to prove the Power Chain Rule,

$$\text{PCR:} \qquad [\,u^n(t)]' = n\,u^{n-1}(t) \times u'(t).$$

We proved PCR for all positive integers, n. In Exercises 4.3.3 and 4.3.4 you showed PCR to be true for all rational numbers, n. In fact, PCR is true for all numbers, n. We then used the power chain rule to solve some problems.

Exercises for Chapter 4, Continuity and the Power Chain Rule.

Chapter Exercise 4.1 Compute the derivative of P.

a.	$P(t) = 3t^2 - 2t + 7$		b.	$P(t) = t + \frac{2}{t}$		c.	$P(t) = \sqrt{t+2}$	
d.	$P(t) = (\,t^2+1\,)^5$		e.	$P(t) = \sqrt{2t+1}$		f.	$P(t) = 2t^{-3} - 3t^{-2}$	
g.	$P(t) = \frac{5}{t+5}$		h.	$P(t) = \left(1+\sqrt{t}\right)^{-1}$		i.	$P(t) = (\,1+2t\,)^5$	
j.	$P(t) = (\,1+3t\,)^{1/3}$		k.	$P(t) = \frac{1}{1+\sqrt{t}}$		l.	$P(t) = \sqrt[3]{1+2t}$	

Chapter Exercise 4.2 In "Natural History", March, 1996, Neil de Grass Tyson discusses the discovery of an astronomical object called a "brown dwarf".

> "We have suspected all along that brown dwarfs were out there. One reason for our confidence is the fundamental theorem of mathematics that allows you to declare that if you were once 3'8" tall and are now 5'8" tall, then there was a moment when you were 4'8" tall (or any other height in between). An extension of this notion to the physical universe allows us to suggest that if round things come in low-mass versions (such as planets) and high-mass versions (such as stars) then there ought to be orbs at all masses in between *provided a similar physical mechanism made both.*

What fundamental theorem of mathematics is being referenced in the article about the astronomical objects called brown dwarfs? What implicit assumption is being made about the sizes of astronomical objects? (For future consideration: Is the number of 'orbs' countable?)

Chapter Exercise 4.3 In a square field with sides of length 1000 feet that are already fenced a farmer wants to fence two rectangular pens of equal area using 400 feet of new fence and the existing fence around the field. What dimensions of lots will maximize the area of the two pens?

Chapter Exercise 4.4 You must cross a river that is 50 meters wide and reach a point on the opposite bank that is 1 km up stream. You can travel 6 km per hour along the river bank and 1 km per hour in the river. Describe a path that will minimize the amount of time required for your trip. Neglect the flow of water in the river.

Chapter Exercise 4.5 Find the point of intersection of the tangents to the ellipse $x^2/224 + y^2/128 = 1/7$ at the points (2,4) and $(5, -2)$.

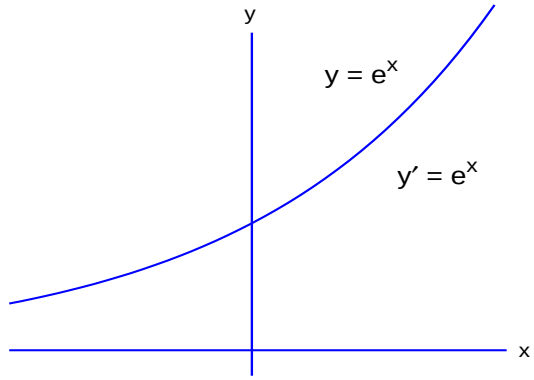
Chapter 5

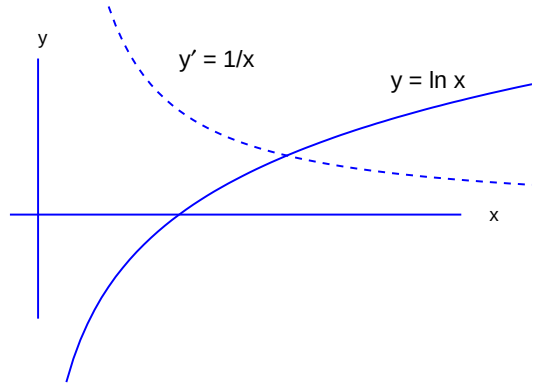
Derivatives of Exponential and Logarithmic Functions

Where are we going?

In this chapter you will learn how to compute the rates of change of exponential and logarithmic functions.

We continue the use of rate of change to model biological and physical processes. The derivatives of exponential and logarithm functions greatly expand the class of biological processes that we describe with equations.





5.1 Derivatives of Exponential Functions.

Exponential functions are often used to describe the growth or decline of biological populations, distribution of enzymes over space, and other biological and chemical relations. The rate of change of exponential functions describes population growth rate, decay of chemical concentration with space, and rates of change of other biological and chemical processes.

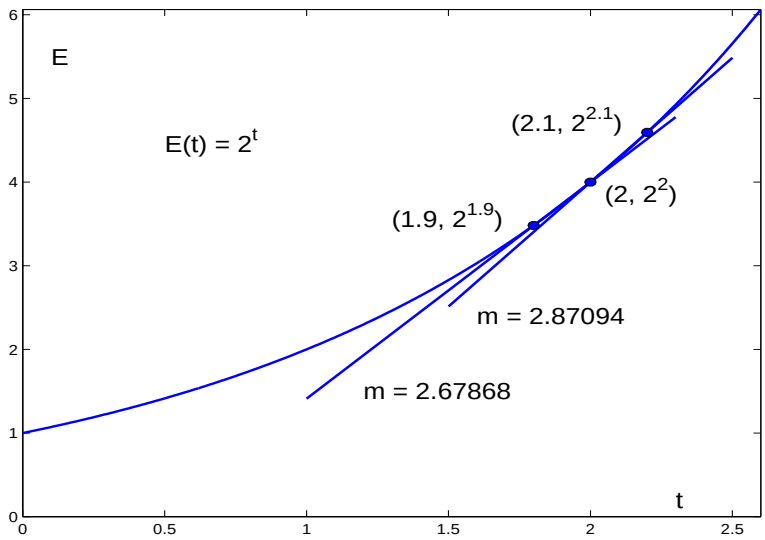
The exponential function $E(t) = 2^t$ (where the base is 2 and the exponent is t) is quite different from the algebraic function, $P(t) = t^2$ (where the base is t and the exponent is 2). $P(t) = t^2$ is well defined for all numbers t in terms of multiplication, $P(t) = t \times t$. However, in elementary courses 2^t is defined only for t a rational number. For the irrational number $\sqrt{2} = 1.4142135\cdots$, for example, $2^{\sqrt{2}}$ is the number to which the sequence $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, $2^{1.4142}$, $\cdots$ approaches. We will not formalize this idea, but will assume that 2^t is meaningful for all numbers t.

Shown in Table 5.1 are computations and a graph directed to finding the rate of growth of $E(t) = 2^t$ at $t = 2$. We wish to find a number m_2 so that

$$\lim_{b\to 2} = \frac{2^b - 2^2}{b-2} = m_2.$$

Table 5.1: Table and Graph of $E(t) = 2^t$ near $t = 2$.

b	$\frac{2^b-2^2}{b-2}$
1.9	2.67868
1.95	2.72509
1.99	
1.995	2.76778
1.9999	2.77249
$\downarrow$	$\downarrow$
2	$E'(2)$
$\uparrow$	$\uparrow$
2.0001	2.77268
2.005	2.77740
2.01	
2.05	2.82119
2.1	2.87094



Explore 5.1.1 Compute the two entries corresponding to $b = 1.99$ and $b = 2.01$ that are omitted from Table 5.1. ■

Within the accuracy of Table 5.1, you may conclude that m_2 should be between 2.77249 and 2.77268. The average of 2.77249 and 2.77268 is a good estimate of m_2.

$$\frac{m_{[2,1.9999]} + m_{[2,2.0001]}}{2} = \frac{2.7724926 + 2.7726848}{2} = 2.7725887$$

As approximations to $E'(2)$, 2.7724926 and 2.7726848 are correct to only five digits, but their average, 2.7725887, is correct to all eight digits shown. Such improvement in accuracy by averaging left and right difference quotients (defined next) is common.

For P a function, the fraction,

$$\frac{P(b) - P(a)}{b - a},$$

is called a *difference quotient for* P. If $h > 0$ then the backward, and centered, and forward difference quotients at a are

Backward	Centered	Forward
$\dfrac{P(a) - P(a-h)}{h}$	$\dfrac{P(a+h) - P(a-h)}{2h}$	$\dfrac{P(a+h) - P(a)}{h}$

Assuming the interval size h is the same in all three, the centered difference quotient is the average of the backward and forward difference quotients.

$$\begin{aligned}
\frac{\dfrac{P(a-h) - P(a)}{-h} + \dfrac{P(a+h) - P(a)}{h}}{2} &= \frac{\dfrac{-P(a-h) + P(a)}{h} + \dfrac{P(a+h) - P(a)}{h}}{2} \\
&= \frac{-P(a-h) + P(a) + P(a+h) - P(a)}{2h} \\
&= \frac{P(a+h) - P(a-h)}{2h} \qquad (5.1)
\end{aligned}$$

The graphs in Figure 5.1 suggest, and it is generally true, that the centered difference quotient is a better approximation to the slope of the tangent to P at $(a, P(a))$ than is either the forward or backward difference quotients. Formal analysis of the errors in the two approximations appears in Example 9.7.2, Equations 9.21 and 9.22. We will use the centered difference quotient to approximate $E'(a)$ throughout this chapter.

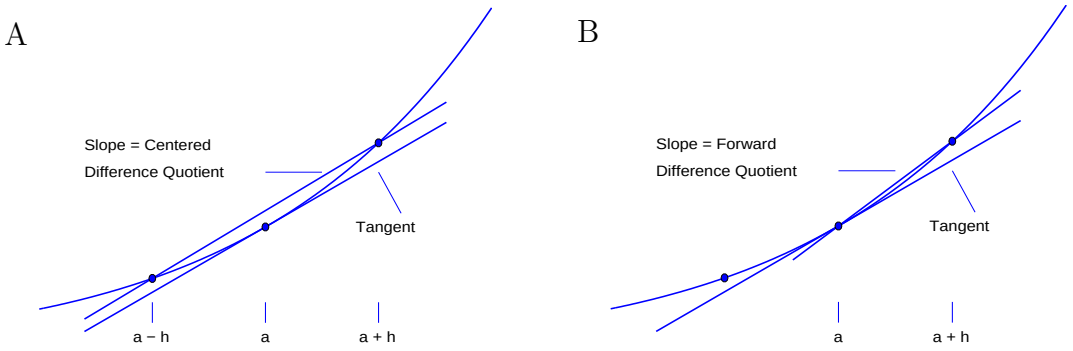


Figure 5.1: The centered difference quotient shown in A is closer to the slope of the tangent at $(a, f(a))$ than is the forward difference quotient shown in B.

There is not a simple integer or rational number that is $E'(2)$, the rate of change of $E(t) = 2^t$ at the time $t = 2$. There is however, an irrational number that is $E'(2)$; correct to twelve digits it is 2.77258872224.

Using the centered difference approximation, we approximate $E'(t)$ for $E(t) = 2^t$ and five different

value of t, -1, 0, 1, 2, and 3.

		Centered diff quot		$E'(t) \doteq$	$E'(t)$	$E(t)$
$E'(-1)$	$\doteq$	$\dfrac{2^{-1+0.0001} - 2^{-1-0.0001}}{0.0002}$	$=$	0.34657359	0.35	$\frac{1}{2}$
$E'(0)$	$\doteq$	$\dfrac{2^{0+0.0001} - 2^{0-0.0001}}{0.0002}$	$=$	0.69314718	0.69	1
$E'(1)$	$\doteq$	$\dfrac{2^{1+0.0001} - 2^{1-0.0001}}{0.0002}$	$=$	1.3862944	1.39	2
$E'(2)$	$\doteq$	$\dfrac{2^{2+0.0001} - 2^{2-0.0001}}{0.0002}$	$=$	2.7725887	2.77	4
$E'(3)$	$\doteq$	$\dfrac{2^{3+0.0001} - 2^{3-0.0001}}{0.0002}$	$=$	5.5451774	5.55	8

Explore 5.1.2 Do This. An elegant pattern emerges from the previous computations. To help you find it the last two columns contain truncated approximations to $E'(t)$ and the values of $E(t)$. Spend at least two minutes looking for the pattern (or less if you find it). ■

The pattern we hope you see is that

$$E'(t) \doteq E'(0) \times E(t)$$

We will soon show that for $E(t) = 2^t$, $E'(t) = E'(0) \times E(t)$, exactly. Meanwhile we observe that $E'(0) \doteq 0.69314718$ and

$$\begin{aligned} E'(0) \times E(-1) &\doteq 0.69314718 \times \frac{1}{2} = 0.34657359 &&\doteq E'(-1), \\ E'(0) \times E(1) &\doteq 0.69314718 \times 2 = 1.38629436 &&\doteq E'(1), \\ E'(0) \times E(2) &\doteq 0.69314718 \times 4 = 2.77258872 &&\doteq E'(2), \quad \text{and} \\ E'(0) \times E(3) &\doteq 0.69314718 \times 8 = 5.54517744 &&\doteq E'(3), \end{aligned}$$

which supports the pattern.

Explore 5.1.3 Let $E(t) = 3^t$. Use the centered difference quotient to approximate $E'(t)$ for $t = -1$, 0, 1, 2, 3. Test your numbers to see whether $E'(t) \doteq E'(0) \times E(t)$. ■

The previous work suggests a general rule:

Theorem 5.1.1 *If $E(t) = B^t$ where $B > 0$, then*

$$E'(t) = E'(0) \times E(t). \tag{5.2}$$

Proof: For $E(t) = B^t$, $B > 0$,

$$\begin{aligned} E'(t) &= \lim_{h\to 0} \frac{B^{t+h} - B^t}{h} \\ &= \lim_{h\to 0} \frac{B^t \times B^h - B^t}{h} \\ &= \lim_{h\to 0} \frac{B^t\left(B^h - 1\right)}{h} \\ &= B^t \lim_{h\to 0} \frac{B^h - B^0}{h} \\ &= E(t) \times E'(0) \end{aligned}$$

End of proof.

The preceding argument follows the pattern of all computations of derivatives using Definition 3.21. We write the difference quotient, $\frac{F(t+h)-F(t)}{h}$, balance h in the denominator with some term in the numerator, and let $h \to 0$. In Chapters 3 and 4 we always could factor an h from the numerator that algebraically canceled the h in the denominator. In this case, there is no factor, h, in the numerator, but

$$\text{as } h \to 0, \qquad \frac{B^h - B^0}{h} \to E'(0),$$

and h in the denominator is neutralized.

Exercises for Section 5.1 Derivatives of Exponential Functions.

Exercise 5.1.1 (a) Compute the centered difference $\dfrac{P(a+h) - P(a-h)}{2h}$ which is an approximation to $P'(a)$ for $P(t) = t^2$ and compare your answer with $P'(a)$. (b) Compute the centered difference $\dfrac{P(a+h) - P(a-h)}{2h}$ for $P(t) = 5t^2 - 3t + 7$ and compare your answer with $P'(a)$.

Exercise 5.1.2 Sketch the graphs of $y = 2^t$ and $y = 4 + 2.7725887(t-2)$

a. Using a window of $0 \le x \le 2.5$, $0 \le y \le 6$.

b. Using a window of $1.5 \le x \le 2.5$, $0 \le y \le 6$.

c. Using a window of $1.8 \le x \le 2.2$, $3.3 \le y \le 4.6$.

Mark the point (2,4) on each graph.

Exercise 5.1.3 Let $E(t) = 10^t$.

a. Approximate $E'(0)$ using the centered difference quotient on $[-0.0001, 0.0001]$.

b. Use your value for $E'(0)$ and $E'(t) = E'(0) \times E(t)$ to approximate $E'(-1)$, $E'(1)$, and $E'(2)$.

c. Sketch the graphs of $E(t)$ and $E'(t)$.

Exercise 5.1.4 Let $E(t) = \left(\frac{1}{2}\right)^t$.

a. Approximate $E'(0)$ using the centered difference quotient on $[-0.0001, 0.0001]$.

b. Use your value for $E'(0)$ and $E'(t) = E'(0) \times E(t)$ to approximate $E'(-1)$, $E'(1)$, and $E'(2)$.

c. Sketch the graphs of $E(t)$ and $E'(t)$.

Exercise 5.1.5 Find equations of the lines (approximately) tangent to the graphs of

a.	y	$=$	1.5^t	at the points	$(-1, 2/3)$,	$(0, 1)$,	and	$(1, 3/2)$
b.	y	$=$	2^t	at the point	$(-1, 1/2)$,	$(0, 1)$,	and	$(1, 2)$
c.	y	$=$	3^t	at the point	$(-1, 1/3)$,	$(0, 1)$,	and	$(1, 3)$
d.	y	$=$	5^t	at the point	$(-1, 1/5)$,	$(0, 1)$,	and	$(1, 5)$

Exercise 5.1.6 Suppose a bacterium *Vibrio natrigens* is growing in a beaker and cell concentration C at time t in minutes is given by

$$C(t) = 0.87 \times 1.02^t \qquad \text{million cells per ml}$$

a. Approximate $C(t)$ and $C'(t)$ for $t = 0$, 10, 20, 30, and 40 minutes.

b. Plot a graph of $C'(t)$ *vs* $C(t)$ using the five pairs of values you just computed.

Exercise 5.1.7 Suppose penicillin concentration in the serum of a patient t minutes after a bolus injection of 2 g is given by

$$P(t) = 200 \times 0.96^t \qquad \mu\text{g/ml}$$

a. Approximate $P(t)$ and $P'(t)$ for $t = 0$, 5, 10, 15, and 20 minutes.

b. Plot a graph of $P'(t)$ *vs* $P(t)$ using the five pairs of values you just computed.

5.2 The number e.

The implication

$$\text{For} \qquad E(t) = B^t \qquad \Longrightarrow \qquad E'(t) = E'(0) \times E(t)$$

would be even simpler if we find a value for B so that $E'(0) = 1$. We next find such a value for B. It is universally denoted by e and

$$e \quad \text{is approximately} \quad 2.71828182845904523536; \quad e \quad \text{is not a rational number.}$$

Then for $E(t) = e^t$, $E'(0) = 1$ so that $E'(t) = E'(0)E(t) = 1 \times E(t) = e(t)$.

$$\text{Thus for} \qquad E(t) = e^t \qquad E'(t) = e^t$$

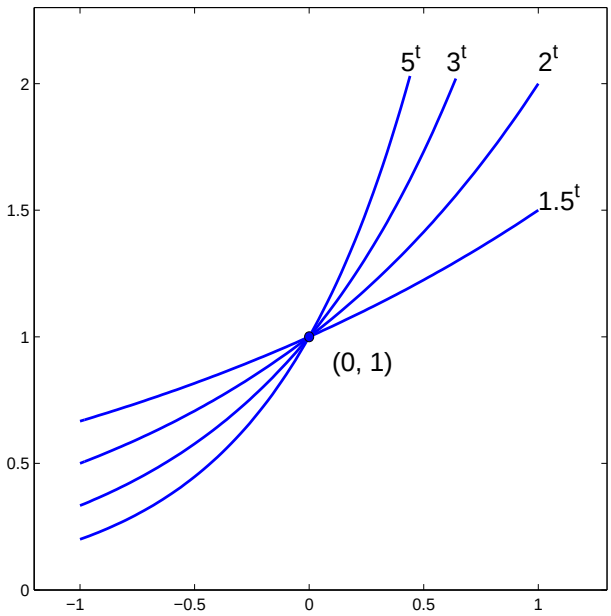


Figure 5.2: Graphs of exponential functions, $E_B(t) = B^t$, for $B = 1.5$, 2, 3, and 5.

We approximate $E'_B(0)$ for $B = 1.5$, 2, 3, and 5.

$$\begin{array}{lrcllrcl}
\text{If} & E_{1.5}(t) & = & 1.5^t & \text{then} & E'_{1.5}(0) & \doteq & 0.405465 \\
\text{If} & E_2(t) & = & 2^t & \text{then} & E'_2(0) & \doteq & 0.693417. \\
\text{If} & E_3(t) & = & 3^t & \text{then} & E'_3(0) & \doteq & 1.098612 \\
\text{If} & E_5(t) & = & 5^t & \text{then} & E'_5(0) & \doteq & 1.609438
\end{array}$$

The values of $E'_B(0)$ change with the base B. Shown in Figure 5.2 are graphs of the preceding four exponential functions.

Explore 5.2.1 For $E(t) = e^t$, $E(0) = 1$ and $E'(0) = 1$, and the line of slope 1 through (0,1) is tangent to E. Draw a line of slope 1 through the point (0,1). Then draw a graph of an exponential function whose tangent at (0,1) is the line you drew. Compare your graph with those in Figure 5.2. ■

Our goal is to find a base, e, so that for $E(t) = e^t$, $E'(0) = 1$. Because

$$\begin{array}{lrcllcl}
\text{For} & E_2(t) & = & 2^t, & E'_2(0) & \doteq & 0.693417 < 1 \\
\text{and for} & E_3(t) & = & 3^t, & E'_3(0) & \doteq & 1.098612 > 1,
\end{array}$$

we think the base e that we seek is between 2 and 3 and perhaps closer to 3 than to 2.

We will find that

$$e = \lim_{h \to 0} (1 + h)^{\frac{1}{h}} \tag{5.3}$$

Heuristic analysis. Assume there is a number e for which $E(t) = e^t$ implies that $E'(0) = 1$. Then we know that

$$\lim_{h\to 0} \frac{e^h - e^0}{h} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Intuitively then, for h close to zero,

$$\begin{aligned} \frac{e^h - 1}{h} &\doteq 1 \\ e^h - 1 &\doteq h \\ e^h &\doteq 1 + h \\ e &\doteq (1+h)^{\frac{1}{h}} \end{aligned}$$

Values of $(1+h)^{\frac{1}{h}}$ for progressively smaller values of h are shown in Table 5.2. You should see that as $h > 0$ decreases toward 0, $(1+h)^{1/h}$ increases. We show in Subsection 5.2.1 that $(1+h)^{\frac{1}{h}}$ approaches a number as h approaches 0, and we denote that number by e.

Table 5.2: Approximations to the number e. As h moves toward 0, $B = (1+h)^{1/h}$ increases toward e.

h	$B = (1+h)^{1/h}$
1.000	2.00000
0.500	2.25000
0.200	2.48832
0.100	2.59374
0.050	2.65329
0.010	2.70481
0.005	2.71152
0.001	2.71692
$\downarrow$	$\downarrow$
0	e

Explore 5.2.2 Compute $B = (1+h)^{1/h}$ for $h = 0.0001$, $h = 0.00001$, and $h = 0.000001$. ■

Your last estimate of B should be approximately 2.71828047, which is an estimate correct to 6 digits of the irrational number e that we are seeking.

Definition 5.2.1 The number e The number e is defined by

$$\lim_{h\to 0} \left(1+h\right)^{\frac{1}{h}} = e$$

Correct to 21 digits, $e = 2.71828\ 18284\ 59045\ 23536$

We show in Subsection 5.2.1 that if $h = \frac{1}{n}$ for n an integer greater than 2

$$(1+h)^{\frac{1}{h}} < e < \left(1+\frac{h}{1-h}\right)^{\frac{1}{h}}, \tag{5.4}$$

and we assume the inequalities are valid for all $0 < h < 1/2$. Using these inequalities it is easy to show that the function $E(t) = e^t$ has the property that $E'(0) = 1$.

We write

$$\begin{array}{rcccl}
 & (1+h)^{\frac{1}{h}} & < & e & < & \left(1+\frac{h}{1-h}\right)^{\frac{1}{h}} \\
 & 1+h & < & e^h & < & 1+\frac{h}{1-h} \\
 & h & < & e^{\frac{1}{n}}-1 & < & \frac{h}{1-h} \\
 & 1 & < & \frac{e^h-1}{h} & < & \frac{1}{1-h} \\
\text{As } h\to 0 & \downarrow & & \downarrow & & \downarrow \\
 & 1 & \leq & E'(0) & \leq & 1
\end{array}$$

The difference quotient, $\left(e^h-1\right)/h$, is bounded between 1 and a number $1/(1-h)$ that is approaching 1 as h approaches zero. This insures that $\left(e^h-1\right)/h$ also approaches 1 as h approaches zero. Thus $E'(0) = 1$.

Because of Theorem 5.1.1, $E'(t) = E'(0) \times E(t) = e^t$, we have another Primary Formula for computing derivatives:

$$\left[e^t\right]' = e^t \qquad \textbf{Exponential Rule} \tag{5.5}$$

Strategy for computing derivatives: Now we can use three Primary Formulas (Constant, Power, Exponential) and three Composition Formulas (Sum, Constant Factor, Power Chain) to compute derivatives. In finding derivatives of functions with many terms, students sometimes ask what derivative rule to use first and in subsequent steps. We think of the derivative procedure as peeling the layers off of an onion – outside layer first, etc. For example, to compute the derivative of $F(t) = \left(2+3t^2+5e^t\right)^3$ we

write:

$$\left.\begin{aligned}
F'(t) &= \left[(\,2+3t^2+5e^t\,)^3\right]' && \text{Symbolic Identity}\\
&= 3\,(\,2+3t^2+5e^t\,)^2\,[\,2+3t^2+5e^t\,]' && \text{Power Chain Rule}\\
&= 3\,(\,2+3t^2+5e^t\,)^2\left([2]'+[3t^2]'+[5e^t]'\right) && \text{Sum Rule}\\
&= 3\,(\,2+3t^2+5e^t\,)^2\left(0+[3t^2]'+[5e^t]'\right) && \text{Constant Rule}\\
&= 3\,(\,2+3t^2+5e^t\,)^2\left(3\,[t^2]'+5\,[e^t]'\right) && \text{Constant Factor Rule}\\
&= 3\,(\,2+3t^2+5e^t\,)^2\left(3\,(2t)+5\,[e^t]'\right) && \text{Power Rule}\\
&= 3\,(\,2+3t^2+5e^t\,)^2\left(6t+5e^t\right) && \text{Exponential Rule}
\end{aligned}\right\} \tag{5.6}$$

Think how the expression for F in the previous example, $F(t) = (2+3t^2+5e^t)^3$, is evaluated. Given a value for t you would compute t^2 and multiply it by 3 and you would compute e^t and multiply it by 5, and then you would sum the three terms. The *last step* (outside layer) in the evaluation is to cube the sum. The *first step* in computing the derivative is to 'undo' that cube (use the Power Chain Rule).

The next step is to undo the sum with the Sum Rule. The next step is not uniquely determined; we worked from left to right and chose to evaluate $[2]'$.

In evaluating $3t^2$ and $5e^t$, the last step would be to multiply by 3 or 5 – the first step in finding the derivative of $3t^2$ and $5e^t$ is to factor 3 and 5 from the derivative (Constant Factor Rule),

$$\left[3t^2\right]' = 3\left[t^2\right]' \qquad\qquad \left[5e^t\right]' = 5\left[e^t\right]'$$

Finally the Primary Formulas for $[t^2]'$ and $[e^t]'$ are used.

Usual course of events.

In computing derivatives, use the Composition Formulas first and the Primary Formulas last.

5.2.1 Proof that $\lim\limits_{\mathbf{h\to 0}} (1+h)^{1/h}$ exists.

We prove that $(1+h)^{1/h}$ approaches a number as h approaches 0 (we denoted that number by e). **You may accept this fact without proof and delay or omit study of this Subsection.**

The argument is based on

Axiom 1 Completeness property of the number system. *If S_1 and S_2 are two sets of numbers and*

1. *Every number belongs to either S_1 or S_2, and*
2. *Every number of S_1 is less than every number of S_2,*

then there is a number C such that C is either the largest number of S_1 or C is the least number of S_2.

The statement of the Completeness Axiom by Richard Dedekind in 1872 greatly increased our understanding of the number system. In this text, the word 'set' means a nonempty set.

If A and B are numbers with $A < B$, the open interval (A, B) consists of all the numbers between A and B; the closed interval $[A, B]$ consists of A, B, and all of the numbers between A and B. That a sequence of numbers s_1, s_2, $s_3 \cdots$ is bounded means that an open interval (A, B) contains every number in s_1, s_2, $s_3 \cdots$; A is called a lower bound and B is called an upper bound of s_1, s_2, $s_3 \cdots$.

The sequence $\{1, 4, 9, 16, 25, \cdots\}$ has no upper bound and is not bounded. The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ is bounded by the numbers 0 and 2. The sequence $\{1, -2, 3, -4, 5, -6, \cdots\}$ has neither a lower bound nor an upper bound and is not bounded.

For this section we prove:

Theorem 5.2.1 *If $s_1 \leq s_2 \leq s_3 \leq \cdots$ is a bounded nondecreasing sequence of numbers there is a number s such that s_1, s_2, $s_3 \cdots$ approaches s.*

To prove Theorem 5.2.1 we need a clear definition of 'approaches'.

Definition 5.2.2 *The number sequence s_1, s_2, $s_3 \cdots$ approaches s means that if (u, v) is an open interval containing s there is a positive integer N such that if n is an integer greater than N, s_n is in (u, v).*

The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ approaches zero. If (u, v) contains 0, v is greater than zero and there is[1] a positive integer N such that N is greater than $1/v$. Therefore, if $n > N$, $n > 1/u$ so that $1/n < u$, and $u < 0 < 1/n < v$ ($\frac{1}{n}$ is in (u, v)).

Proof of Theorem 5.2.1. Suppose s_1, s_2, $s_3 \cdots$ is a nondecreasing sequence bounded by A and B. Let S_1 be the set of numbers x for which some number of s_1, s_2, $s_3 \cdots$ is greater than x. A is a member of S_1 and B is not a member of S_1. Let S_2 denote all of the numbers not in S_1 Clearly every number is in either S_1 or S_2.

To use the Completeness Axiom we must show that every number of S_1 is less than every number of S_2. Suppose not; suppose there is a number x of S_1 that is greater than a number y of S_2. Then x has the property that some number, s_n, say of s_1, s_2, $s_3 \cdots$ is greater than x and y is not in S_1 so no number of s_1, s_2, $s_3 \cdots$ is greater than y. However, the supposition that $y < x$ and $x < s_n$, leads to $y < s_n$ which is a contradiction. Therefore every number of S_1 is less than every number of S_2.

[1] This is called the Archimedean property of the integers and may be treated as an axiom also. It is, however, a consequence of the Completeness Axiom, Exercise 5.2.9

By the Completeness Axiom, either S_1 has a largest number or S_2 has a least number. Suppose S_1 has a largest number L_1. By definition of S_1 there is a member s_n of s_1, s_2, $s_3 \cdots$ greater than L_1. Now $\frac{L_1 + s_n}{2}$ is less than s_n so $\frac{L_1 + s_n}{2}$ is in L_1. But $\frac{L_1 + s_n}{2}$ is greater than L_1 so L_1 is not the largest member of S_1, which is a contradiction; S_1 does not have a largest number. Therefore S_2 must have a least number L_2.

We prove that s_1, s_2, $s_3 \cdots$ approaches L_2. Suppose (u, v) is an open interval containing L_2. Then u is less than L_2 so belongs to S_1 and there is a number, S_N in s_1, s_2, $s_3 \cdots$ that is greater than u. Because s_1, s_2, $s_3 \cdots$ is increasing, if n is greater than N, $S_N \leq s_n$. No number of s_1, s_2, $s_3 \cdots$ is greater than L_2. Therefore if n is greater than N,

$$u < s_N \leq s_n < L_2 < v$$

and the definition that s_1, s_2, $s_3 \cdots$ approaches L_2 is satisfied.
End of proof.

Example 5.2.1 We can use Theorem 5.2.1 to show a useful result that if a is a number and $0 < a < 1$, then the sequence $x_n = a^n$ converges to zero.

Proof. We assume the alternate version of Theorem 5.2.1 that If $s_1 \leq s_2 \leq s_3 \leq \cdots$ is a bounded nonincreasing sequence of numbers there is a number s such that s_1, s_2, $s_3 \cdots$ approaches s.

Because $0 < a < 1$ and $x_{n+1} = a \cdot x_n$, $x_{n+1} < x_n$, and it follows that x_n is a nonincreasing (actually decreasing) sequence. Then $\{x_n\}$ approaches the greatest lower bound, s of $\{x_n\}$. If $s < 0$, there is a number, x_m in $\{x_n\}$ that is between s and 0. Then $x_m = a^m$ is negative which is a contradiction.

Suppose $s > 0$. Because $0 < a < 1$), the number s/a is greater than s and there is a number, x_m in $\{x_n\}$ that is between s and s/a. Then

$$s < x_m < s/a, \qquad s < m^a < s/a, \qquad s \cdot a < m^{a+1} < s, \qquad x_{m+1} < s.$$

But this contradicts the condition that s is a lower bound on $\{x_n\}$. ■

Now to the number, e. We are to show that $(1 + h)^{1/h}$ approaches a number as h approaches 0, but only consider the values of h for $h = \frac{1}{n}$ where n an integer greater than 2. This restriction is acceptable because $(1 + h)^{1/h}$ strictly increases as h decreases on $0 < h < \frac{1}{2}$, a fact that we assume without proof. We will show that

1. The sequence $s_n = \left(1 + \frac{1}{n}\right)^n$ is an increasing sequence.
2. The sequence $t_n = \left(1 + \frac{1}{n-1}\right)^n$, $n > 1$, is a decreasing sequence.
3. For every $n > 1$, $s_n < t_n$.
4. As n increases without bound, $t_n - s_n$ approaches 0.

Conditions 1 and 3 show that $s_1 < s_2 < s_3 < \cdots$ is a bounded increasing sequence and therefore there is a number s such that $s_1 < s_2 < s_3 < \cdots$ approaches s. That number s is the number we denote by e. Conditions 2 and 3 show that $t_1 > t_2 > t_3 > \cdots$ is a bounded decreasing sequence and it follows from

Theorem 5.2.1[2] that there is a number t such that $t_1 > t_2 > t_3 > \cdots$ converges to t By condition 4 above, $t = s$.

The proofs of conditions 3 and 4 above are left as exercises. Our argument for conditions 1 and 2 is that of N. S. Mendelsohn[3] based on the following theorem. The theorem is of general interest and we prove it in Subsection 8.1.2, independently of the work in this section.

Theorem 5.2.2 *If a_1, a_2, $\cdots$, a_n is a sequence of n positive numbers then*

$$\frac{a_1 + a_2 + \cdots a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \tag{5.7}$$

with equality only when $a_1 = a_2 = \cdots = a_n$.

The left side of inequality 5.7 is the *arithmetic mean* and the right side is the *geometric mean* of a_1, a_2, $\cdots$, a_n. The theorem states that the arithmetic mean is greater than or equal to the geometric mean.

Proof that $\left(1 + \frac{1}{n}\right)^n$ is increasing. Consider the set of $n + 1$ numbers

$$1, \quad 1 + \frac{1}{n}, \qquad 1 + \frac{1}{n} \quad \cdots \quad 1 + \frac{1}{n}$$

They are not all equal and they have an arithmetic mean of $1 + 1/(n + 1)$ and a geometric mean of $(\,1 \times (1 + 1/n)^n)^{1/(n+1)}$. Then

$$1 + \frac{1}{n+1} > (\,(1 + 1/n)^n)^{1/(n+1)} \qquad \text{or} \qquad \left(1 + \frac{1}{n+1}\right)^{n+1} > (1 + 1/n)^n$$

Hence $s_{n+1} > s_n$ and the sequence s_1, s_2, $\cdots$ is increasing.

Proof that $\left(1 + \frac{1}{n-1}\right)^n$ is decreasing. Consider the set of $n + 1$ numbers

$$1, \quad \frac{n-1}{n}, \qquad \frac{n-1}{n} \quad \cdots \quad \frac{n-1}{n}$$

They have an arithmetic mean of $n/(n + 1)$ and a geometric mean of $(\,1 \times (\,(n - 1)/n\,)^n\,)^{1/(n+1)}$ and

$$\frac{n}{n+1} > \left(\frac{n-1}{n}\right)^{n/(n+1)}$$

By taking reciprocals this becomes

$$\frac{n+1}{n} < \left(\frac{n}{n-1}\right)^{n/(n+1)} \qquad \text{or} \qquad \left(1 + \frac{1}{(n+1)-1}\right)^{n+1} < \left(1 + \frac{1}{n-1}\right)^n$$

It follows that $t_{n+1} < t_n$ and t_1, t_2, $\cdots$ is decreasing. End of proof.

We have denoted the common number $t = s$ by e and from 1 and 2 above $s_n < e < t_n$. To relate the notation to the Definition 5.2.1

$$e = \lim_{h \to 0} \; (1 + h)^{\frac{1}{h}}$$

[2]Observe that $-t_1 < -t_2 < -t_3 < \cdots$ is a bounded increasing sequence and there is a number $-t$ such that $-t_1 < -t_2 < -t_3 < \cdots$ approaches $-t$.

[3]American Mathematical Monthly, **58** (1951) p. 563.

we substitute $h = 1/n$ and $n = 1/h$ into s_n and t_n and write

$$\begin{array}{rcccl} s_n & < & e & < & t_n \\ \left(1+\frac{1}{n}\right)^n & < & e & < & \left(1+\frac{1}{n-1}\right)^n \\ (1+h)^{\frac{1}{h}} & < & e & < & \left(1+\frac{h}{1-h}\right)^{\frac{1}{h}} \end{array}$$

Exercises for Section 5.2, The number e.

Exercise 5.2.1 Derivatives of functions are computed below. Identify the rule used in each step. In a few steps the rule is an algebraic rule of exponents and not a derivative rule.

a.
$$\begin{array}{l} [\,5t^4 - 7e^t\,]' \\ [\,5t^4]' - [7e^t]' \\ 5\,[\,t^4]' - 7\,[e^t]' \\ 5 \times 4t^3 - 7\,[e^t]' \\ 5 \times 4t^3 - 7 \times e^t \end{array}$$

b.
$$\begin{array}{l} \left[\,(1+e^t)^8\,\right]' \\ 8\,(1+e^t)^7\,[\,1+e^t\,]' \\ 8\,(1+e^t)^7\left([\,1\,]' + [\,e^t\,]'\right) \\ 8\,(1+e^t)^7\left(0 + [\,e^t\,]'\right) \\ 8\,(1+e^t)^7\,(0+e^t) \\ 8e^t\,(1+e^t)^7 \end{array}$$

c.
$$\begin{array}{l} [\,e^{3t}\,]' \\ \left[\,(e^t)^3\,\right]' \\ 3\,(e^t)^2\,[\,e^t\,]' \\ 3\,(e^t)^2 \times e^t \\ 3e^{2t} \times e^t \\ 3e^{3t} \end{array}$$

Exercise 5.2.2 Differentiate (means compute the derivative of) P. Use one rule for each step and identify the rule as, C (Constant Rule), t^n (t^n Rule), S (Sum Rule), CF (Constant Factor Rule), PC (Power Chain Rule), or E (Exponential Rule). For example,

$$\begin{array}{rcll} [\pi t^{-2} - 5(e^t)^7]' & = & [\pi t^{-2}]' - [5(e^t)^7]' & \text{S} \\ & = & \pi\,[t^{-2}]' - 5\,[(e^t)^7]' & \text{CF} \\ & = & \pi \times (-2)t^{-3} - 5\,[(e^t)^7]' & t^n \\ & = & -2\pi t^{-3} - 5(7)(e^t)^6\,[e^t]' & \text{PC} \\ & = & -2\pi t^{-3} - 35(e^t)^6 \times e^t & \text{E} \\ & = & -2\pi t^{-3} - 35(e^t)^7 & \text{Algebra} \end{array}$$

a. $P(t) = 5t^2 + 32e^t$ b. $P(t) = 3\left(e^t\right)^5 - 6t^5$

c. $P(t) = 7 - 8\left(e^t\right)^{-1}$ d. $P(t) = \frac{2}{5} + \frac{t}{3}$

e. $P(t) = t^{25} + 3e$ f. $P(t) = \frac{4}{e} + \frac{t^5}{7}$

g. $P(t) = 1 + t + \frac{t^2}{2} - e^t$ h. $P(t) = \frac{\left(e^t\right)^2}{2} + \frac{t^3}{3}$

Careful, some students have made errors on the next two.

i. $P(t) = 5e^3$ j. $P(t) = 10 \times \pi^2$

Exercise 5.2.3 On a graphing calculator or computer, draw the graphs of

$$y_1(t) = e^t \qquad y_2(t) = t^2 \qquad y_3(t) = t^3 \qquad y_4(t) = t^4 \qquad -1 \le t \le 5$$

The graphs are close together near $t = 0$ and increase as t increases. Which one grows the most as t increases? Expand the domain and range to $-1 \le t \le 10$, $0 \le y \le 25,000$, and answer the same question.

Exercise 5.2.4 On a graphing calculator or computer, draw the graphs of

$$y(t) = e^t \qquad \text{and} \qquad p(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}$$

Set the domain and range to $-1 \le t \le 2$. and $0 \le y \le 8$.

Exercise 5.2.5 We found a base e so that $E(t) = e^t$ has the property that the rate of change of E at 0 is 1. Suppose we had searched for a number B so that the *average* rate of change of $E_B(t) = B^t$ on $[0, 0.01]$ is 1:

$$m_{0,0.01} = \frac{E_B(0.01) - E_B(0)}{0.01} = \frac{B^{0.01} - B^0}{0.01} = \frac{B^{0.01} - 1}{0.01} = 1.$$

a. Solve the last equation for B.

b. Solve for B in each of the equations:

$$\frac{B^{0.001} - 1}{0.001} = 1 \qquad \frac{B^{0.00001} - 1}{0.00001} = 1 \qquad \frac{B^{0.0000001} - 1}{0.0000001} = 1$$

Exercise 5.2.6 Let $y(x) = e^x$. Compute $y'(x)$, $y''(x) = \left(y'\right)'$, and $y'''(x)$.

Exercise 5.2.7 We introduced the power chain rule $[(u(x))^n]' = n(u(x))^{n-1}[u(x)]'$ for fractional and negative exponents, n, in Section 4.3.1 (see Exercises 4.3.3 and 4.3.4). Use this rule when necessary in the following exercise.

Compute $y'(x)$ and $y''(x)$ for

a. $y(x)$	$=$	$x^2 + e^x$	b. $y(x)$	$=$	$3x^2 + 2e^x$
c. $y(x)$	$=$	$(1 + e^x)^2$	d. $y(x)$	$=$	$(e^x)^2$
e. $y(x)$	$=$	$e^{2x} \quad = (e^x)^2$	f. $y(x)$	$=$	$e^{-x} \quad = (e^x)^{-1}$
g. $y(x)$	$=$	$e^{3x} \quad = (e^x)^3$	h. $y(x)$	$=$	$e^x \times e^{2x}$
i. $y(x)$	$=$	$(5 + e^x)^3$	j. $y(x)$	$=$	$\frac{1}{1+e^x} \quad = (1 + e^x)^{-1}$
k. $y(x)$	$=$	$\sqrt{e^x} \quad = (e^x)^{\frac{1}{2}}$	l. $y(x)$	$=$	$e^{\frac{1}{2}x}$
m. $y(x)$	$=$	$e^{0.6x} \quad = (e^x)^{0.6}$	n. $y(x)$	$=$	$e^{-0.005x}$

Exercise 5.2.8 Identify the errors in the following derivative computations.

a. $\left[(t^4 + t^{-1})^7\right]'$	b. $[5t^7 + 7t^{-5}]'$	c. $[10t^8 + 8e^{5t}]'$
$7(t^4 + t^{-1})^6 [t^4 + t^{-1}]'$	$[5t^7]' + [7t^{-5}]'$	$[10t^8]' + [8e^{5t}]'$
$7(t^4 + t^{-1})^6 [t^4]' + [t^{-1}]'$	$5[t^7]' + 7[t^{-5}]'$	$10[t^8]' + 8[e^{5t}]'$
$7(t^4 + t^{-1})^6 4t^3 + (-1)t^{-2}$	$5 \times 7t^6 + 7 \times (-5)t^{-4}$	$10 \times 8t^7 + 8 \times 5e^{4t}$
$28t^3 (t^4 + t^{-1})^6 - t^{-2}$	$35(t^6 - t^{-4})$	$40(2t^7 + e^{4t})$

Exercise 5.2.9 Use the Completeness Axiom 1 to show that the positive integers do not have an upper bound. (This is called the Archimedean Axiom).

Exercise 5.2.10 Argue that if S_1 and S_2 are two sets of numbers and every number is in either S_1 or S_2 and every number in S_1 is less than every number in S_2 then it is not true that there are numbers L_1 and L_2 such that L_1 is the greatest number in S_1 and L_2 is the least number in S_2. Is this a contradiction to the Completeness Axiom?

Exercise 5.2.11 Let S_2 denote the points of the X-axis that have positive x-coordinate and S_1 denote the points of the X-axis that do not belong to S_2. Does S_2 have a left most point?

Exercise 5.2.12 Suppose S_2 is the set of numbers to which x belongs if and only if x is positive and $x^2 > 2$ and S_1 consists of all of the other numbers.

1. Give an example of a number in S_2.
2. Give an example of a number in S_1.

3. Argue that every number in S_1 is less than every number in S_2.
4. Which of the following two statements is true?
   (a) There is a number C which is the largest number in S_1.
   (b) There is a number C which is the least number in S_2.
5. Identify the number C in the correct statement of the previous part.

Exercise 5.2.13 Suppose your number system is that of Early Greek mathematicians and includes only rational numbers. Does it satisfy the Axiom of Completion?

Exercise 5.2.14 Show that if $n \geq 2$ and $s_n = \left(1 + \frac{1}{n}\right)^n$ and $t_n = \left(1 + \frac{1}{n-1}\right)^n$ then

a. For every n $s_n < t_n$.

b. Justify the steps 1 to 4 in

$$\begin{aligned} t_n - s_n &\overset{1}{=} \left(1 + \frac{1}{n-1}\right)^n - \left(1 + \frac{1}{n}\right)^n \\ &\overset{2}{=} \left[\left(1 + \frac{1}{n-1}\right) - \left(1 + \frac{1}{n}\right)\right] \times \\ &\left[\left(1 + \frac{1}{n-1}\right)^{n-1} + \left(1 + \frac{1}{n-1}\right)^{n-2}\left(1 + \frac{1}{n}\right) + \cdots + \left(1 + \frac{1}{n-1}\right)\left(1 + \frac{1}{n}\right)^{n-2} + \left(1 + \frac{1}{n}\right)^{n-1}\right] \\ &\overset{3}{\leq} \frac{1}{n(n-1)} \times n \times \left(1 + \frac{1}{n-1}\right)^{n-1} \\ &\overset{4}{\leq} \frac{1}{n-1} \times 4 \end{aligned}$$

c. As n increases without bound, $t_n - s_n$ approaches zero.

5.3 The natural logarithm

The *natural* logarithm function, ln, is defined for $u > 0$ by

$$\lambda = \ln u = \log_e u \Longleftrightarrow u = e^\lambda \tag{5.8}$$

The natural logarithm is the logarithm to the base e and the properties of logarithms for all bases apply:

$$\ln(A \times B) = \ln(A) + \ln(B) \tag{5.9}$$

$$\ln(A/B) = \ln(A) - \ln(B) \tag{5.10}$$

$$\ln(A^c) = c \times \ln(A) \tag{5.11}$$

$$\text{If } d > 0,\ d \neq 1, \text{ then} \qquad \log_d A = \frac{\ln A}{\ln d} \tag{5.12}$$

$$u = e^{\ln u} \tag{5.13}$$

$$\ln(e^\lambda) = \lambda \tag{5.14}$$

The natural logarithm is keyed on your calculator as ln or LN. Equation 5.12 shows that all logarithms may be calculated using just the natural logarithm. However, $\log_{10}$ is also keyed on most calculators and the key may be labeled Log or LOG.

Using Equation 5.13 we can express all exponential functions in terms of the base e. To see this, suppose $B > 0$ and $E(t) = B^t$. Equation 5.13 states that

$$B = e^{\ln B}$$

so we may write $E(t) = B^t$ as

$$E(t) = B^t = \left(e^{\ln B}\right)^t = e^{t \ln B}$$

For example, $\ln 2 \doteq 0.631472$. For $E(t) = 2^t$ we have

$$E(t) = 2^t = \left(e^{\ln 2}\right)^t = e^{t\ \ln 2} \doteq e^{0.631472\, t}$$

As a consequence, functions of the form $E(t) = e^{kt}$, k a constant, are very important and we compute their derivative in the next section.

Example 5.3.1 We found in Chapter 1 that cell density (measured by light absorbance, Abs) of *Vibrio natrigens* growing in a flask at pH 6.25, data in Table 1.1, was described by (Equation 1.5)

$$\text{Abs} = 0.0174 \times 1.032^{\text{Time}}$$

Using natural logarithm, we write this in terms of e as

$$\begin{aligned}
\text{Abs} &= 0.0174 \times 1.032^{\text{Time}} \\
\text{Abs} &= 0.0174 \times \left[e^{\ln 1.032}\right]^{\text{Time}} && \text{Equation 5.13} \\
\text{Abs} &= 0.0174 \times \left[e^{0.03150}\right]^{\text{Time}} && \ln 1.022 \doteq 0.02176 \\
\text{Abs} &= 0.0174 \times e^{0.03150 \times \text{Time}} && \text{Equation 5.11.} \ \blacksquare
\end{aligned}$$

Exercises for Section 5.3, The natural logarithm.

Exercise 5.3.1 You should have found in Explore 1.6.1 that plasma penicillin during 20 minutes following injection of two grams of penicillin could be computed as

$$P(T) = 200 \times 0.77^T \qquad T = \text{index of five minute intervals.}$$

a. Write P in the form of $P(T) = 200\, e^{-k_1 \times T}$.

b. Write P in the form of $P(t) = 200\, e^{-k_2 \times t}$ where t measures time in minutes.

Exercise 5.3.2 Use Equation 5.12, $\log_d A = \ln A / \ln d$ to compute $\log_2 A$ for $A = 1, 2, 3, \cdots, 10$.

Exercise 5.3.3 We found in Section 1.3 that light intensity, I_d, as a function of depth, d was given by

$$I_d = 0.4 \times 0.82^d$$

Find k so that $I_d = 0.4e^{k \times d}$.

Exercise 5.3.4 Write each of the following functions in the form $f(t) = Ae^{kt}$.

a.	$f(t) = 5 \times 10^t$		b.	$f(t) = 5 \times 10^{-t}$		c.	$f(t) = 7 \times 2^t$	
d.	$f(t) = 5 \times 2^{-t}$		e.	$f(t) = 5 \times \left(\frac{1}{2}\right)^t$		f.	$f(t) = 5 \times \left(\frac{1}{2}\right)^{-t}$	

Exercise 5.3.5 Use the Properties of Logarithms, Equations 5.9 - 5.14 to write each of the following functions in the form $f(t) = A + B \ln t$.

a.	$f(t) = 5 \log_{10} t$		b.	$f(t) = 5 \log_2 t^3$		c.	$f(t) = 7 \log_5 5t$	
d.	$f(t) = 5 \log_{10} 3t$		e.	$f(t) = 3 \log_4 (t/2^3)$		f.	$f(t) = 3 \log_8 (16t^10)$	

5.4 The derivative of e^{kt}.

We have found the derivative of e^t. Often, however, the function of interest is of the form $C \times e^{kt}$ where C and k are constants. In Example 5.3.1 of bacterial growth,

$$\text{Abs} = 0.0174 \times e^{0.02176 \times \text{Time}}$$

the constant $C = 0.0174$ and the function $kt = 0.02176 \times \text{Time}$. We develop a formula for $\left[\, e^{kt} \,\right]'$.

$$\left[\, e^{kt} \,\right]' \stackrel{a}{=} \left[\, (\, e^t)^k \,\right]' \stackrel{b}{=} k\ (\, e^t)^{k-1}\ [\, e^t \,]' \stackrel{c}{=} k\ (\, e^t)^{k-1}\ e^t \stackrel{d}{=} k\, e^{kt}. \tag{5.15}$$

Because Equation 5.15 $\left[e^{k\,t}\right]' = k\, e^{k\,t}$, is used so often, we call it another Primary Formula even though we developed it without direct reference to the Definition of Derivative. **Should you be limited to a single derivative rule, in the life sciences choose the e^{kt} Rule** – exponential functions are pervasive in biology.

$\mathbf{e^{kt}}$ Rule

$$E(t) = e^{k\,t} \quad \Rightarrow \quad E'(t) = e^{k\,t} \times k \qquad \left[e^{k\,t}\right]' = k\, e^{k\,t} \tag{5.16}$$

Explore 5.4.1 Were we to derive $\left[\, e^{kt} \,\right]' = k\, e^{kt}$ from the Definition of Derivative, we would write:

$$\left[\, e^{kt} \,\right]' \stackrel{a}{=} \lim_{b \to a} \frac{e^{kb} - e^{ka}}{b - a} \stackrel{b}{=} \lim_{b \to a} \frac{e^{kb} - e^{ka}}{kb - ka}\, k \stackrel{c}{=} e^{ka}\, k \tag{5.17}$$

The assertion in step c that

$$\lim_{b \to a} \frac{e^{kb} - e^{ka}}{kb - ka} = e^{ka}$$

is correct, puzzles some students, and is worth your thought. ■

We can now differentiate functions like $P(t) = 5t^7 + 3e^{2\,t}$.

$$\begin{aligned}
P'(t) &= [5t^7 + 3e^{2\,t}]' && \text{A symbolic identity.} \\
&= [5t^7]' + [3e^{2\,t}]' && \text{Sum Rule} \\
&= 5\,[t^7]' + 3\,[e^{2\,t}]' && \text{Constant Factor Rule} \\
&= 5 \times 7t^6 + 3\,[e^{2\,t}]' && \text{Power Rule} \\
&= 5 \times 7t^6 + 3e^{2\,t} \times 2 && e^{kt} \text{ Rule} \\
&= 35t^6 + 6e^{2\,t}
\end{aligned}$$

Example 5.4.1 We can also compute $E'(t)$ for $E(t) = 2^t$.

$$[2^t]' = \left[\left(e^{\ln 2}\right)^t\right]' = \left[e^{(\ln 2)\times t}\right]' = e^{(\ln 2)\,t} \times \ln 2 = 2^t \times \ln 2$$

We have an exact solution for the first problem of this Chapter, which was to find $E'(2)$ for $E(t) = 2^t$. The answer is $2^2 \times \ln 2 = 4 \ln 2$. Also, $E'(0) = 2^0 \times \ln 2 = \ln 2$ which answers another question from early in the chapter. ■

More generally, for $b > 0$,

$$[\, b^t \,]' \stackrel{a}{=} \left[\, \left(e^{\ln b}\right)^t \right]' \stackrel{b}{=} \left[\, e^{(\ln b)\times t} \,\right]' \stackrel{c}{=} e^{(\ln b)\,t} \times \ln b \stackrel{d}{=} b^t \times \ln b \tag{5.18}$$

We summarize this information:

$$\boxed{\left[\, b^t \,\right]' = b^t\, \ln b \qquad \text{for } b > 0. \qquad (5.19)}$$

Explore 5.4.2 This is very important. Show that if C and k are constants and $P(t) = Ce^{kt}$ then $P'(t) = kP(t)$. ■

Exercises for Section 5.4, The derivative of e^{kt}.

Exercise 5.4.1 Give reasons for the steps a - c in Equation 5.15 showing that $\left[e^{kt}\right]' = e^{kt}\ k$.

Exercise 5.4.2 Give reasons for the steps a - d in Equation 5.18 showing that $[b^t]' = b^t \ln b$.

Exercise 5.4.3 The function b^t for $b = 1$ is a special exponential function. Confirm that the derivative equation $[\,b^t\,]' = b^t \ln b$ is valid for $b = 1$.

Exercise 5.4.4 Use one rule for each step and identify the rule to differentiate

a. $P(t)$	$=$	$3e^{5\,t} + \pi$	b. $P(t)$	$=$	$\frac{e^2}{2} + \frac{t^3}{3}$
c. $P(t)$	$=$	5^t	d. $P(t)$	$=$	$e^{2\,t} \times e^{3\,t}$

Simplify Part d before differentiating.

Exercise 5.4.5 Compute $y'(x)$ **or assert that you do not yet have forumlas to compute** $\mathbf{y'(x)}$ for

a. $y(x)$	$=$	e^{5x}	b. $y(x)$	$=$	e^{-3x}
c. $y(x)$	$=$	$e^{\sqrt{x}}$	d. $y(x)$	$=$	$(\,e^x\,)^2$
e. $y(x)$	$=$	$(e^{\sqrt{x}})^2$	f. $y(x)$	$=$	$(e^{-x})^2$
g. $y(x)$	$=$	$\frac{e^x + e^{-x}}{2}$	h. $y(x)$	$=$	$\frac{e^x - e^{-x}}{2}$
i. $y(x)$	$=$	$5e^{-0.06x} + 3e^{-0.1x}$	j. $y(x)$	$=$	$e^{(x^2)}$
k. $y(x)$	$=$	$\sqrt{e^x}$	l. $y(x)$	$=$	$8e^{-0.0001x} - 16e^{-0.001x}$
m. $y(x)$	$=$	e^5	n. $y(x)$	$=$	$\sqrt{e}$
o. $y(x)$	$=$	10^x	p. $y(x)$	$=$	10^{-x}
q. $y(x)$	$=$	$x^2 + 2^x$	r. $y(x)$	$=$	$(e^{5x} + e^{-3x})^5$

Exercise 5.4.6 Argue that

$$\lim_{b \to a} \frac{e^{(b^2)} - e^{(a^2)}}{b^2 - a^2} \quad = \quad e^{(a^2)}$$

What is the ambiguity in the notation e^{a^2}. (Consider 4^{3^2}.) Use parenthesis, they are cheap. However, common practice is to interpret e^{t^2} as $e^{(t^2)}$.

Exercise 5.4.7 Argue that

$$\lim_{b \to a} \frac{e^{\sqrt{b}} - e^{\sqrt{a}}}{\sqrt{b} - \sqrt{a}} = e^{\sqrt{a}}$$

Exercise 5.4.8 Review the method in Explore 5.4.1 and the results in Exercises 5.4.6 and 5.4.7. Use Definition 3.20,

$$F'(a) = \lim_{b \to a} \frac{F(b) - F(a)}{b - a}, \quad \text{to compute } E'(a) \text{ for}$$

a.	$E(t)$	$= e^{2t}$	b.	$E(t)$	$= e^{-t}$	c.	$E(t)$	$= e^{2\sqrt{t}}$
d.	$E(t)$	$= e^{2}$	e.	$E(t)$	$= e^{\frac{1}{t}}$	f.	$E(t)$	$= e^{-t^2}$

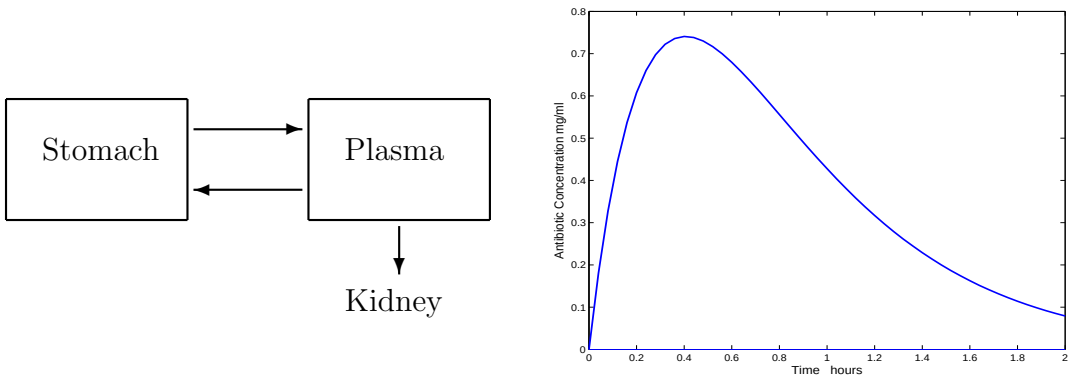
Exercise 5.4.9 Consider the kinetics of penicillin that is taken as a pill in the stomach. The diagram in Figure Ex. 5.4.9(a) may help visualize the kinetics. We will find in Chapter 16 that a model of plasma concentration of antibiotic t hours after ingestion of an antibiotic pill yields an equation similar to

$$C(t) = 5e^{-2t} - 5e^{-3t} \qquad \mu\text{g/ml} \tag{5.20}$$

A graph of C is shown in Figure Ex. 5.4.9. At what time will the concentration reach a maximum level, and what is the maximum concentration achieved?

As we saw in Section 3.5.2 and may be apparent from the graph in Figure Ex. 5.4.9, the highest concentration is associated with the point of the graph of C at which $C' = 0$. The question, then, is at what time t is $C'(t) = 0$ and what is $C(t)$ at that time?

Figure for Exercise 5.4.9 (a) Diagram of compartments for oral ingestion of penicillin. (b) Graph of $C(t) = 5e^{-2t} - 5e^{-3t}$ representative of plasma penicillin concentration t minutes after ingestion of the pill.



(a) (b)

Exercise 5.4.10 Plasma penicillin concentration is

$$P(t) = 5 \times e^{-0.3\,t} - 5 \times e^{-0.4\,t}$$

t hours after ingestion of a penicillin pill into the stomach. A small amount of the drug diffuses into tissue and the tissue concentration, $C(t)$, is

$$C(t) = -e^{-0.3t} + 0.5e^{-0.4t} + 0.5e^{-0.2t} \qquad \mu\text{g/ml}$$

a. Use your calculator to find the time at which the concentration of the drug in tissue is maximum and the value of C at that time.

b. Compute $C'(t)$ and solve for t in $C'(t) = 0$. This is really bad, for you must solve for t in

$$0.3e^{-0.3t} - 0.2e^{-0.4t} - 0.1e^{-0.2t} = 0$$

Try this:

$$\text{Let} \qquad Z = e^{-0.1t} \qquad\qquad \text{then solve} \qquad 0.3Z^3 - 0.2Z^4 - 0.1Z^2 = 0.$$

c. Solve for the possible values of Z. Remember that $Z = e^{-0.1t}$ and solve for t if possible using the possible values of Z.

d. Which value of t solves our problem?

5.5 The derivative equation $P'(t) = k \times P(t)$

A crucial property of exponential functions established by the e^{kt} Rule is

Property 5.5.1 Proportional Growth or Decay. If P is a function defined by

$$P(t) = Ce^{kt}$$

where C and k are numbers, then

$$P'(t) = k \times P(t)$$

Proof that $P(t) = Ce^{kt}$ implies that $P'(t) = kP(t)$:

$$\begin{aligned} P'(t) &= \left[Ce^{kt}\right]' \\ &= C\left[e^{kt}\right]' \\ &= Ce^{kt}k = k \times Ce^{kt} \\ &= k \times P(t) \end{aligned}$$

The reverse implication is also true, and is shown to be true in Section 16.7:

Property 5.5.2 Exponential Growth or Decay
If P is a function and there is a number k for which

$$P'(t) = k \times P(t) \qquad \text{for all } t \geq 0$$

then there is a number C for which

$$P(t) = Ce^{kt}$$

Furthermore,

$$C = P(0) \qquad \text{so that} \qquad P(t) = P(0)e^{kt}$$

In the preceding equations, k can be either positive or negative. When k is negative, it is more common to emphasize this and write $-k$ and write $P(t) = e^{-kt}$, where in this context it is understood that k is a positive number.

In Chapter 1, we examined models of population growth, light decay, and penicillin clearance, all of which were of the form

$$P_{t+1} - P_t = R \times P_t$$

and found that

$$P_t = P_0 \times R^t$$

These are *discrete time* models in which the *average rate of change* of P_t is proportional to P_t. The exponential Growth or Decay Property 5.5.2 is simply a *continuous time* model in which the *rate of change* of $P(t)$ is proportional to $P(t)$, and would be preferred in many instances. Bacterial populations may be visualized as growing continuously (and not in twenty minute bursts), the kidneys clear penicillin continuously (and not in five minute increments), and light decays continuously with depth (and not in one meter increments). Discrete time models are easy to comprehend and with short data intervals give good replications of data, but now that we know the definition of rate of change we can use continuous time or space models.

The equation

$$P' = kP \qquad \text{or} \qquad P'(t) = kP(t) \qquad \text{or} \qquad \frac{dP}{dt} = kP$$

derives from many models of biological and physical processes including population growth, drug clearance, chemical reaction, decay of radio activity – any system which can be described by:

Mathematical Model 5.5.1 Proportional change. The rate of change of a quantity is proportional to the amount of the quantity.

For example in population studies, we commonly assume that

Mathematical Model 5.5.2 Simple population growth. The growth rate of a population is proportional to the size of the population.

Let $P(t)$ be the size of a population at time, t. The component parts of the sentence in the Mathematical Model of simple population growth are symbolized by

a:	The growth rate of a population	:	$P'(t)$
b:	is proportional to	:	$= k\times$
c:	the size of the population	:	$P(t)$

The sentence of the Mathematical Model is then written

$$\underbrace{P'(t)}_{a} \quad \underbrace{= k\times}_{b} \quad \underbrace{P(t)}_{c}$$

From the property of Exponential Growth and Decay

$$P(t) = C \times e^{kt} \qquad \text{and if } P(0) = P_0 \text{ is known} \qquad P(t) = P_0 e^{kt} \tag{5.21}$$

In the event that the rate of **decrease** of a quantity, $P(t)$, is proportional to the size of $P(t)$, then because $-P'(t)$ is the rate of decrease of $P(t)$,

$$-P' = k \times P(t), \qquad P' = -k \times P(t), \qquad \text{and} \qquad P(t) = P_0 e^{-kt},$$

where k is a positive number.

Example 5.5.1 A distinction between discrete and continuous models. Suppose in year 2000 a population is at 5 million people and the population growth rate (excess of births over deaths) is 6 percent per year. One interpretation of this is to let $P(t)$ be the population size in millions of people at time t measured in years after 2000 and to write

$$P(0) = 5 \qquad P'(t) = 0.06 \times P(t)$$

Then, from the property of Exponential Growth or Decay 5.5.2, we may write

$$P(t) = P(0)e^{0.06t} = 5e^{0.06t}$$

$P(t) = 5e^{0.06t}$ does not exactly match the hypothesis that 'population growth rate is 6 percent per year', however. By this equation, after one year,

$$P(1) = 5e^{0.06\times 1} = 5e^{0.06} \doteq 5 \times 1.0618$$

The consequence is that during the first year (and every year) there would be a 6.18 percent increase, a contradiction.

The discrepancy lies with the model equation $P'(t) = 0.6P(t)$. Instead, we may write

$$P(0) = 5 \qquad P'(t) = k \times P(t)$$

where k is to be determined. Then from Exponential Growth or Decay 5.5.2 we may write

$$P(t) = 5e^{kt}$$

Now impose that $P(1) = 5 \times 1.06$, a 6 percent increase during the first year, and write

$$P(1) = 5e^{k\times 1} = 5 \times 1.06$$

This leads to

$$e^{k\times 1} = 1.06$$

We take the natural logarithm of both numbers and get

$$\begin{aligned} \ln\left(e^k\right) &= \ln 1.06 \\ k &= \ln 1.06 \\ &\doteq 0.05827 \end{aligned}$$

Then

$$P(t) = 5e^{0.05827t}$$

gives a description of the population t years after 2000. Each *annual* population is 6 percent greater than that of the preceding year. The continuous model of growth is actually

$$P(0) = 5 \qquad P'(t) = 0.05827P(t)$$

▬

The growth of a bank savings account is similar to this simplified model of population growth. If you deposited \$5000 in 2000 at a true 6 percent annual interest rate, it may amount to

$$P(t) = 5000e^{0.05827t}$$

dollars t years after 2000. On the other hand, some banks advertise and compute interest on the basis of 6% interest with 'instantaneous' compounding, meaning that their model is

$$P'(t) = 0.06 \times P(t)$$

leading to

$$P(t) = 5000e^{0.06t}$$

They will say that their 'APR' (annual percentage rate) is $100 \times e^{0.06} = 6.18$ percent.

Example 5.5.2 Geologists in the early nineteenth century worked out the sequential order of geological layers well before they knew the absolute dates of the layers. Their most extreme estimates of the age of the earth was in the order of 400 million years[4], about 1/10 of today's estimates based on decay of radioactive material. Early applications of radiometric dating used the decomposition of uranium-238 first to thorium-234 and subsequently to lead-206. More recently Potassium-40 decomposition has been found to be useful (and zircon decay is currently the best available).

[4]Charles Darwin wrote in the Origin of Species that Earth was several hundred million years old, but he was opposed in 1863 by a dominant physical scientist, William Thompson (later to become Lord Kelvin) who estimated that Earth was between 24 and 400 million years old. His estimate was based on his calculation of the time it would take for Earth to cool from molten rock to today's temperatures in the upper layers of the Earth. See article by Philip England, Peter Molnar, and Frank Richter, GSA Today, 17, 1(January 1, 2007).

Potassium-40 decomposes to both argon-40 and calcium-40 according to

$$9\,(^{40}\mathrm{K}) \quad \longrightarrow \quad {}^{40}\mathrm{Ar} + 8\,(^{40}\mathrm{Ca})$$

When deposited, volcanic rock contains significant amounts of ^{40}K but is essentially free of ^{40}Ar because ^{40}Ar is a gas that escapes the rock under volcanic conditions. Once cooled, some volcanic rock will become essentially sealed capsules that contain ^{40}K and retain the ^{40}Ar that derives from decomposition of the ^{40}K.

Mathematical Model 5.5.3 Potassium-40 decomposition. The rate of disintegration of ^{40}K is proportional to the amount of ^{40}K present.

If we let $K(t)$ be the amount of ^{40}K present t years after deposition of rock of volcanic origin and K_0 the initial amount of ^{40}K present, then

$$K(0) = K_0, \qquad K'(t) = -rK(t)$$

where r is a positive constant. The minus sign reflects the disintegration of ^{40}K. From the equation we may write

$$K(t) = K_0 e^{-rt}$$

The half-life of ^{40}K is 1.28×10^9 years, meaning that 1.28×10^9 years after deposition of the volcanic rock, the amount of ^{40}K in the rock will be $\frac{1}{2}K_0$. We use this information to evaluate r.

$$\begin{aligned}
\frac{1}{2}K_0 &= K_0 e^{-r\times 1280000000} \\
\frac{1}{2} &= e^{-r\times 1280000000} \\
\ln\frac{1}{2} &= -r \times 1280000000 \\
r &= \frac{\ln 2}{1280000000} \\
K(t) &= K_0 e^{-\frac{\ln 2}{1280000000}t}
\end{aligned}$$

Problem. Suppose a rock sample is found to have 5×10^{14} ^{40}K atoms and 2×10^{13} ^{40}Ar atoms. What is the age of the rock?
Solution. It is necessary to assume[5] that all of the ^{40}Ar derives from the ^{40}K, and that there has been no leakage of ^{40}K or ^{40}Ar into or out of the rock. Assuming so, then the number of ^{40}K atoms that have decomposed (to either ^{40}Ca or ^{40}Ar) must be nine times the number of ^{40}Ar atoms, or $9 \times (2 \times 10^{13}) = 1.8 \times 10^{14}$ atoms. Therefore

$$K_0 = 5 \times 10^{14} + 1.8 \times 10^{14} = 6.8 \times 10^{14}$$

and

$$K(t) = 6.8 \times 10^{14} e^{-\frac{\ln 2}{1280000000}t}$$

[5]Because ^{40}Ar is a gas at the temperatures that the rock was formed, no ^{40}Ar is originally in the rock.

We want the value of t for which $K(t) = 5 \times 10^{14}$. Therefore,

$$\begin{aligned} 5 \times 10^{14} &= 6.8 \times 10^{14} e^{-\frac{\ln 2}{1280000000} t} \\ \frac{5}{6.8} &= e^{-\frac{\ln 2}{1280000000} t} \\ \ln \frac{5}{6.8} &= -\frac{\ln 2}{1280000000} t \\ t &= 568,000,000 \end{aligned}$$

The rock is about 568 million years old. ▬

5.5.1 Two primitive modeling concepts.

Primitive Concept 1. Suppose you have a barrel (which could just as well be a blood cell, stomach, liver, or lake or ocean or auditorium) and $A(t)$ liters is the amount of water (glucose, plasma, people) in the barrel at time t minutes. If water is running into the barrel at a rate R_1 liters/minute and leaking out of the barrel at a rate R_2 liters/minute then

Rate of change of water in the barrel	=	Rate water enters the barrel	−	Rate water leaves the barrel
$A'(t)$	=	R_1	−	R_2
$\frac{\text{L}}{\text{min}}$		$\frac{\text{L}}{\text{min}}$		$\frac{\text{L}}{\text{min}}$

Primitive Concept 2. Similar to Primitive Concept 1 except that there is salt in the water. Suppose $S(t)$ is the amount in grams of salt in the barrel and C_1 is the concentration in grams/liter of salt in the stream entering the barrel and C_2 is the concentration of salt in grams/liter in the stream leaving the barrel. Then

Rate of change of salt in the barrel	=	Rate salt enters the barrel	−	Rate salt leaves the barrel
$S'(t)$	=	$C_1 \times R_1$	−	$C_2 \times R_2$
$\frac{\text{g}}{\text{min}}$		$\frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}}$		$\frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}}$

Observe that the units are g/m on both sides of the equation. Maintaining a balance in units often helps to find the correct equation.

Example 5.5.3 Suppose a runner is exhaling at the rate of 2 liters per second. Then the amount of air in her lungs is decreasing at the rate of two liters per second. If, furthermore, the CO_2 partial pressure in

the exhaled air is 50 mm Hg (approx 0.114 g CO_2/liter of air at body temperature of 310 K)[6], then she is exhaling CO_2 at the rate of 0.114 g/liter $\times$2 liters/sec = 0.228 g/sec. ■

Example 5.5.4 Classical Washout Curve. A barrel contains 100 liters of water and 300 grams of salt. You start a stream of pure water flowing into the barrel at 5 liters per minute, and a compensating stream of salt water flows from the barrel at 5 liters per minute. The solution in the barrel is 'well stirred' so that the salt concentration is uniform throughout the barrel at all times. Let $S(t)$ be the amount of salt (grams) in the barrel t minutes after you start the flow of pure water into the barrel.

Explore 5.5.1 Draw a graph of what you think will be the graph of $S(t)$. In doing so consider

- What is $S(0)$?
- Does $S(t)$ increase or decrease?
- Will there be a time, t_*, for which $S(t_*) = 0$? If so, what is t_*?

■

Solution. First let us analyze S. We use Primitive Concept 2. The concentration of salt in the water flowing into the barrel is 0. The concentration of salt in the water flowing out of the barrel is the same as the concentration $C(t)$ of salt in the barrel which is

$$C(t) = \frac{S(t)}{100}\text{g/L}$$

Therefore

$$\begin{array}{ccccc} \text{Rate of change of salt} & = & \text{Rate salt enters} & - & \text{Rate salt leaves} \\ \text{in the barrel} & & \text{the barrel} & & \text{the barrel} \\ S'(t) & = & C_1 \times R_1 & - & C_2 \times R_2 \\ S'(t) & = & 0 \times 5 & - & \frac{S(t)}{100}\frac{\text{gr}}{\text{L}} \times 5\frac{\text{L}}{\text{min}} \end{array}$$

Furthermore, $S(0) = 300$. Thus

$$\begin{aligned} S(0) &= 300 \\ S'(t) &= -0.05S(t). \end{aligned}$$

From the Exponential Growth and Decay property 5.5.2,

$$S(t) = 300e^{-0.05t}$$

6

$$\frac{\frac{50}{760}\text{A} \times 1\text{liter} \times 44\text{mol wt}}{0.08206\text{gas const} \times 310\text{K}} = 0.114\text{g}$$

Explore 5.5.2 . Draw the graph of $S(t) = 300e^{-0.05t}$ on your calculator and compare it with the graph you drew in the previous Explore. ■ ■

Example 5.5.5 Classical Saturation Curve.
Problem. Suppose a 100 liter barrel is full of pure water and at time $t = 0$ minutes a stream of water flowing at 5 liters per minute and carrying 3 g/liter of salt starts flowing into the barrel. Assume the salt is well mixed in the barrel and water overflows at the rate of 5 liters per minute. Let $S(t)$ be the amount of salt in the barrel at time t minutes after the salt water starts flowing in.

Explore 5.5.3 Draw a graph of what you think will be the graph of $S(t)$. In doing so consider

- What is $S(0)$?
- Does $S(t)$ increase or decrease?
- Is there an upper bound on $S(t)$, the amount of salt in the barrel that will be in the barrel?

■

Solution: We analyze S; again we use Primitive Concept 2. The concentration of salt in the inflow is 3 g/liter. The concentration $C(t)$ of salt in the tank a time t minutes is

$$C(t) = \frac{S(t)}{100}$$

The salt concentration in the outflow will also be $C(t)$. Therefore

$$\begin{array}{ccccc} \text{Rate of change of salt} & = & \text{Rate salt enters} & - & \text{Rate salt leaves} \\ \text{in the barrel} & & \text{the barrel} & & \text{the barrel} \\ S'(t) & = & C_1 \times R_1 & - & C_2 \times R_2 \\ S'(t) & = & 3 \times 5 & - & \frac{S(t)}{100} \times 5 \\ \frac{\text{g}}{\text{min}} & & \frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}} & & \frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}} \end{array}$$

Initially the barrel is full of pure water, so

$$S(0) = 0$$

We now have

$$\begin{array}{rcl} S(0) & = & 0 \\ S'(t) & = & 15 - 0.05S(t) \end{array} \tag{5.22}$$

This equation is not in the form of $P'(t) = kP(t)$ because of the 15. Proceed as follows.
Equilibrium. Ask, 'At what value, E, of $S(t)$ would $S'(t) = 0$?' That would require

$$0 = 15 - 0.05E = 0, \text{or} E = 300\text{g}.$$

$E = 300$ g is the *equilibrium level* of salt in the barrel. We focus attention on the difference, $D(t)$, between the equilibrium level and the current level of salt. Thus

$$D(t) = 300 - S(t) \qquad \text{and} \qquad S(t) = 300 - D(t)$$

Now,

$$D(0) = 300 - S(0) = 300 - 0 = 300$$

Furthermore,

$$S'(t) \quad = \quad [300 - D(t)]' \quad = \quad -D'(t)$$

We substitute into Equations 5.22

$$\begin{array}{rcl} S(0) & = & 0 \\ S'(t) & = & 15 - 0.05S(t) \end{array} \qquad \begin{array}{rcl} D(0) & = & 300 \\ -D'(t) & = & 15 - 0.05(300 - D(t)) \end{array}$$

The equations for D become

$$\begin{array}{rcl} D(0) & = & 300 \\ D'(t) & = & -0.05D(t) \end{array}$$

This is in the form of the Exponential Growth and Decay Property 5.5.2, and we write

$$D(t) = 300e^{-0.05t}$$

Returning to $S(t) = 300 - D(t)$ we write

$$S(t) = 300 - D(t) = 300 - 300e^{-0.05t}$$

The graph of $S(t) = 300 - 300e^{-0.05t}$ is shown in Figure 5.3. Curiously, the graph of $S(t)$ is also called an *exponential decay curve.* $S(t)$ is not decaying at all; $S(t)$ is increasing. What is decaying exponentially is $D(t)$, the remaining salt capacity. ■

Explore 5.5.4 Show that if $S(t) = 300 - 300e^{-0.05t}$, then

$$S(0) = 0, \qquad \text{and} \qquad S'(t) = 15 - 0.05S(t).$$

■

5.5.2 Continuous-space analysis of light depletion.

As observed in Chapter 1, light intensity decreases as one descends from the surface of a lake or ocean. There we was divided the water into discrete layers and it was assumed that each layer absorbs a fixed fraction, F, of the light that enters it from above. This hypothesis led to the difference equation

$$I_{d+1} - I_d = -F \times I_d$$

Light is actually absorbed continuously as it passes down through a (homogeneous) water medium, not in discrete layers. We examine the light intensity, $I(x)$, at a distance, x meters, below the surface of a lake or ocean, assuming that the light intensity penetrating the surface is a known quantity, I_0.

We start by testing an hypothesis about light transmission in water that appears different from the hypothesis we arrived at in Chapter 1:

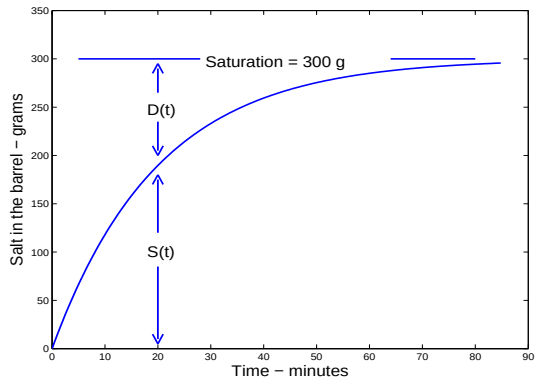


Figure 5.3: The graph of $S(t) = 300 - 300e^{-0.05t}$ depicting the amount of salt in a barrel initially filled with 100 liters of pure water and receiving a flow of 5 L/m carrying 3 g/L. $D(t) = 300 - S(t)$.

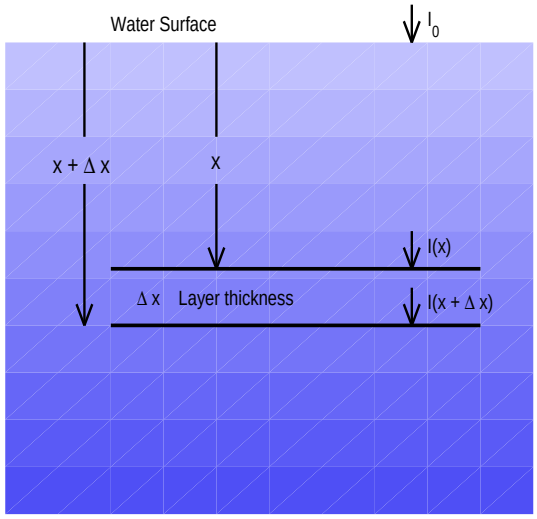


Figure 5.4: Diagram of light depletion below the surface of a lake or ocean. $I(x)$ is light intensity at depth x due to light of intensity I_0 just below the surface of the water.

Mathematical Model 5.5.4 Light Absorbance: The amount of light absorbed by a (horizontal) layer of water is proportional to the thickness of the layer and to the amount of light entering the layer (see Figure 5.4).

The mathematical model of light absorbance implies, for example, that

1. The light absorbed by a water layer of thickness 2Δ is twice the light absorbed by a water layer of thickness Δ and

2. A layer that absorbs 10% of a dim light will absorb 10% of a bright light.

We know from experimental evidence that implication (1) is approximately true for thin layers and for low levels of turbidity. Implication (2) is valid for a wide range of light intensities.

Double Proportionality. The mathematical model of light absorbance the light absorbed by a layer is proportional to two things, the thickness of the layer and the intensity of the light entering the layer. We handle this double proportionality by assuming that the amount of light absorbed in a layer is proportional to the **product** of the thickness of the layer and the intensity of the light incident to the layer. That is, there is a number, K, such that if $I(x)$ is the light intensity at depth x and $I(x+\Delta x)$ is the light intensity at depth $x+\Delta x$, then

$$I(x+\Delta x) - I(x) \doteq -K \times \Delta x \times I(x) \tag{5.23}$$

The product, $K \times \Delta x \times I(x)$, has the advantage that

1. For fixed incident light intensity, $I(x)$, the light absorbed, $I(x+\Delta x) - I(x)$, is proportional to the thickness, Δx, (proportionality constant $= -K \times I(x)$) and
2. For fixed thickness Δx, the light absorbed is proportional to the incident light, $I(x)$ (proportionality constant $= -K \times \Delta x$).

Equation 5.23 can be rearranged to

$$\frac{I(x+\Delta x) - I(x)}{\Delta x} \doteq -KI(x)$$

The approximation ($\doteq$) improves as the layer thickness, Δx, approaches zero.

$$\text{As } \Delta x \to 0 \qquad \frac{I(x+\Delta x) - I(x)}{\Delta x} \to I'(x)$$

and we conclude that

$$I'(x) = -KI(x) \tag{5.24}$$

The Exponential Growth and Decay Property 5.5.2 implies that

$$\text{because} \qquad I'(x) = -KI(x), \qquad I(x) = I_0 e^{-Kx} \tag{5.25}$$

Example 5.5.6 Assume that 1000 w/m^2 of light is striking the surface of a lake and that 40% of that light is reflected back into the atmosphere. We first solve the initial value problem

$$\begin{aligned} I(0) &= 600 \\ I'(x) &= -KI(x) \end{aligned}$$

to get

$$I(x) = 600e^{-Kx}$$

If we have additional information that, say, the light intensity at a depth of 10 meters is 400 W/m^2 we can find the value of K. It must be that

$$I(10) = 600e^{-K\times 10} = 400$$

The only unknown in the last equation is K, and we solve

$$\begin{aligned} 600e^{-K\times 10} &= 400 \\ e^{-K\times 10} &= 400/600 = 5/6 \\ \ln\left(e^{-K\times 10}\right) &= \ln(2/3) \\ -K\times 10 &= \ln(2/3) \\ K &\doteq 0.040557 \end{aligned}$$

Thus we could say that

$$I(x) = 600 \times e^{-0.040557x}$$

If we know, for example, that 30 W/m^2 of light are required for a certain species of coral to grow, we can ask for the maximum depth, $\overline{x}$, at which we might find that species. We would solve

$$\begin{aligned} I(\overline{x}) &= 30 \\ 600 \times e^{-0.040557\overline{x}} &= 30 \\ \ln\left(e^{-0.040557\overline{x}}\right) &= \ln(30/600) \\ -0.040557\overline{x} &= \ln(1/20) \\ \overline{x} &= 73.9 \quad \text{meters} \end{aligned}$$

5.5.3 Doubling time and half-life

Suppose k and C are positive numbers. The *doubling time* of $F(t) = Ce^{kt}$ is a number t_{dbl} such that

$$\text{for any time } t \qquad F(t + t_{dbl}) = 2 \times F(t).$$

That there is such a number follows from

$$\begin{aligned} F(t + t_{dbl}) &= 2 \times F(t) \\ Ce^{k\times(t+t_{dbl})} &= 2 \times Ce^{kt} \\ Ce^{kt} \times e^{kt_{dbl}} &= 2 \times Ce^{kt} \\ e^{kt_{dbl}} &= 2 \\ t_{dbl} &= \frac{\ln 2}{k}. \end{aligned} \tag{5.26}$$

The *half-life* of $F(t) = Ce^{-kt}$ is a number t_{half}, usually written as $t_{1/2}$, such that

$$\text{for any time } t \qquad F(t + t_{1/2}) = \frac{1}{2} \times F(t).$$

Using steps similar to those for the doubling time you can find that

$$t_{1/2} = \frac{\ln 2}{k}. \tag{5.27}$$

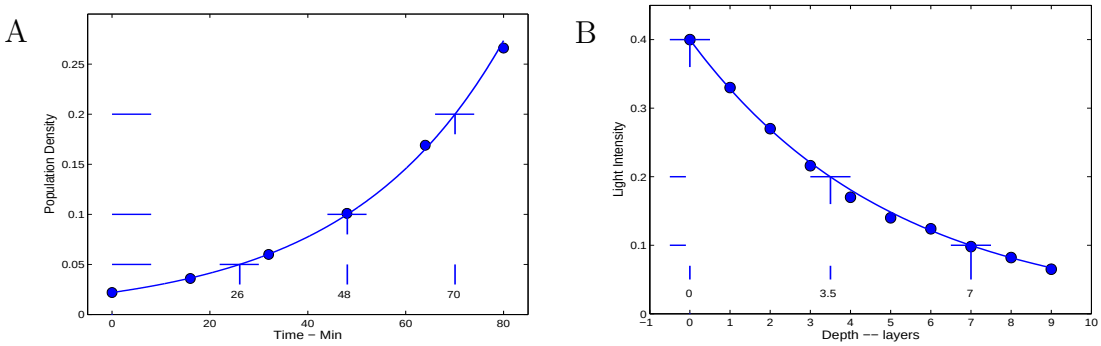


Figure 5.5: A. Bacterial population density and ABS=$0.022e^{0.0315t}$, which has doubling time of 22 minutes. B. Light depletion and $I_d = 0.4e^{-0.196d}$ which has a 'half life' of 3.5 meters.

Explore 5.5.5 Write the steps similar to those for the doubling time to show that $t_{1/2} = (ln2)/k$. ■

In Figure 5.5A is a graph of the bacterial density from Table 1.1 and of the equation

$$\text{Abs} = 0.022e^{0.0315}t, \qquad t_{dbl} = \frac{\ln 2}{0.0315} = 22.0 \quad \text{minutes.}$$

The bacterial density doubles every 22 minutes, as illustrated for the intervals [26,48] minutes and [48,70] minutes.

In Figure 5.5B is a graph of light intensity decay from Figure 1.10 (repeated in Figure 5.6) and of the Equation 1.15

$$I_d = 0.400 \times 0.82^d$$

Because $0.82 = e^{\ln 0.82} = e^{-0.198}$,

$$I_d = 0.400 \times e^{-0.198d} \qquad \text{and} \qquad d_{1/2} = \frac{\ln 2}{0.198} = 3.5$$

Every 3.5 layers of muddy water the light intensity decays by one-half. $d_{1/2}$ is a distance and might be called 'half depth' rather than 'half life.' 'Half life' is the term used for all exponential decay, however, and you are well advised to use it.

Example 5.5.7 *Problem.* Suppose a patient has taken 80 mg of Sotolol, a drug that regularizes heart beat, once per day for several days. Sotolol has a half-life in the body of 12 hrs. There will be an historical accumulation, H, of Sotolol in the body due to previous days' pills, and each day when the 80 mg pill is taken there will be $H + 80$ mg of Sotolol in the body that will decrease according to the 12 hr half-life. Find the historical accumulation, H, and the daily fluctuation.

Solution. Let A_t be the amount of Sotolol in the body at time t hours during the day, with $t = 0$ being the time the pill is taken each day. Then

$$A_0^- = H \qquad \text{and} \qquad A_0^+ = H + 80.$$

Furthermore, because the half-life is 12 hours, A_{24}^-, the amount left in the body just before the pill is taken the next day, is

$$A_{24}^- = \frac{1}{4}\,A_0^+ = \frac{1}{4}\,(H + 80),$$

and

$$A_{24}^{-} = H$$

Therefore,

$$\frac{1}{4}(H + 80) = H, \qquad H = \frac{4}{3}20 = 26.7\text{mg}$$

Furthermore, $A_0^+ = H + 80 = 106.7$mg so the amount of Sotolol in the body fluctuates from 26.7 mg to 106.7 mg, a four to one ratio. You are asked to compare this with taking 40 mg of Sotolol twice per day in Exercise 5.5.5

5.5.4 Semilogarithm and LogLog graphs.

Functions $P(t)$ that satisfy an equation $P'(t) = k \times P(t)$ may be written $P(t) = P_0 e^{kt}$ and will satisfy the relation $\ln P(t) = (\ln P_0) + kt$. The graph of $\ln P(t)$ *vs* t is a straight line with intercept $\ln P_0$ and slope k. Similarly, if $P'(t) = -kP(t)$ the graph of $\ln P(t)$ *vs* t is a straight line with slope (in rectilinear coordinates) $-k$. A scientist with data $t, P(t)$ that she thinks is exponential may plot the graph of $\ln P(t)$ *vs* t. If the graph is linear, then a fit of a line to that data will lead to an exponential relation of the form $P(t) = Ae^{kt}$ or $P(t) = Ae^{-kt}$. She may then search for a biological process that would justify a model $P'(t) = \pm kP(t)$.

Example 5.5.8 In Section 1.3 we showed the results of an experiment measuring the light decay as a function of depth. The data and a semilog graph of the data are shown in Figure 5.6.

Depth Layer	I_d mW/cm^2
0	0.400
1	0.330
2	0.270
3	0.216
4	0.170
5	0.140
6	0.124
7	0.098
8	0.082
9	0.065

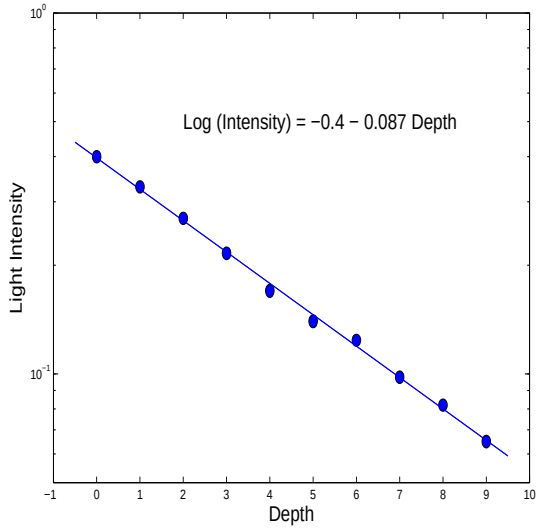


Figure 5.6: Data and a semilog graph of the data showing experimental results of measuring light decrease with depth of water.

As shown in the figure,

$$\log_{10} I_d = -0.4 - 0.087 \times d$$

is a good approximation to the data. Therefore

$$\begin{aligned} I_d &\doteq 10^{-0.4-0.087d} \\ &= 0.4 \times 0.82^d \end{aligned}$$

which is the same result obtained in Section 1.3. As shown in the previous subsection, the relation

$$I'(d) = -k \times I(d)$$

corresponds to a process underlying light depletion in water.

5.5.5 Relative Growth Rates and Allometry.

If y is a positive function of time, the relative growth rate of y is

$$\frac{y'(t)}{y(t)} \qquad \text{Relative Growth Rate.} \tag{5.28}$$

The relative growth rate of y is sometimes called the fractional growth rate or the logarithmic growth rate.

Definition 5.5.1 Allometry. . Two functions, x and y, of time are said to be allometrically related if there are numbers C and a such that

$$y(t) = C \times (\, x(t)\,)^a\,. \tag{5.29}$$

If x and y are allometric then

$$\log y = \log C\, x^a = \log C + a \log x, \tag{5.30}$$

for any base of log. Therefore if $\log y$ is plotted *vs* $\log x$ the graph should be a straight line.

Explore 5.5.6 Show that if x and y satisfy Equation 5.30, $\log y(t) = \log C + a \log x(t)$, then

$$\frac{y'(t)}{y(t)} = a \times \frac{x'(t)}{x(t)}.$$

Conclude that if x and y are allometric then the relative growth rate of y is proportional to the relative growth rate of x. ■

Shown in Figure 5.7 is a graph of $\log_{10}$ of the weight of large mouth bass *vs* $\log_{10}$ of their length[7]. The data appear linear and we conclude that the weight is allometric to the length. An equation of a line close to the data is

$$\frac{\log y - 1.05}{\log x - 2.0} = \frac{2.6 - 1.05}{2.5 - 2.0}, \qquad \log y = -5.75 + 3.1 \log x$$

Then

$$y = 10^{-5.75}\, x^{3.1}$$

The weights of the bass are approximately proportional to the cube of the lengths. This is consistent with the fact that the volume of a cube is equal to the cube of the length of an edge. Many interesting allometric relations are not supported by underlying models, however (Exercises **??**, 5.5.26, and 5.5.25).

[7]Data from Robert Summerfelt, Iowa State University.

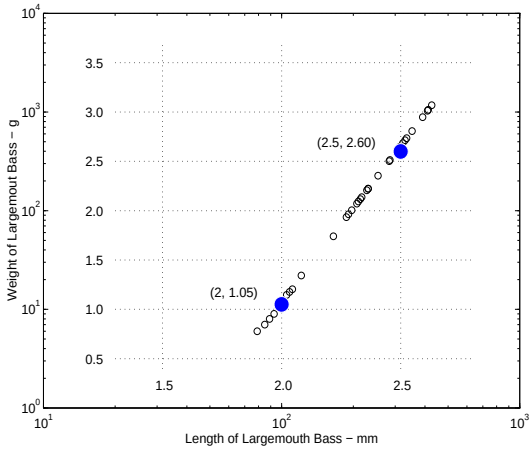


Figure 5.7: Weight *vs* Length for large mouth bass plotted on a log-log graph.

Exercises for Section 5.5, The derivative equation $P'(t) = k \times P(t)$

Exercise 5.5.1 Write a solution for each of the following derivative equations. Sketch the graph of the solution. For each, find the doubling time, t_{dbl}, or half life, $t_{1/2}$, which ever is applicable.

a. $P(0) = 5 \quad P'(t) = 2P(t)$ b. $P(0) = 5 \quad P'(t) = -2P(t)$

c. $P(0) = 2 \quad P'(t) = 0.1P(t)$ d. $P(0) = 2 \quad P'(t) = -0.1P(t)$

e. $P(0) = 10 \quad P'(t) = P(t)$ f. $P(0) = 10 \quad P'(t) = -P(t)$

g. $P(0) = 0 \quad P'(t) = 0.01P(t)$ h. $P(0) = 0 \quad P'(t) = -0.01P(t)$

Exercise 5.5.2 Write a solution for each of the following derivative equations. Sketch the graph of the solution. For each, find the half life, $t_{1/2}$, which is the time required to 'move half way toward equilibrium.'

Recall the solution in Example 5.5.5 to solve $S'(t) = 15 - 0.05S(t)$.

a. $S(0) = 0 \quad S'(t) = 10 - 2S(t)$ b. $S(0) = 2 \quad S'(t) = 10 - 2S(t)$

c. $S(0) = 5 \quad S'(t) = 10 - 2S(t)$ d. $S(0) = 10 \quad S'(t) = 10 - 2S(t)$

e. $S(0) = 0 \quad S'(t) = 20 - S(t)$ f. $S(0) = 10 \quad S'(t) = 20 - S(t)$

g. $S(0) = 20 \quad S'(t) = 20 - S(t)$ h. $S(0) = 30 \quad S'(t) = 20 - S(t)$

Exercise 5.5.3 Find values of C and k so that $P(t) = Ce^{kt}$ matches the data.

a. $P(0) = 5 \quad P(2) = 10$ b. $P(0) = 10 \quad P(2) = 5$

c. $P(0) = 2 \quad P(5) = 10$ d. $P(0) = 10 \quad P(5) = 10$

e. $P(0) = 5 \quad P(2) = 2$ f. $P(0) = 8 \quad P(10) = 6$

g. $P(1) = 5 \quad P(2) = 10$ h. $P(2) = 10 \quad P(10) = 20$

Exercise 5.5.4 Suppose a barrel has 100 liters of water and 400 grams of salt and at time $t = 0$ minutes a stream of water flowing at 5 liters per minute and carrying 3 g/liter of salt starts flowing into the barrel, the barrel is well mixed, and a stream of water and salt leaves the barrel at 5 liters per minute. What is the amount of salt in the barrel t minutes after the flow begins? Draw a candidate solution to this problem before computing the solution.

Exercise 5.5.5 In Example 5.5.7 it was shown that in a patient who takes 80 mg of Sotolol once per day, the daily fluctuation of Sotolol is from 26.7 mg to 106.7 mg. Sotolol has a half-life in the body of 12 hours. What is the fluctuation of Sotolol in the body if the patient takes two 40 mg of Sotolol at 12 hour intervals in the day? Would you recommend two 40 mg per day rather than one 80 mg pill per day?

Exercise 5.5.6 A patient takes 10 mg of coumadin once per day to reduce the probability that he will experience blood clots. The half-life of coumadin in the body is 40 hours. What level, H, of coumadin will be accumulated from previous ingestion of pills and what will be the daily fluctuation of coumadin in the body.

Exercise 5.5.7 Plot semilog graphs of the data sets in Table Ex. 5.5.7 and decide which ones appear to be approximately exponential. For those that appear to be exponential, find numbers, C and k, so that

$$P(t) = Ce^{kt}$$

approximates the the data.

Table for Exercise 5.5.7 Data sets for Exercise 5.5.7

a.

t	$P(t)$
0	2.0
1	2.1
2	2.8
3	4.7
4	8.4
5	14.5

b.

t	$P(t)$
0	6.00
2	4.35
4	2.97
6	2.17
8	1.65
10	1.25

c.

t	$P(t)$
0	2.00
5	2.83
10	4.00
15	5.67
20	8.00
25	11.31

d.

t	$P(t)$
0	2.00
3	1.00
6	0.67
9	0.50
12	0.40
15	0.33

e.

t	$P(t)$
0	2.00
3	1.00
6	0.50
9	0.25
12	0.13
15	0.06

Exercise 5.5.8 Shown in Table Ex. 5.5.8 data from *V. natrigens* growth reported in Chapter 1 on page 5. Find numbers, C and k, so that

$$P(t) = Ce^{kt}$$

approximates the the data. Use your values of C and k and compute $P(0)$, $P(16)$, $P(32)$, $P(48)$, and $P(64)$ and compare them with the observed values in the Table Ex. 5.5.8.

Table for Exercise 5.5.8 Cell density of *V. natrigens* measured as light absorbance at 16-minute time increments.

pH 6.25	
Time (min)	Population Density
0	0.022
16	0.036
32	0.060
48	0.101
64	0.169

Exercise 5.5.9 David Ho and colleagues[8] published the first study of HIV-1 dynamics within patients following treatment with an inhibitor of HIV-1 protease, ABT-538 which stops infected cells from producing new viral particles. Shown in Exercise Figure 5.5.9A is a graph of plasma viral load before and after ABT-treatment was begun on day 1 for patient number 409 and in Exercise Figure 5.5.9B is a semi-log graph of CD4 cell count following treatment.

a. By what percent is viral load diminished from day-1 to day-12?

b. The line in Figure 5.5.9A has an equation, $y = 5.9 - 0.19x$. Remember that $y = \log_{10} V$ and x is days. Find V_0 and m so that the graph of $V(t) = V_0 e^{-mt}$ in semilog coordinates is the line drawn in Exercise Figure 5.5.9A.

c. What is the half-life of the viral load?

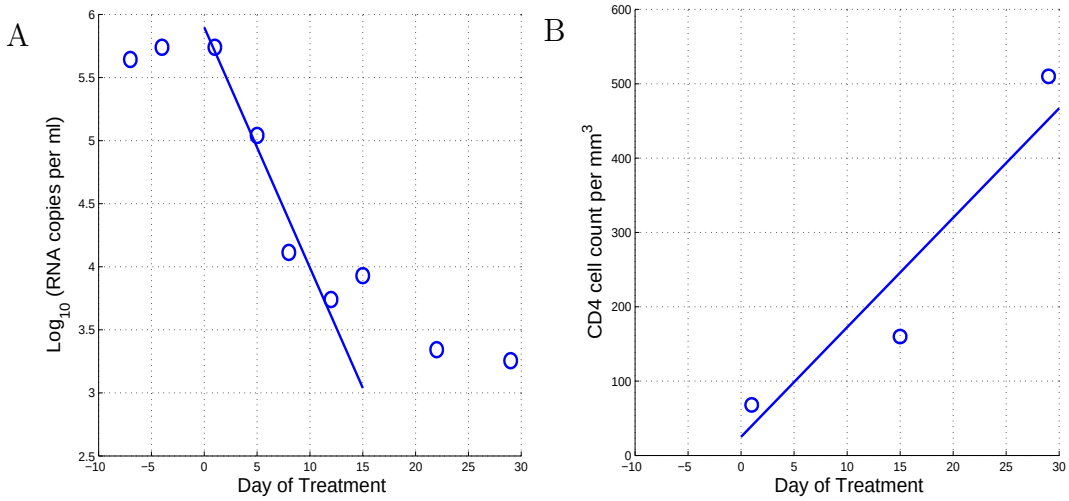
d. From the previous step, $V'(t) = -0.43V(t)$. Suppose ABT-538 totally eliminates viral production during days 1 to 12. At what rate is the immune system of patient 409 eliminating virus before treatment.

e. Assume that CD4 cell counts increase linearly and the equation in Figure 5.5.9B is y = 9.2 + 15.8 x. At what rate are CD4 cells being produced? Remember that y is CD4 count per mm^3 and there are about 6×10^6 mm^3 of blood in the human body.

After about 35 days, the HIV virus mutates into a form resistant to ABT-538 and pre-treatment viral loads soon return. Treatment with a protease inhibitor together with drugs that inhibit the translation of HIV RNA to DNA can decrease viral loads to levels below detection for the duration of treatment.

Figure for Exercise 5.5.9 A. Count of HIV viral load during administration of ABT-538. B. Count of helper t-cell during the same period. Note that if $\log_{10}$ (RNA copies per ml) = 4, for example, then RNA copies per ml = 10^4. See Exercise 5.5.9 Figures adapted by permission at no cost from Macmillan Publishing Group, Ltd. David D. Ho, Avidan U Neumann, Alan S. Perelson, Wen Chen, John M. Leonard & Martin Markowitz, *Nature* 373 (1995) 123:127 Copyright 1995 http://www.nature.com.

[8] David D. Ho, et al, *Nature* 373 (1995) 123:127.



Exercise 5.5.10 You inject two grams of penicillin into the 6 liter vascular pool of a patient. Plasma circulates through the kidney at the rate of 1.2 liters/minute and the kidneys remove 20 per cent of the penicillin that passes through.

1. Draw a schematic diagram showing the vascular pool and kidneys as separate entities, an artery leading from the vascular pool to the kidney and a vein leading from the kidney back to the vascular pool.

2. Let $P(t)$ be the amount of penicillin in the vascular pool t minutes after injection of penicillin. What is $P(0)$?

3. Use Primitive Concept 2 to write an equation for P'.

4. Write a solution to your equation.

Exercise 5.5.11 Suppose a rock sample is found to have 8.02 μg of ^{40}K and 7.56 μg of ^{40}Ar. What is the age of the rock?

Exercise 5.5.12 Suppose a rock sample is found to have 6.11 mg of ^{40}K and 0.05 mg of ^{40}Ar. What is the age of the rock?

Exercise 5.5.13 Rubidium-87 decomposes to strontium-87 with a half-life of 50×10^9 years. Fortunately, rubidium and potassium occur in the same rock types and in the same minerals, usually in the ratio of 1 ^{87}Rb atom to approximately 600 ^{40}K atoms. Age determined by rubidium-87 to strontium-87 decomposition is an excellent check of ^{40}K to ^{40}Ar ages. However, ^{87}Sr may be lost from the rock or may be present but not derived from ^{87}Rb so the ^{87}Rb to ^{87}Sr age may not be as accurate as the ^{40}K to ^{40}Ar age.

a. Suppose a rock sample has 2.5×10^{11} atoms of ^{87}Rb and 1.5×10^{10} atoms of ^{87}Sr. What is the age of the rock?

b. Suppose a rock sample has 6.4μg of ^{87}Rb and 0.01μg of ^{87}Sr. What is the age of the rock?

Exercise 5.5.14 A major advancement in archaeology was the development of carbon-14 dating in the 1950's by an American chemist Willard Libby, for which he received the 1960 Nobel Prize in Chemistry. Carbon-14 develops in the upper atmosphere as neutrons bombard nitrogen, and subsequently combines with oxygen to form carbon dioxide. About 1 in 10^{12} CO_2 atoms is formed with ^{14}C in today's atmosphere. Plants metabolize $^{14}CO_2$ (almost) as readily as $^{12}CO_2$, and resulting sugars are metabolized equally by animals that eat the plants. Consequently carbon from living material is 1 part in 10^{12} carbon-14. Upon death, no additional ^{14}C is absorbed into the material and ^{14}C gradually decomposes into nitrogen. Slightly confounding the use of radio carbon dating is the fact that the fraction of atmospheric $^{14}C0_2$ has not been historically constant at 1 molecule per 10^{12} molecules of $^{12}CO_2$.

Carbon-14 decomposes to nitrogen according to

$$ {}^{14}_{6}\mathrm{C} \longrightarrow {}^{14}_{7}\mathrm{N} + \beta^- + \overline{\nu} + \text{energy} \tag{5.31} $$

where β^- denotes an electron and $\overline{\nu}$ denotes an antineutrino. One of the neutrons of ${}^{14}_{6}$C looses an electron and becomes a proton.

Mathematical Model 5.5.5 Carbon-14 decay. The rate of decomposition of ${}^{14}_{6}$C in a sample is proportional to the size of the sample. One-half of the atoms in a sample will decompose in 5730 years.

a. Write and solve a derivative equation that will show for a sample of ^{14}C initially of size C_0 what the size will be t years later.

b. In tissue living today, the amount of ^{14}C in one gram of carbon is approximately 10^{-12} grams. Assume for this problem that the same ratio in living material has persisted for the last 10,000 years. Also assume that upon death the only change in carbon of any form is the decrease in ^{14}C due to decomposition to nitrogen. Suppose a 100 gram sample of carbon from bone is found to have 3×10^{-11} grams of ^{14}C. What is the age of the sample?

c. Suppose that during the time 10,000 years ago until 2,000 years ago the amount of ^{14}C in one gram of carbon in living tissue was approximately 1.05×10^{-12} grams. Suppose a 100 gram sample of carbon from bone is found to have 3×10^{-11} grams of ^{14}C. What is the age of the sample?

Exercise 5.5.15 Suppose solar radiation striking the ocean surface is 1250 W/m^2 and 20 percent of that energy is reflected by the surface of the ocean. Suppose also that 20 meters below the surface the light intensity is found to be 800 W/m^2.

a. Write an equation descriptive of the light intensity as a function of depth in the ocean.

b. Suppose a coral species requires 100 W/m^2 light intensity to grow. What is the maximum depth at which that species might be found?

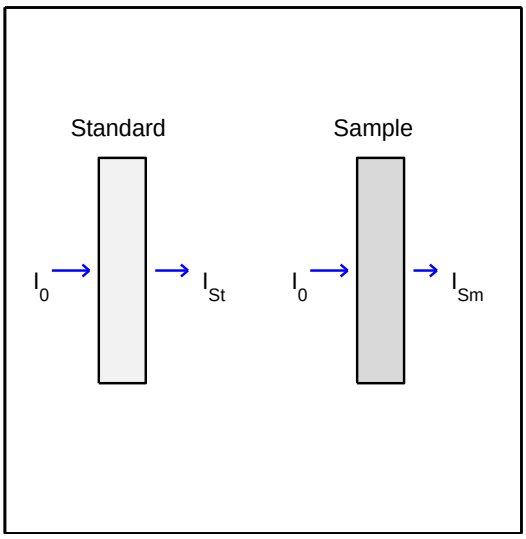
Exercise 5.5.16 In two bodies of water the light intensities $I_1(x)$ and $I_2(x)$ as functions of depth x are measured simultaneously and found to be

$$ I_1(x) = 800e^{-0.04x} \qquad \text{and} \qquad I_2(x) = 700e^{-0.05x} $$

Explain the differences in the two formulas in terms of the properties of water in the two bodies.

Exercise 5.5.17 A spectrophotometer is used to measure bacterial cell density in a growth medium. Light is passed through a sample of the medium and the amount of light that is absorbed by the medium is an indicator of cell density. As cell density increases the amount of light absorbed increases. A standard is established by passing a light beam of intensity I_0 through a 0.5 cm layer of the growth medium without bacteria and the intensity I_{st} of the beam emerging from the medium is measured. See Figure Ex. 5.5.17.

Figure for Exercise 5.5.17 Diagram of spectrophotometer. A light beam of intensity I_0 enters the standard solution and the intensity I_{st} of the emerging beam is measured. A light beam of the same intensity I_0 enters the sample solution and the intensity I_{sm} of the emerging beam is measured. See Exercise 5.5.17.



A light beam of the same intensity I_0 enters the sample solution and the intensity I_{sm} of the emerging beam is measured.

In the mathematical model of light absorbance (the amount of light absorbed by a layer of water is proportional to the thickness of the layer and to the amount of light entering the layer), the proportionality constant K is a measure of the opacity of the water. Recall that the solution Equation 5.25 is $I(x) = I_0 e^{-Kx}$.

The bacteria in the sample placed in the spectrophotometer increase the opacity of the liquid. Explain why cell density is proportional to

$$\ln\left(\frac{I_{sm}}{I_{st}}\right)$$

The number $\ln(I_{sm}/I_{st})$ is called *absorbance.*

Exercise 5.5.18 A patient comes into your emergency room and you start a penicillin infusion into the 6 liter vascular pool of 0.2 gms/min. Plasma circulates through the kidney at the rate of 1.2 liters/minute and the kidneys remove 20 per cent of the penicillin that passes through.

a. Draw a schematic diagram showing the vascular pool and kidneys as separate entities, an artery leading from the vascular pool to the kidney and a vein leading from the kidney back to the vascular pool.

b. Let $P(t)$ be the amount of penicillin in the vascular pool t minutes after injection of penicillin. What is $P(0)$?

c. Use Primitive Concept 2 to write an equation for P'.

d. Compute the solution to your equation and draw the graph of P.

e. The saturation level of penicillin in this problem is critically important to the correct treatment of your patient. Will it be high enough to control the infection you wish to control? If not, what should you do?

f. Suppose your patient has impaired kidney function and that plasma circulates through the kidney at the rate of 0.8 liters per minute and the kidneys remove 15 percent of the penicillin that passes through. What is the saturation level of penicillin in this patient, assuming you administer penicillin the same as before?

Exercise 5.5.19 An egg is covered by a hen and is at 37° C. The hen leaves the nest and the egg is exposed to 17° C air.

a. Draw a graph representative of the temperature of the egg t minutes after the hen leaves the nest.

Mathematical Model 5.5.6 Egg cooling. During any short time interval while the egg is uncovered, the change in egg temperature is proportional to the length of the time interval and proportional to the difference between the egg temperature and the air temperature.

b. Let $T(t)$ denote the egg temperature t minutes after the hen leaves the nest. Consider a short time interval, $[t, t + \Delta t]$, and write an equation for the change in temperature of the egg during the time interval $[t, t + \Delta t]$.

c. Argue that as Δt approaches zero, the terms of your previous equation get close to the terms of

$$T'(t) = -k(T(t) - 17) \tag{5.32}$$

d. Assume T(0) = 37 and find an equation for $T(t)$.

e. Suppose it is known that eight minutes after the hen leaves the nest the egg temperature is 35°C. What is k?

f. If the coldest temperature the embryo can tolerate is 32°C, when must the hen return to the nest?

Exercise 5.5.20 Consider the following osmosis experiment in biology laboratory.

Material: A thistle tube, a 1 liter flask, some 'salt water', and some pure water, a membrane that is impermeable to the salt and is permeable to the water.

The bulb of the thistle tube is filled with salt water, the membrane is place across the open part of the bulb, and the bulb is inverted in a flask of pure water so that the top of the pure water is at the juncture of the bulb with the stem. See the diagram.

Because of osmotic pressure the pure water will cross the membrane pushing water up the stem of the thistle tube until the increase in pressure inside the bulb due to the water in the stem matches the osmotic pressure across the membrane.

Our problem is to describe the height of the water in the stem as a function of time.

Mathematical Model 5.5.7 Osmotic diffusion across a membrane. The rate at which pure water crosses the membrane is proportional to the osmotic pressure across the membrane minus the pressure due to the water in the stem.

Assume that the volume of the bulb is much larger than the volume of the stem so that the concentration of 'salt' in the thistle tube may be assumed to be constant.

Introduce notation and write a derivative equation with initial condition that will describe the height of the water in the stem as a function of time. Solve your derivative equation.

Exercise 5.5.21 2 kilos of a fish poison are mixed into a lake which has a volume of $100 \times 20 \times 2 = 4000$ cubic meters. A stream of clean water flows into the lake at a rate of 1000 cubic meters per day. Assume that it mixes immediately throughout the whole lake. Another stream flows out of the lake at a rate of 1000 cubic meters per day.

a. What is the concentration of poison in the lake at time t days after the poison is applied? Treat the problem as a discrete time problem with one-day time intervals. Solve the difference equation

$$p_0 = 2 \qquad p_{t+1} - p_t = -\frac{1000}{4000} p_t$$

b. Let t denote continuous time and $P(t)$ the amount of poison in the lake at time t. Let $[t, t + \Delta t]$ denote a short time interval (measured in units of days). An equation for the mathematical model is

$$P(t + \Delta t) - P(t) = -\frac{P(t)}{4000} \times \Delta t \times 1000$$

Show that the units on the terms of this equation balance.

c. Argue that

$$P'(t) = -0.25P(t)$$

d. Argue that

$$P(t) = 2e^{-0.25t}$$

e. Compare p_t the solution to the discrete time problem with $P(t)$ the solution to the continuous time problem.

f. On what day, $\bar{t}$ will $P(\bar{t}) = 4$g?

Exercise 5.5.22 Continuous version of Chapter Exercise 1.7. Atmospheric pressure decreases with increasing altitude. Derive a dynamic equation from the following mathematical model, solve the dynamic equation, and use the data to evaluate the parameters of the solution equation.

Mathematical Model 5.5.8 Mathematical Model of Atmospheric Pressure. Consider a vertical column of air based at sea level and assume that the temperature within the column is constant, equal to 20°C. The pressure at any height is the weight of air in the column above that height divided by the cross sectional area of the column. In a 'short' section of the column, by the ideal gas law the the mass of air within the section is proportional to the product of the volume of the section and the pressure within the section (which may be considered constant and equal to the pressure at the bottom of the section). The weight of the air above the lower height is the weight of air in the section plus the weight of air above the upper height.

Sea-level atmospheric pressure is 760 mm Hg and the pressure at 18,000 feet is one-half that at sea level.

Exercise 5.5.23 When you open a bottle containing a carbonated soft drink, CO_2 dissolved in the liquid turns to gas and escapes from the liquid. If left open and undisturbed, the drink goes flat (looses its CO_2). Write a mathematical model descriptive of release of carbon dioxide in a carbonated soft drink. From your model, write a derivative equation descriptive of the carbon dioxide content in the liquid minutes after opening the drink.

Exercise 5.5.24 *Decompression illness in deep water divers.* In the 1800's technology was developed to supply compressed air to under water divers engaged in construction of bridge supports and underwater tunnels. While at depth those divers worked without unusual physical discomfort. Shortly after ascent to the surface, however, they might experience aching joints, numbness in the limbs, fainting, and possible death. Affected divers tended to walk bent over and were said to have the "bends".

It was believed that nitrogen absorbed by the tissue at the high pressure below water was expanding during ascent to the surface and causing the difficulty, and that a diver who ascended slowly would be at less risk. The British Navy commissioned physician and mathematician J. S. Haldane[9] to devise a dive protocol to be followed by Navy divers to reduce the risk of decompression illness. Nitrogen flows quickly between the lungs and the plasma but nitrogen exchange between the plasma and other parts of the body (nerve, brain tissue, muscle, fat, joints, liver, bone marrow, for example) is slower and not uniform. Haldane used a simple model for nitrogen exchange between the plasma and the other parts of the body.

Mathematical Model 5.5.9 Nitrogen absorption and release in tissue. The rate at which nitrogen is absorbed by a tissue is proportional to the difference in the partial pressure of nitrogen in the plasma and the partial pressure of nitrogen in the tissue.

Air is 79 percent nitrogen. Assume that the partial pressure of nitrogen in the lungs and the plasma are equal at any depth. At depth d,

$$\text{Plasma pp } N_2 = \text{Lung pp } N_2 \quad = 0.79 \times \left(1 + \frac{d}{10}\right) \qquad \text{atmospheres.}$$

a. What is the partial pressure of nitrogen in a diver's lungs at the surface?

b. Suppose a diver has not dived for a week. What would you expect to be the partial pressure of nitrogen in her tissue?

[9] J. S. Haldane was the father of J. B. S. Haldane who, along with R. A. Fisher and Sewall Wright developed the field of population genetics.

c. A diver who has not dived for a week quickly descends to 30 meters. What is the nitrogen partial pressure in her lungs after descending to 30 meters?

d. Let $N(t)$ be the partial pressure of nitrogen in a tissue of volume, V, t minutes into the dive. Use the Mathematical Model 5.5.9 Nitrogen absorption and release in tissue and Primitive Concept 2, to write an equation for N'.

e. Check to see whether (k is a proportionality constant)

$$N(t) = 0.79\left(1 + \frac{d}{10}\right) - 0.79\,\frac{d}{10}\,e^{-\frac{k}{V}t} \tag{5.33}$$

solves your equation from the previous step.

f. Assume $\frac{k}{V}$ in Equation 5.33 is 0.0693 and $d = 30$. What is $N(30)$?

g. Haldane experimented on goats and concluded that on the ascent to the surface, $N(t)$ should never exceed two times Lung pp N_2. A diver who had been at depth 30 meters for 30 minutes could ascend to what level and not violate this condition if $\frac{k}{V} = 0.0693$?

Exercise 5.5.25 E. O. Wilson, a pioneer in study of area-species relations on islands, states in *Diversity of Life*, p 221, :

> "In more exact language, the number of species increases by the area-species equation, $S = C\,A^z$, where A is the area and S is the number of species. C is a constant and z is a second, biologically interesting constant that depends on the group of organisms (birds, reptiles, grasses). The value of z also depends on whether the archipelago is close to source ares, as in the case of the Indonesian islands, or very remote, as with Hawaii $\cdots$ It ranges among faunas and floras around the world from about 0.15 to 0.35."

Discuss this statement as a potential Mathematical Model.

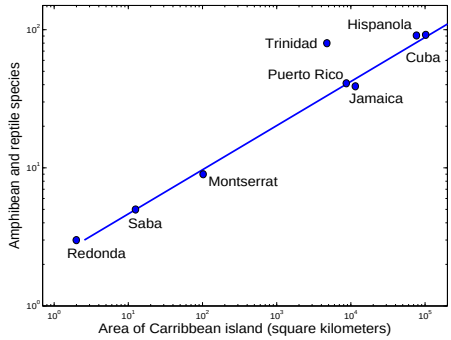
Exercise 5.5.26 The graph of Figure 5.5.26 showing the number of amphibian and reptile species on Caribbean Islands *vs* the areas of the islands is a classic example from P. J. Darlington, *Zoogeography: The Geographical Distribution of Animals*, Wiley, 1957, page 483, Tables 15 and 16.

a. Treat Trinidad as an unexplained outlier (meaning: ignore Trinidad) and find a power law, $S = C\,A^z$, relating number of species to area for this data.

b. Darlington observes that "$\cdots$ within the size range of these islands $\cdots$, division of the area by ten divides the amphibian and reptile fauna by two $\cdots$, but this ratio is a very rough approximation, and it might not hold in other situations." Is your power law consistent with Darlington's observation?

c. Why might Trinidad (4800 km^2) have nearly twice as many reptilian species (80) as Puerto Rico (8700 km^2) which has 41 species?

Figure for Exercise 5.5.26 The number of amphibian and reptile species on islands in the Caribbean *vs* the areas of the islands. The data is from Tables 15 and 16 in P. J. Darlington, *Zoogeography: The Geographical Distribution of Animals*, Wiley, 1957, page 483.



Exercise 5.5.27 The graph in Figure Ex. 5.5.27 relates surface area to mass of a number of mammals. Assume mammal densities are constant (each two mammals are equally dense), so that the graph also relates surface area to volume.

a. Find an equation relating the surface area, S, of a cube to the volume, V, of the cube.

b. Find an equation relating the surface area, $S = 4\pi r^2$, of a sphere of radius r to the volume, $V = \frac{4}{3}\pi r^3$, of the sphere.

c. Find an equation relating the surface area of a mammal to the mass of the mammal, using the graph in Figure Ex. 5.5.27. Ignore the dark dots; they are for beech trees.

d. In what way are the results for the first three parts of this exercise similar?

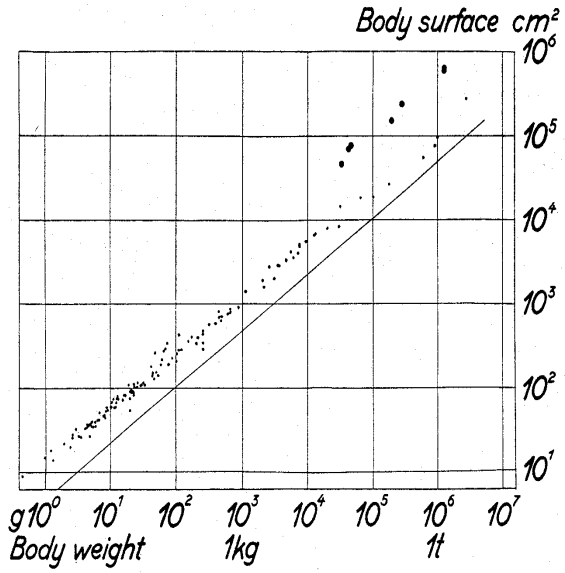


Figure for Exercise 5.5.27

Graph for Exercise 5.5.27 relating surface area to mass of mammals. From A. M. Hemmingson, Energy metabolism as related to body size, and its evolution, Rep. Steno Mem. Hosp. (Copenhagen) 9:1-110. With kind permission from Dr. Peter R. Rossing, Director of Research, Steno Diabetes Center S/A. All rights reserved.

5.6 The exponential chain rule and the logarithm chain rule.

Suppose an increasing function $u(t)$ has a derivative for all t. We establish

$$\left[e^{u(t)}\right]' = e^{u(t)} \times u'(t) \qquad \textbf{Exponential Chain Rule.} \qquad (5.34)$$

The Exponential Chain Rule is used to prove

$$[\ln t]' = \frac{1}{t}, \qquad \textbf{Logarithm Rule.} \qquad (5.35)$$

and, assuming u is positive,

$$[\ln u(t)]' = \frac{1}{u(t)} \times u'(t) \qquad \textbf{Logarithm Chain Rule.} \qquad (5.36)$$

Recall Theorem 4.2.1, The Derivative Requires Continuity. If $u(t)$ is a function and $u'(t)$ exists at $t = a$

$$\lim_{b \to a} u(b) = u(a)$$

Proof of the exponential chain rule. Suppose $u(t)$ has a derivative at $t = a$ and $E(t) = e^{u(t)}$. To simplify the argument we assume that $u(t)$ is a strictly increasing function. That is, if $a < b$, then $u(a) < u(b)$, and, in particular, $u(b) - u(a) \neq 0$.

Then

$$\begin{aligned} E'(a) &= \lim_{b \to a} \frac{e^{u(b)} - e^{u(a)}}{b - a} \\ &= \lim_{b \to a} \frac{e^{u(b)} - e^{u(a)}}{u(b) - u(a)} \times \frac{u(b) - u(a)}{Db - a} \\ &= \lim_{b \to a} \frac{e^{u(b)} - e^{u(a)}}{u(b) - u(a)} \times \lim_{b \to a} \frac{u(b) - u(a)}{Db - a} \\ &= e^{u(a)} \times u'(a) \end{aligned}$$

The limit

$$\lim_{b \to a} \frac{e^{u(b)} - e^{u(a)}}{u(b) - u(a)} = e^{u(a)}$$

requires some explanation. The graph of $y = e^x$ is shown in Figure 5.8. At every point, $(\, x \,,\, e^x \,)$ of the graph, the slope of the tangent is e^x, and specifically at the point $(\, u(a) \,,\, e^{u(a)} \,)$ the slope is $m = e^{u(a)}$. The difference quotient

$$\frac{e^{u(b)} - e^{u(a)}}{u(b) - u(a)}$$

is the slope of a secant to the graph. Because $\lim_{b \to a} u(b) = u(a)$ the slope of the secant approaches the slope of the tangent. Thus

$$\lim_{b \to a} \frac{e^{u(b)} - e^{u(a)}}{u(b) - u(a)} = e^{u(a)}.$$

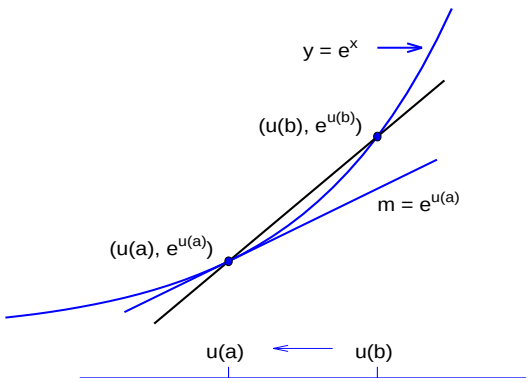


Figure 5.8: Graph of $y = e^x$. Because $u'(a)$ exists, $u(b) \to u(a)$ as $b \to a$. Therefore the slope of the secant, $\frac{u(b)-u(a)}{b-a}$, approaches the slope of the tangent, $e^{u(a)}$.

Example 5.6.1 Find the derivatives of

$$P(x) = 200e^{-3x} \qquad\qquad Q(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Solutions:

$$\begin{aligned}
P'(x) &= [200e^{-3x}]' & Q'(x) &= \left[\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]' && \text{Logical Identity}\\
&= 200\,[e^{-3x}]' & &= \frac{1}{\sqrt{2\pi}}\left[e^{-x^2/2}\right]' && \text{Constant Factor Rule}\\
&= 200e^{-3x}\,[-3x]' & &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,[-x^2/2]' && \text{Exponential Chain Rule}\\
&= 200e^{-3x}(-3) & &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}(-x) && \text{Constant Factor and Power Rules}
\end{aligned}$$

The derivative of $P(x) = 200e^{-3x}$ also can be computed using the e^{kt} Rule. The e^{kt} rule is a special case of the exponential chain rule:

$$\left[\, e^{kt} \,\right]' \quad = \quad e^{kt}\ [\, kt \,]' \quad = \quad e^{kt}\ k.$$

The derivative of $L(t) = \ln t$. The exponential chain rule 5.34 is used to derive the natural logarithm rule. In order to use the formula $\left[\, e^{u(t)} \,\right]' = e^{u(t)}u'(t)$ it is necessary to know that $u'(t)$ exists. We want to compute $\left[\, e^{\ln t} \,\right]'$ and need to know that $[\,\ln t\,]'$ exists. Our argument for this is shown in Figure 5.9. Also observe that L is increasing so that our argument for the exponential chain rule which assumed that $u(t)$ was increasing is sufficient for this use.
Knowing that $[\,\ln t\,]'$ exists, it is easy to obtain a formula for it.

$$\left.\begin{aligned}
e^{\ln t} &= t && \text{a.}\\
\left[\, e^{\ln t} \,\right]' &= [\,t\,]' && \text{b.}\\
e^{\ln t}\,[\,\ln t\,]' &= 1 && \text{c.}\\
t\,[\,\ln t\,]' &= 1 && \text{d.}\\
[\,\ln t\,]' &= \tfrac{1}{t} && \text{Algebra}
\end{aligned}\right\} \tag{5.37}$$

We have now proved the natural logarithm rule, another Primary Formula:

Natural Logarithm Rule

$$[\ln\, t]' = \frac{1}{t} \tag{5.38}$$

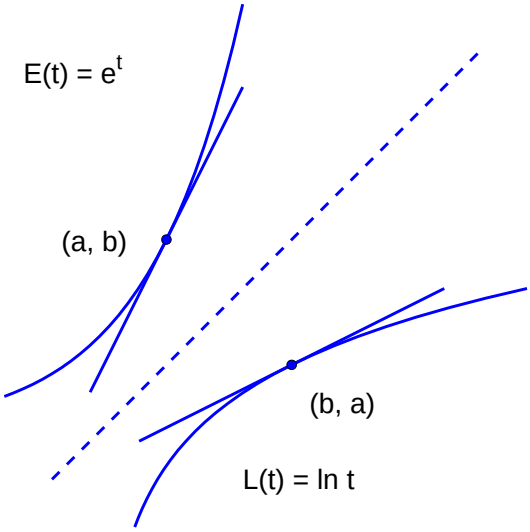


Figure 5.9: Graphs of $E(t) = e^t$ and $L(t) = \ln t$ and tangents at corresponding points. Because L is the inverse of E, the graph of L is the reflection of the graph of E about the line $y = x$. Because E has a tangent at (a, b), L has a tangent at (b, a). Therefore $L'(b)$ exists for every positive value of b.

Using the natural logarithm rule and some properties of logarithms we can differentiate $y(t) = \ln 5t + \ln t^3$:

$$\begin{aligned}
y'(t) &= [\ln 5t + \ln t^3]' && \text{Symbolic identity} \\
&= [\ln 5 + \ln t + 3\ln t]' && \text{Logarithm Prop's} \\
&= [\ln 5]' + [\ln t]' + [3\ln t]' && \text{Sum Rule} \\
&= 0 + [\ln t]' + [3\ln t]' && \text{Constant Rule} \\
&= 0 + [\ln t]' + 3\,[\ln t]' && \text{Constant Factor Rule} \\
&= 0 + \tfrac{1}{t} + 3\tfrac{1}{t} && \text{Natural Logarithm Rule} \\
&= 4\tfrac{1}{t}
\end{aligned}$$

Example 5.6.2 Logarithm functions to other bases have derivatives, but not as neat as $1/t$. We compute $L'(t)$ for $L(t) = \log_b t$ where $b > 0$ and $b \neq 1$.

Solution.

$$[\log_b t]' \overset{a}{=} \left[\frac{\ln t}{\ln b}\right]' \overset{b}{=} \frac{1}{\ln b}[\ln t]' \overset{c}{=} \frac{1}{\ln b}\frac{1}{t} \tag{5.39}$$

We summarize this as

$$\boxed{[\log_b t]' = \frac{1}{\ln b} \times \frac{1}{t} \qquad (5.40)}$$

The Logarithm Chain Rule. We now use the Exponential Chain Rule 5.34 to show that if $u(t)$ is a positive increasing function and $u'(t)$ and $[\ln u(t)]'$ exist for all t, then

$$[\ln u(t)]' = \frac{1}{u(t)} \times [u(t)]'$$

Proof. By the Exponential Chain Rule and $u(t) = e^{\ln u(t)}$,

$$\begin{aligned} \left[e^{\ln u(t)}\right]' &= [u(t)]' \\ e^{\ln u(t)}[\ln u(t)]' &= u'(t) \\ [\ln u(t)]' &= \frac{1}{u(t)}u'(t) \end{aligned}$$

This is the

Logarithm Chain Rule

$$[\ln\ u(t)]' = \frac{1}{u(t)}\,u'(t) \tag{5.41}$$

Example 5.6.3 Find the derivatives of

a. $y = \ln\left(\sqrt{1-t^2}\right)$ b. $y = \ln\left(\dfrac{1-t}{1+t}\right)$

Solutions:

$$\begin{aligned} \left[\ln\left(\sqrt{1-t^2}\right)\right]' &= \left[\tfrac{1}{2}\ln(1-t^2)\right]' && \text{Logarithm Property} \\ &= \tfrac{1}{2}[\ln(1-t^2)]' && \text{Constant Factor} \\ &= \tfrac{1}{2}\frac{1}{1-t^2}[(1-t^2)]' && \text{Generalized Logarithm Rule} \\ &= \tfrac{1}{2}\frac{1}{1-t^2}(-2t) && \text{Sum, Constant, Constant Factor, and Power Rules} \end{aligned}$$

$$
\begin{aligned}
\text{b.}\quad \left[\,\ln\left(\frac{1-t}{1+t}\right)\,\right]' &= [\,\ln(1-t)-\ln(1+t)\,]' && \text{Logarithm Property}\\
&= [\ln(1-t)]' - [\ln(1+t)]' && \text{Sum Rule}\\
&= \frac{1}{1-t}\,[1-t]' - \frac{1}{1+t}\,[1+t]' && \text{Generalized Logarithm Rule}\\
&= \frac{1}{1-t}(-1) - \frac{1}{1+t}(1) && \text{Sum, Constant, and Power Rules}\\
&= \frac{-2}{1-t^2} && \text{Algebra} \quad \blacksquare
\end{aligned}
$$

Example 5.6.4 *Logarithmic Differentiation.* Suppose we are to differentiate

$$y(t) = (t+2)(t+1)(t-1)$$

Proceeding indirectly, we first compute the derivative of the natural logarithm of y.

$$
\begin{aligned}
\ln(\,y(t)\,) &= \ln(\,(t+2)(t+1)(t-1)\,) && \text{Logical Identity}\\
\ln(\,y(t)\,) &= \ln(t+2)+\ln(t+1)+\ln(t-1) && \text{Logarithm Property}\\
[\,\ln(\,y(t)\,)\,]' &= [\,\ln(t+2)+\ln(t+1)+\ln(t-1)\,]' && \text{Logical Identity.}\\
\frac{1}{y(t)}y'(t) &= \frac{1}{t+2}\,[\,t+2\,]' + \frac{1}{t+1}\,[\,t+1\,]' + \frac{1}{t-1}\,[\,t-1\,]' && \text{Logarithm chain rule.}\\
y'(t) &= y(t)\times\left(\frac{1}{t+2}+\frac{1}{t+1}+\frac{1}{t-1}\right) && [t+C]'=1.\\
y'(t) &= (t+2)(t+1)(t-1)\times\left(\frac{1}{t+2}+\frac{1}{t+1}+\frac{1}{t-1}\right) && \text{Definition of } y.\\
y'(t) &= (t+1)(t-1)+(t+2)(t-1)+(t+2)(t+1) && \text{Algebra.} \quad \blacksquare
\end{aligned}
$$

Exercises for Section 5.6, The exponential chain rule and the logarithm chain rule.

Exercise 5.6.1 Use one rule for each step and identify the rule to differentiate

a. $P(t)$	$=$	$3\ln t + e^{3t}$	b. $P(t)$	$=$	$t^2+\ln 2t$
c. $P(t)$	$=$	$\ln 5$	d. $P(t)$	$=$	$\ln(e^{2t})$
e. $P(t)$	$=$	$\ln(t^2+t)$	f. $P(t)$	$=$	e^{t^2-t}
g. $P(t)$	$=$	$e^{1/x}$	h. $P(t)$	$=$	$e^{\sqrt{x}}$
i. $P(t)$	$=$	$\ln((t+1)^2)$	j. $P(t)$	$=$	$e^{-t^2/2}$

Exercise 5.6.2 Compute the derivatives of

a. $P(t) = e^{(t^2)}$	b. $P(t) = \ln(t^2)$	c. $P(t) = (e^t)^2$
d. $P(t) = e^{(2\ln t)}$	e. $P(t) = \ln(e^{3t})$	f. $P(t) = \sqrt{e^t}$
g. $P(t) = e^5$	h. $P(t) = \ln(\sqrt{t})$	i. $P(t) = e^{t+1}$

Exercise 5.6.3 Supply reasons that justify the steps a - d in the equations 5.37.

Exercise 5.6.4 Give reasons for the steps a - c in Equation 5.39 deriving the derivative of the logarithm function $L(t) = \log_b t$.

Exercise 5.6.5 Find a value for x for which $P'(x) = 0$.

a. $P(t) = xe^{-x}$	b. $P(t) = e^{-x} - e^{-2x}$	c. $P(t) = x^2e^{-x}$

Exercise 5.6.6 Use the logarithm chain rule to prove that for all numbers, n:

$$\text{Power chain rule for all } n \qquad [(\,u(t)\,)^n]' = n\,(\,u(t)\,)^{n-1} \times u'(t)$$

Assume that u is an increasing function and $u'(t)$ exists.

Exercise 5.6.7 Use the logarithmic differentiation to compute $y'(t)$ for

a. $y(t) = t^\pi$	b. $y(t) = t^e$	c. $y(t) = (1+t^2)^\pi$

Exercise 5.6.8 Use the logarithmic differentiation to compute $y'(t)$ for

a. $y(t) = \dfrac{(t-1)(t+1)}{t-2}$	b. $y(t) = te^t$	c. $y(t) = e^{-\frac{t^2}{2}}$
d. $y(t) = \sqrt{1+t^2}$	e. $y(t) = \dfrac{t^2}{t^2+1}$	f. $y(t) = 2^t$
g. $y(t) = b^t \quad b > 0$	h. $y(t) = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$	i. $y(t) = \dfrac{\ln t}{e^t}$

5.7 Summary.

The remarkable number e, the exponential function $E(t) = e^t$, and the natural logarithm function $L(t) = \ln t$ are the basic material for this chapter. The number e is defined by

$$\lim_{c\to 0} (1+c)^{\frac{1}{c}} = e$$

It is the only number, b, for which

$$\lim_{h\to 0} \frac{b^h - 1}{h} = 1$$

Because $\lim_{h\to 0}(e^h - 1)/h = 1$, the function $E(t) = e^t$ has the property that $E'(t) = e^t$. For any other base, $b > 0$, the exponential function $B(t) = b^t$ has a derivative, but $B'(t) = b^t \times \ln b$ has the extra factor $\ln b$. The natural logarithm function, $L(t) = \log_e t = \ln t$ also has a simple derivative, $1/t$, and its derivative is the simplest among all logarithm functions. For $L(t) = \log_b t$, $L'(t) = (1/t) \times (\ln b)$.

The function $E(t) = e^{kt}$ has the property

$$E'(t) = e^{kt} \times k = k \times E(t)$$

Many mathematical models of biological and physical systems yield equations of the form $y'(t) = ky(t)$, and for that reason we frequently use the exponential function e^{kt} to describe natural phenomena.

When analyzing data thought to be exponential, a semilog graph of the data will often signal whether the data is indeed exponential.

The exponential and logarithm chain rules

$$\left[e^{u(t)}\right]' = e^{u(t)} \times u'(t) \qquad [\ln(u(t))]' = \frac{1}{u(t)} \times u'(t)$$

expand the class of functions for which we can compute derivatives, and the logarithm chain rule is used to extend the power chain rule for integer exponents to all values of the exponent.

Exercises for the Summary of Chapter 5.

Chapter Exercise 5.1 Compute $P'(t)$ for:

a. $P(t) = e^{5t}$ b. $P(t) = \ln 5t$ c. $P(t) = e^{t\sqrt{t}}$

d. $P(t) = e^{\sqrt{2t}}$ e. $P(t) = \ln(\ln t)$ f. $P(t) = e^{\ln t}$

g. $P(t) = 1/(1+e^t)$ h. $P(t) = 1/\ln t$ i. $P(t) = 1/(1+e^{-t})$

j. $P(t) = (1+e^t)^3$ k. $P(t) = \left(e^{\sqrt{t}}\right)^3$ l. $P(t) = \ln\sqrt{t}$

Chapter Exercise 5.2 Use the logarithmic differentiation to compute $y'(t)$ for

a. $y(t) = 10^t$ b. $y(t) = \dfrac{t-1}{t+1}$ c. $y(t) = (t-1)^3(t^3-1)$

Chapter Exercise 5.3 Use a semilog graph to determine which of the following data sets are exponential.

a.

t	P(t)
0	5.00
1	3.53
2	2.50
3	1.77
4	1.25
5	0.88

b.

t	P(t)
0	5.00
1	1.67
2	1.00
3	0.71
4	0.55
5	0.45

c.

t	P(t)
0	5.00
1	3.63
2	2.50
3	1.63
4	1.00
5	0.63

Chapter Exercise 5.4 The function

$$P(t) = \frac{10 \times e^t}{9 + e^t} \quad = \quad \frac{10}{9 \times e^{-t} + 1}$$

is an example of a logistic function that often is used to describe the growth of populations. Use your calculator to plot the graph of this function. Then find how to compute $P'(t)$ on your calculator. It may be dy/dx, or $d(y, x)$, or $differentiate$. Draw the graph of P' and find at what time it is a maximum. Identify the point on the graph of P corresponding to that time.

Chapter Exercise 5.5 A pristine lake of volume 1,000,000 m^3 has a river flowing through it at a rate of 10,000 m^3 per day. A city built beside the river begins dumping 1000 kg of solid waste into the river per day.

1. Write a derivative equation that describes the amount of solid waste in the lake t days after dumping begins.

2. What will be the concentration of solid waste in the lake after one year?

Chapter Exercise 5.6 Estimate the slope of the tangent to the graph of

$$y = \log_{10} x$$

at the point $(3, \log_{10} 3)$ correct to three decimal digits.

Chapter Exercise 5.7 Show that $y = te^{3t}$ satisfies $y'' - 6y\prime + 9y = 0$. Use logarithmic differentiation.

Chapter Exercise 5.8 Show that for any numbers C_1 and C_2, $y = C_1e^t + C_2e^{-t}$ satisfies $y'' - y = 0$.

Chapter 6

Derivatives of Products, Quotients and Compositions of Functions.

Where are we going?

The basic derivative formulas that you need are shown below. Many additional derivative formulas are derived from them.

The trigonometric derivatives are develop in Chapter 7. The last three Combination Rules are developed in this chapter.

Primary Formulas

$$
\begin{aligned}
[\,C\,]' &= 0 & [\,t^n\,]' &= nt^{n-1} \\
[\,e^t\,]' &= e^t & [\,\ln t\,]' &= \frac{1}{t} \\
[\,\sin t\,]' &= \cos t & [\,\cos t\,]' &= -\sin t
\end{aligned}
\tag{6.1}
$$

Combination formulas

$$
\begin{aligned}
[\,u+v\,]' &= [\,u\,]' + [\,v\,]' & [\,C\,u\,]' &= C\,[\,u\,]' \\
[\,u\,v\,]' &= [\,u\,]'\,v + u\,[\,v\,]' & \left[\,\frac{u}{v}\,\right]' &= \frac{v\,[\,u\,]' - u\,[\,v\,]'}{u^2} \\
[\,G(u)\,]' &= G'(\,u\,)\,u'
\end{aligned}
$$

6.1 Derivatives of Products and Quotients.

Derivatives of products. We determine the derivative of a function, P, that is a product of two functions, $P(t) = u(t) \times v(t)$ in terms of the values of $u(t)$, $u'(t)$, $v(t)$ and $v'(t)$; all four are required. The formula we obtain is

$$[\,u(t) \times v(t)\,]' \quad = \quad u'(t) \times v(t) + u(t) \times v'(t) \tag{6.2}$$

It is not a very intuitive formula. The derivative of a sum of two functions is the sum of the derivatives of the functions. One might expect derivative of the product of two functions to be the product of the derivatives of the two functions. Alas, this is seldom correct, but were it correct your learning of calculus would be notably simplified.

The correct formula is used as follows. Let

$$P(t) \quad = \quad t^3 \times e^{-2t}$$

Then with $u(t) = t^3$ and $v(t) = e^{-2t}$,

$$\begin{aligned} P'(t) &= \left[t^3\right]' \times e^{-2t} + t^3 \times \left[e^{-2t}\right]' \\ &= 3t^2 \times e^{-2t} + t^3 \times e^{-2t} \times (-2) \end{aligned}$$

A product of functions is useful, for example, in examining the factors affecting total corn production. Total production, $P(t)$, is the product of the number of acres planted, $A(t)$, and the average yield per acre, $Y(t)$. The factors that affect $A(t)$ and $Y(t)$ are distinct. The acres planted, $A(t)$, is affected mostly by government programs and anticipated price of corn; the yield, $Y(t)$ is affected mostly by natural events such as weather and insect prevalence and by improved genetics and farming practices. Government economists often try to maintain total production, $P(t)$, at a fairly constant level, but can affect only $A(t)$, the number of acres planted.

Other instances in which a function is inherently a product of component parts include

1. In simple epidemiological models, the number of newly infected is proportional to the product of the number of infected and the number of susceptible.

2. The rate of a binary chemical reaction $\mathrm{A} + \mathrm{B} \longrightarrow \mathrm{AB}$ is usually proportional to the product of the concentrations of the two constituents of the reaction.

We prove the following theorem:

Theorem 6.1.1 *Suppose u and v are two functions. Then for every number a for which $u'(a)$ and $v'(a)$ exist,*

$$[\,u(t) \times v(t)\,]'_{t=a} = u'(a) \times v(a) + u(a) \times v'(a) \tag{6.3}$$

The proof uses Theorem 4.2.1, The Derivative Requires Continuity:

In symbols: $\displaystyle\lim_{b \to a} \frac{u(b) - u(a)}{b - a} = u'(a)$ exists implies that $\displaystyle\lim_{b \to a} u(b) = u(a)$.

Proof of Theorem 6.1.1.

$$\left.\begin{aligned}
[\,u(t)\times v(t)\,]'_{t=a} &\stackrel{\text{a}}{=} \lim_{b\to a}\frac{u(b)\times v(b)-u(a)\times v(a)}{b-a}\\
&\stackrel{\text{b}}{=} \lim_{b\to a}\frac{u(b)\times v(b)-u(a)\times v(b)+u(a)\times v(b)-u(a)\times v(a)}{b-a}\\
&\stackrel{\text{c}}{=} \lim_{b\to a}\left(\frac{u(b)\times v(b)-u(a)}{b-a}\times v(b)+u(a)\times\frac{v(b)-v(a)}{b-a}\right)\\
&\stackrel{\text{d}}{=} \lim_{b\to a}\left(\frac{u(b)-u(a)}{b-a}\times v(b)\right)+u(a)\times\lim_{b\to a}\frac{v(b)-v(a)}{b-a}\\
&\stackrel{\text{e}}{=} \lim_{b\to a}\frac{u(b)-u(a)}{b-a}\times v(a)+u(a)\times\lim_{b\to a}\frac{v(b)-v(a)}{b-a}\\
&\stackrel{\text{f}}{=} u'(a)\times v(a)+u(a)\times v'(a)
\end{aligned}\right\}\tag{6.4}$$

End of proof.

Explore 6.1.1 In which step of Equations 6.4 was Theorem 4.2.1, The Derivative Requires Continuity, used? ■

Derivatives of quotients. The $\tan x = \frac{\sin x}{\cos x}$ is the quotient of two functions, $\sin x$ and $\cos x$. The logistic function, $L(x) = \frac{10e^x}{9+e^x}$, used to describe population growth and chemical reactions is the quotient of two exponential functions. There is a formula for computing the rate of change of quotients:

Theorem 6.1.2 Suppose u and v are functions and $u'(a)$ and $v'(a)$ exist and $v(a) \neq 0$. Then

$$\left[\frac{u(t)}{v(t)}\right]'_{t=a} = \frac{u'(a)\times v(a)-u(a)\times v'(a)}{v^2(a)}\tag{6.5}$$

Note: $v^2(a)$ is $(v(a))^2$.

Proof of Theorem 6.1.2.

$$\left.\begin{aligned}
\left[\frac{u(t)}{v(t)}\right]'_{t=a} &\overset{\text{a}}{=} \lim_{b\to a} \frac{\dfrac{u(b)}{v(b)} - \dfrac{u(a)}{v(a)}}{b-a} \\
&\overset{\text{b}}{=} \lim_{b\to a} \frac{\dfrac{u(b)v(a) - u(a)v(b)}{b-a}}{v(b)v(a)} \\
&\overset{\text{c}}{=} \lim_{b\to a} \frac{\dfrac{u(b)v(a) - u(a)v(a) + u(a)v(a) - u(a)v(b)}{b-a}}{v(b)v(a)} \\
&\overset{\text{d}}{=} \lim_{b\to a} \frac{\dfrac{u(b)-u(a)}{b-a}v(a) - u(a)\dfrac{v(b)-v(a)}{b-a}}{v(b)v(a)} \\
&\overset{\text{e}}{=} \frac{\left(\lim\limits_{b\to a} \dfrac{u(b)-u(a)}{b-a}\right) v(a) - u(a) \lim\limits_{b\to a} \dfrac{v(b)-v(a)}{b-a}}{\left(\lim\limits_{b\to a} v(b)\right) v(a)} \\
&\overset{\text{f}}{=} \frac{u'(a)\times v(a) - u(a)\times v'(a)}{v(a)v(a)}
\end{aligned}\right\} \tag{6.6}$$

End of proof.

Explore 6.1.2 Was Theorem 4.2.1, The Derivative Requires Continuity, used in Equations 6.6? ■

Example 6.1.1 The logistic function and its derivative. The logistic function

$$P(t) = \frac{P_0 M e^{rt}}{M - P_0 + P_0 e^{rt}} \tag{6.7}$$

describes the size of a population of initial size P_0 with low density relative growth rate r growing in an environment with limited carrying capacity M. After the function e^{kt}, the logistic function is the most important function in population biology. The graph of a typical logistic curve is shown in Figure 6.1. Obviously, population growth rate is important, and we use the quotient rule to compute $P'(t)$.

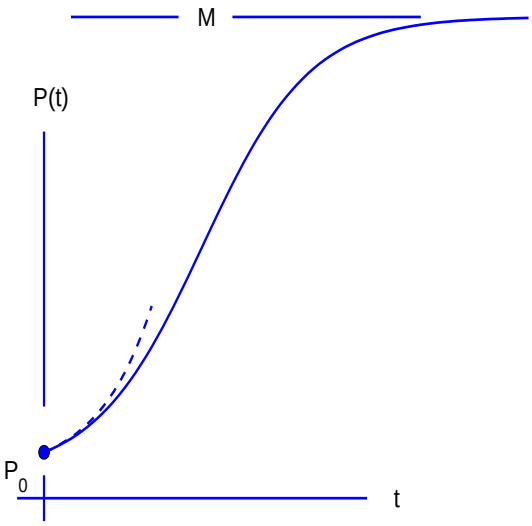


Figure 6.1: The graph of a logistic curve $P(t) = P_0Me^{rt}/(M-P_0+P_0e^{rt})$. The dashed curve is the graph of $p(t) = P_0e^{rt}$ showing the close approximation to exponential growth for $P(t)/M$ small (low density).

$$\left.\begin{aligned}
P'(t) &\stackrel{a}{=} \left[\frac{P_0Me^{rt}}{M-P_0+P_0e^{rt}}\right]' \\
&\stackrel{b}{=} P_0M\left[\frac{e^{rt}}{M-P_0+P_0e^{rt}}\right]' \\
&\stackrel{c}{=} P_0M\frac{(M-P_0+P_0e^{rt})\left[e^{rt}\right]' - e^rt\left[M-P_0+P_0e^{rt}\right]'}{\left(M-P_0+P_0e^{rt}\right)^2} \\
&\stackrel{d}{=} P_0M\frac{(M-P_0+P_0e^{rt})e^{rt}\times r - e^{rt}\times P_0e^{rt}\times r}{\left(M-P_0+P_0e^{rt}\right)^2} \\
&\stackrel{e}{=} P_0M\frac{(M-P_0)e^{rt}\times r}{\left(M-P_0+P_0e^{rt}\right)^2} \\
&\stackrel{f}{=} r\frac{P_0Me^{rt}}{M-P_0+P_0e^{rt}}\frac{(M-P_0)}{M-P_0+P_0e^{rt}} \\
&= rP(t)\left(1-\frac{P(t)}{M}\right)
\end{aligned}\right\} \tag{6.8}$$

Step e. shows P'. Steps f. and g. characterize the population growth rate as

$$P'(t) = rP(t)\left(1-\frac{P(t)}{M}\right) \tag{6.9}$$

The fraction $P(t)/M$ is the *density* of the population. If the density is small (population size, $P(t)$, is small compared to the environmental carrying capacity, M), the factor $1 - P(t)/M$ is close to 1 and 'almost' $P'(t) = rP(t)$. Almost $P'(t)/P(t) = r$ and for that reason r is called the *low density relative growth rate* of P. We compare $P(t)$ with the function $p(t)$ which satisfies the simpler equation

$$p(0) = P_0, \qquad p'(t) = rp(t)$$

We know from Section 5.5 that

$$p(t) = P_0 e^{rt}$$

The graph of $p(t)$ is shown as the dashed curve in Figure 6.1 where it is seen that $p(t)$ is close to $P(t)$ while $P(t)$ is small.

The number $M - P(t)$ is the unused environment, or the *residual environmental capacity.* When $P(t)$ is almost as large as M (the density is large), the residual capacity $M - P(t)$ is close to zero and the factor $(1 - P(t)/M)$ is close to zero. From Equation 6.9 the growth rate of the population $P'(t)$ is also close to zero. Equation 6.9 is consistent with:

Mathematical Model 6.1.1 Mathematical model of logistic growth. The growth rate of a population is proportional to the size of the population and is proportional to the residual capacity of the environment in which the population is growing.

We acknowledge that we have reversed the usual role of modeling. We began with a reported solution equation, obtained a derivative equation, and then wrote the model. The steps are reversed with respect to the accepted order in Chapter 1, and with respect to Pierre Verhulst's development of the model in 1838.

Example 6.1.2 Examples of computing the derivatives of products and quotients.

a.
$$\begin{aligned} P(t) &= e^{2t}\ln t & P'(t) &= [e^{2t}\ln t]' \\ & & &= [e^{2t}]'\ln t + e^{2t}[\ln t]' \\ & & &= e^{2t}\,2\,\ln t + e^{2t}\frac{1}{t} \end{aligned}$$

b.
$$\begin{aligned} P(t) &= \frac{3t-2}{4+t^2} & P'(t) &= \left[\frac{3t-2}{4+t^2}\right]' \\ & & &= \frac{(4+t^2)[3t-2]' - (3t-2)[4+t^2]'}{(4+t^2)^2} \\ & & &= \frac{(4+t^2)3 - (3t-2)(0+2t)}{(4+t^2)^2} \\ & & &= \frac{12-6t^2}{(4+t^2)^2} \end{aligned}$$

c.
$$\begin{aligned} P(t) &= \frac{e^{2t}}{\ln t} & P'(t) &= \left[\frac{e^{2t}}{\ln t}\right]' \\ & & &= \frac{\ln t[e^{2t}]' - e^{2t}[\ln t]'}{(\ln t)^2} \\ & & &= \frac{(\ln t)e^{2t}2 - e^{2t}\frac{1}{t}}{(\ln t)^2} \\ & & &= e^{2t}\frac{2t(\ln t)-1}{t(\ln t)^2} \end{aligned}$$

Exercises for Section 6.1, Derivatives of Products and Quotients.

Exercise 6.1.1 The word *differentiate* means 'find the derivative of'.
Differentiate

1. $P(t) = \frac{e^{-3t}}{t^2}$
2. $P(t) = e^{2\ln t}$
3. $P(t) = e^t \ln t$
4. $P(t) = t^2 \times e^{2t}$
5. $P(t) = e^{t\ln 2}$
6. $P(t) = e^2$
7. $P(t) = (e^t)^5$
8. $P(t) = \dfrac{t-1}{t+1}$
9. $P(t) = \dfrac{3t^2-2t-1}{t^-1}$

Exercise 6.1.2 Compute P' for:

a.	$P(t)$	$=$	t^2e^t	b.	$P(t)$	$=$	$\sqrt{t}e^{\sqrt{t}}$
c.	$P(t)$	$=$	$\dfrac{t}{1+t^2}$	d.	$P(t)$	$=$	$\dfrac{t+1}{t-1}$
e.	$P(t)$	$=$	$e^t\sqrt{1+t}$	f.	$P(t)$	$=$	$t\ln t - t$
g.	$P(t)$	$=$	$te^t - e^t$	h.	$P(t)$	$=$	$t^2e^t - 2te^t + 2e^t$
i.	$P(t)$	$=$	$\dfrac{\sqrt{t}}{\ln t}$	j.	$P(t)$	$=$	$e^t \ln t$
k.	$P(t)$	$=$	$\dfrac{1}{\ln t}$	l.	$P(t)$	$=$	$e^{(t\ln t)}$
m.	$P(t)$	$=$	$10\dfrac{e^{0.2t}}{9+e^{0.2t}}$	n.	$P(t)$	$=$	$\dfrac{20}{1+19e^{-0.1t}}$

Exercise 6.1.3 Give reasons for steps a - e in Equations 6.4 proving Theorem 6.1.1, the derivative of a product formula.

Exercise 6.1.4 Give reasons for steps a - f in Equations 6.6 proving Theorem 6.1.2, the derivative of a quotient formula.

Exercise 6.1.5 Write an equation that interprets the mathematical model of logistic growth, Mathematical Model 6.1.1, and show that it can be written in the form of Equation 6.9.

Exercise 6.1.6 Is there an example of two functions, $u(x)$ and $v(x)$, for which $[u(x) \times v(x)]' = u'(x) \times v'(x)$?

Exercise 6.1.7 Is there an example of two functions, $u(x)$ and $v(x)$, for which $\left[\dfrac{u(x)}{v(x)}\right]' = \dfrac{u'(x)}{v'(x)}$?

Exercise 6.1.8 An examination of 1000 people showed that 41 were carriers (heterozygotic) of the gene for cystic fibrosis. Let p be the proportion of all people who are carriers of cystic fibrosis. We can not say with certainty that $p = 41/1000$. For any number p in [0,1], let $L(p)$ be the *likelihood* of the event that 41 of 1000 people are carriers of cystic fibrosis given that the probability of being a carrier is p. Then

$$L(p) = \binom{1000}{41} p^{41} \times (1-p)^{959}$$

where $\binom{1000}{41}$ is a constant[1] approximately equal to 1.3×10^{73}.

a. Compute $L'(p)$.

[1] $\binom{n}{r}$ is the number of r member subsets of a set with n elements, and is equal to $\frac{n!}{r!(n-r)!}$.

b. Find the value $\hat{p}$ of p for which $L'(p) = 0$.

The value $L(\hat{p})$ is the maximum value of $L(p)$ and $\hat{p}$ is called the maximum likelihood estimator of p.

Exercise 6.1.9 An examination of 1000 people showed that 41 were carriers (heterozygotic) of the gene for cystic fibrosis. In a second, independent examination of 2000 people, 79 were found to be carriers of cystic fibrosis. Let p be the proportion of all people who are carriers of cystic fibrosis. For any number p in [0,1], let $L(p)$ be the likelihood of finding that 41 of 1000 people in one study and 79 out of 2000 people in a second independent study are carriers of cystic fibrosis given that the probability of being a carrier is p. Then

$$L(p) = \binom{1000}{41} p^{41} \times (1-p)^{959} \times \binom{2000}{79} p^{79} \times (1-p)^{1921}$$

where $\binom{1000}{41} \doteq 1.3 \times 10^{73}$ and $\binom{2000}{79} \doteq 1.4 \times 10^{143}$ are constants.

a. Simplify $L(p)$.

b. Compute $L'(p)$.

c. Find the value $\hat{p}$ of p for which $L'(p) = 0$.

The value $L(\hat{p})$ is the maximum value of $L(p)$ and $\hat{p}$ is called the maximum likelihood estimator of p.

Exercise 6.1.10 A bird searches bushes in a field for insects. The total weight of insects found after t minutes of searching a single bush is given by $w(t) = \frac{2t}{t+4}$ grams. Draw a graph of w. From your graph, does it appear that a bird should search a single bush for more than 10 minutes? It takes the bird one minute to move from one bush to another. How long should the bird search each bush in order to harvest the most insects in an hour of feeding?

Exercise 6.1.11 Van der Waal's equation for gasses at high pressure (20 to 1000 atmospheres, say) is

$$(P + \frac{n^2 \times a}{V^2})(V - n * b) = n * R * T \tag{6.10}$$

where n and R are number of moles and the ideal gas law constant, and a and b are constants specific to the gas under study.

1. Find $\frac{dP}{dT}$ under the assumption that the volume, V, is constant.
2. Find $\frac{dP}{dV}$ under the assumption that the temperature, T, is constant.

Exercise 6.1.12 Let

$$P(t) = \frac{u(t)}{v(t)} = u(t) \times (v(t))^{-1}$$

Use the product rule and power chain rule to show that

$$P'(t) = \frac{u'(t) \times v(t) - u(t) \times v'(t)}{v^2(t)}$$

Exercise 6.1.13 Let $P(t) = u(t) \times v(t)$. Then

$$\ln P(t) = \ln(u(t) \times v(t)) = \ln u(t) + \ln v(t) \tag{6.11}$$

Compute the derivative of the two sides of Equation 6.11 using the logarithm chain rule and show that

$$P'(t) = u(t)v'(t) + u'(t)v(t)$$

Exercise 6.1.14 Let $P(t) = u(t)/v(t)$. Then

$$\ln P(t) = \ln\left(\frac{u(t)}{v(t)}\right) = \ln u(t) - \ln v(t) \tag{6.12}$$

Compute the derivative of the two sides of Equation 6.12 using the logarithm chain rule and show that

$$P'(t) = \frac{u(t)v'(t) - u'(t)v(t)}{v^2(t)}$$

Exercise 6.1.15 A useful special case of the quotient formula is the *reciprocal formula*: If $u(t)$ has a derivative and $u(t) \neq 0$ and

$$P(t) = \frac{1}{u(t)}$$

then

$$P'(t) = \frac{1}{u^2(t)}\, u'(t)$$

Prove the formula using logarithmic differentiation. That is, write

$$\ln P(t) = \ln\left(\frac{1}{u(t)}\right) = -\ln u(t)$$

and compute the derivatives of both sides using the logarithm chain rule.

We write the formula as

$$\left[\frac{1}{u(t)}\right]' = \frac{1}{u^2(t)} u'(t) \tag{6.13}$$

Exercise 6.1.16 Use Equations 6.2 and 6.13 to compute the derivative of

$$P(t) = \frac{t^2 - 1}{t^2 + 1} = (t^2 - 1) \times \frac{1}{t^2 + 1}$$

Exercise 6.1.17 Provide reasons for steps a - g in Equations 6.8 computing the derivative of the logistic function. To prove step g you should first compute $1 - P(t)/M$ where $P(t) = P_0 M e^{kt}/(M - P_0 + P_0 e^{kt})$.

Exercise 6.1.18 Sketch the graphs of the logistic curve

$$P(t) = \frac{P_0 M e^{rt}}{M - P_0 + P_0 e^{rt}}$$

for

a. $r = 0.5 \quad M = 20 \quad P_0 = 1, \quad 15, \quad 20, \quad \text{and} \quad 30 \qquad 0 \leq t \leq 20$

b. $r = 0.1 \quad M = 20 \quad P_0 = 1, \quad 15, \quad 20, \quad \text{and} \quad 30 \qquad 0 \leq t \leq 70$

c. $P_0 = 1 \quad M = 20 \quad r = 0.1, \quad 0.3, \quad 0.5, \quad \text{and} \quad 0.7 \qquad 0 \leq t \leq 50$

d. $P_0 = 1 \quad r = 0.2 \quad M = 10, \quad 15, \quad 20, \quad \text{and} \quad 25 \qquad 0 \leq t \leq 50$

Exercise 6.1.19 For what population size is the growth rate P' of the logistic population function the greatest? The equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{M}\right)$$

provides an answer. Observe that $y = rp(1 - p/M) = r\,p - (r/M)\,p^2$ is a quadratic whose graph is a parabola.

The answer to this question is important, for the population size for which P' is greatest is that population that wildlife managers may wish to maintain to provide maximum growth.

Exercise 6.1.20 Crows on the west coast of Canada feed on a mollusk called a whelk (shown in Figure 6.2)[2]. The crows break the whelk shell to obtain the meat inside by lifting the whelk to a height of about 5 meters and dropping it onto a rock.

Copyright permission to use the figures from Reto Zach's papers was kindly given by Brill Publishers, PO Box 9000, 2300 PA Leiden, The Netherlands.

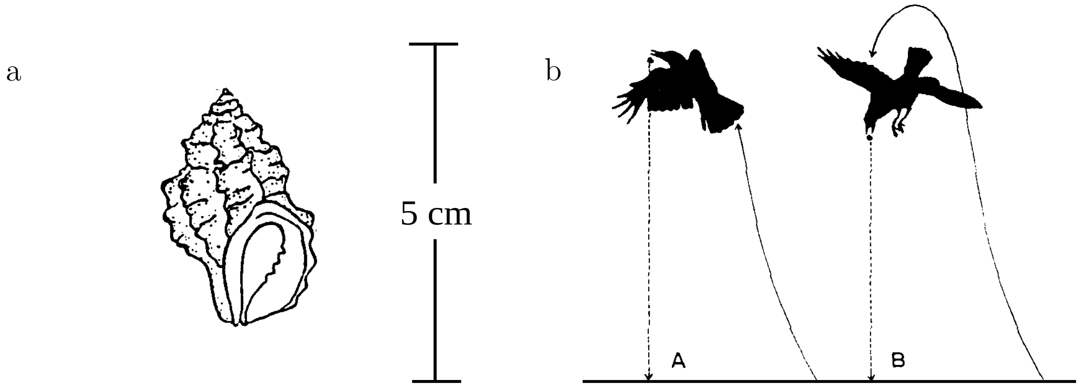


Figure 6.2: a. Schematic drawing of a whelk (Zach, 1978, Figure 1). b. "Flights during dropping. Some crows release whelk at highest point of flight and are unable to see whelk fall (A). Most crows lose some height before dropping but are able to see whelk fall (B)." (*ibid.*, Fig 6.)

Reto Zach (1978,1979) investigated the behavior of crow feeding as an example of decision making while foraging for food, and concluded that crows break the whelk in a manner that minimized their effort (optimal foraging). Crows find whelks in the intertidal zone near the water, carry it towards the

---

[2] Reto Zach, Selection and dropping of whelks by northwestern crows, *Behaviour* **67** (1978), 134-147. Reto Zach, Shell dropping: Decision-making and optimal foraging in northwestern crows, *Behaviour* **68** (1979), 106-117.

land, fly vertically and drop it from a height for breaking. The vertical ascent and drop are repeated until the whelk breaks. Zach made two interesting observations:

1. The crows fed only on large whelk. When large whelk were not available, crows selected another food source.

2. Consistently the crows dropped the whelk from a height of about 5 meters.

Zach gathered whelks from the intertidal zone, separated them into small, medium, and large categories, and dropped them repeatedly at a given height until they broke. He repeated this at varying heights, and his results are shown in Figure 6.3.

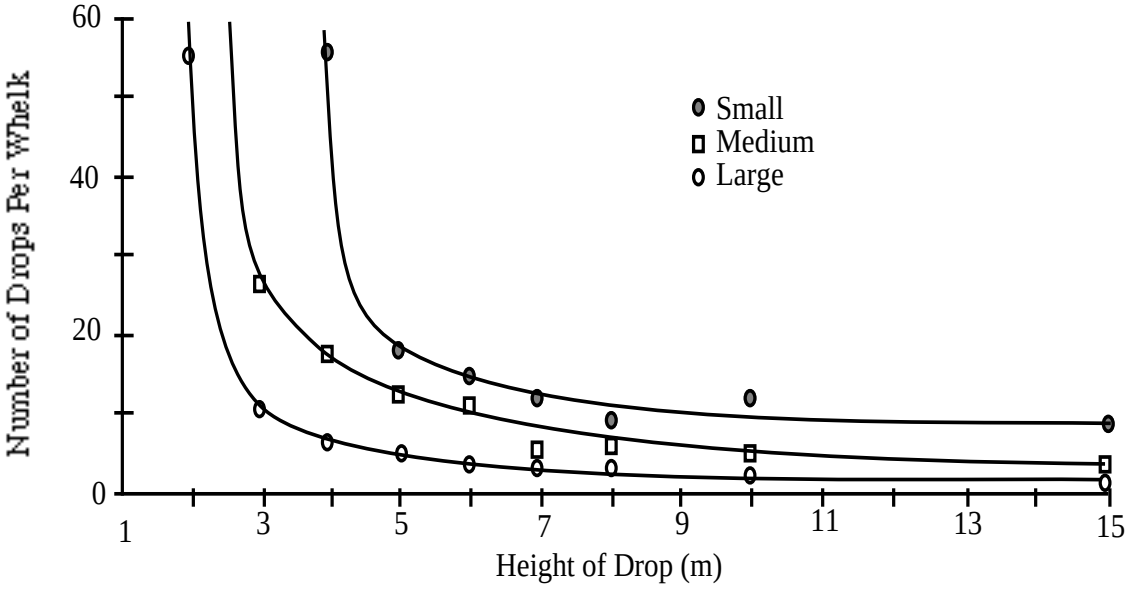


Figure 6.3: Mean number of drops required for breaking large, medium and small whelks dropped from various heights. Curves fitted by eye. (Zach, 1979, Figure 2.)

We read data from the graph for N the number of drops required to break a medium sized whelk from a height H and find that the following hyperbola matches the data:

$$N = 1 + \frac{1}{-0.103 + 0.0389 \times H} \qquad H \geq 2. \tag{6.14}$$

Zach reasoned that the work, W, done to break a whelk by dropping it N times from a height H was equal to $N \times H$. For a medium sized whelk,

$$W = N \times H = \left(1 + \frac{1}{-0.103 + 0.0389 \times H}\right) \times H \qquad H \geq 2. \tag{6.15}$$

a. Use a TI-86 calculator to graph Equation 6.15. Use FMIN to find the value of H for which W is a minimum.

b. Compute $\frac{dW}{dH}$ from Equation 6.15. Find the value H_0 of H for which $\frac{dW}{dH} = 0$ and the value W_0 of W corresponding H_0.

c. Explain why the answers to the two previous parts are equal (or very close).

d. Interpret the quotient W_0/H_0.

e. Data for large whelk (read from an enlargement of Figure 6.3) are shown in Table 6.1. From the graph in Figure 6.3, read the number of drops required to break a large whelk for Height= 2m and Height= 3m and complete the table.

Table 6.1:

Large Size Whelk

Height of Drop	Number of Drops	[Number of Drops - 1]$^{-1}$
2		0.019
3		
4	6.7	0.18
5	4.8	
6	3.8	0.36
7	3.2	0.45
8	2.5	0.67
10	2.6	0.63
15	1.9	1.1

f. Find an equation of a hyperbola that matches the data for a large whelk. Note: The number of drops is clearly at least 1, use the equation

$$N = 1 + \frac{1}{a + bH}$$

and find a and b to match the data. The previous equation can be changed to

$$N - 1 = \frac{1}{a + bH}, \qquad \frac{1}{N-1} = a + bH$$

Therefore a graph of $\frac{1}{N-1}$ versus H should be approximately linear, and the coefficients of line fit to that data will be good values for a and b. Find a and b.

g. Find the value of H_0 of H that minimizes the work required to break a large whelk.

h. Find the minimum amount of work required to break a large whelk. On average, how many drops does it take to break a large whelk from the optimum height, H_0?

Summary. The work required to break a medium sized whelk is twice that require to break a large whelk, and the optimum height from which to drop a large whelk is 6.1 meters, reasonably close to the 5.58 meters obtained by Zach.

A similar behavior is observed in sea gulls feeding on mussels. A large mussel broken on the second drop by a gull is shown in Figure 6.4; attached to the large mussel shell is a small mussel that the gull did not bother to break.

Figure 6.4: A large mussel shell broken by a gull on the second drop. Attached to it is a small mussel that the gull did not break. (Picture by JLC)

6.2 The chain rule.

The Power Chain Rule, the Exponential Chain Rule, and the Logarithm Chain Rule have a common pattern and we list all three to show the similarity:

If a function, $u(t)$, has derivative then

$$\begin{array}{rcll} [\,u(t)^n\,]' & = & nu(t)^{n-1} \times [\,u(t)\,]' & \text{Power Chain Rule} \\ \left[e^{u(t)}\right]' & = & ne^{u(t)} \times [\,u(t)\,]' & \text{Exponential Chain Rule} \\ [\ln u(t)\,]' & = & \dfrac{1}{u(t)} \times [\,u(t)\,]' & \text{Logarithm Chain Rule} \end{array}$$

All three of these are of the form

$$[\,G\,(\,u(t)\,)\,]' = G'\,(\,u(t)\,) \times [u(t)]' \qquad \textbf{Chain Rule} \tag{6.16}$$

where

$G(u)$	$G'(u)$	$G(u(t))$	$G'(u(t))$	$\times$	$[u(t)]'$	Chain Rule
u^n	nu^{n-1}	$(u(t))^n$	$nu(t)^{n-1}$	$\times$	$[\,u(t)\,]'$	Power
e^u	e^u	$e^{u(t)}$	$e^{u(t)}$	$\times$	$[\,u(t)\,]'$	Exponential
$\ln u$	$\frac{1}{u}$	$\ln(u(t))$	$\frac{1}{u(t)}$	$\times$	$[\,u(t)\,]'$	Logarithm

$G'(u)$ in the second column is the derivative with respect to u, the independent variable of G, and will normally be computed using a Primary Formula.

Example 6.2.1 Compute $F'(t)$ for $F(t) = (1-t^2)^3$. Let

$$G(z) = z^3 \qquad \text{and} \qquad u(t) = 1 - t^2. \qquad \text{Then} \qquad F(t) = G(u(t)).$$

$$G'(z) = 3z^2 \qquad \text{and} \qquad [u(t)]' = -2t,$$
$$G'(\,u(t)\,) = 3(\,u(t)\,)^2 = 3(\,1 - t^2\,)^2,$$

and

$$F'(t) = G'(\,u(t)\,) \times [u(t)]' = 3(\,1 - t^2)^2 \times (-2t)$$

In the form $G(u(t))$, $G(u)$ is often called the 'outside' function and $u(t)$ is called the inside function. Consider

For	Outside	Inside
$\sqrt{1+t^2}$	$G(u) = \sqrt{u}$	$u(t) = 1+t^2$
$\dfrac{1}{1+e^t}$	$G(u) = \dfrac{1}{u}$	$u(t) = 1+e^t$
$e^{-t^2/2}$	$G(u) = e^u$	$u(t) = -t^2/2$
$\ln(e^t+1)$	$G(u) = \ln u$	$u(t) = e^t+1$

Theorem 6.2.1 Chain Rule. Suppose G and u are functions that have derivatives and $G(\,u(t)\,)$ is defined for all numbers t. Then $G(\,u(t)\,)$ has a derivative for all t and

$$[\,G(\,u(t)\,)\,]' = G'\,(u(t)) \times [u(t)]' \tag{6.17}$$

Proof: The argument is similar to that for the exponential chain rule. The difference is that we now have a general function $G(u)$ rather than the specific functions e^u. We argue only for u an increasing function, and we need Theorem 4.2.1, The Derivative Requires Continuity.

Let $F = G \circ u$ ($F(t) = G(u(t))$ for all t).

$$\begin{aligned}
F'(a) &= \lim_{b\to a} \frac{F(b)-F(a)}{b-a} \\
&= \lim_{b\to a} \frac{G(\,u(b)\,) - G(\,u(a)\,)}{b-a} \\
&= \lim_{b\to a} \frac{G(\,u(b)\,) - G(\,u(a)\,)}{u(b)-u(a)} \lim_{b\to a} \frac{u(b)-u(a)}{b-a} \qquad u(b)-u(a) \neq 0 \\
&= G'(u(a)) \times u'(a)
\end{aligned}$$

The conclusion that

$$\lim_{b\to a} \frac{G(\,u(b)\,) - G(\,u(a)\,)}{u(b)-u(a)} = G'(u(a))$$

requires some support.

$$\text{In Figure 6.5, the slope of the secant is} \qquad \frac{G(\,u(b)\,) - G(\,u(a)\,)}{u(b) - u(a)}.$$

Because $u'(a)$ exists, $u(b) \to u(a)$ as $b \to a$. The slope of the secant approaches the slope of the tangent as $u(b) \to u(a)$, and

$$\lim_{b \to a} \frac{G(\,u(b)\,) - G(\,u(a)\,)}{u(b) - u(a)} = G'(u(a)).$$

End of proof.

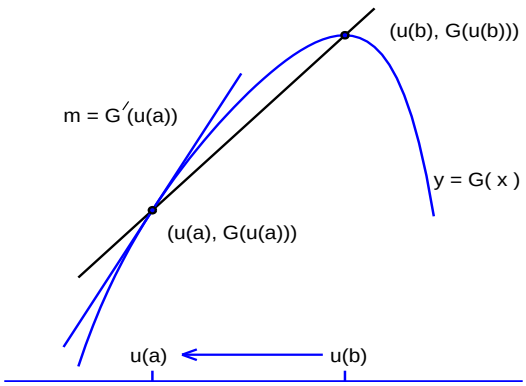


Figure 6.5: Graph of a function $y = G(x)$. As $b \to a$, $(G(\,u(b)\,) - G(\,u(a)\,))/(u(b) - u(a)) \to G'(\,u(a)\,)$.

Example 6.2.2 Repeated use of the chain rule allows computation of derivatives of some quite complex functions.

Problem. Compute the derivative of

$$F(t) = e^{\sqrt{\ln t}}$$

Solution. We peel the layers off from the outside. $F(t)$ can be thought of as

$$F(t) = G\left(H(K(t))\right), \qquad \text{where} \qquad G(z) = e^z, \qquad H(x) = \sqrt{x}, \qquad \text{and} \qquad K(t) = \ln t$$

$$\begin{aligned}
\left[\, e^{\sqrt{\ln t}} \,\right]' &= e^{\sqrt{\ln t}} \left[\, \sqrt{\ln t} \,\right]' && G(z) = e^z, \qquad G'(z) = e^z \\
&= e^{\sqrt{\ln t}} \frac{1}{2\sqrt{\ln t}} \left[\, \ln t \,\right]' && H(x) = \sqrt{x}, \qquad H'(x) = \frac{1}{2\sqrt{x}} \\
&= e^{\sqrt{\ln t}} \frac{1}{2\sqrt{\ln t}} \frac{1}{t} && K(t) = \ln t, \qquad K'(t) = \frac{1}{t}
\end{aligned}$$

Extreme Problem. **Argh!** Compute the derivative of

$$F(t) = \left(1 + \sqrt{\ln \frac{1-t}{1+t}}\right)^4$$

$$\begin{aligned}
\left[\left(1+\sqrt{\ln\frac{1-t}{1+t}}\right)^4\right]' &= 4\left(1+\sqrt{\ln\frac{1-t}{1+t}}\right)^3\left[1+\sqrt{\ln\frac{1-t}{1+t}}\right]' \\
&= 4\left(1+\sqrt{\ln\frac{1-t}{1+t}}\right)^3\left(0+\frac{1}{2}\frac{1}{\sqrt{\ln\frac{1-t}{1+t}}}\left[\ln\frac{1-t}{1+t}\right]'\right) \\
&= 4\left(1+\sqrt{\ln\frac{1-t}{1+t}}\right)^3\frac{1}{2}\frac{1}{\sqrt{\ln\frac{1-t}{1+t}}}\left(\frac{1}{\frac{1-t}{1+t}}\left[\frac{1-t}{1+t}\right]'\right) \\
&= 4\left(1+\sqrt{\ln\frac{1-t}{1+t}}\right)^3\frac{1}{2}\frac{1}{\sqrt{\ln\frac{1-t}{1+t}}}\frac{1}{\frac{1-t}{1+t}}\frac{(1+t)(-1)-(1-t)1}{(1+t)^2}
\end{aligned}$$

The chain rule is an investment in the future. It does not immediately expand the collection of functions for which we can compute the derivative. To use the chain rule on $G(\,u(t)\,)$ we need $G'(u)$ which requires a Primary derivative formula. The relevant Primary derivative formulas so far developed are the power, exponential and logarithm Primary formulas, for which we have already developed chain rules. In the next chapter, we develop the Primary formula

$$[\,\sin t\,]' = \cos t$$

Then from the chain rule of this section, we immediately have the chain rule

$$[\,\sin(\,u(t)\,)\,]' = \cos(\,u(t)\,)\times u'(t)$$

Using this we can, for example, compute $[\,\sin(\,\pi t\,)\,]'$ as

$$\begin{aligned}
[\,\sin(\,\pi t\,)\,]' &= \cos(\,\pi t\,)\times[\,\pi t\,]' \\
&= \cos(\,\pi t\,)\times\pi
\end{aligned}$$

Leibnitz notation. The Leibnitz notation makes the chain rule look deceptively simple. For $G(u(t))$ one has

$$[\,G(\,u(t)\,)\,]' = \frac{dG}{dt} \qquad G'\,(u(t)) = \frac{dG}{du} \qquad [u(t)]' = \frac{du}{dt}.$$

Then the chain rule is

$$\frac{dG}{dt} = \frac{dG}{du}\frac{du}{dt}$$

Example 6.2.3 Find $\frac{dy}{dt}$ for $y(t) = (1+t^4)^7$. $y(t)$ is the composition of $G(u) = u^7$ and $u(t) = 1+t^4$. Then

$$\begin{aligned}
\frac{dG}{du} &= \frac{d}{du}u^7 &&= 7u^6 \\
\frac{du}{dt} &= \frac{d}{dt}(1+t^4) &&= 4t^3 \\
\frac{dG}{dt} &= \frac{dG}{du}\frac{du}{dt} &&= 7u^6\times 4t^3 = 7(1+t^4)\times 4t^3
\end{aligned}$$

Exercises for Section 6.2, The chain rule.

Exercise 6.2.1 Use the chain rule to differentiate $P(t)$ for

a. $P(t) = e^{(-t^2)}$

b. $P(t) = (e^t)^2$

c. $P(t) = e^{2\ln t}$

d. $P(t) = \ln e^{2t}$

e. $P(t) = \ln(2\sqrt{t})$

f. $P(t) = \sqrt{2\ln t}$

g. $P(t) = \sqrt{e^{2t}}$

h. $P(t) = \sqrt{e^{(-t^2)}}$

i. $P(t) = (t + e^{-2t})^4$

j. $P(t) = \left(1 + e^{(-t^2)}\right)^{-1}$

k. $P(t) = \frac{3}{4}\left(1 - x^2/16\right)^{1/2}$

l. $P(t) = (t + \ln(1 + 2t))^2$

Exercise 6.2.2 Use the Leibnitz notation for the chain rule to find $\dfrac{dy}{dt}$ for

a. $y(t) = e^{(-t^2)}$

b. $y(t) = (e^t)^2$

c. $y(t) = e^{2\ln t}$

d. $y(t) = (\dfrac{e^t}{1+e^t})^2$

e. $y(t) = \sqrt{2t^2 - t + 1}$

f. $y(t) = (t^2 + 1)^3$

Compare your answers for a - c with those of Exercise 6.2.1 a - c.

Exercise 6.2.3 In Chapter 7 we show that $[\cos(t)]' = -\sin(t)$. Use this formula and $[\sin t]' = \cos t$ written earlier to differentiate:

a. $y(t) = \cos(2t)$

b. $y(t) = \sin(\frac{\pi}{2}t)$

c. $y(t) = e^{\cos t}$

d. $y(t) = \cos(e^t)$

e. $y(t) = \sin(\cos t)$

f. $y(t) = \sin(\cos e^t)$

g. $y(t) = \ln(\sin t)$

h. $y(t) = \sec t = (\cos t)^{-1}$

i. $y(t) = \ln(\cos(e^t))$

j. $y(t) = \ln(\cos(e^{\sin t}))$

Exercise 6.2.4 The Doppler effect. You are standing 100 meters south of a straight train track on which a train is traveling from west to east at the speed 30 meters per second. See Figure 6.2.4. Let $y(t)$ be the distance from the train to you and $|x(t)|$ be the distance from the train to the point on the track nearest you; $x(t)$ is negative when the train is west of the point on the track nearest you.

a. Write $y(t)$ in terms of $x(t)$.

b. Find $y'(t)$ for a time t at which $x(t) = -200$

c. Time is measured so that $x(0) = 0$. Write an equation for $x(t)$.

d. Write an equation for $y'(t)$ in terms of t.

e. The whistle from the train projects sound waves at frequency f cycles per second. The frequency, f_L, of the sound reaching your ear is

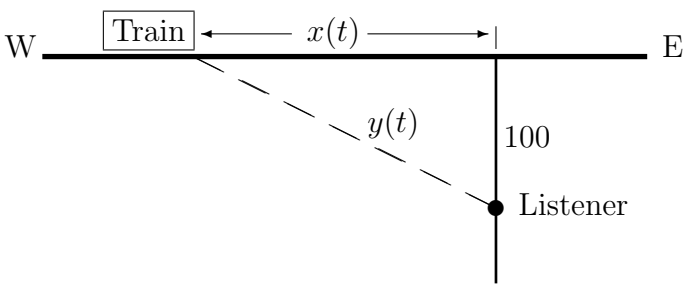
$$f_L = \frac{331.4}{331.4 + y'(t)} \times f \frac{\text{cycles}}{\text{second}} \tag{6.18}$$

331.4 m/s is the speed of sound in air. Draw a graph of f_L assuming $f = 500$.

Derivation of Equation 6.18 for the Doppler effect. A sound of frequency f traveling in still air has wave length (331.4 m/s)/(f cycles/s) = 331.4/f m/cycle. If the source of the sound is moving at a velocity v with respect to a listener, the wave length of the sound reaching the listener is $((331.4 + v)/f)$ m/cycle. These waves travel at 331.4 m/sec, and the frequency f_L of these waves reaching the listener

$$f_L = \frac{331.4 \text{ m/s}}{((331.4 + v)/f) \text{ m/cycle}} = \frac{331.4}{331.4 + v} \times f \frac{\text{cycles}}{\text{second}}$$

Figure for Exercise 6.2.4 A train and track with listener location. As drawn, $x(t)$ is negative



Exercise 6.2.5 Air is being pumped into a spherical balloon at the rate of 1000 cm^3/min. At what rate is the radius of the balloon increasing when the volume is 3000 cm^3? Note: $V(t) = \frac{4}{3}\pi r^3(t)$.

Exercise 6.2.6 Consider a spherical ice ball that is melting. A reasonable model is:

Mathematical Model.

1. The rate at which heat is transferred to the ice ball is proportional to the surface area of the ice ball.
2. The rate at which the ball melts is proportional to the rate at which heat is transferred to the ball.

The volume, V, of a sphere of radius r is $\frac{4}{3}\pi r^3$ and its surface area, S, is $4\pi r^2$. From 1 and 2 we conclude that *the rate of change of volume of the ice ball is proportional to the surface area of the ice ball.*

a. Write an equation representative of the previous italicized statement.

b. As the ball melts, V, r, and S change with time. Differentiate $V(t) = \frac{4}{3}\pi r^3(t)$ to obtain

$$V'(t) = 4\pi r^2(t)r'(t)$$

c. Use your equation from (a) and the equation from (b) to show that

$$r'(t) = K \qquad \text{where } K \text{ is a constant}$$

d. Why should K be negative?

e. Because K should be negative, we write

$$r'(t) = -K$$

A good candidate for $r(t)$ is

$$r(t) = -Kt + C \qquad \text{where } C \text{ is a constant}$$

Why?

f. Only discussion included in this part. With $r(t) = -KT + C$ we find that

$$W(t) = A(1 - t/B)^3 \qquad \text{where } A \text{ and } B \text{ are constants}$$

and $W(t)$ is the weight of the ball. In order to test this conclusion, we filled a plastic ball about the size of a volley ball with water and froze it to -14° C (Figure 6.2.6). One end of a chord (knotted) was frozen into the center of the ball and the other end extended outside the ball as a handle. We removed the plastic and placed the ball in a 10° C water bath, held below the surface by a weight attached to the ball. At four minute intervals we removed the ball and weighed it and returned to the bath. The data from one of these experiments is shown in Table 6.2.6 and a plot of the data and of a cubic, $y = 3200(1 - t/120)^3$, is shown in the figure of Table 6.2.6. The cubic looks like a pretty good fit to the data, and we might argue that the data is consistent with our model.

There are some flaws with the fit of the cubic, however. The cubic departs from the data at both ends. $y(0) = 3200$, but the ball only weighed 3020 g; the cubic is also above the data at the right end.

g. We found that we could fit the data more closely with an equation of the form

$$W(t) = A(1 - t/100)^\alpha$$

where α is closer to 2 than to 3. Find values for A and α. Note: $\ln W(t) = \ln A + \alpha \ln(1 - t/100)$. Then reasonable estimates of $\ln A$ and α are the coefficients of a line fit to the graph of $\ln w(t)$ versus $\ln(1 - t/100)$.

h. If the data is not consistent with the model, in what way might the model be deficient?

Figure for Exercise 6.2.6 (h) Pictures of an ice ball used in the experiments described in Exercise 6.2.6.

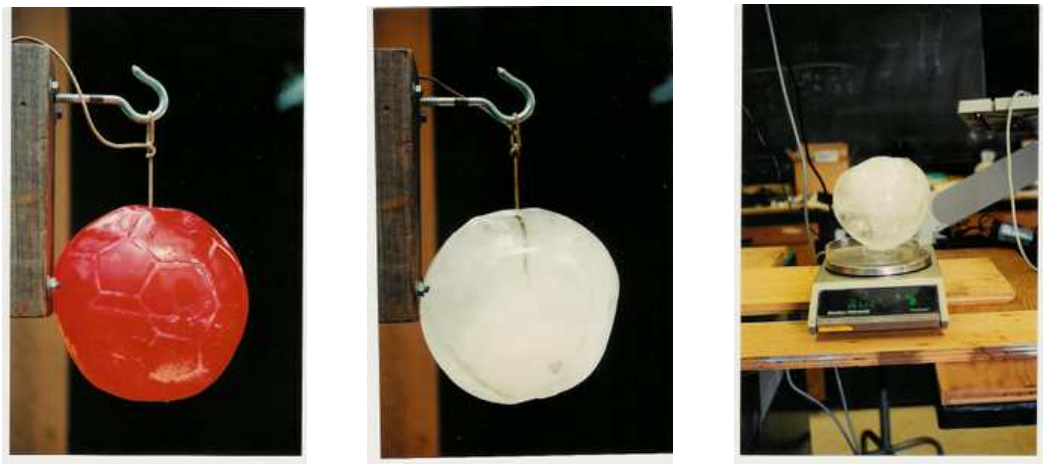
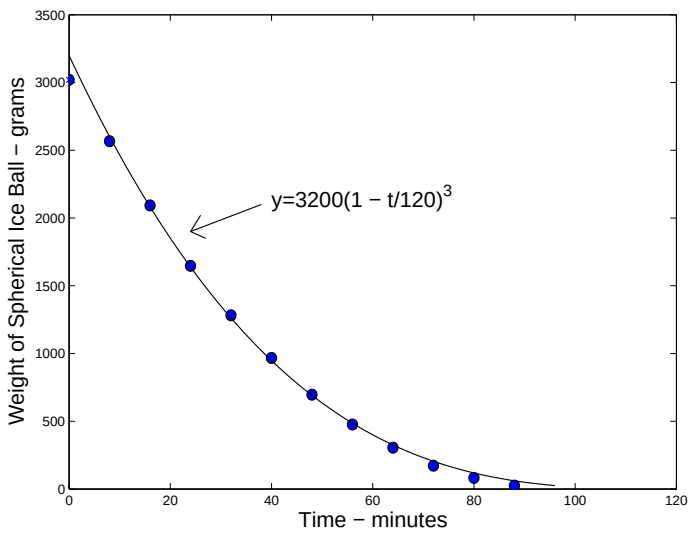


Table for Exercise 6.2.6 Data from an ice ball melt experiment described in Exercise 6.2.6.

Time (m)	Weight (g)
0	3020
4	2567
8	2093
12	1647
16	1282
20	967
24	696
28	476
32	305
36	172
40	82
44	25



6.3 Derivatives of inverse functions.

The inverse of a function was defined in Definition 2.6.2 in Section 2.6.2. The natural logarithm function is the inverse of the exponential function, $f(t) = \sqrt{t}$ is the inverse of $g(t) = t^2$, $t \geq 0$, and $f(t) = \sqrt[3]{t}$ is the inverse of $g(t) = t^3$, are familiar examples. We show here that 'the derivative of the inverse f^{-1} of a function f is the reciprocal of the derivative of f,' but this phrase has to be carefully explained.

Example 6.3.1 The linear functions

$$y_1(x) = 1 + \frac{3}{2}x \quad \text{and} \quad y_2(x) = -\frac{2}{3} + \frac{2}{3}x$$

are each inverses of the other, and their slopes (3/2 and 2/3) are reciprocals of each other.

Explore 6.3.1 Show that in the previous example, $y_1(y_2(x)) = x$ and $y_2(y_1(x)) = x$.

The crucial point is shown in the graphs of y_1 and y_2 in Figure 6.6. Each graph is the image of the other by a reflection about the line $y = x$. One line contains the points (a, b) and (c, d) and another line contains the points (b, a) and (d, c).

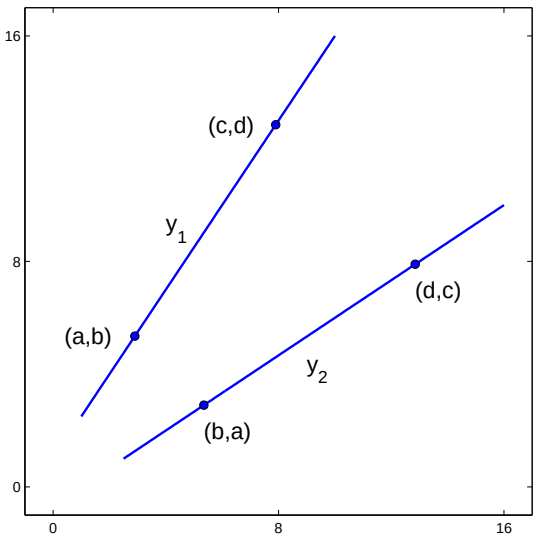


Figure 6.6: Graphs of $y = 1 + (3/2)x$ and $y = -2/3 + (2/3)x$; each is the inverse of the other and the slopes, 3/2 and 2/3 are reciprocals.

The relation important to us is that their slopes are reciprocals, a general property of a function and its inverse. Specifically,

$$m_1 = \frac{d-b}{c-a} \qquad\qquad m_2 = \frac{d-b}{c-a} = \frac{1}{m_1}$$

Example 6.3.2 Figure 6.7 shows the graph of

$$E(x) = e^x \qquad \text{and its inverse} \qquad L(x) = \ln x$$

The point $(x_3, 3)$ has y-coordinate 3. Because $E'(x) = E(x)$ the slope of E at $(x_3, 3)$ is 3. The point $(3, x_3)$ is the reflection of $(x_3, 3)$ about $y = x$ and the graph of L has slope 1/3 at $(3, x_3)$ because $L'(t) = [\ln t]' = 1/t$. More generally

$$L'(t) = \frac{1}{E'(L(t))} \qquad \text{and} \qquad E'(t) = \frac{1}{L'(E(t))}$$

Explore 6.3.2 (a.) Evaluate x_3 in Figure 6.7A.

(b.) Evaluate x_2 in Figure 6.7B and find the slope at $(x_2, 2)$ and at $(2, x_2)$. ■

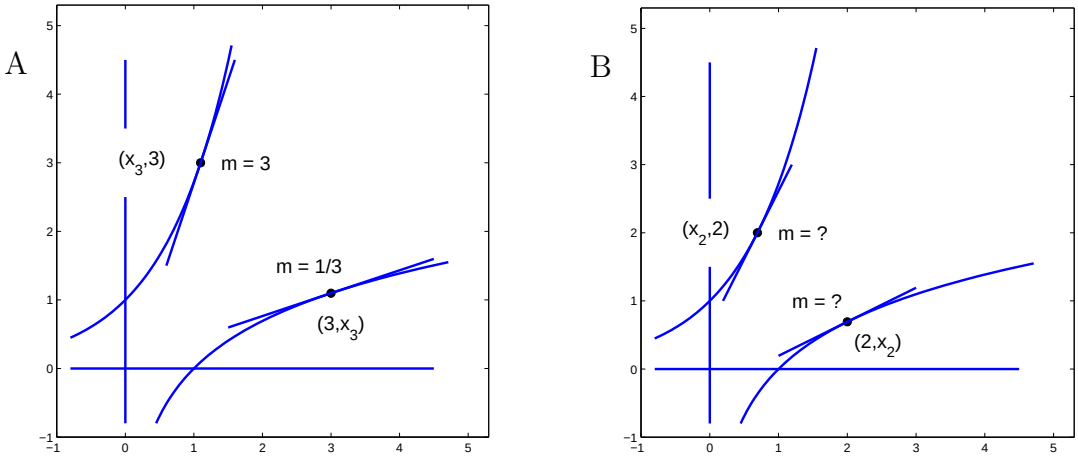


Figure 6.7: Graphs of $E(x) = e^x$ and $L(x) = \ln x$. Each is the inverse of the other. The point A has coordinates $(2, x_2)$ and the slope of L at A is $1/2$.

The derivative of the inverse of a function. If g is an invertible function that has a nonzero derivative and h is its inverse, then for every number, t, in the domain of g,

$$g'(h(t)) = \frac{1}{h'(t)} \qquad \text{and} \qquad g'(t) = \frac{1}{h'(g(t))}.$$

If g is an invertible function and h is its inverse, then for every number, t, in the domain of g,

$$g(h(t)) = t$$

We differentiate both sides of this equation.

$$\begin{aligned} [\,g(h(t))]' &= [\,t\,]' \\ g'(h(t))\,h'(t) &= 1 && \text{Uses the Chain Rule} \\ h'(t) &= \frac{1}{g'(h(t))} && \text{Assumes } g'(h(t)) \neq 0 \end{aligned}$$

Explore 6.3.3 Begin with $h(g(t)) = t$ and show that

$$g'(t) = \frac{1}{h'(g(t))}$$

Example 6.3.3 The function, $h(t) = \sqrt{t}$, $t > 0$ is the inverse of the function, $g(x) = x^2$, $x > 0$.

$$g'(x) = 2x$$

$$\begin{aligned} h'(t) &= \frac{1}{g'(h(t))} \\ &= \frac{1}{2h(t)} \\ &= \frac{1}{2\sqrt{t}}, \end{aligned}$$

a result that we obtained directly from the definition of derivative.

Leibnitz notation. The Leibnitz notation for the derivative of the inverse is deceptively simple. Let $y = g(x)$ and $x = h(y)$ be inverses. Then $g'(x) = \frac{dy}{dx}$ and $h'(y) = \frac{dx}{dy}$. The equation

$$h'(t) = \frac{1}{g'(h(t))} \qquad \text{becomes} \qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

Exercises for Section 6.3 Derivatives of inverse functions.

Exercise 6.3.1 Find formulas for the inverses of the following functions. See Section 2.6.2 for a method. Then draw the graphs of the function and its inverse. Plot the point listed with each function and find the slope of the function at that point; plot the corresponding point of the inverse and find the slope of the inverse at that corresponding point.

a. $P(t) = 4 - \frac{1}{2}t \quad 0 \le t \le 4 \quad (2, 3)$

b. $P(t) = \frac{1}{1+t} \quad 0 \le t \le 2 \quad (1, \frac{1}{2})$

c. $P(t) = \sqrt{4-t} \quad 0 \le t \le 4 \quad (2, \sqrt{2})$

d. $P(t) = 2^t \quad 0 \le t \le 2 \quad (0, 1)$

e. $P(t) = t^2 + 1 \quad -2 \le t \le 0 \quad (-1, 2)$

f. $P(t) = \sqrt{4-t^2} \quad 0 \le t \le 2 \quad (1, \sqrt{3})$

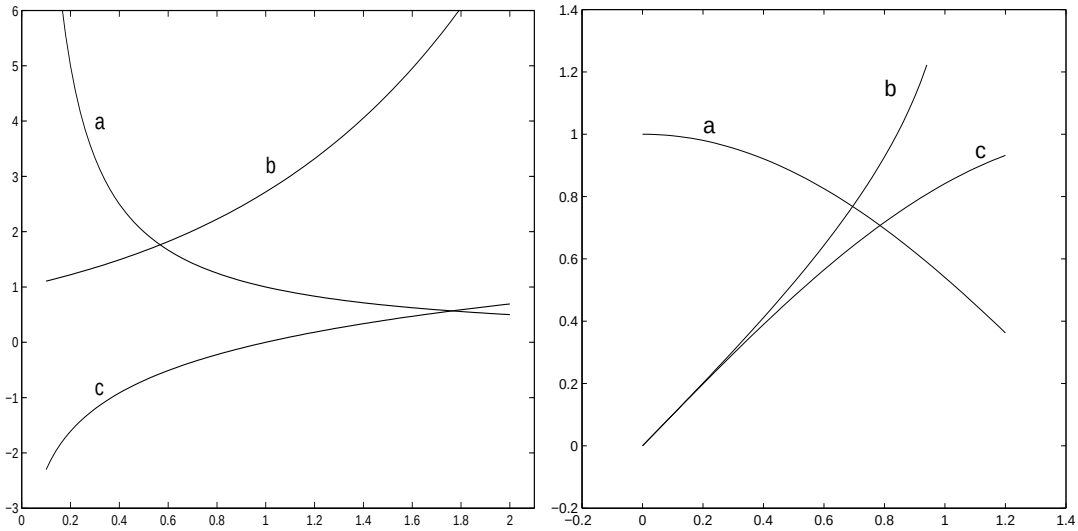
Exercise 6.3.2 The function, $h(t) = t^{1/3}$ is the inverse of the function $g(x) = x^3$. Use steps similar to those of Example 6.3.3 to compute $h'(t)$.

Exercise 6.3.3 The graphs of a function F its inverse F^{-1} and its derivative F' are shown in each of Figure 6.3.3(a) and (b).

a. Identify each graph in Figure 6.3.3(a) as F, F^{-1} or F'.

b. Identify each graph in Figure 6.3.3(b) as F, F^{-1} or F'.

Figure for Exercise 6.3.3 Graphs of a function F, F^{-1}, and F'. See Exercise 6.3.3.



(a) (b)

6.4 Summary of Chapter 6, Derivatives of Products, Quotients and Compositions of Functions.

The thrust of this chapter is to expand your ability to compute derivatives of functions. We have now introduced all of the combination derivative formula that you will need. Together with the Primary derivative formula already introduced and two others to be presented in the next chapter, Chapter 7, Derivatives of the Trigonometric Functions, you will be able to compute the derivatives of all of the functions you will meet in ordinary work. The basic formulas that you need are shown below. You need to be able to use them forward *and backward*. That is, given a function, find its derivative, and given a derivative of a function, identify the function, or several such functions. The *backward* process is crucial to the application of the Fundamental Theorem of Calculus, introduced in Chapter 14

The complete list of derivative formulas that you need is:

Primary Formulas

$$[C]' = 0 \qquad [t^n]' = nt^{n-1}$$

$$[e^t]' = e^t \qquad [\ln t]' = \frac{1}{t}$$

$$[\sin t]' = \cos t \qquad [\cos t]' = -\sin t$$

Combination formulas

$$[u+v]' = [u]' + [v]' \qquad [C\,u]' = C\,[u]'$$

$$[u\,v]' = [u]'\,v + u\,[v]' \qquad \left[\frac{u}{v}\right]' = \frac{v\,[u]' - u\,[v]'}{u^2}$$

$$[G(u)]' = G'(u)\,u'$$

Chapter Exercise 6.1 Differentiate

a. $P(t) = t^4 + e^{2t}$

b. $P(t) = t^4 e^{2t}$

c. $P(t) = \dfrac{t^4}{e^{2t}}$

d. $P(t) = (e^{2t})^4$

e. $P(t) = e^{2t^4}$

f. $P(t) = (t^2+1)^4\,(5t+1)^7$

g. $P(t) = (\ln t)^3$

h. $P(t) = (e^{3t}\ln 2t)^4$

i. $P(t) = \dfrac{\ln t}{t}$

j. $P(t) = t\ln t - t$

k. $P(t) = \dfrac{1}{\ln t}$

l. $P(t) = e^{(\ln t)}$

m. $P(t) = \dfrac{5t^2 - 2t - 7}{t^2+1}$

n. $P(t) = \dfrac{(t+2)^2}{t^2+2}$

Chapter Exercise 6.2 Data from another of the ice ball experiments (see Exercise 6.2.6) are shown in Table 6.1.

a. Find a number A so that $W(t) = 3200(1 - t/A)^3$ is close to the data.

b. Find a number B so that $W(t) = 3100(1 - t/B)^2$ is close to the data.

c. Which of the two functions is closer to the data?

Two potential procedures to solve a.

1. Compute
$$SA = \left|w_1 - 3200(1 - t_1/A)^3\right| + \left|w_2 - 3200(1 - t_2/A)^3\right| + \cdots + \left|w_{21} - 3200(1 - t_{21}/A)^3\right|$$
or
$$SS = \left(w_1 - 3200(1 - t_1/A)^3\right)^2 + \left(w_2 - 3200(1 - t_2/A)^3\right)^2 + \cdots + \left(w_{21} - 3200(1 - t_{21}/A)^3\right)^2$$
for several values of A, and select the value of A for which SA or SS is the smallest.

2. Let $\alpha = 1/A$ and write
$$W(t) = 3200(1 - \alpha t)^3 \qquad\qquad (W(t))/3200)^{1/3} - 1 + \alpha t = 0$$
Define SS by
$$\begin{aligned} SS &= \left((W_1/3200)^{1/3} - 1 + \alpha t_1\right)^2 + \\ &\qquad \left((W_2/3200)^{1/3} - 1 + \alpha t_2\right)^2 + \cdots + \left((W_{21}/3200)^{1/3} - 1 + \alpha t_{21}.\right)^2 \end{aligned}$$
Compute $\dfrac{dSS}{d\alpha}$ and show that the value α_0 of α for which $\dfrac{dSS}{d\alpha} = 0$ is
$$\alpha_0 = \frac{t_1(1 - (W_1/3200)^{1/3}) + t_2(1 - (W_2/3200)^{1/3}) + \cdots + t_{21}(1 - (W_{21}/3200)^{1/3})}{t_1^2 + \cdots + t_{21}^2}.$$
Compute α_0 and let $A = 1/\alpha_0$ in $W(t) = 3200(1 - t/A)^3$.

Table for Chapter Exercise 6.1 Weight of an ice ball in 8° C water.

Time m	0	4	8	12	16	20	24	28	32	36	
Wt g	3085	2855	2591	2337	2085	1855	1645	1436	1245	1097	
Time m	40	44	48	52	56	60	64	68	72	76	80
Wt g	908	763	634	513	407	316	216	164	110	66	34

Chapter 7

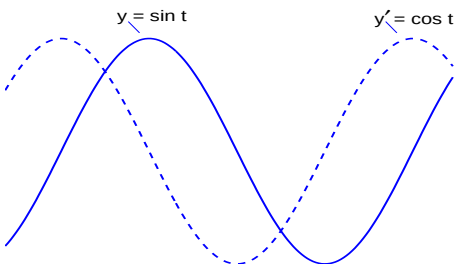
Derivatives of the Trigonometric Functions.

Where are we going?

The trigonometric functions, sine and cosine, are useful for describing periodic variation in mechanical and biological systems. The sine and cosine functions and their derivatives are interrelated:

$$[\sin t]' = \cos t$$

$$[\cos t]' = -\sin t$$



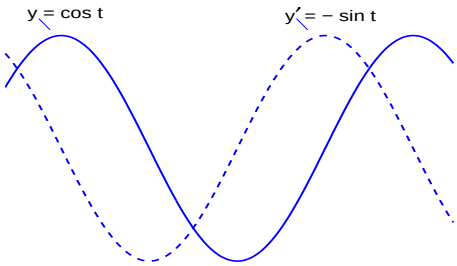
The derivative equation

$$y''(t) + y(t) = 0$$

is basic to mathematical models of oscillating processes, and every solution to this equation can be written in the form

$$y(t) = A\cos t + B\sin t$$

where $A = y(0)$ and $B = y'(0)$ are constants.



7.1 Radian Measure.

Calculus with trigonometric functions is easier when angles are measured in radians. Radian measure of an angle and the trigonometric measures of that angle are all scaled by the length of the radius of a defining circle. For use in calculus you should put your calculator in RADIAN mode.

The circle in Figure 7.1 has radius 1. For the angle z' (the angle $\angle AOC$) the radian measure, the sine, and the cosine are all dimensionless quantities:

$$\begin{aligned}
\text{Radian measure of } z' &= \frac{\text{length of arc } \overset{\frown}{AC}}{\text{length of the radius}} = \frac{z \quad \text{cm, m, in, ?}}{1 \quad \text{cm, m, in, ?}} = z \\
\text{sine of } z' &= \frac{\text{length of the segment } \overline{AB}}{\text{length of the radius}} = \frac{y \quad \text{cm, m, in, ?}}{1 \quad \text{cm, m, in, ?}} = y \\
\text{cosine of } z' &= \frac{\text{length of the segment } \overline{OB}}{\text{length of the radius}} = \frac{x \quad \text{cm, m, in, ?}}{1 \quad \text{cm, m, in, ?}} = x
\end{aligned}$$

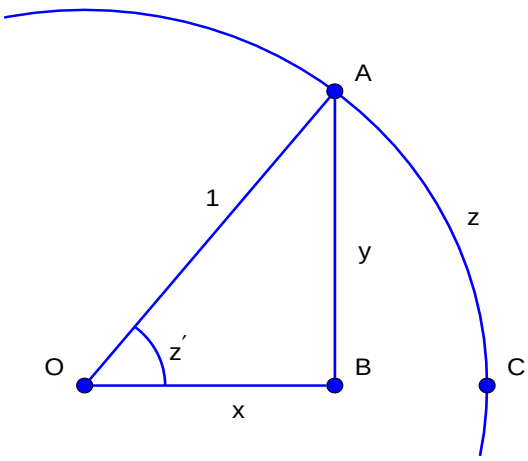


Figure 7.1: A circle with radius 1. The radian measure of the angle z' is z, the length of the arc AC divided by the length of the radius, 1.

It is obvious[1] from the figure that for z' in the first quadrant

$$\begin{aligned}
0 &< \text{Length of } \overline{AB} < \text{Length of } \overset{\frown}{AC} \\
0 &< \sin z' < z
\end{aligned}$$

The inequality, $\sin z' < z$, is read, 'the sine of angle z' is less than z, the radian measure of z'.' We intentionally blur the distinction between the angle z' and z, the radian measure of z', to the point that they are used interchangeably. The inequality

$$\sin z' < z \qquad \text{is usually replaced with} \qquad \sin z < z, \tag{7.1}$$

[1]Should this not be obvious then reflect the figure about the horizontal line through O, B, and C and let A' be the image of A under the reflection. The length of the chord $A'BA$ is less than the length of the arc, $A'CA$ (the straight line path is the shortest path between two points).

the sine of z is less than z, where z is a positive number.

The definitions of $\sin z$ and $\cos z$ for angles that are not accute are extended by use of the unit circle, the circle with center at (0,0) and of radius 1. For z positive, consider the arc of length z counterclockwise along the unit circle from (1,0) to a point, (x, y), in Figure 7.2A. For z negative consider the arc of length $|z|$ clockwise along the unit circle from (1,0) to a point, (x, y), in Figure 7.2B. In either case

$$\sin z = \frac{y}{1} = y \qquad \cos z = \frac{x}{1} = x \tag{7.2}$$

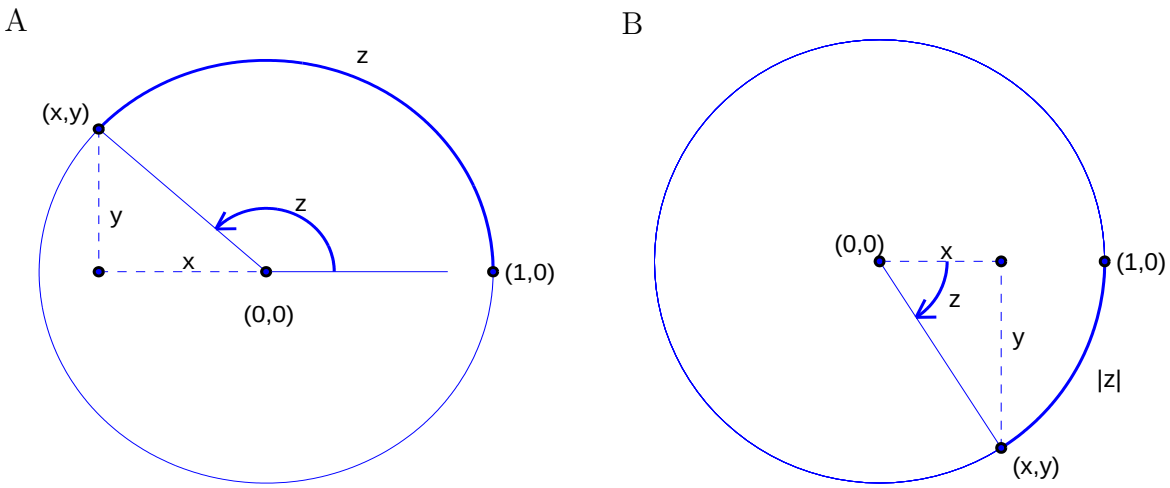


Figure 7.2: The unit circle with A. an arc of length z between 0 and 2π and B. an arc of length $|z|$ for z a negative number.

From Figure 7.2B it can be seen that if z is a negative number then

$$z < \sin z < 0 \tag{7.3}$$

A single statement that combines Equations 7.1 and 7.3 is written:

$$|\sin z| < |z| \qquad \text{for all numbers } z \tag{7.4}$$

We need this inequality for computing $[\sin t]'$ in the next section, and we also need the inequality

$$|z| < |\tan z| \qquad \text{for} \qquad \frac{\pi}{2} < z < \frac{\pi}{2}. \tag{7.5}$$

For $z > 0$, examine the circle with radius 1 in Figure 7.3,

$$z = \frac{\overset{\frown}{AC}}{1} = \overset{\frown}{AC}, \qquad \text{and} \qquad \tan z = \frac{\overline{CD}}{\overline{OC}} = \overline{CD}.$$

We need to show that $\overset{\frown}{AC} < \overline{CD}$ which appears reasonable from the figure, but we present a proof. The area of the sector of the circle OAC is equal to the area of the whole circle times the ratio of the length of the arc $\overset{\frown}{AC}$ to the circumference of the whole circle. Thus

$$\text{Area of sector } OAC = \pi\, 1^2 \times \frac{\overset{\frown}{AC}}{2\pi \times 1} = \frac{\overset{\frown}{AC}}{2}$$

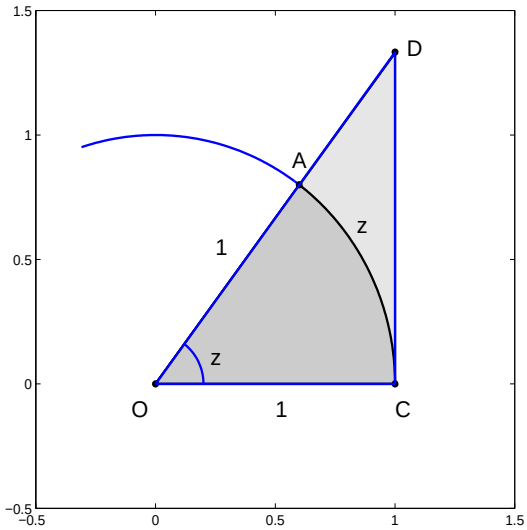


Figure 7.3: The unit circle and an angle z, $0 < z < \pi/2$.

The area of the triangle $\triangle OCD$ is

$$\text{Area } \triangle OCD = \frac{1}{2} \times 1 \times \overline{CD} = \frac{\overline{CD}}{2}$$

The sector OAC is contained in the triangle $\triangle OCD$ so that the area of sector OAC is less than the area of triangle $\triangle OCD$. Therefore

$$\frac{\overset{\frown}{AC}}{2} < \frac{\overline{CD}}{2}, \qquad \overset{\frown}{AC} < \overline{CD}, \qquad \text{and} \qquad z < \tan z.$$

An argument for Equation 7.5 for $z < 0$ can be based on the reflection of Figure 7.3 about the interval OC.

In addition to the basic trigonometric identities, ($\sin^2 t + \cos^2 t = 1$, $\tan t = \sin t/\cos t$, etc.) the double angle trigonometric formulas are critical to this chapter:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B. \tag{7.6}$$

From these you are asked to prove in Exercises 7.1.3

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

Exercises for Section 7.1 Radian Measure.

Exercise 7.1.1 For small positive values of z, $\sin z < z$ (Equation 7.4), but 'just barely so'.

a. Compute

$$z - \sin z \qquad \text{for} \quad z = 0.1 \qquad \text{for} \quad z = 0.01 \qquad \text{and for} \quad z = 0.001.$$

b. Compute

$$\frac{\sin z}{z} \qquad \text{for} \quad z = 0.1 \qquad \text{for} \quad z = 0.01 \qquad \text{and for} \quad z = 0.001.$$

c. Note that the slope of the tangent to $y = \sin t$ at (0,0) is

$$[\sin t]'_{t=0} = \lim_{h \to 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h}.$$

What is your best estimate of

$$[\sin t]'_{t=0}\,?$$

Exercise 7.1.2 For small positive values of z, $z < \tan z$ (Equation 7.5), but 'just barely so'.

a. Compute

$$\tan z - z \qquad \text{for} \quad z = 0.01 \qquad \text{for} \quad z = 0.001 \qquad \text{and for} \quad z = 0.0001.$$

b. Compute

$$\frac{\tan z}{z} \qquad \text{for} \quad z = 0.01 \qquad \text{for} \quad z = 0.001 \qquad \text{and for} \quad z = 0.0001.$$

c. Note that the slope of the tangent to $y = \tan t$ at (0,0) is

$$[\tan t]'_{t=0} = \lim_{h \to 0} \frac{\tan(0+h) - \tan 0}{h} = \lim_{h \to 0} \frac{\tan h}{h}.$$

What is your best estimate of

$$[\tan t]'_{t=0}\,?$$

Exercise 7.1.3 We need the identities

$$\begin{aligned} \sin x - \sin y &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \\ \cos x - \cos y &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \end{aligned}$$

in for the next two exercises and in the next section. It is unlikely that you remember them from a trigonometry class. We hope you do remember, however, the double angle formulas 7.6,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B.$$

a. Use $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and the identities, $\sin(-A) = -\sin A$ and $\cos(-A) = \cos A$, to show that

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

b. Use the equations

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B\end{aligned}$$

to show that

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B \tag{7.7}$$

c. Solve for A and B in

$$\begin{aligned}A+B &= x\\ A-B &= y\end{aligned}$$

d. Substitute the values for $A+B$, $A-B$, A, and B into Equation 7.7 to obtain

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

e. Use $\cos(A+B) = \cos A\cos B - \sin A\sin B$ to show that

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

The argument will be similar to the previous steps.

Exercise 7.1.4 Use steps a - d below to show that at all numbers z,

$$\lim_{h\to 0}\sin(z+h) = \sin z, \tag{7.8}$$

and therefore conclude that the sine function is continuous.

a. Write the trigonometric identity,

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

with $x = z+h$ and $y = z$.

b. Justify the inequality in the following statement.

$$|\sin(z+h) - \sin z| = 2\times\left|\cos\left(\frac{2z+h}{2}\right)\right|\times\left|\sin\left(\frac{h}{2}\right)\right| < 2\times 1\times\frac{|h|}{2} = |h|$$

c. Suppose ϵ is a positive number. Find a positive number δ so that

$$\text{if}\qquad |(z+h)-z| = |h| < \delta \qquad\text{then}\qquad |\sin(z+h) - \sin z| < \epsilon$$

d. Is the previous step useful?

Exercise 7.1.5 Use the Inequality 7.4 $|\sin z| < |z|$ and the trigonometric identity,

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

to argue that

$$\lim_{h\to 0}\cos(z+h) = \cos z,$$

and therefore conclude that the cosine function is continuous.
Hint: Look at the steps a - d of Exercise 7.1.5.

7.2 Derivatives of trigonometric functions

We will show that the derivative of the sine function is the cosine function, or

$$[\sin t]' = \cos t \tag{7.9}$$

From this formula and the Combination Derivative formulas 6.1 shown at the beginning of Chapter 6, the derivatives of the other five trigonometric functions are easily computed.

We first show that

$$[\sin t]'_{t=0} \;=\; \sin'(0) \;=\; 1$$

Assume h is a positive number less than $\pi/2$. We know from Inequalities 7.4 and 7.5 that

$$\sin h < h < \tan h$$

We write

$$\left.\begin{array}{rcccl} \sin h & < & h & < & \tan h \\ \sin h & < & h & < & \dfrac{\sin h}{\cos h} \\ 1 & < & \dfrac{h}{\sin h} & < & \dfrac{1}{\cos h} \qquad \text{a.} \\ 1 & > & \dfrac{\sin h}{h} & > & \cos h \qquad \text{b.} \end{array}\right\} \tag{7.10}$$

The inequalities

$$1 > \frac{\sin h}{h} > \cos h$$

present an opportunity to reason in a rather clever way. We wish to know what $\frac{\sin h}{h}$ approaches as the positive number h approaches 0 (as h approaches 0^+). Because $\cos x$ is continuous (Exercise 7.1.5) and $\cos 0 = 1$, $\cos h$ approaches 1 as h approaches 0^+. Now we have $\frac{\sin h}{h}$ 'sandwiched' between two quantities, 1 and a quantity that approaches 1 as h approaches 0^+. We conclude that $\frac{\sin h}{h}$ also approaches 1 as h approaches 0^+, and illustrate the result in the array:

$$\text{As} \quad h \to 0^+ \qquad \left\{\begin{array}{ccccc} 1 & < & \dfrac{\sin h}{h} & < & \cos h \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \leq & 1 & \leq & 1 \end{array}\right.$$

The argument can be formalized with the ϵ, δ definition of limit, but we leave it on an intuitive basis.

We have assumed $h > 0$ in the previous steps. A similar argument can be made for $h < 0$.

We now know that the slope of the graph of the sine function at (0,0) is 1. It is this result that makes radian measure so useful in calculus. For any other angular measure, the slope of the sine function at 0 is not 1. For example, the sine graph plotted in degrees has slope of $\pi/180$ at (0,0).

Because of the continuity of the composition of two functions, Equation 4.4, the equation

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \tag{7.11}$$

may take a variety of forms:

$$\lim_{h\to 0} \frac{\sin 2h}{2h} = 1 \qquad \lim_{h\to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} = 1 \qquad \lim_{h\to 0} \frac{\sin h^2}{h^2} = 1$$

We write a general form:

$$\boxed{\text{If} \quad \theta(h) \neq 0 \quad \text{for} \quad h \neq 0 \quad \text{and} \quad \lim_{h\to 0} \theta(h) = 0 \quad \text{then} \quad \lim_{h\to 0} \frac{\sin \theta(h)}{\theta(h)} = 1 \qquad (7.12)}$$

We use

$$\lim_{h\to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} = 1.$$

in the next paragraph. We also use the fact that the cosine function is continuous, Exercise 7.1.5.

Now we compute $[\sin t]'$ for any t. By Definition 3.20,

$$[\sin t]' = \lim_{h\to 0} \frac{\sin(t+h) - \sin t}{h}.$$

With the trigonometric identity

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

we get

$$
\begin{aligned}
[\sin t]' &= \lim_{h\to 0} \frac{\sin(t+h)-\sin t}{h} && \text{a.}\\
&= \lim_{h\to 0} \frac{2\cos\left(\dfrac{t+h+t}{2}\right)\sin\left(\dfrac{t+h-t}{2}\right)}{h} && \text{b.}\\
&= \lim_{h\to 0} \frac{\cos\left(t+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)}{\dfrac{h}{2}} \\
&= \lim_{h\to 0} \cos\left(t+\frac{h}{2}\right) \times \lim_{h\to 0} \frac{\sin\left(\dfrac{h}{2}\right)}{\dfrac{h}{2}} && \text{c.}\\
&= \cos t \ \times \lim_{h\to 0} \frac{\sin\left(\dfrac{h}{2}\right)}{\dfrac{h}{2}} && \text{d.}\\
&= \cos t \times 1 = \cos t && \text{e.}
\end{aligned}
\tag{7.13}
$$

We have now shown that $[\sin t]' = \cos t$. The derivatives of the other five trigonometric functions are easily computed using the derivative formulas 6.1 shown at the beginning of Chapter 6.

Derivative of the cosine. We will show that

$$[\cos t]' = -\sin t \tag{7.14}$$

Observe that

$$\sin(t+\frac{\pi}{2}) = \sin t\cos\frac{\pi}{2} + \cos t\sin\frac{\pi}{2} = \sin t \times 0 + \cos t \times 1 = \cos t$$

Therefore

$$\cos t = \sin(t+\frac{\pi}{2}) \qquad \text{and} \qquad [\cos t]' = \left[\sin(t+\frac{\pi}{2})\right]'.$$

We use the Chain Rule Equation 6.16

$$[G(u(t))]' = G'(u(t)) \times u'(t) \qquad \text{with} \qquad G(u) = \sin u \qquad \text{and} \qquad u(t) = t + \frac{\pi}{2}.$$

Note that $G'(u) = \cos u$ and $u'(t) = 1$.

$$
\begin{aligned}
[\cos t]' &= \left[\sin(t+\frac{\pi}{2})\right]' \\
&= \cos(t+\frac{\pi}{2}) \times \left[t+\frac{\pi}{2}\right]'
\end{aligned}
$$

$$
\begin{aligned}
&= \cos(t + \frac{\pi}{2}) \times 1 \\
&= \cos t \cos \frac{\pi}{2} - \sin t \sin \frac{\pi}{2} \\
&= -\sin t
\end{aligned}
$$

We have established Equation 7.14, $[\cos t]' = -\sin t$.
Derivative of the tangent function. We will show that

$$[\tan t]' = \sec^2 t \tag{7.15}$$

using

$$\tan t = \frac{\sin t}{\cos t} \qquad \text{and} \qquad [\sin t]' = \cos t \qquad \text{and} \qquad [\cos t]' = -\sin t$$

and the quotient rule for derivatives from Equations 6.1,

$$\left[\frac{u}{v}\right]' = \frac{v\,u' - uv'}{v^2}.$$

$$
\begin{aligned}
[\tan t]' &= \left[\frac{\sin t}{\cos t}\right]' \\
&= \frac{\cos t[\sin t]' - \sin t[\cos t]'}{(\cos t)^2} \\
&= \frac{\cos t \cos t \ - \sin t(-\sin t)}{\cos^2 t} \\
&= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} \\
&= \sec^2 t
\end{aligned}
$$

In summary we write the formulas

$[\sin t]' = \cos t$	7.9	$[\cos t]' = -\sin t$	7.14
$[\tan t]' = \sec^2 t$	7.15	$[\cot t]' = -\csc^2 t$	7.16
$[\sec t]' = \sec t \tan t$	7.17	$[\csc t]' = -\csc t \cot t$	7.18

You are asked to compute $[\cot t]'$, $[\sec t]'$, and $[\csc t]'$ in Exercise 7.2.5.

Example 7.2.1 We illustrate the use of the derivative formulas for sine, cosine, and tangent by computing the derivatives of

$$\text{a. } y = 3\sin t\cos t \quad \text{b. } y = \sin^4 t \quad \text{c. } y = \ln(\tan t) \quad \text{d. } y = e^{\sin t}$$

$$\begin{aligned}
\text{a.} \qquad [3\sin t\cos t]' &= 3\left(\sin t\,[\cos t]' + [\sin t]'\cos t\right) \\
&= 3\left(\sin t \times (-\sin t) + \cos t \times \cos t\right) \\
&= -3\sin^2 t + 3\cos^2 t
\end{aligned}$$

$$\text{b.} \qquad \left[\sin^4 t\right]' = 4\left(\sin^3 t\right) \times [\sin t]' = 4\left(\sin^3 t\right) \times \cos t.$$

$$\begin{aligned}
\text{c.} \quad [\ln\tan t]' &= [(\ln(\sin t)) - (\ln(\cos t))]' = [\ln(\sin t)]' - [\ln(\cos t)]' \\
&= \frac{1}{\sin t}[\sin t]' - \frac{1}{\cos t}[\cos t]' = \frac{1}{\sin t}\cos t - \frac{1}{\cos t}(-\sin t) \\
&= \frac{\cos^2 t + \sin^2 t}{\cos t\sin t} = \sec t\csc t.
\end{aligned}$$

$$\text{d.} \qquad \left[e^{\sin t}\right]' = e^{\sin t}[\sin t]' = e^{\sin t}\cos t$$

Example 7.2.2 The function $F(t) = e^{-t/10}\sin t$ is an example of 'damped oscillation,' an important type of vibration. Its graph is shown in Figure 7.4. The peaks and valleys of the oscillation are marked by values of t for which $F'(t) = 0$. We find them by

$$\begin{aligned}
\left[e^{-t/10}\sin t\right]' &= e^{-t/10}[\sin t]' + \left[e^{-t/10}\right]'\sin t \\
&= e^{-t/10}\cos t - \frac{1}{10}e^{-t/10}\sin t = e^{-t/10}\left(\cos t - \frac{1}{10}\sin t\right)
\end{aligned}$$

Now $e^{-t/10} > 0$ for all t; $F'(t) = 0$ implies that

$$\cos t - \frac{1}{10}\sin t = 0, \qquad \tan t = 10, \qquad t = \arctan 10 + n\pi \qquad \text{for } n \text{ an integer.}\blacksquare$$

Exercises for Section 7.2 Derivatives of trigonometric functions

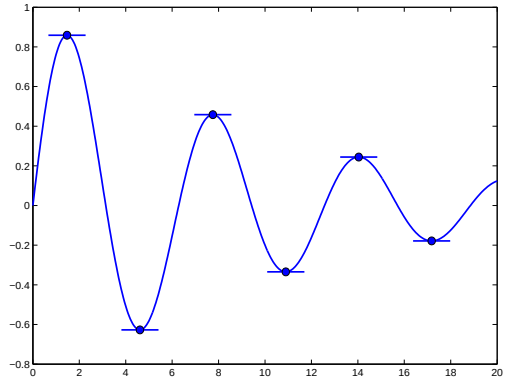


Figure 7.4: Graph of $y = e^{-t/10} \sin t$. The relative high and low points are marked by horizontal tangents and occur at $t = \arctan 10 + n\pi$ for n an integer.

Exercise 7.2.1 The difference quotient

$$\frac{F(t+h) - F(t)}{h} \qquad \text{approximates} \qquad F'(t) \qquad \text{when } h \text{ is 'small.'}$$

Make a plot of

$$y = \cos t \qquad \text{and of} \qquad \frac{\sin(t+0.2) - \sin t}{0.2} \qquad -\frac{\pi}{2} \le t \le 2\pi.$$

Repeat, using $h = 0.05$ instead of $h = 0.2$.

Exercise 7.2.2 Compute the derivative of

$$y(t) = 1 + 2t - 5t^7 + 2e^{3t} - \ln 6t + 2\sin t - 3\cos t$$

Exercise 7.2.3 Compute the derivatives of

a.	$y = 2\sin t \cos t$	b.	$y = \sin^2 t + \cos^2 t$	c.	$y = \sec t = \dfrac{1}{\cos t}$
d.	$y = \cot t$	e.	$y = \ln \cos t$	f.	$y = \sin^2 t - \cos^2 t$
g.	$y = \csc t = \dfrac{1}{\sin t}$	h.	$y = \sec^2 t$	i.	$y = e^{\cos t}$
j.	$y = \ln(\sec t)$	k.	$y = e^{-t} \sin t$	l.	$y = \tan^2 t$
m.	$y = \dfrac{e^{2t}}{400}$	n.	$y = \ln(\cos^{20} t)$	o.	$y = \dfrac{\ln t^2}{30}$

Exercise 7.2.4 Compute y' and solve for t in $y'(t) = 0$. Sketch the graphs and find the highest and the lowest points of the graphs of:

a.	$y = \sin t + \cos t$	$0 \le t \le \pi$		b.	$y = e^{-t} \sin t$	$0 \le t \le \pi$
c.	$y = \sqrt{3}\sin t + \cos t$	$0 \le t \le \pi$		d.	$y = \sin t \times \cos t$	$0 \le t \le \pi$
e.	$y = e^{-t} \cos t$	$0 \le t \le \pi$		f.	$y = e^{-\sqrt{3}t} \cos t$	$0 \le t \le \pi$

Note: $\cos^2 t - \sin^2 t = \cos(2t)$.

Exercise 7.2.5 a. Use $\cot x = \dfrac{\cos x}{\sin x}$ and the quotient rule to show that

$$[\cot x]' = -\csc^2 x. \tag{7.16}$$

b. Use $\sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$ and the power chain rule to show that

$$[\sec x]' = \sec x \tan x. \tag{7.17}$$

c. Show that

$$[\csc x]' = -\csc x \cot x. \tag{7.18}$$

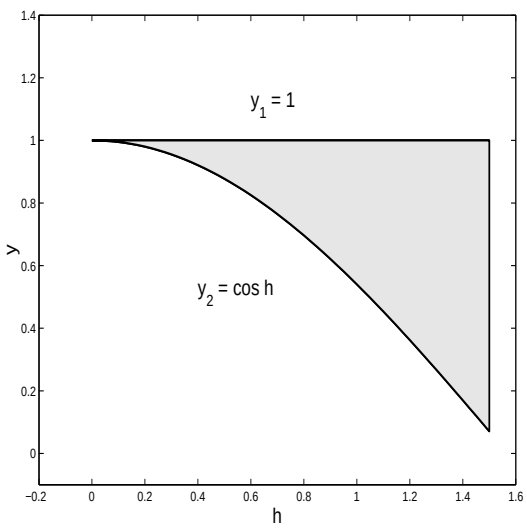
Exercise 7.2.6 The graphs of $y_1 = \cos h$ and $y_2 = 1$ are shown in Figure Ex. 7.2.6. The inequality

$$1 < \frac{\sin h}{h} < \cos h$$

implies that the graph of $y = \dfrac{\sin h}{h}$, $0 < h < \pi/2$ is 'sandwiched' between y_1 and y_2. Let F be any function defined on $0 < z \leq 1$ whose graph lies above the graph of y_1 and below the graph of y_2.

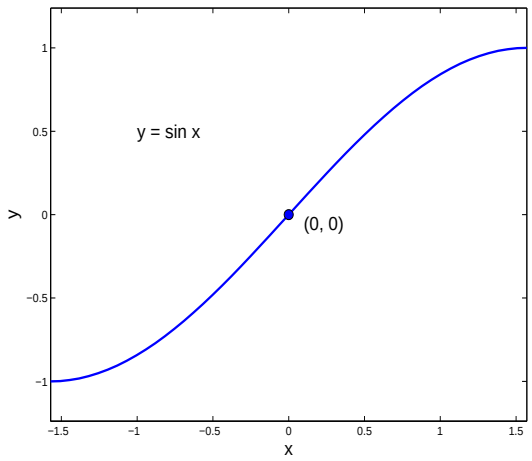
1. Draw the graph of one such function, F.
2. What number does $F(h)$ approach as h approaches 0?

Figure for Exercise 7.2.6 Graphs of $y_1 = 1$ and $y_2 = \cos h$. See Exercise 7.2.6.



Exercise 7.2.7 Draw a tangent to the graph of the sine function at the point (0,0) in Figure Ex. 7.2.7. Choose two points of the tangent, measure the coordinates of the two points, and use those coordinates to compute the slope of the tangent and $[\sin t]'\big|_{t=0}$.

Figure for Exercise 7.2.7 Graph of $y = \sin x$. See Exercise 7.2.7.



Exercise 7.2.8 Give reasons for steps a and b in Equation 7.10 leading to the inequalities

$$1 > \frac{\sin h}{h} > \cos h$$

It is important that $h > 0$; why?

Exercise 7.2.9 The derivative of $y = \cos x$ is defined by

$$[\cos x]' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

Make a plot of

$$y = -\sin t \qquad \text{and of} \qquad \frac{\cos(t+0.2) - \cos t}{0.2} \qquad -\frac{\pi}{2} \le t \le 2\pi.$$

Repeat, using $h = 0.05$ instead of $h = 0.2$.

Exercise 7.2.10 Show that $[\cos t]' = -\sin t$. To do so, use

$$[\cos t]' = \lim_{h \to 0} \frac{\cos(t+h) - \cos t}{h}$$

and the trigonometric identity

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

The steps will be similar to those of Equation 7.13.

Exercise 7.2.11 We have shown that the slope of the graph of the sine function at (0,0) is 1. Use symmetry of the graph of $y = \sin x$ to find the slope of the graph at the point $(\pi, 0)$. Find an equation of the line tangent to the graph of $y = \sin x$ at the point $(\pi, 0)$. Draw the graph of your line and a graph of $y = \sin x$.

Exercise 7.2.12 Alternate derivation of $[\cos x]' = -\sin x$ using implicit differentiation and $[\sin x]' = \cos x$.

Compute the derivatives of both sides of the identity

$$\sin^2 x + \cos^2 x = 1 \quad \text{and obtain} \quad 2\sin x \cos x + 2\cos x\,[\cos x]' = 0$$

Use the last equation to argue that if $\cos x \neq 0$ then

$$[\cos x]' = -\sin x$$

7.3 The Chain Rule with trigonometric functions.

The Chain Rule 6.16 states that if G and u are functions that have derivatives and the composition of G with u is well defined then

$$[G(u(t))]' = G'(u(t)) \times [u(t)]'$$

With $G(u) = \sin u$ or $\cos u$ or $\tan u$ we have

$$\begin{aligned}
[\sin(u(t))]' &= \cos(u(t)) \times u'(t), \\
[\cos(u(t))]' &= -\sin(u(t)) \times u'(t), \qquad \text{and} \\
[\tan(u(t))]' &= \sec^2(u(t)) \times u'(t)
\end{aligned}$$

Example 7.3.1 For $u(t) = kt$ where k is a constant,

$$[\sin(kt)]' = \cos(kt) \times [kt]' = \cos(kt) \times k$$

Example 7.3.2 With repeated use of the chain rule, derivatives of some rather difficult and exotic functions can be computed. For example, find y' for

$$y(t) = \ln\left(\sin\left(e^{\cos t}\right)\right)$$

You may find it curious that $y(t)$ is meaningful for all values of t. In the following, we 'peel one outside layer at a time'.

$$\begin{aligned}
y'(t) &= [\ln(\sin(e^{\cos t}))]' && \text{Outside layer} \\
&= \frac{1}{\sin\left(e^{\cos t}\right)} \times [\sin(e^{\cos t})]' && G(u) = \ln u, \qquad G'(u) = \frac{1}{u} \\
&= \frac{1}{\sin\left(e^{\cos t}\right)} \times \cos(e^{\cos t}) \times [e^{\cos t}]' && G(u) = \sin u, \qquad G'(u) = \cos u \\
&= \frac{1}{\sin\left(e^{\cos t}\right)} \times \cos(e^{\cos t}) \times e^{\cos t} \times [\cos t]' && G(u) = e^u, \qquad G'(u) = e^u \\
&= \frac{1}{\sin\left(e^{\cos t}\right)} \times \cos(e^{\cos t}) \times e^{\cos t} \times (-\sin t) && G(u) = \cos u, \qquad G'(u) = -\sin u \\
&= -\sin t \times e^{\cos t} \times \cot(e^{\cos t})
\end{aligned}$$

Example 7.3.3 We will find in later sections that the dynamics of some physical and biological systems are described by equations similar to

$$y''(t) + 2y'(t) + 37y(t) = 0. \tag{7.19}$$

Some of these equations describe functions, $y(t)$, that are defined using sine and cosine functions together with exponential functions. This specific equation has a solution

$$y(t) = e^{-t}\cos 6t \tag{7.20}$$

the graph of which is shown in Figure 7.5. The function is called a 'damped cosine' function. It is a cosine function whose amplitude is e^{-t} which decreases with time.

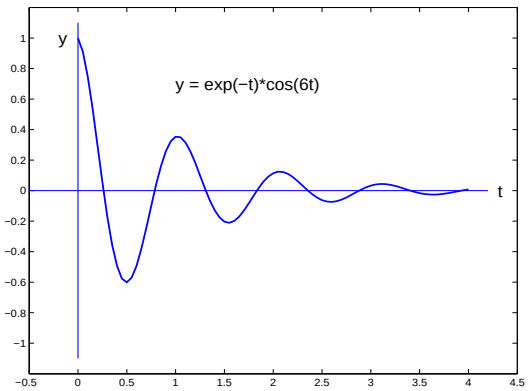


Figure 7.5: The graph of the damped cosine function $y = e^{-t}\cos 6t$.

We show that $y(t) = e^{-t}\cos 6t$ solves $y''(t) + 2y'(t) + 37y(t) = 0$.

$$\begin{aligned} y'(t) &= \left[e^{-t}\cos t\right]' \\ &= \left[e^{-t}\right]'\cos 6t + e^{-t}\left[\cos 6t\right]' && \text{a.} \\ &= \left(-e^{-t}\right)\cos 6t + e^{-t}\left(-\sin 6t\right)\times 6 && \text{b.} \end{aligned} \tag{7.21}$$

$$\begin{aligned} y''(t) &= \left[-e^{-t}\cos 6t - 6e^{-t}\sin 6t\right]' \\ &= \left[-e^{-t}\cos 6t\right]' - 6\left[e^{-t}\sin 6t\right]' && \text{c.} \\ &= -\left[e^{-t}\right]'\cos 6t - e^{-t}\left[\cos 6t\right]' \\ &\quad -6\left(\left[e^{-t}\right]'\sin 6t + e^{-t}\left[\sin 6t\right]'\right) && \text{d.} \\ &= e^{-t}\cos 6t + 6e^{-t}\sin 6t + 6e^{-t}\sin 6t - 36e^{-t}\cos 6t && \text{e.} \\ &= 12e^{-t}\sin 6t - 35e^{-t}\cos 6t \end{aligned} \tag{7.22}$$

We next substitute $y(t) = e^{-t}\cos 6t$, and the computed values for $y'(t)$ and $y''(t)$ into $y'' + 2y' + 37y$ and confirm the solution.

$$\begin{array}{ccccc} y'' & + & 2y' & + & 37y \\ 12e^{-t}\sin 6t - 35e^{-t}\cos 6t & + & 2\left(-e^{-t}\cos 6t - 6e^{-t}\sin 6t\right) & + & 37e^{-t}\cos 6t \\ = (12 - 2\times 6)e^{-t}\sin 6t & + & (-35 - 2 + 37)e^{-t}\cos 6t & = 0 & \end{array}$$

Example 7.3.4 A searchlight is 400 m from a straight beach and rotates at a constant rate once in two minutes. How fast is the beam moving along the beach when the beam is 600 m from the nearest point, A, of the beach to the searchlight.

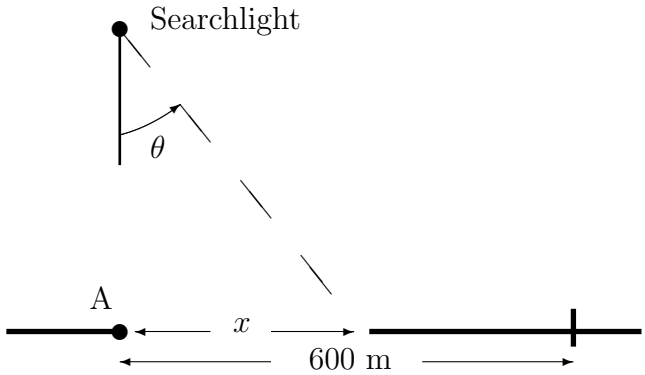


Figure 7.6: Searchlight shining along a beach.

Solution. See Figure 7.6. Let θ measure the rotation of the light with $\theta = 0$ when the light is pointing toward A. Let x be the distance from A to the point where the beam strikes the beach. We have

$$\tan(\theta) = \frac{x}{400}$$

Both θ and x are functions of time and we write.

$$\tan\theta(t) = \frac{x(t)}{400}$$

and differentiate both sides of the equation with respect to t. We get

$$\begin{aligned} [\tan(\theta(t))]' &= \left[\frac{x(t)}{400}\right]' \\ \sec^2(\theta(t))\,[\theta(t)]' &= \frac{1}{400}[x(t)]' \qquad \text{Tangent Chain,Constant Factor} \\ \sec^2(\theta(t))\theta'(t) &= \frac{1}{400}x'(t) \end{aligned}$$

The problem is to find $x'(t)$ when $x(t) = 600$. When $x(t) = 600$, $\tan(\theta(t)) = \frac{600}{400} = \frac{3}{2}$, and $\sec^2(\theta(t)) = 1 + \left(\frac{3}{2}\right)^2$. The light rotates once every two minutes, so $\theta'(t) = \frac{2\pi \text{ radians}}{2 \text{ minutes}} = \pi$ radians/minute. Therefore when $x(t) = 600$,

$$x'(t) = 400 \times \frac{13}{4}\pi \qquad \text{meters/minute}$$

Note: Radian is a dimensionless measurement.

Exercises for Section 7.3 The Chain rule with trigonometric functions.

Exercise 7.3.1 Find y' for

a.	y	$=$	$2\cos t$	b.	y	$=$	$\cos 2t$	c.	y	$=$	$\sin t^2$
d.	y	$=$	$\sin(t+\pi)$	e.	y	$=$	$\cos(\pi t - \pi/2)$	f.	y	$=$	$(\sin t) \times (\cos t)$
g.	y	$=$	$\sin^2(t^4)$	h.	y	$=$	$\cos(\ln(t+1))$	i.	y	$=$	$\sin(\cos t)$
j.	y	$=$	$\tan(\frac{\pi}{2}t)$	k.	y	$=$	$\tan^2(t^2)$	l.	y	$=$	$\tan(\cos t)$
m.	y	$=$	$-\ln(\cos t)$	n.	y	$=$	$e^{\sin t}$	o.	y	$=$	$\ln(\sin t)$

Exercise 7.3.2 Provide reasons for the differentiation steps in Equations 7.21 and Equations 7.22.

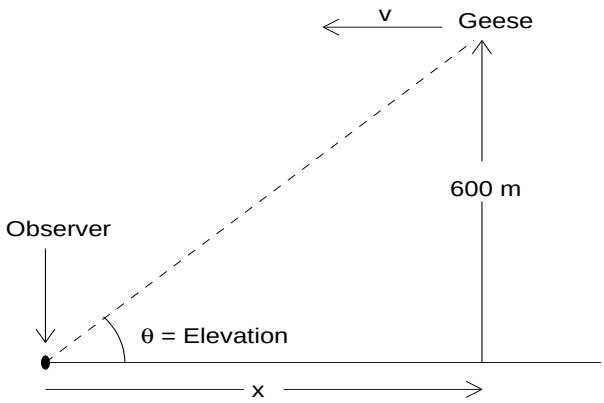
Exercise 7.3.3 Show that the suggested solutions solve the associated equations.

a.	y	$=$	$\cos t$	$y'' + y$	$=$	0
b.	y	$=$	$\cos 2t$	$y'' + 4y$	$=$	0
c.	y	$=$	$3\sin t + 2\cos t$	$y'' + y$	$=$	0
d.	y	$=$	$-3\sin 2t + 5\cos t$	$y'' + 4y$	$=$	0
e.	y	$=$	e^{-t}	$y'' + 2y' + y$	$=$	0
f.	y	$=$	$e^{-t}\sin t$	$y'' + 2y' + 2y$	$=$	0

Exercise 7.3.4 In Example 7.3.4 of the rotating light with beam shining along the beach, where is the motion of the beam along the beach the smallest?

Exercise 7.3.5 A flock of geese is flying toward you on a path that will be directly above you and at a height of 600 meters above you. During a ten second interval, you measure the angles of elevation, $\theta(t)$, of the flock three times and obtain the data shown. How fast are the geese flying[2]?

Time, t Sec	Elevation, $\theta(t)$ Degrees
0	40°
5	45°
10	51°



Let $x(t)$ denote the horizontal distance from you to the geese. Then

$$\tan\theta(t) = \frac{600}{x(t)} = 600\,(x(t))^{-1}\,.$$

a. Show that

$$\left(\sec^2\theta(t)\right) \times \theta'(t) = -600\,(x(t))^{-2} \times x'(t) \tag{7.23}$$

b. The derivative formulas require angles to be measured in radians. Convert the values of θ in the table to radians.

c. Estimate $\theta'(5)$ in radians per second.

d. Compute $x(5)$

e. Use Equation 7.23 to estimate $x'(t)$ in meters per second for $t = 5$ seconds.

Exercise 7.3.6 In Exercise 7.3.5, the centered difference estimate of $\theta'(5)$ is

$$\theta'(5) \doteq \frac{\theta(10) - \theta(0)}{10 - 0} = \frac{51 \times \dfrac{\pi}{180} - 40 \times \dfrac{\pi}{180}}{10 - 0} = 0.0192 \quad \text{radians/second.}$$

a. Estimate x' using this value of θ'.

b. What error in the estimate of $x'(t)$ meters/second might be caused by an error of 0.001 in $\theta'(5)$?

c. With some confidence (and looking at the data), we might argue that

$$\frac{\theta(5) - \theta(0)}{5 - 0} \le \theta'(5) \le \frac{\theta(10) - \theta(5)}{10 - 5}$$

Compute these bounds for $\theta'(5)$ and use them in similar computations of $x'(5)$ to compute bounds on the estimate of $x'(5)$.

[2]http://north.audubon.org/facts.html#wea shows snow geese migrate 3000 miles at 2952 foot altitude and at an average speed of 50 mph

d. Show that an error of size ϵ in $\theta'(5)$ causes an error of $1200 \times \epsilon$ in $x'(5)$.

Exercise 7.3.7 A piston is linked by a 20 cm tie rod to a crank shaft which has a 5 cm radius of motion (see Figure 7.3.7). Let $x(t)$ be the distance from the rotation center of the crank shaft to the end of the tie rod and $\theta(t)$ be the rotation angle of the crank shaft, measured from the line through the centers of the crank shaft and piston. The crank shaft is rotating at 100 revolutions per minute. The goal is to locate the point of the cylinder at which the piston speed is the greatest.

Note: You may prefer to think of this as fly fishing; the crank shaft (rescaled) is your fly rod rotating about your wrist, the tie rod is your fly line, and the piston is a fish.

a. Find an equation relating $x(t)$ and $\theta(t)$. You will find the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$, useful, where a, b, and c are the lengths of the sides of a triangle and C is the angle opposite side c.

b. What is $\theta'(t)$?

c. What values of $x(t)$ are possible?

d. Differentiate your equation from part (a) to find an equation relating $x'(t)$ to $x(t)$, $\theta(t)$, and $\theta'(t)$.

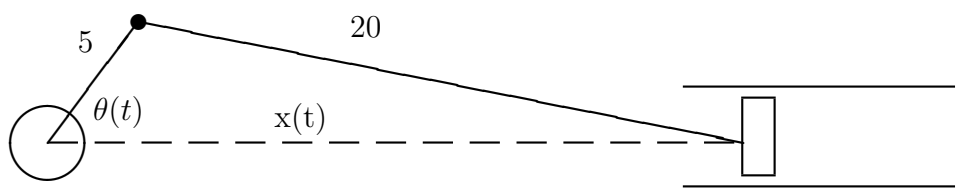
e. Evaluate $x(t)$ for $\theta(t) = 0 + 2\pi$. Evaluate $x'(t)$ for $\theta(t) = 0 + 2\pi$.

f. Evaluate $x(t)$ for $\theta(t) = \frac{\pi}{2} + 2\pi$. Evaluate $x'(t)$ for $\theta(t) = \frac{\pi}{2} + 2\pi$.

g. Evaluate $x(t)$ for $\theta(t) = 1.35 + 2\pi$. Evaluate $x'(t)$ for $\theta(t) = 1.35 + 2\pi$.

The last two results may seem a bit surprising, but would be intuitive to fly fishermen and fly fisherwomen. We will return to this problem in Example 8.4.3.

Figure for Exercise 7.3.7 Crank shaft, tie rod, and piston for Exercise 7.3.7.



7.4 The Equation $y'' + \omega^2 y = 0$.

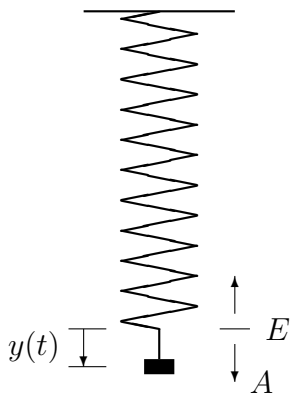
In this section:

A principal use of the sine and cosine functions are in the descriptions of *Harmonic Oscillations*. A mass suspended on a spring that is displaced from equilibrium, E, and released is an example of a simple harmonic oscillator. This spring-mass system is modeled by the equations

$$\begin{aligned} y(0) &= A \quad \text{Initial displacement.} \\ y'(0) &= 0 \quad \text{Initial velocity.} \\ y''(t) + \omega^2 y(t) &= 0 \end{aligned}$$

The equations are solved by

$$y(t) = A\cos(\omega t)$$



Harmonic oscillations are ubiquitous in the material world. The sine and cosine functions are called the harmonic functions and at least to first approximation are descriptive of sound waves, light waves, planetary motion, tidal motion, ear drum oscillations, swinging pendula, alternating electrical current, earthquake waves, flutter of a leaf, (the list is quite long).

The most simple equation that applies to oscillating systems in which the resistance to motion is negligible is:

$$y''(t) + \omega^2 y(t) = 0 \tag{7.24}$$

Generally $y(t)$ is the displacement from equilibrium of some measure of the system. The constant ω measures the strength of the force that restores the system to equilibrium. Solutions to the equation are of the form

$$y(t) = A\sin(\omega t) + B\cos(\omega t) \tag{7.25}$$

where A and B are constants that are determined from information about the state of the system at time 0 (for example, $y(0) = 1$, $y'(0) = 0$ implies that $A = 0$ and $B = 1$). All such functions satisfy $y''(t) + \omega^2 y(t) = 0$. That there are no other solutions follows from the uniqueness of solutions to linear differential equations usually established in differential equation courses.

We first establish that if $y(t) = A\sin(\omega t)$ then $y''(t) + \omega^2 y(t) = 0$.

$$y(t) = A\sin(\omega t)$$

$$\begin{aligned} y'(t) &= A\cos(\omega t) \times [\omega t]' \\ &= A\cos(\omega t) \times \omega \end{aligned}$$

$$\begin{aligned} y''(t) &= [A\omega\cos(\omega t)]' \\ &= A\omega(-\sin(\omega t)) \times [\omega t]' \\ &= -A\omega^2\sin(\omega t) \end{aligned}$$

It is immediate then that $y''(t) + \omega^2 y(t) = 0$ for

$$y''(t) + \omega^2 y(t) = \left(-A\omega^2\sin(\omega t)\right) + \omega^2 \times A\sin(\omega t) = 0$$

You are asked to show in Exercises 7.4.4 and 7.4.5 that $y(t) = B\cos(\omega t)$ and $y(t) = A\sin\omega t + B\cos(\omega t)$ solve $y''(t) + \omega^2 y(t) = 0$.

It is easy to visualize the motion of a mass suspended on a spring and we begin there. However, the mathematics involved is the same in many other systems; one of the powers of mathematics is that a single mathematical formulation may be descriptive of many systems.

It is shown in beginning physics courses that if $y(t)$ measures the displacement from the rest position of a body of mass m suspended on a spring (see Figure 7.7) with spring constant k, then

$$my''(t) + ky(t) = 0$$

. The equation is derived by equating the two forces on the mass,

$$\begin{aligned} \text{Newton's Second Law of Motion} \qquad F_1 &= \text{mass} \times \text{acceleration} &&= m \times y'' \\ \text{Hooke's Spring Law} \qquad F_2 &= -k \times \text{spring elongation} &&= -k \times y \end{aligned}$$

$$F_1 = F_2 \qquad \text{implies that} \qquad my'' = -ky, \qquad \text{or} \qquad my'' + ky = 0. \tag{7.26}$$

Assume that the mass is held motionless a distance A below the equilibrium point and at time $t = 0$ the mass is released. Then

$$y(0) = A \qquad \text{and} \qquad y'(0) = 0$$

If we let $\omega = \sqrt{\frac{k}{m}}$ so that $\omega^2 = \frac{k}{m}$ we have

$$y(0) = A \qquad y'(0) = 0 \qquad y''(t) + \omega^2 y(t) = 0 \tag{7.27}$$

You will show in Exercise 7.4.4 that the function

$$y(t) = A\cos\omega t.$$

satisfies the three conditions of Equations 7.27. It gives a good description of the motion of a body suspended on a spring.

The number ω is important. On the time interval $[0, 2\pi/\omega]$, $\cos\omega t$ completes one cycle and the position, $y(t)$, of the body progresses from A to $-A$ and back to A. Thus

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{is the } \textbf{period} \text{ of one oscillation.}$$

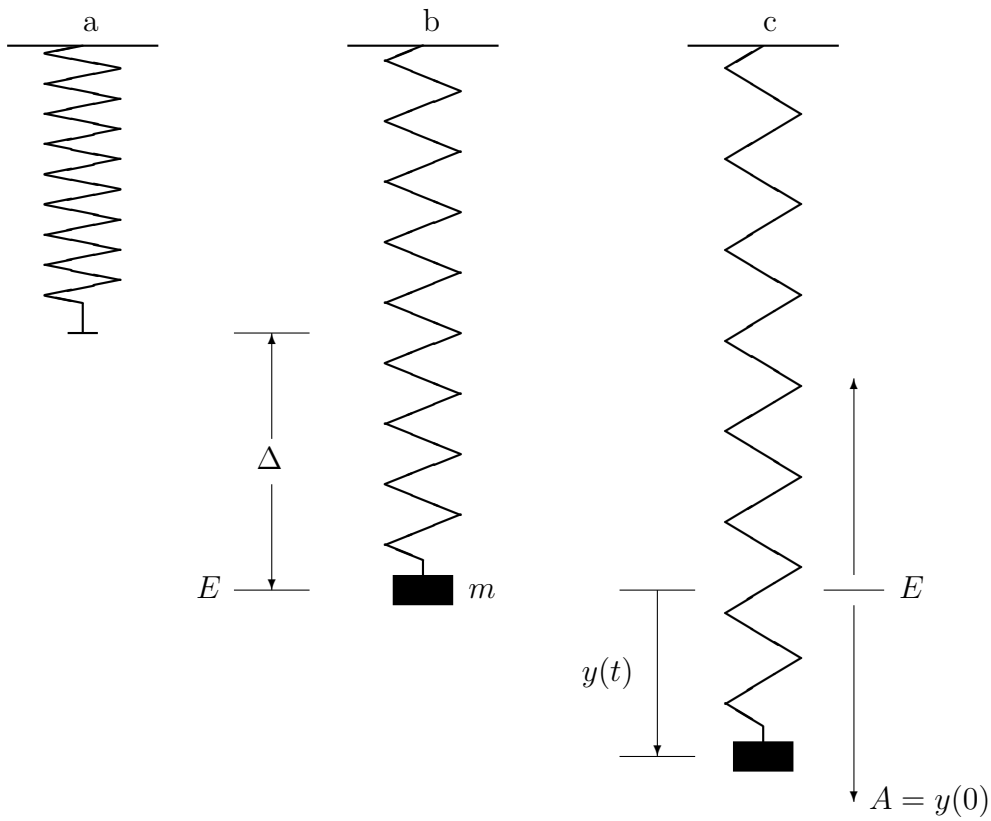


Figure 7.7: Oscillation of a mass attached to a spring. a. Relaxed spring with no mass attached. b. A body of mass m is attached and stretches the spring a distance Δ to an equilibrium position E. c. The body is displaced a distance A below the equilibrium point E and released. $y(t)$ is the displacement of the body from E at time t and is positive when the body is below E.

One time unit divided by the length of a period of oscillation gives the number of oscillations per unit time and

$$\frac{1}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{is the } \textbf{frequency} \text{ of oscillations.}$$

If k/m is 'large' (stiff spring or small mass) the period will be 'short' and the body oscillates rapidly. If k/m is 'small' (weak spring or large mass) the period will be long, the frequency will be low and the body oscillates slowly.

Example 7.4.1 Suppose a body of mass m is suspended from a spring with spring constant k.

1. If $m = 20$ gm $= 0.020$ Kg and $k = 0.125$ Newtons/meter and the initial displacement, $y_0 = 5$ cm $=$ 0.05 m, then

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.125 \text{ Kg}}{0.020 \text{ Kg-m/s}^2\text{/m}}} = 2.5/\text{s}$$

and

$$y(t) = 0.05\cos(2.5t)$$

The period of oscillation is

$$\frac{2\pi}{\omega} = \frac{2\pi}{2.51/\text{s}} = 2.51 \text{ s.}$$

and the frequency of oscillation is approximately

$$60/2.51 = 23.9 \qquad \text{oscillations per minute.}$$

2. If $m = 5$ gm $= 0.005$ Kg (one-fourth the previous mass) and $k = 0.125$ Newtons/meter then the period and frequency of oscillation would be

$$\text{Period} = \quad 2\pi\sqrt{\frac{m}{k}} = \quad 2\pi\sqrt{\frac{0.005}{0.125}} = 2\pi \times 0.2 \doteq 1.25 \quad \text{seconds per oscillation}$$

$$\text{Frequency} = \quad \frac{1}{1.25} \quad \text{oscillations per second} \quad = \quad 48 \quad \text{oscillations per minute.}$$

Thus one-fourth the mass oscillates twice as fast.

3. If $m = 20gm$ and the spring extends 16 cm when the body is attached to it, then the spring constant, k is

$$k = \frac{\text{Force}}{\text{Extension}} = \frac{0.02 \times 9.8 \text{ Kg-Force}}{0.16 \text{ meter}} \times \frac{1 \text{ Newton}}{9.8 \text{ Kg-Force}} = 0.125 \qquad \text{Newton/meter}$$

It is a curious consequence of the previous analysis that the magnitude of the gravitational field, g, is not reflected in the model equation nor in the solution equation. The role of g is to determine equilibrium location, E. The period and frequency of the oscillations would be the same on the Moon as on Earth.

There are two important omissions in the previous analysis. We have ignored the mass of the spring (which will also be moving) and we have ignored resistance to movement (by the air and in the spring).

7.4.1 Resistance.

In most systems, the amplitudes of the oscillations decrease with time due to resistance to the movement or *friction* in the system. Resistance is a force directed opposite to the direction of motion and may be modeled by

$$\text{Resistance} = \qquad -r \times y'(t)$$

Including the force of resistance with the force of the spring, Equation 7.26 is modified to

$$my''(t) = -ky(t) - ry'(t)$$

or

$$my''(t) + ry'(t) + ky(t) = 0 \tag{7.28}$$

This is referred to as the equation of *damped* motion.

Example 7.4.2 Suppose $m = 20$ gm $= 0.020$ kg, r $= 0.06$ Newtons/(meter/sec) and $k = 0.125$ Newtons/meter. Then the equation of damped motion is

$$0.02y''(t) + 0.06y'(t) + 0.125y(t) = 0 \tag{7.29}$$

We show that a solution to this equation is

$$y(t) = e^{-1.5t}\cos(2t)$$

$$\begin{aligned}
y'(t) &= \left[e^{-1.5t}\cos(2t)\right]' \\
&= \left[e^{-1.5t}\right]'\cos(2t) + e^{-1.5t}\left[\cos(2t)\right]' \\
&= e^{-1.5t}(-1.5)\cos 2t + e^{-1.5t}(-\sin 2t)\,2 \\
&= -1.5e^{-1.5t}\cos 2t - 2e^{-1.5t}(\sin 2t) \\
y''(t) &= \left[-1.5e^{-1.5t}\cos 2t - 2e^{-1.5t}(\sin 2t)\right]' \\
&= -1.5\left(\left[e^{-1.5t}\right]'\cos 2t + e^{-1.5t}\left[\cos 2t\right]'\right) - 2\left(\left[e^{-1.5t}\right]'\sin 2t + e^{-1.5t}\left[\sin 2t\right]'\right) \\
&= (-1.5)^2e^{-1.5t}\cos 2t + 2(-1.5)(-2)e^{-1.5t}\sin 2t - 2^2e^{-1.5}\cos 2t \\
&= -1.75e^{-1.5t}\cos 2t + 6e^{-1.5t}\sin 2t
\end{aligned}$$

Now we set up a table of coefficients and terms of Equation 7.29.

0.02	$y''(t)$	$-1.75e^{-1.5t}\cos 2t$	$+$	$6e^{-1.5t}\sin 2t$
0.06	$y'(t)$	$-1.5e^{-1.5t}\cos 2t$	$-$	$2e^{-1.5t}\sin 2t$
0.125	$y(t)$	$e^{-1.5t}\cos 2t$		

After substitution into Equation 7.29 the coefficients of $e^{-1.5t}\cos 2t$ and $e^{-1.5t}\sin 2t$ are

$$0.02 \times (-1.75) + 0.06 \times (-1.5) + 0.125 \times 1 = 0.0 \qquad \text{and} \qquad 0.02 \times 6 + 0.06 \times (-2) = 0.0$$

so

$$y(t) = e^{-1.5t}\cos(2t) \qquad \text{solves} \qquad 0.02y''(t) + 0.06y'(t) + 0.125y(t) = 0.$$

In the previous problem, the resistance, $r = 0.06$ was selected to make the numbers in the solution (-1.5 and 2) reasonably tractable. The resistance, $r = 0.06$ is so great, however, that the oscillations are imperceptable after only two or three oscillations, as illustrated in Figure 7.8A.

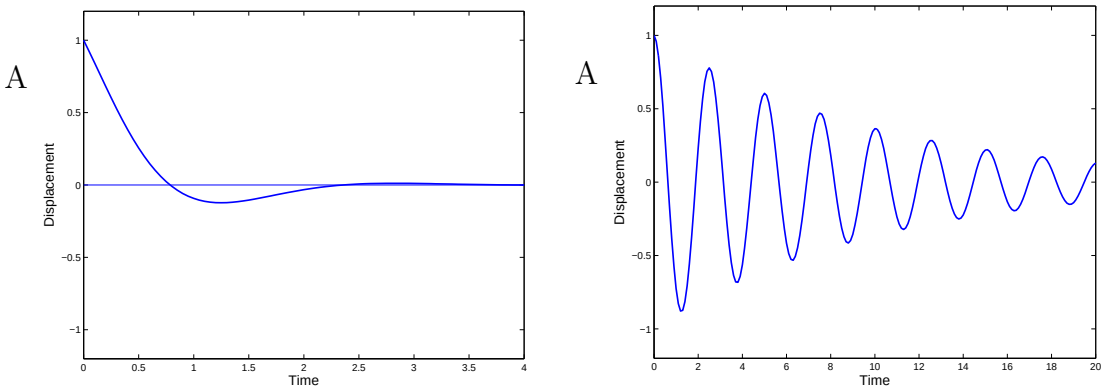


Figure 7.8: A. Graph of $y(t) = e^{-1.5t}\cos(2t)$. B. Graph of $y(t) = e^{-0.1t}\cos(2.4995t)$.

If $r = 0.002$ then the solution is

$$\begin{aligned} y(t) &= e^{-0.1t}\cos(\sqrt{6.2475}t) \\ &\doteq e^{-0.1t}\cos(2.4995t) \end{aligned}$$

A graph of $y(t) = e^{-0.1t}\cos(2.4995t)$ is shown in Figure 7.8B.

If the resistance in a vibrating system is quite large (system is buried in molasses), the system may not vibrate at all but may just ooze back to equilibrium after a displacement. Equation 7.28, $my''(t) + ry'(t) + ky(t) = 0$, may be written (divide by m)

$$y''(t) + 2by'(t) + cy(t) = 0 \tag{7.30}$$

The solutions to Equation 7.30 are **Bolt of Lightning**

$$\left.\begin{array}{lll} 1. \quad y(t) = A\,e^{(-b+\sqrt{b^2-c})t} + B\,e^{(-b-\sqrt{b^2-c})t} & \text{if} & b^2 - c > 0 \\ 2. \quad y(t) = e^{-bt}(A + Bt) & \text{if} & b^2 - c = 0 \\ 3. \quad y(t) = A\,e^{-bt}\sin\sqrt{c-b^2}t + B\,e^{-bt}\cos\sqrt{c-b^2}t & \text{if} & b^2 - c < 0 \end{array}\right\} \tag{7.31}$$

where A and B are constants determined by $y(0)$ and $y'(0)$. All three can be shown by substitution to solve Equation 7.30.

A system with $b^2 - c > 0$ is 'over damped' and does not oscillate. For this condition, $b = \frac{r}{2m}$, $c = \frac{k}{m}$, and

$$b^2 - c = \left(\frac{r}{2m}\right)^2 - \frac{k}{m} = \frac{r^2 - 4km}{4m^2}.$$

The condition for overdamping, no oscillation in the system, is $r^2 > 4km$ – the square of the resistance is greater than 4 times the spring constant times the mass.

If the formulas above remind you of the roots to a quadratic polynomial, it is not an accident; the connection is shown in Exercise 7.4.6.

The body suspended on a spring is easy to experiment with and typifies many oscillations that occur throughout nature. Other mechanical systems that have similar oscillations include the swinging

pendulum and a rotating disc (as in the flywheel of a watch). Less apparent oscillating systems include diatomic molecules in which the distance between the two atoms oscillates very rapidly but can be approximated with the harmonic equations. In the next section we will give a simplified biological example of oscillations in predator-prey systems.

Exercises for Section 7.4, The Equation $y'' + \omega^2 y = 0$.

Exercise 7.4.1 Show that the proposed solutions satisfy the equations and initial conditions.

	Solution	Derivative Equation	Initial conditions
a.	$y(t) = 2\sin t + \cos t$	$y'' + y = 0$	$y(0) = 1$, $y'(0) = 2$
b.	$y(t) = 4\cos 2t$	$y'' + 4y = 0$	$y(0) = 4$, $y'(0) = 0$
c.	$y(t) = \cos 3t - \sin 3t$	$y'' + 9y = 0$	$y(0) = 1$, $y'(0) = -3$
d.	$y(t) = -20\sin 5t + 15\cos 5t$	$y'' + 25y = 0$	$y(0) = 15$, $y'(0) = -100$
e.	$y(t) = 4\cos(3t + \pi/3)$	$y'' + 9y = 2$	$y(0) = 2$, $y'(0) = -6\sqrt{3}$
f.	$y(t) = \sin 2t - 2\cos 2t$	$y'' + 4y = 0$	$y(0) = -2$, $y'(0) = 2$
g.	$y(t) = 2\sin 3t + 3\cos 3t$	$y'' + 9y = 0$	$y(0) = 3$, $y'(0) = 6$
h.	$y(t) = 3\sin \pi t + 4\cos \pi t$	$y'' + \pi^2 y = 0$	$y(0) = 4$, $y'(0) = 3\pi$
i.	$y(t) = e^{-t}\sin t$	$y'' + 2y' + 2y = 0$	$y(0) = 0$, $y'(0) = 1$
j.	$y(t) = e^{-0.1t}\cos 2t$	$y'' + 0.2y' + 4.01y = 0$	$y(0) = 1$, $y'(0) = -0.1$
k.	$y(t) = e^{-0.1t}(0.1\sin 2t + 2\cos 2t)$	$y'' + 0.2y' + 4.01y = 0$	$y(0) = 2$, $y'(0) = 0$

Exercise 7.4.2 Find a number ω so that the proposed solution satisfies the derivative equation.

	Solution	Derivative equation
a.	$y(t) = 3\cos 5t$	$y'' + \omega^2 y = 0$
b.	$y(t) = 2\sin 3t + 5\cos 3t$	$y'' + \omega^2 y = 0$
c.	$y(t) = -4\cos \pi t$	$y'' + \omega^2 y = 0$
d.	$y(t) = 3e^{-t}\cos 5t$	$y'' + 2y' + \omega^2 y = 0$
e.	$y(t) = -4e^{-2t}\sin 3t$	$y'' + 4y' + \omega^2 y = 0$

Exercise 7.4.3 Find a number k so that the proposed solution satisfies the derivative equation.

	Solution	Derivative equation
a.	$y(t) = e^{-2t}\cos 5t$	$y'' + ky' + 29y = 0$
b.	$y(t) = 5e^{-3t}\sin t$	$y'' + ky' + 10y = 0$
c.	$y(t) = 3e^{-2t}\cos 3t$	$y'' + ky' + 13y = 0$
d.	$y(t) = e^{-0.1t}\cos t$	$y'' + ky' + 1.01y = 0$
e.	$y(t) = \cos 5t$	$y'' + ky' + 25y = 0$

Exercise 7.4.4 Show that if B and ω are constants and $y(t) = B\cos(\omega t)$, then

$$y(0) = B \qquad y'(0) = 0 \qquad \text{and} \qquad y''(t) + \omega^2 y(t) = 0.$$

Exercise 7.4.5 Show that if A, B and ω are constants and $y(t) = A\sin(\omega t) + B\cos(\omega t)$, then

$$y(0) = B \qquad y'(0) = \omega A \qquad \text{and} \qquad y''(t) + \omega^2 y(t) = 0.$$

Exercise 7.4.6 Recall Equation 7.30

$$y''(t) + 2by'(t) + cy(t) = 0$$

and suppose that m is a number such that $y(t) = e^{mt}$ solves this equation. Compute $y'(t) = [\,e^{mt}\,]'$ and $y''(t)$. Substitute them into the equation, observe that e^{mt} is never zero, and conclude that

$$m^2 + 2bm + c = 0 \qquad \text{and} \qquad m = -b + \sqrt{b^2 - c} \qquad \text{or} \qquad m = -b - \sqrt{b^2 - c}$$

From this we conclude that

$$y(t) \qquad \text{is either} \qquad e^{(-b+\sqrt{b^2-c})t} \qquad \text{or} \qquad e^{(-b-\sqrt{b^2-c})t}$$

Table 7.1: Part of the data read from D. A. MacLulich, *ibid.*

Hare Lynx Data								
Year	Hare	Lynx	Year	Hare	Lynx	Year	Hare	Lynx
1909	23	9	1919	20	10	1929	1	15
1911	52	20	1921	62	20	1931	21	15
1913	69	40	1923	80	25			
1915	24	50	1925	23	40			
1917	11	25	1927	8	40			

Both are solutions to Equation 7.30. If $b^2 - c > 0$, these are the terms in solution 1 of Equations 7.31. The exact condition $b^2 - c$ seldom occurs in nature. The issue for $b^2 - c < 0$ is how to interpret

$$e^{(-b+\sqrt{b^2-c})\,t} = e^{(-b+i\sqrt{c-b^2})t} = e^{-bt}e^{i\sqrt{c-b^2}\,t}$$

where $i = \sqrt{-1}$. **Warning: Incoming Bolt from the Blue.** The answer is that[3]

$$e^{i\sqrt{c-b^2}t} = \cos\sqrt{c-b^2}\,t + i\sin\sqrt{c-b^2}\,t.$$

This suggests (to some people at least) that

$$y(t) = e^{-bt}\cos\sqrt{c-b^2}\,t + ie^{-bt}\sin\sqrt{c-b^2}\,t$$

is a solution to Equation 7.30 **BANG**. Because Equation 7.30 has real number coefficients, some people think that the real and imaginary parts of $y(t)$ should each solve Equation 7.30. They do, and with tenacity you can show that they do.

Exercise 7.4.7 At least, show that

$$y(t) = e^{-bt}\cos(\sqrt{c-b^2}\;\; t)$$

solves

$$y''(t) + 2by'(t) + cy(t) = 0 \qquad \text{for} \qquad c - b^2 > 0$$

7.5 Elementary predator-prey oscillation.

Predator-prey systems are commonly cited examples of periodic oscillation in biology. Data from trapping records of the snowshoe hare and lynx gathered by trappers and sold to the Hudson Bay Company are among the most popular first introduction. Shown in Figure 7.9 is a graph showing the numbers of pelts purchased by the Hudson Bay Company for the years 1845 to 1935, and in Table **??** are values read from the graph. [4]

[3] We will try to convince you that this is reasonable in Chapter 9.

[4] Recent studies also demonstrate the fluctuations as shown on the web site http://lynx.uio.no/catfolk/sp-accts.htm. "Lynx density fluctuates dramatically with the hare cycle (Breitenmoser et al. Oikos 66 (1993), pp. 551-554). An ongoing long-term study of an unexploited population in good quality habitat in the Yukon found densities of 2.8 individuals (including kittens) per 100 km2 during the hare low, and 37.2 per 100 km2 during the peak (G. Mowat and B. Slough, unpubl. data)."

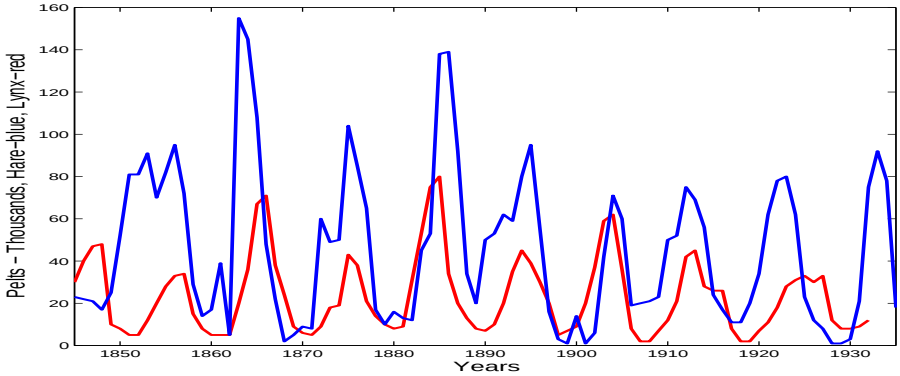
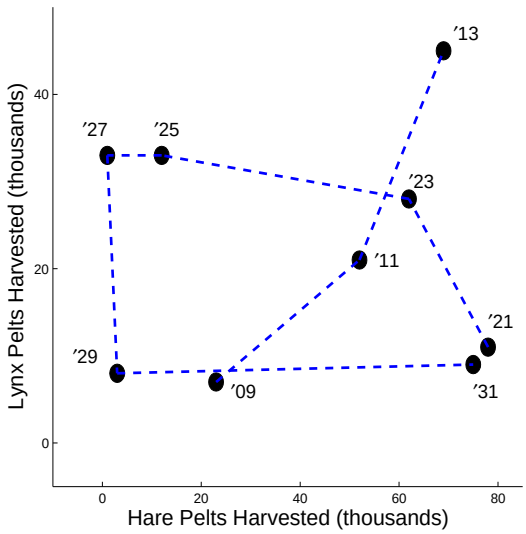


Figure 7.9: Graph of snowshoe shoe hare and lynx pelts purchased by the Hudson Bay company for the years 1845 to 1945. Data read from Figures 3 (hare) and 16 (lynx) of D. A. MacLulich, Fluctuations in the number of varying hare, University of Toronto Studies, Biological Sciences, No. 43, 1937.

Explore 7.5.1 Do this. Complete the phase graph shown in Figure 7.5.1 using data from Table 7.1. Plot the points for the years 1915, 1917, and 1919 and draw the missing lines. ■

Explore Figure 7.5.1 Phase graph axes for hare and lynx data.



The *phase graph* that you just drew is a good way to display the interaction between two populations. You should see a general counter clockwise direction to the graph. When you are in the right-most portion of the region with large hare population, the lynx population is increasing (the curve goes up). As you get to the upper right corner the lynx population has increased sufficiently that the hare population decreases (the curve goes to the left). And the pattern continues. We will return to the lynx-hare data in Exercise 7.5.6 and find that there are exceptions to this pattern in the data.

Nils Chr. Stenseth, Wilhelm Falck, Ottar N. Bjrnstad, and Charles J. Krebs, argue in Population regulation in snowshoe hare and Canadian lynx: Asymmetric food web configurations between hare and lynx, *Proc. Nat. Acad. Sci. USA* **95**, 5147-5152, that

" $\cdots$the classic view of a symmetric harelynx interaction is too simplistic. Specifically, we argue that the classic food chain structure is inappropriate: the hare is influenced by many predators other than the lynx, and the lynx is primarily influenced by the snowshoe hare."

A Predator-Prey Model. Assume that there are two populations that interact as predator and prey in a reasonably isolated environment. Let $U(t)$ denote the number of prey and $V(t)$ denote the number of predators, and assume there are equilibrium values, U_e and V_e, so that U_e prey would provide enough food for V_e predators to just maintain their numbers (predator birth rate = predator death rate) and V_e predators would just balance the often excess birth rate of the prey (prey birth rate = prey death rate).

Shown in Figure 7.10 is an axis system where the horizontal axis is $U(t)$ and the vertical axis is $V(t)$. An equilibrium point, (U_e, V_e), is plotted. If for some time, t, the populations are not at equilibrium, we let

$$\begin{aligned} u(t) &= U(t) - U_e \\ v(t) &= V(t) - V_e \end{aligned}$$

measure the departures from equilibrium.

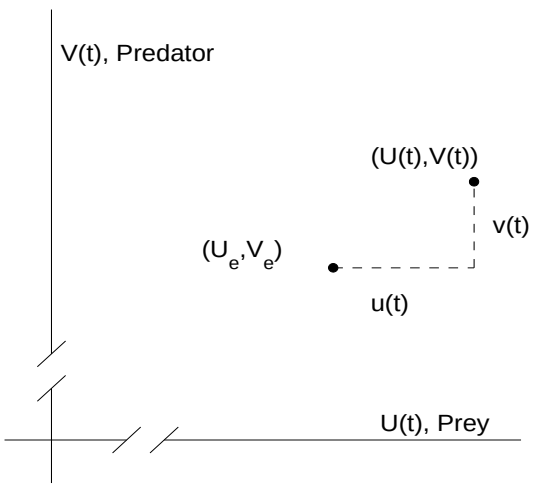


Figure 7.10: Axes for a predator prey phase graph. The gaps in the axes allow $u(t)$ to be small compared to U_e and $v(t)$ to be small compared to V_e.

Suppose the populations $U(t)$ and $V(t)$ are in equilibrium and the predator population increases (perhaps some predators immigrate into the system). The excess predators would increase capture of prey, and we could expect the prey population to *decrease*. Alternatively, if the prey should become more numerous, the predators would have a greater food supply and their numbers may *increase*.

Mathematical Model 7.5.1 For *small* deviations, $u(t)$ and $v(t)$, from equilibrium, we assume that

1. The rate of prey population *decrease*, $-U'(t)$, is proportional to the excess predator population, $v(t)$.

2. The rate of predator population *increase*, $V'(t)$, is proportional to the excess prey population, $u(t)$.

Thus from Part 1 we write

$$-U'(t) = a \times v(t) \qquad \text{or} \qquad U'(t) = -a \times v(t)$$

By the model, if the predator population exceeds equilibrium, $V(t) > V_e$ ($v(t) > 0$), then $U'(t) < 0$ and the prey population will decrease. However, if the predator population is less than normal, $V(t) < V_e$ ($v(t) < 0$), then $U'(t) > 0$ and the prey population increases. For this model, both populations must be assumed to be close to equilibrium. For example, a prey population greatly exceeding equilibrium, U_e might support a predator population slightly above equilibrium V_E and still grow.

Because $u(t) = U(t) - U_e$, $U(t) = U_e + u(t)$ and

$$U'(t) = u'(t).$$

We write

$$u'(t) = -a \times v(t)$$

Similarly,

$$v'(t) = b \times u(t)$$

Explore 7.5.2 Show that Part 2 of the Mathematical Model 7.5.1 leads to the equation

$$v'(t) = bu(t)$$

where b is a proportionality constant. ■

The two equations

$$\begin{aligned} u'(t) &= -av(t) \\ v'(t) &= bu(t) \end{aligned}$$

describe the dynamics of the predator – prey populations. There are two unknown functions, u and v, and the equations are linked, because u' is related to v and v' is related to u. There is a general procedure to obtain a single equation involving only u, as follows:

$$\begin{aligned} u'(t) &= -av(t) && \text{First Original Equation.} \\ [u'(t)]' &= [-av(t)]' && \text{Differentiate First Eq.} \\ u''(t) &= -av'(t) \\ u''(t) &= -a(bu(t)) && \text{Substitute Second Eq.} \\ u''(t) + (ab)u(t) &= 0 \end{aligned}$$

Now we let $\omega = \sqrt{ab}$ so that $\omega^2 = ab$ and write

$$u''(t) + \omega^2 u(t) = 0 \tag{7.32}$$

and see that it is equivalent to the dynamic equation in Equations 7.27. To complete the analogy, we need $u(0)$ and $u'(0)$.

Suppose we have a predator-prey pair of populations and because of some disturbance to the environment (rain, cold, or fire, for example) at a time, $t = 0$, the populations are at (U_0, V_0), close to but different from the equilibrium values (U_e, V_e). Let the departures from equilibrium be

$$u_0 = U_0 - U_e \qquad \text{and} \qquad v_0 = V_0 - V_e$$

Then clearly we will use $u(0) = u_0$. Also, from $u'(t) = -av(t)$ we will get $u'(0) = -av(0) = -av_0$. Thus we have the complete system

$$u(0) = u_0 \qquad u'(0) = -av_0 \qquad u''(t) + \omega^2 u(t) = 0 \tag{7.33}$$

From Equation 7.24 $y''(t) + \omega^2 y(t) = 0$, and its solution, Equation 7.25, $y(t) = A\sin(\omega t) + B\cos(\omega t)$, we conclude that $u(t)$ will be of the form

$$u(t) = A\sin(\omega t) + B\cos(\omega t)$$

where A and B are to be determined. Observe that

$$u'(t) = A\omega\cos(\omega t) - B\omega\sin(\omega t)$$

Now,

$$\begin{aligned} u(0) &= A\sin(\omega 0) + B\cos(\omega 0) \\ &= A \times 0 + B \times 1 = B \\ u'(0) &= A\omega\cos(\omega 0) - B\omega\sin(\omega 0) \\ &= A\omega \times 1 - B\omega \times 0 = A\omega \end{aligned}$$

It follows that

$$B = u_0 \qquad \text{and} \qquad A\omega = -av_0, \qquad \text{so that} \qquad A = -\frac{a}{\omega}v_0$$

and the solution is

$$u(t) = -\frac{a}{\omega}v_0\sin(\omega t) + u_0\cos(\omega t)$$

Remembering that $\omega = \sqrt{ab}$ we may write

$$\begin{aligned} u(t) &= -\frac{a}{\sqrt{ab}}v_0\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t) \\ &= -v_0\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t) \end{aligned} \tag{7.34}$$

Explore 7.5.3 Also remember the first equation, $u'(t) = -av(t)$ and show that

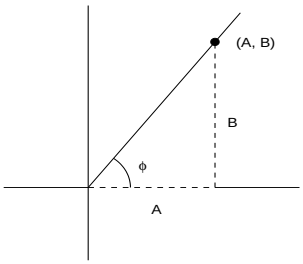
$$v(t) = v_0 \cos(\sqrt{ab}\, t) + u_0 \sqrt{\frac{b}{a}} \sin(\sqrt{ab}\, t) \tag{7.35}$$

■

A little trigonometry It often happens that a periodic oscillation is the sum of two oscillations of the same frequency as in Equations 7.34 and 7.35. When that happens, the two can be combined into a single sine function. Suppose $F(t) = A\sin(\omega t) + B\cos(\omega t)$ where A, B, and ω are numbers.

Let ϕ be an angle such that

$$\cos\phi = \frac{A}{\sqrt{A^2+B^2}}$$
$$\sin\phi = \frac{B}{\sqrt{A^2+B^2}}$$



Then

$$\begin{aligned} F(t) &= A\sin(\omega t) + B\cos(\omega t) \\ &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin(\omega t) + \frac{B}{\sqrt{A^2+B^2}}\cos(\omega t)\right) \\ &= \sqrt{A^2+B^2}\,(\cos\phi\sin(\omega t) + \sin\phi\cos(\omega t)) \\ &= \sqrt{A^2+B^2}\sin(\omega t + \phi) \end{aligned}$$

The last step uses $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Exercises for Section 7.5, Elementary predator-prey oscillation.

Exercise 7.5.1 Suppose that $a = b = 1$ and $u_0 = 3$ and $v_0 = 4$ in the prey equation 7.34 so that

$$u(t) = -4\sin(t) + 3\cos(t).$$

(u_0 and v_0 are 'small' disturbances. We might suppose, for example, that the equilibrium populations are $U_e = 300$, $V_e = 200$, with 3 and 4 'small' with respect to 300 and 200).

a. Sketch the graph of $u(t) = -4\sin(t) + 3\cos(t)$.

b. Let ϕ (the Greek letter denote the angle between 0 and 2π whose sine is $\frac{3}{5}$ and whose cosine is $\frac{-4}{5}$. Show that

$$\begin{aligned} u(t) &= 5\,(\cos\phi \sin t + \sin\phi \cos t) \\ &= 5\sin(t+\phi) \end{aligned}$$

c. Plot the graph of $5\sin(t+\phi)$ and compare it with the graph of $-4\sin t + 3\cos t$.

Exercise 7.5.2 Compare Equation 7.34, $u(t) = -v_0\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t)$ for the two cases:
Case 1: $a = b = 1$ and $u_0 = 3$ and $v_0 = 4$ (in the previous problem).
Case 2: $a = 4$, $b = 1$ and $u_0 = 3$ and $v_0 = 4$
What is the biological interpretation of the change from $a = 1$ to $a = 4$?

Exercise 7.5.3 a. Find the formula for the *predator* population Equation 7.35 using the parameters, $a = b = 1$, $u_0 = 3$ and $v_0 = 4$.

b. Let ψ (Greek letter *psi*) be the angle between 0 and 2π for which

$$\cos\psi = \frac{3}{5} \qquad \text{and} \qquad \sin\psi = \frac{4}{5}$$

and show that

$$v(t) = 5\sin(t+\psi)$$

Exercise 7.5.4 Develop the *predator* harmonic equation.

a. Examine the steps leading to Equations 7.33 and show that

$$v(0) = v_0 \qquad v'(0) = bu_0 \qquad v''(t) + \omega^2 v(t) = 0 \qquad \text{with} \quad \omega^2 = ab \tag{7.36}$$

b. Rewrite this system for $a = b = 1$, $u_0 = 3$ and $v_0 = 4$ and conclude that the solution is

$$v(t) = 3\sin t + 4\cos t$$

c. Find a formula for $v(t)$ (predator) using the formula from Exercise 7.5.1 $u(t) = -4\sin t + 3\cos t$ (prey) and the equation $u'(t) = -av(t)$ $(a = 1)$.

Exercise 7.5.5 The previous three exercises show that for $a = b = 1$, $u_0 = 3$ and $v_0 = 4$

$$\begin{aligned} u(t) &= -4\sin(t) + 3\cos(t) &&= 5\sin(t + 2.498) \\ v(t) &= 3\sin t + 4\cos t &&= 5\sin(t + 0.927) \end{aligned}$$

Graphs of u and v are displayed in two ways in Figure 7.11. In Figure 7.11A are the conventional graphs $u(t)$ *vs* t and $v(t)$ *vs* t. In Figure 7.11B is a $V(t) = V_e + v(t)$ *vs* $U(t) = U_e + u(t)$ (U_e and V_e not specified).
At time $t = 0$ the excess predator and prey populations are both positive ($u_0 = 3$ and $v_0 = 4$).

a. Replicate Figure 7.11 on your paper.

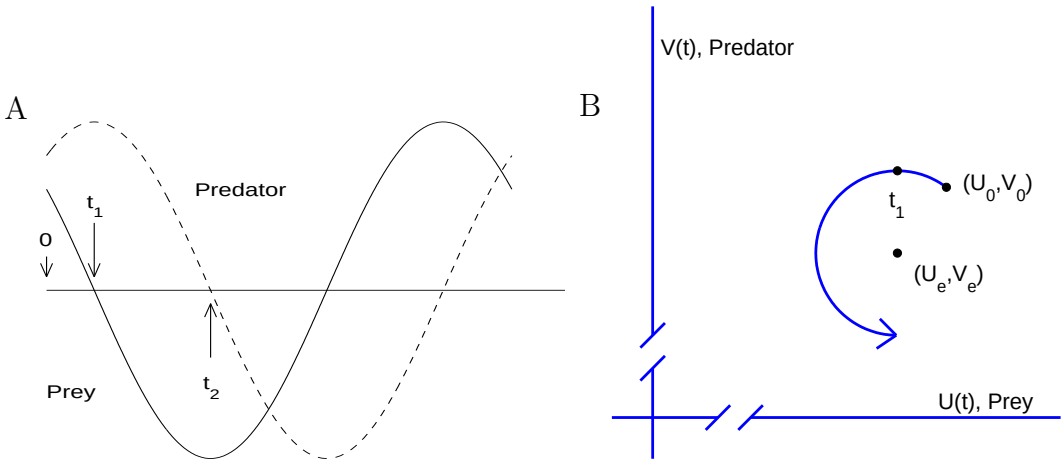


Figure 7.11: A. Conventional and B. phase graphs of predator prey functions. For scale, the distance from (U_e, V_e) to (U_0, V_0) is 5.

b. What are $u'(0)$ and $v'(0)$?

c. On your replica of Figure 7.11A, draw tangents to the graphs of u and v at $u(0)$ and $v(0)$.

d. On your replica of Figure 7.11B draw a line from (U_0, V_0) $(U_0 + u'(0), V_0 + v'(0))$. Note that the distance from (U_e, V_e) to (U_0, V_0) is 5.

e. In Figure 7.11A, at the time $t = 0$ the prey curve has negative slope and the predator curve has positive slope. The graph in Figure 7.11B moves from (U_0, V_0) to the left (prey is decreasing) and upward (predator is increasing).

1. Discuss the dynamics of the predator and prey populations at time t_1.

2. Discuss the dynamics of the predator and prey populations at time t_2 marked on Figure 7.11A and mark the corresponding point on Figure 7.11B.

f. Show that

$$(\,u(t)\,)^2 + (\,v(t)\,)^2 = u_0^2 + v_0^2$$

What is the significance of this equation?

Exercise 7.5.6 J. D. Murray *Mathematical Biology*, Springer, New York, 1993, p 66 observes an exception to the phase graph for the Lynx-Hare during years 1874-1904. The data are shown in Figure 7.12 along with a table of part of the data.

a. Read data points in Figure **??** for the years 1887 and 1888.

b. Use the data in Table **??** and your two data points to make a phase plot for the years 1875 - 1888.

c. Discuss the peculiarity of this predator-prey phase plot.

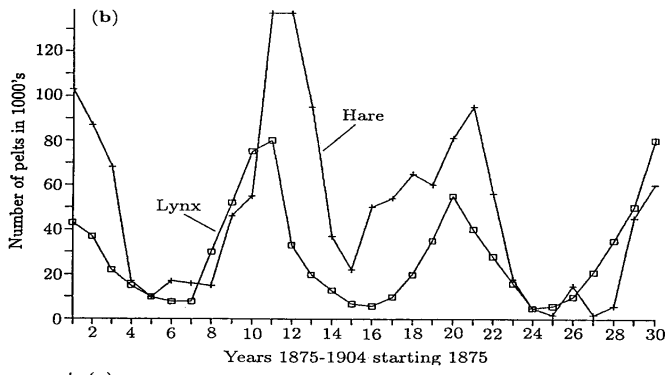


Figure 7.12: Snowshoe Hare and Lynx data for the years 1874-1904.

Table 7.2: Lynx-Hare data read Figures 3 (hare) and 16 (lynx) of D. A. MacLulich, Fluctuations in the number of varying hare, University of Toronto Studies, Biological Sciences, No. 43, 1937.

Hare Lynx Data Read From the Graph								
Year	Hare	Lynx	Year	Hare	Lynx	Year	Hare	Lynx
1875	104	43	1879	10	10	1883	45	53
1876	85	38	1880	16	8	1884	53	75
1877	65	21	1881	13	9	1885	138	80
1878	17	14	1882	12	31	1886	139	34

Murray notes that the 1875 - 1887 data seems to show that the 'hares are eating the lynx', and cites some explanations that have been offered, including a possible hare disease that could kill the lynx (no such disease is known) and variation in trapping practice in years of low population density.

Exercise 7.5.7 For the algebraically robust. For Equations 7.34 and 7.35,

$$u(t) = -v_0\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t) \qquad v(t) = v_0\cos(\sqrt{ab}\,t) + u_0\sqrt{\frac{b}{a}}\sin(\sqrt{ab}\,t)$$

show that

$$\sqrt{\frac{b}{a}}\,(\,u(t)\,)^2 + \sqrt{\frac{a}{b}}\,(\,v(t)\,)^2 = \sqrt{\frac{b}{a}}\,(\,u_0\,)^2 + \sqrt{\frac{a}{b}}\,(\,v_0\,)^2\,.$$

What is the significance of this equation?

Exercise 7.5.8 Suppose the prey population may be affected by over crowding or under crowding even without predators present.

Mathematical Model 7.5.2 For *small* deviations, $u(t)$ and $v(t)$, from equilibrium, we assume that

1. The rate of prey population *decreases*, $-U'(t)$, when there is excess predator population $v(t)$ and when there is excess prey population, $u(t)$.

2. The rate of predator population *increase*, $V'(t)$, is proportional to the excess prey population, $u(t)$.

Step 1 can be interpreted at least two ways:
A. $-U'(t)$ could be proportional to the *product* of $u(t)$ and $v(t)$, or
B. $-U'(t)$ could be proportional to the *sum* of $av(t) + cu(t)$ where a and c are numbers.
Both interpretations are relevant. Here we choose interpretation B. Step 2 is the same as for Model7.37 and write

$$\begin{aligned} -U'(t) &= av(t) + cu(t) \\ V'(t) &= bu(t) \end{aligned}$$

Again $U'(t) = u'(t)$ and $V'(t) = v'(t)$, so that

$$\begin{aligned} u'(t) &= -av(t) - cu(t) \\ v'(t) &= bu(t) \end{aligned} \tag{7.37}$$

a. Use Equations 7.37 to show that

$$v''(t) + cv'(t) + abv(t) = 0 \tag{7.38}$$

Hint: Compute

$$[\,v'(t)\,]' = [\,bu(t)\,]', \qquad \text{use} \qquad u'(t) = -av(t) - cu(t) \qquad \text{and} \qquad bu(t) = v'(t)$$

b. Equation 7.38 may be compared with Equation 7.28 for damped harmonic motion (harmonic motion with resistance). For $a = 2.02$, $b = 0.5$, and $c = 0.2$ the equation becomes

$$v''(t) + 0.2v'(t) + 1.01v(t) = 0 \tag{7.39}$$

Show that

$$v(t) = e^{-t/10} \sin t$$

is a solution to this equation.

c. Show that if $v(t) = e^{-t/10} \sin t$ then

$$u(t) = 2e^{-t/10} \cos t - 0.2e^{-t/10} \sin t$$

d. Show that

$$u^2 + 0.4uv + 4.04v^2 = 4e^{-t/5}$$

e. Plot a graph of $u(t)$ *vs* $v(t)$. It is of interest that $u^2 + 0.4uv + 4.04v^2 = 4$ is an ellipse in the u-v plane.

7.6 Periodic systems.

Many biological and physical systems exhibit periodic variation governed by feed back of information from the state of the system to the driving forces of the system. An excess of predators (state of the system) drives down (driving force) the prey population. An elongation of a spring (state of the system) causes (driving force) the suspended mass to move up toward the equilibrium position.

Examples of periodically varying feed back systems are presented. Exercises are distributed through the three subsections.

Explore 7.6.1 Chamelons are a group of lizards that change their color to match the color of their environment. What is the color of a chamelon placed on a mirror? ■

7.6.1 Control switches.

Some street lights and household night lights have photosensitive switches that turn the lights on at sunset and turn them off at sunrise. In Figure 7.6.1 is a household night light with a photosensitive switch and a mirror. The mirror can be adjusted so that the light from the bulb is reflected back to the photosensitive switch. What will be the behavior of the switch at night? With sunlight shining on it?

Explore 7.6.2 You will find it interesting to perform an experiment. In a dark room, hold a mirror about 3 inches from a night light so that it reflects light from a night light back to the photosensitive switch. Move the mirror about 8 inches from the night light and note the change in the activity of the night light. ■

We propose the following mathematical model for the system.

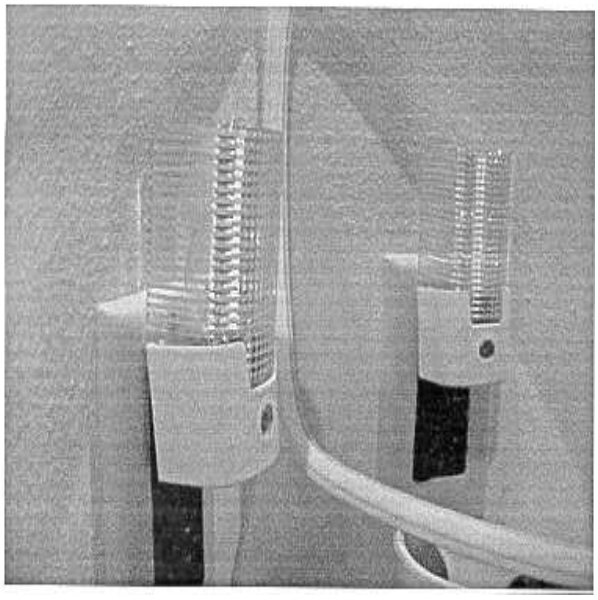
Mathematical Model 7.6.1 Night lights. I There is a voltage in the photosensitive switch that increases at a rate proportional to the intensity of light striking the switch and the intensity of the light leaving the bulb decreases at a rate proportional to the voltage.

The intensity of the reflected light that strikes the switch is proportional to the intensity of the light leaving the bulb and inversely proportional to the square of the distance from the bulb to the mirror.

Exercise 7.6.1 a. The first paragraph should remind you of a predator prey system. Assuming so, is the voltage the predator or the prey?

b. Write equations for the mathematical Model 7.6.1.

Figure for Exercise 7.6.1 A night light with a mirror that can be positioned to reflect light back to the photosensitive switch.



We propose a second mathematical model for night lights.

Mathematical Model 7.6.2 Night lights. II There is a voltage in the photosensitive switch that increases at a rate proportional to the intensity of light striking the switch and dissipates at a constant rate when no light strikes the switch. The light is either on or off; it turns on when the voltage falls below a certain threshold and turns off when the voltage exceeds another threshold.

The intensity of the reflected light that strikes the switch is proportional to the intensity of the light leaving the bulb and inversely proportional to the square of the distance from the bulb to the mirror.

Exercise 7.6.2 Let $v(t)$ be the voltage in the photosensitive switch at time t and $i(t)$ be the illumination striking the photosensitive switch at time t.

a. Write an equation descriptive of

> There is a voltage in the photosensitive switch that increases at a rate proportional to the intensity of light striking the switch and dissipates at a constant rate when no light strikes the switch.

b. Because the light is either on or off, it is easiest to treat the mathematical model 7.6.2 Night Light II as a discrete system. Choose an increment time $\delta > 0$ and for $n = 0, 1, 2, \cdots N$, let

$$v_k = v(k \times \delta) \qquad \text{and} \qquad i_k = i(k \times \delta),$$

and assume that

$$v'(k \times \delta) \doteq \frac{v_{k+1} - v_k}{\delta}.$$

Write a discrete analog of your previous equation

c. Let $v_{on} < v_{off}$ be threshold values and $sw(v)$ be a 'switch' function defined by

$$sw(v(t)) = \begin{array}{lll} 1 & \text{for} & v(t) \leq v_{on} \\ (1 + sign(v'(t)))/2 & \text{for} & v_{on} < v(t) < v_{off} \\ 0 & \text{for} & v_{off} \leq v(t) \end{array}$$

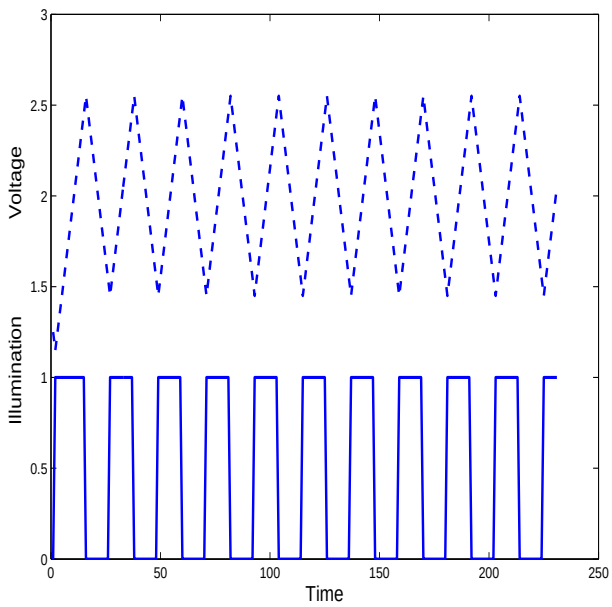
Figure 7.6.2 illustrates the solutions to the equation

$$\begin{aligned} v_0 &= 1.25 \\ i_0 &= 0 \\ v_{k+1}+ &= v + k + (2 \times i_k - 1) \times \delta \\ \\ i_{k+1} &= sw(v_k), \end{aligned} \tag{7.40}$$

where $v_{on} = 1.5$ and $v_{off} = 2.5$, and $N = 230$.

If you have adequate computing, replicate Figure 7.6.2. Else, compute (v_1, i_1), (v_2, i_2), (v_3, i_3), (v_4, i_4), and (v_5, i_5).

Figure for Exercise 7.6.2 Solutions to Equations 7.40. Light intensity is the solid curve and voltage is the dashed curve.



Exercise 7.6.3 Thermostats control the furnaces on houses. They turn the furnace on when the temperature falls below a temperature set by the home owner and turn the furnace off when the temperature exceeds a temperature set by the home owner.

The control system of a thermostat is shown in Figure 7.6.3. There is a bi-metalic coil with a closed glass tube that tilts according to the temperature of the coil. Two wires are at one end of the tube and a small amount of mercury is inside the tube. When the temperature is "low" the tube tilts so that the mercury completes the connection between the two wires. When the temperature is "high" the coil tilts so that the mercury is at the opposite end of the tube from the wires and the connection is broken. As the temperature moves between the "low" and "high" temperatures the mercury slowly moves towards the center of the tube until a threshold angle is reached and it flows to the opposite end of the tube.

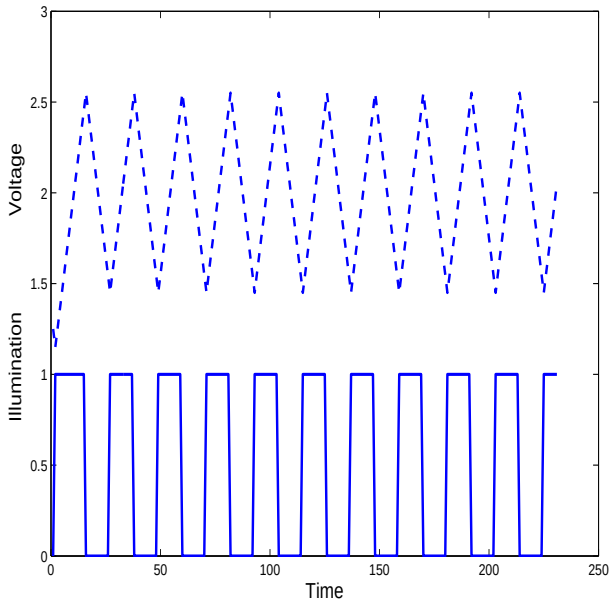
a. Write a mathematical model descriptive of temperature inside a house when the furnace is not running and the outside temperature is below the temperature inside the house. You may wish to review Exercises 1.5 and 5.5.19 and 7.6.2.

b. Write equations that describe your mathematical model.

c. Draw a graph descriptive of the temperature inside a house in northern Minnesota for one day in January.

d. Draw a graph descriptive of the temperature inside a house in Virginia for one day in January.

Figure for Exercise 7.6.3 The bi-metalic coil inside a thermostat with glass tube on top.



7.6.2 Earthquakes.

The San Andreas fault in California is an 800 mile zone of contact between two tectonic plates, with the continental crust on the east and the oceanic crust on the west. As the oceanic crust moves north and rubs against the continental crust, at some points along the fault faces of the crust lock together and the earth bends — until a threshold distortion is surpassed. Then the faces of the crusts abruptly slide past one another sending shock waves out across the earth, and the crusts returns to a more relaxed condition. The maximum slippage recorded between two crusts is a 21 foot displacement of a road during the 1906 earthquake in the San Francisco region.

A simple model of this system was described by Steven Gao of Kansas State University. Consider a body of mass m on a horizontal platform, a spring with one end attached to the body and the other end moving along the platform at a rate v. There are two frictions associated with the body, the starting friction, F_{start}, and the sliding friction, F_{slide},

$$F_{slide} < F_{start}.$$

If the body is not moving relative to the platform, F_{start} is the force required to initiate movement. If the body is in motion along the platform, F_{slide} is the force required to continue motion.

The spring has a spring constant k; an elongation of length E in the spring causes a force of magnitude $k \times E$ on the body. Let L be the length of the spring when there is no tension on the spring.

Figure 7.13: San Andreas Fault. Photograph from Wallace, Robert E., ed., 1990, The San Andreas fault system, California: U.S. Geological Survey Professional Paper 1515, 283 p. [http://pubs.usgs.gov/pp/1988/1434/].

In this model, the horizontal platform is the continental crust and the body and spring are the oceanic crust.

Mark a point on the platform as the zero point, let $x(t)$ be the distance from zero to the forward face of the body, and let $y(t)$ be the distance from zero to the forward end of the spring. Let $E(t)$ be the extension of the spring.

Assume the initial conditions:

$$x(0) = 0, \qquad y(0) = L, \qquad \text{so that} \qquad E(0) = 0$$

Exercise 7.6.4 1. What is $y'(t)$ for all t?

2. Write a formula for $y(t)$.

3. The force of the spring on the body will be $k \times E(t) = k \times (y(t) - L - x(t))$. What is the force at time $t = 0$?

4. At what time, t_1, will the body first move (will the force on the body $= F_{start}$)?

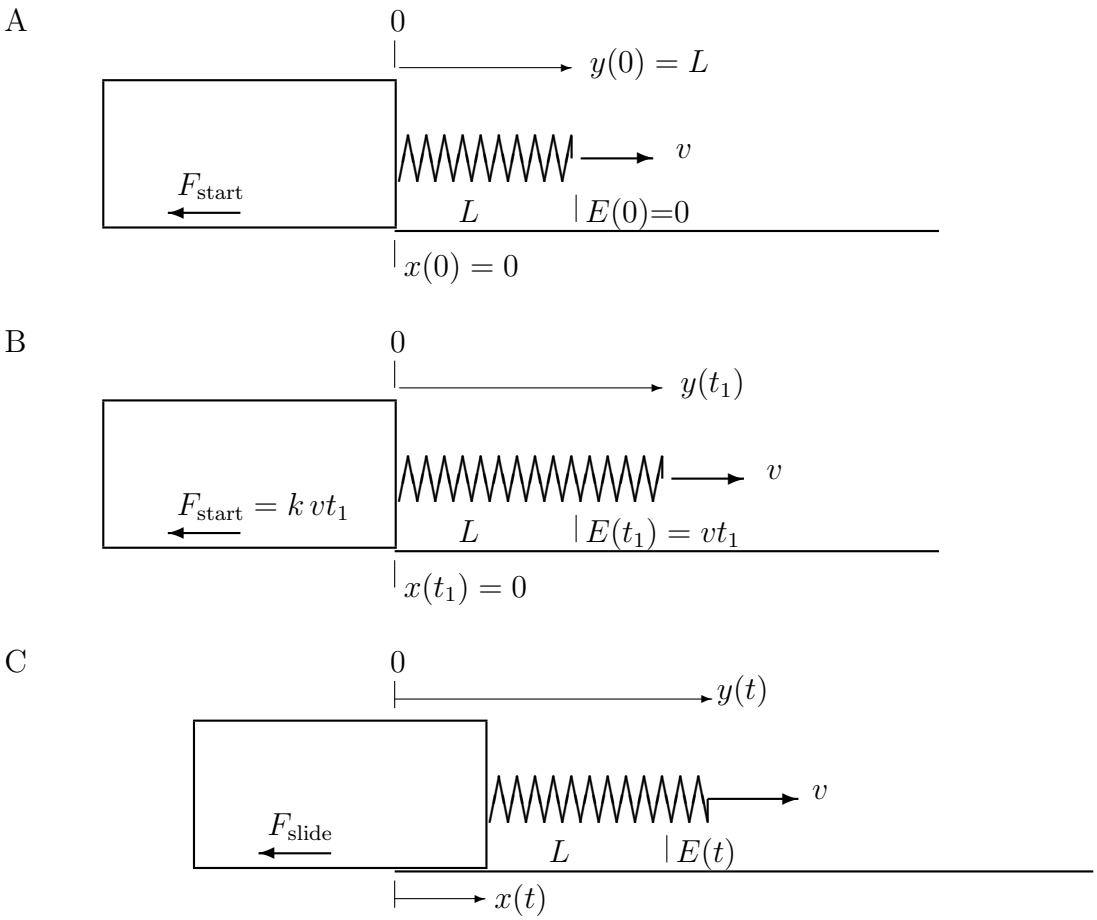


Figure 7.14: A. Initial condition of a block with spring pull and starting friction. B. The block will first move at time t_1 when the force of the spring elongation, $k\,vt_1$, exactly matches the starting friction, F_{Start}. C. Motion of a block with spring pull and sliding friction.

During the first motion of the body, the net force, F, on the body will be

$$\begin{aligned} F &= k \times (y(t) - L - x(t)) - F_{slide} \\ &= k \times (L + vt - L - x(t)) - F_{slide} \\ &= k \times (vt - x(t)) - F_{slide} \end{aligned}$$

Newton's second law of motion gives

$$F = ma = mx'',$$

so that

$$mx'' = k \times (vt - x(t)) - F_{slide}$$

or

$$x'' + \omega^2 x = \frac{k}{m} vt - \frac{F_{slide}}{m} \qquad \omega^2 = \frac{k}{m} \tag{7.41}$$

Let t_1 be the time at which the first motion starts. Then

$$x(t_1) = 0 \tag{7.42}$$

$$x'(t_1) \quad = \quad 0 \tag{7.43}$$

Reader Beware: Incoming Lightning Bolt! In Chaper 17, Second order and systems of two first order differential equations, you will learn how to find the function, $x(t)$, that satisfies equations 7.41, 7.42, and 7.43:

$$\begin{aligned} x(t) \quad = \quad & \frac{-F_{start} + F_{slide}}{k} \cos\left(\omega\left(t - t_1\right)\right) \\ & -\frac{v}{\omega} \sin\left(\omega(t - t_1)\right) + vt - \frac{F_{slide}}{k} \end{aligned} \tag{7.44}$$

Exercise 7.6.5 Show that the function, $x(t)$ defined in Equation 7.44 satisfies equation 7.42, 7.43, and 7.41.

Exercise 7.6.6 Equation 7.44 is valid until $x'(t)$ next equals to zero. Use the parameters

$$F_{start} = 5, \qquad F_{slide} = 4, \qquad k = 1, \qquad v = 0.1, \qquad \text{and} \qquad m = 1$$

and draw the graph of x. An elongation of $E = F_{start}/k = 5$ initiates motion at time $t_1 = F_{\text{start}}/kv = 50$, and $\omega^2 = k/m = 1$ with these parameters. Find the time and value of x at which $x' = 0$.

Using the parameters of the previous exercise, the body moved 2.33 units in a time of 3.33 units; the velocity of the forward end of the spring is 0.1 so the forward end moved 0.33 units during the motion. At the time the body stopped moving, the elongation of the spring was 5 - 2.33 + 0.33 = 3. After 20 more time units, the elongation will again reach 5 and the motion of the body will be repeated. A graph of the motion of the body is shown in Figure 7.15

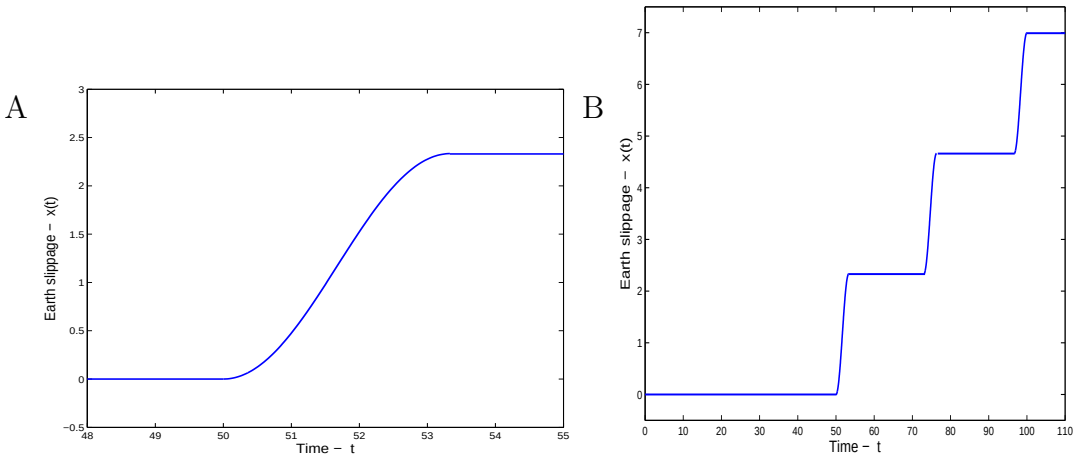


Figure 7.15: A. Graph of the first slippage, $x(t)$, for Equation 7.44 and the parameter values of Exercise 7.6.6. B. Graph of three slippage events with quiescence between events.

The model we describe exhibits periodic 'relief' of the tension in the spring. Do earth quakes exhibit periodicity in their recurrence? If so it would greatly simplify the prediction of earth quakes! The dates of significant earth quakes and their magnitudes (4.8 - 6.7) that have occurred in the Los Angeles area fault since 1920 are shown in Table 7.3. You may search the table for periodicity. As emphasized by Dr. Gao, the fault is a complex system; many small earthquakes occur every month; relief in on section may increase strain at other points. His model does, however, suggest the nature of the mechanics of an earth quake. The U.S. Geological Survey web site, http://pubs.usgs.gov/gip/earthq3/, contains very interesting discussion of earthquakes.

Table 7.3: Significant earthquakes in the Los Angeles, CA area 1920 - 1994. Mgtd = magnitude. http://pasadena.wr.usgs.gov/info/index.html

Year	Mgtd.	Year	Mgtd.	Year	Mgtd.	Year	Mgtd.
1920	4.9	1941	4.8	1973	5.2	1989	5.0
1930	5.2	1952	5.2	1979	5.2	1990	5.3
1933	6.4	1969	5.1	1981	6.0	1991	5.8
1938	5.2	1970	5.2	1987	5.9	1994	6.7
1941	4.8	1971	6.7	1988	5.0		

7.6.3 The circadian clock.

Exciting current studies in molecular biology and genetics are illuminating the molecular and neural systems that control the daily rhythm of our lives, wakes us up in the morning, puts us to sleep at night, and causes 'jet lag' when we travel to different time zones or even without travel when daylight savings time is initiated in the Spring. The molecular clocks that control circadian rhythms have been identified in fruit flies, mammals, *Neurospora* (fungus), *Arabidopsis* (plant), and cyanobacteria. They are markedly similar and It appears that there has been multiple evolution of the same basic mechanism. The *Neurospora* mechanism is described here from a review, J. C. Dunlap, *Ann. Rev. Genet.* **30** (1996), 579.

In *Neurospora*, there is a gene, *frq* (frequency), that is transcribed into a mRNA also called *frq* that codes for a protein, denoted FRQ. FRQ stimulates metabolic activities associated with daylight. A high level of FRQ also acts to shut off transcription of the gene *frq* so that the concentration of mRNA *frq* decreases. There is a potentially oscillating system: *frq* increases and causes an increase in FRQ that causes a decrease in *frq* so that FRQ decreases (is no longer transcribed from *frq* and is naturally degraded as are most proteins). However, there is resistance in the system and periodic stimulus from daylight is necessary to keep the system active and to entrain it to the daily 24 hour rhythm.

Circadian time (CT) begins with 0 at dawn, 6 is noon, 12 is dusk, 18 is midnight, and 24 = 0 is dawn. The *Neurospora* circadian cycle begins at midnight, CT 18, at which time both *frq* and FRQ are at low levels, but transcription of *frq* begins, say at a fixed rate. After a 3 hour time lag translation of *frq* to create FRQ begins. At dawn, CT 0, there is a marked increase in transcription of *frq* and an almost immediate increase of translation to FRQ. A high level of FRQ inhibits the transcription of *frq* and *frq* levels peak during CT 2 - 6 and declines steadily until CT 18. FRQ levels peak during CT 6 - 10 and also decline until CT 18 (the protein FRQ is constantly degraded and with decreasing levels of *frq* the turnover is more rapid than production).

Exercise 7.6.7 a. Draw a graph representative of the concentration of the mRNA *frq* as a function of time (use CT).

b. Draw a graph representative of the concentration of the protein FRQ as a function of time (use CT).

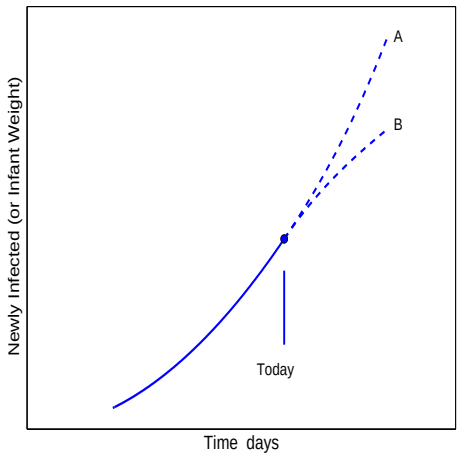
c. Draw a phase diagram with concentration of *frq* on the horizontal axis and concentration of FRQ on the vertical axis covering one 24 hour period.

Chapter 8

Applications of Derivatives.

Where are we going?

The derivative is a powerful tool for analysis of curves, selection of optimal parameter values, and measuring the influence of one variable on another. If you are a doctor in the midst of a flu epidemic, you know that the number of newly infected people is increasing every day. What is important to you today is whether the rate at which newly infected people appear is increasing suggesting a future similar to A or is slowing down as in B. B is better.



If the graphs represent the daily weight of an infant, however, B. could be a signal of inadequate nutrition or disease. A is better

8.1 Some geometry of the derivative

It is often important to know whether a population, atmospheric CO_2 concentration, average temperature, white blood count, number of flu cases, etc. is increasing or decreasing with time or location, and whether the rate of increase is itself increasing or decreasing.

The relation between positive, negative, and zero slopes and increasing, nondecreasing, nonincreasing, decreasing, highest points and lowest points is examined. The graph in Figure 8.1

illustrates three of these concepts. The graph has positive slope at every point, and is increasing. The slopes increase between points A and B, and the tangents lie below the graph. The slopes decrease between points C and D, and the tangents lie above the graph.

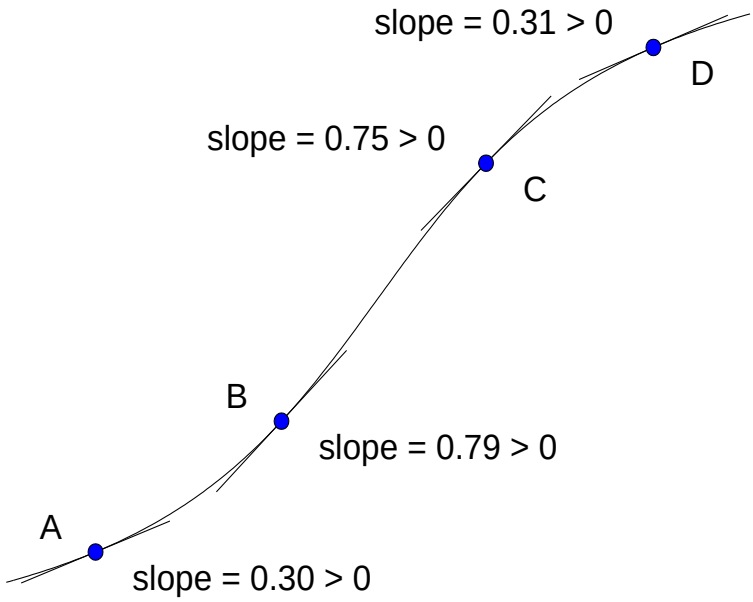


Figure 8.1: Graph of an increasing function. The slope is positive at every point. Between points A and B the slopes increase from 0.3 to 0.79 and the tangents lie below the curve. Between point C and D the slopes decrease from 0.75 to 0.31 and the tangents lie above the curve.

In Definition 3.1.2, the derivative $P'(a)$ of a function $P(t)$ at a number a is the slope of the tangent to the the graph of P at the point $(a, P(a))$.

The next definition of *increasing function* should agree with your intuitive notions that a population is growing, or that a chemical is accumulating, or a that temperature is increasing.

Definition 8.1.1 *A function P, is said to be* **increasing** *if for any two numbers, a and b in the domain of P,*

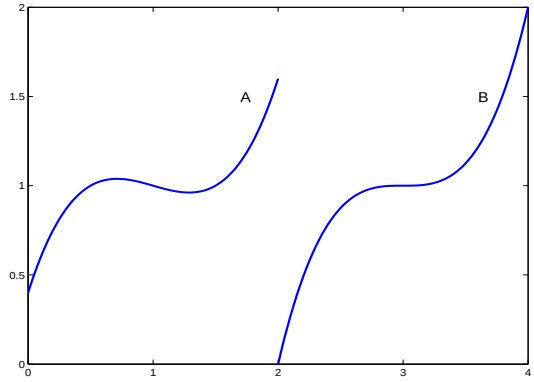
$$\textit{if} \qquad a < b, \qquad \textit{then} \qquad P(a) < P(b).$$

Explore 8.1.1 Do this.

1. Only one of the graphs shown in Figure 8.1.1 is the graph of an increasing function. Which one?
2. For the function that is not increasing, explain why it fails to satisfy Definition 8.1.1.
3. Does every tangent to the graph of the increasing functions have positive slope?

■

Explore Figure 8.1.1 Graphs of an increasing function and a function that is not increasing.

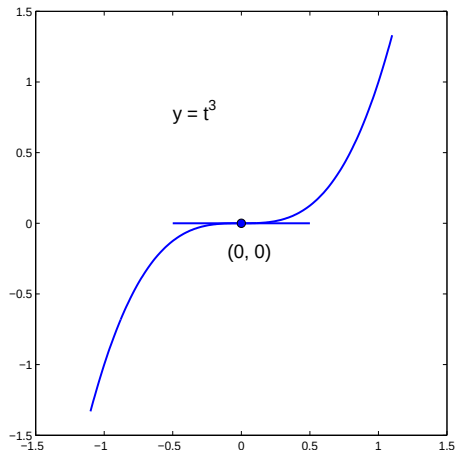


It may seem intuitive that if a population is growing then the growth rate is always positive. This is not true – as you may have found out from the preceding Explore 8.1.1 and is shown in the next example.

Example 8.1.1 A graph of the cubic function, $P(t) = t^3$ is shown in Figure 8.1.1.1 (and in Explore Figure 8.1.1), together with the tangent to the graph at the point (0,0). $P(t) = t^3$ is an increasing function: If $a < b$ then $a^3 < b^3$.

However, the tangent to the graph of P at (0,0) is horizontal. $P'(0) = 3 \times t^2|_{t=0} = 3 \times 0^2 = 0$. So that P' is not everywhere positive.

Figure for Example 8.1.1.1 The graph of $y = t^3$ and its tangent at (0,0).



The previous example is a bit of a nuisance. We would like increasing function and positive slope to be equivalent, and they are not. The fall back position is to define *non-decreasing* functions which will be found to be equivalent to *non-negative* slopes.

Definition 8.1.2 *A function P, is said to be* **non-decreasing** *if for any two numbers a and b in the domain of P,*

$$\textit{if} \qquad a < b, \qquad \textit{then} \qquad P(a) \leq P(b).$$

Note: Compare

$$\text{If} \qquad a < b \qquad \text{then} \qquad P(a) < P(b). \qquad \textbf{Increasing}$$

and

$$\text{If} \qquad a < b \qquad \text{then} \qquad P(a) \leq P(b). \qquad \textbf{Non-decreasing}$$

Every increasing function is also non-decreasing, but, for example, a constant function, $P(t) = C$, is non-decreasing but is not an increasing function.

Explore 8.1.2 Suppose P is a non-decreasing function.

a. Argue that every difference quotient, $\frac{P(b)-P(a)}{b-a}$, is greater than or equal to zero. (Consider two cases: $a < b$ and $b < a$.)

b. Suppose a is a number for which $P'(a)$ exists. Argue that $P'(a)$ is not a negative number.■

Thus if P is a non-decreasing function, then $P'(t) \geq 0$ for all t (P' is nonnegative).
This result is often used in reverse:

If $P'(t) \geq 0$ for all t then P is non-decreasing.

That is a true statement and is Theorem 9.2.1 of Section 9.2. We use it in this chapter, but we have not yet shown it to be true.

The two results may be written as a theorem to be proved in Section 9.2.

Theorem 8.1.1 Suppose P is a function defined on an interval $[a, b]$. Then P is nondecreasing on $[a, b]$ if and only if $P'(t) \geq 0$ for all t in $[a, b]$.

Explore 8.1.3 Write definitions of *decreasing* function and of *non-increasing* function and state a theorem about nonincreasing functions analogous to Theorem 8.1.1. ■

We also state a related theorem to be proved in Section 9.2:

Theorem 8.1.2 Suppose P is a continuous function defined on an interval $[a, b]$. If $P'(t) > 0$ for $a < t < b$ then P is increasing on $[a, b]$.

That the converse of this theorem is not a theorem follows from the example of $P(t) = t^3$, $[a, b] = [-1, 1]$.

Example 8.1.2 The function, ln, is an increasing function.

By the previous theorem, ln is increasing if $\ln'$ is positive. $\ln x$ is only defined for $x > 0$, and

$$[\ln x]' \quad = \quad \frac{1}{x} \quad > \quad 0 \qquad \text{for} \qquad x > 0$$

■

Explore 8.1.4 Is it true that ln is a nondecreasing function?

Example 8.1.3 Following ingestion of a penicillin pill, the concentration, C, of penicillin in the blood is approximated by the function

$$C(t) = 5e^{-0.2t} - 5e^{-0.3t} \quad \frac{\mu g}{ml}$$

where t is time measured in hours (Figure 8.2).

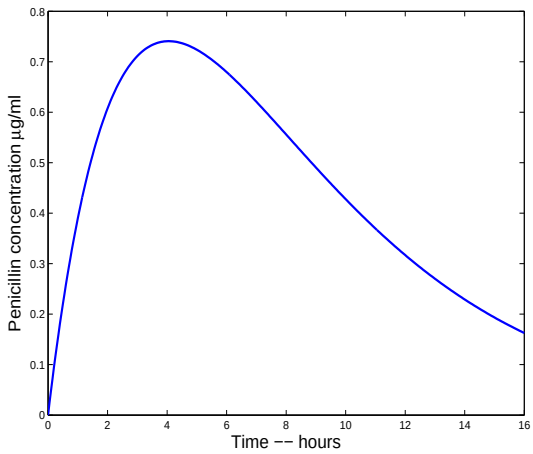


Figure 8.2: The graph of $C(t) = 5e^{-0.2t} - 5e^{-0.3t}$ showing a period of increasing blood penicillin concentration followed by a period of decreasing blood penicillin concentration.

Problem: During what time interval is the concentration of penicillin in the blood increasing? What is the maximum concentration of penicillin?
Solution: Compute

$$\begin{aligned} C'(t) &= \left[5e^{-0.2t} - 5e^{-0.3t}\right]' \\ &= 5e^{-0.2t} \times (-0.2) - 5e^{-0.3t} \times (-0.3) \\ &= 5 \times e^{-0.3t} \times \left(-0.2e^{0.1t} + 0.3\right) \end{aligned}$$

Because $5 > 0$ and $e^{-0.3t} > 0$ for all t, $C'(t)$ is positive if $-0.2e^{0.1t} + 0.3$ is positive. The following inequalities are equivalent:

$$\begin{aligned} -0.2e^{0.1t} + 0.3 &> 0 \\ 1.5 &> e^{0.1t} \\ \ln 1.5 &> \ln(e^{0.1t}) \qquad (\ln t \quad \text{is an increasing function.}) \\ \ln 1.5 &> 0.1t \\ 10 \ln 1.5 &> t \end{aligned}$$

We can conclude that $C'(t)$ is positive if $0 \le t < 10\ln 1.5 \doteq 4.05$. Thus serum penicillin is increasing for about four hours after ingestion of the penicillin pill into the intestinal track. ■

You were asked in Explore 8.1.3 to define **decreasing** and **nonincreasing**.

As in Explore 8.1.2, you can show that if P is nonincreasing, then $P'(t) \le 0$. By considering $Q(t) = -P(t)$, it follows from Theorem 8.1.1 that if $P'(t) \le 0$ then P is nonincreasing, and from Theorem 8.1.2 that if $P'(t) < 0$ for $a < t < b$ then P is decreasing on $[a, b]$.

From the preceding example of blood penicillin concentration, with $C'(t) = 5 \times e^{-0.3t} \times (-0.2e^{0.1t} + 0.3)$ we observe that

$$5 > 0 \qquad e^{-0.3t} > 0 \quad \text{for all } t \text{ and} \qquad -0.2e^{0.1t} + 0.3 < 0 \quad \text{for} \quad 10\ln 1.5 < t$$

and we conclude that penicillin concentration is decreasing for $t \ge 10\ln 1.5 \doteq 4.05$ hours.

Because the penicillin concentration is increasing during 0 to $10\ln 1.5$ hours and decreasing afterward, the maximum concentration must occur at $10\ln 1.5$ hours. That maximum concentration is $C(10\ln 1.5)$ and

$$\begin{aligned} C(10\ln 1.5) &= 5e^{-0.2\times 10\ln 1.5} - 5e^{-0.3\times 10\ln 1.5} \\ &= 5e^{-2\ln 1.5} - 5e^{-3\ln 1.5} \\ &= 5\left(e^{\ln 1.5}\right)^{-2} - 5\left(e^{\ln 1.5}\right)^{-3} \\ &= 5\frac{1}{1.5^2} - 5\frac{1}{1.5^3} \quad \doteq \quad 0.74074. \end{aligned}$$

Hence, the maximum concentration is approximately 0.74 μg/ml.

Explore 8.1.5 Locate the point (4.05, 0.74) on the graph in Figure 8.2. ■

8.1.1 Convex up, concave down, and inflection points.

Now we examine when P' is increasing (signaled by $P'' > 0$). In such regions, the graph of P is *convex up* or simply *convex*. Similarly, when $P'' < 0$, P' is decreasing and the graph of P is *concave down* or simply *concave*.

Information provided by the second derivative is illustrated by the graph of $P(x) = x(x-2)(x-4) = x^3 - 7x^2 + 10x$ in Figure 8.3.

$$\begin{aligned} P(x) &= x^3 - 7x^2 + 10x, \qquad P'(x) = 3x^2 - 14x + 10 \\ P''(x) &= 6x - 14 = 6(x - 7/3) \end{aligned}$$

In Figure 8.3, the slope of P decreases on the interval $[A, a]$. The slope of P increases on the interval $[b, B]$.

If $x < 7/3$, $P''(x) < 0$, and $P'(x)$ is decreasing and $P(x)$ is concave down.

If $x < 7/3$, $P''(x) > 0$, and $P'(x)$ is increasing and $P(x)$ is convex up.

The tangents to the graph at A and a lie above the graph except at the points of tangency, consistent with P being concave down on $x < 7/3$. The tangents to the graph at b and B lie below the graph except at the points of tangency, consistent with P being convex up on $7/3 < x$. The point at (7/3,19/3) is an inflection point. $P''(7/3) = 0$ and the tangent at (7/3,19/3) crosses the graph.

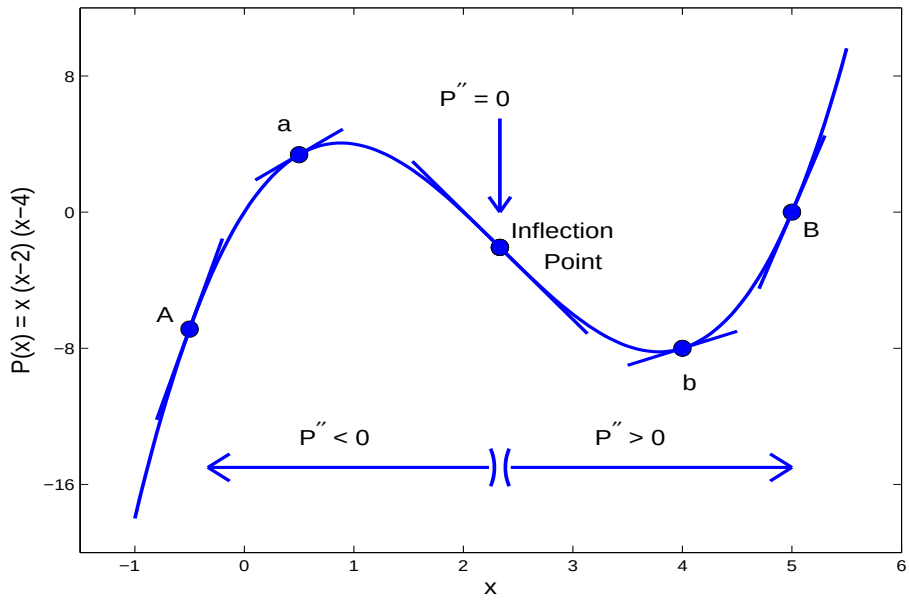


Figure 8.3: Graphs of $P(x) = x(x-2)(x-4)$. a. The slope is decreasing on $(-\infty, 7/3)$; the tangents lie above the graph except for the points of tangency and the graph is concave down. b. (7/3, 19/3) is an inflection point; the tangent crosses the graph at that point. The slope is increasing on $(7/3, -\infty)$; the tangents lie below the graph except for the points of tangency and the graph is convex up.

8.1.2 The arithmetic mean is greater than or equal to the geometric mean.

We prove here Theorem 5.2.2 which was stated without proof in Subsection 5.2.1 in which the number e was shown to be $\lim_{h\to 0}(1+h)^{1/h}$.

Theorem 5.2.2. If $a_1, a_2, \cdots, a_n$ is a sequence of n positive numbers then

$$\frac{a_1 + a_2 + \cdots a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \tag{8.1}$$

with equality only when $a_1 = a_2 = \cdots = a_n$.

In the proof we use the facts that for any integer $n > 1$

$$1 + (n-1)\sqrt[n-1]{t} = n\sqrt[n]{t} \qquad \text{for} \qquad t = 1 \tag{8.2}$$

$$1 + (n-1)\sqrt[n-1]{t} > n\sqrt[n]{t} \qquad \text{for} \qquad t > 1 \tag{8.3}$$

Proof. Suppose $n > 1$ is an integer and

$$F(t) = 1 + (n-1)\sqrt[n-1]{t} - n\sqrt[n]{t} = 1 + (n-1)t^{\frac{1}{n-1}} - nt^{\frac{1}{n}}$$

Then $F(1) = 1 + (n-1) \times 1 - n \times 1 = 0$ and Equation 8.2 is satisfied. Next $F'(t)$ is computed as

$$\begin{aligned} F'(t) &= \left[1 + (n-1)t^{\frac{1}{n-1}} - nt^{\frac{1}{n}}\right]' \\ &= 0 + (n-1)\frac{1}{n-1}t^{\frac{1}{n-1}-1} - n\frac{1}{n}t^{\frac{1}{n}-1} \\ &= t^{-\frac{n-2}{n-1}} - t^{-\frac{n-1}{n}} \\ &= t^{-\frac{n-2}{n-1}}\left(1 - t^{-\frac{1}{n(n-1)}}\right) \end{aligned}$$

For $t > 1$ both factors are greater than zero and $F'(t) > 0$ for $t > 1$. It follows from Theorem 8.1.2 that F is increasing for $t > 1$. Because $F(1) = 0$ and F is increasing for $t > 1$, for $t > 1$

$$\begin{aligned} F(t) &> 0 \\ 1 + (n-1)\sqrt[n-1]{t} - n\sqrt[n]{t} &> 0 \\ 1 + (n-1)\sqrt[n-1]{t} &> n\sqrt[n]{t} \end{aligned}$$

Equation 8.3 is satisfied. ■ *End of proof of Equations 8.2 and 8.3.*

Proof of Theorem 5.2.2. We proceed by induction. First we prove that if a_1 and a_2 are two positive numbers then $(a_1 + a_2)/2 \geq \sqrt[2]{a_1 a_2}$.

$$\begin{aligned} (a_1 - a_2)^2 &\geq 0 \\ a_1^2 - 2a_1a_2 + a_2^2 &\geq 0 \\ a_1^2 + 2a_1a_2 + a_2^2 &\geq 4a_1a_2 \\ \frac{(a_1 + a_2)^2}{4} &\geq a_1a_2 \\ \frac{a_1 + a_2}{2} &\geq \sqrt[2]{a_1a_2} \end{aligned}$$

Furthermore, $(a_1 - a_2)^2 = 0$ only when $a_1 = a_2$ and equality holds in each expression of the previous array only when $a_1 = a_2$. The statement in Theorem 5.2.2 is valid with $n = 2$.

Now suppose Equation 8.1 is valid for sequences of length $n-1$ and $a_1, a_2, \cdots a_n$ is a sequence of positive numbers of length n. We assume without loss of generality that a_n is the smallest number in $a_1, a_2, \cdots a_n$, and consider the sequence $b_k = a_k/a_n$, $k = 1, n$. Then $b_k \geq 1$ and $b_n = 1$. Consequently,

$$t = b_1b_2 \cdots b_{n-1}b_n = b_1b_2 \cdots b_{n-1} \geq 1$$

with equality only for $b_1 = b_2 = \cdots = b_{n-1} = 1$. Observe that $b_1 = b_2 = \cdots = b_{n-1} = 1$ only if $a_1 = a_2 = \cdots = a_n$.

From Equations 8.2 and 8.3 we know that $1 + (n-1)\sqrt[n-1]{t} \geq n\sqrt[n]{t}$ for $t \geq 1$ with equality only for $t = 1$. Therefore

$$1 + (n-1)\sqrt[n-1]{b_1 b_2 \cdots b_{n-1}} \geq n\sqrt[n]{b_1 b_2 \cdots b_n}$$

By the induction hypothesis

$$b_1 + b_2 + \cdots b_{n-1} \geq (n-1)\sqrt[n-1]{b_1 b_2 \cdots b_{n-1}}$$

so that

$$1 + b_1 + b_2 + \cdots b_{n-1} \geq n\sqrt[n]{b_1 b_2 \cdots b_n}$$

Multiply all all terms by a_n and we get

$$a_n + a_1 + a_2 + \cdots a_{n-1} \geq n\sqrt[n]{a_1 a_2 \cdots a_n}$$

or

$$\frac{a_1 + a_2 + \cdots a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Equality holds in the previous five expressions only if $b_1 = b_2 = \cdots = b_{n-1} = 1$, in which case $a_1 = a_2 = \cdots = a_n$.
End of proof.

Exercises for Section8.1, Some geometry of the derivative.

Exercise 8.1.1 Argue that e^x is an increasing function.

Exercise 8.1.2 Argue that $\sin x$ is an increasing function on $0 \leq x \leq \pi/2$.

Exercise 8.1.3 Suppose penicillin concentration is given by $C(t) = 8e^{-0.2t} - 8e^{-0.4t}$ μgm/ml t hours after ingestion of a penicillin pill. For what time period is the concentration increasing? What is the maximum penicillin concentration?

Exercise 8.1.4 Suppose penicillin concentration is given by $C(t) = 0.4te^{-0.5t}$ μgm/ml t hours after ingestion of a penicillin pill. For what time period is the concentration decreasing? What is the maximum penicillin concentration?

Exercise 8.1.5 Identify the intervals, if any, on which $f(x)$ is increasing and intervals, if any, on which f' is increasing.

a.	$f(x) = x^2 - 1$	$-2 \leq x \leq 2$		b.	$f(x) = x^2 - x$	$-2 \leq x \leq 2$
c.	$f(x) = x^3 - x^2$	$-2 \leq x \leq 2$		d.	$f(x) = \dfrac{x}{x^2+1}$	$-2 \leq x \leq 2$
e.	$f(x) = e^{-x}$	$-2 \leq x \leq 2$		f.	$f(x) = xe^{-x}$	$0 \leq x \leq 3$
g.	$f(x) = e^{-x^2}$	$-2 \leq x \leq 2$		h.	$f(x) = e^{-2x} - e^{-x}$	$0 \leq x \leq 4$
i.	$f(x) = \ln x$	$0 < x \leq 2$		j.	$f(x) = x \ln x$	$0 < x \leq 1$
k.	$f(x) = \cos x$	$-\pi \leq x \leq \pi$		l.	$f(x) = \sin^2 x$	$-\pi \leq x \leq \pi$

Exercise 8.1.6 We show in Chapter 13 that if n is an integer then

$$1 + 2 + \cdots (n-1) + n = \frac{n(n+1)}{2}.$$

Use this and Theorem 5.2.2 to show that if n is an integer

$$\left(\frac{n+1}{2}\right)^n \geq n!$$

where $n! = 1 \cdot 2 \cdot 3 \cdot (n-1) \cdot n$.

Exercise 8.1.7 Trout, Moose, and Bear lakes are connected into a chain by a stream that runs into Trout Lake, out of Trout Lake into Moose Lake, out of Moose Lake and into Bear Lake and out of Bear Lake. The volumes of all of the three lakes are the same, and stream flow is constant into and out of all lakes. A load of waste is dumped into Trout Lake. With t measured in days and concentration measured in mg/l, the concentration of wastes in the three lakes is projected to be

$$\begin{array}{lrcl} \text{Trout Lake:} & C_{\mathrm{T}}(t) & = & 0.01e^{-0.05t} \\ \text{Moose Lake:} & C_{\mathrm{M}}(t) & = & 0.005te^{-0.05t} \\ \text{Bear Lake:} & C_{\mathrm{B}}(t) & = & 0.000025\dfrac{t^2}{2}e^{-0.05t} \end{array}$$

For each lake, find the time interval, if any, on which the concentration in the lake is increasing.

8.2 Some Traditional Max-Min Problems.

We present a few of the large number of interesting optimization problems that are part of the culture of calculus. The basis for solving these problems is the following definition and theorem.

Definition 8.2.1 Suppose f is a function with domain, D, and c is a number in D.

1. The point $(c, f(c))$, is a *maximum* for f means that

$$\text{for all numbers } x \text{ in } D \qquad f(x) \leq f(c)$$

2. The point $(c, f(c))$ is a *local maximum* for f means that there is an open interval (p, q) that contains c and

$$\text{for all numbers } x \text{ in } D \text{ and in } (p, q) \qquad f(x) \leq f(c)$$

3. The point $(c, f(c))$ is an *interior local maximum* for f if D contains an open interval (p, q) that contains c and

$$\text{for all numbers } x \text{ in } (p, q) \qquad f(x) \leq f(c)$$

4. Similar definitions are made for *minima*.

In Figure 8.4, the point B is a maximum for the graph and an interior maximum. The points A and C are local minima for the graph; A is an endpoint local minimum and C is an interior local minimum. The point D is an interior local maximum and E is an endpoint minimum.

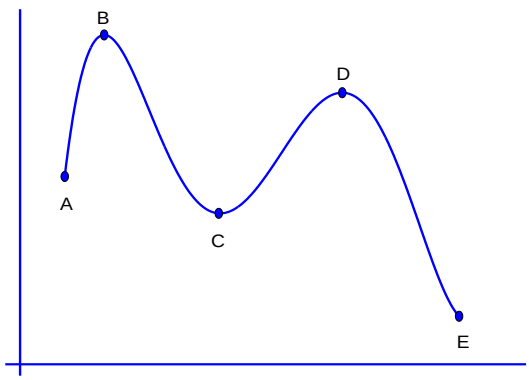


Figure 8.4: Maxima and minima. A is an endpoint local minimum, B is an interior maximum, C is an interior local minimum, D is an interior local maximum, E is an endpoint minimum.

If $(c, f(c))$ is a maximum for f then we may say that f attains its maximum value at c, or the maximum value of f occurs at c, or that the maximum value of f is $f(c)$. Thus, the sine function attains its maximum value at $\frac{\pi}{2}$ and the maximum value of the sine function is 1. Also the sine function attains its minimum value at $\frac{3\pi}{2}$ and the minimum value of the sine function is -1. The function $f(t) = t^3$ does not have a maximum value and does not have a minimum value. The word, 'value', is often omitted in

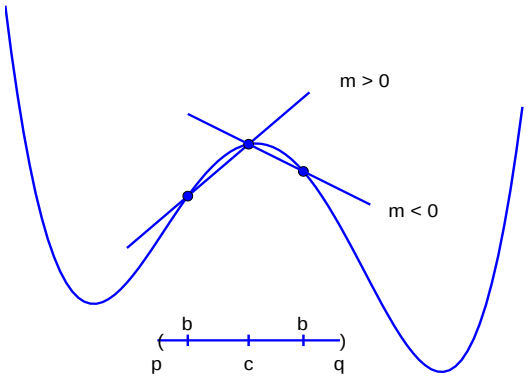


Figure 8.5: A local maximum and left and right secants.

these statements, and we may say, for example, that 1 is the maximum of the sine function and $f(t) = t^3$ has neither a maximum nor a minimum. Similar language is used with local and interior local maxima and minima.

Theorem 8.2.1 If $(c, f(c))$ is an interior local maximum for a function, f, and

$$\text{if } f'(c) \text{ exists, then} \qquad f'(c) = 0.$$

(Equivalently, if the graph of f has a tangent at an interior local maximum $(c, f(c))$, then that tangent is horizontal.)

Proof. Suppose $(c, f(c))$ is an interior local maximum for f, and (p, q) is an interval in D containing c for which $f(x) \leq f(c)$ for all x in (p, q). We wish to show that $f'(c) = 0$. See Figure 8.5.

Suppose $p < b < c$. Then $b - c < 0$, and because c is a local maximum

$$f(b) \leq f(c) \quad \text{and} \quad f(b) - f(c) \leq 0 \qquad \text{and} \qquad \frac{f(b) - f(c)}{b - c} \geq 0$$

It follows that

$$f'^{-}(c) = \lim_{b \to c^-} \frac{f(b) - f(c)}{b - c} \geq 0.$$

Similar analysis shows that

$$f'^{+}(c) = \lim_{b \to c^+} \frac{f(b) - f(c)}{b - c} \leq 0.$$

It follows that $f'(c) = 0$. *End of Proof.*

Summary: Critical Points.

If c is a local maximum or local minimum for a function, f, defined on an interval $[a, b]$ then

1. $f'(c) = 0$, or
2. $f'(c)$ does not exist, or
3. c is either a or b.

The collection of all points c satisfying either 1, 2, or 3 is called the **Critical Points** for f on $[a, b]$.

In the examples considered in this book, there are usually only a few (no more than five, say) critical points. Then finding the maximum and minimum values only involves selecting from among no more than five values of f.

Example 8.2.1 Suppose you are going to make a rectangular box with open top from a 3 meter by 4 meter sheet of tin. One procedure for doing so would be to cut squares of side x from each corner as shown in Figure 8.6A, and to fold the 'tabs' up. Four pieces of area x^2 would be discarded. What value of x will maximize the volume of the box you construct in this way? What will be the volume of the largest such box?

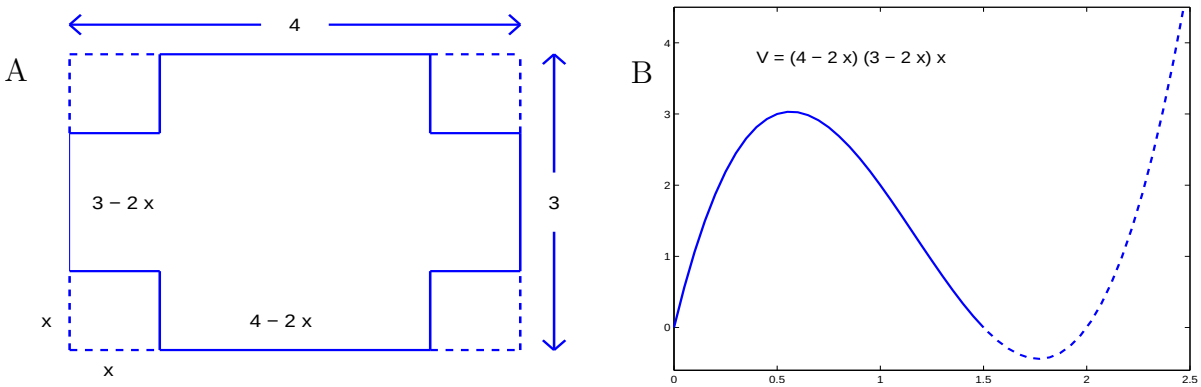


Figure 8.6: A. Diagram and B. graph (solid line) for Example 8.2.1

After the corners are cut, the 'core' of the tin that will make the bottom of the box will be of length $4 - 2x$ and width $3 - 2x$. The height of the box will be x and the volume of the box formed will be

$$V = (4 - 2x)(3 - 2x)x \qquad 0 \le x \le \frac{3}{2}$$

$0 \le x \le \frac{3}{2}$ insures that no side of the box is a negative number. A graph of V appears in Figure 8.6B. The dashed line shows an extension of the graph of $y = (4 - 2x)(3 - 2x)x$ that is not part of the graph of

V. Compute V':

$$\begin{aligned} V' &= [(4-2x)(3-2x)x]' \\ &= [(4-2x)]'(3-2x)x + (4-2x)[(3-2x)x]' \\ &= [(4-2x)]'(3-2x)x + (4-2x)[(3-2x)]'x + (4-2x)(3-2x)[x]' \\ V' &= 8x^2 - 28x + 12 \end{aligned}$$

Find the critical points:

- **$\mathbf{V' = 0}$**
$$V' = 0 \qquad \text{implies} \qquad 8x^2 - 28x + 12 = 0 \qquad \text{implies} \qquad x = 3 \quad \text{or} \quad x = \frac{1}{2}$$
$x = \frac{1}{2}$ is in the domain of V but $x = 3$ is not.
- **$\mathbf{V'}$ does not exist.** No such points. V is a cubic function and has derivatives at every point.
- **End points.** The end points are $x = 0$ and $x = \frac{3}{2}$

The three critical values of x are 0, $\frac{1}{2}$ and $\frac{3}{2}$

Find the maximum V:

$$\begin{aligned} V(0) &= (4 - 2\times 0)(3 - 2\times 0)\times 0 = 0 \\ V\left(\frac{1}{2}\right) &= (4 - 2\times\frac{1}{2})(3 - 2\times\frac{1}{2})\times\frac{1}{2} = 3 \\ V\left(\frac{3}{2}\right) &= (4 - 2\times\frac{3}{2})(3 - 2\times\frac{3}{2})\times\frac{3}{2} = 0 \end{aligned}$$

The maximum volume is 3 and occurs with $x = \frac{1}{2}$. ■

In the preceding Example, we were given a surface area and asked to find the dimensions that will maximize the volume. A dual problem is to be given a required volume, find the dimensions that will minimize the required surface area.

Example 8.2.2 Suppose a box with a square base and closed top and bottom is to have 8 cubic meters. What dimensions of the box will minimize the surface area of the box?

Solution. Let x be the length of one side of the square base and y be the height of the box. Then the volume and surface area of the box are

$$V = x^2 y \qquad S = 2x^2 + 4xy$$

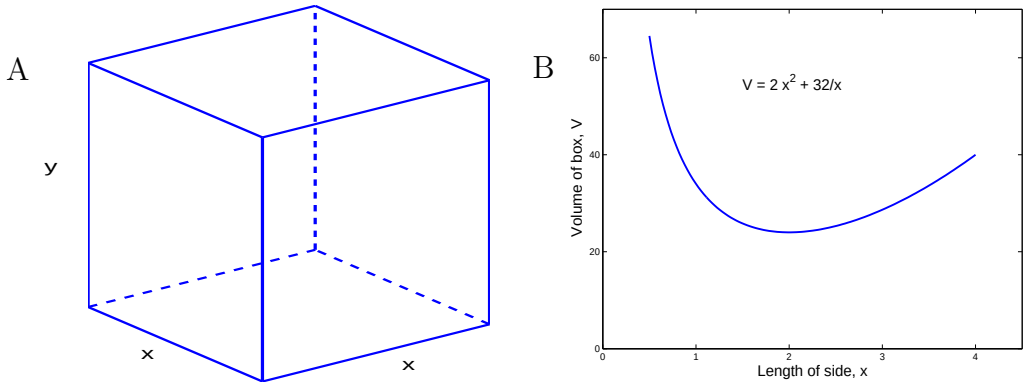
Because V is specified to be 8 cubic meters

$$8 = x^2 y \qquad \text{so that} \qquad y = \frac{8}{x^2}$$

We substitute $\frac{8}{x^2}$ for y in the expression for S and get

$$S = 2x^2 + 4x\frac{8}{x^2} = 2x^2 + \frac{32}{x}$$

Figure for Example 8.2.2.2 A. Box with square bottom and volume $= 8\ m^3$. B. Graph of the function, $S(x) = 2x^2 + \dfrac{32}{x}$.



The domain of S is $x > 0$ (there are no endpoints, $x = 0$ is not allowed by the x in the denominator and there is no upper limit on x).

Then

$$\begin{aligned} S'(x) &= \left[2x^2 + \frac{32}{x}\right]' \\ &= \left[2x^2\right]' + \left[32x^{-1}\right]' \\ &= 4x - 32x^{-2} \end{aligned}$$

$S'(x)$ exists for all $x > 0$ and $S'(x) = 0$ yields

$$4x - 32x^{-2} = 0, \qquad 4x^3 - 32 = 0, \qquad x = 2 \qquad \blacksquare$$

Thus we conclude that the base of the box should be 2 by 2 and because $x^2y = 8$ the height y of the box should also be 2. Examination of the graph of $S(x) = 2x^2 + \dfrac{32}{x}$ in Example Figure 8.2.2.2B suggests it is a minimum (and not, for example, a maximum!).

8.2.1 The Second Derivative Test.

There is a clever way of distinguishing local maxima from local minima using the second derivative. The following theorem is proved in Section 9.2.

Theorem 9.2.3. Suppose f is a function with continuous first and second derivatives throughout an interval $[a, b]$ and c is a number between a and b for which $f'(c) = 0$. Under these conditions:

1. If $f''(c) > 0$ then c is a local minimum for f (see Figure 8.7A).
2. If $f''(c) < 0$ then c is a local maximum for f (see Figure 8.7B).

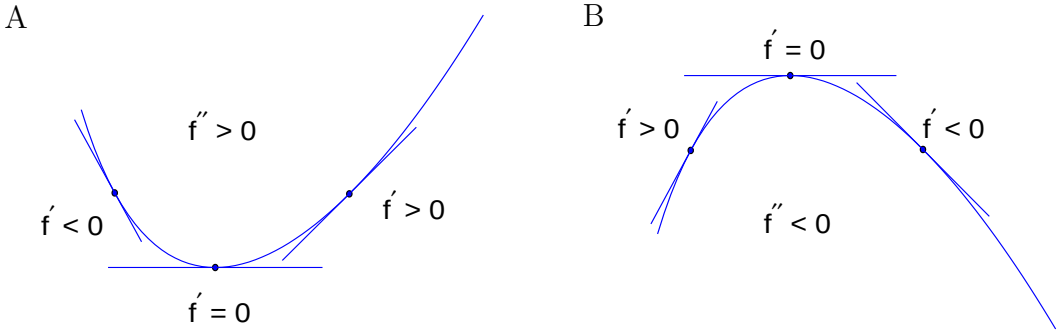


Figure 8.7: A. An interior local minimum, $f''(c) > 0$. B. An interior local maximum, $f''(c) < 0$.

In Example 8.2.2, $S'(x) = 4(x - 8x^{-2})$, so that

$$S''(x) \quad = \quad \left[4\left(x - 8x^{-2}\right)\right]' \quad = \quad 4\left(1 + 16x^{-3}\right)$$

Now $S'(2) = 0$, and $S''(2) = 12 > 0$, so by the second derivative test, $x = 2$ is a local minimum for S, as we previously concluded.

Example 8.2.3 The salmon and tuna cans shown in Example Figure 8.2.3.3 both contain fish. Why are they shaped so differently?

Suppose the criterion for making cans is to minimize the amount of metal required to hold a fixed amount of fish (the cans shown are 14.5 oz and 16 oz; we did not find cans holding equal weights of fish). Which can most closely meets the criterion?

Figure for Example 8.2.3.3 A. Salmon and tuna cans.

Question: Of all cans of volume equal to 1, which has the smallest surface area?

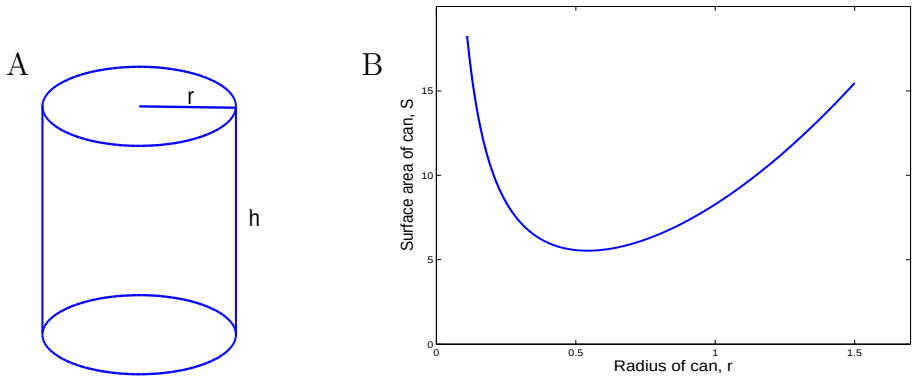
Assume that the 'can' (cylinder) in Example Figure 8.2.3.3 has volume equal to 1. The area of the top end of the can is πr^2 and the circumference of the top lid is $2\pi r$. The volume, V, of the can is

$$V = \pi r^2 \times h$$

and the surface area, S, of the can is

$$S = 2\pi r^2 + 2\pi r \times h$$

Figure for Example 8.2.3.3 A. A can of volume 1. B. A graph of $S(r) = 2\pi r^2 + \frac{2}{r}$



The requirement that the volume be 1 yields

$$1 = \pi r^2 \times h \qquad \text{and solving for } h \text{ yields} \qquad h = \frac{1}{\pi r^2}$$

We substitute this value into the expression for S and obtain

$$S = 2\pi r^2 + 2\pi r \times \frac{1}{\pi r^2} \qquad\qquad S = 2\pi r^2 + \frac{2}{r}$$

The domain for S is $r > 0$ (there are no endpoints).

From the graph of S in Figure 8.2.3.3B it appears that there is a single minimum at about $r = 0.5$. We find the critical points.

$$S'(r) \quad = \quad \left[2\pi r^2 + \frac{2}{r}\right]' \quad = \quad \left[2\pi r^2\right]' + \left[2r^{-1}\right]' \quad = \quad 4\pi r - 2r^{-2}$$

The requirement $S'(r) = 0$ yields

$$4\pi r - 2r^{-2} \quad = \quad 0, \qquad 4\pi r^3 - 2 \quad = \quad 0, \qquad r \quad = \quad \frac{1}{\sqrt[3]{2\pi}} \doteq 0.542$$

Recall that $h = \frac{1}{\pi r^2}$ so that the ratio of h to r (height to radius ratio) that gives minimum surface area is

$$\frac{h}{r} \quad = \quad \left.\frac{1/\pi r^2}{r}\right|_{r=1/\sqrt[3]{2\pi}} \quad = \quad \left.\frac{1}{\pi r^3}\right|_{r=1/\sqrt[3]{2\pi}} \quad = \quad 2$$

Thus the height should be twice the radius, or equal to the diameter. ■

8.2.2 How to solve these problems.

There are some procedures we have been using that will help you solve these problems. They are listed in rough order of use and applied to an example.

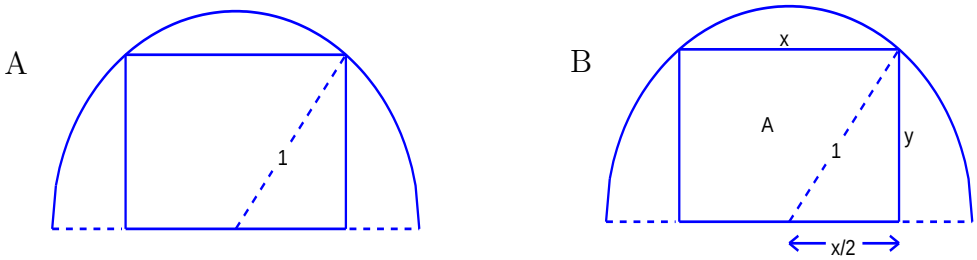


Figure 8.8: A. A semicircle of radius 1 with inscribed rectangle. B. Labels for the diagram in A.

- **Read the problem!**

  Problem: Find the rectangle of largest area that can be inscribed in a semicircle of radius 1.

- **Draw a picture.** (Representative of the problem!)

  Picture should have a semicircle of radius 1 and a rectangle inscribed in it. Also draw a radius to a corner of the rectangle. Figure 8.8A.

- **Label parts of the picture.** This will introduce symbols for important parameters of the problem.

  x and y should label horizontal and vertical sides of the rectangle, and A is the area of the rectangle. Figure 8.8B.

- **Write relations between the parameters.**

  $\left(\frac{x}{2}\right)^2 + y^2 = 1$ *and* $A = xy$.

- **Write a function of a single variable.** Write the parameter to be optimized in terms of a single adjustable parameter (variable).

  Solve for y in $\left(\frac{x}{2}\right)^2 + y^2 = 1$ *and substitute into* $A = xy$.

$$A = x\sqrt{1 - x^2/4}$$

- **Draw a graph.** It is usually easy to draw a graph of your function on a calculator and the graph will often reveal the maximum and minimum points. You may end the process at this step with a calculator based estimate of the answer. A graph of $A = x\sqrt{1 - x^2/4}$ appears in Figure 8.9. The TI-86 shows the maximum to be at $x = 1.41421347$. One suspects that x is $\sqrt{2}$ and may continue with an analytical solution.

- **Look for a clever simplification.**

  The value of x that minimizes A also minimizes A^2 and $A^2 = x^2 - x^4/4$. You have your choice: Compute the derivative of $A = x\sqrt{1 - x^2/4}$ or compute the derivative of $A^2 = x^2 - x^4/4$. (Choose A^2!!)

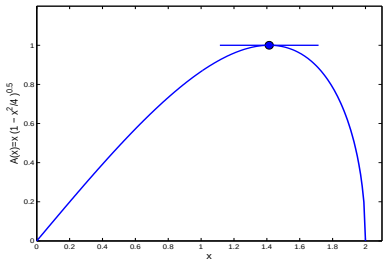


Figure 8.9: The graph of $A = x\sqrt{1 - x^2/4}$; the calculator maximum is at $x = 1.4142317$ and $\sqrt{2} = 1.41421356\cdots$.

- **Find the critical points.** Compute the first derivative; find where it is zero or fails to exist; examine the end points.

$$\left[A^2\right]' = 2x - x^3 \qquad \text{exists for all } x$$

$$\left[A^2\right]' = 0 \Longrightarrow x = 0 \quad \text{or} \quad x = \sqrt{2}$$

The critical points are $x = \sqrt{2}$ and the end points $x = 0$ and $x = 2$.
$A^2(0) = 0$, $A^2(\sqrt{2}) = 1$, $A^2(2) = 0$. At this stage we know that $x = \sqrt{2}$ maximizes $A^2(x)$ because the maximum has to occur at one of the critical points. For illustrations, we also:

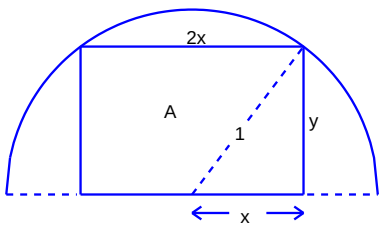
- **Use the second derivative test, if needed and applicable.**

$$\left[A^2\right]'' = 2 - 3x^2 \qquad \left[A^2\right]''\Big|_{x=\sqrt{2}} = -4$$

$\left[A^2\right]''\Big|_{x=\sqrt{2}}$ *is negative, so* $x = \sqrt{2}$ *is a local maximum. To see that it is actually a maximum, we have to check the other critical points.*

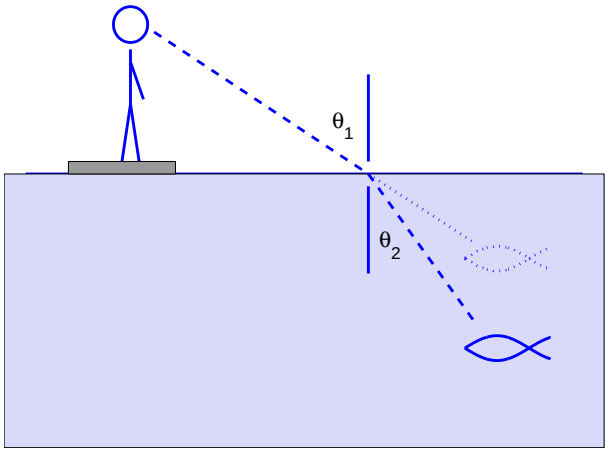
Explore 8.2.1 Selection of labels on a figure can simplify or complicate the equations you derive. The previous figure is shown with the horizontal side of the rectangle being $2x$ instead of x. Write the equations that correspond to $\left(\frac{x}{2}\right)^2 + y^2 = 1$, $A = xy$ and $A = x\sqrt{1 - x^2/4}$ from the previous analysis.

Figure for Exercise 8.2.0 Improved labels for Figure 8.8A.



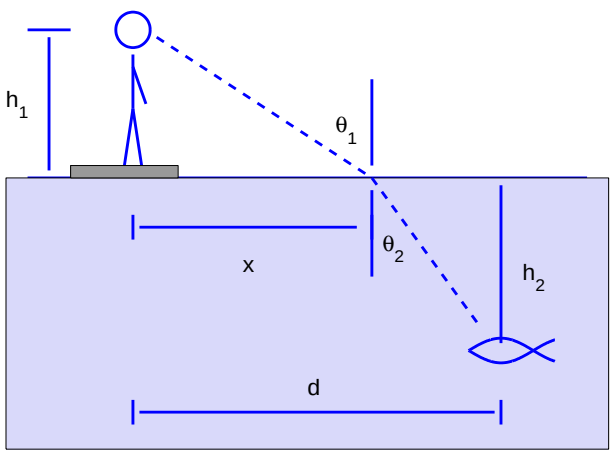
Example 8.2.4 Snell's Law When you see a fish in a lake it typically is below where it appears to be. A spear, arrow, bullet, rock or other projectile launched toward the image of the fish that you see will pass above the fish. The different speeds of light in air and in water cause the light beam traveling from the fish to your eye to bend at the surface of the lake. The apparent location of the fish is marked as a dotted fish in Figure 8.2.4.4.

Figure for Example 8.2.4.4 A. Light ray from fish to observer. The dotted fish is the apparent location of the fish.



Pierre Fermat asserted in 1662 that the path of the light beam will be that path that minimizes the total time of travel in the two media. The speed of light in water, v_2, is about 0.75 times the speed of light in air, v_1. Suppose d is the horizontal distance between your eye and the fish, h_1 is the height of your eye above the water and h_2 is the depth of the fish below the surface of the lake. Finally let x be the horizontal distance between your eye and the point at which the beam passes through the surface of the lake.

Figure for Example 8.2.4.4 B. Light ray from fish to observer with labels.



The distances the light ray travels in air and water are

$$\text{Air distance:} \quad \sqrt{h_1^2 + x^2} \qquad \text{Water distance:} \quad \sqrt{h_2^2 + (d-x)^2}$$

The times that the light spends traversing air and traversing water are (distance/velocity)

$$\text{Air time:} \quad \frac{\sqrt{h_1^2 + x^2}}{v_1} \qquad \text{Water time:} \quad \frac{\sqrt{h_2^2 + (d-x)^2}}{v_2}$$

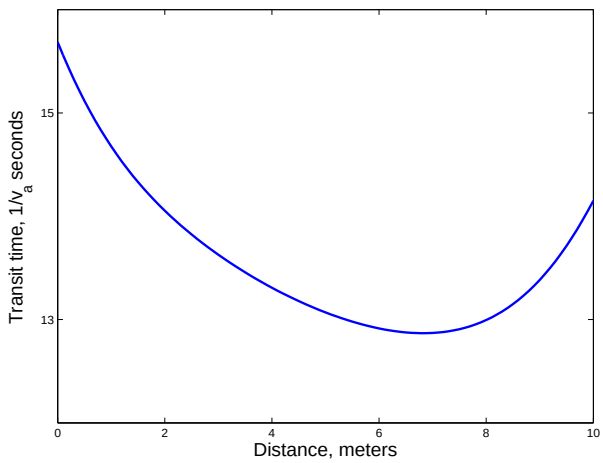
The total time, T, for the ray to travel from the fish to your eye is

$$T = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (d-x)^2}}{v_2} \qquad 0 \le x \le d$$

Our task is to find the value of x that minimizes T. A graph of T is shown for values $v_1 = 1$, $v_2 = 0.75$, $d = 10$ meters, $h_1 = 2$ meters, $h_2 = 3$ meters. As tempting as it may be, it is not the time to get clever and square each term in the previous equation, for it is not generally true that $(a+b)^2 = a^2 + b^2$. Instead we compute T' directly and find that

$$T' = \frac{\dfrac{x}{\sqrt{h_1^2 + x^2}}}{v_1} - \frac{\dfrac{d-x}{\sqrt{h_2^2 + (d-x)^2}}}{v_2}$$

Figure for Example 8.2.4.4 Transit time for a light ray from fish to observer for parameter values $v_1 = 1$, $v_2 = 0.75$, $d = 10$ meters, $h_1 = 2$ meters, $h_2 = 3$ meters. The distance, x, for which the transit time is minimum is a bit less that 7 meters. v_a is the velocity of light in air.



Setting $T' = 0$ and solving for x is not advised. We take a qualitative approach instead, and return to the geometry and identify two angles, θ_1 and θ_2, the angles the light ray makes with a vertical line through the point of intersection with the surface. They are marked in both figures 8.2.4.4 and 8.2.4.4. It may be seen that

$$\frac{x}{\sqrt{h_1^2 + x^2}} = \sin\theta_1 \qquad \text{and} \qquad \frac{d-x}{\sqrt{h_2^2 + (d-x)^2}} = \sin\theta_2$$

and

$$T' = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

Observe from the geometry that as x moves from 0 to d, θ_1 increases from 0 to a positive number and θ_2 decreases from a positive number to 0. There is a single value of x where the graphs of

$$\frac{\sin\theta_1}{v_1} \qquad \text{and} \qquad \frac{\sin\theta_2}{v_2}$$

cross and at that point $T' = 0$ and

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{8.4}$$

Equation 8.4 is referred to as Snell's law and applies to many problems of optics.

Because for water and air, $v_2 = 0.75\ v_1$, and we would have

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{0.75 v_1} \qquad \text{or} \qquad \frac{\sin\theta_1}{1} = \frac{\sin\theta_2}{0.75} = \frac{4}{3}\sin\theta_2$$

we would have

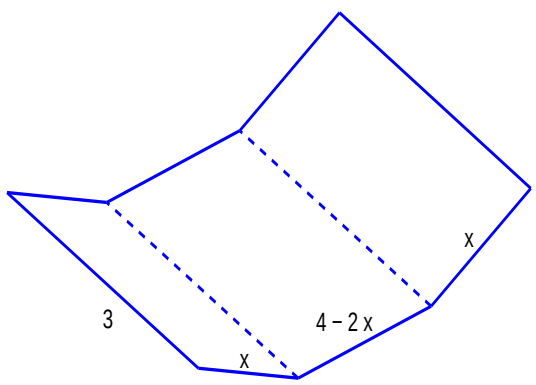
$$\sin\theta_1 > \sin\theta_2 \qquad \text{or} \qquad \theta_1 > \theta_2$$

which implies that the light ray bends down into the water as shown, and the fish is actually below its apparent position.

Exercises for Section 8.2, Some Traditional Max-Min Problems.

Exercise 8.2.1 Suppose you have a 3 meter by 4 meter sheet of tin and you wish to make a box that has tin on the bottom and on two opposite sides. The other two sides are of wood that is in plentiful supply. You are going to make a rectangular box by folding up panels of width x across the ends that are 3 meters wide as shown in Figure 8.2.1. What value of x will maximize the volume of the box and what is the volume?

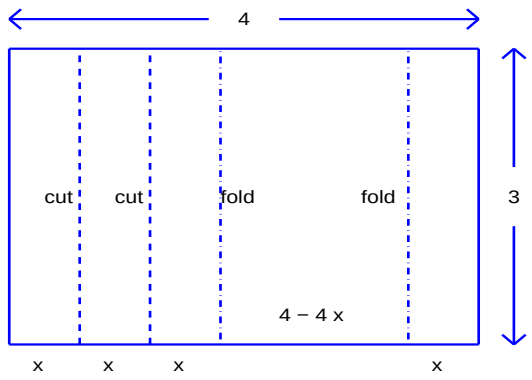
Figure for Exercise 8.2.1 Diagram for Exercise 8.2.1.



Exercise 8.2.2 In Exercise 8.2.1, would it be better (make a box of larger volume) to fold up panels of width x across the sides that are 4 meters long?

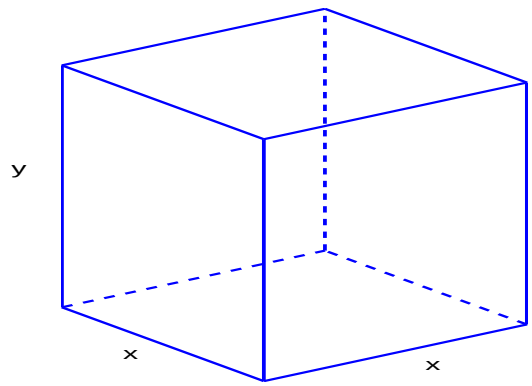
Exercise 8.2.3 Dissatisfied with having to discard the four corners as in Example 8.2.1, you decide to take another approach. From the 3 by 4 meter panel, you will cut two strips of width x across an end that is 3 meters wide, fold up two similar strips of width x and use the first two strips to make the other sides. See Figure 8.2.3. The two strips you cut may be too long, so that you may still have to discard some tin. What value of x will make a box of the largest volume and what is the volume?

Figure for Exercise 8.2.3 Diagram for Exercise 8.2.3.



Exercise 8.2.4 A box with a square base and open top has surface area of 12 square meters (Figure 8.2.4. What dimensions of the box will maximize the volume?

Figure for Exercise 8.2.4 Box with square base and surface area = 12 $m^2$4 for Exercise 8.2.4.



Exercise 8.2.5 Electron micrographs of diatoms are shown in Exercise Figure 8.2.5. Diatoms are phytoplankton that originated some 200 million years ago and are found in marine, fresh water and moist terrestrial environments, and contribute some 45 percent of the oceans organic production. The shells of the diatoms illustrated are like pill boxes (a cylindrical cup caps another cylindrical cup and a 'girdle' surrounds the overlap of the shells) and some parts are permeable and other parts appear to be impermeable. (Other diatoms have quite different structures, oblong and pennate, for example.) Typically the structure of such shells optimize some aspect of the organism, subject to functional constraints. Make measurements of the diatoms and discuss the optimality of the use of materials to make the shell, with the constraint that light must penetrate the shell.

Figure for Exercise 8.2.5 Electron micrographs of diatoms for Exercise 8.2.5. A. *Cyclotella comta* is the dominant component of the spring diatom bloom in Lake Superior. http://www.glerl.noaa.gov/seagrant/GLWL/Algae/Diatoms/Diatoms.html B. *Thalassiosira pseudonana*, the first eukaryotic marine phytoplankton for which the whole genome was sequenced. Micrograph by Nils Kroger, Universitat Regensburg, from http://www.jgi.doe.gov/News/news_9_30_04.html or from http://gtresearchnews.gatech.edu/newsrelease/diatom-structure.htm

A 

B. See the Department of Energy or Georgia Tech web page.

Exercise 8.2.6 From Example 8.2.4. Show that

$$\left[\frac{\sqrt{h_1^2+x^2}}{v_1}+\frac{\sqrt{h_2^2+(d-x)^2}}{v_2}\right]'=\frac{\dfrac{x}{\sqrt{h_1^2+x^2}}}{v_1}-\frac{\dfrac{d-x}{\sqrt{h_2^2+(d-x)^2}}}{v_2}$$

Exercise 8.2.7 Find the area of the largest rectangle that can be inscribed in a right triangle with sides of length 3 and 4 and hypotenuse of length 5.

Exercise 8.2.8 Two rectangular pens of equal area are to be made with 120 meters of fence. An exterior fence surrounding a rectangle is first constructed and then an interior fence that partitions the rectangle into two equal areas is constructed. What dimensions of the pens will maximize the total area of the two pens?

Exercise 8.2.9 A dog kennel with four pens each of area 7 square meters is to be constructed. An exterior fence surrounding a rectangular area is to be built of fence costing \$20 per meter. That rectangular area is then to be partitioned by three fences that are all parallel to a single side of the original rectangle and using fence that costs \$10 per meter. What dimensions of pens will minimize the cost of fence used?

Exercise 8.2.10 A ladder is to be put against a wall that has a 2 meter tall fence that is 1 meter away from the wall. What is the shortest ladder that will reach from the ground to the wall and go above the fence?

Exercise 8.2.11 a. What is the length of the longest ladder than can be carried horizontally along a 2 meter wide hallway and turned a corner into a 1 meter hallway? Suppose the floor to ceiling height in both hallways is 3 meters. b. What is the longest pipe that can be carried around the corner?

Exercise 8.2.12 A box with a square base and with a top and bottom and a shelf entirely across the interior is to be made. The total surface area of all material is to be 9 m^2. What dimensions of the box will maximize the volume?

Exercise 8.2.13 A rectangular box with square base and with top and bottom and a shelf entirely across the interior is to have 12 m^3 volume. What dimensions of the box will minimize the material used?

Exercise 8.2.14 A box with no top is to be made from a 22 cm by 28 cm piece of card board by cutting squares of equal size from each corner and folding up the 'tabs'. What size of squares should be cut from each corner to make the box of largest volume?

Exercise 8.2.15 A shelter is to be made with a 3 meter by 4 meter canvas sheet. There are equally spaced grommets at one meter intervals along the 4-meter edges of the canvas. Ropes are tied to the four grommets that are one meter from a corner and stretched, so that there is a two-meter by three-meter horizontal sheet with two one-meter by three-meter flaps on two sides. The other two sides are open. The two flaps are to be staked at the corners so that the flaps slope away from the region below the horizontal portion of the canvas. How high should the horizontal section be in order to maximize the volume of the shelter?

Exercise 8.2.16 An orange juice can has volume of 48π cm^3 and has metal ends and cardboard sides. The metal costs 3 times as much as the card board. What dimensions of the can will minimize the cost of material?

Exercise 8.2.17 The 'strength' of a rectangular wood beam with one side vertical is proportional to its width and the square of its depth. A sawyer is to cut a single beam from a 1 meter diameter log. What dimensions should he cut the beam in order to maximize the strength of the beam?

Exercise 8.2.18 A rectangular wood beam with one side vertical has a 'stiffness' that is proportional to its width and the cube of its depth. A sawyer is to cut a single beam from a 1 meter diameter log. What dimensions should he cut the beam in order to maximize the stiffness of the beam?

Exercise 8.2.19 A life guard at a sea shore sees a swimmer in distress 70 meters down the beach and 30 meters from shore. She can run 4 meters/sec and swim 1 meter per second. What path should she follow in order to reach the swimmer in minimum time?

Exercise 8.2.20 You stand on a bluff above a quiet lake and observe the reflection of a mountain top in the lake. Light from the mountain top strikes the lake and is reflected back to your eye, the path followed, by Fermat's hypothesis, being that path that takes the least time. Show that the angle of incidence is equal to the angle of reflection. That is, show that the angle the beam from the mountain top to the point of reflection on the lake makes with the horizontal surface of the lake (the angle of incidence) is equal to the angle the beam from the point of reflection to your eye makes with the horizontal surface of the lake (the angle of reflection). Let v_a denote the velocity of light in air.

Exercise 8.2.21 Two light bulbs of different intensities are a distance, d, apart. At any point, the light intensity from one of the bulbs is proportional to the intensity of the bulb and inversely proportional to the square of the distance from the bulb. Find the point between the two bulbs at which the sum of the intensities of light from the two bulbs is minimum.

Exercise 8.2.22 An equation for continuous logistics population growth is

$$P' = RP\left(1 - \frac{P}{M}\right)$$

where P is population size, R is the low density growth rate, and M is the carrying capacity of the environment. For what value of P will the growth rate, P' be the greatest?

Exercise 8.2.23 Ricker's model for population growth is

$$P' = RPe^{-\frac{P}{\alpha}}$$

where P is population size, R is the low density growth rate, and α reflects the carrying capacity of the environment. For what value of P will the growth rate, P' be greatest?

Exercise 8.2.24 A comet follows the parabolic path, $y = x^2$ and Earth is at (3,8). How close does the comet come to Earth?

Exercise 8.2.25 A tepee is to be covered with 30 buffalo skins. What should be the angle at the base of the tepee that will maximize the volume inside the tepee?

Note: The volume of a right circular cone with base radius r and height h is $\pi r^2 h/3$ and its lateral surface area is $\pi r\sqrt{r^2 + h^2}$.

Exercise 8.2.26 A very challenging exercise. A tepee is to be covered with 30 buffalo skins. The skins are a little longer than the height of the Indians that will be inside and not quite as wide as the height of the Indians. Thus the areas of the skins are approximately the square of the height of the Indians. The Indians wish to maximize the area inside the teepee in which they can walk standing upright. What angle at the base of the teepee will maximize that area?

8.3 Life Sciences Optima

Natural selection constantly optimizes life forms for reproductive success. Consequently, optima are endemic in living organisms and groups of organisms, but they are typically difficult to describe and analyze and vary with the organism. Biology optima are never as simple as the geometry problems of the previous sections, such as, "What is the size of the largest cube that can fit in a sphere?" An otherwise square cell confined to live in a sphere is very likely to become a sphere. Typically the optimum is a balance between opposing requirements as in the simplified model[1] of cell size included here. A few such problems are supplied for your analysis.

Example 8.3.1 Mathematical Model 8.3.1 Consider a bacterium that grows as a sphere, such as *streptococcus*. Its reproductive success is proportional to the energy that it produces.

1. Energy production is proportional to cell volume which provides space for processing and storage of nutrients.

[1] A much more satisfactory model and analysis is presented in Kohei Yoshiyama and Christopher A. Klausmeier, Optimal cell size for resource uptake in fluids: A new facet of resource competition. *American Naturalist* 171 (2008), pp. 59-79

2. Energy production is proportional to the concentration of nutrients inside the cell, which in turn is proportional to the ratio of the surface area of the cell to the volume of the cell.

If we concentrate only on component 1, we might write

$$\text{Energy Production}_1 = A\,V$$

where A is a positive constant. Energy Production$_1$ will be large if V is large and *streptococcus* cell size should increase without bound.

If we concentrate only on component 2, we might write

$$\text{Energy Production}_2 = B_0\,\frac{S}{V} = B\,\frac{1}{\sqrt[3]{V}}$$

where B is a positive constant (for *streptococcus* cells which grow in the shape of a sphere, $S = \sqrt[3]{36\pi}V^{2/3}$). Energy Production$_2$ will be large if V is small and *streptococcus* cell size should decrease – to zero with no nutrient processing organs!

Large cells have large capacity for energy production, but are inefficient because of low nutrient concentration. Small cells have high nutrient concentration, but may have limited capacity to use the nutrients.

Caution: Smoke and mirrors ahead. How do we combine the two expressions of Energy Production? We choose the *harmonic mean* of the two. The harmonic mean of n numbers $\{a_1,\, a_2,\, \cdots,\, a_n\}$ is defined by

$$\text{Harmonic mean} \quad = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots \dfrac{1}{a_n}} \tag{8.5}$$

Explore 8.3.1 The harmonic mean is useful for combining rates. Suppose you travel from city A to city B at rate r mph and travel from B back to A at a rate s mph. Show that the rate of the round trip is the harmonic mean of r and s.

Now lets write the harmonic mean of the two Energy Productions as

$$\text{Energy Production} = \frac{2}{\dfrac{1}{A\,V} + \dfrac{1}{B/V^{1/3}}} = \frac{2A\,V\,B/V^{1/3}}{A\,V + B\,/V^{1/3}}$$

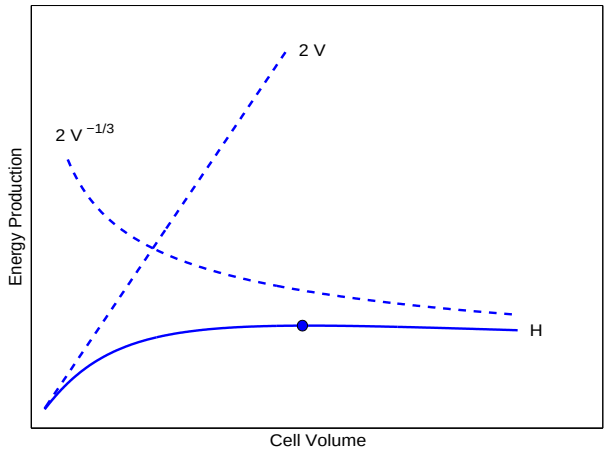
For illustration purposes, suppose $A = B = 1$ and examine the function

$$H(V) = \frac{2V\,V^{-1/3}}{V + V^{-1/3}}.$$

Shown in Figure 8.3.1.1 are the graphs of $y = 2V$, $y = 2V^{-1/3}$ and H. When $V \ll 1$, $V^{-1/3} \gg V$ and the denominator is $\approx V^{-1/3}$ and cancels the factor $V^{-1/3}$ in the numerator. Consequently, when $V \ll 1$, $H(V) \approx 2V$ and close to zero. Similarly, when $V \gg 1$, $H(V) \approx V^{-1/3}$ and is close to zero. There is an intermediate value of V for which H is maximum. You are asked to find that value in Exercise 8.3.1.

We have shown that the competing requirements of large capacity and high nutrient concentrations can be balanced to achieve an intermediate optimum cell size. We are ignorant of the values of A and B and have selected the harmonic mean only as a reasonably unbiased means of combining the two functions.

Figure for Example 8.3.1.1 Graphs of $y = 2V$, $y = 2V^{-1/3}$ and $H(V) = 2V\,V^{-1/3}/(V + V^{-1/3})$.



Exercises for Section 8.3, Life Sciences Optima

Exercise 8.3.1 Find the maximum value of $H(V) = 2V\,V^{-1/3}/(V + V^{-1/3})$ for $V > 0$.

Exercise 8.3.2 The *island body size rule* states that when an ecosystem becomes isolated on an island, say by rising sea levels, the species of large body size tend to evolve to a smaller body size, and species of small body size tend to increase in size. Craig McClain et al[2] found a similar contrast in the sea between shallow water where nutrient levels are high and deep water (depth greater than 200 meters) where nutrient levels are low. Gastropod genera that have large shallow-water species tend to have smaller deep-water representatives; those that have small shallow water species tend to have larger deep-water species.

Suppose the ratio of nutrient conversion to body size *vs* body size is similar to the graph in Figure 8.3.2 and that at high nutrient concentrations, nutrient conversion is not the limiting factor — predation and mate finding, for example, may be more important. Suppose further that a low nutrient concentrations, nutrient conversion becomes more important. How is the graph consistent with the observed differences in body size?

Figure for Exercise 8.3.2 Hypothetical graph of the ratio of nutrient production to gastropod body size *vs* body size.

[2]Craig R. McClain, Alison G. Boyer and Gary Rosenberg, The island rule and the evolution of body size in the deep sea, Journal of Biogeography 33 (2008), pp 1578-1584.

Exercise 8.3.3 Sickle cell anemia is an inherited blood disease in which the body makes sickle-shaped red blood cells. It is caused by a single mutation from glutamic acid to valine at position 6 in the protein Hemoglobin B. The gene for hemoglobin B is on human chromosome 11; a single nucleotide change in the codon for the glutamic acid, GAG, to GTG causes the change to valine. The location of a genetic variation is called a *locus* and the different genetic values (GAG and GTG) at the location are called *alleles*.

People who have GAG on one copy of chromosome 11 and GTG on the other copy do not have sickle cell anemia and have elevated resistance to malaria over those who have GAG on both copies of chromosome 11. Those who have GTG on both copies of chromosome 11 have sickle cell anemia – the hydrophobic valine allows aggregation of hemoglobin molecules within the blood cell causing a sickle-like deformation that does not move easily through blood vessels.

Let A denote presence of GAG and a denote presence of GTG on chromosome 11 and let AA, Aa and aa denote the various presences of those codons on the two chromosomes of a person (either egg or adult), referred to as the genotype of the person (note: $Aa = aA$). It is necessary to assume *non-overlapping generations*, meaning that all are born and grow to sexual maturity, mate, leave offspring and die. Let p and q denote the frequencies of A and a, respectively in a breeding population and P, Q and R denote the frequencies of AA, Aa and aa genotypes in the same population. As an example, in a mating of AA with Aa adults, the chromosome in the fertilized egg (zygote) obtained from AA must be A and the with probability 1/2 the chromosome obtained from Aa will be A and with probability 1/2 will be a. Therefore, the zygote will be AA with probability 1/2 and will be Aa with probability 1/2.

a. Show that $p = P + \frac{1}{2}Q$ and $q = \frac{1}{2}Q + R$.

b. Complete the table showing probabilities of zygote type for the various mating possibilities, the frequencies of the mating possibilities, and the zygote genotype frequencies. Include zeros with the zygote type probabilities but omit the zeros in the zygote genotype frequencies. Random mating assumes that the selection of mating partners is independent of the genotypes of the partners.

Adult Male	×	Female	Zygote types and probabilities			Random mating frequency	Zygote Genotype frequencies		
			AA	Aa	aa		AA	Aa	aa
AA	×	AA	1	0	0	P^2	P^2		
AA	×	Aa	1/2	1/2	0	PQ	$\frac{1}{2}PQ$	$\frac{1}{2}PQ$	
AA	×	aa	0	1	0	PR		PR	
Aa	×	AA							
Aa	×	Aa				QQ	$\frac{1}{4}QQ$	$\frac{1}{2}QQ$	$\frac{1}{4}QQ$
Aa	×	aa				QR		$\frac{1}{2}QR$	$\frac{1}{2}QR$
aa	×	AA				PR		PR	
aa	×	aA							
aa	×	aa							
	Sum					1	Σ_{AA}	Σ_{Aa}	Σ_{aa}

c. When the table is complete, you should see that

$$\begin{aligned}
\Sigma_{Aa} &= \frac{1}{2}PQ + PR + \frac{1}{2}QP + \frac{1}{2}Q^2 + \frac{1}{2}QR + RP + \frac{1}{2}QR \\
&= PQ + 2PR + \frac{1}{2}Q^2 + QR \\
&= 2P\left(\frac{1}{2}Q + R\right) + Q\left(\frac{1}{2}Q + R\right) \\
&= (2P + Q)\left(\frac{1}{2}Q + R\right) \\
&= 2\left(P + \frac{1}{2}Q\right)\left(\frac{1}{2}Q + R\right) \\
&= 2pq
\end{aligned}$$

Show that

$$\Sigma_{AA} = p^2 \qquad \text{and} \qquad \Sigma_{aa} = q^2$$

This means that under the random mating hypothesis, the zygote genotype frequencies are determined by the allele frequencies of the adults. This is referred to as the Hardy-Weinberg theorem. If the probability of an egg growing to adult and contributing to the next generation of eggs is the same for all eggs, independent of genotype, then the allele frequencies are constant after the first generation.

Random mating does not imply the promiscuity that might be imagined. It means that the selection of mating partner is independent of the genotype of the partner. In the United States, blood type would be a random mating locus; seldom does a United States young person inquire about the blood type of an attractive partner. In Japan, however, this seems to be a big deal, to the point that dating services arranging matches to also match blood type. The major histocompatibility complex (MHC) of a young person would seem to be fairly neutral; few people even know their MHC type. It has been demonstrated, however, that young women are repulsed by the smell of men of the same MHC type as their own.

d. Suppose that because of malaria, an AA type egg, either male or female, has probability 0.8 of reaching maturity and mating and because of sickle cell anemia an aa type has only 0.2 probability of mating, but that an Aa type has 1.0 probability of mating.

Then the distribution of genotypes in the egg and the mating populations will be

Genotype	AA	Aa	aa	
Egg	p^2	$2pq$	q^2	
Adult	$0.8p^2/F$	$2pq/F$	$0.2q^2/F$	where $F = 0.8p^2 + 2pq + 0.2q^2$

Find the frequency of A in the adult population. Note: This will also be the frequency of A in the next egg population.

e. F is called the *fitness* of the population and because $p + q = 1$

$$F = F(p) = 0.8p^2 + 2p(1-p) + 0.2(1-p)^2$$

You will be asked in Exercise 11.4.4 to show that during succeeding generations, allele frequency, p, moves toward the value p that maximizes F.

Find the value $\hat{p}$ of p that maximizes $F(p)$.

f. Show that if the frequency of A in an egg generation is $\hat{p}$, then the frequency of A in the next egg generation will also be $\hat{p}$.

Exercise 8.3.4 Consider two alleles A and a at a locus of a random mating population and the fractions of AA, Aa and aa zygotes that reach maturity and mate are in the ratio $1 + s_1 : 1 : 1 + s_2$ where s_1 and s_2 can be positive, negative, or zero. The fitness function is

$$F(p) = (1+s_1)p^2 + 2pq + (1+s_2)q^2 = (1+s_1)p^2 + 2p(1-p) + (1+s_2)(1-p)^2$$

where p and q are the frequencies of A and a among the zygotes.

a. Sketch the graphs of F and find the values $\hat{p}$ of p in [0,1] for which $F(p)$ is a maximum for

1. $s_1 = 0.2$ and $s_2 = -0.3$.
2. $s_1 = 0$ and $s_2 = -0.2$.
3. $s_1 = -0.2$ and $s_2 = -0.3$.
4. $s_1 = 0.2$ and $s_2 = -0.2$.

b. Show that if $s_1 + s_2 \neq 0$ and $0 \leq s_1/(s_1 + s_2) \leq 1$, then the value $\hat{p}$ of p in [0,1] for which $F(p)$ is a maximum is $s_1/(s_1 + s_2)$.

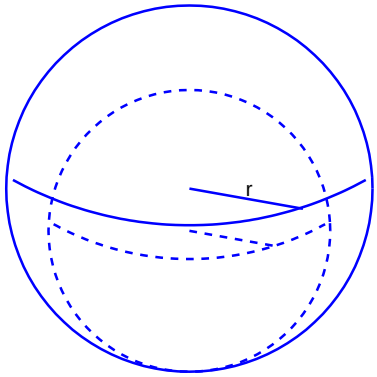
Exercise 8.3.5 Read the amazing story of the kakapo at

http://evolution.berkeley.edu/evolibrary/news/060401_kakapo and the Wikipedia entry on kakapo. You might assume that in really good times, only males would be born and in really bad times only females would be born. Why is it that both males and females are retained in the natural population?

8.4 Related Rates

Typical Problem and Solution:

Air is pumped into a spherical balloon at the rate of 6 liters per minute. At what rate is the radius of the balloon expanding when the volume of the balloon is 36 liters?



Two variables, volume V and radius r, **are intrinsically related** by the relation

$$V = \frac{4}{3}\pi r^3.$$

One variable (V) is changing with time. The other variable (r) must also change. Almost always the chain rule is used.

$$V'(t) \;=\; \frac{4}{3}\pi\left[(r(t))^3\right]' \;=\; \frac{4}{3}\pi\, 3\,(r(t))^2\; r'(t) \;=\; 4\pi\,(r(t))^2\; r'(t).$$

Evaluation of variables. For all time, $V' = 6$ liters/minute $= 6000$ cm^3/min. **At the given instant**, $V = 36$ liters $= 36000$ cm^3, and

$$\text{from} \qquad 36000 = \frac{4}{3}\pi r^3 \qquad \text{we get} \qquad r = \frac{30}{\sqrt[3]{\pi}} \quad \text{cm.}$$

Then we write

$$6000 = 4\pi\left(\frac{30}{\sqrt[3]{\pi}}\right)^2 r' \qquad \text{and compute} \qquad r' = \frac{10}{12\sqrt[3]{\pi}} \doteq 0.57 \quad \frac{\text{cm}}{\text{min}}.$$

The solutions to all of the problems of this section follow the pattern of the solution just illustrated. In each of the problems, there are two or more variables intrinsically related (by one or more equations); the variables are changing with time; at a given instant, values of some of the variables and some of the rates of change are given and the problem is to evaluate the remaining variables and rates of change. Some more examples follow.

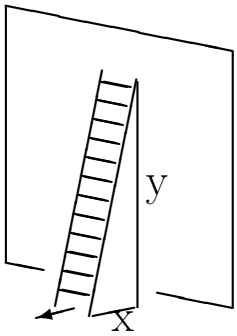
Example 8.4.1 A 10 meter ladder leans against a wall. The foot of the ladder slips horizontally at the rate of 1 meter per minute. At what rate does the top of the ladder descend when the top is 6 meters from the ground?

Draw a picture. See Figure 8.4.1.1. Let x be the distance from the wall to the foot of the ladder and let y be the distance from the ground to the top of the ladder. We are asked to find y' at a certain instant. Because the top of the ladder is descending, we expect our answer (y') to be negative.

x and y are intrinsically related.

$$x^2 + y^2 = 10^2$$

Figure for Example 8.4.1.1 Ladder leaning against a wall and sliding down the wall.



x and y are changing with time, and

$$(x(t))^2 + (y(t))^2 = 10^2$$

Differentiate with respect to t (use the chain rule):

$$2x(t)x'(t) + 2y(t)y'(t) = 0$$

'When' The instant, t_0, specified in the problem is defined by $y = 6$. At that instant $x^2 + 6^2 = 10^2$, so that $x = 8$. The problem also specifies $x'(t) = 1$ for all t. The actual value of t_0 is not required; we know that $x(t_0) = 8$, $y(t_0) = 6$, and $x'(t_0) = 1$. Therefore

$$(2 \times 8 \times 1) + (2 \times 6 \times y') = 0 \qquad \text{and} \qquad y' = -\frac{4}{3} \quad \text{meter/min}$$

Example 8.4.2 In an aqueous solution the concentrations of H^+ and OH^- ions satisfy

$$[H^+]\,[OH^-] = 10^{-14}$$

If in a certain lake the pH is 6 ($[H^+]$=10^{-6}) and is decreasing at the rate of 0.1 pH units per year, at what rate is the hydroxyl concentration, $[OH^-]$, increasing?

Solution: It is useful to take the log_{10} of the two sides of the previous equation:

$$\log_{10}([H^+]\,[OH^-]) = \log_{10} 10^{-14}$$

$$\log_{10}[H^+] + \log_{10}[OH^-] = -14$$

$$\begin{aligned} \text{pH} + \log_{10}[\text{OH}^-] &= -14 \\ \text{pH} + \frac{\ln[\text{OH}^-]}{\ln 10} &= -14 \end{aligned}$$

Now we take derivatives of each term:

$$\begin{aligned} [\text{pH}]' + \left[\frac{\ln[\text{OH}^-]}{\ln 10}\right]' &= [-14]' \\ [\text{pH}]' + \frac{1}{\ln 10}\frac{1}{[\text{OH}^-]}[\text{OH}^-]' &= 0 \end{aligned}$$

At the given instant, ($[\text{H}^+]=10^{-6}$) so that

$$[10^{-6}]\,[\text{OH}^-] = 10^{-14} \qquad \text{and} \qquad [\text{OH}^-] = 10^{-8}$$

Also, $[\text{pH}]' = -0.1$. Therefore

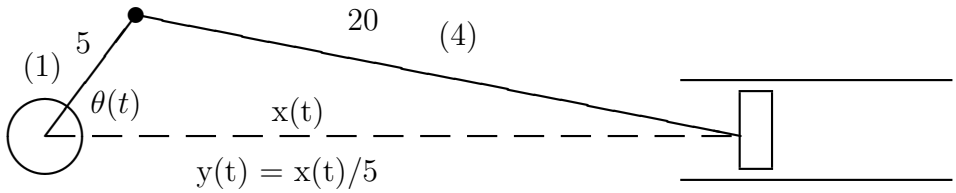
$$\begin{aligned} [\text{pH}]' + \frac{1}{\ln 10}\frac{1}{[\text{OH}^-]}[[\text{OH}^-]]' &= 0 \\ = -0.1 + \frac{1}{\ln 10}\frac{1}{10^{-8}}[[\text{OH}^-]]' &= 0 \\ [[\text{OH}^-]]' &= 0.1 \times \ln 10 \times 10^{-8} \doteq 0.23 \times 10^{-8}\frac{\text{ions}}{\text{liter}}\text{per second} \end{aligned}$$

Example 8.4.3 In Exercise 7.3.7 we discussed the following problem.

A piston is linked by a 20 cm tie rod to a crank shaft which has a 5 cm radius of motion (see Figure 8.4.3.3). Let $x(t)$ be the distance from the rotation center of the crank shaft to the end of the tie rod and $\theta(t)$ be the rotation angle of the crank shaft, measured from the line through the centers of the crank shaft and piston. The crank shaft is rotating at 100 revolutions per minute. The goal is to locate the point of the cylinder at which the piston speed is the greatest.

We present an analysis of this problem that illustrates some common steps in such problems.

Figure for Example 8.4.3.3 Crank shaft, tie rod, and piston (or fly rod, fly line, and fish).



Step 1. We rescale the problem by dividing dimensions by 5. We are more accurate manipulating 1 and 5 than manipulating 4 and 20. Thus the parenthetical entries (1), (4), and ($y(t) = x(t)/5$) in Figure 8.4.3.3. Observe that $3 \le y \le 5$.

By the Law of Cosines,

$$4^2 = 1^2 + y^2(t) - 2 \times 1 \times y(t) \times \cos\theta(t). \tag{8.6}$$

Differentiation of this equation yields

$$0 = 0 + 2y(t)y'(t) - 2y'(t)\cos\theta(t) - 2y(t)(-\sin\theta(t))\theta'(t)$$

$$\frac{y'(t)}{\theta'(t)} = \frac{y'(t)}{200\pi} = \frac{y(t)\sin\theta(t)}{\cos\theta(t) - y(t)} \tag{8.7}$$

(100 rotations per minute implies that $\theta' = 200\pi$) From Equation 8.6

$$\cos\theta = \frac{y^2 - 15}{2y} \qquad 3 \le y \le 5,$$

and from $\sin^2\theta + \cos^2\theta = 1$ we find

$$\sin\theta = \pm\sqrt{1 - \left(\frac{y^2 - 15}{2y}\right)^2} = \pm\frac{\sqrt{4y^2 - (y^2 - 15)^2}}{2y} \qquad 3 \le y \le 5.$$

Now use Equation 8.7 to write $y'(t)$ in terms of $y(t)$, and suppress the (t).

$$\frac{y'}{200\pi} = \pm\frac{y\dfrac{\sqrt{4y^2 - (y^2 - 15)^2}}{2y}}{\dfrac{y^2 - 15}{2y} - y} = \mp\frac{y\sqrt{-y^4 + 34y^2 - 225}}{y^2 + 15}$$

Note the change from $\pm$ to $\mp$. Now to find the value of y for which y' is maximum, we might wish to compute $[\,y'\,]' = \dfrac{d\,(y'\,)}{dy}$ and find where it is zero. Computation of $[\,y'\,]'$ from the previous equation appears to be messy.

Step 2. Instead we note that where y' is a maximum, $[\,y'\,]^2$ is also a maximum and we look at

$$z = [\frac{y'}{200\pi}]^2 = \left(\mp\frac{y\sqrt{-y^4 + 34y^2 - 225}}{y^2 + 15}\right)^2 = \frac{-y^6 + 34y^4 - 225y^2}{(y^2 + 15)^2}$$

This ratio of polynomials is easier to differentiate.

$$\begin{aligned}\frac{dz}{dy} &= \frac{(y^2 + 15)^2\left[-y^6 + 34y^4 - 225y^2\right]' - \left(-y^6 + 34y^4 - 225y^2\right)\left[(y^2 + 15)^2\right]'}{(y^2 + 15)^4} \\ &= (-2y)\frac{y^6 + 45y^4 - 1245y^2 + 3375}{(y^2 + 15)^3}, \qquad 3 \le y \le 5\end{aligned}$$

Explore 8.4.1 You should check the previous computation.

Step 3. We need to know the value of y for which $z' = 0$. That will only occur when $y = 0$ (outside of $3 \leq y \leq 5$) and when the numerator, $y^6 + 45y^4 - 1245y^2 + 3375 = 0$. It is helpful that the denominator is never zero. So solve

$$y^6 + 45y^4 - 1245y^2 + 3375 = 0$$

Clever Step 4. Only even powers of y occur in this sixth degree polynomial. Let $w = y^2$ and solve

$$w^3 + 45w^2 - 1245w + 3375 = 0$$

Just as the roots to a quadratic equation are given by the formula $\frac{-b \pm \sqrt{b^2 - 4zc}}{2a}$, there are formulas for the roots to cubics, but they are seldom useful. Instead we use a polynomial solver on a calculator and compute the roots

$$w_1 = -64.96412 \qquad w_2 = 16.88784 \qquad w_3 = 3.07628$$

The root $w_2 = 16.88784$ is relevant to our problem. We remember that $w = y^2$ and $x = 5y$ and compute

$$y = \sqrt{16.88784} = 4.10948 \qquad x = 5y = 5 \times 4.10948 = 20.55$$

From the dimensions of the crank shaft and tie rod, x is between 15 and 25. The highest speed of the piston occurs at x = 20.55, to the right of midpoint of its motion.

Exercises for Section 8.4 Related Rates

Exercise 8.4.1 A pebble dropped into a pond makes a circular wave that travels outward at a rate 0.4 meters per second. At what rate is the area of the circle increasing 2 seconds after the pebble strikes the pond?

Exercise 8.4.2 A boat is pulled toward a dock by a rope through a pulley that is 5 meters above the water. The rope is being pulled at a constant rate of 15 meters per minute. At the instant when the boat is 12 meters from the dock, how fast is the boat approaching the dock?

Exercise 8.4.3 Corn is conveyed up a belt at the rate of 10 m^3 per minute and dropped onto a conical pile. The height of the pile is equal to twice its radius. At what rate is the top of the pile increasing when the volume of the pile is 1000 m^3? (Note: Volume of a cone is $\frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is the height of the cone.)

Exercise 8.4.4 A light house beacon makes one revolution every two minutes and shines a beam on a straight shore that is one kilometer from the light house. How fast is the beam of light moving along the shore when it is pointing toward the point of the shore closest to the light house? How fast is the beam of light moving along the shore when it is pointing toward a point that is one kilometer from the closest point of the shore to the light house?

Exercise 8.4.5 Two planes are traveling toward an airport. One plane is flying at 450 kilometers per hour from a position due North of the airport and the other plane is traveling at 300 kilometers per hour from a position due East of the airport. At what rate is the distance between the planes decreasing when the first plane is 10 km North of the airport and the second plane is 5 km East of the airport?

Exercise 8.4.6 Two planes are traveling toward an airport. One plane is flying at 450 kilometers per hour from a position due North of the airport and the other plane is traveling at 300 kilometers per hour from a position due East of the airport. Assuming the planes continue on a path over and beyond the airport, what will be the shortest distance between them?

Exercise 8.4.7 A woman 1.7 meters tall walks under a street light that is 10 meters above the ground. She is walking in a straight line at a rate of 30 meters per minute. How fast is the tip of her shadow moving when she is 5 meters beyond the street light?

Exercise 8.4.8 A gas in a perfectly insulated container and at constant temperature satisfies the gas law $pv^{1.4} =$ constant. When the pressure is 20 Newtons per cm^2 the volume is 3 liters. The gas is being compressed at the rate of 0.2 liters per minute. How fast is the pressure changing at the instant at which the volume is 2 liters?

Exercise 8.4.9 Find x' at the instant that $x = 2$ if $y' = 5$ and

a. $xy + y = 9$ b. $xe^y = 2e$ c. $x^2y + xy^2 = 2$

Exercise 8.4.10 A point is moving along the parabola $y = x^2$ and its y coordinate increases at a constant rate of 2. At what rate is the distance from the point to (4,0) changing at the instant at which $x = 2$?

Exercise 8.4.11 Accept as true that if a particle moves along a graph of $y = f(x)$ then the speed of the particle along the graph is $\sqrt{(x')^2 + (y')^2}$.

(The notation means that x' is the rate at which the x-coordinate is increasing, y' denotes the rate at which the y-coordinate is increasing and the speed of the particle is the rate at which the particle is moving along the curve.)

a. Suppose a particle moves along the graph of $y = x^2$ so that $y' = 2$. What is x' when $y = 2$?

b. Suppose a particle moves along the circle $x^2 + y^2 = 1$ so that its speed is 2π. Find x' when $x = 1$.

c. Suppose a particle moves along the circle $x^2 + y^2 = 1$ so that its speed is 2π. Find x' when $x = 0$. (Two answers.)

8.5 Finding roots to $f(x) = 0$.

It is unfortunately frequent to encounter simple equations such as $xe^{-x} = a$ for which no amount of algebraic manipulation yields a solution such as $x =$ an expression in a. Three numerical schemes are commonly used to solve specific instances of such equations: iteration, the bisection method, and Newton's method.

Example 8.5.1 *Problem.* Solve $xe^{-x} = a$ for $a = 0.2$ and $a = 2$.

Solution. Graphs help explore the problem. Shown in Figure 8.10A is a graph of $y = xe^{-x}$. It is apparent that the highest point of the graph has y-coordinate about $y = 0.37$. We are relieved of solving $xe^{-x} = 2$; there is no solution. The dashed line at $y = 0.2$ does intersect the graph of $y = xe^{-x}$; at two points so there are two solutions to $xe^{-x} = 0.2$, r_1 at about $x = 0.3$ and r_2 at about $x = 2.5$.

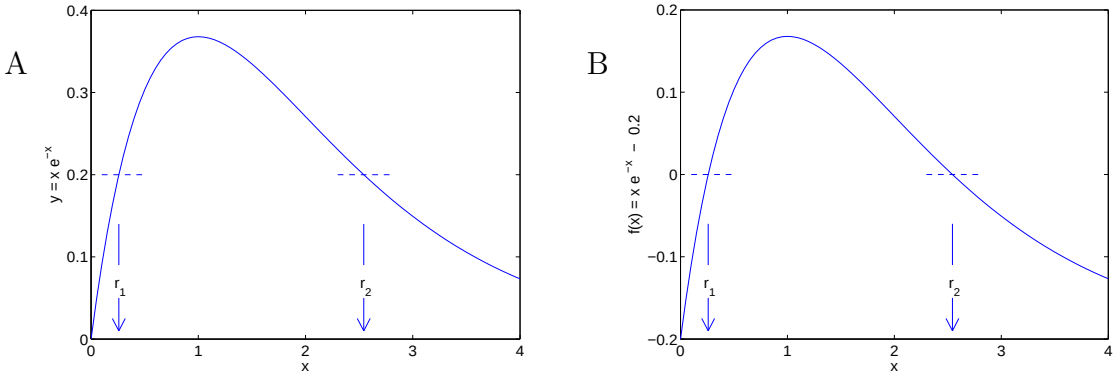


Figure 8.10: A. Graph of $y = xe^{-x}$. B.Graph of $y = xe^{-x} - 0.2$.

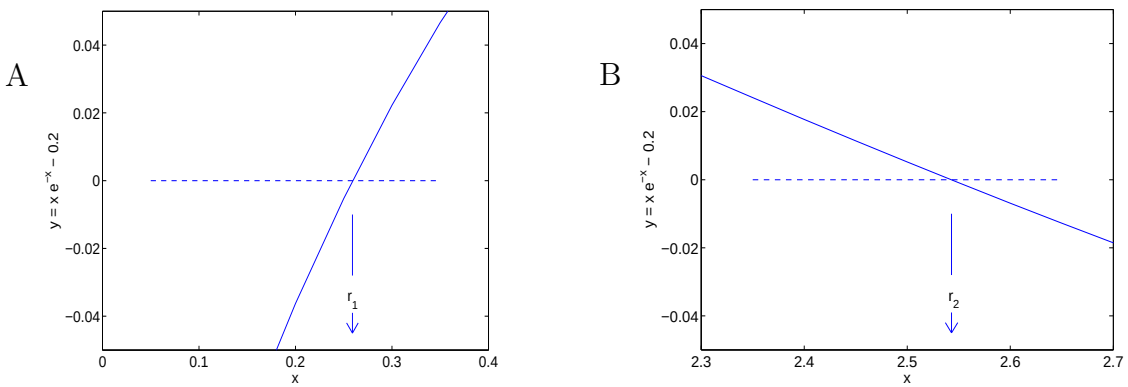


Figure 8.11: A. Graph of $f(x) = xe^{-x} - 0.2$ on $0 \le x \le 0.4$. B. Graph of $f(x) = xe^{-x} - 0.2$ on $2.35 \le x \le 2.37$.

Iteration. There is a simple scheme that sometimes works. $xe^{-x} = 0.2$ is equivalent to $x = 0.2e^{x}$. We can guess $x_0 = 0.3$ as an estimate of r_1, and compute a new estimate $x_1 = 0.2e^{x_0} = 0.2e^{0.3} = 0.2700$. Then compute $x_2 = 0.2e^{x_1} = 0.2e^{0.2700} = 0.2620$. Continue and we find that

$$x_3 = 0.2599, \qquad x_4 = 0.2594, \qquad x_5 = 0.2592, \qquad x_6 = 0.2592.$$

Correct to four decimal places, $x = 0.2592$ is a solution to $xe^{-x} = 0.2$. This scheme fails for finding the root r_2 near $x = 2.5$ and an alternate scheme is suggested in Exercise 8.5.1. Conditions for iteration to succeed are given in Theorem 11.4.1.

Another approach. It is common to modify the problem and examine the function,

$$f(x) = x \times e^{-x} - 0.2$$

the graph of which is shown in Figure 8.11B. We are searching for points where the graph of f crosses the x-axis (that is, $f(x) = 0$).

Bisection. A rather plodding, but sure way to find r_1 is called the *bisection method.* We observe from Figure 8.11A that r_1 appears to be between 0.2 and 0.3. We check

$$f(0.2) = .2 \times e^{-.2} - .2 = -0.03625 \qquad \text{and} \qquad f(0.3) = .3 \times e^{-.3} - .2 = 0.22245$$

Because $f(0.2)$ is negative and $f(0.3)$ is positive, we think f should be zero somewhere between 0.2 and 0.3.

Next we check the value of f at the midpoint of [0.2, 0.3] and

$$f(0.25) = -0.0052998$$

Because $f(0.25)$ is negative and $f(0.3)$ is positive, we think f should be zero somewhere between 0.25 and 0.3.

Now we check the value of f at the midpoint of [0.25, 0.3] and

$$f(0.275) = 0.0088823$$

Because $f(0.25)$ is negative and $f(0.275)$ is positive, we think f should be zero somewhere between 0.25 and 0.275.

Obviously this can be continued as long as one has patience, and the interval containing the root decreases in length by a factor of 0.5 each step. Computation of seven decimal places in $r_1 = 0.2591711$ requires

$$0.1 \times 0.5^n < 0.5 \times 10{-}8 \qquad \text{which implies that} \qquad n > 24.25$$

or 25 steps.

Newton's Method. It may not be a surprise to find that Isaac Newton, in the days of quill and parchment and look up table for exponential and trigonometric functions, developed a very efficient method for finding roots to equations. We suppose, as before, that a function, f, is defined on an interval, $[a, b]$, that for some number, r, in $[a, b]$ $f(r) = 0$, and we are to compute an approximate value of r. It is also necessary to know the derivative, f' of f (which Newton was also good at).

Newton began with an approximate value x_0 to r. Then he found the equation of the tangent to the graph of f at the point, $(x_0, f(x_0))$. He reasoned that the tangent was close to the graph of f and should cross the horizontal axis at a number, x_1, close to r, where the graph of f crosses the axis. x_1 is easy to compute, and is an approximate value to r. Newton repeated the process as necessary, but we will see that not many repetitions are required.

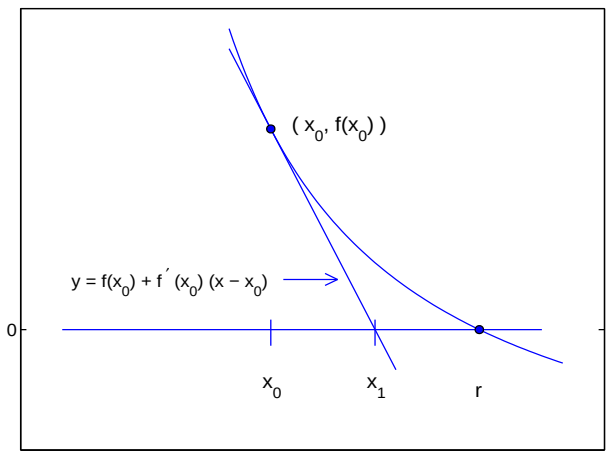


Figure 8.12: Diagram of Newton's root finding method.

An equation of the tangent to the graph of f at the point $(x_0, f(x_0))$ is

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0) \qquad \textbf{Equation of Tangent}$$

The tangent crosses the horizontal axis at $(x_1, 0)$. By substitution,

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$

We solve for x_1 and find

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If we repeat this process, we will find a next approximation, x_2 to r, where

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The general formula is

$$x_0 \quad \text{given} \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \cdots$$

In the special case,

$$f(x) = x \times e^{-x} - 0.2 \qquad f'(x) = e^{-x} - xe^{-x}$$

$$\text{so that} \qquad x_{n+1} = x_n - \frac{x_n \times e^{-x_n} - 0.2}{e^{-x_n} - x_n \times e^{-x_n}}$$

We choose $x_0 = 0.3$. Then

$$x_1 = 0.3 - \frac{0.3 \times e^{-0.3} - 0.2}{e^{-0.3} - 0.3 \times e^{-0.3}} = 0.25710251645$$

$$x_2 = 0.25710251645 - \frac{0.25710251645 \times e^{-0.25710251645} - 0.2}{e^{-0.25710251645} - 0.25710251645 \times e^{-0.25710251645}} = 0.25916608777$$

A summary of the computations is shown in Table 8.1. The remarkable thing is that after only four iterations, we obtain 11 digit accuracy. It is typical that the number of correct digits doubles on each step (underlined digits). By contrast, the bisection gets one new correct digit in roughly three steps. The bisection method requires 38 steps for comparable accuracy.

It would not have escaped Newton that the iteration

$$x_{n+1} = x_n - \frac{x_n \times e^{-x_n} - 0.2}{e^{-x_n} - x_n \times e^{-x_n}}$$

can be simplified to

$$x_{n+1} = x_n - \frac{x_n - 0.2e^{x_n}}{1 - x_n}$$

The 12 digits shown appear a bit tedious, but your calculator makes it easy. Newton's method is a simple iteration and can be computed using the previous answer key, ANS, on your calculator. Keystrokes for the simplified form and results are shown in Table 8.1.

Table 8.1: Calculator keystrokes to compute Newton iterates to solve $xe^{-x} - 0.2 = 0$. The underlined digits are accurate and the number of accurate digits doubles with each iteration.

Keystroke	Keystroke	Display
0.3	ENTER	0.30000000000
ANS - (ANS - 0.2 × e^ANS)/(1 - ANS)	ENTER	0.<u>25</u>710251645
	ENTER	0.<u>2591</u>6608777
	ENTER	0.<u>25917110</u>179
	ENTER	0.<u>25917110182</u>

Newton's method to solve $f(x) = 0$ requires three things. First it requires the derivative of f, but there is an alternate method that uses an average rate of change rather than the rate of change and works almost as well. Secondly, a good approximation to the root is required, and there is no substitute for that. Finally, in order to eliminate some unpleasant pathology, it is sufficient to assume that f, f' and f'' are continuous and that

$$\left| \frac{f(s) \times f''(s)}{(f'(s))^2} \right| \quad < \quad 1$$

These conditions are often met.

Exercises for Section 8.5, Finding roots to $f(x) = 0$.

Exercise 8.5.1 Try to solve $xe^{-x} = 0.2$ by iteration of $x_{n+1} = 0.2e^{x_n}$ beginning with $x_0 = 2.5$. This is easily done with your calculator and the ANS key. Enter 2.5. Then type $0.2 \times e^{\text{ANS}}$, ENTER, ENTER, $\cdots$. Describe the result. Try again with $x_0 = 2.6$. Describe the result.

An alternate procedure is to solve for x as follows.

$$\begin{aligned} xe^{-x} &= 0.2 \\ e^{-x} &= \frac{0.2}{x} \\ -x &= \ln \frac{0.2}{x} \\ x &= -\ln \frac{0.2}{x} \quad = \quad \ln(5 \times x) \end{aligned}$$

Now let $x_0 = 2.5$ and iterate $x_{n+1} = \ln(5 \times x_n)$ and describe your results.

Exercise 8.5.2 Begin with $x_0 = 0.25917110182$ and compute the iteration steps ($x_{n+1} = x_n e^{-x_n} - 0.2$). Describe your results.

Exercise 8.5.3 Refer to Figure 8.11B and use the bisection method to find the root to $f(x) = xe^{-x} - 0.2$ near $x = 2.5$.

Exercise 8.5.4 Use three steps of both the bisection method and Newton's method to find the a value s in the indicated interval for which $f(s) = 0$ for

a.	$f(x) = x^2 - 2$	$[1, 2]$	b.	$f(x) = x^3 - 5$	$[1, 2]$
c.	$f(x) = x^2 + x - 1$	$[0, 2]$	d.	$f(x) = (x - \sqrt{2})^{1/3}$	$[0, 1]$

Exercise 8.5.5 Suppose you are trying to solve $f(x) = 0$ and for your first guess, x_0, $f'(x_0) = 0$. Remember that x_1 is defined to be the point where the tangent to f at $(x_0, f(x_0))$ intersects the X-axis. What, if anything, is x_1 in this case?

Exercise 8.5.6 (A common example.) Use Newton's Method to find the root of

$$f(x) = x^{\frac{1}{3}}$$

(Overlook the fact that the root is obviously 0!) Show that if $x_0 = 1$ the successive 'approximations' are 1, -2, 4, -8, 10, $\cdots$.

Exercise 8.5.7 Ricker's equation for population growth with proportional harvest is presented in Exercise 11.3.4 as

$$P_{t+1} - P_t = \alpha \times P_t \times e^{-P_t/\beta} - R \times P_t$$

If a fixed number is harvested each time period, the equation becomes

$$P_{t+1} - P_t = \alpha \times P_t \times e^{-P_t/\beta} - H$$

For the parameter values $\alpha = 1.2$, $\beta = 3$ and $H = 0.1$, calculate the positive equilibrium value of P_t.

8.6 Harvesting of whales.

The sei whale (pronounced 'say') is a well studied example of over exploitation of a natural resource. Because they were fast swimmers, not often found near shore, and usually sank when killed, they were not hunted until modern methods of hunting and processing at sea were developed. Reasonably accurate records of sei whale harvest have been kept. "$\cdots$sei whale catches increased rapidly in the late 1950s and early 1960s (Mizroch et al., 1984c). The catch peaked in 1964 at over 20,000 sei whales, but by 1976 this number dropped to below 2,000 and the species received IWC protection in 1977."
http://spo.nwr.noaa.gov/mfr611/mfr6116.pdf

Data shown in Figure 8.13 are from Joseph Horwood, The Sei Whale: Population Biology, Ecology & Management, Croom Helm, London, 1987.

The International Convention for Regulating Whaling was convened in 1946 and gradually became a force so that by 1963/64 effective limits on catches of blue and fin whales were in place because of depletion of those populations. As is apparent from the data, whalers turned to the sei whale in 1964-65, catching 22,205 southern hemisphere sei whales. Catches declined in the years 1965-1979 despite continued effort to harvest them, indicating depletion of the southern hemisphere stock. A moratorium on sei whale harvest was established in 1979. The **Maximum Sustainable Yield** (MSY) is an important concept in population biology, being an amount that can be harvested without eliminating the population.

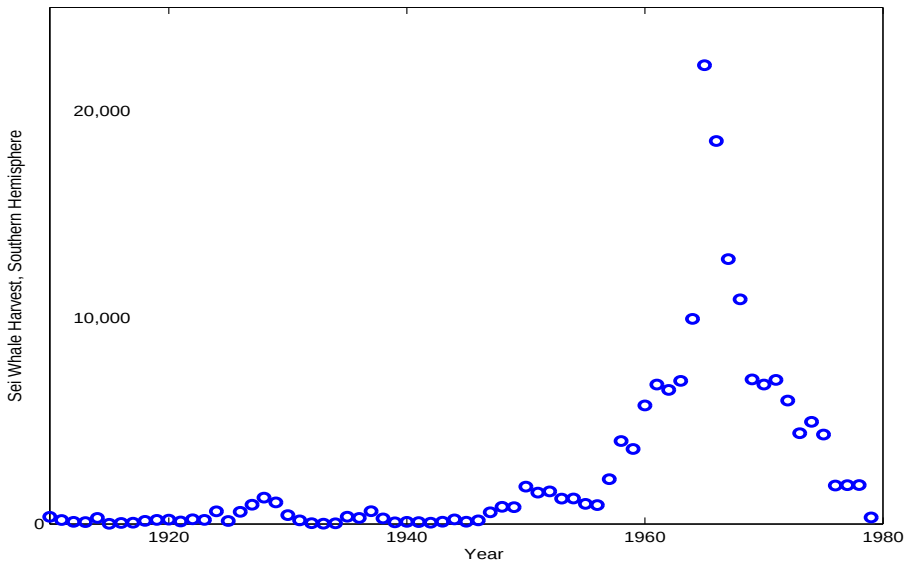


Figure 8.13: Harvest of Sei whales, southern hemisphere, 1910 - 1979.

Explore 8.6.1 Would you estimate that the southern hemisphere sei whale would sustain a harvest of 6000 whales per year?

The actual size of a whale population is difficult to measure. A standard technique is the 'Mark and Recapture Method', in which a number, N, of the whales is marked at a certain time, and during a subsequent time interval the number m of marked whales among a total of M whales sited is recorded.

Exercise 8.6.1 Suppose 100 fish in a lake are caught, marked, and returned to the lake. Suppose that ten days later 100 fish are caught, among which 5 were marked. How many fish would you estimate were in the lake?

Another method to estimate population size is the 'Catch per Unit Effort Method', based on the number of whale caught per day of hunting. As the population decreases, the catch per day of hunting decreases.

In the *Report of the International Whaling Commission (1978)*, J. R. Beddington refers to the following model of Sei whale populations.

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t \tag{8.8}$$

N_t, N_{t+1} and N_{t-8} represent the adult female whale population subjected to whale harvesting in years t, $t+1$, and $t-8$, respectively. C_t is the number of female whales harvested in year t. There is an assumption that whales become subject to harvesting the same year that they reach sexual maturity and are able to reproduce, at eight years of age. The whales of age less than 8 years are not included in N_t. N_* is the number of female whales that the environment would support with no harvesting taking place.

Exercise 8.6.2 Show that if there is no harvest ($C_t = 0$) and both N_t and N_{t-8} are equal to N_* then $N_{t+1} = N_*$.

If we divide all terms of Equation 8.8 by N_*, we get

$$\frac{N_{t+1}}{N_*} = 0.94\frac{N_t}{N_*} + \frac{N_{t-8}}{N_*}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94\frac{C_t}{N_*}$$

We might define new variables, $D_t = \frac{N_t}{N_*}$ and $E_t = \frac{C_t}{N_*}$ and have the equation

$$D_{t+1} = 0.94D_t + D_{t-8}\left[0.06 + 0.0567\left\{1 - D_{t-8}^{2.39}\right\}\right] - 0.94E_t \tag{8.9}$$

Equation 8.9 is simpler by one parameter (N_*) than Equation 8.8 and yet illustrates the same dynamical properties. Rather than use new variables, it is customary to simply rewrite Equation 8.8 with new interpretations of N_t and C_t and obtain

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - N_{t-8}^{2.39}\right\}\right] - 0.94C_t \tag{8.10}$$

N_t now is a fraction of N_*, the number supported without harvest, and C_t is a fraction of N_* that is harvested.

Equation 8.8 says that the population in year $t+1$ is affected by three things: the number of female whales in the previous year (N_t), the "recruitment" of eight year old female whales into the population subject to harvest, and the harvest during the previous year (C_t).

Exercise 8.6.3 a. Suppose $N_* = 250,000$ and a quota of 500 female whales harvested each year is established. Change both Equations 8.8 and 8.10 to reflect these parameter values.

b. Suppose 2% of the adult female whale population is harvested each year. Change both Equations 8.8 and 8.10 to reflect this procedure.

c. What is the meaning of 0.94 at the two places it enters Equation 8.10?

The term

$$N_{t-8}\left[0.06 + 0.0567\left\{1 - (N_{t-8})^{2.39}\right\}\right]$$

represents the number of eight year old females recruited into the adult population in year, t. The factor,

$$\left[0.06 + 0.0567\left\{1 - (N_{t-8})^{2.39}\right\}\right]$$

represents the fecundity of the females in year $t-8$. It has been observed that as whale numbers decrease, the fecundity increases. This term was empirically determined by Beddington.

Exercise 8.6.4 Draw the graph of fecundity *vs* N_{t-8}.

Exercise 8.6.5 Suppose harvest level, C_t is set at a constant level, C for a number of years. Then the whale population should reach an equilibrium level, N_e, and approximately

$$N_{t+1} = N_t = N_{t-8} = N_e$$

We would like to have N_e as a function of the harvest level, C. This is a little messy (actually quite a bit messy), but it is rather easy to write the inverse function, C as a function of N_e.

a. Let $C_t = C$ and $N_{t+1} = N_t = N_{t-8} = N_e$ in Equation 8.10 and solve the resulting equation for C. Draw the graph of C *vs* N_e.

b. Find a point on the graph of $C(N_e)$ at which the tangent to the graph is horizontal.

You may wish to graph $C(N_e)$ on your calculator and use FMAX.

c. If you did not do it in the previous step, compute $C'(N_e) = \frac{dC}{dN_e}$ and find a value of N_e for which $C'(N_e)$ is zero.

d. Give an interpretation of the point $(N_e, C(N_e)) = (0.60001, 0.025507)$. What do you suppose happens if the constant harvest level is set to $C = 0.03$?

e. N_* for the southern hemisphere sei whale has been estimated to be 250,000. If N_* is 250,000, what is the maximum harvest that will lead to equilibrium? Ans: 6376.75. This suggests that the maximum supportable yield of the southern hemisphere sei whale is about 6000 whales per year.

Exercise 8.6.6 In this problem, you will gain some experience with the solution to Equation 8.10 for several values of the parameters by computing the solutions on your calculator. You should record the behavior of the solutions.

On your calculator, press PRGM EDIT NAME = (TYPE IN SEI) ENTER. Then type in the following program.

```
:PROGRAM:SEI
:0.0255 -> C                  | -> is the STO-> key
:0.61  -> S                   | S is the initial population size.
:9 -> dim N                   | Store population sizes N_(t-8),
:For(K,1,9):S->N(K):End       | N_(t-7),...,N_t in N(1),N(2),...,N(9)
:For(I,1,30)                  |
   :For(J,1,5)                | Display only every fifth iterate.
      :NEW = 0.94*N(9) + N(1)*(0.06 + 0.0567*(1-N(1)^2.39)) - 0.94*C
      For(K,1,8):N(K+1)->N(K):End
      :NEW -> N(9)
   :END
   :PAUSE NEW
:END
```

a. Press EXIT Type SEI Press ENTER **repeatedly**. What happens to the whale population?

b. Change 0.61 → S to 0.59 → S, and run again. What happens to the whale population?

c. Change 0.0255 → C to 0.023 → C and run again. What happens to the whale population?

d. Find the values of N_e that correspond to $C = 0.023$. Suggestion: a. Iterate $x_0 = 0.5$, $x_{n+1} = 0.3813 + x_n^{3.39}$. b. Iterate $x_0 = 0.5$, $x_{n+1} = (x_n - 0.3813)^{(1/3.39)}$.

e. Use 0.0230 → C and 0.43 → S and run again. What happens to the whale population?

f. Change 0.43 → S to 0.45 → S and run again. What happens to the whale population?

g. Change 0.45 → S to 0.78 → S and run again. What happens to the whale population?

8.7 Summary and review of Chapters 3 - 8.

Definitions. We have introduced the concepts of *difference quotient* of a function and *rate of change* of a function, and applied the concepts to polynomial, exponential and logarithm functions and trigonometric functions. The difference quotient of a function is:

$$\text{Difference quotient of } P \text{ on } [a,b] = \frac{P(b)-P(a)}{b-a}$$

where P is a function whose domain contains at least the numbers a and b.

To compute the rate of change of P at a number a in the domain of P, we compute the difference quotient of P over intervals, $[a,b]$, having a as one endpoint and decreasing in size to zero. Then there is that mysterious step of deciding what those difference quotients get close to as b gets close to a. Assuming the difference quotients get close to some number, that number is called the rate of change of P at a and is denoted by $P'(a)$.

The symbol, P', denotes the function defined by

$$P'(a) = \text{the rate of change of } P \text{ at } a$$

for every number a in the domain for which the rate of change exists.

The words 'get close to' are sometimes replaced with 'approaches', and the word 'limit' is used for the number that $(P(b)-P(a))/(b-a)$ gets close to. This can reduce a lot of words to a simple symbol,

$$P'(a) = \lim_{b\to a} \frac{P(b)-P(a)}{b-a} \tag{8.11}$$

Chapter Exercise 8.1 Let $P(t)=\sqrt{t}$ and a be a positive number. Give reasons for the following steps to find $P'(a)$.

$$\begin{aligned} P'(a) &\overset{a}{=} \lim_{b\to a} \frac{P(b)-P(a)}{b-a} \\ &\overset{b}{=} \lim_{b\to a} \frac{\sqrt{b}-\sqrt{a}}{b-a} \\ &\overset{c}{=} \lim_{b\to a} \frac{1}{\sqrt{b}+\sqrt{a}} \\ &\overset{d}{=} \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\times\sqrt{a}} \end{aligned}$$

Chapter Exercise 8.2 Suppose $P(t)=t^4$.

a. Write an expression for $(P(b)-P(a))/(b-a)$, the difference quotient of P on the interval, $[a,b]$.

b. Simplify your expression.

c. Use your simplified expression to show that the rate of change of P at a is $4\,a^3$.

Chapter Exercise 8.3 Use the definition of rate of change to find the rate of change of $P(t) = \frac{1}{t}$ at $a = 5$. Repeat for a unspecified. Complete the formula

$$P(t) = \frac{1}{t} \qquad \Rightarrow \qquad P'(t) =$$

The functions F and G of the next two exercises present interesting challenges.

Chapter Exercise 8.4 Nine points of the graph of the function F defined by

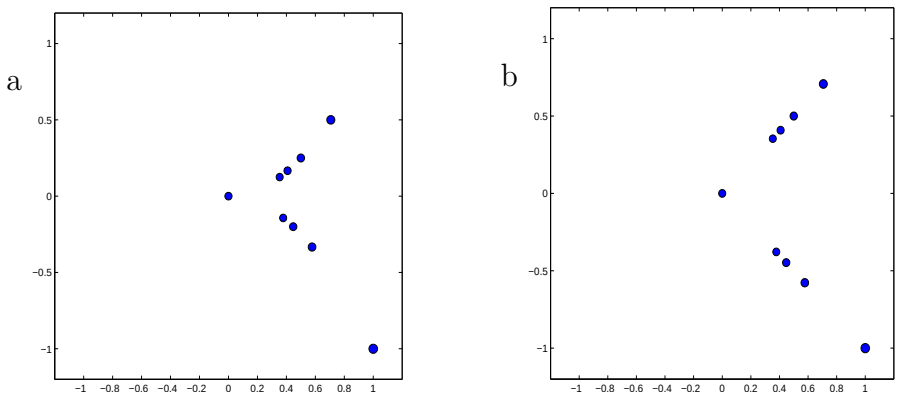
$$\begin{aligned} F\left(\tfrac{1}{\sqrt{n}}\right) &= \frac{(-1)^n}{n} \quad \text{for} \quad n = 1, 2, 3, \cdots \\ F(0) &= 0 \end{aligned} \tag{8.12}$$

are shown in Chapter Exercise Figure 8.4a.

a. Plot two more points of the graph of F.

b. Does $F'(0)$ exist?

Figure for Chapter Exercise 8.4 Graphs of F and G for Chapter Exercises 8.4 and 8.5.



Chapter Exercise 8.5 Nine points of the graph of the function G defined by

$$\begin{aligned} G\left(\tfrac{1}{\sqrt{n}}\right) &= \frac{(-1)^n}{\sqrt{n}} \quad \text{for} \quad n = 1, 2, 3, \cdots \\ G(0) &= 0 \end{aligned} \tag{8.13}$$

are shown in Chapter Exercise Figure 8.4b.

a. Plot two more points of the graph of G.

b. Does $G'(0)$ exist?

8.8 Derivative Formulas.

Using the definition of rate of change, we obtained formulas for derivatives of six functions.

Primary Formulas

$$[C]' = 0 \quad (8.14)$$
$$[t^c]' = c\,t^{c-1} \quad (8.15)$$
$$\left[e^t\right]' = e^t \quad (8.16)$$
$$[\ln t]' = \frac{1}{t} \quad (8.17)$$
$$[\sin t]' = \cos t \quad (8.18)$$
$$[\cos t]' = -\sin t \quad (8.19)$$

Then we obtained rules for derivatives of combinations of functions.

Combination Rules

$$[C \times u(t)]' = C \times u'(t) \quad (8.20)$$
$$[u(t) + v(t)]' = u'(t) + v'(t) \quad (8.21)$$
$$[u(t) \times v(t)]' = u'(t) \times v(t) + u(t) \times c'(t) \quad (8.22)$$
$$\left[\frac{u(t)}{v(t)}\right]' = \frac{v(t) \times u'(t) - u(t) \times v'(t)}{(v(t))^2} \quad (8.23)$$
$$[G(u(t))]' = G'(u(t)) \times u'(t) \quad (8.24)$$

Formula 8.24, ($[G(u(t))]' = G'(u(t)) \times u'(t)$, called the chain rule) is a bit difficult for students, and special cases were presented.

Chain Rule Special Cases

$$[(\,u(t)\,)^n]' = n\,(\,u(t)\,)^{n-1} \times u'(t) \quad (8.25)$$
$$\left[e^{u(t)}\right]' = e^{u(t)} \times u'(t) \quad (8.26)$$
$$[\ln(\,u(t)\,)]' = \frac{1}{u(t)} \times u'(t) \quad (8.27)$$
$$[\sin(\,u(t)\,)]' = \cos(\,u(t)\,) \times u'(t) \quad (8.28)$$
$$[\cos(\,u(t)\,)]' = -\sin(\,u(t)\,) \times u'(t) \quad (8.29)$$

One special case of the chain rule is so important to the biological sciences that it requires individual attention.

$$\left[e^{kt}\right]' = e^{kt} \times k \quad (8.30)$$

Example 8.8.1 Usually more than one derivative formula may be used in a single step toward computing a derivative. It is important to know which steps are being used, and a slow but sure way to insure proper use is to use but a single formula for each step. We illustrate with $P(t) = \left[e^{t^2} + e^{-t}\right]^4$.

$$\begin{aligned}
P'(t) &= \left[\left[e^{t^2} + \sin t\right]^4\right]' && \text{Notational identity.}\\
&= 4\left[e^{t^2} + \sin t\right]^3 \times \left[e^{t^2} + \sin t\right]' && \text{Equation 8.25}\\
&= 4\left[e^{t^2} + \sin t\right]^3 \times \left(\left[e^{t^2}\right]' + [\sin t]'\right) && \text{Equation 8.21}\\
&= 4\left[e^{t^2} + \sin t\right]^3 \times \left(e^{t^2}\,[t^2]' + [\sin t]'\right) && \text{Equation 8.26}\\
&= 4\left[e^{t^2} + \sin t\right]^3 \times \left(e^{t^2} \times (2t) + [\sin t]'\right) && \text{Equation 8.15}\\
&= 4\left[e^{t^2} + \sin t\right]^3 \times \left(e^{t^2} \times (2t) + \cos t\right) && \text{Equation 8.18.}
\end{aligned}$$

Observe that the last two steps involved primary formulas, which is common.

Chapter Exercise 8.6 Use the derivative formulas, one formula for each step, to find $P'(t)$ for

a.	$P(t)$	$=$	$t^5 + 3t^4 - 5t^2 + 7$	b.	$P(t)$	$=$	$t^3 \times e^{6t}$
c.	$P(t)$	$=$	$[1 + t^2]^3$	d.	$P(t)$	$=$	$\sin t^2$
e.	$P(t)$	$=$	$\sin^2 t$	f.	$P(t)$	$=$	$\ln t^7$
g.	$P(t)$	$=$	$e^{\sin t}$	h.	$P(t)$	$=$	$\cos 5t$
i.	$P(t)$	$=$	$\dfrac{\sin t}{\cos t}$	j.	$P(t)$	$=$	$\dfrac{t}{t^2+1}$

Chapter Exercise 8.7 Compute the derivatives of

a.	$P(t)$	$=$	$t^5 + 3t^4 - 4t^2 + 1$	b.	$P(t)$	$=$	t^{-3}
c.	$P(t)$	$=$	$t^4 \times e^t$	d.	$P(t)$	$=$	$(t + e^t)^4$
e.	$P(t)$	$=$	$\frac{1}{t^2+1}$	f.	$P(t)$	$=$	$\ln(t^2 + 1)$
g.	$P(t)$	$=$	$e^{-3t} - e^{-5t}$	h.	$P(t)$	$=$	e^{-t^2}

Chapter Exercise 8.8 The word *differentiate* means 'find the derivative of'. Differentiate

a.	$P(t)$	$=$	$\frac{e^{-3t}}{t^2}$	b.	$P(t)$	$=$	$e^{2\ln t}$
c.	$P(t)$	$=$	$e^t \ln t$	d.	$P(t)$	$=$	$t^2 \times e^{2t}$
e.	$P(t)$	$=$	$e^{t\ln 2}$	f.	$P(t)$	$=$	e^2
g.	$P(t)$	$=$	$[e^t]^5$	h.	$P(t)$	$=$	$\ln e^{3t}$
k.	$P(t)$	$=$	$\frac{1}{\ln t}$	l.	$P(t)$	$=$	$[\sin \pi t]^5$
m.	$P(t)$	$=$	$\sin(\cos t)$n. $P(t)$	$=$	$\frac{t}{t^2+1}$		

Geometry. The rate of change of $P(t)$ at a is the slope of the tangent to the graph of P at the point $(a, P(a))$. With only one exception, the existence of a tangent to the graph of P at $(a, P(a))$ is equivalent to the existence of the rate of change of $P(t)$ at a. The exception is illustrated by

Chapter Exercise 8.9 Let $P(t) = t^{\frac{1}{3}}$.

a. Draw the graph of P on $-1 \le t \le 1$.

b. Draw the tangent to the graph of P at (0,0).

c. Compute $(P(b) - P(a))/(b - a)$.

d. Discuss $\lim_{b\to 0} (P(b) - P(a))/(b - a)$.

Chapter Exercise 8.10 Find an equation of the line tangent to the graph of $P(t) = e^t$ at the point $(1, e^1) = (1, e)$.

The tangent line lies close to the graph near the point of tangency, and is often used as an easily computable approximation to the graph. The next problem illustrates this.

Chapter Exercise 8.11 In Problem 8.10 you found the equation of the line tangent to the graph of $P(t) = e^t$ at the point (1,e). Let y_x denote the y-coordinate of the point on the tangent whose x-coordinate is x.

a. Find $y_{1.8}$.

b. Compute $e^{1.8}$.

c. Compute the relative error

$$\frac{y_{1.8} - e^{1.8}}{e^{1.8}}$$

d. Mark the points $(1.8, P(1.8))$ and $(1.8, y_{1.8})$ on a copy of the graph in Figure 8.11A.

e. Find $y_{1.1}$.

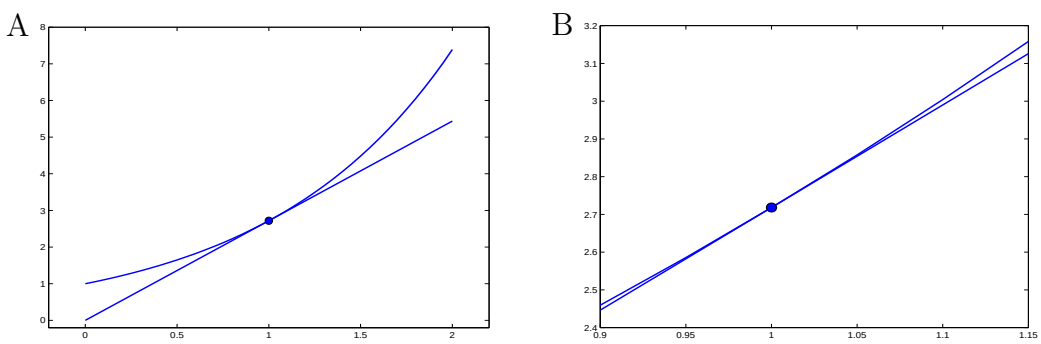
f. Compute $e^{1.1}$.

g. Compute the relative error

$$\frac{y_{1.1} - e^{1.1}}{e^{1.1}}$$

h. Mark the points $(1.1, P(1.1))$ and $(1.1, y_{1.1})$ on a copy of the graph in Figure 8.11B.

Figure for Chapter Exercise 8.11 Graphs of $y = e^x$ and its tangent at (1,e) at two scales.



Chapter Exercise 8.12 Let $P(t) = t^3 + 6t^2 + 9t + 5$. Find the intervals on which $P'(t) > 0$. If you draw the graph of P on your calculator, you will observe that P is increasing on these same intervals.

Chapter Exercise 8.13 Find an equation of the line tangent to the graph of $P(t) = t^3 - 3t^2 + 5$ at the point (1,3). Find a point on the graph of P at which the tangent to the graph is horizontal.

Chapter Exercise 8.14 We will find in Chapter 16 that a population of size, $P(t)$, growing in a limited environment of size, M, is modeled by an equation of the form

$$P'(t) = k \times P(t) \times \left(1 - \frac{P(t)}{M}\right),$$

and that $P(t)$ can be written explicitly as

$$P(t) = \frac{P_0 M e^{kt}}{M - P_0 + P_0 e^{kt}}$$

where P_0 is the population size at time $t = 0$.

Assume $P_0 = 1$, $M = 10$, and $k = 0.1$.

a. Use the first equation and draw a graph of P' *vs* P.

b. What value of P maximizes P'?

c. Use the second equation and draw the graph of $P(t)$.

d. Draw a tangent to the graph of P at the point for which P' is a maximum. Is the point an inflection point?

e. Rewrite your equation to be

$$P(t) = \frac{10}{9e^{-0.1t} + 1} = 10\left(9e^{-0.1t} + 1\right)^{-1}$$

and compute $P'(t)$.

f. At what time t and population size, $P(t)$, is $P'(t)$ a maximum?

Chapter Exercise 8.15 Heavy rainfall in the drainage area of a lake causes runoff into the lake at the rate of $100e^{-t}$ acre-feet per day. The accumulated runoff into the lake due to the rain after t days is $100(1 - e^{-t})$ acre feet. Water is released from the lake at the rate of 20 acre-feet per day and the volume of the lake at the time of the rain was 400 acre-feet. Flood stage is 450 acre-feet.

a. Write an equation for the volume, $V(t)$ of water in the lake at time t days after the rain, accounting for the initial volume, the volume accumulated due to the rain, and the total volume released during the t days.

b. Does the lake flood?

c. At what time does the lake reach its maximum volume and what is the maximum volume?

Chapter Exercise 8.16 This exercise explores the optimum clutch size for a bird. Most biological parameters are compromise values that optimize some feature of the system. In this model, we assert that clutch size of birds is adapted to maximize the number of birds that survive for three months past hatching. The data is for a chickadee, *Parus major*, as observed in woods near Oxford, England[3]. This is part of a historic study of the chickadee population in those woods begun in 1947 and from which data are still being gathered. Only two of the several factors influencing clutch size are considered here, the weight, WT of the young at age 15 days in a clutch of size n, and the percent of fledglings recovered 3 months past fledging. The data is for the year 1961. In order to have several young survive to age three months, a large clutch should be produced. However, if the clutch is too large the adults can not adequately feed the young, they are small at fledging, and the probability of survival is reduced. The following exercises explore the relations between the parameters.

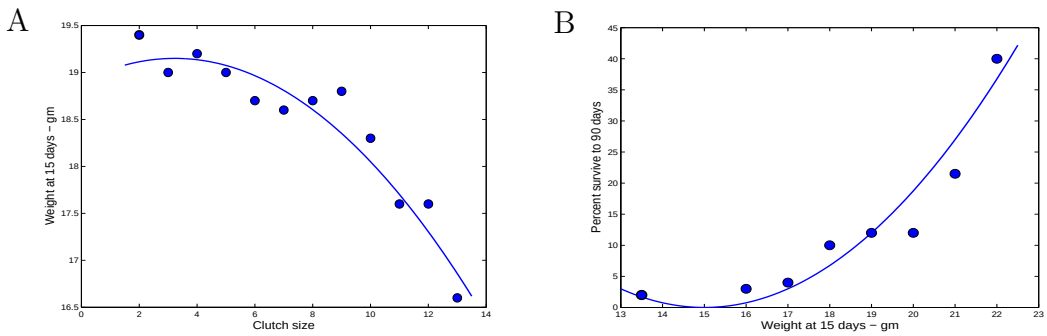
The basic data is displayed in Chapter Exercise Figure 8.16 along with the following parabolas that approximate the data:

$$WT = -0.024(BD - 3.25)^2 + 19.2 \tag{8.31}$$

$$PCT = 0.75(W - 15)^2 \tag{8.32}$$

WT is the average weight at 15 days of chicks in a clutch of size BD, and PCT is the percent of chicks of weight W at 15 days that survive to 90 days.

Figure for Chapter Exercise 8.16 Data and quadratic approximations for chickadees. A. Graph of average body weight *vs* clutch size. B. Graph of percent survival to 90 days *vs* body weight of chick. $WT \doteq -0.024(BD - 3.25)^2 + 19.2$ and $PCT \doteq 0.75(W - 15)^2$.



We make an assumption that the expected number, XP, of chicks that survive to age 90 days from a clutch size of BD is

$$XP = BD \times PCT/100. \tag{8.33}$$

The distribution of weights of chicks at 15 days within a clutch is unknown to us; we are assuming that $W = WT$, an assumption we would prefer to avoid.

a. Assume that $W = WT$ and use Equations 8.31 and 8.32 to write a single equation for PCT in terms of BD.

[3] C. M. Perrins, Population fluctuations and clutch-size in the great tit, *PARUS MAJOR L.*, *J. Animal Ecol.* **34** 601-647 (1965)

b. Now use Equation 8.33 to write XP in terms of BD.

c. Use simpler notation.

$$y = x \times 0.0075 \times 0.024^2 \times \left[(x - 3.25)^2 - 175\right]^2$$

Use derivative formulas to compute y'.

d. One factor of y' is $5x^2 - 19.5x - 164.4375$. Find a value of x between 6 and 10 for which $y'(x) = 0$.

e. How many chicks from a brood of optimum size would be expected to survive?

You should have found that a clutch size of 8.0 will maximize the expected number of chicks that will survive to 90 days. From table 14 of Perrins, 383 broods had an average brood size of 8.1 chicks.

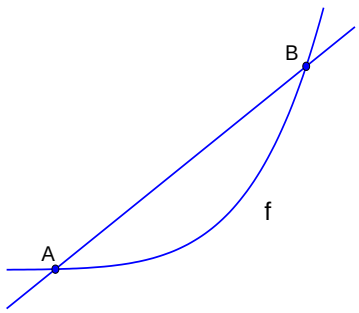
Chapter 9

The Mean Value Theorem and Taylor Polynomials.

Where are we going?

The graph of the function, f, has a tangent that is parallel to the secant line $\overline{AB}$.

The Mean Value Theorem is a careful statement supporting this assertion and is at the heart of a surprising number of explanations about functions, including, for example, the second derivative tests for maxima and minima.



Explore 9.0.1 An eagle flies from its aerie, remains in flight for 30 minutes, and returns to the aerie. Must there have been some instant in which the eagle's flight was horizontal? ■

9.1 The Mean Value Theorem.

Theorem 9.1.1 Mean Value Theorem. Suppose f is a function defined on a closed interval $[a, b]$ and

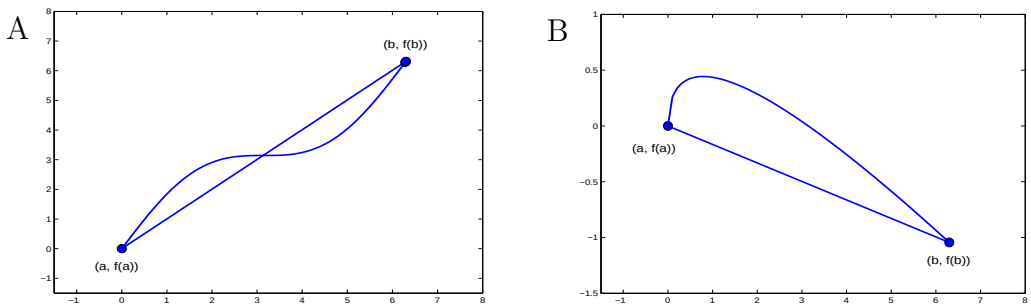
1. f is continuous on the interval, $[a, b]$.
2. $f'(x)$ exists for every number $a < x < b$.

Then there is a number, c, with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{9.1}$$

Explore 9.1.1 Do this. Locate (approximately) a point on each graph in Explore Figure 9.1.1A and B at which the tangent is parallel to the secant. ■

Explore Figure 9.1.1 Graphs with secants; find points at which tangents to the graphs will be parallel to the secants.



The Mean Value Theorem relates an average rate of change to a rate of change. It is surprisingly useful, despite there being uncertainty about the number c asserted to exist.

Uncertainty: Suppose a motorist travels on a turnpike which is 150 miles between the entry and exit, and the ticket records that it was 1:20 pm on entry and 3:20 pm on exit. The operator at the exit observes that the motorist was speeding, and the motorist protests that he never exceeded the 70 mph speed limit. The operator looks the car over and responds, "I see no evidence of possible discontinuous motion (as might occur in extra-terrestrial vehicles) and I am certain that at some time during your trip you were traveling 75 mph." His reasoning is that the average speed was 75 mph so there must have been an instant at which the speed was 75 mph. He cannot say exactly when that speed was attained, only that it occurred between 1:20 and 3:20 pm.

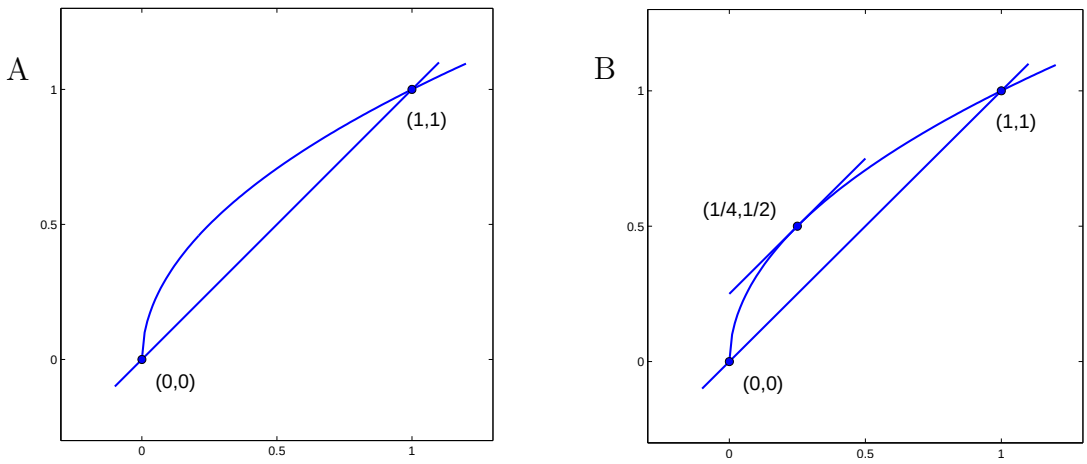
Example 9.1.1 In special cases, we can actually find the value of c.

The Mean Value Theorem asserts that there is a tangent to the parabola $y^2 = x$ that is parallel to the secant containing (0,0) and (1,1). Let

$$f(x) = \sqrt{x} \qquad 0 \le x \le 1.$$

See Example Figure 9.1.0.2A.

Figure for Example 9.1.0.2 A. Graph of $f(x) = \sqrt{x}$ and the secant through (0,0) and (1,1) B. The tangent at (1/4, 1/2) is parallel to the secant.



The function, f is continuous on the closed interval [0,1] and

$$f'(x) = \left[\sqrt{x}\right]' = \left[x^{\frac{1}{2}}\right]' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \qquad 0 < x \le 1$$

The parabola has a vertical tangent at (0,0) and $f'(0)$ is not defined. Even so, the hypothesis of the Mean Value Theorem is satisfied; the requirement is that $f'(x)$ exist for $0 < x < 1$ – only required for points *between* 0 and 1.

We seek a tangent parallel to the secant through (0,0) and (1,1); the secant has slope $\dfrac{f(1) - f(0)}{1 - 0} = 1$, so we look for a tangent of slope 1.

The Mean Value Theorem asserts that there is a number c such that $f'(c) = 1$. We solve for c in

$$f'(c) = 1 \qquad \frac{1}{2\sqrt{c}} = 1 \qquad \frac{1}{2} = \sqrt{c} \qquad c = \frac{1}{4}. \qquad \text{Also,} \qquad f\left(\frac{1}{4}\right) = \frac{1}{2}$$

The tangent to f at (1/4, 1/2) has slope 1 and is parallel to the secant from (0,0) to (1,1) (Example Figure 9.1.0.2B.) ■

We will find good uses for the Mean Value Theorem in the next section. First, though, we tend to its proof. The strategy is to prove a special case where $f(a) = f(b) = 0$, called Rolle's Theorem, published by the French mathematician Michel Rolle in 1691. Then a proof of the general theorem is easy.

In Theorem 8.2.1 on page 353 we showed that for an interior local maximum, $(c, f(c))$, if the graph of f has a tangent at $(c, f(c))$ then that tangent is horizontal. The same assertion applies to interior local minima. See Figure 9.1

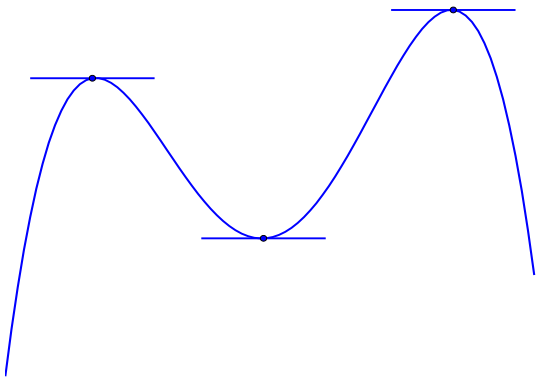


Figure 9.1: The tangents to interior local minimum and maxima are horizontal.

Theorem 9.1.2 Rolle's Theorem. If a function, f, defined on a closed interval, $[a, b]$, satisfies

1. $f(a) = f(b)$.
2. f is continuous on $[a, b]$.
3. $f'(c)$ exists for every number c, $a < c < b$.

then there is a number c between a and b for which $f'(c) = 0$.

The proof is seemingly harmless. We are looking for a horizontal tangent and therefore only have to look at a high point $(u, f(u))$ and a low point $(v, f(v))$ of f on $[a, b]$.

If $a < u < b$, as in Figure 9.2, then $(u, f(u))$ is an interior local maximum and the tangent at $(u, f(u))$ is horizontal. Choose $c = u$. Then $f'(c) = 0$.

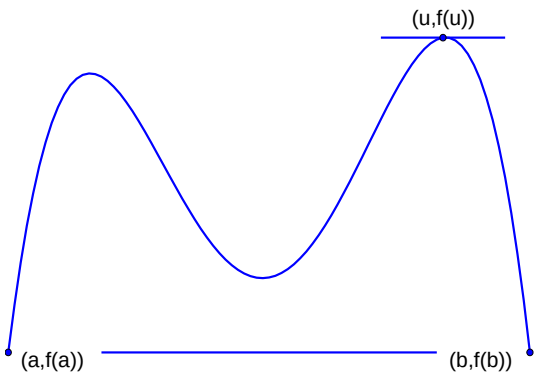


Figure 9.2: Graph with interior maximum (and endpoint minimum).

Similarly, if $a < v < b$, $(v, f(v))$ is an interior local minimum, the tangent at $(v, f(v))$ is horizontal. Choose $c = v$. Then $f'(c) = 0$.

If u and v are both end points of $[a, b]$ (meaning u is a or b and v is a or b) then for x in $[a, b]$, because $f(a) = f(b)$, the largest value of $f(x)$ is $f(a) = f(b)$ and the least value of $f(x)$ is $f(a) = f(b)$ so

the only value of $f(x)$ is $f(a) = f(b)$. See Figure 9.3. Thus the graph of f is a horizontal interval and every tangent is horizontal. For the conclusion of the theorem we can choose c any point between a and b and $f'(c) = 0$.

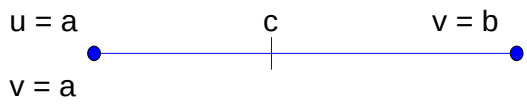


Figure 9.3: Graph with endpoint minimum, and endpoint maximum. u and v may be the same endpoint or at opposite ends of $[a, b]$.

A weakness in the preceding discussion is the presumption that the graph of f has a highest point. Under the assumptions of Theorem 9.1.2, f is continuous at every number in $[a, b]$, and there is a theorem that the graphs of such functions have high points (and low points). We ask you to accept the following theorem as true.

Theorem 9.1.3 High Point Theorem. If f is a continuous function defined on a closed interval, $[a, b]$, then there is a number c in $[a, b]$ such that for all x in $[a, b]$, $f(x) \leq f(c)$.

We ask that you accept the High Point Theorem as true without proof. It is interesting that the High Point Theorem may be used as an alternative to the Completion Property as an axiom of the number system (see Exercise 9.1.8). With the High Point Theorem accepted as true, the proof of Rolle's Theorem is complete. *End of Proof.*

Example 9.1.1 The function, f, defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad x = 0 \\ \frac{1}{x} & \text{for} \quad 0 < x \leq 1 \end{cases}$$

is defined on [0,1] but does not have a highest point; nor is f continuous. See Figure 9.4 ■

Example 9.1.2 Rolle's Theorem is useful in showing some functions are invertible. Let F be the function defined by $F(x) = x + x^3$ for all numbers, x. We will show that no horizontal line intersects the graph of F at two distinct points, which means that F is invertible. Observe that

$$F'(x) = [\, x + x^3 \,]' = 1 + 3x^2 \qquad > 0 \qquad \text{for all } x.$$

By Rolle's Theorem, if some horizontal line contains two points of the graph of F then at an intervening point there is a horizontal tangent and at that point $F' = 0$. This is impossible because $F'(x) = 1 + 3x^2 > 0$ for all x. Thus F is an invertible function. ■

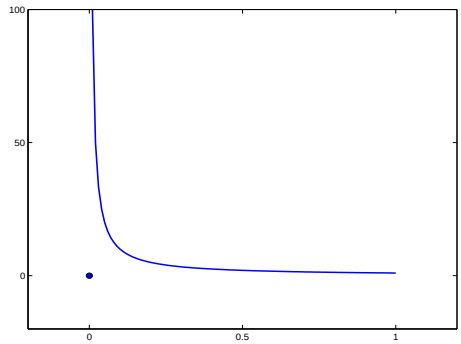


Figure 9.4: The graph of $f(0) = 0$, $f(x) = 1/x$ for $0, x \leq 1$.

Proof of the Mean Value Theorem. Suppose f is a continuous function on an interval $[a, b]$ and $f'(t)$ exists for $a < t < b$. An equation of the secant line through $(a, f(a))$ and $(b, f(b))$ is

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Let $D(x) = f(x) - L(x)$. Then D is the difference between two continuous functions f and L that have derivatives between a and b, so D is continuous and has a derivative between a and b. Furthermore

$$D(a) = f(a) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(a - a)\right) = 0$$

and

$$D(b) = f(b) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(b - a)\right) = 0$$

(These zeros are not a surprise because f and the secant intersect at $(a, f(a))$ and $(b, f(b))$.)

Therefore D satisfies the hypothesis of Rolle's Theorem so there is a number c, $a < c < b$, such that $D'(c) = 0$. Now

$$\begin{aligned} D'(x) &= \left[f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right)\right]' \\ &= [f(x)]' - [f(a)]' - \left[\frac{f(b) - f(a)}{b - a}(x - a)\right]' \\ &= f'(x) - 0 - \frac{f(b) - f(a)}{b - a} \end{aligned}$$

and $D'(c) = 0$ leads to

$$\begin{aligned} D'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ 0 &= f'(c) - \frac{f(b) - f(a)}{b - a} \end{aligned}$$

$$f'(c) \quad = \quad \frac{f(b) - f(a)}{b - a},$$

which is the conclusion of the Mean Value Theorem. *End of proof.*

Exercises for Section 9.1, The Mean Value Theorem.

Exercise 9.1.1 For each of the functions, F, draw a graph of F and the secant through $(a, F(a))$ and $(b, F(b))$ and a tangent to the graph of F that is parallel to the secant.

a. $F(x) = \frac{1}{x}, \quad [a, b] = [1/2, 2]$ b. $F(x) = \dfrac{1}{x^2 + 1}, \quad [a, b] = [-1, 1]$

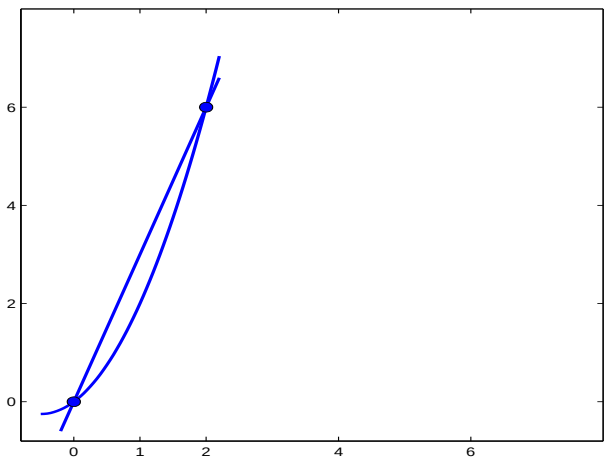
c. $F(x) = e^x, \quad [a, b] = [0, 2]$ d. $F(x) = \sin x, \quad [a, b] = [0, \pi/2]$

Exercise 9.1.2 For $f(x) = x^2 + x$ find a value of c for which

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} \qquad \text{and} \qquad 0 < c < 2$$

Locate the point $(c, f(c))$ on the graph in Ex. Figure 9.1.2 and draw its tangent.
Compute $f'(c)$.

Figure for Exercise 9.1.2 Graph of $f(x) = x^2 + x$ and the secant through (0,f(0)) and (2,f(2))



Exercise 9.1.3 Find a number c between a and b for which $f'(c) = f(b) - f(a)/(b - a)$.

a. $f(t) \quad = \quad t^4 \qquad [a, b] = [0, 1],$

b. $f(t) \quad = \quad t^3 - 3t^2 + 3t \qquad [a, b] = [1, 2]$

c. $f(t) \quad = \quad t^3 - 3t^2 + 3t \qquad [a, b] = [1, 2]$

d. $f(t) \quad = \quad \ln t \qquad [a, b] = [1, e]$

Exercise 9.1.4 Show that the following functions are invertible.

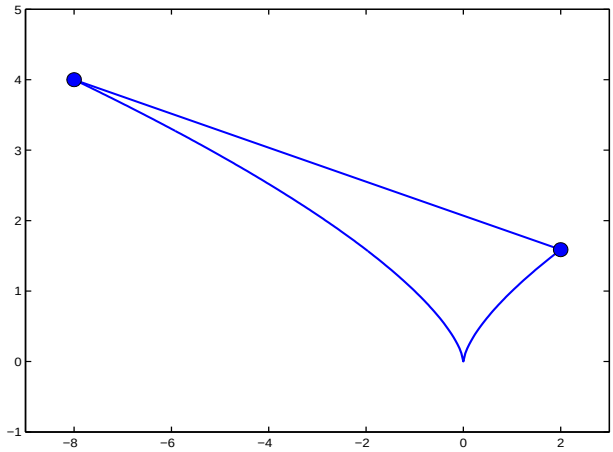
$$\begin{array}{llll} \text{a.} & F(t) & = & e^t \\ \text{b.} & F(t) & = & \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2} \\ \text{c.} & F(t) & = & \ln t, \qquad 0 < t \\ \text{d.} & F(t) & = & \sqrt{5-t}, \qquad t \leq 5 \end{array}$$

Exercise 9.1.5 The graph of the function F defined by

$$F(x) = \sqrt[3]{x^2}$$

is shown in Figure 9.1.5 together with the secant to the graph through (-1,1) and (8,4). Is there a tangent to the graph at a point between (-1,1) and (8,4) at which tangent is parallel to the secant between these two points? Does the Mean Value Theorem assert that there is such a tangent? Explain your answer.

Figure for Exercise 9.1.5 Graph of $y = \sqrt[3]{x^2}$, $-8 \leq x \leq 2$.

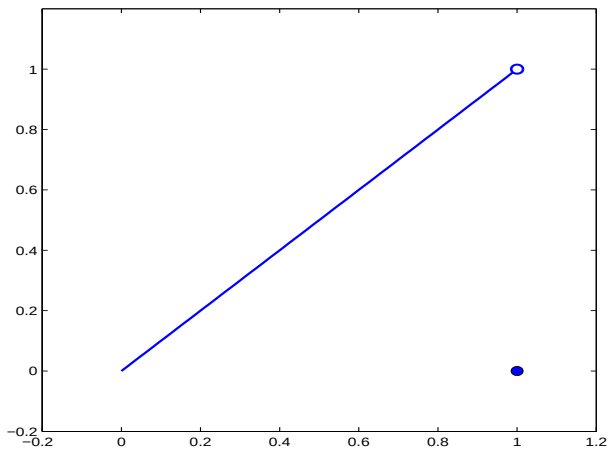


Exercise 9.1.6 The graph of the function f defined by

$$f(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1 \\ 0 & \text{for} \quad t = 1 \end{cases} \tag{9.2}$$

is shown in Figure Ex. 9.1.6. It does not have a high point. Furthermore, it does not have a horizontal tangent; yet it satisfies all but one of the hypotheses of Rolle's Theorem. Which hypothesis of Rolle's Theorem does it not satisfy?

Figure for Exercise 9.1.6 Graph of Equation 9.2.



Exercise 9.1.7 . Draw the graph of a continuous function F with domain [1,5] that has a a non-negative derivative at every point between 1 and 5, and for which 5 is a local minimum.

Exercise 9.1.8 Only for the really curious. Complete the proof of Theorem 9.1.4:

> **Theorem 9.1.4** If the usual properties of addition, multiplication and order of the number system are assumed, one may accept the statement of the High Point Theorem as an axiom and prove the statement of the Completion Axiom as a theorem.

Proof. Assume the usual properties of addition, multiplication, and order of the number system and the statement of the High Point Theorem. Suppose the statement of the Completion Axiom is not true. Then there are sets of numbers, S_1 and S_2 such that every number is in either S_1 or S_2 and every number of S_1 is less than every number in S_2 and S_1 does not have a largest number and S_2 does not have a least number. Let a be a number in S_1 and b be a number in S_2. Consider the function f defined on $[a, b]$ by

$$f(x) = \begin{cases} x - a & \text{for} \quad a \leq x \qquad \text{and} \qquad x \quad \text{in } S_1 \\ \\ x - b & \text{for} \quad x \leq b \qquad \text{and} \qquad x \quad \text{in } S_2 \end{cases} \tag{9.3}$$

a. Show that f is continuous. (Remember that S_1 does not have a largest number and S_2 does not have a least number.)

b. There is a number c such that if x is in $[a, b]$ then $f(x) \leq f(c)$. In the event that $c = b$, because $f(a) = 0 = f(b)$, we can assume that $c = a$ instead of $c = b$. Show that c is the largest number in S_1.

c. The assumption that the statement of the Completion Axiom is false has lead to a contradiction.

End of Proof.

9.2 Nondecreasing and increasing functions; second derivative test for high points.

With the Mean Value Theorem we can easily prove three results left from Sections 8.1 and 8.2.1. In Section 8.1 we stated that we would prove:

Theorem 9.2.1 Suppose P is a continuous function defined on an interval $[d, e]$ and at every point t in (d, e) $P'(t)$ exists and $P'(t) \geq 0$. Then P is nondecreasing on $[d, e]$.

Proof. Suppose a and b are numbers in $[d, e]$ and $a < b$. Then by the Mean Value Theorem, there is a number, c, between a and b such that

$$\frac{P(b) - P(a)}{b - a} = P'(c) \qquad \text{so that} \qquad P(b) - P(a) = P'(c)(b - a)$$

By hypothesis, $P'(c) \geq 0$. Also $a < b$ so that $b - a > 0$. It follows that $P(b) - P(a) \geq 0$, so that $P(a) \leq P(b)$. Therefore P is nondecreasing on $[e, f]$. End of proof.

We also stated in Section 8.1 that we would prove:

Theorem 9.2.2 Suppose P is a continuous function defined on an interval $[d, e]$ and at every point t in (d, e) $P'(t)$ exists and $P'(t) > 0$. Then P is increasing on $[d, e]$.

Proof: The argument is the similar to that for Theorem 8.1.1 and is left as Exercise 9.2.3

Example 9.2.1 Theorems 9.2.1 and 9.2.2 can be used in the following way.
Problem. Let $P(x) = 3x^4 - 20x^3 - 6x^2 + 60x - 37$. On what intervals is P increasing? On what intervals is P decreasing?
Solution. We compute

$$P'(x) = 12x^3 - 60x^2 - 12x + 60 = 12(x^3 - 5x^2 - x + 5) = 12(x + 1)(x - 1)(x - 5)$$

Confession. $P(x)$, was selected so that the cubic, $P'(x)$, would factor nicely.

Now $P'(-1) = P'(1) = P'(5) = 0$ which suggests that -1, 1, and 5 might be x-coordinates of local maxima and minima. We form a chart:

$(-)(-)(-)$	$(+)(-)(-)$	$(+))(+)(-)$	$(+)(+)(+)$
-1	1	5	
$P' < 0$	$P' > 0$	$P' < 0$	$P' > 0$

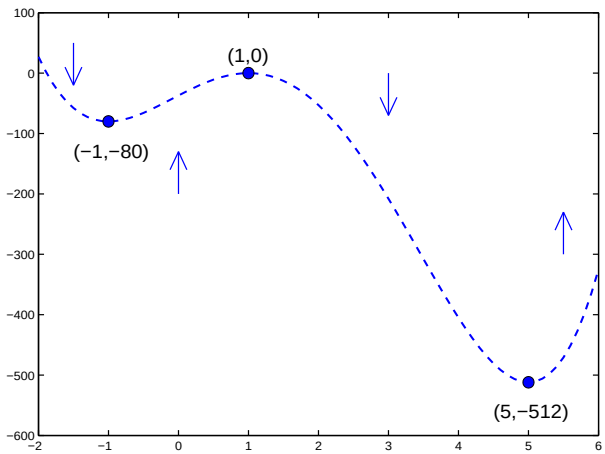
The chart means that for $x < -1$ the three factors in $P' = 12(x+1)(x-1)(x-5)$ are all negative so that P' is negative for all $x < -1$. This means that P is decreasing through out $(-\infty, -1)$. By similar reasoning, P is increasing on $(-1, 1)$, decreasing on $(1, 5)$ and increasing on $(5, \infty)$.

We evaluate P at three (critical) points.

$$\begin{aligned} P(-1) &= 3(-1)^4 - 20(-1)^3 - 6(-1)^2 + 60(-1) - 37 = -80 \\ P(1) &= 0 \\ P(5) &= -512 \end{aligned}$$

These three points are plotted on the graph in Figure 9.2.0.2 and arrows indicate the increasing and decreasing character of P between the data points. A candidate for the graph of P could be drawn with just this information. In fact, the dashed line in Figure 9.2.0.2 is the graph of P ■

Figure for Example 9.2.0.2 The points (-1,-80), (1,0), and (5,-512) are plotted with arrows showing the increasing and decreasing character of $P(x) = 3x^4 - 20x^3 - 6x^2 + 60x - 37$ on the intervals between the data points. The dashed line is the graph of P.



The second derivative test. In Section 8.2.1 we stated the second derivative test:

Theorem 9.2.3 Suppose f is a function with continuous first and second derivatives throughout an interval $[a, b]$ and c is a number between a and b for which $f'(c) = 0$. Under these conditions:

1. If $f''(c) > 0$ then c is a local minimum for f.
2. If $f''(c) < 0$ then c is a local maximum for f.

Proof. We prove case 1, for c to be a local minimum of f. It will be helpful to look at the numbers on the number lines shown in Figure 9.5. Because f'' is continuous and positive at c, there is an interval, (u, v), containing c so that f'' is positive throughout (u, v)[1]. See the top number line in Figure 9.5. Suppose x is in (u, v) (the second number line in Figure 9.5). We must show that $f(c) \leq f(x)$.

We consider the case $x < c$.

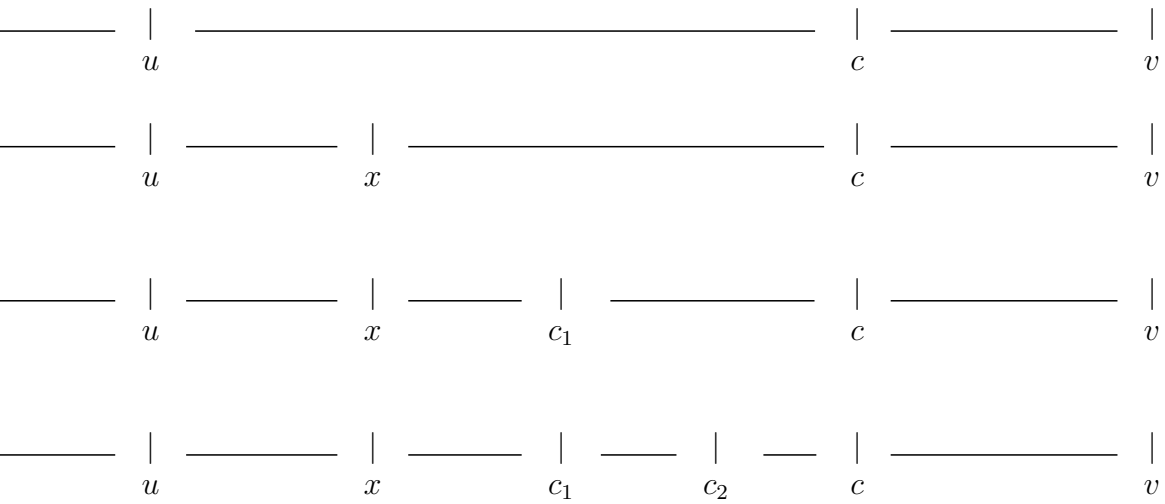


Figure 9.5: Number lines for the argument to Theorem 9.2.3.

By the Mean Value Theorem, there is a number, c_1 between x and c for which

$$\frac{f(c) - f(x)}{c - x} = f'(c_1) \qquad \text{so that} \qquad f(c) - f(x) = f'(c_1)(c - x) \tag{9.4}$$

See the third number line in Figure 9.5.

Watch this! By the Mean Value Theorem applied to the function f' and its derivative f'', there is a number c_2 between c_1 and c so that

$$\frac{f'(c) - f'(c_1)}{c - c_1} = f''(c_2) \qquad \text{and} \qquad f'(c) - f'(c_1) = f''(c_2)(c - c_1) \tag{9.5}$$

[1]This is the Locally Positive Theorem 4.1.1 of Exercise 4.1.13.

See the bottom number line in Figure 9.5.

Remember that $f'(c) = 0$ so from the last equation (Equation 9.5)

$$f'(c_1) = -f''(c_2)(c - c_1)$$

and from this and Equation 9.4 we get

$$f(c) - f(x) = -f''(c_2)(c - c_1)(c - x)$$

Now $c > x$ and $c > c_1$ so that $c - x$ and $c - c_1$ are positive. Furthermore, $f''(c_2) > 0$ because f'' is positive throughout (u, v). It follows that $f(c) - f(x)$ is negative, and that $f(c) < f(x)$.

The event $c < x$ is similar.

The argument for Case 2. of Theorem 9.2.3 is similar. *End of proof.*

Exercises for Section 9.2, Nondecreasing and increasing functions; second derivative test for high points.

Exercise 9.2.1 Show that the following functions are invertible.

a. $f(x) = x^2$	$0 \le x$		b. $f(x) = \frac{1}{x}$	$0 < x$
c. $f(x) = \sin x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$		d. $f(x) = e^x$	$-\infty < x < \infty$
e. $f(x) = \frac{x}{x^2+1}$	$-1 \le x \le 1$		f. $f(x) = \frac{x}{x^2+1}$	$1 \le x$
g. $f(x) = x^3 - 6x^2 + 9x - 4$	$1 \le x \le 3$		h. $f(x) = x\lvert x\rvert$	$-\infty < x < \infty$

Exercise 9.2.2 Find the intervals on which f is increasing. Identify the local minima and local maxima. Plot the local minima and local maxima on a graph, and sketch a candidate graph of the function.

a. $f(x) = x^2 - 3x + 7$	b. $f(x) = -x^2 + 5x + 16$
c. $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 17$	d. $f(x) = x^5 - 5x + 2$
e. $f(x) = x^4 + 4x + 8$	f. $f(x) = x \ln x$
g. $f(x) = xe^{-x}$	h. $f(x) = x^2 e^{-x}$

Exercise 9.2.3 Prove Theorem 9.2.2. The proof will be similar to that for Theorem 8.1.1 except that by hypothesis $P'(c) > 0$ instead of $P'(c) \ge 0$, which leads to subsequent changes.

Exercise 9.2.4 Complete the argument for Case 1, of Theorem 9.2.3 by treating the event that $c < x$.

9.3 Approximating functions with quadratic polynomials.

Three Questions.

1. Pat is looking for Mike. Sean tells Pat that three minutes ago he saw Mike at the corner of First and Main. Where should Pat go to find Mike?

2. Pat is looking for Mike. Sean tells Pat that three minutes ago he saw Mike at the corner of First and Main and he was jogging North on Main at the rate of 2 blocks per minute. Where should Pat go to find Mike?

3. Pat is looking for Mike. Sean tells Pat that three minutes ago he saw Mike at the corner of First and Main and he was jogging North on Main at the rate of 2 blocks per minute, but Mike looked like he was getting tired. Where should Pat go to find Mike?

The first two questions above have reasonably defensible answers, and you likely worked them out as you read them.

1. If Pat only knows that Mike was at First and Main three minutes ago, it seems the best he could do is to go to First and Main and look for Mike.

2. If in addition, Pat knows that Mike was jogging North at 2 blocks per minute, Pat should look for Mike 6 blocks north of First and Main, which we will take to be Seventh and Main. This assumes that Mike's jogging remains constant, and that he is not, for example, running laps around the court house at First and Main.

3. If it looked like Mike was getting tired, we are not sure, but Mike might also be slowing down. Perhaps Pat should look for Mike at about Fifth or Sixth and Main. The question hinges on how rapidly Mike was slowing down as a result of being tired.

Of course there might be other useful information. For example, Mike might be eating a quick energy bar and regaining some of his strength!

To make the questions more specific we pose the following similar questions.

1. Suppose f is a function and $f(0) = 1$. Ten points of your next examination depends on your estimate of $f(3)$. What is your best estimate?

2. Suppose f is a function and
$$f(0) = 1 \qquad \text{and} \qquad f'(0) = 2$$
What is your best estimate of $f(3)$? (Another 10 points.)

3. Suppose f is a function and
$$f(0) = 1 \qquad \text{and} \qquad f'(0) = 2 \qquad \text{and} \qquad f''(0) = -\frac{1}{4}$$
What is your best estimate of $f(3)$? (Good for 15 points.) (Note: The condition, $f''(0) = -\frac{1}{4}$ or $f''(0)$ negative implies that $f'(t)$ is decreasing (Pat is slowing down).)

Again the first two questions have fairly defensible answers, and we wish to develop a rational approach to the third question,

1. Knowing only that $f(0) = 1$ our best guess of $f(3)$ is also 1. Without knowledge of how f may change in [0,3], our guess is that it does not change, and for all t between 0 and 3 our guess of $f(t)$ is 1.

2. If we know that $f(0) = 1$ and $f'(0) = 2$, clearly the value of f is increasing, but we assume that $f'(t)$ is unchanging from $f'(0)$ (no reason to suppose that it is either higher or lower). If so, then the graph of f is a straight line, $L(t)$, between 1 and 3 and
$$L(t) = 1 + 2 \times t \qquad \text{so that} \qquad L(3) = 1 + 2 \times 3 = 7.$$
Note that the straight line $f(t) = 1 + 2t$ is perhaps the simplest function we can find for which $f(0) = 1$ and $f'(0) = 2$.

3. Now suppose that
$$f(0) = 1 \qquad \text{and} \qquad f'(0) = 2 \qquad \text{and} \qquad f''(0) = -\frac{1}{4}$$
We assume that $f''(t) = -\frac{1}{4}$ **for all** $0 \le t \le 3$, and our strategy is to find the most simple function, f, with these properties. We choose polynomials as 'simple' and in particular look for a parabola,
$$p(t) = a_0 + a_1 t + a_2 t^2$$
with the stated properties. We need to decide what to choose for a_0, a_1 and a_2. In order to **match** the information about f we will insist that
$$p(0) = 1 \qquad \text{and} \qquad p'(0) = 2 \qquad \text{and} \qquad p''(0) = -\frac{1}{4}$$
It is easy, as the array in Table 9.1 shows.

Thus we conclude that
$$p(t) = 1 + 2t - \frac{1}{8}t^2$$

Table 9.1: Array for matching coefficients of quadratics.

Analysis of $p(t)$	Value at 0	Required value	Coefficient
$p(t) = a_0 + a_1 t + a_2 t^2$	$p(t)\big\|_{t=0} = a_0$	$p(0) = 1$	$a_0 = 1$
$p'(t) = a_1 + 2a_2 t$	$p'(t)\big\|_{t=0} = a_1$	$p'(0) = 2$	$a_1 = 2$
$p''(t) = 2a_2$	$p''(t)\big\|_{t=0} = 2a_2$	$p''(0) = -\frac{1}{4}$	$a_2 = -\frac{1}{8}$

has the required properties and is simple. Our guess is that

$$f(3) \qquad \text{should be} \qquad p(3) = 1 + 2 \times 3 - \frac{1}{8}3^2 = 5\frac{7}{8}$$

For future reference, note that

$$p(t) = 1 + 2t - \frac{1}{8}t^2 \qquad \text{is} \qquad L(t) = 1 + 2t \qquad \text{plus an additional term.}$$

Exercises for Section 9.3, Approximating functions with quadratic polynomials.

Exercise 9.3.1 If you know that the temperature range was from a low of 60°F to a high of 84°F on Monday, what is your best estimate for the temperature range on Tuesday?

Exercise 9.3.2 Suppose you are measuring the growth of a corn plant, and observe that

a. The plant is 14 cm tall at 10:00 am Monday and 15.5 cm tall at 10:00 am on Tuesday. What is your best guess for the height of the plant at 10:00 am on Thursday?

b. The plant is 14 cm tall at 10:00 am on Monday, 15.5 cm tall at 10:00 am on Tuesday, and 17.5 cm tall at 10:00 am on Wednesday. What is your best guess for the height of the plant at 10:00 am on Thursday?

Exercise 9.3.3 In Table 9.1 we computed two derivatives, p' and p'', that you should be sure you understand.

Find $p'(t)$, $p''(t)$ (the derivative of p'), $p'''(t) = P^{(3)}$ (the derivative of p''), $p''''(t) = P^{(4)}$ and $p^{(5)}(t)$. In parts f - g it is easiest to retain the binomial form and use

$$[(t-a)^n]' = n(t-a)^{n-1}.$$

a. $p(t) = 2 + 5t - 3t^2$ b. $p(t) = 3 - 2t + t^2 + 7t^3$

c. $p(t) = -7 + t + 3t^2 - 5t^3 - t^4$ d. $p(t) = a + bt + ct^2 + dt^3$

e. $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ f. $p(t) = \; = 2 + 3(t-2) + 4(t-2)^2$

g. $p(t) = p(t) = -3 + 4(t-5) - 2(t-5)^2 + \frac{1}{2}(t-5)^3 - \frac{1}{3}(t-5)^4$

h. $p(t) = a_0 + a_1(t-3) + a_2(t-3)^2 + a_3(t-3)^3$

Exercise 9.3.4 Find a_0, a_1, and a_2 for which the polynomial, $p(t) = a_0 + a_1 t + a_2 t^2$, satisfies

$$\begin{array}{lllllllllll}
\text{a.} & p(0) & = & 5, & p'(0) & = & -2, & \text{and} & p''(0) & = & \frac{1}{3}. \\
\text{b.} & p(0) & = & 1, & p'(0) & = & 0, & \text{and} & p''(0) & = & -\frac{1}{2}. \\
\text{c.} & p(0) & = & 0, & p'(0) & = & 1, & \text{and} & p''(0) & = & 0. \\
\text{d.} & p(0) & = & 1, & p'(0) & = & 0, & \text{and} & p''(0) & = & -1. \\
\text{e.} & p(0) & = & 1, & p'(0) & = & 1, & \text{and} & p''(0) & = & 1. \\
\text{f.} & p(0) & = & 17, & p'(0) & = & -15, & \text{and} & p''(0) & = & 12.
\end{array}$$

9.4 Polynomial approximation centered at 0.

The procedure we introduced in the previous section, Section 9.3, extends to polynomials of all degrees. It is remarkably accurate when used to approximate functions. We use it to find a cubic polynomial that approximates $\sin x$ close to the central point, $c = 0$. To do so, we let $f(x) = \sin x$ and compute $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$. Then we seek a cubic polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

that matches these values, meaning that

$$p(0) = f(0) \qquad p'(0) = f'(0) \qquad p''(0) = f''(0) \qquad p'''(0) = f'''(0)$$

The information is organized in Table 9.2.

Table 9.2: Array to show the match of cubic coefficients. .

Analysis of $f(x) = \sin(x)$		Analysis of $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$		Match at $c = 0$
$x) = \sin x$	$f(0) = \sin 0 = \mathbf{0}$	$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$	$p(0) = \mathbf{a_0}$	$a_0 = 0$
$(x) = \cos x$	$f'(0) = \cos 0 = \mathbf{1}$	$p'(x) = a_1 + 2a_2 x + 3a_3 x^2$	$p'(0) = \mathbf{a_1}$	$a_1 = 1$
$(x) = -\sin x$	$f''(0) = -\sin 0 = \mathbf{0}$	$p''(x) = 2a_2 + 3 \times 2a_3 x$	$p''(0) = \mathbf{2a_2}$	$a_2 = 0$
$'(x) = -\cos x$	$f'''(0) = -\cos 0 = \mathbf{-1}$	$p'''(x) = 3 \times 2a_3$	$p'''(0) = \mathbf{3 \times 2a_3}$	$a_3 = -\frac{1}{3\times 2}$

The **match** of derivatives in the last column is obtained by equating boldfaced entries to the left, 0,1, 0, and -1 with the values a_0, a_1, $2 \times a_2$ and $3 \times 2a_3$. We conclude that

$$p(x) = x - \frac{x^3}{2 \times 3} = x - \frac{x^3}{3 \times 2}$$

$p(x) = x - \frac{x^3}{3\times 2}$ is said to be the cubic polynomial that matches $f(x) = \sin x$ at the central point, $c = 0$.

To see how well $p(x) = x - \frac{x^3}{3\times 2}$ approximates $f(x) = \sin x$, the graphs of both are drawn in Figure 9.6, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The graphs are indistinguishable on $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$. The values at $\frac{\pi}{4}$ are

$$\sin\frac{\pi}{4} = 0.7071 \qquad p(\frac{\pi}{4}) = 0.7047$$

so that

$$\begin{aligned} \text{Absolute Error} &= |0.7047 - 0.7071| = 0.0024 \\ \text{Relative Error} &= \frac{|0.7047-0.7071|}{0.7071} = 0.0034 \end{aligned} \tag{9.6}$$

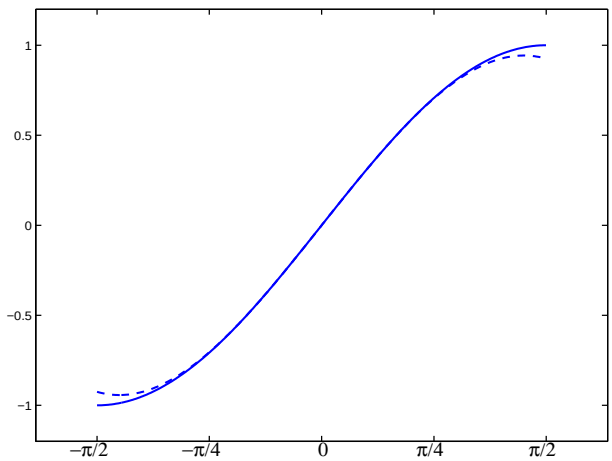


Figure 9.6: Graphs of $f(x) = \sin x$ and $p(x) = x - \dfrac{x^3}{3\times 2}$ (dashed line), the cubic polynomial that matches the sine graph at the central point $x = 0$.

n! The Meaning of n Factorial.

$0! = 1$ and $1! = 1$ If n is an integer greater than 1

$$n! = 1 \times 2 \times \cdots \times (n-1) \times n$$

The symbol, $n!$, is read 'n factorial.' Thus 5! is read '5 factorial' and is $1 \times 2 \times 3 \times 4 \times 5$ which is 120.

We can use more information about $f(x) = \sin x$ at $c = 0$ and find a quintic (fifth degree) polynomial that matches f at $c = 0$ and even more closely approximates f.

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ and find a_0, a_1, a_2, a_3, a_4 and a_5 so that

$$p(0) = f(0) \qquad p'(0) = f'(0) \qquad p''(0) = f''(0)$$

$$p^{(3)}(0) = f^{(3)}(0) \qquad p^{(4)}(0) = f^{(4)}(0) \qquad p^{(5)}(0) = f^{(5)}(0)$$

We could use f''', f'''', and f''''', but this mode of counting is a bit primitive. It is useful to define

$$f^{(0)}(t) = f(t) \qquad f^{(1)}(t) = f'(t) \qquad f^{(2)}(t) = f''(t)$$

and $f^{(k)}(t)$ is the $k\underline{th}$ derivative of $f(t)$.

Analysis of $f(x) = \sin x$

$$\begin{array}{rclrclcr}
f(x) & = & \sin x & f(0) & = & \sin 0 & = & \mathbf{0} \\
f'(x) & = & \cos x & f'(0) & = & \cos 0 & = & \mathbf{1} \\
f''(x) & = & -\sin x & f''(0) & = & -\sin 0 & = & \mathbf{0} \\
f^{(3)}(x) & = & -\cos x & f^{(3)}(0) & = & -\cos 0 & = & \mathbf{-1} \\
f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & \sin 0 & = & \mathbf{0} \\
f^{(5)}(x) & = & \cos x & f^{(5)}(0) & = & \cos 0 & = & \mathbf{1}
\end{array}$$

Analysis of $p(x)$		Match $\mathbf{p^{(k)}(0) = f^{(k)}(0)}$
$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$	$p(0) = \mathbf{a_0}$	$a_0 = 0$
$p'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4$	$p'(0) = \mathbf{a_1}$	$a_1 = 1$
$p''(x) = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + 5 \times 4a_5x^3$	$p''(0) = \mathbf{2a_2}$	$a_2 = 0$
$p^{(3)}(x) = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + 5 \times 4 \times 3a_5x^2$	$p^{(3)}(0) = \mathbf{3 \times 2a_3}$	$a_3 = -\frac{1}{3!}$
$p^{(4)}(x) = 4 \times 3 \times 2a_4 + 5 \times 4 \times 3 \times 2a_5x$	$p^{(4)}(0) = \mathbf{4 \times 3 \times 2a_4}$	$a_4 = 0$
$p^{(5)}(x) = 5 \times 4 \times 3 \times 2a_5$	$p^{(5)}(0) = \mathbf{5 \times 4 \times 3 \times 2a_5}$	$a_5 = \frac{1}{5!}$

Observed that $a_0 = 0$, $a_1 = 1$, $a_2 = 0$ and $a_3 = -\frac{1}{3!}$ are the same as for the cubic approximation to

the $\sin x$. The new terms are $a_4 = 0$ and $a_5 = \frac{1}{5!}$. Our quintic polynomial approximation to $\sin x$ is

$$p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

which is the previous cubic polynomial plus two terms, one of which is zero.

> $p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is said to be the fifth degree polynomial that matches $f(x) = \sin x$ at $c = 0$.

Exercises for Section 9.4, Polynomial approximation centered at 0.

Exercise 9.4.1 On your calculator, draw the graphs of

$$f(x) = \sin x \qquad \text{and} \qquad p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad \text{for} \qquad -7\frac{\pi}{10} \le x \le 7\frac{\pi}{10}$$

The graphs should be indistinguishable on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

The largest separation on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ occurs at $x = \frac{\pi}{2}$ (and $x = -\frac{\pi}{2}$). Compute $f(\frac{\pi}{2})$, $p(\frac{\pi}{2})$, and the relative error in the approximation $f(\frac{\pi}{2}) \doteq p(\frac{\pi}{2})$.

Exercise 9.4.2 Find a cubic polynomial, $p(x) = a_0 + a_1x + x_2x^2 + a_3x^3$ that matches $f(x) = e^x$ at $c = 0$. To do so, you should complete the following table.

Analysis of $f(x) = e^x$		Analysis of $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$		Match $\mathbf{p^{(k)}(0) = f^{(k)}(0)}$
$) = e^x$	$f(0) = e^0 = \mathbf{1}$	$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$	$p(0) = \mathbf{a_0}$	$a_0 = 0$
$) = e^x$	$f'(0) = e^0 =$ ___	$p'(x) = a_1 + 2a_2x + 3a_3x^2$	$p'(0) = \mathbf{a_1}$	$a_1 =$ ___
$x) =$ ___	$f''(0) =$ ___ $=$ ___	$p''(x) =$ _____	$p''(0) =$ ___	$a_2 =$ ___
$(x) =$ ___	$f^{(3)}(0) =$ ___ $=$ ___	$p^{(3)}(x) =$ _____	$p^{(3)}(0) =$ _____	$a_3 =$ ___

You should conclude from the table that

$$p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

is the cubic polynomial that matches $f(x) = e^x$ at $c = 0$.

On your calculator draw the graphs of $f(x) = e^x$ and $p(x) = 1 + x + x^2/2 + x^3/6$ on $-1 \le x \le 1$. You should find a pretty good match. The maximum separation occurs at $x = 1$ with a maximum

$$\text{Relative error} \qquad = \frac{|1 + 1 + 1/2 + 1/6 - e^1|}{e^1} \doteq \frac{|2.6666 - 2.71828|}{e^1} \doteq 0.02$$

There is about a 2% relative error at $x = 1$.

Compute the relative error in the approximation, $e^{-1} \doteq p(-1)$.

Exercise 9.4.3 Find a fourth degree polynomial, $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, that matches $f(x) = e^x$ at the central point, $c = 0$. You did most of the work in the previous exercise.

Compute the relative errors in the approximations, $e^1 \doteq p(1)$ and $e^{-1} \doteq p(-1)$.

Exercise 9.4.4 Find the quadratic polynomial, $p(x) = a_0 + a_1x + a_2x^2$, that matches $f(x) = \cos x$ at the central point, $c = 0$. Draw the graphs of $f(x)$ and $p(x)$ and discuss the accuracy of the approximation, $\cos(\frac{\pi}{4}) \doteq p(\frac{\pi}{4})$.

Exercise 9.4.5 Find the fourth degree polynomial, $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, that matches $f(x) = \cos x$ at the central point, $c = 0$. Draw the graphs of $f(x)$ and $p(x)$ and discuss the accuracy of the approximation, $\cos(\frac{\pi}{4}) \doteq p(\frac{\pi}{4})$.

Exercise 9.4.6 Assume that the sixth degree polynomials, $S(x)$, $C(x)$, and $E(x)$ that match respectively $\sin x$, $\cos x$, and e^x are

$$\begin{aligned} S(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ C(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ E(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \end{aligned}$$

a. Compute $S'(x)$ and compare it with $C(x)$.

b. Compute $C'(x)$ and compare it with $S(x)$.

c. Compute $E'(x)$ and compare it with $E(x)$.

d. (Only for the adventurous.) Let $i = \sqrt{-1}$. Note that $i^2 = -1$, $i^3 = i^2 \times i = -i$, and $i^4 = i^2 \times i^2 = 1$, and continue this sequence. Compute $E(i * x)$ and write it in terms of $S(x)$ and $C(x)$.

9.5 Polynomial approximation to solutions of differential equations.

The following differential equations appear in the biological literature and in this section we compute polynomial approximations to their solutions.

a.	y'	$=$	ky	b.	y'	$=$	$-ky$
c.	y'	$=$	$ky(1-y/M)$	d.	y''	$=$	$-\omega^2 y$
e.	y'	$=$	$kye^{-y/\beta}-\alpha y$	f.	y''	$=$	$-ky'-\omega^2 y$

You can find polynomials that approximate solutions to differential equations and we illustrate this for some simple equations for which you either already know or soon will know exact solutions.

Example 9.5.1 *Problem*: Find a polynomial, $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$ that approximates the solution to

$$y'(t) = y(t), \qquad y(0) = 1.$$

Solution: We need to find a_0, a_1, a_2, a_3, and a_4. We will insist that '$p(t)$ match $y(t)$ at $t = 0$' meaning that

$$p(0) = y(0),\ p'(0) = y'(0),\ p^{(\prime\prime)}(0) = y^{(\prime\prime)}(0),\ p^{(3)}(0) = y^{(3)}(0) \text{ and } p^{(4)}(0) = y^{(4)}(0),$$

Because

$$p(0) = a_0,\ p'(0) = a_1,\ p''(0) = 2 \cdot a_2,\ p^{(3)}(0) = 3 \cdot 2 \cdot a_3, \text{ and } p^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot a_4,$$

it is sufficient to find values for

$$y(0), \quad y'(0), \quad y''(0), \quad y^{(3)}(0), \quad \text{and} \quad y^{(4)}(0).$$

We are given $y(0) = 1$.

$$\text{Because} \quad y'(t) = y(t), \qquad y'(0) = y(0) = 1.$$

$$\text{Because} \qquad y'(t) = y(t), \qquad y''(t) = y'(t), \qquad \text{and} \qquad y''(0) = y'(0) = 1.$$

Continuing we get

$$\begin{aligned} y''(t) &= y'(t) & y^{(3)}(t) &= y''(t) & y^{(3)}(0) &= y''(0) = 1 \\ y^{(3)}(t) &= y''(t) & y^{(4)}(t) &= y^{(3)}(t) & y^{(4)}(0) &= y^{(3)}(0) = 1 \end{aligned}$$

By insisting that

$$p(0) = y(0),\ p'(0) = y'(0),\ p^{(\prime\prime)}(0) = y^{(\prime\prime)}(0),\ p^{(3)}(0) = y^{(3)}(0) \text{ and } p^{(4)}(0) = y^{(4)}(0),$$

we conclude that

$$a_0 = 1,\ a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3!},\ \text{and } a_4 = \frac{1}{4!},$$

and that

$$p(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4.$$

You know from Chapter 5 that $y = e^t$ solves $y'(t) = y(t)$, $y(0) = 1$. The polynomial that we have found is the polynomial approximation to e^t that you (may have) found in Exercise 9.4.3. Furthermore, we are usually interested in

$$y'(t) = ky(t), \quad y(0) = y_0 \qquad \text{or} \qquad y'(t) = -ky(t) \quad y(0) = y_0$$

for which the solution is

$$y(t) = y_0 e^{kt} \qquad \text{or} \qquad y(t) = y_0 e^{-kt}$$

In the first case

$$P(t) = p(kt) = 1 + kt + \frac{1}{2}(kt)^2 + \frac{1}{3!}(kt)^3 + \frac{1}{4!}(kt)^4$$

approximates the solution and in the second case

$$P(t) = p(-kt) = 1 - kt + \frac{1}{2}(kt)^2 - \frac{1}{3!}(kt)^3 + \frac{1}{4!}(kt)^4$$

approximates the solution.

Example 9.5.2 *Problem*: Find a polynomial, $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$ that approximates the solution to

$$y''(t) + y(t) = 0, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{9.7}$$

Solution: We need to find a_0, a_1, a_2, a_3, and a_4. We will insist that '$p(t)$ match $y(t)$ at $t = 0$' meaning that

$$p(0) = y(0),\ p'(0) = y'(0),\ p^{(\prime\prime)}(0) = y^{(\prime\prime)}(0),\ p^{(3)}(0) = y^{(3)}(0) \text{ and } p^{(4)}(0) = y^{(4)}(0),$$

Because

$$p(0) = a_0,\ p'(0) = a_1,\ p''(0) = 2 \cdot a_2,\ p^{(3)}(0) = 3 \cdot 2 \cdot a_3,\ \text{and } p^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot a_4,$$

it is sufficient to find values for

$$y(0), \quad y'(0), \quad y''(0), \quad y^{(3)}(0), \quad \text{and} \quad y^{(4)}(0).$$

We are given $y(0) = 1$ and $y'(0) = 0$.

$$\text{Because} \quad y''(t) + y(t) = 0, y''(t) = -y(t), \qquad \text{and} \qquad y''(0) = -y(0) = -1.$$

Furthermore,

$$\begin{aligned} y^{(3)}(t) &= -y'(t) \qquad & y^{(3)}(0) &= -y'(0) = 0 \\ y^{(4)}(t) &= -y''(t) \qquad & y^{(4)}(0) &= -y''(0) = 1 \end{aligned}$$

By matching $p(t)$ to $y(t)$ at $t-0$ we get

$$a_0 = 1, \qquad a_1 = 0, \qquad a_2 = -1/2, \qquad a_3 = 0, \qquad \text{and} \qquad a_4 = -1/4!$$

so that

$$p(t) = 1 - \frac{t^2}{2} + \frac{t^4}{4!}$$

The solution to Equation 9.7 is $y = \cos t$ and $p(t)$ is a close approximation to $y(t)$ on the interval $[-\pi/2, \pi/2]$. For

$$Y''(t) + \omega^2 y(t) = 0, \qquad y(0) = 1, \qquad y'(0) = 0,$$

the solution is $y(t) = \cos \omega t$ and is approximated by

$$P(t) = p(\omega t) = 1 - \frac{(\omega t)^2}{2} + \frac{(\omega t)^4}{4!}.$$

Example 9.5.3 If your stomach turns queazy when dissecting a frog, you should skip this example.

Problem: Find a polynomial, $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$ that approximates the solution to the logistic equation

$$y'(t) = y(t)(1 - y(t)), \qquad y(0) = 1/2. \tag{9.8}$$

Solution: As in Examples 9.5.1 and 9.5.2 we will match derivatives at $t = 0$ to find the coefficients of p.

$$\begin{aligned}
y'(t) &= y(t) - y^2(t) && y'(0) = \frac{1}{4} \\
y''(t) &= y\prime(t) - 2y(t)y'(t) && y''(0) = 0 \\
y^{(3)} &= y'' - 2yy'' - 2(y')^2 && y^{(3)}(0) = -\frac{1}{8} \\
y^{(4)} &= y^{(3)} - 2yy^{(3)} - 2y'y'' - 4y'y'' \\
&= y^{(3)} - 6y'y'' - 2yy^{(3)} && y^{(4)}(0) = 0 \\
y^{(5)} &= y^{(4)} - 6y'y^{(3)} - 6(y'')^2 - 2yy^{(4)} - 2y'y^{(3)} && y^{(5)}(0) = \frac{1}{4}
\end{aligned}$$

By insisting that

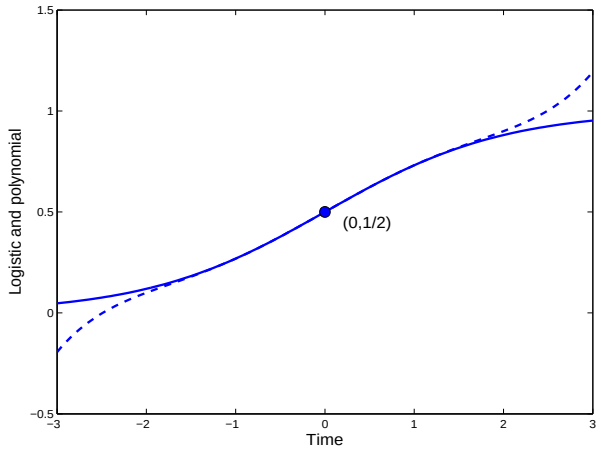
$$a_k = \frac{y^{(k)}(0)}{k!}, \qquad k = 1, 2, \cdots, 5$$

we get

$$p(t) = \frac{1}{2} + \frac{1}{4}t - \frac{1}{48}t^3 + \frac{1}{480}t^5$$

The actual solution to Equation 9.8 is $y(t) = e^t/(1 + e^t)$ and graphs of y and p are shown in Figure 9.5.3.3. Equation 9.8 is an equation for logistic growth with the initial population size equal to one-half the maximum supportable population (equal to 1), and (0,1/2) is an inflection point of the curve. Polynomials do not have horizontal asymptotes and must eventually increase without bound or decrease without bound.

Figure for Example 9.5.3.3 Graphs of $y(t) = e^t/(1+e^t)$ and $p(t) = \frac{1}{2} + \frac{1}{4}t - \frac{1}{48}t^3 + \frac{1}{480}t^5$.



Exercises for Section 9.5, Polynomial approximation to solutions of differential equations.

Exercise 9.5.1 Find polynomials of degree specified that approximate solutions to the equations shown.

							Degree
a.	$y'(t)$	$=$	$-y(t)$	$y(0)$	$=$	1	4
b.	$y''(t)$	$=$	$-y(t)$	$y(0)$	$=$	0	5
				$y'(0)$	$=$	1	
c.	$y''(t)$	$=$	$-y(t) - y'(t)$	$y(0)$	$=$	1	5
				$y'(0)$	$=$	0	
d.	$y'(t)$	$=$	$y(t)e^{-y(t)}$	$y(0)$	$=$	1	3
e.	$y'(t)$	$=$	$y(t)e^{-y(t)} - \frac{1}{2}y(t)$	$y(0)$	$=$	$\ln 2$	3

Exercise 9.5.2 Suppose $y(t)$ solves

$$y' = y(t)(1 - y(t)) \qquad \text{and} \qquad z(t) = M \times y(kt).$$

Show that $z(t)$ solves

$$z'(t) = kz(t)\left(1 - \frac{z(t)}{M}\right).$$

9.6 Polynomial approximation at center $c \neq 0$.

All of the polynomials approximations we have found so far have matched data at the central point, $c = 0$. If we wish to find a polynomial approximation to $f(x) = \ln x$, because $\ln 0$ is not defined, we will have to choose a central point other than $c = 0$. We can choose $c = 1$ as our central point, and as before, look for $p(x) = a_0 + a_1 x + a_2 x^2$ that matches $f(x) = \ln x$ at the central point, $c = 1$, as follows.

Analysis of f		Analysis of $p(x) = a_0 + a_1 x + a_2 x^2$		Match $\mathbf{p^{(k)}(0) = f^{(k)}(0)}$
$f(x) = lnx$	$f(1) = \ln 1 = \mathbf{0}$	$p(x) = a_0 + a_1 x + a_2 x^2$	$p(1) = \mathbf{a_0 + a_1 + a_2}$	$a_0 + a_1 + a_2 = 0$
$f'(x) = \frac{1}{x}$	$f'(1) = \frac{1}{1} = \mathbf{1}$	$p'(x) = a_1 + 2a_2 x$	$p'(1) = \mathbf{a_1 + 2a_2}$	$a_1 + 2a_2 = 1$
$f''(x) = \frac{-1}{x^2}$	$f''(1) = -\frac{1}{1^2} = \mathbf{-1}$	$p''(x) = 2a_2$	$p''(1) = \mathbf{2a_2}$	$2a_2 = -1$

Now we solve a system of equations,

$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_1 + 2a_2 &= 1 \\ 2a_2 &= -1 \end{aligned} \qquad \begin{aligned} a_0 + 2 + (-\tfrac{1}{2}) &= 0 \\ a_1 + 2 \times (-\tfrac{1}{2}) &= 1 \\ a_2 &= \tfrac{-1}{2} \end{aligned} \qquad \begin{aligned} a_0 &= -\tfrac{3}{2} \\ a_1 &= 2 \\ a_2 &= -\tfrac{1}{2} \end{aligned}$$

They are easy to solve, from the 'bottom up'. That is, find $a_2 = -\frac{1}{2}$ from the last equation. Substitute $-\frac{1}{2}$ for a_2 in the second equation and compute $a_1 = 2$. Then use these two values in the first equation to solve $a_1 = \frac{3}{2}$

Now our quadratic polynomial is

$$p(x) = -\frac{3}{2} + 2x - \frac{1}{2}x^2 \tag{9.9}$$

and it can be seen in Figure 9.7 that it gives a fair fit on the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Without showing the arithmetic, we claim that the cubic that matches $f(x) = \ln x$ at the center $c = 1$ is

$$p(x) = -\frac{11}{6} + 3x - \frac{3}{2}x^2 + \frac{1}{3}x^3. \tag{9.10}$$

Observe that the terms of the cubic in Equation 9.10 have no obvious relation to the terms of the quadratic in Equation 9.9, contrary to our experience when the central point was $c = 0$. When the center of the data is $c = 1$, the polynomial form

$$p(x) = \alpha_0 + \alpha_1(x - 1) + \alpha_2(x - 1)^2$$

is preferred for two reasons. First equations for the coefficients each involve only one unknown coefficient and are very easy to solve. More importantly, after a quadratic approximation has been obtained, if later

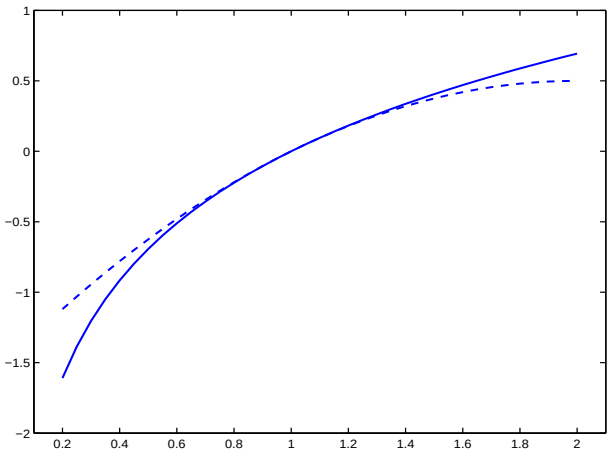


Figure 9.7: Graphs of $f(x) = \ln x$ and $p(x) = -\frac{3}{2} + 2x - \frac{1}{2}x^2$. (dashed line), the quadratic polynomial that matches the natural logarithm graph at the central point $x = 1$.

we wish to compute a cubic approximation, its first three terms are the terms of the quadratic and we only have to compute one new coefficient. We illustrate this idea.

Compute α_0, α_1, and α_2 so that the quadratic

$$p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2$$

matches $f(x) = \ln x$ at the central point, $c = 1$.

Analysis of $f(x) = \ln x$

$f(x) = \ln x \qquad f(1) = \ln 1 = \mathbf{0}$

$f'(x) = \frac{1}{x} \qquad f'(1) = \frac{1}{1} = \mathbf{1}$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -\frac{1}{1^2} = \mathbf{-1}$

Analysis of $p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2$		Match $\mathbf{p^{(k)}(0) = f^{(k)}(0)}$
$p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2$	$p(1) = \alpha_{\mathbf{0}}$	$\alpha_0 = 0$
$p'(x) = \alpha_1 + 2\alpha_2(x-1)$	$p'(1) = \alpha_{\mathbf{1}}$	$\alpha_1 = 1$
$p''(x) = 2\alpha_2(x-1)$	$p''(1) = \mathbf{2}\alpha_{\mathbf{2}}$	$\alpha_2 = -\frac{1}{2}$

Then

$$p(x) = 0 + 1 \times (x-1) - \frac{1}{2}(x-1)^2$$

is the quadratic polynomial that matches $f(x) = \ln x$ at the central point, $c = 1$. This is the same

quadratic polynomial as first computed. We expand $(x-1)^2$ and collect terms.

$$\begin{aligned} p(x) &= (x-1) - \frac{1}{2}(x-1)^2 \\ &= x - 1 - \frac{1}{2}(x^2 - 2x + 1) \\ &= x - 1 - \frac{1}{2}x^2 + x - \frac{1}{2} \\ &= -\frac{3}{2} + 2x - \frac{1}{2}x^2 \end{aligned}$$

which is the earlier quadratic form.

It is relatively easy now to compute a cubic

$$p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2 + \alpha_3(x-1)^3$$

that matches $f(x) = \ln x$ at the central point, $c = 1$.

Analysis of $f(x) = \ln x$

$f(x) = \ln x$	$f(1) = \ln 1 = \mathbf{0}$
$f'(x) = \frac{1}{x}$	$f'(1) = \frac{1}{1} = \mathbf{1}$
$f''(x) = -\frac{1}{x^2}$	$f''(1) = -\frac{1}{1^2} = \mathbf{-1}$
$f'''(x) = \frac{1}{2x^3}$	$f'''(1) = \frac{1}{2\times 1^2} = \mathbf{\frac{1}{2}}$

Analysis of $p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2 + \alpha_3(x-1)^3$		Match $\mathbf{p^{(k)}(1) = f^{(k)}(1)}$
$p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2 + \alpha_3(x-1)^3$	$p(1) = \alpha_{\mathbf{0}}$	$\alpha_0 = 0$
$p'(x) = \alpha_1 + 2\alpha_2(x-1) + 3\alpha_3(x-1)^2$	$p'(1) = \alpha_{\mathbf{1}}$	$\alpha_1 = 1$
$p''(x) = 2\alpha_2(x-1) + 3 \times 2\alpha_3(x-1)^3$	$p'(1) = \mathbf{2}\alpha_{\mathbf{2}}$	$\alpha_2 = -\frac{1}{2}$
$p'''(x) = 3 \times 2\alpha_3$	$p'''(1) = \mathbf{3} \times \mathbf{2}\alpha_{\mathbf{3}}$	$\alpha_3 = \frac{1}{3}$

As for the quadratic, $\alpha_0 = 0$, $\alpha_1 = 1$, and $\alpha_2 = -\frac{1}{2}$. The only new term $\alpha_3(x-3)^3 = \frac{1}{3}(x-3)^3$ and the cubic is

$$p(x) = 0 + 1 \times (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

General form of the coefficients. There is a general pattern to the coefficients for an approximating polynomial with central point c when the polynomial is expanded in powers of $(x-c)$.

We treat the case of a third degree polynomial to illustrate the pattern.

Assume we are given a function, f, and a number, c, and know

$$f(c) \qquad f'(c) \qquad f^{(2)}(c) \qquad \text{and} \qquad f^{(3)}(c)$$

Then the third degree polynomial that matches f at c is

$$p(x) = f(c) + f'(c)(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

The lower and higher degree polynomials that match f at c have a similar form.

We show the validity for the third degree polynomial. Let

$$p(x) = \alpha_0 + \alpha_1(x-c) + \alpha_2(x-c)^2 + \alpha_3(x-c)^3$$

and observe that

$$\begin{aligned}
p(x) &= \alpha_0 + \alpha_1(x-c) + \alpha_2(x-c)^2 + \alpha_3(x-c)^3 & p(c) &= \alpha_0 \\
p'(x) &= \alpha_1 + 2\alpha_2(x-c) + 3\alpha_3(x-c)^2 & p'(c) &= \alpha_1 \\
p''(x) &= 2\alpha_2 + 3 \times 2\alpha_3(x-c) & p''(c) &= 2! \times \alpha_2 \\
p^{(3)}(x) &= 3 \times 2\alpha_3 & p^{(3)}(c) &= 3! \times \alpha_3
\end{aligned}$$

Now the requirement that $p^{(k)}(c) = f^{(k)}(c)$ leads to

$$\begin{aligned}
p(c) &= f(c) & \alpha_0 &= f(c) & \alpha_0 &= f(c) \\
p'(c) &= f'(c) & \alpha_1 &= f'(c) & \alpha_1 &= f'(c) \\
p''(c) &= f''(c) & 2! \times \alpha_2 &= f''(c) & \alpha_2 &= \frac{f''(c)}{2!} \\
p^{(3)}(c) &= f^{(3)}(c) & 3! \times \alpha_3 &= f^{(3)}(c) & \alpha_3 &= \frac{f^{(3)}(c)}{3!}
\end{aligned}$$

The general form for the cubic approximating polynomial with central point, c, is

$$p(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

Lower and higher order polynomials have similar forms.

Example 9.6.1 Suppose you are monitoring a rare avian population and you have the data shown. What is your best estimate for the number of adults on January 1, 2003?

Adult Avian Census		
Jan 1, 2000	Jan 1, 2001	Jan 1, 2002
1000	1050	1080

Let t measure years after the year 2000, so that $t = 0$, $t = 1$, and $t = 2$ correspond to the years 2000, 2001, and 2002, respectively, and let $A(t)$ denote adult avian population at time t. Our problem is to estimate $A(3)$.

Solution 1. We know $A(1) = 1050$. If we knew $A'(1)$ and $A''(1)$ we could compute the degree-2 polynomial, $P(t)$, that matches $A(t)$ at the central point $c = 1$ and use $P(3)$ as our estimate of $A(3)$. We use the following estimates:

$$A'(c) \doteq \frac{A(c+h) - A(c-h)}{2h} \qquad A''(c) \doteq \frac{A(c+h) - 2A(c) + A(c-h)}{h^2}$$

where for our problem, $c = 1$ and $h = 1$. The first estimate is what we have called the centered difference quotient estimate of the derivative; we call the second estimate the centered difference quotient estimate of the second derivative. A rationale for these two estimates appears in the discussion of Solution 2. With these estimates, we have

$$\begin{aligned} A'(1) &\doteq \frac{A(2) - A(0)}{2} \\ &= \frac{1080 - 1000}{2} = 40 \\ A''(1) &\doteq \frac{A(2) - 2A(1) + A(0)}{1^2} \\ &= \frac{1080 - 2 \times 1050 + 1000}{1} = -20 \end{aligned}$$

Therefore,

$$\begin{aligned} P(t) &= A(1) + A'(1)(t-1) + \frac{1}{2}A''(1)(t-1)^2 \\ &= 1050 + 40(t-1) + \frac{1}{2}(-20)(t-1)^2 \\ P(3) &= 1050 + 40 \times 2 + \frac{1}{2}(-20)(2)^2 = 1090 \end{aligned}$$

Our estimate of the January 1, 2003 adult population is 1090.

Solution 2. We might compute the second degree polynomial, $Q(t)$, that satisfies

$$Q(0) = A(0), \qquad Q(1) = A(1), \qquad \text{and} \qquad Q(2) = A(2). \tag{9.11}$$

Then $Q(3)$ would be our estimate of the January 1, 2003 adult population.

It is an interesting fact that $Q(t)$ is the degree-2 polynomial, $P(t) = 1050 + 40(t-1) + \frac{1}{2}(-20)(t-1)^2$, that we computed in Solution 1. We can easily check this by substitution:

$$\begin{aligned} P(0) &= 1050 + 40 \times (-1) + \frac{1}{2}(-20)(-1)^2 \\ &= 1050 - 40 - 10 = 1000 \end{aligned}$$

$$\begin{aligned} P(1) &= 1050 \\ P(2) &= 1050 + 40 \times (1) + \frac{1}{2}(-20)(1)^2 = 1080 \end{aligned}$$

There is only one quadratic polynomial satisfying the three conditions 9.11, $P(t)$ is one such, so $P(t)$ is $Q(t)$. Our estimate of the January 1, 2003 adult population is again 1090.

It is a general result that if $(c-h, A(c-h))$, $(c, A(c))$, and $(c+h, A(c+h))$ are equally space data points then

$$P(t) = A(c) + \frac{A(c+h) - A(c-h)}{2h}(t-c) + \frac{1}{2}\frac{A(c+h) - 2A(c) + A(c-h)}{h^2}(t-c)^2 \tag{9.12}$$

is the quadratic polynomial that matches the three data points. Furthermore,

$$P'(c) \doteq \frac{A(c+h) - A(c-h)}{2h} \qquad P''(c) \doteq \frac{A(c+h) - 2A(c) + A(c-h)}{h^2} \tag{9.13}$$

These results can be confirmed by substitution and differentiation. ■

Exercises for Section 9.6, Polynomial approximation at center $c \neq 0$.

Exercise 9.6.1 a. Compute the fourth degree polynomial,

$$p(x) = \alpha_0 + \alpha_1(x-1)^2 + \alpha_2(x-1)^2 + \alpha_3(x-1)^3 + \alpha_4(x-1)^4$$

that matches $f(x) = \ln x$ at the central point $c = 1$.

b. From the pattern of coefficients in p, guess the fifth degree term of a fifth degree polynomial that matches $f(x) = \ln x$ at the central point $c = 1$.

Exercise 9.6.2 Compute the coefficients of a quadratic polynomial, $p(x) = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2$ that matches the function, $f(x) = \sqrt{x}$ at the central point, $c = 1$.

Exercise 9.6.3 Compute the coefficients of quadratic polynomials that match the following functions at the indicated central points:

	Function	Central Point		Function	Central Point
a.	$f(x) = \sin x$	$c = \frac{\pi}{2}$	b.	$f(x) = \sin x$	$c = \frac{\pi}{4}$
c.	$f(x) = e^x$	$c = 1$	d.	$f(x) = x^2$	$c = 1$

Exercise 9.6.4 What is your best estimate of the adult avian population for January 1, 2003 based on the following data? Which of these estimates seems unlikely.

Adult Avian Census			
	Jan 1, 2006	Jan 1, 2007	Jan 1, 2008
Case 1.	1000	1050	1110
Case 2.	1000	1050	1100
Case 3.	1000	950	920
Case 4.	1000	980	1010

Exercise 9.6.5 Compute $P(c-h)$, $P(c)$, and $P(c+h)$ from Equation 9.12 and show that $P(c-h) = A(c-h)$, $P(c) = A(c)$, and $P(c+h) = A(c+h)$.

Exercise 9.6.6 Compute $P'(c)$ and $P''(c)$ from Equation 9.12 and confirm Equations 9.13.

9.7 The Accuracy of the Taylor Polynomial Approximations.

The polynomials we have been studying are called Taylor polynomials after an English mathematician Brooke Taylor (1685-1731) who wrote a good expository account of them in 1715. In some texts, the polynomials for the important case where the central point is $c = 0$ are called McLaurin polynomials after Colin McLaurin (1698-1746), who contributed to their development. We refer to them all as Taylor polynomials.

We have seen that the approximations by polynomials can be quite 'good'. It is better to be able to quantify 'good' and there is a theorem to that effect.

Theorem 9.7.1 The Remainder Theorem for Taylor's Polynomials. *Suppose f is a function with continuous first, second, third, and fourth derivatives on an interval, $[a, b]$. Then there are numbers c_0, c_1, c_2, and c_3 such that*

$$f(b) = f(a) + \mathbf{f'(c_0)(b-a)} \tag{9.14}$$

$$f(b) = f(a) + f'(a)(b-a) + \mathbf{\frac{f^{(2)}(c_1)}{2!}(b-a)^2} \tag{9.15}$$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \mathbf{\frac{f^{(3)}(c_2)}{3!}(b-a)^3} \tag{9.16}$$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \mathbf{\frac{f^{(4)}(c_3)}{4!}(b-a)^4} \tag{9.17}$$

Each formula states that the actual value of $f(b)$ is the matching polynomial value, $p(b)$, plus the remainder term that appears in bold face. There are similar formulas for higher order polynomial matches.

The numbers c_0, c_1, c_2, and c_3 share the mystery of the number c in Equation 9.1 of the Mean Value Theorem. In fact, you are asked in Exercise 9.7.2 to show that Equation 9.14 is the same as Equation 9.1.

Lets look at the second formula, Equation 9.15, in the context of Pat looking for Mike. Suppose Mike is at First and Main, is jogging north at a *constant rate* of two blocks per minute. Let $f(t)$ be the position of Mike measured in city blocks numbered First, Second, $\cdots$, Seventh, going north. Let t be time in minutes after Sean sees Mike at First and Main, and let $a = 0$ and $b = 3$ minutes. Because Mike's speed is constant during the three minutes, $f'(t)$ is constant ($= 2$) and therefore $f''(t) = 0$ for all t in [0,3].

In the formula

$$f(b) = f(a) + f'(a)(b-a) + \frac{\mathbf{f^{(2)}(c_1)}}{\mathbf{2!}}(\mathbf{b-a})^{\mathbf{2}},$$

$b = 3$, $a = 1$, $f'(a) = 2$, and $f^{(2)}(c_1) = 0$ so that

$$\begin{aligned} f(3) &= f(0) + f'(0)(3-0) + \frac{0}{2!}(3-0) \\ &= 1 + 2(3-0) \\ &= 7 \end{aligned}$$

Thus Pat should look for Mike at Seventh and Main, as we thought.

Now lets look at the last formula, Equation 9.17 for the case that $f(x) = \sin x$ and $a = 0$ and $b = \frac{\pi}{4}$. Remember that

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x \end{aligned}$$

According to Taylor's Theorem 9.7.1 and Equation 9.17, there is a number, c_3 in $[0, \frac{\pi}{4}]$ for which

$$\sin\frac{\pi}{4} = 0 + 1(\frac{\pi}{4} - 0) + \frac{0}{2!}\left(\frac{\pi}{4} - 0\right)^2 + \frac{-1}{3!}\left(\frac{\pi}{4} - 0\right)^3 + \frac{\sin(\mathbf{c_3})}{\mathbf{4!}}\left(\frac{\pi}{\mathbf{4}} - \mathbf{0}\right)^{\mathbf{4}}$$

or

$$\sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{3!}\left(\frac{\pi}{4}\right)^3 + \frac{\sin(\mathbf{c_3})}{\mathbf{4!}}\left(\frac{\pi}{\mathbf{4}} - \mathbf{0}\right)^{\mathbf{4}}.$$

What is the error in

$$\sin\frac{\pi}{4} \doteq \frac{\pi}{4} - \frac{1}{3!}\left(\frac{\pi}{4}\right)^3?$$

The answer is

$$\text{exactly} \qquad \frac{\sin(\mathbf{c_3})}{\mathbf{4!}}(\frac{\pi}{\mathbf{4}} - \mathbf{0})^{\mathbf{4}}.$$

Some fuzz appears. We do not know the value of c_3. We only know that it is a number between 0 and $\frac{\pi}{4}$. We do a **worst case analysis.** The largest value of $\sin c_3$ for c_3 in $[0, \frac{\pi}{4}]$ is for $c_3 = \frac{\pi}{4}$ and

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \doteq 0.7071$. Therefore, we say that the error in

$$\sin\frac{\pi}{4} \doteq \frac{\pi}{4} - \frac{1}{3!}(\frac{\pi}{4})^3 \qquad \textbf{is no bigger than} \qquad \frac{\frac{\sqrt{2}}{2}}{4!}(\frac{\pi}{4})^3 \doteq 0.0112$$

We computed the actual error in Equation 9.6 and found it to be 0.0024. Therefore, our 'worst case analysis' over estimated the error by $\frac{0.0112}{0.0024}$ which is about a factor of 5. This is common in a robust analysis that guarantees the error is no bigger than the computed value.

We will prove the second formula, Equation 9.15, in Theorem 9.7.1. You are asked to prove that the first formula is the Mean Value Theorem in Exercise 9.7.2. It will be seen that the proof of the second formula is just an extension of the proof of the Mean Value Theorem. The remaining formulas of Theorem 9.7.1 have similar proofs.

We must prove that there is a number, c_1, in $[a, b]$ for which

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(c_1)}{2!}(b-a)^2$$

Claim: There is a number, M, such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{M}{2!}(b-a)^2 \tag{9.18}$$

Just solve the previous equation for M to find the needed value. We let

$$q(x) = f(a) + f'(a)(x-a) + \frac{M}{2!}(x-a)^2 \tag{9.19}$$

and show in Figure 9.8 a graph of f and q. You are asked in Exercise 9.7.3 to show that

$$\left.\begin{array}{rcl} q(a) & = & f(a) \\ q'(a) & = & f'(a) \\ q(b) & = & f(b) \end{array}\right\} \tag{9.20}$$

The difference D in f and q defined by

$$D(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{M}{2!}(x-a)^2\right)$$

is also shown in Figure 9.8.

Now we observe that $D(a) = 0$ and $D(b) = 0$.

$$D(a) = f(a) - \left(f(a) + f'(a)(a-a) + \frac{M}{2!}(a-a)^2\right) = 0, \qquad \text{and}$$

$$D(b) = f(b) - \left(f(a) + f'(a)(b-a) + \frac{M}{2!}(b-a)^2\right) = 0$$

by the definition of M in Equation 9.18. By Rolle's Theorem, there is a number c, $a < c < b$, so that $D'(c) = 0$. We have to compute D'.

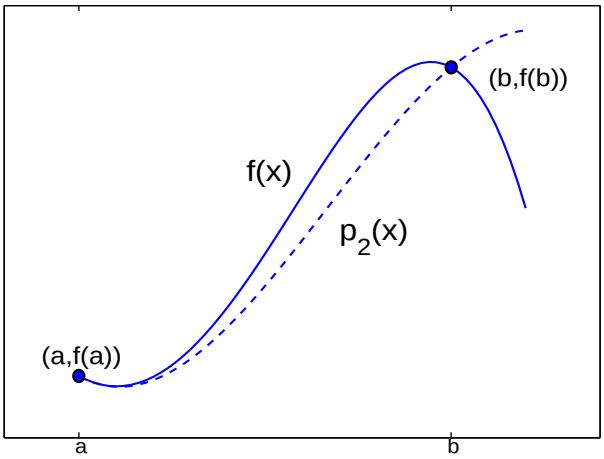


Figure 9.8: Graph of a function f and a quadratic polynomial q for which $q(a) = f(a)$, $q'(a) = f'(a)$ and $q(b) = f(b)$.

$$\begin{aligned} D'(x) &= \left[f(x) - \left(f(a) + f'(a)(x-a) + \frac{M}{2!}(x-a)^2\right)\right]' \\ &= [f(x)]' - [f(a)]' - [f'(a)(x-a)]' - \left[\frac{M}{2!}(x-a)^2\right]' \\ &= f'(x) - 0 - f'(a) \times 1 - M \times (x-a). \end{aligned}$$

Watch this! We know that $D'(c) = 0$, but we next show that $D'(a) = 0$ also. By the previous equation,

$$\begin{aligned} D'(a) &= f'(x) - 0 - f'(a) \times 1 - M \times (x-a)|_{x=a} \\ &= f'(a) - 0 - f'(a) \times 1 - M \times (a-a) \\ &= 0. \end{aligned}$$

Now, by Rolle's Theorem (again), because $D'(a) = 0$ and $D'(c) = 0$, there is a number (call it c_1) such that

$$D''(c_1) = 0$$

We have to compute $D''(x)$.

$$\begin{aligned} D''(x) &= [f'(x) - f'(a) - M \times (x-a)]' \\ &= [f'(x)]' - [f'(a)]' - [M \times (x-a)]' \\ &= f''(x) - M. \end{aligned}$$

$$D'(c_1) = 0 \qquad \text{implies that} \qquad f''(c_1) - M = 0 \qquad \text{or} \qquad M = f''(c_1) = f^{(2)}(c_1).$$

We substitute this value for M in Equation 9.18 and get

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(c_1)}{2!}(b-a)^2$$

which is what was to be proved. Whew!

Example 9.7.1 Taylor's formula

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(c_1)}{2!}(b-a)^2$$

provides a short proof of the second derivative test for a maximum:

$$\text{If} \qquad f'(a) = 0 \qquad \text{and} \qquad f''(a) < 0 \qquad \text{then } (a, f(a)) \text{ is a local maximum for the graph of } f.$$

We suppose that $f^{(2)}$ is continuous so that $f^{(2)}(a) < 0$ implies that there is an interval, (p, q) surrounding a so that $f^{(2)}(x) < 0$ for all x in (p, q).

Suppose b is in (p, q). Then because $f'(a) = 0$

$$f(b) = f(a) + \frac{f^{(2)}(c_1)}{2!}(b-a)^2.$$

c_1 is between b and a and therefore is in (p, q) so that

$$f(b) = f(a) + \quad \text{a negative number.}$$

Thus, $(a, f(a))$ is a local maximum for f. ■

Example 9.7.2 Taylor's Theorem provides a good explanation as to why the centered difference quotient is usually a better approximation to $f'(a)$ than is the forward difference quotient. Directly from Taylor's formula with $b = a + h$, $b - a = h$ is

$$f(a+h) = f(a) + f'(a)h + \frac{f^{(2)}(c_1)}{2!}h^2$$

we can solve for $f'(a)$ and get

$$f'(a) = \frac{f(a+h) - f(a)}{h} - \frac{f^{(2)}(c_1)}{2!}h \tag{9.21}$$

The equation states that $f'(a)$ is the forward difference quotient plus a number times h. Taylor's formula for $n = 2$ can be written twice:

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + f''(a)\frac{h^2}{2} + f^{(3)}(c_1)\frac{h^3}{6} \\ f(a-h) &= f(a) + f'(a)(-h) + f''(a)\frac{(-h)^2}{2} + f^{(3)}(c_2)\frac{(-h)^3}{6} \end{aligned}$$

Subtraction of the last two equations yields

$$f(a+h) - f(a-h) \;=\; f'(a)(2h) + \Big(f^{(3)}(c_1) + f^{(3)}(c_2) \Big) \frac{h^3}{6}$$

Now we solve for $f'(a)$ and get

$$\begin{aligned} f'(a) &= \frac{f(a+h) - f(a-h)}{2h} - \frac{f^{(3)}(c_1) + f^{(3)}(c_2)}{2} \frac{h^2}{6} \\ &= \frac{f(a+h) - f(a-h)}{2h} - f^{(3)}(c) \frac{h^2}{6} \end{aligned} \tag{9.22}$$

This equation states that $f'(a)$ is the centered difference quotient plus a number times h^2.

The number h used in computing difference quotients should be small, say 0.01, so that h^2 is smaller, (0.0001), suggesting that the error term in the centered difference quotient is smaller than the error in the forward difference quotient. This advantage may be nullified, however, if $f^{(3)}(c)/6$ greatly exceeds $f^{(2)}(c_2)/2$. ■

Exercises for Section 9.7, The Accuracy of the Taylor Polynomial Approximations.

Exercise 9.7.1 Find a bound on the error of the Taylor's polynomial approximation centered at $c = 0$ and of indicated degree to

a. $f(x) = \cos x$, of degree 2, on $[0, \pi/4]$.

b. $f(x) = e^x$, of degree 4 on $[0, 1]$.

c. $f(x) = \cos x$, of degree 4 on $[0, \pi/2]$.

d. $f(x) = \sin x$, of degree 5, on $[0, \pi/2]$.

e. $f(x) = \ln(1 + x)$ of degree 5 on $[0, 1/2]$.

Exercise 9.7.2 Examine the formula, Equation 9.14. Show that it is just a statement of the Mean Value Theorem.

Exercise 9.7.3 Show that the function, q defined in Equation 9.19 has the properties stated in Equation 9.20.

Exercise 9.7.4 Add the two versions of Taylor's third order polynomial with error, Equation 9.17:

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + \frac{f^{(2)}(a)}{2!}h^2 + \frac{f^{(3)}(a)}{3!}h^3 + \frac{f^{(4)}(c_1)}{3!}h^4 \\ f(a-h) &= f(a) - f'(a)h + \frac{f^{(2)}(a)}{2!}h^2 - \frac{f^{(3)}(a)}{3!}h^3 + \frac{f^{(4)}(c_2)}{4!}h^4, \end{aligned}$$

and solve for $f^{(2)}(a)$. You should find that (after a slight alteration)

$$f^{(2)}(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} - \frac{f^{(4)}(c_1) + f^{(4)}(c_2)}{2}\frac{h^2}{12}$$

Use the intermediate value property to argue that there is a number c such that

$$f^{(2)}(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} - f^{(4)}(c)\frac{h^2}{12} \tag{9.23}$$

This provides a good formula for approximating second derivatives.

Chapter 10

Partial derivatives of functions of two variables.

Where are we going?

Most measurable biological quantities are dependent on more than one variable; they are functions of two or more variables. The concept of derivative of a function of one variable is extended to functions of two variables in this chapter. Conditions for local maxima and minima of functions of two variables are presented.

Functions of two variables may describe diffusion of disease, invasive species, heat, or chemicals in space and time dimensions. An equation that relates partial derivatives with respect to to a space variable and with respect to time is introduced and used to quantify diffusion processes.

10.1 Partial derivatives of functions of two variables.

Most measurable biological quantities are dependent on more than one variable. Corn yield is measurably dependent on rainfall, number of degree-days and available nitrogen, potassium, and phosphorus. Brain development in children is dependent on several nutritional factors as well as environmental factors such as rest and sociological experiences.

A function of two variables is a rule that assigns to each ordered number pair in a set called its domain a number in a set called its range. Examples include

$$F(x,y) = x^2 + y^2 \qquad F(x,y) = xe^{-y^2} \qquad F(x,y) = \sin(x+y) \qquad F(x,y) = \sqrt{x}\ln y$$

The domains of the first three functions implicitly are all number pairs (x, y). The first function assigns to the number pair (x, y) the number $x^2 + y^2$; it assigns to (-2,3) the number 13, for example. The domain of $F(x, y) = \sqrt{x}\ln y$ implicitly is the set of number pairs (x, y) for which $x \geq 0$ and $y > 0$. In each example, the first variable of F is x and the second variable of F is y.

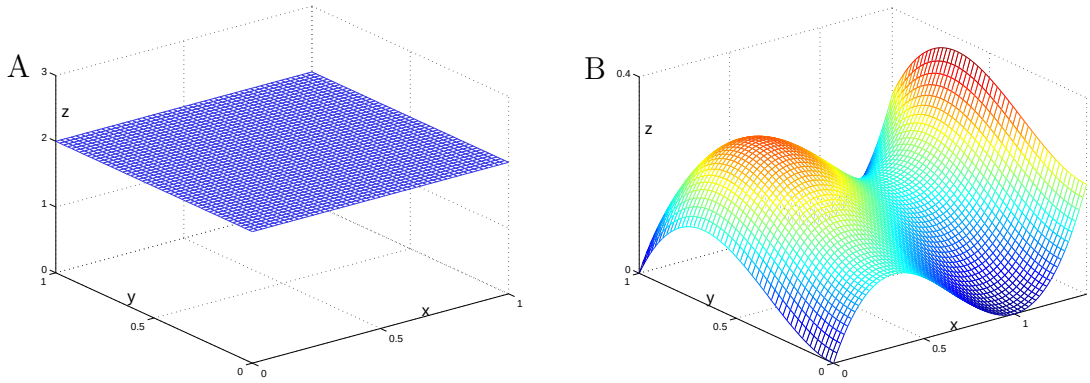


Figure 10.1: Graph in 3-dimensional space of A. $F(x, y) = 2$ and B. $F(x, y) = x(1 - x)^2 + y^2(1 - y)$.

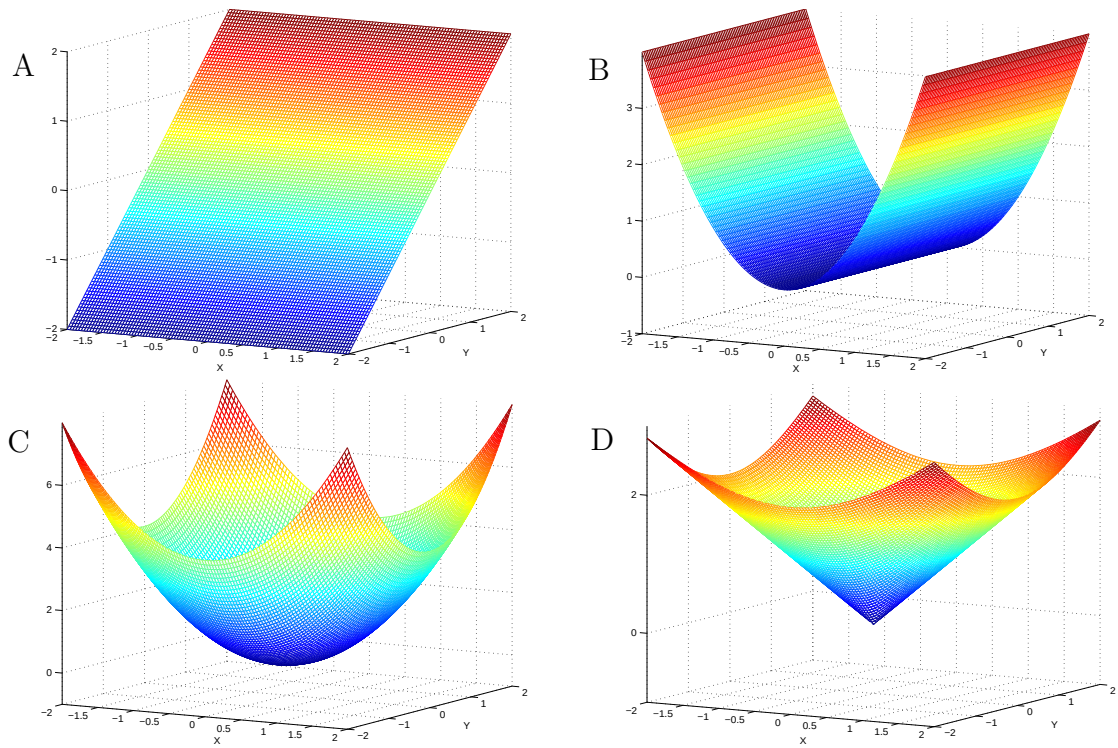


Figure 10.2: Graphs in 3-dimensional space of A. $F(x, y) = y$, B. $F(x, y) = x^2$, C. $F(x, y) = x^2 + y^2$, C. $F(x, y) = \sqrt{x^2 + y^2}$.

Graphs of functions of two variables can be visualized in three-dimensional space (3-space) with the domain D lying in a horizontal x, y plane and the vertical axis being $z = F(x, y)$. Shown in Figure 10.1A is the graph of $F(x, y) = 2$ which is a horizontal plane a distance 2 above the x, y plane. Shown in Figure 10.1B is a graph of the function $F(x, y) = x(1 - x)^2 + y^2(1 - y)$. Additional graphs of functions of two variables are shown in Figure 10.2. In drawing such graphs, it is customary to use a **"right-handed axis system."** Visualize your right hand aligned so that your thumb lies on and points in the direction of the positive z-axis. Then in a right-handed system, your fingers will point from the positive x-axis to

the positive y-axis.

A linear function of two variables is a function of the form $F(x, y) = ax + by + c$ where a, b, and c are numbers. For example, $F(x, y) = 2x + 3y - 6$ is a linear function and the graph of $z = 2x + 3y - 6$ in Figure 10.3 is a plane in three-dimensional space. The portion of the graph of $z = 2x + 3y - 6$ that lies in the plane $y = 0$ (marked A) is the line $z = 2x - 6$ in the x, z plane. The x, z plane is the set of all points $(x, 0, z)$ in 3-space. The portion of the graph of $z = 2x + 3y - 6$ that lies in the plane $x = 0$ (marked B) is the line $z = 3y - 6$ in the y, z plane. The portion of the graph of $z = 2x + 3y - 6$ that lies in the plane $z = 0$ (marked C) is the line $0 = 2x + 3y - 6$ in the $(x, y, 0)$ plane.

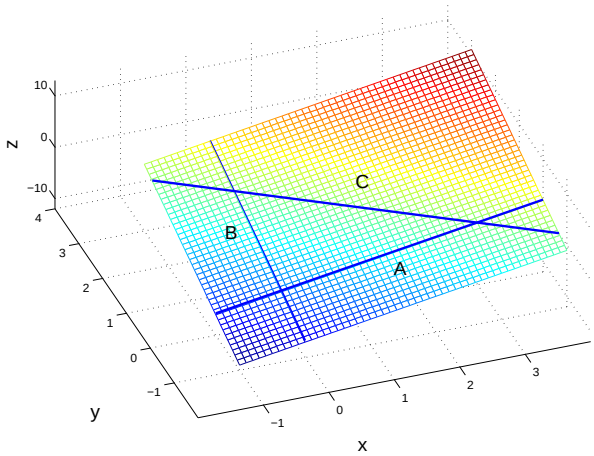


Figure 10.3: Graph of the plane $z = 2x + 3y - 6$ in three dimensional space. The line marked 'A' is the slice of that plane with $y = 0$. The line marked 'B' is the slice with $x = 0$, and the line marked 'C' is the 'level curve' with $z = 0$.

Definition 10.1.1 Partial Derivative. Suppose F is a function of two variables and (a, b) is in the domain of F. The partial derivative of F with respect to its first variable, denoted by F_1, is the ordinary derivative of F with respect to its first variable with the second variable held constant. Similarly, the partial derivative of F with respect to its second variable, denoted by F_2, is the ordinary derivative of F with respect to its second variable with the first variable held constant.

Second order derivatives are denoted by $F_{1,1}$, $F_{1,2}$, $F_{2,1}$, and $F_{2,2}$ where $F_{i,j}$ is the derivative of F_i with respect to the $\underline{jth}$ variable.

The limit definitions of partial derivatives are

$$F_1(a, b) = \lim_{h \to 0} \frac{F(a + h, b) - F(a, b)}{h} \qquad F_2(a, b) = \lim_{h \to 0} \frac{F(a, b + h) - F(a, b)}{h}. \tag{10.1}$$

The Leibnitz notation can be particularly helpful in writing partial derivatives of $F(x, y)$.

$$F_1(a,b) = \frac{\partial F}{\partial x}(a,b), \qquad F_2(a,b) = \frac{\partial F}{\partial y}(a,b),$$

$$F_{1,1}(a,b) = \frac{\partial^2 F}{\partial x^2}(a,b), \qquad F_{1,2}(a,b) = \frac{\partial^2 F}{\partial y \partial x}(a,b), \qquad F_{2,2}(a,b) = \frac{\partial^2 F}{\partial y^2}(a,b).$$

When notations for the domain variables of F are clear, as in $F = F(x, y)$ it is helpful to write, for examples,

$$\begin{aligned} F_1(x,y) &= F_x(x,y), & F_{1,1}(x,y) &= F_{xx}(x,y), \\ F_2(x,y) &= F_y(x,y), \quad \text{and} & F_{2,1}(x,y) &= F_{yx}(x,y). \end{aligned}$$

Some examples of partial derivatives are:

$$F(x,y) = x^2 + y^2$$

$$\begin{aligned} F_1(x,y) &= 2x & F_{1,2}(x,y) &= 0 & F_2(x,y) &= 2y \\ F_{1,1}(x,y) &= 2 & F_{2,1}(x,y) &= 0 & F_{2,2}(x,y) &= 2 \end{aligned}$$

$$F(x,y) = xe^{-y^2}$$

$$\begin{aligned} F_1(x,y) &= e^{-y^2} & F_{1,2}(x,y) &= e^{-y^2} \times (-2y) & F_2(x,y) &= xe^{-y^2} \times (-2y) \\ F_{1,1}(x,y) &= 0 & F_{2,1}(x,y) &= e^{-y^2} \times (-2y) & F_{2,2}(x,y) &= -2xe^{-y^2} + 4xy^2e^{-y^2} \end{aligned}$$

$$F(x,y) = \sin(x+y)$$

$$\begin{aligned} F_x(x,y) &= \cos(x+y) & F_{xy}(x,y) &= -\sin(x+y) & F_y(x,y) &= \cos(x+y) \\ F_{xx}(x,y) &= -\sin(x+y) & F_{yx}(x,y) &= -\sin(x+y) & F_{yy}(x,y) &= -\sin(x+y) \end{aligned}$$

$$F(x,y) = x^{1/2} \ln y$$

$$\begin{aligned} \frac{\partial F(x,y)}{\partial x} &= (1/2)x^{-1/2} \ln y & \frac{\partial^2 F(x,y)}{\partial x \partial y} &= (1/2)x^{-1/2}y^{-1} & \frac{\partial F(x,y)}{\partial y} &= x^{1/2}y^{-1} \\ \frac{\partial^2 F(x,y)}{\partial^2 x} &= -(1/4)x^{-3/2} \ln y & \frac{\partial^2 F(x,y)}{\partial y \partial x} &= (1/2)x^{-1/2}y^{-1} & \frac{\partial^2 F(x,y)}{\partial y^2} &= -x^{1/2}y^{-2} \end{aligned}$$

In all of these cases, and usually, $F_{1,2} = F_{2,1}$. *Always when $F_{1,2}$ and $F_{2,1}$ are continuous they are equal.*

Definition 10.1.2 Limit and continuity of a function of two variables. Suppose F is a function of two variables with domain D and (a, b) is a point of D and L is a number. Then

$$\lim_{(x,y)\to(a,b)} F(x, y) = L$$

means that if ϵ is a positive number, there is a positive number δ such that if (x, y) is in D and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|F(x, y) - L| < \epsilon$. The statement that F is continuous at (a, b) means that

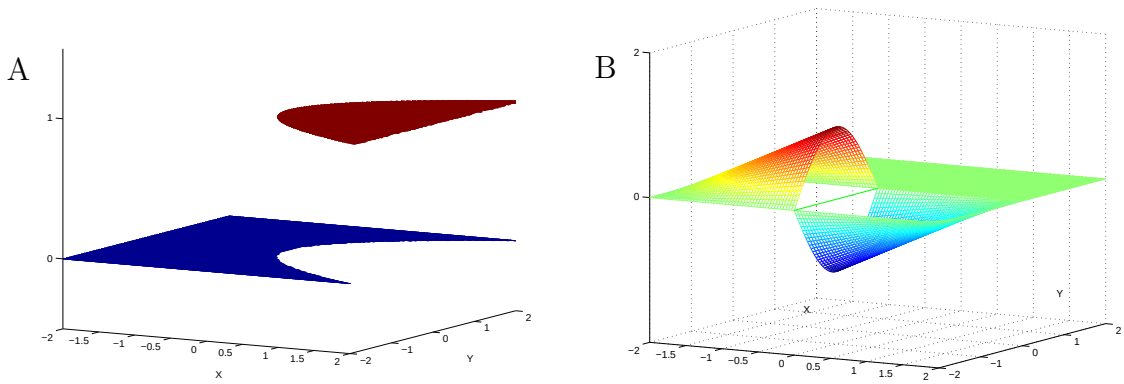
$$\lim_{(x,y)\to(a,b)} F(x, y) = F(a, b)$$

or that there is a number $\delta_0 > 0$ such that for every point (x, y) of D distinct from (a, b), $\sqrt{(x-a)^2 + (y-b)^2} > \delta_0$.

Explore 10.1.1 Identify the points of the graphs in Explore Figure 10.1.1 A and B at which the graphs are discontinuous.

Explore Figure 10.1.1 A. $F(x, y) = 1$ if $x^2 \le y$; else $F(x, y) = 0$.

B. $F(x, y) = -(1 + x/2)\sin(\pi y/2)$ for $x < 0$ and $-2 < y < 0$, $F(x, y) = (1 - x/2)\sin(\pi y/2)$ for $x > 0$ and $-2 < y < 0$, else $F(x, y) = 0$.



In Figure 10.4, slices of the graph of

$$F(x, y) = x(1-x)^2 + y^2(1-y)$$

corresponding to $x = 0.2$ and $y = 0.4$ are shown. The tangent to the slice $x = 0.2$ at the point $(0.2, 0.7, 0.191)$ is drawn. Its slope is

$$F_2(0.2, 0.7) = 0 + 2y(1-y) - y^2\Big|_{(x,y)=(0.2,0.7)} = -0.07.$$

The tangent to the slice $y = 0.4$ at the point $(0.3, 0.4, 0.243)$ is also drawn. Its slope is

$$F_1(0.3, 0.4) = (1-x)^2 - 2x(1-x)\Big|_{(x,y)=(0.3,0.4)} = 0.07.$$

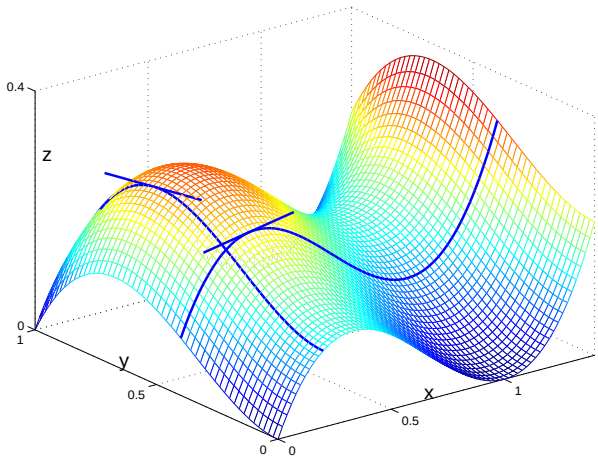


Figure 10.4: Graph in 3-dimensional space of $F(x, y) = x(1-x)^2 + y^2(1-y)$ and slices through the graph at $x = 0.2$ and $y = 0.4$.

Definition 10.1.3 Local linear approximation, tangent plane. Suppose F is a function of two variables, (a, b) is a number pair in the domain of F, and F_1 and F_2 exist and are continuous on the interior of a circle with center (a, b). Then the local linear approximation to F at (a,b) is the linear function

$$L(x, y) = F(a, b) + F_1(a, b) \times (x - a) + F_2(a, b) \times (y - b) \tag{10.2}$$

The graph of L is the *tangent plane* to the graph of F at $(a, b, F(a, b))$, and F is said to be differentiable at (a, b).

For $F(x, y) = x^2 + y^2$, $F_1(x, y) = 2x$, and $F_2(x, y) = 2y$. At the point (1,2), F_1 and F_2 are continuous on the circle of radius 1 and center (1,2) (actually continuous everywhere), and

$$F(1, 2) = 5, \qquad F_1(1, 2) = 2x|_{x=1,y=2} = 2, \qquad F_2(1, 2) = 4,$$

The local linear approximation to F at (1,2) is

$$L(x, y) = 5 + 2(x - 1) + 4(y - 2).$$

A graph of F and L appear in Figure 10.5. The graph of L is below the graph of F except at the point of tangency, (1,2,5).

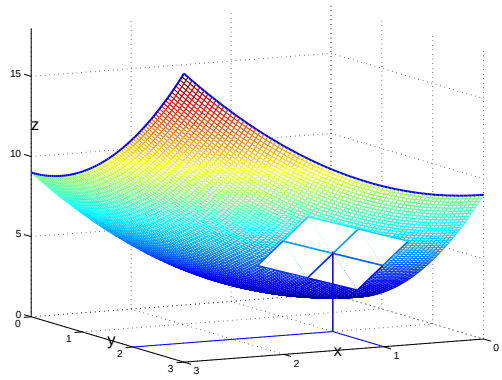


Figure 10.5: Graph in 3-dimensional space of $z = F(x, y) = x^2 + y^2$ and a square lying in the tangent plane at (1,2,5), $L(x, y) = 5 + 2(x - 1) + 4(y - 2)$, which is below the graph of F except at the point of tangency, (1,2,5). View is from below the graph.

For functions f of one variable, the tangent line to the graph of f at $(a, f(a))$ is simply the line $y = f(a) + f'(a)(x - a)$. For f to have a tangent at $(a, f(a))$, there is no requirement that $f'(x)$ be continuous. The example,

$$F(x, y) = \sqrt{|xy|}$$

illustrates the need for conditions beyond the existence of $F_1(a, b)$ and $F_2(a, b)$ in order that there be a tangent plane at $(a, b, F(a, b))$. See Figure 10.6.

$$\begin{array}{rrcl@{\qquad}rcl} \text{For} & F(x,y) & = & \sqrt{|xy|}, & F(0,0) & = & 0 \\ & F(x,0) & = & 0, & F_1(0,0) & = & 0, \\ & F(0,y) & = & 0, & F_2(0,0) & = & 0. \end{array}$$

so the local linear approximation at (0,0) might be

$$L(x, y) = 0 + 0 \times (x - 0) + 0 \times (y - 0) = 0,$$

the graph of which is the horizontal plane $z = 0$. However, the slice of the graph through $y = x$ for which $z = |x|$ shown in Figure 10.6B is a line that pierces the $z = 0$ plane at an angle of about 35 degrees. Thus over the line $y = x$, $L(x, y) = 0$ is not tangent to the graph of F. We do not accept the graph of $L(x, y)$ as tangent to the graph of F. Also, F is not differentiable at (0,0). In this case

$$F_1(x, y) = \frac{1}{2}\sqrt{\frac{|y|}{|x|}} \quad \text{for} \quad x > 0, \quad \text{and} \quad -\frac{1}{2}\sqrt{\frac{|y|}{|x|}} \quad \text{for} \quad x < 0,$$

$$F_1(0, 0) = 0, \quad \text{and} \quad F_1(0, y) \quad \text{does not exist for} \quad y \neq 0$$

F_1 is neither continuous nor even defined throughout the interior of any circle with center (0,0). The conditions of Definition 10.1.3 for a local linear approximation are not met.

Explore 10.1.2 Compute $F_2(x, y)$ for $F(x) = \sqrt{|xy|}$. Is F_2 continuous on the interior of a circle with center (0,0)?

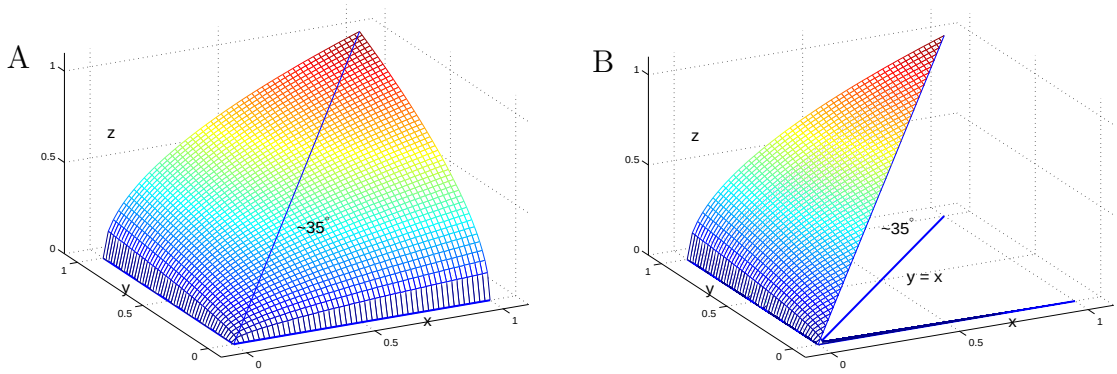


Figure 10.6: A. Graph of $F(x,y) = \sqrt{|xy|}$ on $x \geq 0$, $y \geq 0$. The rest of the graph is obtained by rotation of the portion shown about the z-axis 90, 180 and 270 degrees. There is no local linear approximation to F at (0,0) and no tangent plane to the graph of F at (0,0,0). B. Cut away of the graph of A showing the angle between the surface and the horizontal plane above the line $y = x$.

Property 10.1.1 A property of tangents to functions of one variable. Suppose f is a function of one variable and at a number a in its domain, $f'(a)$ exists. The graph of $L(x) = f(a) + f'(a)(x-a)$ is the tangent to the graph of f at $(a, f(a))$. Then

$$\begin{aligned} \lim_{x\to a} \frac{|f(x) - L(x)|}{|x-a|} &= \lim_{x\to a} \left| \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} \right| \\ &= \left| \lim_{x\to a} \frac{f(x) - f(a)}{x-a} - f'(a) \right| \\ &= 0. \end{aligned} \tag{10.3}$$

It is sometimes said that the numerator, $f(x) - L(x)$, in $\dfrac{|f(x) - L(x)|}{|x-a|}$ 'goes to zero faster' than does the denominator, $|x-a|$. *It is this property that is the defining characteristic of 'tangent'.* Only the existence of $f'(a)$ is required. It is not required that $f'(x)$ be continuous.

As was apparent in Figure 10.5B for the function $F(x,y) = \sqrt{|xy|}$, something more than existence of $F_1(a,b)$ and $F_2(a,b)$ is required in order to have a tangent plane for $F(x,y)$ at a point (a,b). A sufficient condition is that F_1 and F_2 exist and be continuous on a circle with center (a,b).

Property 10.1.2 A property of local linear approximations to functions of two variables. Suppose F is a function of two variables, (a, b) is a number pair in the domain of F, and F_1 and F_2 exist and are continuous on the interior of a circle with center (a, b). Then $L(x, y) = F(a, b) + F_1(a, b)(x - a) + F_2(a, b)(y - b)$ is the local linear approximation to F at (a, b), and

$$\lim_{(x,y)\to(a,b)} \frac{|F(x, y) - L(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0 \tag{10.4}$$

The proof of Property 10.1.2 involves some interesting analysis that you can understand, but is long enough that we have not included it. We use this property in proving asymptotic stability of systems of difference equations in Section 12.8.

Exercises for Section 10.1, Partial derivatives of functions of two variables.

Exercise 10.1.1 Draw three dimensional graphs of

a.	$F(x, y) = 2$	b.	$F(x, y) = x$
c.	$F(x, y) = x^2$	d.	$F(x, y) = x + y$
e.	$F(x, y) = 2x + 3y$	f.	$F(x, y) = x^2 + y^2$
g.	$F(x, y) = xe^{-y}$	h.	$F(x, y) = \sin y$
i.	$F(x, y) = x + \sin y$	j.	$F(x, y) = x \sin y$
k.	$F(x, y) = \sqrt{x^2 + y^2}$	l.	$F(x, y) = xy$
m.	$F(x, y) = \frac{x}{y}$	n.	$F(x, y) = \lvert xy \rvert$

Exercise 10.1.2 Find the partial derivatives, F_1, F_2, $F_{1,1}$, $F_{1,2}$, $F_{2,1}$ and $F_{2,2}$ of the following functions.

a.	$F(x, y) = 3x - 5y + 7$	b.	$F(x, y) = x^2 + 4xy + 3y^2$
c.	$F(x, y) = x^3y^5$	d.	$F(x, y) = \sqrt{xy}$
e.	$F(x, y) = \ln(x \times y)$	f.	$F(x, y) = \frac{x}{y}$
g.	$F(x, y) = e^{x+y}$	h.	$F(x, y) = x^2e^{-y}$
i.	$F(x, y) = \sin(2x + 3y)$	j.	$F(x, y) = e^{-x} \cos y$

Exercise 10.1.3 Is the plane $z = 0$ a tangent plane to the graph of $F(x, y) = \sqrt{x^2 + y^2}$ shown in Figure 10.2D.

Exercise 10.1.4 Find $F_1(a, b)$ and $F_2(a, b)$ for

$$\begin{array}{llcl} \text{a.} & F(x,y) = \dfrac{x}{1+y^2} & (a,b) = & (1,0) \\ \text{b.} & F(x,y) = \sqrt{x^2+y^2} & (a,b) = & (1,2) \\ \text{c.} & F(x,y) = e^{-xy} & (a,b) = & (0,0) \\ \text{d.} & F(x,y) = \sin x \cos y & (a,b) = & (\pi/2, \pi) \end{array}$$

Exercise 10.1.5 Find the local linear approximation, $L(x, y)$, to $F(x, y)$ at the point (a,b). For each case, compute

$$\frac{|F(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} \quad \text{for} \quad (x,y) = (a + 0.1, b), \quad \text{and} \quad (x,y) = (a + 0.01, b + 0.01).$$

$$\begin{array}{llcl} \text{a.} & F(x,y) = 4x + 7y - 16 & (a,b) = & (3,2) \\ \text{b.} & F(x,y) = xy & (a,b) = & (2,1) \\ \text{c.} & F(x,y) = \dfrac{x}{y+1} & (a,b) = & (1,0) \\ \text{d.} & F(x,y) = xe^{-y} & (a,b) = & (1,0) \\ \text{e.} & F(x,y) = \sin \pi(x+y) & (a,b) = & (1/2, 1/4) \end{array}$$

Exercise 10.1.6 For $P = nRT/V$, find $\frac{\partial}{\partial V}P$ and $\frac{\partial}{\partial T}P$. How does P change as V increases? How does P change as T increases?

Exercise 10.1.7 Draw the graph of $F(x, y)$ and the graph of the plane tangent to the graph of F at the point (a,b).

$$\begin{array}{llcl} \text{a.} & F(x,y) = x + y + 1 & (a,b) = & (1,2) \\ \text{b.} & F(x,y) = xy & (a,b) = & (0,0) \\ \text{c.} & F(x,y) = x^2 + y^2 & (a,b) = & (1,1) \\ \text{d.} & F(x,y) = \frac{y}{x} & (a,b) = & (1,0) \\ \text{e.} & F(x,y) = \sqrt{25 - x^2 - y^2} & (a,b) = & (3,4) \end{array}$$

Exercise 10.1.8 Let F be defined by

$$\begin{aligned} F(x,y) &= x^2 \quad \text{for} \quad y > 0 \\ &= 0 \quad \text{for} \quad y \leq 0 \end{aligned}$$

1. Sketch a graph of F in three dimensional space.
2. Is $F_1(x,y)$ continuous interior to a circle with center $(0,0)$?
3. Let $L(x,y) = 0$ for all (x,y). Is it true that

$$\lim_{(x,y)\to(0,0)} \frac{F(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0 \quad ?$$

4. Are you willing to call the plane $z = 0$ a tangent plane to the graph of F?

10.2 Maxima and minima of functions of two variables.

Figure 10.7B shows a slice of the graph of $F(x,y) = x(1-x)^2 + y^2(1-y)$ through $x = 1/3$ and a slice through $y = 2/3$. The point $(1/3, 2/3, 8/27)$ at which these slices intersect appears to be and is a local maximum for F. The two tangents to these slices are horizontal as would be expected at an interior local maximum for a function of a single variable. These two slices were selected by solving

$$F_1(x,y) = 1 - 4x + 3x^2 = 0 \qquad \text{and} \qquad F_2(x,y) = 2y - 3y^2 = 0$$

$$x = 1/3 \quad (\text{or} \quad x = 1), \qquad \text{and} \qquad y = 2/3 \quad (\text{or} \quad y = 0).$$

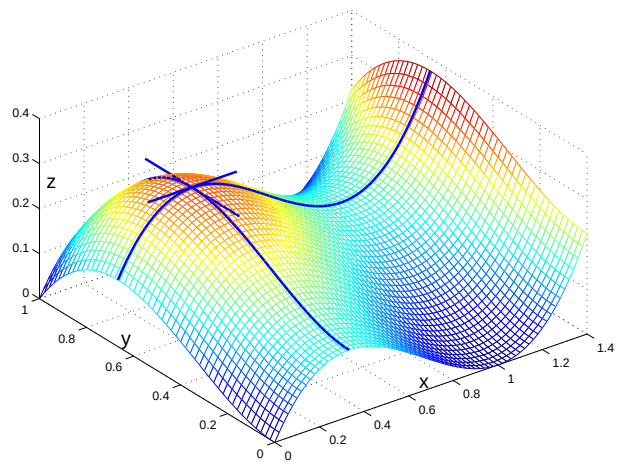


Figure 10.7: Slices through the graph of $F(x,y) = x(1-x)^2 + y^2(1-y)$ through $x = 1/3$ and $y = 2/3$. (1/3,2/3,8/27) is a local maximum of F.

Definition 10.2.1 Critical points. An interior point (a, b) of the domain of a function of two variables, F, is a critical point of F means that

$$F_1(a, b) = 0 \qquad \text{and} \qquad F_2(a, b) = 0$$

or that one of $F_1(a, b)$ or $F_2(a, b)$ fails to exist.

A point of a domain is an *interior point* of the domain if it is the center of a circle, the interior of which lies in the domain. Else it is called a *boundary point*.

There are four critical points of $F(x, y) = x(1 - x)^2 + y^2(1 - y)$,

$$(1/3, 2/3), \qquad (1, 2/3), \qquad (1/3, 0), \qquad \text{and} \qquad (1, 0).$$

The critical point (1/3,2/3) is discussed above and shown in Figure 10.7.

The critical point (1,2/3) is illustrated in Figure 10.8 and is neither a high point nor a low point, it is a *saddle* point. The surface is convex upward in the x-direction and is concave downward in the y-direction. The tangent at (1,2/3,4/27) parallel to the y-axis is above the surface; the tangent parallel to the x-axis is below the surface and would normally not be visible.

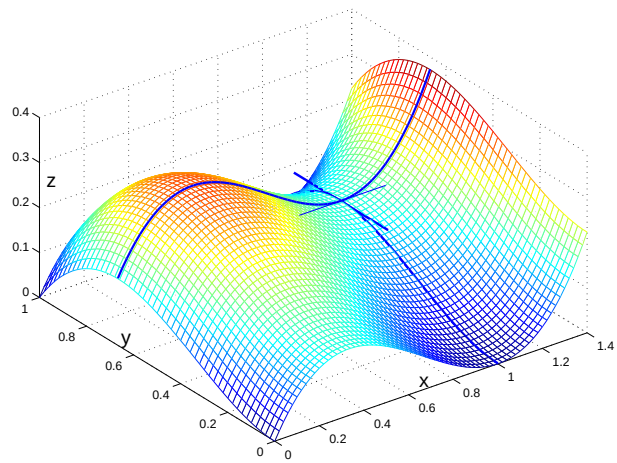


Figure 10.8: Slices through the graph of $F(x, y) = x^2(1 - x) + y(1 - y)^2$ through $x = 1$ and $y = 2/3$. (1,2/3,4/27) is a saddle point of F.

The critical point (1,0) is illustrated in Figure 10.9 and is a local minimum for F and is illustrated in Figure 10.9. The domain for y is [-0.4,1]; in Figures 10.7 and 10.8 the domain for x is [0,1]. Also the viewpoint is lower in Figure 10.9 than in Figures 10.7 and 10.8 in order to look underneath the graph at the low point.

Explore 10.2.1 Locate the critical point (1/3,0) of $F(x, y) = x(1 - x)^2 + y^2(1 - y)$ in Figure 10.9 and classify it as a local maximum, local minimum or saddle point. ■

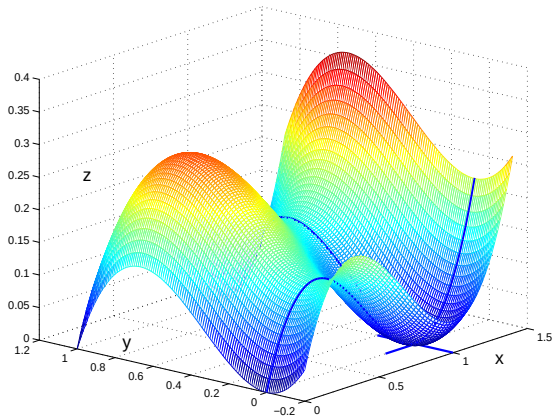


Figure 10.9: Slices through the graph of $F(x,y) = x(1-x)^2 + y^2(1-y)$ through $x = 1$ and $y = 0$. (1,0,0) is a local minimum of F.

Example 10.2.1 The graphs of $F(x,y) = x^2 + y^2$, $G(x,y) = x^2 - y^2$ and $H(x,y) = -x^2 - y^2$ shown in Figure 10.10 illustrate three important options. The origin, (0,0), is a critical point of each of the graphs and the $z = 0$ plane is the tangent plane to each of the graphs at (0,0). For F, for example,

$$\begin{array}{rclrclrcl} F(x,y) & = & x^2+y^2 & F_1(x,y) & = & 2x & F_2(x,y) & = & 2y \\ \\ F(0,0) & = & 0 & F_1(0,0) & = & 0 & F_2(0,0) & = & 0 \end{array}$$

The origin, (0,0), is a critical point of F, the linear approximation to F at (0,0) is $L(x,y) = 0$, and the tangent plane is $z = 0$, or the x, y plane. The same is true for G and H; the tangent plane at (0,0,0) is $z = 0$ for all three examples. This seemingly monotonous information is saved by the observations of the relation of the tangent plane to the graphs of the three functions. For F, (0,0,0) is the lowest point of the graph of F, and for H, (0,0,0) is the highest point of the graph of H. This is similar to the horizontal lines associated with high and low points of graphs of functions of one variables. The graph of G is different. The point (0,0,0) is called a saddle point of the graph of G. The portion of G in the x, z plane ($y = 0$) has a low point at (0,0,0) and the portion of G in the y, z plane ($x = 0$) has a high point at (0,0,0).

For a function f of a single variable and a number a for which $f'(a) = 0$, there is a simple second derivative test that distinguishes whether $(a, f(a))$ is a locally high point ($f''(a) < 0$) or a locally low point ($f''(a) > 0$). There is also a second derivative test for functions of two variables.

Definition 10.2.2 Definition of Local Maxima and Minima. If (a,b) is a point in the domain of a function F of two variables, $F(a,b)$ is a local maximum for F means that there is a number $\delta_0 > 0$ such that if (x,y) is in the domain of F and $\sqrt{(x-a)^2 + (y-b)^2} < \delta_0$ then $F(x,y) \leq F(a,b)$.

The definition of local minimum is similar.

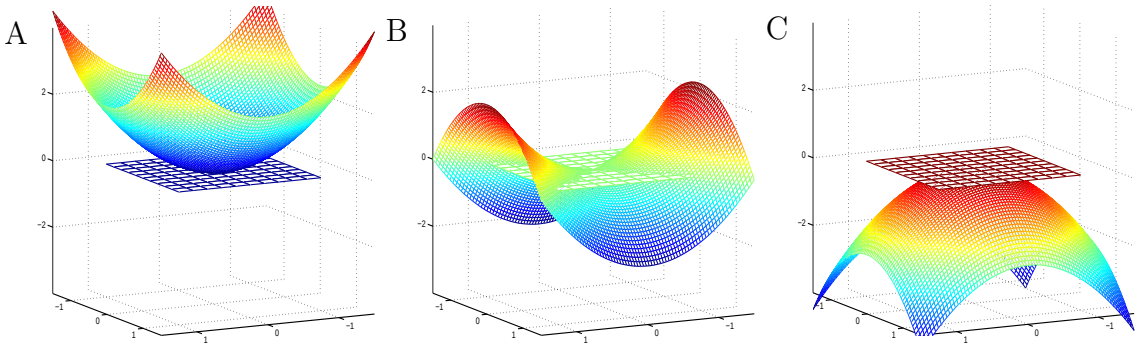


Figure 10.10: Graphs of A. $z = x^2+y^2$, B. $z = x^2-y^2$, and C. $z = -x^2-y^2$. The local linear approximation to each is the horizontal plane $z = 0$.

Theorem 10.2.1 Local Maxima and Minima of functions of two variables. Suppose (a, b) is a critical point of a function F of two variables that has continuous first and second partial derivatives in a circle with center at (a, b) and

$$\Delta = F_{1,1}(a, b)F_{2,2}(a, b) - (F_{1,2}(a, b))^2.$$

Case 1. If $\Delta > 0$ and $F_{1,1}(a, b) > 0$ then $F(a, b)$ is a local minimum of F.

Case 2. If $\Delta > 0$ and $F_{1,1}(a, b) < 0$ then $F(a, b)$ is a local maximum of F.

Case 3. If $\Delta < 0$ then $(a, b, F(a, b))$ is a saddle point of F.

Case 4. If $\Delta = 0$, punt, use another supplier.

We omit the proof of Theorem 10.2.1, but illustrate its application to the functions F, G, and H of Example 10.2.1 and shown in Figure 10.10.

$$\text{For} \quad F(x, y) = x^2 + y^2, \quad F_{1,1}(0, 0) = 2 > 0, \quad \Delta = 2 \times 2 - 0 = 4 > 0,$$

and the origin is a local minimum for F.

$$\text{For} \quad G(x, y) = x^2 - y^2, \quad \Delta(0, 0) = 2 \times (-2) - 0 = -4 < 0,$$

and the origin is a saddle point for G. We have not defined a saddle point. For our purposes, a saddle point is a critical point for which $\Delta < 0$.

$$\text{For} \quad H(x, y) = -x^2 - y^2, \quad H_{1,1}(0, 0) = -2, \quad \Delta = -2 \times (-2) - 0 = 4 > 0,$$

and the origin is a local maximum for H.

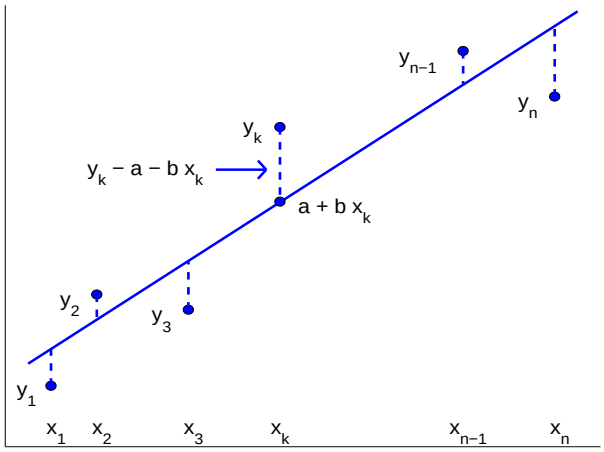
Example 10.2.2 Least squares fit of a line to data. A two variables minimization problem crucial to the sciences is the fit of a linear function to data.

Problem. Given points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), where $x_i \neq x_j$ if $i \neq j$, find numbers a_0 and b_0 so that $(a, b) = (a_0, b_0)$ minimizes

$$SS(a, b) = \sum_{k=1}^{n} (y_k - a - bx_k)^2 \tag{10.5}$$

In Figure 10.2.2.2, $SS(a, b)$ is the sum of the squares of the lengths of the dashed lines.

Figure for Example 10.2.2.2 A graph of data and a line. $SS(a, b)$ is the sum of the squares of the lengths of the dashed lines.



Solution. The critical points of SS are the solutions to the equations

$$SS_1(a, b) = \frac{\partial}{\partial a} SS = 0 \qquad SS_2(a, b) = \frac{\partial}{\partial b} SS = 0. \tag{10.6}$$

$$\begin{aligned}
SS_1(a, b) &= \frac{\partial}{\partial a} \left[\sum_{k=1}^{n} (y_k - a - bx_k)^2 \right] \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial a} (y_k - a - bx_k)^2 \\
&= \sum_{k=1}^{n} 2 (y_k - a - bx_k) (-1) \\
&= 2 \sum_{k=1}^{n} (a + bx_k - y_k) \\
&= 2 \left(a \sum_{k=1}^{n} 1 + b \sum_{k=1}^{n} x_k - \sum_{k=1}^{n} y_k \right)
\end{aligned}$$

$$SS_2(a, b) = \frac{\partial}{\partial b} \left[\sum_{k=1}^{n} (y_k - a - bx_k)^2 \right]$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{\partial}{\partial b} \left(y_k - a - bx_k\right)^2 \\
&= \sum_{k=1}^{n} 2\left(y_k - a - bx_k\right)\left(-x_k\right) \\
&= 2\sum_{k=1}^{n} \left(ax_k + bx_k^2 - x_k y_k\right) \\
&= 2\left(a\sum_{k=1}^{n} x_k + b\sum_{k=1}^{n} s_k^2 - \sum_{k=1}^{n} x_k y_k\right)
\end{aligned}
$$

Imposing the conditions $SS_1(a,b) = 0$ and $SS_2(a,b) = 0$ and simplifying leads to the *Normal Equations*:

$$
\begin{array}{rcrcl}
an & + & b\sum_{k=1}^{n} x_k & = & \sum_{k=1}^{n} y_k \\
a\sum_{k=1}^{n} x_k & + & b\sum_{k=1}^{n} x_k^2 & = & \sum_{k=1}^{n} x_k y_k
\end{array}
\tag{10.7}
$$

Some notation is useful:

$$
S_x = \sum_{k=1}^{n} x_k, \quad S_y = \sum_{k=1}^{n} y_k, \quad S_{xx} = \sum_{k=1}^{n} x_k^2, \quad S_{xy} = \sum_{k=1}^{n} x_k\, y_k, \quad S_{yy} = \sum_{k=1}^{n} y_k^2.
$$

The solution to the Normal Equations 10.7 is

$$
\begin{aligned}
a_0 &= \frac{S_{xx}\, S_y - S_x S_{xy}}{\Delta} \\
b_0 &= \frac{nS_{xy} - S_x S_y}{\Delta} \\
\Delta &= nS_{xx} - (S_x)^2
\end{aligned}
\tag{10.8}
$$

It is important that $\Delta \neq 0$. The proof that Δ is actually positive involves some clever and not very intuitive algebra. Form the sum

$$
S = \sum_{k=1}^{n} \left(S_{xx} - X_x x_k\right)^2 .
$$

Because the x_k are distinct, at least one of $S_{xx} - S_x x_k \neq 0$ and $S > 0$.

$$
\begin{aligned}
S &= \sum_{k=1}^{n} \left(S_{xx} - S_x x_k\right)^2 \\
&= \sum_{k=1}^{n} \left((S_{xx})^2 - 2S_{xx}S_x x_k + (S_x)^2 x_k^2\right) \\
&= \sum_{k=1}^{n} (S_{xx})^2 - 2S_{xx}S_x \sum_{k=1}^{n} x_k + (S_x)^2 \sum_{k=1}^{n} x_k^2 \\
&= n(S_{xx})^2 - 2S_{xx}S_x \times S_x + (S_x)^2 \times S_{xx}
\end{aligned}
$$

$$= \quad S_{xx}(nS_{xx} - (S_x)^2)$$

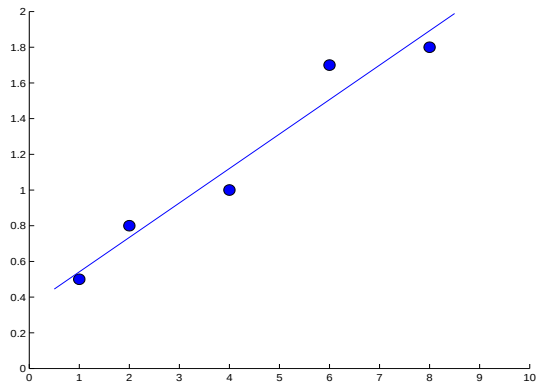
$$= \quad S_{xx} \times \Delta$$

Now $S_{xx} > 0$ and $S = S_{xx} \times \Delta > 0$, so $\Delta > 0$.

Example 10.2.3 Fit a line to the data in Example Figure 10.2.3.3.

Figure for Example 10.2.3.3 Data, a graph of the data and a line $y = a + bx$ fit to the data:

x_t	y_t
1	0.5
2	0.8
4	1.0
6	1.7
8	1.8



$$\begin{aligned}
S_x &= 1 + 2 + 4 + 6 + 8 = 21, \\
S_y &= 0.5 + 0.8 + 1.0 + 1.7 + 1.8 = 5.8 \\
S_{xx} &= 1^2 + 2^2 + 4^2 + 6^2 + 8^2 = 121, \\
S_{xy} &= 1 \times 0.5 + 2 \times 0.8 + 4 \times 1.0 + 6 \times 1.7 + 8 \times 1.8 = 30.7
\end{aligned}$$

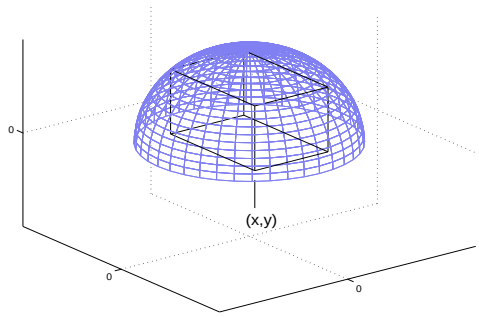
$$\begin{aligned}
\Delta &= nS_{xx} - (S_x)^2 = 5 \times 121 - (21)^2 = 164 \\
a &= (S_{xx}\, S_y - S_x S_{xy})/\Delta = (121 \times 5.8 - 21 \times 30.7)/164 = 0.348 \\
b &= (nS_{xy} - S_x S_y)/\Delta = (5 \times 30.7 - 21 \times 5.8)/164 = 0.193
\end{aligned}$$

The line $y = 0.348 + 0.193x$ is the closest to the data in the sense of least squares and is drawn in Example Figure 10.2.3.3.

You would systematically organize the arithmetic if you fit very many lines to data as we just did. Better than that, however, is that your calculator does all of this arithmetic for you. In STAT, CALC you have LinR. Enter your data in xStat and yStat and on the screen type LinR xStat,yStat. LinR is found in STAT, CALC and xStat and yStat are found in LIST, NAMES.

Example 10.2.4 *Problem:* Find the dimensions of the largest box (rectangular solid) that will fit in a hemisphere of radius R.
Solution. Assume the hemisphere is the graph of $z = \sqrt{R^2 - x^2 - y^2}$ and that the optimum box has one face in the x, y-plane and the other four corners on the hemisphere (see Figure 10.2.4.4).

Figure for Example 10.2.4.4 A box in a sphere. One corner of the box is at (x, y) in the x, y-plane.



The volume, V of the box is

$$V(x, y) = 2x \times 2y \times z = 2x \times 2y \times \sqrt{R^2 - x^2 - y^2}$$

Before launching into partial differentiation, it is perhaps clever, and certainly useful, to observe that the values of x and y for which V is a maximum are also the values for which $V^2/16$ is a maximum.

$$\frac{V^2(x, y)}{16} = W(x, y) = \frac{16x^2y^2(R^2 - x^2 - y^2)}{16} = R^2x^2y^2 - x^4y^2 - x^2y^4.$$

It is easier to analyze $W(x, y)$ than it is to analyze $V(x, y)$.

$$\begin{aligned} W_1(x, y) &= 2R^2xy^2 - 4x^3y^2 - 2xy^4 \\ &= 2xy^2(R^2 - 4x^2 - 2y^2) \\ W_2(x, y) &= 2R^2x^2y - 2x^4y^2 - 4x^2y^3 \\ &= 2x^2y(R^2 - 2x^2 - 4y^2) \end{aligned}$$

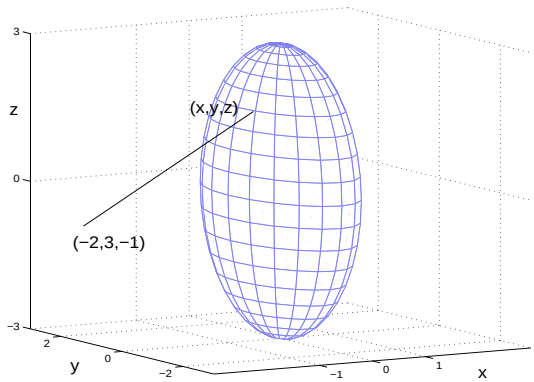
Solving for $W_1(x, y) = 0$ and$W_2(x, y) = 0$ yields $x = R/\sqrt{3}$ and $y = R/\sqrt{3}$, for which $z = R/\sqrt{3}$. The dimensions of the box are $2R/\sqrt{3}$, $2R/\sqrt{3}$, and $R/\sqrt{3}$,

Example 10.2.5 Warning: Obnubilation Zone. *Problem.* Find the point of the ellipsoid

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1 \qquad \text{that is closest to} \qquad (-2, 3, -1) \tag{10.9}$$

See Example Figure 10.2.5.5

Figure for Example 10.2.5.5 The ellipsoid $x^2/1 + y^2/4 + z^2/9 = 1$ and a line from (-2,3,-1) to a point (x,y,z) of the ellipsoid.



Solution. Claim without proof: The point (x, y, z) of the ellipsoid that is closest to $(-2, 3, -1)$ will have negative x, positive y and negative z coordinates.

The distance between (a, b, c) and (p, q, r) in 3-dimensional space is

$$\sqrt{(p-a)^2 + (q-b)^2 + (r-c)^2}.$$

The distance from (-2,3,-1) to (x, y, z) on the ellipsoid is

$$D(x, y, z) = \sqrt{(x+2)^2 + (y-3)^2 + (z+1)^2}$$

We solve for z in Equation 10.9 and write

$$z = -3\sqrt{1 - x^2 - y^2/4} \qquad z \text{ is negative.} \tag{10.10}$$

$$E(x, y) = \sqrt{(x+2)^2 + (y-3)^2 + (1 - 3\sqrt{1 - x^2 - y^2/4})^2} \tag{10.11}$$

Define $F(x, y) = (E(x, y))^2$ and write

$$\begin{aligned}
F(x, y) &= (x+2)^2 + (y-3)^2 + \left(1 - 3\sqrt{1 - x^2 - y^2/4}\right)^2 \\
F_1(x, y) &= 2(x+2) + 2\left(1 - 3\sqrt{1 - x^2 - y^2/4}\right) \frac{\partial}{\partial x}\left(1 - 3\sqrt{1 - x^2 - y^2/4}\right) \\
&= 2(x+2) + 2\left(1 - 3\sqrt{1 - x^2 - y^2/4}\right) \frac{(-3)(-2x)}{2\sqrt{1 - x^2 - y^2/4}} \\
F_2(x, y) &= 2(y-3) + 2\left(1 + 3\sqrt{1 - x^2 - y^2/4}\right) \frac{3(-2y/4)}{2\sqrt{1 - x^2 - y^2/4}}
\end{aligned}$$

Now we have a mess. We need to (and can!) solve for (x, y) in

$$\begin{aligned}
F_1(x, y) &= 2(x+2) + 6x\left(\frac{1}{\sqrt{1 - x^2 - y^2/4}} - 3\right) = 0 \\
F_2(x, y) &= 2(y-3) + \frac{6y}{4}\left(\frac{1}{\sqrt{1 - x^2 - y^2/4}} - 3\right) = 0
\end{aligned}$$

First we write

$$\begin{aligned} 2(x+2) &= -6x\left(\frac{1}{\sqrt{1-x^2-y^2/4}}-3\right) \qquad (10.12)\\ 2(y-3) &= -\frac{6y}{4}\left(\frac{1}{\sqrt{1-x^2-y^2/4}}-3\right), \end{aligned}$$

divide corresponding sides of the two equations, and find that

$$y = \frac{12x}{3x-2}. \qquad (10.13)$$

Substitute this expression for y in Equation 10.12, simplify, and find that

$$8x-2 = \frac{\mp 3x(3x-2)}{\sqrt{(1-x^2)(9x^2-12x+4)-36x^2}}$$

Square both sides of this equation and clear fractions.

$$(8x-2)^2\left[(1-x^2)(9x^2-12x+4)-36x^2\right] = 9x^2(9x^2-12x+4)$$

Multiply and collect.

$$576x^6-1056x^5+2485x^4-380x^3-480x^2+176x-16=0 \qquad (10.14)$$

Now we have it! Go to POLY on your calculator, enter order = 6, and enter the coefficients. Press SOLVE and wait 20 seconds. The two real (not complex) answers are $x=-0.482870022$ and $x=0.165385675$. Use $x=-0.482870022$ and compute $y=1.680224831$ and $z=-0.726205610$ from Equations 10.13 and 10.10, and the distance from (-2,3,-1) to the ellipsoid is 2.029. **Whew!**

Explore 10.2.2 For the algebraically strong, fill in the algebra omitted in Example 10.2.5

Exercises for Section 10.2 Maxima and minima of functions of two variables.

Exercise 10.2.1 Find the critical points, if any, of F.

a.	$F(x,y)$	$=$	$2x+5y+7$	b.	$F(x,y)$	$=$	$x^2+4xy+3y^2$
c.	$F(x,y)$	$=$	$x^3(1-x)+y$	d.	$F(x,y)$	$=$	$xy(1-xy)$
e.	$F(x,y)$	$=$	$(x-x^2)(y-y^2)$	f.	$F(x,y)$	$=$	$\frac{x}{y}$
g.	$F(x,y)$	$=$	e^{x+y}	h.	$F(x,y)$	$=$	$\sin(x+y)$
i.	$F(x,y)$	$=$	$\frac{x^2}{1+y^2}$	j.	$F(x,y)$	$=$	$\cos x \sin y$

Exercise 10.2.2 For each of the following functions, find the critical points and use Theorem 10.2.1 to determine whether they are local maxima, local minima, or saddle points or none of these.

$$\begin{aligned}
\text{a.} \quad F(x,y) &= -x^2 + xy - y^2 \\
\text{b.} \quad F(x,y) &= x^2 + xy - y^2 \\
\text{c.} \quad F(x,y) &= x^2 + y^2 - 2xy + 2x - 2y \\
\text{d.} \quad F(x,y) &= -x^2 - 5y^2 + 2xy - 10x + 6y + 20
\end{aligned}$$

Exercise 10.2.3 Find C and b so that Ce^{bx} closely approximates the data

x	0	1	2	3	4
y	2.18	5.98	16.1	43.6	129.7

Observe that for $y = Ce^{bx}$, $\ln y = \ln C + bx$. Therefore, fit $a + bx$ to the number pairs, $(x, \ln y)$ using linear least squares. Then $\ln y_k \doteq a + bx_k$, and

$$y_k \doteq e^{a+bx_k} = e^a \times e^{bx_k} = Ce^{bx_k}, \quad \text{where} \quad C = e^a.$$

Exercise 10.2.4 a. Find a, b, and c so that $y = a + bx + cx^2$ is the least squares approximation to data, (x_1, y_1), $x_2, y_2)$, $\cdots$, (x_n, y_n). To do so you will need to minimize

$$SS = \sum_{k=1}^{n} \left(y_k - (a + bx_k + cx_k^2)\right)^2.$$

This is a three-variable minimization problem. The solution will be similar to the least squares line approximation to data of Example 10.2.2.
b. In Exercise Table 10.2.4 are data showing the height of a ball above a Texas Instruments CBL motion detector falling in air. Find the parabola that is the least squares fit to the data.
c. Check your answer by using P2reg on your calculator.

Table for Exercise 10.2.4 (h!) Height of a ball above a Texas Instruments CBL motion detector falling in air.

Time$_i$ sec	Position$_i$ cm
0.232	245.9
0.333	235.8
0.435	214.3
0.537	184.1
0.638	146.0
0.739	99.3
0.840	45.5

Exercise 10.2.5 Find a and b so that $\sin(ax+b)$ closely approximates the data

x	0	1	2	3	4
y	0.97	0.70	0.26	-0.26	-0.5

Observe that for $y = \sin(ax+b)$, $\arcsin y = ax + b$. Therefore, fit $ax+b$ to the number pairs, $(x, \arcsin y)$ using linear least squares.

Exercise 10.2.6 Interpret the real root $x = 0.165385675$ of Equation 10.14 related to the ellipsoid example.

Exercise 10.2.7 Find the largest box that will fit in the positive octant ($x \geq 0$, $y \geq 0$, and $z \geq 0$) and underneath the plane $z = 12 - 2x - 3y$.

Exercise 10.2.8 Find the largest box that will fit in the positive octant and underneath the hemisphere $z = \sqrt{25 - x^2 - y^2}$.

Exercise 10.2.9 Find the point of the plane $z = 2x + 3y - 12$ that is

1. closest to the origin.
2. closest to (4,5,6)

Exercise 10.2.10 Find the point of the sphere $x^2 + y^2 + z^2 = 25$ that is closest to (3,4,5).

Exercise 10.2.11 Find the point of the ellipsoid of Equation 10.9

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1 \qquad \text{that is farthest from} \qquad (-2, 3, -1) \tag{10.15}$$

10.3 The diffusion equation.

Partial derivatives may appear in equations and when they do the equations are called partial differential equations. This is a vast field of study. We give one very important example, the diffusion equation, and suggest a numerical method for its solution. The diffusion equation is

$$\begin{aligned} u_t(x,t) &= c^2 u_{xx}(x,t), \qquad a \leq x \leq b, \qquad 0 \leq t, \\ u(x,0) &= g(x), \qquad a < x < b, \\ u(a,t) &= A, \qquad \text{and} \qquad u(b,t) = B, \qquad 0 < t. \end{aligned}$$

You may think of a brass rod of diameter, d, that is small compared to its length, L. Let $u(x,t)$ be the temperature in the rod at position x and at time t and suppose that at time $t = 0$ the temperature in the rod is $g(x)$. Alternatively, you may think of a glass tube of diameter, d, that is small compared to its length, L, and is filled with distilled water. At time $t = 0$ a drop of salt water is placed in one end of the rod and the salt spreads (diffuses) along the water in the rod. Let $u(x,t)$ be the concentration of salt at position x and time t.

A single equation describes $u(x,t)$ for both of these problems and a host of other problems. Molecules diffuse in intracellular fluids; diseases diffuse in a population; an invasive species diffuses over an extended range. 'Diffuse' is also used to describe the transfer of molecules across a cellular membrane due to osmotic pressure. In this case the concentration is essentially discontinuous across the thin membrane; the problems in this section assume the concentration is continuous along an extended region.

We derive the diffusion equation in terms of the diffusion of salt. Consider a circular glass rod of cross sectional area A and length L filled with water and assume that at time $t = 0$ the concentration of the salt along the tube is $g(x)$. Let $u(x,t)$ be the salt concentration at distance x from one end of the rod at time t. Figure 10.11A. Assume that the diameter of the rod is small enough that the salt concentration at any position x along the rod depends only on x.

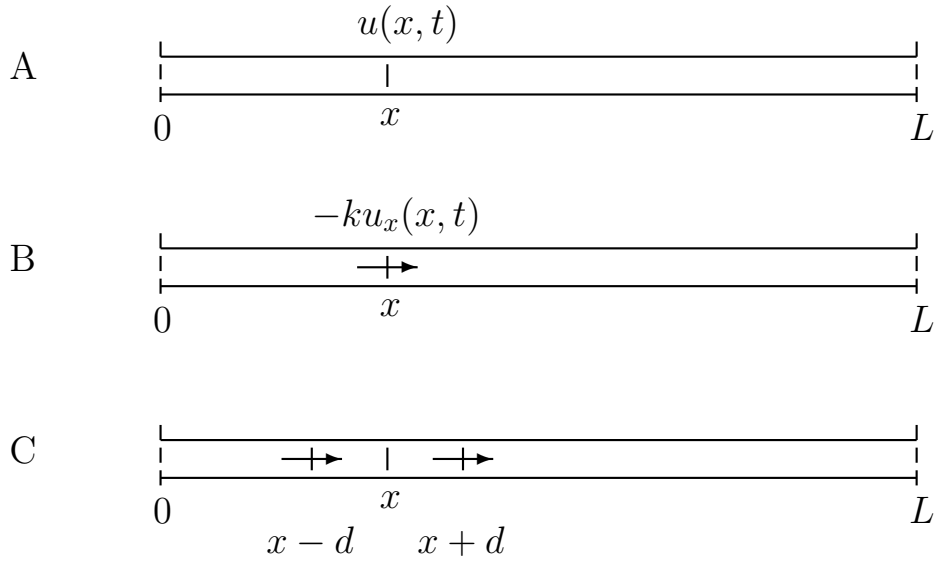


Figure 10.11: A. Glass rod, salt concentration at x at time t is $u(x,t)$. B. The rate at which salt flows past position x is proportional to $-u_x(x,t)$. C. The amount of salt between $x-d$ and $x+d$ at time t is approximately $u(x,t) \times A \times 2d$.

Mathematical model 1. Salt tends to flow from regions of high concentration to regions of low concentration. The rate at which salt flows past position x in the direction of increasing x at time t is proportional to $-u_x(x,t)$ and to A, the cross sectional area of the rod. Figure 10.11B.

From this model we write

$$R(x,t) = -kAu_x(x,t) \tag{10.16}$$

where $R(x,t)$ is the rate at which salt diffuses past position x at time t and k is a proportionality constant that is a property of the solvent.

Units. The units on u, u_x, A and R are, respectively,

$$\begin{array}{rcll} u & : & \dfrac{\text{gm}}{\text{cm}^3} & \dfrac{\text{mass}}{\text{distance}^3} \\[2ex] u_x & : & \dfrac{\text{gm}}{\text{cm}^3 \times cm} & \dfrac{\text{mass}}{\text{distance}^4} \\[2ex] A & : & \text{cm}^2 & \text{distance}^2 \\[2ex] R & : & \dfrac{\text{gm}}{\text{sec}} & \dfrac{\text{mass}}{\text{time}} \end{array}$$

In order to balance the units on Equation 10.16, the units on k must be cm^2/sec.

$$R = kAu_x, \qquad \frac{\text{gm}}{\text{sec}} = k \times \text{cm}^2 \times \frac{\text{gm}}{\text{cm}^3 \times cm}, \qquad \frac{1}{\text{sec}} = k \times \frac{1}{\text{cm}^2}, \qquad k = \frac{\text{cm}^2}{\text{sec}}.$$

Now consider a section of the rod between $x - d$ and $x + d$, Figure 10.11C. During a time interval t to $t + \delta$ salt flows past $x - d$, flows past $x + d$, and accumulates in the section.

> **Mathematical model 2.** The amount of salt in any small region of the tube is approximately the concentration of salt at some point in the region times the volume of the region.

From Mathematical Model 2 we write that the amount of salt in the section from $x - d$ to $x + d$ at time t is approximately $u(x, t) \times A \times 2d$.

> **Mathematical model 3.** During a time interval from t to $t + \delta$, the amount of salt that flows into a region minus the amount of salt that flows out of the region is the the accumulation within the region.

Now $R(x - d, t)$ is the rate at which salt flows past $x - d$ in the direction of increasing x. The amount of salt that flows into the section $x - d$ to $x + d$ during the time t to $t + \delta$ is approximately $R(x - d, t) \times \delta$. The amount of salt that flows out of the section during time t to $t + \delta$ is approximately $R(x + d, t) \times \delta$, Both rates may be negative and the actual flows negative.

The accumulation in the section during a time interval from t to $t + \delta$ is approximately $u(x, t + \delta) \times A \times 2d$ minus $u(x, t) \times A \times 2d$.

For Mathematical Model 3, we write

$$\begin{aligned} u(x, t + \delta) \times A \times 2d - u(x, t) \times A \times 2d &\doteq R(x - d, t) \times \delta - R(x + d, t) \times \delta \\ &= -kAu_x(x - d, t) \times \delta - (-kAu_x(x + d, t)) \times \delta \\ \frac{u(x, t + \delta) - u(x, t)}{\delta} &\doteq k\frac{u_x(x + d, t) - u_x(x - d, t)}{2d} \end{aligned} \tag{10.17}$$

Using the Mean Value Theorem 9.1.1 twice, there are numbers τ in $(t, t + \delta)$ and ξ in $(x - d, x + d)$ such that

$$\frac{u(x, t + \delta) - u(x, t)}{\delta} = u_t(x, \tau) \qquad \text{and} \qquad \frac{u_x(x + d, t) - u_x(x - d, t)}{2d} = u_{xx}(\xi, t)$$

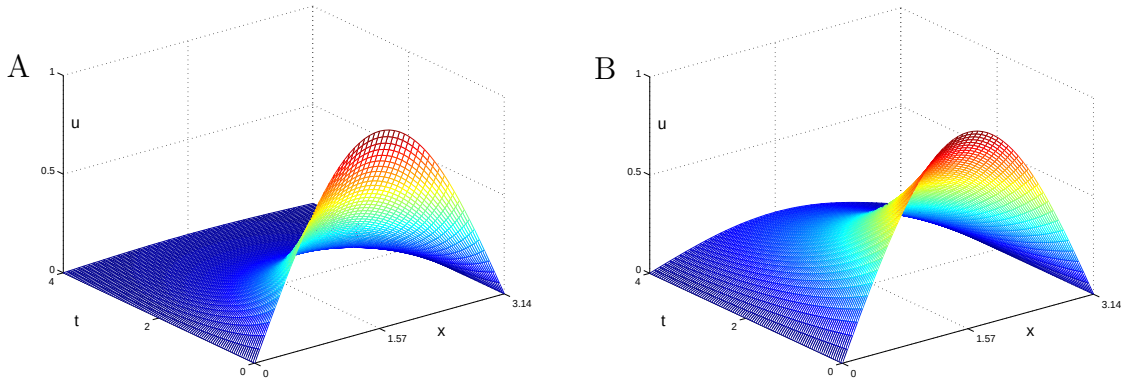


Figure 10.12: A. Graph in 3-dimensional space of $u(x,t) = e^{-t}\sin x$. B. Graph of $u(x,t) = e^{-0.5t}\sin x$.

Then

$$u_t(x,t) \doteq u_t(x,\tau) \doteq k u_{x,x}(\xi,t) \doteq k u_{xx}(x,t)$$

As $\delta \to 0$ and $d \to 0$, all of the errors reduce (we suppose to zero) and we write

$$u_t(x,t) = k u_{xx}(x,t) \qquad \text{Diffusion equation.} \tag{10.18}$$

The proportionality constant k is positive and is usually written as c^2 to signal this and to simplify analytical solutions. As noted above, the units on k are cm^2/sec. The size of k reflects how rapidly the salt moves in water or the heat moves in a rod or a disease spreads in a population or generally how rapidly a substance diffuses in its medium. If k is large, $u(x,t)$ changes rapidly; if k is small, $u(x,t)$ changes slowly. For example,

$$u(x,t) = e^{-kt}\sin x$$

is a solution to Equation 10.18:

$$u_t(x,t) = -ke^{-kt}\sin x, \qquad u_x(x,t) = e^{-kt}\cos x, \qquad u_{xx}(x,t) = -e^{-kt}\sin x,$$

$$u_t(x,t) = k u_{xx}(x,t).$$

Graphs of $e^{-t}\sin x$ and $e^{-0.5t}\sin x$ appear in Figure 10.12A and B respectively. It can be seen that the graph in B with smaller k changes more slowly than the graph in A.

The initial concentration of salt in the rod is required in order to compute the concentration at later times. Assume that there is a known function g such that at time $t = 0$ the concentration at position x is $g(x)$. Then

$$u(x,0) = g(x) \qquad 0 < x < L \qquad \text{Initial condition.} \tag{10.19}$$

Finally we need some knowledge about the ends of the rod, referred to as *boundary conditions*. The ends may be sealed so that no salt diffuses past either end. This is expressed as

$$u_x(0,t) = u_x(L,t) = 0, \qquad 0 \le t, \qquad \text{Insulated boundary conditions.} \tag{10.20}$$

Alternatively, we might assume the rod connects two reservoirs in which the salt concentration is constant, but there is no actual flow of solvent through the rod. Then there will be two concentrations, C_0 and C_L, such that

$$u(0,t) = C_0, \qquad U(L,t) = C_L, \qquad 0 \le t \qquad \text{Fixed boundary conditions.} \tag{10.21}$$

Explore 10.3.1 Suppose the $x = 0$ end of the tube is attached to a reservoir with salt concentration $c_0 = 1$ and the $x = L$ end of the tube is sealed. What would be the boundary conditions? What would $u(x, t)$ be for 'large' values of t?

Equation 10.18 for which $u(x, t)$ is concentration of salt also describes the temperature, $u(x, t)$ in a rod of length L that is insulated along its sides. In this case, the initial condition $g(x)$ would be the temperature distribution along the rod at time $t = 0$. The rod may also be insulated at each end and boundary condition 10.20 would apply, and this accounts for the name 'Insulated boundary condition.' Alternatively, one end of the rod may be exposed to, say, steam and the other end exposed to ice water, and boundary condition 10.21 would apply.

Explore 10.3.2 Suppose on a flat sandy beach the temperature at a depth of 2 meters is constant, equal to 20°C, and the temperature at the surface of the beach is $27 + 4\sin((2\pi/24)t)$ °C. Suppose the sand temperature varies only vertically. What equations would you like to solve if you were interested in a nest of turtle eggs buried 80 cm?

There are analytical solutions to the diffusion equation 10.18 with initial condition 10.19 with either of the boundary conditions 10.20 or 10.21. Only a few of them are simple enough for our use. We describe one example and include two examples in Exercises 10.3.4 and 10.3.9.

Example 10.3.1 *Problem.* Let

$$u(x, t) = 30 * (1 - e^{-t} \cos \pi * x) \qquad 0 \le x \le 2 \qquad 0 \le t. \tag{10.22}$$

Show that

$$u_t(x, t) = \frac{1}{\pi^2} u_{xx}(x, t), \quad u(x, 0) = 30 * (1 - \cos \pi x), \quad u_x(0, t) = 0, \quad \text{and} \quad u_x(2, t) = 0. \tag{10.23}$$

Assuming Equations 10.23 are correct, $u(x, t)$ would describe the temperature in a rod of length 2 that is perfectly insulated along its side, had an initial temperature of $30 * (1 - cos\pi x)$ at position x, and was perfectly insulated on each end. The diffusion coefficient of the material in the rod is $k = 1/\pi^2$. Alternatively, the equations would describe the salt concentration in a tube closed at each end when the initial salt concentration at position x was $30 * (1 - cos\pi x)$.

Explore 10.3.3 What will be the 'eventual' temperature distribution (or salt concentration) in the rod (tube)?

A graph of the initial temperature distribution appears in Figure 10.13A, and graphs of the temperature at times 0, 0.5, 1, 1.5 and 2 appear in Figure 10.13B.

Solution. First compute some partial derivatives.

$$u(x, t) = 30(1 - e^{-t} \cos \pi x) \tag{10.24}$$

$$u_t(x, t) = 30(0 - (e^{-t})(-1) \cos \pi x)$$

$$= 30e^{-t} \cos(\pi x) \tag{10.25}$$

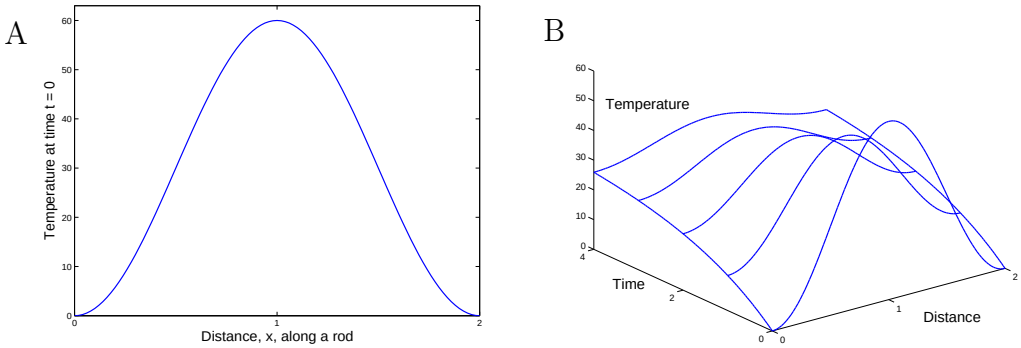


Figure 10.13: Partial graphs of Equation 10.22. A. Graph of temperature at time $t = 0$. B. Graphs of the temperature at times 0, 0.5, 1.0, 1.5 and 2.0. The graphs of temperature at the ends $x = 0$ and $x = 2$ are included.

$$
\begin{aligned}
u_x(x,t) &= 30(0 - e^{-t}(-\sin \pi x)(\pi) \\
&= 30\pi e^{-t} \sin \pi x \\
u_{xx}(x,t) &= 30\pi^2 e^{-t} \cos \pi x \qquad (10.26) \\
& \qquad (10.27)
\end{aligned}
$$

From Equations 10.25 and 10.26

$$u_t(x,t) = 30e^{-t}\cos(\pi x) = \frac{1}{\pi^2}30\pi^2 e^{-t}\cos(\pi x) = \frac{1}{\pi^2}u_{xx}.$$

From Equation 10.22

$$u(x,0) = 30 * (1 - e^{-t}\cos \pi * x)\Big|_{t=0} = 30(1 = \cos \pi x).$$

From Equation 10.26,

$$u_x(0,t) = 30\pi e^{-t}\sin \pi x\Big|_{x=0} = 0 \qquad \text{and} \qquad u_x(2,t) = 30\pi e^{-t}\sin \pi x\Big|_{x=2} = 0$$

Thus all of Equations 10.23 are satisfied.

Numerical solutions. Finding analytic solutions is beyond the scope of this text. However, a numerical scheme for approximating a solution is well within reach.

Partition the tube into n intervals of length $d = L/n$, and partition time into intervals of length δ, as shown in Figure 10.14 for $n = 5$ and the space time grid shown above the tube.

Begin with Equation 10.18,

$$u_t(x,t) = ku_{xx}(x,t)$$

From Equations 9.21 and 9.23

$$u_t(x,t) \doteq \frac{u(x,t+\delta) - u(x,t)}{\delta} \qquad \text{and} \qquad u_{xx}(x,t) \doteq \frac{u(x+d,t) - 2u(x,t) + u(x-d,t)}{d^2}.$$

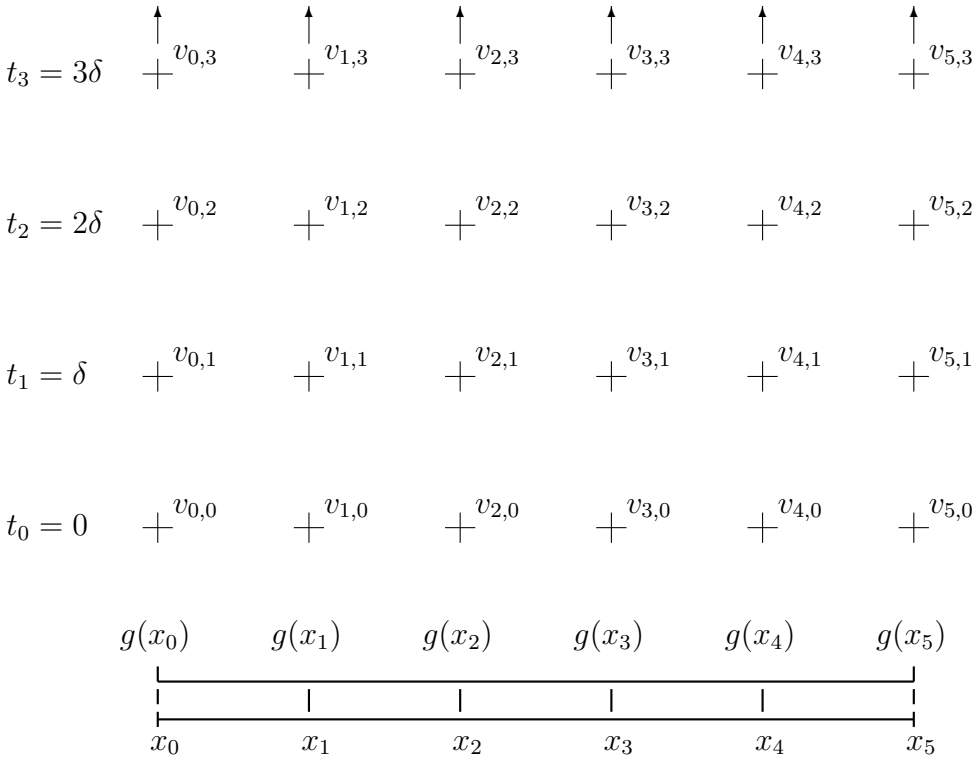


Figure 10.14: Glass tube and grid for diffusion computation.

Using these in the previous equation leads to

$$\frac{u(x,t+\delta)-u(x,t)}{\delta} \doteq k\frac{u(x+d,t)-2u(x,t)+u(x-d,t)}{d^2}. \tag{10.28}$$

Now we write an exact equation

$$\frac{v_{i,j+1}-v_{i,j}}{\delta} = k\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{d^2}. \tag{10.29}$$

where $v_{i,j}$ is an approximation to $u(i*d, j*\delta)$. Equations 10.29 can be solved as is illustrated in the next example. Using this notation in Equation 10.29 and rearranging leads to

$$v_{i,j+1} = v_{i,j} + \hat{k}\left(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}\right) \qquad \text{where} \qquad \hat{k} = \frac{k\times\delta}{d^2}. \tag{10.30}$$

Equation 10.30 defines $v_{i,j+1}$ (at time value $j+1$)in terms of $v_{i-1,j}$, $v_{i,j}$ and $v_{i+,j}$, values of v at the immediately preceding time value j. See Figure 10.15. The computation is started at time $j=0$ with values of $v_{i,0}$ equal to values of the initial condition $g(x)$ and progresses 'upward' in time, one layer at a time.

Example 10.3.2 Assume there is a glass tube of length $L = 1$ meter and of cross sectional area $A = 1$ cm filled with water; initially there is no salt in the tube; the left end is attached to a salt water reservoir

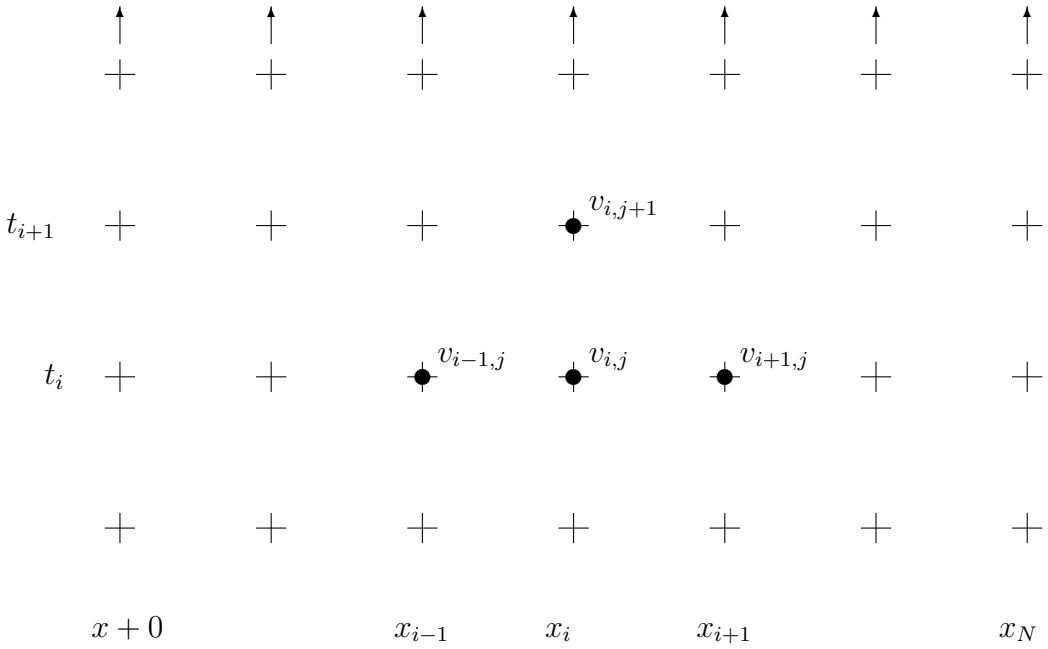


Figure 10.15: Geometric arrangement of the grid points of Equation 10.30.

with salt concentration = 1 and the right end is attached to a pure water reservoir. Let $u(x,t)$ be the salt concentration in the tube at position x and time t.

Explore 10.3.4 What do you expect the 'eventual' salt concentration along the tube to be?

The diffusion equation is $u_t(x,t) = ku_{xx}(x,t)$. Because initially there is no salt in the tube, $g(x) = 0$ for $0 < x < 1$, and the reservoirs at the ends of the tube imply the fixed boundary conditions 10.21, $u(0,t) = 1$ and $u(1,t) = 0$, $0 \le t$.

Partition the tube into 5 equal intervals. There is in Figure 10.14 an array of points horizontally distributed with position x along the tube and distributed vertically in time t.

In this example,

$$v_{i,j} \doteq u(i \times 1/5, j \times \delta), \qquad i = 1, 5, \qquad j = 1, \cdots.$$

and from Equation 10.29

$$v_{i,j+1} = v_{i,j} + \hat{k}\,(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}) \qquad \text{where} \qquad \hat{k} = \frac{\delta \times k}{d^2}. \tag{10.31}$$

The boundary conditions 10.21 with $u(0,t) = 1$ and $u(1,t) = 0$ lead to

$$v_{0,j} = 1 \qquad\qquad v_{5,j} = 0 \tag{10.32}$$

The initial condition and equations 10.31 determine the $v_{i,j}$ one horizontal layer at a time for the interior grid points, $1 < i < 5$.

Begin with the initial condition, Equation 10.19:

$$v_{i,0} = g(x_i) = 0, \qquad i = 1, 4 \tag{10.33}$$

Then for the bottom layer of the grid in Figure 10.14

$$v_{0,0} = 1 \qquad \text{and} \qquad v_{j,0} = 0 \qquad j = 1, \cdots, 5.$$

Then compute the next layer up for $t = \delta$:

$$\begin{aligned}
v_{1,1} &= v_{1,0} + \hat{k}(v_{0,0} - 2v_{1,0} + v_{2,0}) \\
v_{2,1} &= v_{2,0} + \hat{k}(v_{1,0} - 2v_{2,0} + v_{3,0}) \\
v_{3,1} &= v_{3,0} + \hat{k}(v_{2,0} - 2v_{3,0} + v_{4,0}) \\
v_{4,1} &= v_{4,0} + \hat{k}(v_{3,0} - 2v_{4,0} + v_{5,0}) \\
v_{0,1} &= v_{1,1} \\
v_{5,1} &= v_{4,1}
\end{aligned}$$

In a similar way as many layers as necessary can be computed. The computations for $\hat{k} = 0.2$ are shown in Table 10.1.

Remember that $\hat{k} = \dfrac{\delta \times k}{d^2}$ incorporates the time step, δ, and dimension step, d, as well as the diffusion constant, k.

These computations may be visualized as salt (blue color) migrating to the right in a tube as illustrated in the graphic below the data.

There is a severe constraint on $\hat{k} = \dfrac{k \times \delta}{d^2}$ in order that Equations 10.30 yield values of $v_{i,j}$ that reasonably approximate the target function $u(x,t)$. We must have

$$\hat{k} = \frac{k \times \delta}{d^2} < \frac{1}{2}. \tag{10.34}$$

The consequence of $\hat{k} > 1/2$ is illustrated in Exercise 10.3.1b, where $\hat{k} = 0.6$. In Example 10.3.2 the tube partition is $d = 0.2$ meters. If one wanted a smaller partition in order to more accurately approximate the salt concentration, say $d = 0.01$, one centimeter or 1/20th of the space step of the example, then the time step δ would have to be 1/400th of the first time dimension. It would take 400 iterations of the resulting equations in order to move one of the original time steps. There is an interesting alternate procedure that is not so constrained that will be found in numerical analysis books.

Exercises for Section 10.3, The diffusion equation.

Exercise 10.3.1 1. Enter the program of Table 10.1 into your calculator, run it and confirm the computations shown in Table 10.1.

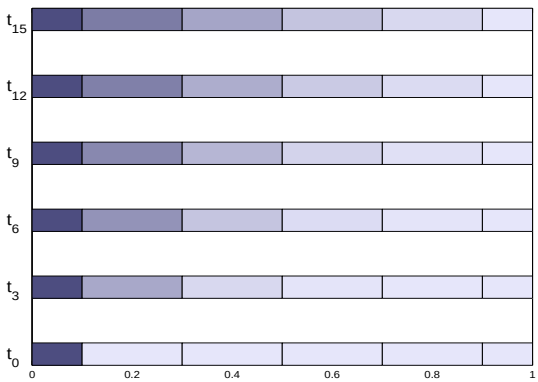
2. The program is written for $\hat{k} = 0.2$. Alter the program so that $\hat{k} = 0.6$ and run it. Do the computed approximations match what you think will be the salt concentrations?

Exercise 10.3.2 a. Enter the program of Table 10.1 into your calculator and alter it to accomodate 10 intervals in [0,1] for x. Retain $\hat{k} = 0.2$.

Table 10.1: A calculator program and approximations computed from Equations 10.31-10.33 with $\hat{k} = 0.2$

```
:Fix 3
:6->dimL V
:6->dimL VN
:For(i,1,6)
:0->V(I)
:End
:For(J,1,15)
:For(I,2,5)
:V(I)+.2*(V(I+1)-2*V(
I)+V(I-1))->VN(I)
:End
:1->VN(1)
:0->VN(6)
:Pause VN
:For(I,1,6)
:VN(I)->V(I)
:End
:End
:Fix 9
```

	x_0	x_1	x_2	x_3	x_4	x_5
t_{15}	1.000	0.688	0.421	0.224	0.092	0.000
t_{14}	1.000	0.678	0.406	0.209	0.084	0.000
t_{13}	1.000	0.667	0.389	0.194	0.075	0.000
t_{12}	1.000	0.655	0.371	0.178	0.066	0.000
t_{11}	1.000	0.642	0.351	0.160	0.056	0.000
t_{10}	1.000	0.626	0.329	0.142	0.047	0.000
t_9	1.000	0.609	0.305	0.123	0.037	0.000
t_8	1.000	0.588	0.278	0.102	0.028	0.000
t_7	1.000	0.565	0.248	0.082	0.019	0.000
t_6	1.000	0.536	0.214	0.061	0.011	0.000
t_5	1.000	0.501	0.177	0.041	0.005	0.000
t_4	1.000	0.458	0.134	0.022	0.002	0.000
t_3	1.000	0.400	0.088	0.008	0.000	0.000
t_2	1.000	0.320	0.040	0.000	0.000	0.000
t_1	1.000	0.200	0.000	0.000	0.000	0.000
t_0	1.000	0.000	0.000	0.000	0.000	0.000



b. Observe that the x-interval $d = 0.1$, is now one-half of the previous value of 0.2, and that

$$\hat{k} = \frac{\delta \times k}{d^2}.$$

Assume that the conductance coefficient, k, has not changed. How must the time increment, δ change?

c. Run your program and report the result.

Exercise 10.3.3 Note: March 22, 2010 I was not successful in getting the program to run. There is an excellent diffusion simulator written by Michael Karweit of Johns Hopkins University Department of Chemical Engineering at the web site

$$\text{www.jhu.edu/ virtlab/diffusion} - \text{processes}$$

The program simulates diffusion in two dimensions which we have not analyzed. However, on the rectangular domain you may set all parameters to be constant in the vertical direction and observe variation in the horizontal direction. It is also interesting to see two-dimensional diffusion. Run three experiments on the simulator and report the results.

Exercise 10.3.4 a. Show that

$$u(x,t) = 20e^{-t}\sin \pi x, \qquad 0 \le x \le 1, \qquad 0 \le t \tag{10.35}$$

solves

$$u_t(x,t) = \frac{1}{\pi^2}u_{xx}(x,t), \quad u(x,0) = 20\sin \pi x, \quad \text{and} \quad u(0,t) = u(1,t) = 0 \tag{10.36}$$

b. Describe a physical problem for which this is a solution.

c. What is the 'eventual' value of $u(x,t)$ (what is $\lim_{t\to\infty} u(x,t)$)?

d. At what time, t, will the maximum value of $u(x,t)$ for $0 \le x \le 1$ be 20?

Exercise 10.3.5 In Example 10.3.2, what is

$$\lim_{t\to\infty} u(x,t)\,?$$

Alternatively, what is

$$\lim_{j\to\infty} v_{i,j}\,?$$

The columns of Table 10.1 may suggest an answer.

Exercise 10.3.6 a. How can the calculator program in Table 10.1 be modified if initially the salt concentration in the tube were 0.5?

b. If initially the salt concentration at position x in the tube is x?

Exercise 10.3.7 For the insulated ends boundary condition 10.21

$$u_x(0,t) = 0 \qquad \text{and} \qquad u_x(L,t) = 0,$$

you might approximate these partial derivatives with the difference quotients

$$u_x(0,t) \doteq \frac{u(0+d,t) - u(0,t)}{d} \qquad \text{and} \qquad u_x(L,t) \doteq \frac{u(L,t) - u(L-d,t)}{d}.$$

Then $u_x(0,t) = 0$ and $u_x(L,t) = 0$ would lead to

$$u(0,t) = u(0+d,t) \qquad \text{and} \qquad u(L,t) = u(L-d,t).$$

Suppose the initial condition is

$$u(x,0) = x$$

a. Modify the calculator program in Table 10.1 (better, modify the calculator program of Exercise 10.3.2) to use this boundary condition and initial condition.

b. Run the program and report the result.

c. What do you expect

$$\lim_{t\to\infty} u(x,t) \qquad \text{to be?}$$

Exercise 10.3.8 This exercise is worth quite a bit of your time.

a. For any $g(x)$, how much salt is in the rod (of length 1 meter and cross section 1cm^2) at time $t = 0$? Consider some special cases such as $g(x) \equiv 1\text{g/cm}^3$ and $g(x) = x\text{g/cm}^3$. Then for general g, think, approximately how much salt is in the first 10 cm of the rod, the second 10 cm of the rod, $\cdots$, the last 10 cm of the rod, and add those quantities.

b. For the insulated end boundary condition 10.21 and any initial condition $g(x)$ and rod length L what is

$$\lim_{t\to\infty} u(x,t)\,?$$

Exercise 10.3.9 Suppose there is an infinitely long tube containing water lying along the X-axis from $-\infty$ to ∞ and at time $t = 0$ a bolus injection of one gram of salt is made at the origin. Let $u(x,t)$ be the concentration of salt at position x in the tube at time t.

Considering $t = 0$ is a bit of stressful: $u(x,0) = 0$ for $x \neq 0$; but the bolus injection of one gm at the origin causes the concentration at $x = 0$ and $t = 0$ to be rather large; $u(0,0) = \infty$.

Moving on, for $t > 0$ we may assume that

$$u_t(x,t) = k u_{xx}(x,t) \tag{10.37}$$

where the diffusion coefficient, k, describes the rate at which salt diffuses in water.

a. Show that

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)} \tag{10.38}$$

is a solution to Equation 10.37.

b. Suppose $k = 1/4$. Sketch the graphs of $u(x, 1)$, $u(x, 4)$, and $u(x, 8)$.

c. Suppose $k = 1/4$. Sketch the graphs of $u(x, 1)$, $u(x, 1/2)$, and $u(x, 1/4)$.

d. Estimate the areas under the previous curves. For any time, t_0, what do you expect to be the area under the curve of $u(x, t_0)$, $\infty < x < \infty$. In Chapter 13 the area is shown to be the value of

$$\int_{-\infty}^{\infty} u(x, t_0)\, dx \quad ?$$

Exercise 10.3.10 Diffusion in two dimensions is similar to that in one dimension. The two-dimensional diffusion equation is

$$u_t(x, y, t) = k(u_{xx}(x, y, t) + u_{yy}(x, y, t)). \tag{10.39}$$

1. Suppose a square thin copper plate is embedded in perfect thermal insulation with only one edge exposed. Initially the plate is at 0°C. Then 100°C steam passes over the exposed edge. Describe how you might approximate the temperature distribution within the plate as time progresses.

2. A single instance of a highly contagious influenza occurs at the center of a square city and diffuses through the uniformly distributed population according to Equation 10.39, with $u(x, y, t)$ being the fraction of the population at location (x, y) that is infected at time t. Describe how you may approximate the progress of the disease as a function of time.

Chapter 11

First Order Difference Equations

Where are we going?

The difference equation models of population growth and penicillin clearance introduced in Chapter 1 are refined to account for additional factors (limited population environments and continuous penicillin infusion). Formulas for solutions to linear systems are developed. Qualitative analysis of nonlinear systems shows important biological and mathematical concepts of **equilibrium point** and **asymptotically stable** and **unstable** equilibria.

11.1 Difference Equations and Solutions.

The equation

$$\begin{array}{lcccc} \text{Initial condition} & \text{Change per time interval} & & \text{Removal rate} & \text{Environmental input} \\ P_0 = 0 & P_{t+1} - P_t & = & -0.23P_t \quad + & 0.5 \end{array} \tag{11.1}$$

might well represent continuous infusion of penicillin into a patient. The penicillin kinetics suggested by Equation 11.1 are that each 5 minutes, 23% of the penicillin in the serum is removed and 0.5 μgm/ml is added[1].

We found in Section 1.8 that the solution to the difference equation

$$P_{t+1} - P_t = r\,P_t \,+\, b \tag{11.2}$$

is

$$P_t = -\frac{b}{r} + \left(P_0 + \frac{b}{r}\right)(1+r)^t. \tag{11.3}$$

[1]You should doubt that removal of 23% of serum penicillin each 5 minutes is realistic, and we ask you to read the footnote on page 34.

Thus the response to continuous penicillin infusion, Equation 11.1, is

$$P_t = \frac{0.5}{0.23} + \left(0 - \frac{0.5}{0.23}\right)(1-0.23)^t = 2.17 - 2.17 \times 0.77^t.$$

Important to the analysis (and to the nurse) is the equilibrium state $E = -b/r$ ($= 2.17$ μgm/ml). After two hours ($t = 24$) the penicillin concentration will be 2.17 μgm/ml and will continue at that level. The nurse must decide whether 2.17 μgm/ml will prevent infection, or perhaps be toxic.

In the *Report of the International Whaling Commission (1978)*, J. R. Beddington refers to the following model of Sei whale populations.

$$N_{t+1} - N_t = -0.06N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t \tag{11.4}$$

N_t, N_{t+1} and N_{t-8} represent the adult female whale population subjected to whale harvesting in years t, $t+1$, and $t-8$, respectively. C_t is the number of female whales harvested in year t. There is an assumption that whales become subject to harvesting the same year that they reach sexual maturity and are able to reproduce, at eight years of age. The whales of age less than 8 years are not included in N_t. N_* is the number of female whales that the environment would support with no harvesting taking place.

Explore 11.1.1 Explain the term $-0.06N_t$ in Equation 11.4. ■

In order to solve for progressive values of the whale population, it is necessary to know the first nine values of N_t, for $t = 0, 1, 2, \cdots 8$, and the number, C_t. of whales that would be caught. However, once those values are known, N_t can be computed for any integer value of t. It is a chore to compute very many values of N_t using only paper and pencil, but is rather easy to compute even a thousand values using a calculator program. *There is no formula for the solution to the whale equation as there is in the penicillin example, however.* Our method will be to give qualitative descriptions of the solutions — what happens to the whale population if harvested at a certain intensity for a long period of time.

These are two examples of *difference equations* with *initial data.* They give approximate values of the underlying real world penicillin kinetics or populations. Usually when difference equations arise, there is a property of the system (population, light intensity, chemical concentration, etc.) that can be measured as a quantity, Q, and one of the obvious things that one can say about Q how the change, $Q_{t+1} - Q_t$, depends on time, t, or on Q_t, or perhaps on other values of Q such as Q_{t-8}.

Difference equations are by nature based on discrete independent variables. The discrete characteristic is natural for the whale model because of the annual breeding cycle of whales. It is a minor limitation, however for penicillin kinetics which is a continuous process, and does not proceed in five minute jumps. Division of bacteria is in fact a discrete process, but because of the large number of bacteria (of the order 10^8/ml) and the variety of bacterial ages in the population, the population can be better modeled with a continuous model.

The following equations are, respectively, examples of first order, second order and third order autonomous (explained below) difference equations with initial conditions.

$$\begin{array}{llcll}
Q_0 = 4 & & Q_{t+1} - Q_t & = & 0.05\sqrt{Q_t} & \text{First order.} \\
Q_0 = 1 & Q_1 = 2 & Q_{t+1} - Q_t & = & -0.5Q_t + 0.5Q_{t-1} & \text{Second order} \\
Q_0 = 2 & Q_1 = 1 \quad Q_2 = 1 & Q_{t+1} - Q_t & = & 0.2Q_t - 0.4Q_{t-1}Q_{t-2} & \text{Third order.}
\end{array} \tag{11.5}$$

The first order and third order equations are nonlinear because of the square root and product terms. The second order equation is linear because each term involves only one value of Q_k to the first power. All three equations are *autonomous* because the change, $Q_{t+1} - Q_t$ depends only on values of Q_k.

The following difference equation is not autonomous

$$Q_{t+1} - Q_t = -0.5Q_t + 0.5Q_{t-1} + t^2$$

because of the term t^2. The relation between Q-values changes with time.

The whale equation

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t$$

is not autonomous because of the term C_t, representing the catch in year t which may change from year to year. If the catch were set to be 2 percent of the population, $C_t = 0.02N_t$, however, the resulting equation would be autonomous.

A solution to a difference equation is a sequence, $Q = \{Q_0, Q_1, Q_2, \cdots\}$, of numbers that satisfies the initial conditions and the condition imposed by the difference equation. Usually one can compute at least some of the first terms of the sequence. Consider

$$Q_0 = 2 \qquad Q_{t+1} - Q_t = Q_t + t$$

First, convert to iteration form

$$Q_0 = 0 \qquad Q_{t+1} = 2 \times Q_t + t$$

Then

$$\begin{array}{llllllll} t = 0 & Q_{0+1} & = & 2 \times Q_0 + 0 & = & 2 \times 0 + 0 & = & 0 \\ t = 1 & Q_{1+1} & = & 2 \times Q_1 + 1 & = & 2 \times 0 + 1 & = & 1 \\ t = 2 & Q_{2+1} & = & 2 \times Q_2 + 2 & = & 2 \times 1 + 2 & = & 4 \end{array}$$

Additional values of Q_t can be computed as needed.

Exceptions. It may happen that only a finite sequence of values may be computed from the difference equation or that the values computed become unrealistic (negative, for example, when the numbers in the sequence represent population size). Consider the initial condition and difference equation

$$P_0 = 1 \qquad P_{t+1} - P_t = \sqrt{4 - P_t}$$

One can compute (using iteration form, $P_{t+1} = P_t + \sqrt{4 - P_t}$)

$$\begin{array}{llllllll} t = 0 & P_1 & = & P_0 + \sqrt{4 - P_0} & = & 1 + \sqrt{4 - 1} & = & 2.732 \cdots \\ t = 1 & P_2 & = & P_1 + \sqrt{4 - P_1} & = & 2.732 + \sqrt{1.268} & = & 3.858 \\ t = 2 & P_3 & = & P_2 + \sqrt{4 - P_2} & = & 3.858 + \sqrt{0.142} & = & 4.235 \\ t = 3 & P_4 & = & P_3 + \sqrt{4 - P_3} & = & 4.235 + \sqrt{-0.235} & & \textbf{Punt!} \end{array}$$

Thus P_4 requires the square root of a negative number and is outside our domain of real numbers. We stop with P_3.

Uniqueness of solutions. Difference equations with specified initial conditions of the form

$$Q_{t+1} - Q_t = F(t, Q_t) \qquad Q_0 = Q_{init} \qquad \text{where } Q_{init} \text{ is specified}$$

have unique solutions. Because F is a function; for each pair, (t, Q_t), F assigns a unique number, $F(t, Q_t)$, to the pair. Once Q_0 is known, $Q_1 - Q_0 = F(0, Q_0)$, $Q_1 = Q_0 + F(0, Q_0)$, so that Q_1 is known uniquely, and the rest of the sequence is similarly determined.

As has been noted, some pair (t, Q_t) may fall outside the domain of F so that the computation terminates after a finite number of terms. The point here is that there is no ambiguity in the terms computed, they are unique.

The second order difference equation,

$$Q_{t+2} - Q_{t+1} = Q_t + 0.06Q_{t+1} \times (1 - Q_t), \qquad Q_{t+2} = Q_{t+1} + Q_t + 0.06Q_{t+1} \times (1 - Q_t)$$

states specifically that

$$Q_2 = Q_1 + Q_0 + 0.06Q_1 \times (1 - Q_0).$$

In order to compute Q_2, both Q_0 and Q_1 are required, but once Q_0 and Q_1 are specified, Q_2 is determined. And Q_3 is determined by Q_1 and Q_2. All of the terms Q_2, Q_3, Q_4, $\cdots$ are determined by Q_0, Q_1, and the equation $Q_{t+2} - Q_{t+1} = Q_t + 0.06Q_{t+1} \times (1 - Q_t)$.

For a kth-order difference equation,

$$Q_{t+1} - Q_t = F(t, Q_t, Q_{t-1}, \cdots Q_{t-k+1}),$$

k initial conditions, Q_0, Q_1, $\cdots$, Q_{k-1}, are required to determine Q_k, Q_{k+1}, $\cdots$.

On most calculators it is easy to compute successive iterates of an autonomous first order recursion equation using the 'Previous Answer' key, ANS. For example, to compute iterates of

$$Q_0 = 0.5 \qquad\qquad Q_{t+1} = 1.5Q_t\,(1 - Q_t) \tag{11.6}$$

on your calculator type

0.5, ENTER, 1.5 × ANS × (1 - ANS), ENTER, ENTER, $\cdots$

The following numbers appear on your screen:

0.50000	0.37500	0.35156	0.34194	$\cdots$
$= Q_0$	$= Q_1$	$= Q_2$	$= Q_3$	$\cdots$

Looking ahead. Equation 11.6, is an instance ($\rho = 1.5$) of a renown equation,

$$Q_0 \quad \text{given}, \qquad Q_{t+1} = \rho\, Q_t\,(1 - Q_t). \tag{11.7}$$

which is an important introduction to the exotic subjects of chaos and fractals. We use it in examples and exercises before the thorough description in Section 11.6 of its source, the Verhulst's population growth Equation 11.29,

$$P_{t+1} - P_t = R\,P_t \times \left(1 - \frac{P_t}{M}\right),$$

where P_t is population size at time, t, R is population growth rate in an unlimited environment, and M is the carrying capacity of the environment. The relation between the two equations is

$$\begin{aligned}
P_{t+1} - P_t &= R\,P_t \times \left(1 - \frac{P_t}{M}\right) \\
P_{t+1} &= P_t + R\,P_t - R\,\frac{P_t^2}{M} \\
&= (1+R)P_t\left(1 - \frac{R\,P_t}{M(1+R)}\right) \\
\frac{P_{t+1}}{M(1+R)/R} &= (1+R)\frac{P_t}{M(1+R)/R}\left(1 - \frac{P_t}{M(1+R)/R}\right) \\
Q_{t+1} &= \rho\,Q_t\,(1 - Q_t)
\end{aligned}$$

where

$$\rho = 1 + R, \qquad \text{and} \qquad Q_t = \frac{P_t}{M(1+R)/R}.$$

11.1.1 Formulas for solutions.

Claim:

$$P_t = \frac{1}{3}\,2^t + \frac{2}{3} + t \tag{11.8}$$

is a formula for the solution to

$$P_0 = 1, \qquad P_1 = \frac{7}{3}, \qquad P_{t+2} - 5P_{t+1} + 6P_t = 2t. \tag{11.9}$$

Proof of claim. We must:

1. Check to see whether formula 11.8 yields $P_0 = 1$ and $P_1 = 7/3$. By the formula

$$\begin{aligned}
t = 0 \qquad P_0 &= \frac{1}{3}\,2^0 + \frac{2}{3} + 0 = 1 \\
t = 1 \qquad P_1 &= \frac{1}{3}\,2^1 + \frac{2}{3} + 1 = \frac{7}{3} \qquad \textbf{The initial conditions are satisfied.}
\end{aligned}$$

2. Show that $P_{t+2} - 5P_{t+1} + 6P_t = 2t$. To compute P_{t+1} substitute $t+1$ for each t that appears in $P_t = \frac{1}{3}\,2^t + \frac{2}{3} + t$, and to compute P_{t+2} substitute $t+2$ for each t that appears in the solution.

$$\begin{aligned}
t \qquad P_t &= \frac{1}{3}\,2^t + \frac{2}{3} + t \\
t+1 \qquad P_{t+1} &= \frac{1}{3}\,2^{t+1} + \frac{2}{3} + (t+1) \\
t+2 \qquad P_{t+2} &= \frac{1}{3}\,2^{t+2} + \frac{2}{3} + (t+2)
\end{aligned}$$

Next compute

$$\begin{aligned}
P_{t+2} - 5P_{t+1} + 6P_t &= \left(\frac{1}{3}\,2^{t+2} + \frac{2}{3} + (t+2)\right) - 5\left(\frac{1}{3}\,2^{t+1} + \frac{2}{3} + (t+1)\right) \\
&\qquad +6\left(\frac{1}{3}\,2^{t} + \frac{2}{3} + t\right) \\
&= \left(\frac{1}{3}\,2^{t+2} - \frac{5}{3}\,2^{t+1} + \frac{6}{3}\,2^{t}\right) + (t - 5t + 6t) + \\
&\qquad \left(\frac{2}{3} + 2 - 5 \times \frac{2}{3} - 5 + 6 \times \frac{2}{3}\right) \\
&= \left(\frac{4}{3}\,2^{t} - \frac{10}{3}\,2^{t} + \frac{6}{3}\,2^{t}\right) + (2t) + (0) \\
&= 2t \qquad \textbf{The Difference Equation is satisfied.}
\end{aligned}$$

To check that a proposed formula actually defines the solution of a difference equation, use the two steps of the previous example. It is more difficult to find such a formula. You are asked to find solutions to first order linear difference equations with constant coefficients in Exercises 11.1.15 and 11.1.16. The same procedure works for second order linear difference equations with constant coefficients. Part of the procedure is illustrated by the following two examples.

Problem. Find numbers p and q such that

$$P_t = p + qt \qquad \text{is a solution to} \qquad P_{t+2} - 5P_{t+1} + 6P_t = 2 + t$$

Solution. Substitute $P_t = p + qt$ into the difference equation.

$$\begin{aligned}
p + q(t+2) - 5\,(p + q(t+1)) + 6(p + qt) &= 2 + t \\
p + 2q - 5p - 5q + 6p + (q - 5q + 6q)\,t &= 2 + t \\
2p - 3q + 2qt &= 2 + t
\end{aligned}$$

Match coefficients. That is, require the constant terms and coefficients of t to be the same:

$$2p - 3q = 2, \qquad \text{and} \qquad 2q = 1.$$

Then $q = 1/2$ and $p = 7/4$, and $P_t = 7/4 + (1/2)t$ solves $P_{t+2} - 5P_{t+1} + 6P_t = 2 + t$. There may be initial conditions that are not satisfied by $P_t = 7/4 + (1/2)t$, however, and additional terms (**Lightning Bolt:** of the form $C_1 2^t + C_2 3^t$) may be required

Problem. Find numbers C and R such that

$$P_t = C \times R^t \qquad \text{is a solution to} \qquad P_{t+2} - 5P_{t+1} + 6P_t = 3 \times 5^t$$

Solution. Substitute $P_t = C \times R^t$ into the difference equation.

$$C \times R^{t+2} - 5C \times R^{t+1} + 6C \times R^t = 3 \times 5^t$$

$$C \times R^2 \times R^t - 5C \times R \times R^t + 6C \times R^t = 3 \times 5^t$$

$$\Big(C \times R^2 - 5C \times R + 6C\Big) \times R^t = 3 \times 5^t$$

By inspection, choose $R = 5$ and $C = 3/(5^2 - 5 \times 5 + 6) = 1/2$. Then $P_t = (1/2)\,5^t$ solves $P_{t+2} - 5P_{t+1} + 6P_t = 3 \times 5^t$.

Exercises for Section 11.1, Difference Equations and Solutions.

Exercise 11.1.1 Write the Difference Equations 11.5 in iteration form and compute three terms after the initial conditions.

Exercise 11.1.2 Compute Q_2, Q_3, Q_4 and Q_5 for

a. $Q_0 = 0.1 \qquad Q_1 = 0.1 \qquad Q_{t+2} = Q_{t+1} + 0.06 \times (1 - Q_t)$

b. $Q_0 = 10 \qquad Q_1 = 12 \qquad Q_{t+2} - Q_{t+1} = 0.2 \times Q_t \times \left(1 - \frac{Q_t}{100}\right)$

c. $Q_0 = 0 \qquad Q_1 = 1 \qquad Q_{t+2} = Q_{t+1} + Q_t$

d. $Q_0 = 0.3 \qquad Q_1 = 0.7 \qquad Q_{t+2} = 5 \times Q_{t+1} - 6 \times Q_t$

e. $Q_t = 0.2 \times 2^t + 0.1 \times 3^t$

Exercise 11.1.3 Peroxidase catalyzes the reaction

$$2H_2O_2 \rightarrow 2H_2O + 2O$$

The rate of the reactions is proportional to the concentration of peroxidase times the concentration of hydrogen peroxide, H_2O_2. Because the enzyme peroxidase recycles in the reaction, suppose the concentration of enzyme is constant, $= E$. Assume time is measured in in 0.1 second intervals, and let w_t denote the concentration of H_2O_2 at time t.

1. Assume the proportionality constant for the reaction is k. Write a difference equation showing the change in $H2O_2$ between time t and time $t + 1$.
2. Assume the concentration of H_2O_2 at time $t = 0$ is 0.2 molar. Write an equation for w_t in terms of t.

Exercise 11.1.4 Use your calculator and the 'Previous Answer' key to compute $Q_1, \cdots Q_{50}$ for

$$Q_0 = 10 \qquad\qquad Q_{t+1} = Q_t + 0.2Q_t\left(1 - \frac{Q_t}{50}\right)$$

(Type 10, ENTER, ANS + 0.2 × ANS × (1 − ANS/50), ENTER (50 times))

You should find $Q_{50} = 49.99583$. Approximately what will be the values of $Q_{51}, \cdots Q_{100}$?

Exercise 11.1.5 Use your calculator and the 'Previous Answer' key to compute $Q_1, \cdots Q_{20}$ for

a. $$\begin{aligned} Q_0 &= 5 \\ Q_{t+1} &= Q_t + 0.1Q_t\left(1 - \frac{Q_t}{20}\right) \end{aligned}$$

b. $$\begin{aligned} Q_0 &= 5 \\ Q_{t+1} &= 1.1Q_t\left(1 - \frac{Q_t}{20}\right) \end{aligned}$$

c. $$\begin{aligned} Q_0 &= 5 \\ Q_{t+1} &= Q_t + 0.1Q_t + 0.02 * Q_t^2 \end{aligned}$$

d. $$\begin{aligned} Q_0 &= 0.5 \\ Q_{t+1} &= \cos(Q_t) \end{aligned}$$

e. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= Q_t(2 - Q_t) \end{aligned}$$

f. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= Q_t(3 - Q_t) \end{aligned}$$

g. $$\begin{aligned} Q_0 &= 0.5 \\ Q_{t+1} &= Q_t(3.5 - Q_t) \end{aligned}$$

h. $$\begin{aligned} Q_0 &= 0.6 \\ Q_{t+1} &= Q_t(3.5 - Q_t) \end{aligned}$$

i. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= e^{-Q_t} \end{aligned}$$

j. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= \left(\sqrt{Q_t}\right) \times e^{-Q_t} \end{aligned}$$

k. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= \frac{Q_n + 2/Q_n}{2} \end{aligned}$$

l. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= \frac{Q_n + 3/Q_n}{2} \end{aligned}$$

Exercise 11.1.6 Find the first five terms of the solutions to

a. $Q_0 = 1 \quad Q_1 = 0 \qquad Q_{t+1} - Q_t = Q_{t-1}$

b. $Q_0 = 1 \quad Q_1 = 0 \qquad Q_t - Q_{t-1} = Q_{t-2}$

c. $Q_0 = 1 \quad Q_1 = 1 \qquad Q_{t+1} - Q_t = Q_{t-1}$

d. $Q_0 = 1 \quad Q_1 = 0 \qquad Q_{t+1} = -Q_{t-1}$

e. $Q_0 = 1 \quad Q_1 = 1 \qquad Q_t = -Q_{t-2}$

f. $Q_0 = 1 \quad Q_1 = 0 \quad Q_2 = 2 \qquad Q_{t+1} - Q_t = Q_t - Q_{t-2}$

g. $Q_0 = 1 \quad Q_1 = 1 \quad Q_2 = 1 \qquad Q_{t+1} = Q_t - Q_{t-1} + Q_{t-2}$

h. $Q_0 = 0.6 \quad Q_1 = 0.7 \quad Q_2 = 0.5 \qquad Q_{t+1} = Q_t + 0.6 * Q_{t-2}(1 - Q_{t-2}) - 0.2Q_{t-1}$

Exercise 11.1.7 Compute Q_2, Q_3, and Q_4 for

(a) $Q_0 = 1 \quad Q_1 = 1 \qquad Q_{t+2} - 2Q_{t-1} + Q_t = 0$

(b) $Q_0 = 10 \quad Q_1 = 5 \qquad Q_{t+2} - 1.2Q_{t+1} + 0.32Q_t = 0$

Exercise 11.1.8 Enter the following program on your calculator.

```
Prgm, Edit WH3
:0.5->XM1
:0.6->XM0
:For(I,1,6)
:XN=XM0 + 0.06*XM1*(1-XM1)- 0.02*XM1
:XM0->XM1
:XN->XM0
:Pause Disp XM0
:End
```

Exit Prgm. Type WH3 and ENTER, ENTER, ENTER, ENTER, ENTER, ENTER, ENTER. Interpret the results.

Exercise 11.1.9 For each equation, modify the calculator program in Exercise 11.1.8 to compute P_2, P_3, $\cdots$ P_7.

a. $P_0 = 0.3$ $P_1 = 0.4$ $$P_{t+1} = P_t + 0.6P_{t-1}(1 - P_{t-1}) - 0.02P_{t-1}$$

b. $P_0 = 0.8$ $P_1 = 0.7$ $$P_{t+1} = P_t + 0.6P_{t-1}(1 - P_{t-1}) - 0.02P_{t-1}$$

c. $P_0 = 0.5$ $P_1 = 0.6$ $$P_t = 2P_{t-1} - 0.96P_{t-2}$$

d. $P_0 = 1$ $P_1 = 2$ $$P_{t+2} = \frac{2P_{t+1}}{1 + P_t/50}$$

e. $P_0 = 1$ $P_1 = 2$ $$P_{t+2} = 1.2 * P_{t+1} * \left(1 - \frac{P_t}{10}\right) - 0.1$$

f. $P_0 = 1$ $P_1 = 2$ $P_2 = 1$ $$P_{t+3} = P_{t+2} + P_t\left(1 - \frac{P_t}{50}\right) - 0.1 * P_{t+1}$$

Exercise 11.1.10 Compute solutions until they become negative or imaginary for the systems:

a. $$\begin{aligned} P_0 &= \pi/4 \\ P_{t+1} &= \ln(\tan(P_t)) \end{aligned}$$

b. $$\begin{aligned} P_0 &= 5 \\ P_{t+1} &= P_t - 1 \end{aligned}$$

c. $$\begin{aligned} P_0 &= 0.76 \\ P_{t+1} &= 2\sqrt{P_t}\,(1 - P_t^2) \end{aligned}$$

d. $$\begin{aligned} P_0 &= 2 \\ P_{t+1} &= 0.9P_t - 0.1 \end{aligned}$$

Exercise 11.1.11 The following difference equations, initial data, and solutions have been scrambled.

1. Match each solution to a correct initial condition and difference equation that it satisfies.
2. Compute Q_{50} using the solution.
3. Show algebraically that the solution satisfies the proposed difference equation.

Solutions	Initial Data	Difference Equations
$S_1 \quad P_t = 2+4t$	$ID_1 \quad P_0 = 0$	$DE_1 \quad P_{t+1}-P_t = 4t+8$
$S_2 \quad P_t = 3\times 5^t$	$ID_2 \quad P_0 = 1$	$DE_2 \quad P_{t+1}-P_t = 0$
$S_3 \quad P_t = 5\times 3^t$	$ID_3 \quad P_0 = 2$	$DE_3 \quad P_{t+1}-P_t = 2t+1$
$S_4 \quad P_t = 2t^2+6t$	$ID_4 \quad P_0 = 3$	$DE_4 \quad P_{t+1}-P_t = 2\times P_t$
$S_5 \quad P_t = 4+t^2$	$ID_5 \quad P_0 = 4$	$DE_5 \quad P_{t+1}-P_t = 4$
$S_6 \quad P_t = 1$	$ID_6 \quad P_0 = 5$	$DE_6 \quad P_{t+1}-P_t = 4\times P_t$

Exercise 11.1.12 For each equation find a number E such that $P_t = E$ is a solution.

a. $P_{t+1}-0.9P_t = 2$ b. $P_{t+1}+0.9P_t = 2$

c. $P_{t+1}-0.5P_t = 3$ d. $P_{t+1}+0.9P_t = -5$

e. $P_{t+2}-0.2P_{t+1}-0.6P_t = 2$ f. $P_{t+2}+0.2P_{t+1}-0.4P_t = 2$

Exercise 11.1.13 For each equation find a numbers p and q such that $P_t = p+qt$ is a solution.

a. $P_{t+1}-0.9P_t = t$ b. $P_{t+1}+0.9P_t = 1+2t$

c. $P_{t+1}-0.5P_t = 3+2t$ d. $P_{t+1}+0.9P_t = 3-t$

e. $P_{t+2}-0.2P_{t+1}-0.6P_t = 5+4t$ f. $P_{t+2}+0.2P_{t+1}-0.4P_t = 2-3t$

Exercise 11.1.14 For each equation find a numbers C and R such that $P_t = C\times R^t$ is a solution.

a. $P_{t+1}-0.9P_t = 3\times 2^t$ b. $P_{t+1}+0.9P_t = 5\times 0.5^t$

c. $P_{t+1}-0.5P_t = 0.8^t$ d. $P_{t+1}+0.9P_t = -0.9^t$

e. $P_{t+2}-0.2P_{t+1}-0.6P_t = 2\times 0.9^t$ f. $P_{t+2}+0.2P_{t+1}-0.4P_t = 2\times 1.1^t$

Exercise 11.1.15 How to solve

$$P_0 \quad \text{known} \qquad P_{t+1}-P_t = rP_t+b+c\,t. \tag{11.10}$$

a. Compare this equation with Equation 11.2.

b. Find an 'equilibrium' linear sequence $E_t = p+q\,t$ such that

$$\begin{aligned} E_{t+1}-E_t &= rE_t+b+c\,t \\ p+q(t+1)-(p+q\,t) &= r(p+q\,t)+b+c\,t \\ q-r\,p+-r\,q\,t &= b+c\,t \end{aligned}$$

Equate coefficients

$$q - r\,p = b \qquad -r\,q = c$$

and solve for p and q.

c. Subtract the two equations:

$$\begin{aligned} P_{t+1} - P_t &= rP_t + b + c\,t \\ E_{t+1} - E_t &= rE_t + b + c\,t \end{aligned}$$

to get

$$P_{t+1} - E_{t+1} = (1+r)(P_t - E_t)$$

Argue that $P_t - E_t = (P_0 - E_0)(1+r)^t$.

d. Conclude that

$$P_t = -\left(\frac{b}{r} + \frac{c}{r^2} + \frac{c}{r}t\right) + \left(P_0 + \frac{b}{r} + \frac{c}{r^2}\right)(1+r)^t \tag{11.11}$$

e. Compare this solution with Equation 11.3.

Exercise 11.1.16 How to solve

$$P_0 \quad \text{known} \qquad P_{t+1} - P_t = rP_t + De^{kt}. \tag{11.12}$$

a. Find an 'equilibrium' exponential sequence $E_t = Ce^{kt}$ such that

$$\begin{aligned} E_{t+1} - E_t &= rE_t + De^{kt} \\ Ce^{k(t+1)} - Ce^{kt} &= rCe^{kt} + De^{kt} \\ (Ce^k - C - rC)e^{kt} &= De^{kt} \end{aligned}$$

Choose $C = D/(e^k - 1 - r)$.

b. Subtract the two equations:

$$\begin{aligned} P_{t+1} - P_t &= rP_t + De^{kt} \\ E_{t+1} - E_t &= rE_t + De^{kt} \end{aligned}$$

to get

$$P_{t+1} - E_{t+1} = (1+r)(P_t - E_t)$$

Argue that $P_t - E_t = (P_0 - E_0)(1+r)^t$.

c. Conclude that

$$P_t = \frac{D}{e^k - 1 - r}e^{kt} + \left(P_0 - \frac{D}{e^k - 1 - r}\right)(1+r)^t \tag{11.13}$$

Exercise 11.1.17 Suppose there is a lake of volume 100,000 cubic meters and a stream that runs into the lake and out of the lake at a rate of 2,000 meters cube per day. Suppose further a mining operation is developed in the drainage area to the lake and one kilogram of mercury leaches into the lake each day and is mixed uniformly into the water. How much mercury is in the lake after 1000 days?

Exercise 11.1.18 Use the following formulas to find solutions to the subsequent equations.

DifferenceEquation	Solution
$P_{t+1} - P_t = r\,P_t + b$	$P_t = -\frac{b}{r} + \left(P_0 + \frac{b}{r}\right)(1+r)^t.$
$P_{t+1} - P_t = rP_t + b + c\,t$	$P_t = -\left(\frac{b}{r} + \frac{c}{r^2} + \frac{c}{r}t\right) + \left(P_0 + \frac{b}{r} + \frac{c}{r^2}\right)(1+r)^t$
$P_{t+1} - P_t = rP_t + De^{kt}$	$P_t = \frac{D}{e^k - 1 - r}e^{kt} + \left(P_0 - \frac{D}{e^k - 1 - r}\right)(1+r)^t$

a. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 0$

b. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 1$

c. $P_0 = 2 \quad P_{t+1} - 1.2P_t = 0$

d. $P_0 = 2 \quad P_{t+1} - 1.2P_t = 2$

e. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 3 + 2t$

f. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 3e^t$

g. $P_0 = 2 \quad P_{t+1} - 1.2P_t = -1 + 4t$

h. $P_0 = 2 \quad P_{t+1} - 1.2P_t = 2e^{-t}$

11.2 Graphical methods for difference equations.

The difference equations for which we have found solution formulas have all been linear difference equations. They are one of the forms

$$P_{t+1} - P_t = rP_t + G_t \qquad \text{or} \qquad P_{t+2} + a_1P_{t+1} + a_0P_t = G_t$$

The terms involving the unknown sequence P_t are all of first power. No term is P_{t+1}^2 or $\ln P_t$ or $\frac{1}{P_t}$ or $P_{t+2} \times P_t$. The sequence G_t, can involve linear and exponential terms in t, and even polynomials in t (but do not push this – solutions may be hard to write), but is independent of P_t.

It is usually hard to find formulas for solutions to nonlinear equations. There are no formulas that describe the solutions to

$$\begin{array}{rcl@{\qquad}rcl} P_0 &=& 4 & P_{t+1} - P_t &=& 0.2 \times e^{P_t/50} \\ P_0 &=& 10 & P_{t+1} - P_t &=& 0.2 \times P_t \times \left(1 - \frac{P_t}{500}\right) \end{array}$$

$$\left.\begin{array}{rcl} P_0 &=& 10 \\ P_1 &=& 8 \\ P_2 &=& 12 \end{array}\right\} \qquad P_{t+1} - P_t = -0.2 \times P_t + P_{t-2}\left[0.2 + 0.3\left(1 - \frac{P_{t-2}}{500}\right)\right]$$

But we can write solutions to the nonlinear equation

$$P_0 = 2 \qquad P_{t+1} = \frac{1.2P_t}{1 + P_t/50}$$

(This is actually a linear equation in disguise.)

The previous nonlinear equations all have reasonable population model interpretations, and we look for graphical methods that will show the qualitative behavior of solutions as a good replacement for having a formula for the solution.

For the equation,

$$P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{500}\right)$$

we can select various values for P_0 and compute subsequent values of P_1, P_2, $\cdots$, as is illustrated in the table and graph in Figure 11.1.

Time t	P_t $P_0 = 100$	P_t $P_0 = 400$	P_t $P_0 = 650$
0.0	100.0	400.0	650.0
1.0	116.0	416.0	611.0
2.0	133.8	430.0	583.9
3.0	153.4	442.0	564.3
4.0	174.7	452.3	549.8
5.0	197.4	460.9	538.8
6.0	221.3	468.1	530.5
7.0	246.0	474.1	524.0
8.0	271.0	479.0	519.0
9.0	295.8	483.0	515.0
...	...	...	...

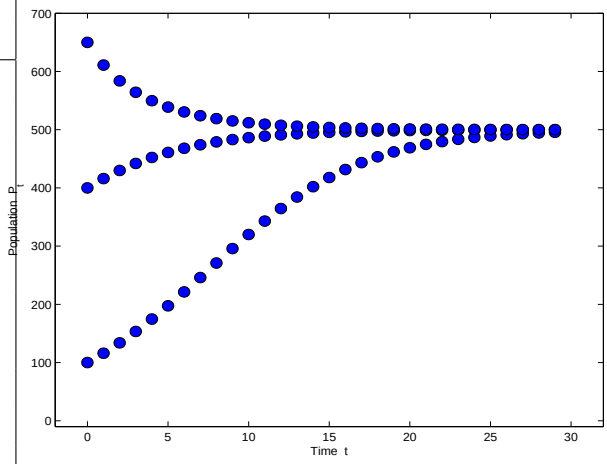


Figure 11.1: Table and graph for $P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{500}\right)$

It is fairly apparent that at least these three solutions have the same horizontal asymptote, at about P= 500. We will find that the number 0.2 regulates the rate at which P_t moves towards the asymptote.

For

$$P_{t+1} - P_t = -0.2 \times P_t + P_{t-2}\left[0.2 + 0.3\left(1 - \frac{P_{t-2}}{500}\right)\right]$$

we have to select values for P_0, P_1 and P_2 in order to initiate the computation, and then additional values can be computed and graphed, as shown in Figure 11.2

Exercises for Section 11.2, Graphical methods for difference equations.

Time t	P_t $P_0 = 100$ $P_1 = 100$ $P_2 = 100$	P_t $P_0 = 400$ $P_1 = 400$ $P_2 = 400$	P_t $P_0 = 650$ $P_1 = 650$ $P_2 = 650$
0.0	100.0	400.0	650.0
1.0	100.0	400.0	650.0
2.0	100.0	400.0	650.0
3.0	124.0	424.0	591.5
4.0	143.2	443.2	544.7
5.0	158.6	458.6	507.3
6.0	179.6	471.0	491.6
7.0	203.0	480.5	487.6
8.0	226.6	487.5	489.4
9.0	251.7	492.4	492.3
...	...	...	...

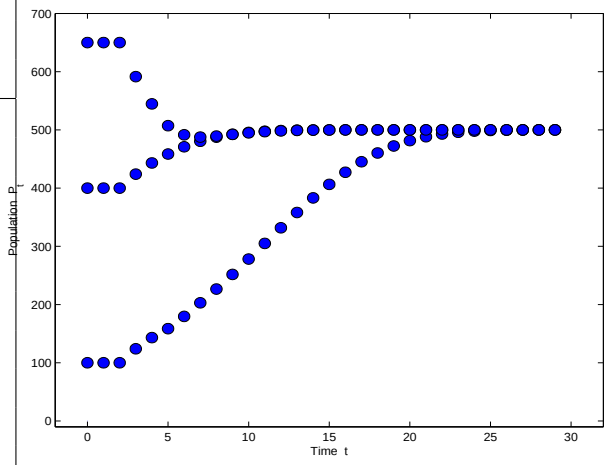


Figure 11.2: Table and graph for $P_{t+1} - P_t = -0.2 \times P_t + P_{t-2}\left[0.2 + 0.3\left(1 - \frac{P_{t-2}}{500}\right)\right]$.

Exercise 11.2.1 Using the graphs in Figures 11.1 and 11.2 as guides, draw graphs of the solutions to

$$\begin{array}{llllll}
\text{a.} & P_0 & = & 100 & P_{t+1} - P_t & = & 0.1P_t \times \left(1 - \frac{P_t}{500}\right) \\
\text{b.} & P_0 & = & 100 & P_{t+1} - P_t & = & 0.2P_t \times \left(1 - \frac{P_t}{200}\right) \\
\text{c.} & P_0 & = & 100 & & & \\
 & P_1 & = & 100 & P_{t+1} - P_t & = & -0.2 \times P_t + P_{t-3}\left[0.2 + 0.3\left(1 - \frac{P_t}{1000}\right)\right] \\
 & P_2 & = & 100 & & &
\end{array}$$

Exercise 11.2.2 Plot the graph of

$$w_0 = 2 \quad w_{t+1} = 5.1 \times \frac{w_t}{5 + w_t}$$

Solution 1. Use the 'Previous Answer' key on your calculator:

2.0, ENTER, 5.1× ANS/(5 + ANS), ENTER, ENTER, $\cdots$.

Your display will show 2.0, 1.45713, 1.50885, 0.954255, $\cdots$, which are the values of w_0, w_1, w_2, w_3, $\cdots$. They can be plotted against the index, 0, 1, 2, 3, $\cdots$ to yield the graph.
Solution 2. Enter the following program on your calculator:

```
PROGRAM:ITER
:21-> dimL T
:21-> dimL W
:0 -> T(1)
:2 -> W(1)
:For(K,1,20)
:min(T)->xMin
:max(T)->xMax
:(xMax-xMin)/5->xScl
:min(W)->yMin
:max(W)->yMax
:(yMax-yMin)/5->yScl
```

```
:K+1 -> T(K+1)
:5.1*W(K)/(5+W(K)) -> W(K+1)
:End
:Disp ''yMin'', yMin
:Disp ''yMax'', yMax
:Pause
:Scatter T,W
:Stop
```

'Unselect' all functions in GRAPH. Run ITER. It will display yMin 0.27726 yMax 2.00000 and pause. Press ENTER and it will display the graph. You can remove the menu from the bottom of the screen with CLEAR. To finish, press EXIT. You need yMin and yMax to draw the graph on paper.

Exercise 11.2.3 Plot graphs of solutions to

a. $w_0 = 2 \quad w_{t+1} = 1.2 \times \frac{w_t}{0.5+w_t}$

b. $w_0 = 0.2 \quad w_{t+1} = 1.2 \times \frac{w_t}{0.5+w_t}$

c. $w_0 = 2 \quad w_{t+1} = 1.2 \times w_t \times e^{-w_t/10}$

d. $w_0 = 0.1 \quad w_{t+1} = 1.2w_t \times \cos(w_t)$

e. $w_0 = 0.001 \quad w_{t+1} = w_t + \sin w_t$

f. $w_0 = 0 \quad w_{t+1} = w_t + \sin w_t$

g. $w_0 = 0 \quad w_{t+1} = w_t + 1$

h. $\begin{aligned} w_0 &= 0 \\ w_1 &= 1 \end{aligned} \quad w_{t+2} = w_{t+1} - w_t$

Exercise 11.2.4 For each equation use $W_0 = 0.2$ and plot N iterates to the equations:

a. $w_{t+1} = 2.8w_t(1 - w_t) \quad N = 20$

b. $w_{t+1} = 3.2w_t(1 - w_t) \quad N = 20$

c. $w_{t+1} = 3.5w_t(1 - w_t) \quad N = 100$

d. $w_{t+1} = 3.56w_t(1 - w_t) \quad N = 200$

e. $w_{t+1} = 3.58w_t(1 - w_t) \quad N = 400$

f. $w_{t+1} = 3.5825w_t(1 - w_t) \quad N = 400$

Exercise 11.2.5 For the iteration equation,

$$w_0 = 0.2, \qquad w_{t+1} = Rw_t(1 - w_t)$$

and for each number, $R = 2.5 + k \times 0.01$, $k = 0, \cdots, 150$, compute 1000 iterates of the equation and plot (R, w_k) for $k = 950, \cdots, 1000$.

11.3 Equilibrium Points, Stable and Nonstable

In This Section:

The first crucial question about difference equations that describe populations is whether the solution progresses to a finite nonzero equilibrium value. The alternative is that the population decreases to extinction or grows beyond environmental bounds or oscillates repeatedly, possibly without periodic repetition.

The second question is, given that the population size is at a nonzero equilibrium value, if the population experiences a small perturbation (by a weather change, for example), will the size return to the equilibrium value?

Methods are given for answering these questions by examination of the difference equation, even without a solution to the difference equation.

The whale equation

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t$$

may be rescaled by dividing each term by N_*, the number of females that would be present without harvesting. Define

$$Y_t = N_t/N_* \qquad \text{and} \qquad D_t = C_t/N_*,$$

and write

$$Y_{t+1} = 0.94Y_t + Y_{t-8}\left[0.06 + 0.0567\left\{1 - Y_{t-8}^{2.39}\right\}\right] - 0.94D_t$$

We set the harvest $D_t = 0.02$ (so that $C_t = 0.2N_*$) which means that 2% of the equilibrium in an unharvested population is harvested without regard to the present population size.

Solutions to

$$Y_{t+1} = 0.94Y_t + Y_{t-8}\left[0.06 + 0.0567\left\{1 - Y_{t-8}^{2.39}\right\}\right] - 0.94 \times 0.02 \tag{11.14}$$

are plotted in Figure 11.3. The harvest is set to be 0.02 (last term in the equation), which means that 2% of the equilibrium in an unharvested population is harvested without regard to the present population size. The initial conditions are set at A: 1.0, B: 0.6, and C: 0.2 of the unharvested equilibrium population.

For both cases A and B it appears that the whale population is moving toward about 0.8, meaning that with harvest of 2% of the equilibrium unharvested population, the harvested population will stabilize at about 80% of the unharvested equilibrium. However, if the initial whale population is only 20% of the unharvested equilibrium population, when 2% harvest is initiated the population will disappear.

In order to explore the interaction of whale growth and harvest more thoroughly we introduce a modification in the model and a new way of graphing the relation. First we rewrite the model to be

$$Y_{t+1} = 0.94Y_t + Y_t\left[0.06 + 0.0567\left\{1 - Y_t^{2.39}\right\}\right] - 0.94 \times 0.02 \tag{11.15}$$

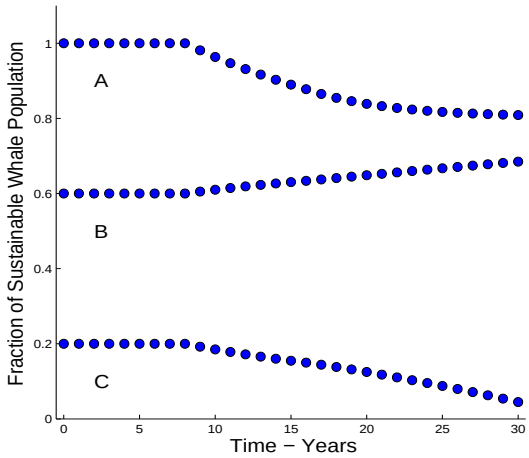


Figure 11.3: Solutions to the whale equation, Equation 11.14, with harvest set at 2 percent of the equilibrium population in the absence of harvest and three sets of initial conditions, A: $Y_i = 1.0, \quad i = 0, 8$, B: $Y_i = 0.6, \quad i = 0, 8$, and C: $Y_i = 0.2, \quad i = 0, 8$.

We have replaced the Y_{t-8} with Y_t which effectively assumes the whales reach sexual maturity in one year rather than in eight years. To compute data we need only a single initial value. A plot of the new whale numbers is shown in Figure 11.4 It can be seen that the dynamics are similar to those in the previous figure, except that the whale population in case C crashes more quickly, in about 16 years.

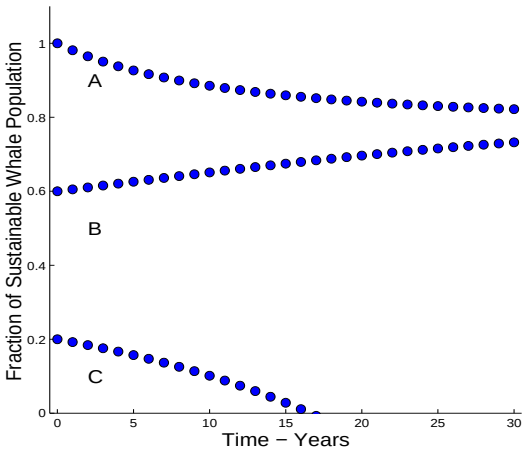


Figure 11.4: Solutions to the modified whale equation, Equation 11.15, with harvest set at 2 percent of the equilibrium population in the absence of harvest and three sets of initial conditions, A, $Y_0 = 1.0$, B, $Y_0 = 0.6$, and C, $Y_0 = 0.2$.

A new way of visualizing the whale growth is to examine Y_{t+1} as a function of Y_t. Thus from

$$Y_{t+1} = 0.94Y_t + Y_t\left[0.06 + 0.0567\left\{1 - Y_t^{2.39}\right\}\right] - 0.94 \times 0.02$$

we write a functional form

$$Y_{t+1} = F(Y_t)$$

where

$$F(Y) = 0.94Y + Y\left[0.06 + 0.0567\left\{1 - Y^{2.39}\right\}\right] - 0.94 \times 0.02$$

which simplifies to

$$F(Y) = Y + 0.0567Y\left(1 - Y^{2.39}\right) - 0.0188$$

For illustration, we first examine the equation

$$F(Y) = Y + 5 \times \left(0.0567Y\left(1 - Y^{2.39}\right) - 0.0188\right) \tag{11.16}$$

that corresponds to a model with low density birth rate and harvest five times that of the previous model. A graph of $y = F(Y)$ is shown in Figure 11.5 together with the graph of $y = x$.

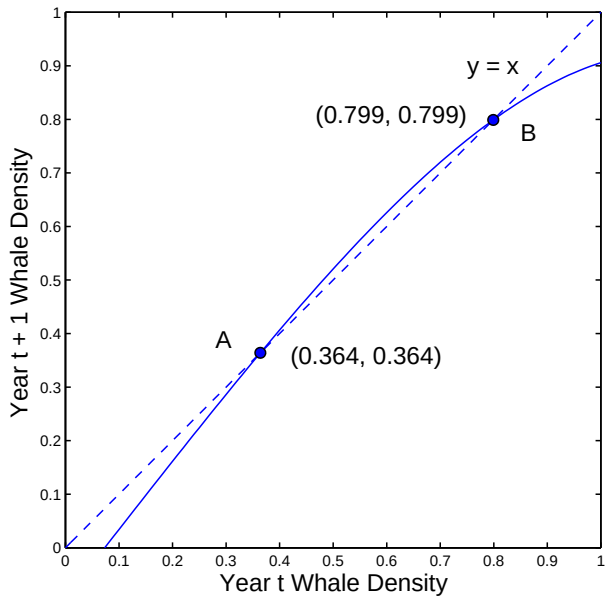


Figure 11.5: Graphs of Equation 11.16 and $y = x$.

Remember that if Y is the population in year t, then $y = F(Y)$ is the population in year $t + 1$. That gives special meaning to the two points, **A** (0.364, 0.364) and **B** (0.799, 0.799), where the graph of $y = F(Y)$ crosses the line $y = x$. If the population in year t is 0.364, then the population in year $t + 1$ is also 0.364. By the same reason, the population in year $t + 2$ is also 0.364, and according to this model, the population will stay at 0.364. Similarly, if the population is ever 0.799, by this model it will always be at 0.799.

The points **A** and **B** are called **equilibrium points** for the iteration $Y_{t+1} = F(Y_t)$. Alternatively, the common coordinates, 0.364 of **A** and 0.799 of **B**, are called **equilibrium values** of $N_{t+1} = F(N_t)$.

Equilibrium points of the iteration

$$Y_{t+1} = F(Y_t) \qquad \text{are found by solving for } Y_e \text{ in} \qquad Y_e = F(Y_e)$$

Explore 11.3.1 Do this. Equation 11.16 can be simplified to

$$Y_{t+1} = 1.2835Y_t - 0.2835Y_t^{3.39} - 0.094.$$

a. Begin with $Y_0 = 0.799$ and compute Y_1, Y_2, $\cdots$, Y_{20}. You should get $Y_{20} = 0.79909107$.)

b. Repeat the calculation with $Y_0 = 0.364$. You should observe that the iterations drift downward from 0.364 and that $Y_{20} = 0.359$. In fact, Y_{43} is negative.

c. Repeat the calculation with $Y_0 = 0.365$. You should find that $Y_{20} = 0.395$.

■

You just found that for $Y_0 = 0.799$, Y_t remains close to 0.799, but for $Y_0 = 0.364$, Y_t moves away from 0.364. Neither 0.364 nor 0.799 are the exact values of the points of intersection. Closer approximations are 0.36412683047 and 0.79909120051, but neither of these is exact either. The exact values are solutions to

$$y = x \qquad \text{and} \qquad y = 1.2835x - 0.2835x^{3.39} - 0.094$$

or

$$x_e = 1.2835x_e - 0.2835x_e^{3.39} - 0.094 \qquad \text{or} \qquad 0.2835x_e - 0.2835x_e^{3.39} - 0.094 = 0$$

The portions of the graph of $y = F(Y)$ that lie below the graph of $y = x$ are informative. Observe the point **C**, (0.260, 0.237) in Figure 11.6. If $Y_t = 0.260$, then in the next generation, $Y_{t+1} = 0.237$ which is less than Y_t.

The portions of the graph of $y = F(x)$ below the diagonal, $y = x$, where $F(x) < x$ and $Y_{t+1} = F(Y_t) < Y_t$, are associated with decreasing population numbers. Similarly, the portions of the graph of $y = F(x)$ above the diagonal, $y = x$, are associated with increasing population numbers, $Y_{t+1} = F(Y_t) > Y_t$. These properties confer quite different characteristics on the two equilibrium points, **A** and **B**.

Briefly, iterates move away from **A** and toward **B**. Let **A** be (a, a) exactly ($a \doteq 0.364$). If Y_0 is just less than a then Y_1 is less than Y_0, Y_2 is less than Y_1, and the values Y_0, Y_1, Y_2, $\cdots$ decrease away from a.

Alternatively, if Y_0 is slightly greater than a, then Y_1 is greater than Y_0, Y_2 is greater than Y_1, and the values Y_0, Y_1, Y_2, $\cdots$ increase away from a. The number, a, is said to be a **nonstable equilibrium** for the iteration, $Y_{t+1} = F(Y_t)$.

By similar reasoning it can be seen that for **B** (b, b), b is a **stable equilibrium** for the iteration $Y_{t+1} = F(Y_t)$. If Y_0 is greater than b Y_0, Y_1, Y_2, $\cdots$ decrease toward b and if Y_0 is slightly less than b Y_0, Y_1, Y_2, $\cdots$ increase toward b.

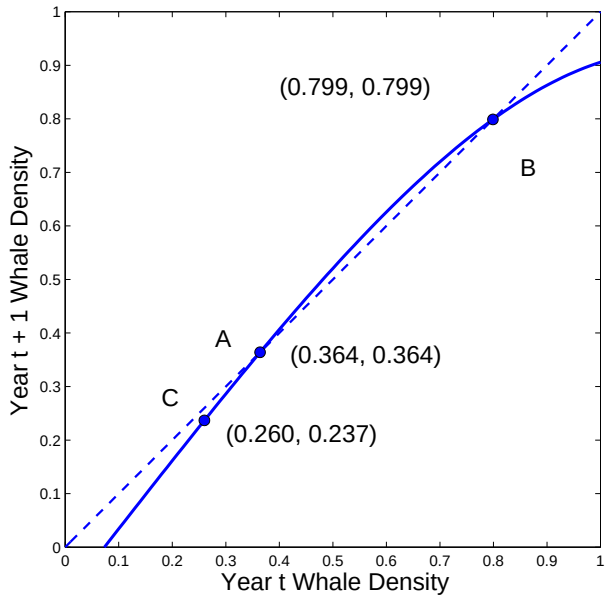


Figure 11.6: The graphs of $F(Y) = 1.2835Y - 0.2835Y^{3.39} - 0.094$ and $y = x$. The points A and B are equilibrium points of the iteration $Y_{t+1} = F(Y_t)$. The point C illustrates that if $Y_t = 0.260$, then $Y_{t+1} = 0.237$ is less than Y_t.

This discussion is repeated and expanded in the next section, but a formal definition of stable and nonstable equilibrium points follows.

Definition 11.3.1 Stable and nonstable equilibrium points. Suppose F is a function defined on $[0, 1]$ and for every x in $[0, 1]$ $F(x)$ is in $[0, 1]$.

An equilibrium point of F is a number a_e in $[0, 1]$ such that $F(a_e) = a_e$.

An equilibrium point a_e of F is *locally stable* means that there is an interval (p, q) that contains a_e and if x_0 is any point in (p, q), a_e is the limit of the sequence x_0, x_1, x_2, $\cdots$ defined by $x_{n+1} = F(x_n)$ for $n = 0, 1, 2, \cdots$.

An equilibrium point is *nonstable* if it is not locally stable.

To return to the original whale equation, Equation 11.14, the same analysis applies as for Equation 11.15. The difference is that the graph of the iteration function is hardly distinct from the graph of the diagonal, $y = x$, as shown in Figure 11.8. The equilibrium points are the same and have the same stability character. The case considered simply gives easier graphs to examine.

Exercises for Section 11.3, Equilibrium Points, Stable and Nonstable.

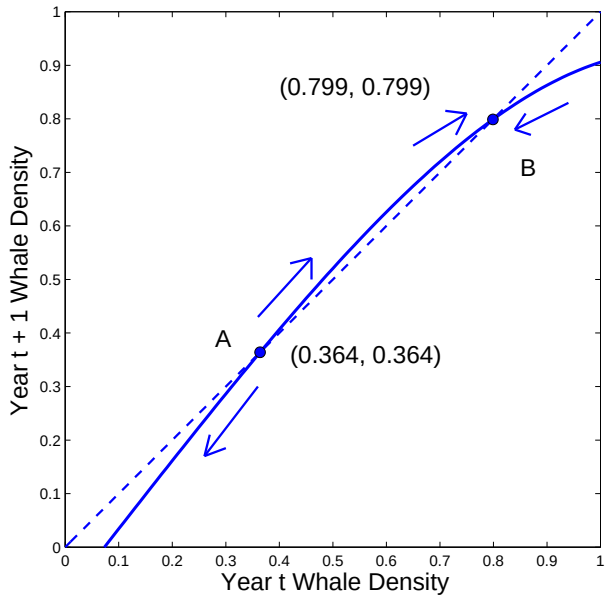


Figure 11.7: Graphs of $F(Y) = 1.2835Y - 0.2835Y^{3.39} - 0.094$ and $y = x$ showing the iteration movement of $Y_{t+1} = F(Y_t)$. Population numbers move away from **A** and toward **B**. **A** is a nonstable equilibrium, **B** is a stable equilibrium.

Exercise 11.3.1 Do Explore 11.3.1.

Exercise 11.3.2 Compute $x_{n+1} = F(x_n)$ for $n = 1, \cdots 20$ for each of the values of x_0. Stop if $x_n < 0$. Either list the values or plot the points (n, x_n) for $n = 0, \cdots 20$ (or the last such point if for some n $x_n < 0$), and describe the trend of each sequence.

a.	$F(x)$	$= 0.7x + 0.2$	$x_0 = 0.5$	$x_0 = 0.8$		
b.	$F(x)$	$= 1.1x - 0.05$	$x_0 = 0.5$	$x_0 = 0.8$		
c.	$F(x)$	$= x^2 + 0.1$	$x_0 = 0.1$	$x_0 = 0.2$	$x_0 = 0.7$	$x_0 = 0.9$
d.	$F(x)$	$= \sqrt{x} - 0.2$	$x_0 = 0.07$	$x_0 = 0.08$	$x_0 = 0.4$	$x_0 = 0.7$
e.	$F(x)$	$= -0.9x^2 + 2x - 0.2$	$x_0 = 0.1$	$x_0 = 0.2$	$x_0 = 0.4$	$x_0 = 0.7$
f.	$F(x)$	$= -0.9x^2 + 2x - 0.1$	$x_0 = 0.1$	$x_0 = 0.2$	$x_0 = 0.4$	$x_0 = 0.7$
g.	$F(x)$	$= x^3 + 0.2$	$x_0 = 0.1$	$x_0 = 0.5$	$x_0 = 0.8$	$x_0 = 0.9$
h.	$F(x)$	$= 8x^3 - 12x^2 + 6x - 1/2$	$x_0 = 0.1$	$x_0 = 0.2$	$x_0 = 0.8$	$x_0 = 0.9$

Exercise 11.3.3 Draw the graph of F and the graph of $y = x$ on a single axes with $0 \le x \le 1$ and $0 \le y \le 1$. The points of intersection of the graphs of F and $y = x$ are listed with F correct to 3 decimal

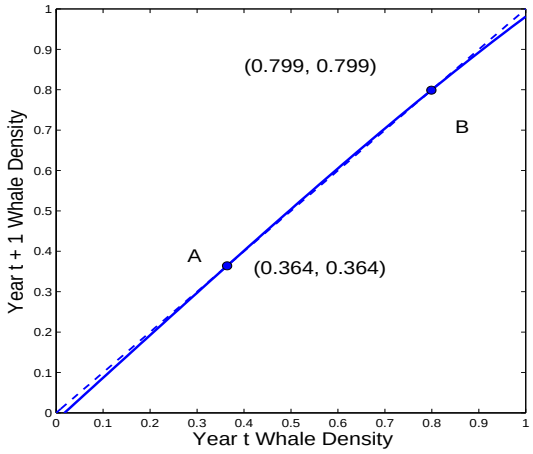


Figure 11.8: The graph of the original 1-year whale iteration. It is hardly distinct from the graph of $y = x$. The equilibrium points are the same as in Figure 11.5.

places. They are the equilibrium points of the iteration $x_{n+1} = F(x_n)$. For each such point, determine whether it is a locally stable equilibrium or an unstable equilibrium.

a.	$F(x)$	$=$	$0.7x + 0.2$	$(0.667, 0.667)$	
b.	$F(x)$	$=$	$1.1x - 0.05$	$(0.500, 0.500)$	
c.	$F(x)$	$=$	$x^2 + 0.1$	$(0.113, 0.113)$	$(0.887, 0.887)$
d.	$F(x)$	$=$	$\sqrt{x} - 0.2$	$(0.076, 0.076)$	$(0.524, 0.524)$
e.	$F(x)$	$=$	$-0.9x^2 + 2x - 0.2$	$(0.262, 0.262)$	$(0.850, 0.850)$
f.	$F(x)$	$=$	$-0.9x^2 + 2x - 0.1$	$(0.111, 0.111)$	$(1.000, 1.000)$
g.	$F(x)$	$=$	$x^3 + 0.2$	$(0.210, 0.210)$	$(0.879, 0.879)$
h.	$F(x)$	$=$	$8x^3 - 12x^2 + 6x - 1/2$	$(0.146, 0.146)$ $(0.854, 0.854)$	$(0.500, 0.500)$

Exercise 11.3.4 Ricker's equation for recruits into a fish population is

$$\text{Recruits in period } t \quad = \alpha \times P_t e^{P_t/\beta}$$

Assuming that a fixed fraction, R, of the adults is lost in the same period, we would have

$$P_{t+1} - P_t = \alpha \times P_t \times e^{-P_t/\beta} - R \times P_t \tag{11.17}$$

a. Find two equilibrium values for Ricker's equation. What condition will insure a positive equilibrium value?

b. Modify Equation 11.17 to account for harvest of H each time period.

c. Divide each term of your equation by β to get

$$Q_{t+1} - Q_t = \alpha \times Q_t \times e^{-Q_t} - R \times Q_t - G$$

What are Q_t and G?

d. Write an equation for the equilibrium $\overline{Q}$. This equation is not solvable with algebraic operations.

e. Suppose $\alpha = 1.2$, $R = 0.1$ and $G = 0.2$. Solve for $\overline{Q}$. Note: to solve the equation $1.1x \times e^{-x} + 0.9x - 0.2 = 0$, solve $x = 0.2/(1.1e^{-x} + 0.9)$. To solve $x = 0.2/(1.1e^{-x} + 0.9)$, enter 0.5 on your calculator and enter $0.2/(1.1e^{-x} + 0.9)$. Press ENTER, ENTER, $\cdots$ until the number in the display repeats the previous number in the display.

11.4 Cobwebbing.

Stability of an equilibrium point of an iteration $P_{t=1} = F(P_t)$ is readily determined by **cobwebbing** the iteration graph. To explain cobwebbing we use an example difference equation

$$P_{t+1} - P_t = 0.8\, P_t \left(1 - \frac{P_t}{100}\right) - 10$$

representative of a population that is limited to size 100 by the environment, has a low density growth rate of 0.8 (high for illustrative purpose), and from which 10 units are removed (harvested) every time interval. We normalize the equation by dividing by 100.

$$\frac{P_{t+1}}{100} - \frac{P_t}{100} = 0.8\, \frac{P_t}{100} \left(1 - \frac{P_t}{100}\right) - \frac{10}{100}$$

Let $p_t = \frac{P_t}{100}$ and write

$$p_{t+1} - p_t = 0.8\, p_t\, (1 - p_t) - 0.1$$
$$p_{t+1} = p_t + 0.8\, p_t\, (1 - p_t) - 0.1$$

Define

$$F(p) = p + 0.8p(1 - p) - 0.1$$

so that the iteration is

$$p_{t+1} = F(p_t)$$

The graph of F is shown in Figure 11.9, together with the graph of $y = x$. The equilibrium points are found by solving

$$p_e = p_e + 0.8p_e(1 - p_e) - 0.1 \qquad -0.8p_e^2 + 0.8p_e - 0.1 = 0$$

$$p_e \quad = \quad \frac{2 \pm \sqrt{2}}{4} \quad \doteq \quad 0.146 \quad \text{or} \quad 0.853$$

Suppose we start with $p_0 = 0.5$ for which $p_1 = 0.6$, $p_2 = 0.692$, $p_3 = 0.763$, $\cdots$. The arrows in Figure 11.10A illustrate these computations.

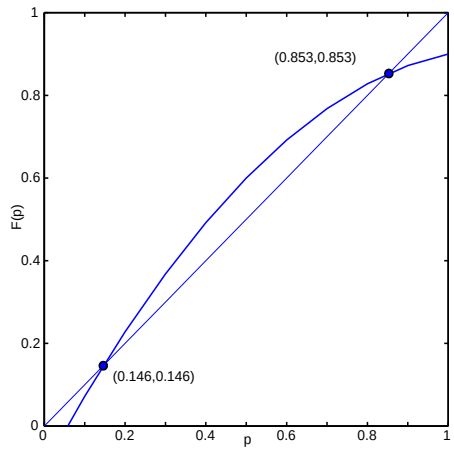


Figure 11.9: Graphs of $F(p) = p + 0.8p(1-p) - 0.1$ and $y = x$.

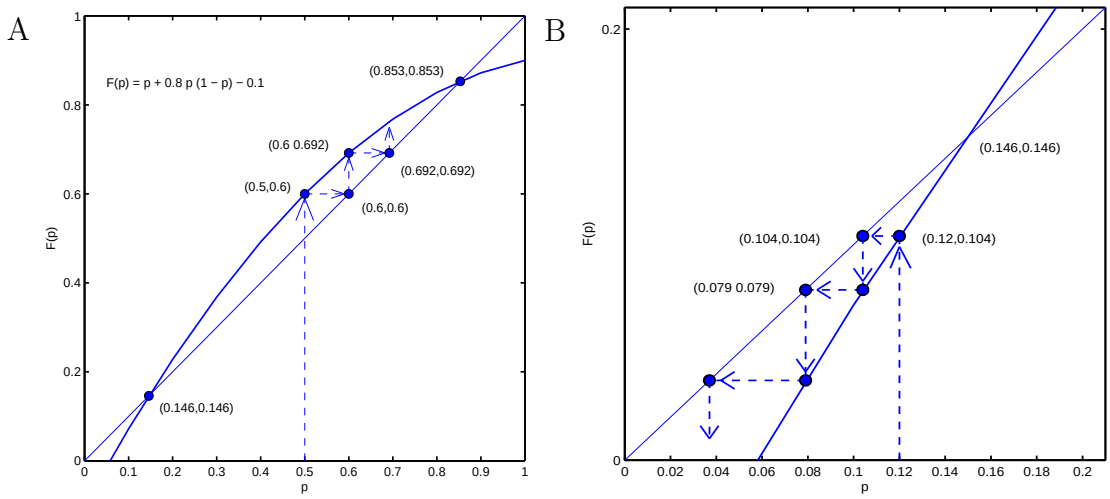


Figure 11.10: A. Cobweb path for $F(p) = p + 0.8p(1-p) - 0.1$. The path begins at $p_0 = 0.5$ and progress to $p_1 = 0.6$, $p_2 = 0.692$, $\cdots$ toward 0.853. B. Enlarged view of the region of Figure 11.10 around the non-stable equilibrium (0.146,0.146). The cobweb path begins at $p_0 = 0.12$ and progresses to $p_1 = 0.104$, $p_2 = 0.079$, $\cdots$ away from 0.146.

Start from 0.5 on the horizontal axis and move **vertically to** (0.5, 0.6) on the **graph**.
Move **horizontally to** (0.6, 0.6) on the **diagonal**.
Move **vertically to** (0.6, 0.692) on the **graph**.
Continue

It is apparent that this path (called a **cobweb**) is channeled between the graph of F and the diagonal line and that it will go to the equilibrium point (0.853, 0.853). Furthermore, any path that begins with p_0 between 0.146 and 0.853 will lead to (0.853, 0.853).

Explore 11.4.1 Start at $p_0 = 0.95$ on the horizontal axis in Figure 11.10A. Move vertically to the graph of F. Move horizontally to the diagonal. Move vertically to the graph of F. Where does this path lead?

The number $\frac{2+\sqrt{2}}{4} \doteq 0.853$ is a locally stable equilibrium. Any sequence with $p_0 > \frac{2-\sqrt{2}}{4} \doteq 0.146$ will go to (0.853, 0.853).

A similar geometric analysis shows that $\frac{2-\sqrt{2}}{4} \doteq 0.146$ is not a locally stable equilibrium point. Shown in Figure 11.10B is an enlarged picture of the region around (0.146,0.146).

If we **Start with $\mathbf{p_0 < 0.146}$** and move **vertically to the graph** of F. Then move **horizontally to the diagonal.** and move **vertically to the graph.** and continue we move to the left and eventually compute a negative number.

Explore 11.4.2 In Figure 11.10B Choose a point p_0 on the X-axis to the right of 0.146 and cobweb the sequence p_1, p_2, p_3. Are you moving away from 0.146?

We seek a general quality of F that will signal whether or not p_e is a locally stable equilibrium point of the iteration $p_{t+1} = F(p_t)$. Four examples in which the function $F(p)$ is linear will motivate our conclusion. Shown in Figure 11.11A is the graph of $F(p) = 0.7p + 0.2$, a line with slope 0.7, less than 1, and the equilibrium point at (2/3,2/3) is locally stable. In Figure 11.11B is the graph of $F(p) = 1.2p - 2/15$, a line with slope 1.2, greater than 1, and the equilibrium point at (2/3,2/3) is not stable.

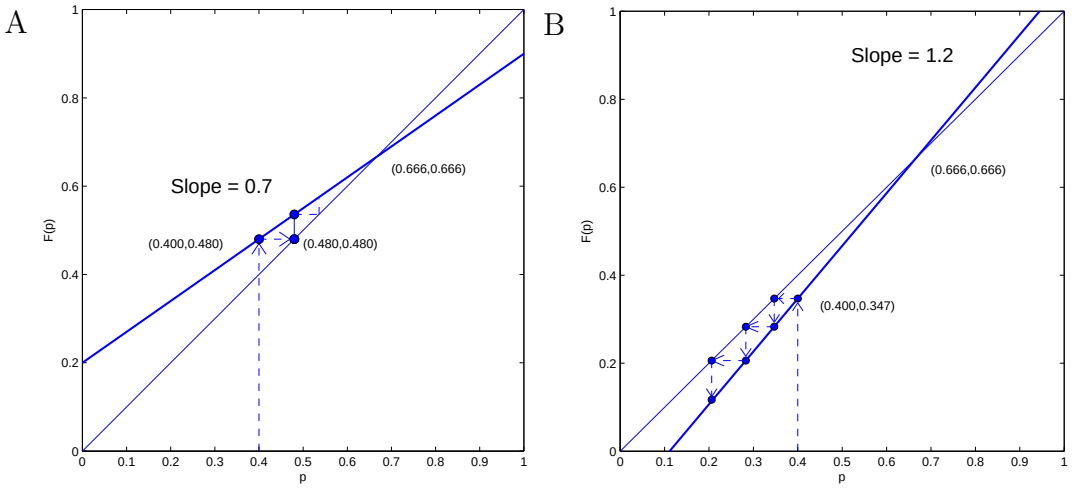


Figure 11.11: Cobwebs for linear iteration graphs. A. The graph of of $F(p) = 0.7p + 0.2$, a line with slope 0.7, less than 1, and the equilibrium point at (2/3,2/3) is locally stable. B. The graph of $F(p) = 1.2p - 2/15$, a line with slope 1.2, greater than 1, and the equilibrium point at (2/3,2/3) is not stable.

From the graphs in Figure 11.11 it appears that p_e is a locally stable equilibrium point if the graph of F has a slope less than one at p_e. From Figure 11.12 it appears that we also have to require that the slope of F at p_e be greater than minus one. The general result is:

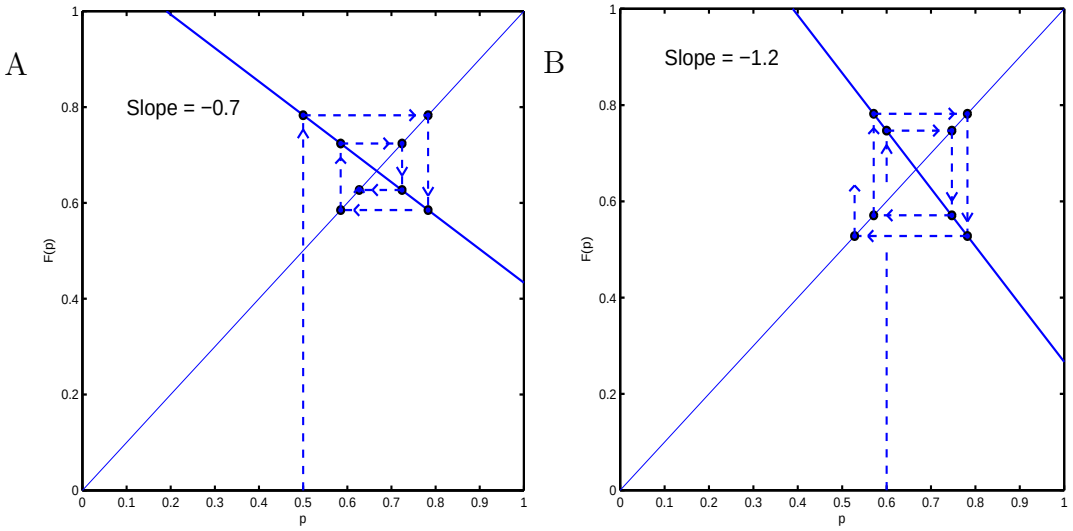


Figure 11.12: Cobwebs with linear $F(p)$. A. $F(p) = -0.7x + 1.133$ with slope = -0.7; the equilibrium point is locally stable. B. $F(p) = -1.2x + 1.467$ with slope of -1.2; the equilibrium point is not stable.

Theorem 11.4.1 If F and F' are continuous on $[0, 1]$ and for x in $[0, 1]$, $F(x)$ is in $[0, 1]$, and p_e is a number in $[0, 1]$ for which $F(p_e) = p_e$. Then the equilibrium point p_e is locally stable if $|\, F'(p_e)\,| < 1$.

Proof. The proof of Theorem 11.4.1 makes good use of the Mean Value Theorem on page 396.

Suppose F satisfies the hypothesis of Theorem 11.4.1. Let $R = (F'(p_e) + 1)/2 < 1$. Because F' is continuous there is an subinterval (a, b) of (0,1) with midpoint p_e for which

$$|F'(x)| < R \qquad \text{for all} \qquad a < x < b \qquad \text{(Similar to Theorem 4.1.1 in Exercise 4.1.13)}$$

Suppose p_0 is in (a, b), and p_0, p_1, p_2, $\cdots$ is the iteration sequence defined by $p_{t+1} = F(p_t)$. We will show that

$$\lim_{t\to\infty} p_t \quad = \quad p_e$$

We first show by induction that every point of p_0, p_1, p_2, $\cdots$ is in (a, b). By hypothesis p_0 is in (a, b). Suppose p_t is in (a, b). Then

$$p_e = F(p_e) \qquad \text{p_e is an equilibrium point.}$$

$$p_{t+1} = F(p_t) \qquad \text{Iteration equation.}$$

$$p_{t+1} - p_e = F(p_t) - F(p_e) \qquad \text{Subtraction.}$$

Then there is a number c between p_e and p_t such that

$$p_{t+1} - p_e = F'(c)(p_t - p_e) \qquad \text{Mean Value Theorem.}$$

$$|p_{t+1} - p_e| = |F'(c)|\,|p_t - p_e| \quad < \quad R\,|p_t - p_e| \qquad |F'(c)| < R$$

The last inequality asserts that p_{t+1} is closer to p_e than is p_t by a factor of $R < 1$. Because p_e is the midpoint of (a, b) it follows that p_{t+1} is also in (a, b). By induction all of p_0, p_1, p_2, $\cdots$ are in (a, b).

The inequalities

$$|p_{t+1} - p_e| \leq R\,|p_t - p_e|\,, \qquad t = 0, 1, \cdots$$

can be cascaded to find that

$$|p_t - p_e| < R^t\,|p_0 - p_e|\,.$$

Because $0 < R < 1$, $R^t \to 0$ as $t \to \infty$, and $p_t \to p_e$ as $t \to \infty$. It follows that p_e is locally stable. End of proof.

Example 11.4.1 The iteration

$$x_{n+1} = x_n \times (2 - a \times x_n) \qquad \text{where} \qquad \frac{1}{2} < a \leq 1$$

was the basis for division in early versions of the very large Cray computers. Primitively, the computer could add, subtract and multiply, and in order to divide, for example B/A, it computed $1/A$ using the iteration and multiplied $B \times 1/A$. Numbers were stored in binary notation and every number stored was written in the form

$$A = \pm\, a \times 2^e \qquad \text{where} \qquad \frac{1}{2} < a \leq 1 \qquad \text{and} \qquad e \qquad \text{is an integer}$$

For example, π might be stored as $0.110010010001 \times 2^2$.

$$\text{Then} \qquad \frac{1}{A} = \frac{1}{\pm a \times 2^e} = \pm\frac{1}{a}2^{-e}$$

so that to compute $\frac{1}{A}$, only $\frac{1}{a}$ where $\frac{1}{2} < a < 1$ must be computed.

We show that the number $1/a$ is the only locally stable equilibrium point of the iteration function

$$F(x) = x \times (2 - a \times x)$$

for the iteration $x_{n+1} = x_n \times (2 - a * x_n)$.

The equilibrium points, x_e, are

$$x_e = F(x_e) \qquad x_e = x_e \times (2 - a \times x_e), \qquad x_e = 0 \quad \text{or} \quad x_e = \frac{1}{a}$$

To examine stability using Theorem 11.4.1 we compute

$$F'(x) = [x \times (2 - a \times x)]' = \left[2x - a \times x^2)\right]' = 2 - 2 \times a \times x$$

$$\text{For} \quad x_e = 0, \quad F'(0) = 2 \qquad \text{and } x_e = 0 \text{ is a nonstable equilibrium.}$$

$$\text{For} \quad x_e = \frac{1}{a}, \quad F'\left(\frac{1}{a}\right) = 2 - 2 \times a \times \frac{1}{a} = 0 \qquad \text{and } x_e = \frac{1}{a} \text{ is a locally stable equilibrium.}$$

An iteration sequence to compute $1/\pi$ is illustrated in Figure 11.13. The sequence is $x_0 = 1.75$, $x_{n+1} = x_n(2 - 0.7854x_n)$; .7854 is approximately $\pi/4$ and π would be stored on the Cray as $a \times 2^2$. a then is $\pi/4 \doteq 0.7854$ and in binary notation $a \doteq 0.110010010001$.

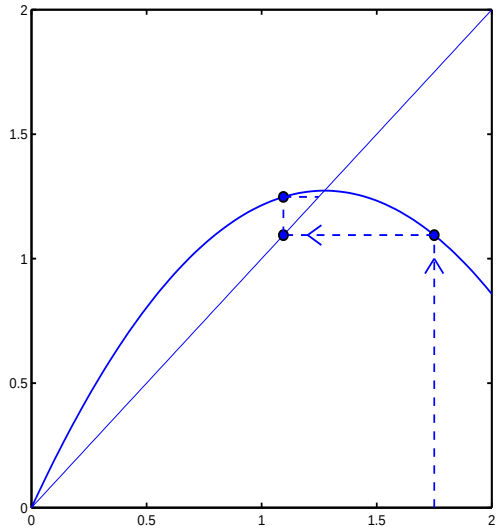


Figure 11.13: Cobweb for the iteration $x_{n+1} = x_n \times (2 - 0.7854 * x_n)$ to compute 1/0.7854.

Because $F'\left(\frac{1}{a}\right) = 0$, convergence is very rapid as can be seen in Figure 11.13 and by

$$x_0 = 1.75 \qquad x_1 = 1.0947 \qquad x_2 = 1.2482 \qquad x_3 = 1.2727 \qquad x_4 = 1.2732 \qquad x_3 = 1.2732$$

The optimum choice for x_0 in the iteration $x_{n+1} = x_n \times (2 - a * x_n)$, is $x_0 = \frac{4}{3}$ for giving rapid convergence for all values of a between $\frac{1}{2}$ and 1. We chose $x_0 = 1.75$ so that the cobweb in Figure 11.13 would be visible.

Exercises for Section 11.4, Cobwebbing.

Exercise 11.4.1 Find the locally stable equilibrium points of the following iteration functions. Draw the graphs of the iteration function $y = f(x)$ and the diagonal $y = x$. Start with $x_0 = 0.5$ and show the paths of the iterates on your graphs. With $x_0 = 0.5$ compute $x_1, \cdots, x_{10}$ (use ANS key on your calculator).

a. $x_{t+1} = x_t \times (1 - x_t)$ b. $x_{t+1} = 1.5 \times x_t \times (1 - x_t)$

c. $x_{t+1} = 2 \times x_t \times (1 - x_t)$ d. $x_{t+1} = 2.5 \times x_t \times (1 - x_t)$

e. $x_{t+1} = 3.25 \times x_t \times (1 - x_t)$ f. $x_{t+1} = 3.5 \times x_t \times (1 - x_t)$

Exercise 11.4.2 Find the locally stable equilibrium points of the following iteration functions. Draw the graphs of the iteration function $y = f(x)$ and the diagonal $y = x$. Start with $x_0 = 1.0$ and show the paths of the iterates on your graphs. With $x_0 = 1.0$ compute $x_1, \cdots, x_{10}$.

a. $x_{t+1} = \dfrac{x_t + 2/x_t}{2}$ b. $x_{t+1} = \dfrac{x_t + 5/x_t}{2}$

Exercise 11.4.3 Draw the graphs of the iteration function $y = f(x)$ and the diagonal $y = x$. For each equilibrium, choose a value of x_0 close to but distinct from that equilibrium and compute $x_1, \cdots, x_{10}$. Conclude whether the equilibrium is stable.

a.	x_{t+1}	$=$	$\cos(x_t)$	b.	x_{t+1}	$=$	$2 \times \cos(x_t)$
c.	x_{t+1}	$=$	$3\ln(x_t)$	d.	x_{t+1}	$=$	e^{-x_t}
e.	x_{t+1}	$=$	$2 \times x \times e^{-x_t}$	f.	x_{t+1}	$=$	$1.2835x_t - 0.2835x_t^{3.39} - 0.094$

Exercise 11.4.4 Continuation of Exercises 8.3.3 and 8.3.4.

Consider a single locus, two allele (A and a) gene in a random mating, non-overlapping generation population with the allele frequencies of A and a in the initial adult population being p and $1-p$. Then the allele frequencies in the offspring will also be p and $1-p$. If the viabilities of AA, Aa and aa are in the ratio $1+s_1 : 1 : 1+s_2$ (where s_1 and s_2 can be positive, negative or zero) then the frequencies of AA, Aa and aa in the following adult generation are $(1+s_1)p^2/F$, $2p(1-p)/F$, $(1+s_2)(1-p)^2/F$, respectively, where

$$F = (1+s_1)p^2 + 2p(1-p) + (1+s_2)(1-p)^2 = 1 + s_1p^2 + s_2(1-p)^2$$

The frequency, p^*, of the A allele in the second adult generation will be

$$p^* = \frac{(1+s_1)p^2 + p(1-p)}{F}$$

a. Find the value $\hat{p}$ of p on [0,1] for which $F(p)$ is a maximum.

b. Show that the change in allele A frequency between adult generations is

$$p^* - p = p(1-p)\frac{(s_1+s_2)p - s_2}{1 + s_1p^2 + s_2(1-p)^2}.$$

c. Find the stable equilibrium of

$$p_{n+1} = p_n + p_n(1-p_n)\frac{(s_1+s_2)p_n - s_2}{1 + s_1p_n^2 + s_2(1-p)^2} \tag{11.18}$$

for the following values of s_1 and s_2:

s_1	$=$	0.2,	s_2	$=$	-0.4;	s_1	$=$	-0.2,	s_2	$=$	-0.4;
s_1	$=$	0.2,	s_2	$=$	-0.2;	s_1	$=$	0,	s_2	$=$	-0.4;

Exercise 11.4.5 Consider the case $s_1 = 0$, $s_2 = -1$ of Equation 11.18, referred to as a lethal recessive, the a allele is lethal if it appears on both chromosomes, but has no deleterious effect if it appears on only one chromosome. Show from Equation 11.18 that

$$p_{n+1} = \frac{1-p_n}{2-p_n}.$$

Use $p_n + q_n = 1$ to show that

$$q_{n+1} = \frac{q_n}{1 + q_n}.$$

Suppose $q_0 = 0.01$ so that the frequency of aa is 0.0001, or 1 in 10,000. For what value of n will q_n be 0.005? You may want to use

$$\frac{1}{q_{n+1}} = \frac{1}{q_n} + 1.$$

Exercise 11.4.6 The equilibrium points for Ricker's equation, $P_{t+1} - P_t = \alpha \times P_t \times e^{-P_t/\beta} - R \times P_t$, presented in Exercise 11.3.4 are 0 and $\beta \ln(\alpha/R)$. Determine the stability of these equilibria.

Exercise 11.4.7 For each of the values of $a = 0.5$, $a = 0.625$, $a = 0.75$, and $a = 0.875$ choose $x_0 = 1.375$ and compute $x_{n+1} = x_n * (2 - a * x_n)$ until 'convergence' (repeat of the first four decimal digits). (1.375 is 1.011 in binary and is close to 4/3.) Record the number of iterations required for convergence. See Example 11.4.1

11.5 Exponential growth and L'Hospital's rule.

In 1798 and subsequent years, Malthus argued that

> The annual increase in human population is proportional to the size of the population at the beginning of the year.
>
> and
>
> The food supply increases a constant amount each year.

With P_t being the human population in year t and F_t the available food each year, the statements lead to

$$\begin{aligned} P_{t+1} - P_t &= R\,P_t \qquad \text{where} \qquad R > 1 \\ F_{t+1} - F_t &= G \qquad \text{and} \qquad G > 0 \end{aligned} \tag{11.19}$$

Malthus argued that the amount of food per person,

$$\frac{F_t}{P_t} \to 0 \qquad \text{as} \qquad t \to \infty$$

except that he believed that calamity would strike long before t gets 'close to' ∞.

The solutions to the iterations

$$\begin{array}{lcl} P_0 = 1 & P_{t+1} - P_t = 0.01\,P_t & \\ & & \text{are} \\ F_0 = 1 & F_{t+1} - F_t = 2 & \end{array} \qquad \begin{array}{lcl} P_t = (1.01)^t & & \\ & & \text{are} \\ F_t = 1 + 2t & & \end{array}$$

are shown in Figure 11.14. P_0 and F_0 are both one unit and may be considered to be the present population and the present food supply. The iteration P_t allows for 1 percent annual growth (less than present world population growth). The iteration F_t allows for a constant increase of 2 'units' per year (in year 1 food production is three times present food production; after 2 years food production is 5 times

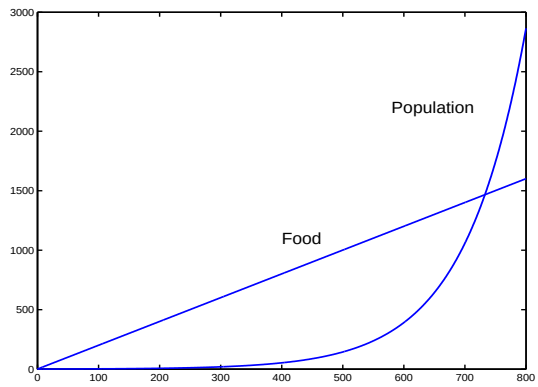


Figure 11.14: Linear growth of Food, $F_t = 1 + 2t$ and exponential growth of population, $P_t = 1.01^t$.

present production). It can be seen in Figure 11.14 that after 750 years the population has exceeded the available food. The exponential growth of P_t overtakes the linear growth of F_t.

By cascading the iterations in Equation 11.19, we know that

$$P_t = P_0 \times (1 + R)^t \qquad \text{and} \qquad F_t = F_0 + G \times t$$

Exponential functions, B^t with $B > 1$, grow faster than all polynomial functions. We show that

$$\text{If } B > 1 \text{ and } n \text{ is a positive integer, then} \qquad \lim_{t\to\infty} \frac{B^t}{t^n} = \infty$$

We first show that

$$\lim_{t\to\infty} \frac{e^t}{t} = \infty \tag{11.20}$$

We assume it to be true that

$$\lim_{t\to\infty} e^t = \infty \tag{11.21}$$

but have included a proof for you to complete in Problem 11.5.6. This is fairly intuitive from comparison with the sequence 2, 4, 8, 16, 32, 64, $\cdots$, 2^n, $\cdots$ and the observation that $e > 2$ so that $e^t > 2^t$.

Now

$$\frac{e^t}{t} > \frac{e^t - e^{\frac{t}{2}}}{t} = \frac{1}{2}\frac{e^t - e^{\frac{t}{2}}}{t - \frac{t}{2}} = \frac{1}{2}e^{c_t} \qquad \text{where} \qquad \frac{t}{2} < c_t < t \tag{11.22}$$

You are asked to give reasons for the steps in Equation 11.22 in Exercise 11.5.1.

$$\text{Because } \tfrac{t}{2} < c_t \qquad \text{as} \qquad t \to \infty, \qquad c_t \to \infty \qquad \text{and} \qquad \frac{1}{2}e^{c_t} \to \infty$$

$$\text{Because } \frac{e^t}{t} > \frac{1}{2}e^{c_t} \text{ it follows that} \qquad \lim_{t\to\infty} \frac{e^t}{t} = \infty$$

Next we show that

$$\text{If } n \text{ is a positive integer, then} \qquad \lim_{t\to\infty} \frac{e^t}{t^n} = \infty \tag{11.23}$$

Observe that

$$\begin{aligned} \frac{e^t}{t^n} &= \frac{\left(e^{\frac{t}{n}}\right)^n}{t^n} \\ &= \frac{1}{n^n}\frac{\left(e^{\frac{t}{n}}\right)^n}{\left(\frac{t}{n}\right)^n} \\ &= \frac{1}{n^n}\left(\frac{e^\tau}{\tau}\right)^n \qquad \text{where} \qquad \tau = \frac{t}{n} \end{aligned}$$

Remember that n is fixed.

$$\text{As} \qquad t \to \infty \qquad \tau \to \infty \qquad \text{and} \qquad \frac{e^\tau}{\tau} \to \infty.$$

Therefore

$$\lim_{t\to\infty} \frac{e^t}{t^n} = \lim_{\tau\to\infty} \frac{1}{n^n}\left(\frac{e^\tau}{\tau}\right)^n = \frac{1}{n^n}(\infty)^n = \infty.$$

A short digression to present L'Hôspital's Rule. A well known theorem about limits of quotients is attributed to Antoine de L'Hôspital who published the first calculus text book in 1691 [2]. One case of the theorem is

Theorem 11.5.1 L'Hospital's Theorem on $(0, \infty)$. If F and G are differentiable functions on $(0, \infty)$ and

$$G' \neq 0, \qquad \lim_{t\to\infty} F(t) = \infty, \qquad \lim_{t\to\infty} G(t) = \infty, \qquad \text{and} \qquad \lim_{t\to\infty} \frac{F'(t)}{G'(t)} = L$$

$$\text{then} \qquad \lim_{t\to\infty} \frac{F(t)}{G(t)} = L. \qquad (L \text{ can be a real number or } +\infty \text{ or } -\infty)$$

[2] From http://mathworld.wolfram.com/LHospitalsRule.html: " Within the book, l'Hospital thanks the Bernoulli brothers for their assistance and their discoveries. An earlier letter by John Bernoulli gives both the rule and its proof, so it seems likely that Bernoulli discovered the rule (Larson et al. 1999, p. 524)."

Larson, R.; Hostetler, R. P.; and Edwards, B. H. Calculus: Early Transcendental Functions, 2nd ed. Boston: Houghton Mifflin, 1999.

To illustrate the Theorem, in our principal Limit 11.20,

$$\text{Evaluate} \qquad \lim_{t\to\infty} \frac{e^t}{t} \qquad \text{choose } F(t) = e^t,\ G(t) = t. \text{ Then}$$

$$F'(t) = e^t \qquad G'(t) = 1 \qquad \text{and}$$

$$G' = 1 \neq 0, \qquad \lim_{t\to\infty} F(t) = \lim_{t\to\infty} e^t = \infty, \qquad \lim_{t\to\infty} G(t) = \lim_{t\to\infty} t = \infty,$$

$$\text{and} \qquad \lim_{t\to\infty} \frac{F'(t)}{G'(t)} = \lim_{t\to\infty} \frac{e^t}{1} = \infty$$

From Theorem 11.5.1 we can conclude

$$\lim_{t\to\infty} \frac{F(t)}{G(t)} = \lim_{t\to\infty} \frac{e^t}{t} = \infty \qquad \text{which is Limit 11.20.}$$

Other forms of L'Hôspital's rule include for a a real number, (all require that $G'(t) \neq 0$)

$$\lim_{t\to\infty} F(t) = 0 \qquad \lim_{t\to\infty} G(t) = 0 \qquad \lim_{t\to\infty} \frac{F'(t)}{G'(t)} = L \implies \lim_{t\to\infty} \frac{F(t)}{G(t)} = L \tag{11.24}$$

$$\lim_{t\to a^+} F(t) = \infty \qquad \lim_{t\to a^+} G(t) = \infty \qquad \lim_{t\to a^+} \frac{F'(t)}{G'(t)} = L \implies \lim_{t\to a^+} \frac{F(t)}{G(t)} = L \tag{11.25}$$

$$\lim_{t\to a^+} F(t) = 0 \qquad \lim_{t\to a^+} G(t) = 0 \qquad \lim_{t\to a^+} \frac{F'(t)}{G'(t)} = L \implies \lim_{t\to a^+} \frac{F(t)}{G(t)} = L \tag{11.26}$$

Fundamental to proving L'Hospital's rule is to prove

Theorem 11.5.2 Extended Mean Value Theorem. If F and G are continuous functions on a closed interval $[a, b]$ and are differentiable on the open interval (a, b) and $G'(x) \neq 0$ for any x in (a, b), then there is a number c in $[a, b]$ such that

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)} \tag{11.27}$$

You are asked to prove Theorem 11.5.2, with suggestions, in Exercise 11.5.11
A proof of implication 11.26 follows. The complete statement is

Theorem 11.5.3 L'Hospital's Theorem on a bounded interval. Suppose $[a, b]$ is a number interval and F and G are continuous functions defined on the half open interval $(a, b]$ and F' and G' are continuous on (a, b) and G' is nonzero on (a, b). If

$$\lim_{t\to a^+} F(t) = 0, \qquad \lim_{t\to a^+} G(t) = 0 \qquad \text{and} \qquad \lim_{t\to a^+} \frac{F'(t)}{G'(t)} = L$$

then

$$\lim_{t\to a^+} \frac{F(t)}{G(t)} = L.$$

Proof. Because $\lim_{t\to a^+} F(t) = \lim_{t\to a^+} G(t) = 0$, the domain of F and G can be extended to include a by defining $F(a) = G(a) = 0$ and the extended F and G are continuous on $[a, b]$. By the Extended Mean Value Theorem 11.5.2, for any t in (a, b) there is a number c_t between a and t such that

$$\frac{F(t)}{G(t)} = \frac{F(t) - F(a)}{G(t) - G(a)} = \frac{F'(c_t)}{G'(c_t)}.$$

As t approaches a, c_t also approaches a and

$$\lim_{t\to a^+} \frac{F(t)}{G(t)} = \lim_{t\to a^+} \frac{F'(c_t)}{G'(c_t)} = L$$

End of proof.

Theorem 11.5.3 may be used to evaluate the limit

$$\lim_{t\to 0^+} \frac{\sin t}{\ln(t+1)}$$

First we note that $\sin t$ and $\ln(t + 1)$ are continuous and differentiable on (0,1] and that $\lim_{t\to 0^+} \sin t = 0$ and $\lim_{t\to 0^+} \ln(t + 1) = 0$. Furthermore

$$[\sin t]' = \cos t, \qquad [\ln(t+1)]' = \frac{1}{t+1} \qquad \text{and} \qquad \lim_{t\to 0^+} \frac{\cos t}{1/(t+1)} = \lim_{t\to 0^+} (\cos t)(t+1) = 1$$

By Theorem 11.5.3

$$\lim_{t\to 0^+} \frac{\sin t}{\ln(t+1)} = 1$$

Exercises for Section 11.5, Exponential growth and L'Hospital's theorem.

Exercise 11.5.1 What is the reason for the three steps that lead to Equation 11.22?

Exercise 11.5.2 Evaluate the limits using Equation 11.23.

a. $\lim_{t\to\infty} \frac{e^{0.5t}}{t}$ Note: $\frac{e^{0.5t}}{t} = 0.5\frac{e^{0.5t}}{0.5t} = 0.5\frac{e^{\tau}}{\tau}$ $\tau = 0.5t$

b. $\lim_{t\to\infty} \frac{2^t}{t}$ Note: $\frac{2^t}{t} = \frac{e^{(\ln 2)t}}{t}$

c. $\lim_{t\to\infty} \frac{e^t}{5t}$

d. $\lim_{t\to\infty} \frac{e^t}{2t+15}$ Note: $\frac{e^t}{2t+15} > \frac{e^t}{3t}$ for $t > 15$

Exercise 11.5.3 Evaluate the limits using Equation 11.23.

a. $\lim_{t\to\infty} \frac{e^t}{t^2+15}$ b. $\lim_{t\to\infty} \frac{e^t}{t^{5,000,000}}$

c. $\lim_{t\to\infty} \frac{e^{\sqrt{t}}}{\sqrt{t}}$ d. $\lim_{t\to\infty} \frac{t}{e^t}$

e. $\lim_{t\to\infty} \frac{t}{2^t}$ f. $\lim_{t\to\infty} t \times 2^{-t}$

Exercise 11.5.4 Evaluate the limits using Equation 11.23.

a. $\lim_{t\to\infty} \frac{t}{R^t}$ where $R > 1$

b. $\lim_{t\to\infty} \frac{F_0 + G \times t}{P_0 \times R^t}$ where $F_0 > 0, \quad G > 0, \quad P_0 > 0, \quad R > 1$

Exercise 11.5.5 If Malthus's two premises are correct, Part b. of the last problem suggests that indeed the amount of food per person will go to zero. Shown in Table 11.5.5 is the record of the world populations since 1900 the world production of cereal grain for 1961 - 2001.

Table for Exercise 11.5.5 Estimates of World Population by the World Health Organization and estimates of world cereal grain production read from a graph in http://www.freedom21.org/alternative/5b-1.jpg

World Population Estimates

Year	10^9 Pop
1900	1.65
1910	1.75
1920	1.86
1930	2.07
1940	2.30
1950	2.52
1960	3.02
1970	3.70
1980	4.45
1990	5.30
2000	6.06

World Cereal Grain Production

Year	Index
1961	0.90
1965	1.10
1969	1.25
1973	1.35
1977	1.45
1981	1.60
1985	1.80
1989	1.88
1993	1.90
1997	2.10
2001	2.12

a. Use your calculator to fit an exponential function to the human population data.

b. Is Malthus's assertion about exponential growth of the human population consistent with the data?

c. Use your calculator to fit both a second degree polynomial and a linear to the food data. Which is the better fit?

d. In what sense is the quadratic fit more pessimistic than the linear fit?

e. Would it change Malthus's assertion if food production increases quadratically?

The questions Malthus raised are with us today. There are optimistic reports[3] and criticisms of optimistic reports[4]. The problem is enormously more complex than can be described by two short tables showing population growth and food production. For example, scientists at Goddard Space Flight Center have estimated the total solar energy that is converted to plant material and estimated that humans consume an astounding 14 to 26 percent of that energy as food, clothing, fuel and timber[5].

Exercise 11.5.6 Show that

$$\lim_{t\to\infty} 2^t = \infty$$

using the following procedure.

Argue using mathematical induction that all of the statements in the sequence $\{S_1, S_2, S_3, \cdots\}$ of statements are true where

S_n is the statement that $\quad n < 2^n \quad$, for $n = 1, 2, 3, \cdots$.

[3]World Bank Report, November 1993, World Food Outlook

[4]David Pimentel, "Exposition on Skepticism", *Bioscience*, March 2002, 52, 295-298

[5]Roger Doyle, "the Lion's Share", Scientific American, April, 2005, p 30.

Your argument should have two parts.
Part 1. Show that S_1 is true.
Part 2. If one of the statements in $\{S_1, S_2, S_3, \cdots\}$ is not true, there must be a first one that is not true. Let m be the subscript of the first statement in $\{S_1, S_2, S_3, \cdots\}$ that is not true. Show that

1. $1 < m$.
2. S_{m-1} is true.
3. Because S_{m-1} is true, S_m is also true.
4. The assumption that some statement in $\{S_1, S_2, S_3, \cdots\}$ is not true is false. Therefore all of the statements are true.

Conclude from this that as $n \to \infty$, $2^n \to \infty$, and $e^n \to \infty$.

Exercise 11.5.7 Recall that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$. What is $\lim\limits_{x\to\infty} \ln x$?

Exercise 11.5.8 Review the argument that $\lim_{t\to\infty} \frac{e^t}{t} = \infty$, Equation11.20. Use similar steps to show that

$$\lim_{t\to\infty} \frac{\ln t}{t} = 0 \tag{11.28}$$

You may find the following algebra helpful.

$$\frac{\ln t}{t} = \frac{2\ln t - \ln t}{t} = \frac{\ln t^2 - \ln t}{t} < \frac{\ln t^2 - \ln t}{t^2 - t} \qquad \text{for} \qquad 2 < t$$

Exercise 11.5.9 Evaluate

a. $\lim\limits_{t\to\infty} \dfrac{\ln\sqrt{t}}{\sqrt{t}}$ b. $\lim\limits_{t\to\infty} \dfrac{\ln t}{\sqrt{t}}$

Exercise 11.5.10 Use L'Hôspital's Rule, where appropriate, to evaluate the limits

a. $\lim\limits_{t\to\infty} \dfrac{e^{2t}}{t}$ b. $\lim\limits_{t\to\infty} \dfrac{e^t}{\sqrt{t}}$ c. $\lim\limits_{t\to\infty} \dfrac{2t^2+1}{5t^2+2}$

d. $\lim\limits_{t\to\infty} \dfrac{\ln t}{e^t}$ e. $\lim\limits_{t\to 0} \dfrac{\ln t}{1/t}$ f. $\lim\limits_{t\to 0} \dfrac{\sin t}{t}$

g. $\lim\limits_{t\to\infty} \dfrac{\ln\sqrt{t}}{\sqrt{t}}$ h. $\lim\limits_{t\to 0} \dfrac{\sin 2t}{\sin 3t}$ i. $\lim\limits_{t\to\infty} \dfrac{\ln t}{\sqrt{t}}$

Exercise 11.5.11 Prove Theorem 11.5.2, the Extended Mean Value Theorem. In doing so, you will find it useful to consider the function,

$$D(x) = (\,F(b) - F(a)\,) \times (\,G(x) - G(a)\,) - (\,G(b) - G(a)\,) \times (\,F(x) - F(a)\,)$$

You may then find it useful to show

a. $D(a) = 0$.

b. $D(b) = 0$.

c. There is a number c such that $D'(c) = 0$.

d. $(\,F(b) - F(a)\,) \times G'(c) = (\,G(b) - G(a)\,) \times F'(c)$

e. $G'(c) \neq 0$.

f. $G(b) - G(a) \neq 0$.

11.6 Environmental carrying capacity.

We examine a difference equation used to describe population growth in limited environments.

$$P_{t+1} - P_t \;=\; R \times P_t \times \left(1 - \tfrac{P_t}{M}\right) \qquad \text{Logistic Growth}$$

An alternate equation is examined in Section 11.8

$$P_{t+1} \;=\; \frac{(1+R)\,P_t}{1 + R\dfrac{P_t}{M}} \qquad \text{Alternate Logistic Growth}$$

In 1838, Verhulst, in response to Malthus, argued that environments have a carrying capacity and as populations near their environmental limits, the growth rates decrease. Something like this is happening today in the human population. In less developed regions of the world total fertility has fallen from 6 to 3 births per woman in the past fifty years[6].

Verhulst's model of population growth was

> **Mathematical Model of population growth in a limited environment.** The annual increase in human population is proportional to the size of the population at the beginning of the year and is proportional to the fraction of the carrying capacity unused by the population.

If the carrying capacity is measured as M, the number of humans the environment will support, and P_t is the population size in year t then

$$\frac{P_t}{M} \qquad \text{is the fraction of the carrying capacity that is used, and}$$

$$1 - \frac{P_t}{M} \qquad \text{is the unused fraction of the carrying capacity.}$$

[6] www.undp.org/popin

Because of the double proportionality, the annual increase is proportional to the *product* of the population size and the unused carrying capacity:

$$P_{t+1} - P_t = R\,P_t \times \left(1 - \frac{P_t}{M}\right) \tag{11.29}$$

R is called the *low density growth rate.* When P_t is 'small' compared to M, the factor, $\left(1 - \frac{P_t}{M}\right)$ is close to 1 and

$$P_{t+1} - P_t \doteq R\,P_t \qquad \text{for} \qquad \frac{P_t}{M} \qquad \text{'small'}$$

As $P_t \to M$, $\left(1 - \frac{P_t}{M}\right)$ approaches 0, so that the population change,

$$P_{t+1} - P_t \doteq R \times P_t \times 0 = 0 \qquad \text{for} \qquad \frac{P_t}{M} \doteq 1$$

For reasons that are unknown to us, the word *logistic* is used to describe Equation 11.29. Equation 11.29 is the discrete logistic equation, and we will study the continuous logistic equation in Chapter 16.

To be concrete, consider a low density growth rate, $R = 0.4$, and a carrying capacity, $M = 1000$ in Equation 11.29. Then

$$P_{t+1} - P_t = 0.4 \times P_t \times \left(1 - \frac{P_t}{1000}\right) \tag{11.30}$$

When $P_t = 10$ (small compared to 1000), then $1 - \frac{P_t}{1000} = 0.99 \doteq 1$ and

$$P_{t+1} - P_t \;\doteq\; 0.4 \times P_t \times 1 = 0.4 \times P_t$$

The population increases approximately 40% each time period, and there is exponential growth. This growth continues until the population size compared to 1000 is significant.

When $P_t = 500$, $1 - \frac{P_t}{1000} = 0.5$ and

$$P_{t+1} - P_t \;\doteq\; 0.4 \times P_t \times 0.5 = 0.2 \times P_t$$

The population is still growing, but only at 20% per time period, one-half of that at low density.

When the populations size, P_t reaches 990 (high density, almost to carrying capacity), $1 - \frac{P_t}{1000} = 0.01$ and

$$P_{t+1} - P_t \;\doteq\; 0.4 \times P_t \times 0.01 = 0.004 \times P_t$$

and the growth rate has fallen to 0.4% per time period.

In Figure 11.15 are the graphs of

$$P_0 = 10 \qquad P_{t+1} = P_t + 0.4P_t\left(1 - \frac{P_t}{1000}\right) \qquad \text{Circles}$$

and

$$Q_0 = 10 \qquad Q_{t+1} = Q_t + 0.4Q_t \qquad \text{X's}$$

It can be seen that Q_t follows P_t until $P_t \doteq 200$ and $1 - \frac{P_t}{1000} \doteq 0.8$.

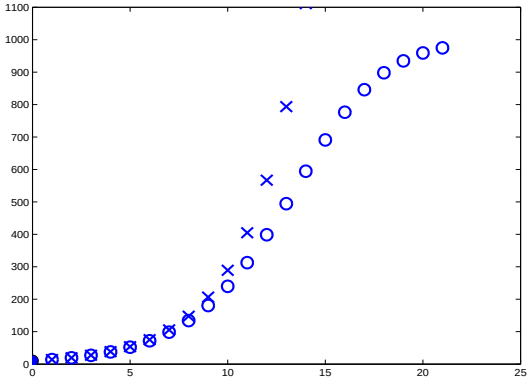


Figure 11.15: The graph of a logistic curve $P_{t+1} = P_t + 0.4P_t\left(1 - \frac{P_t}{1000}\right)$ (circles) and the exponential curve $Q_{t+1} = Q_t + 0.4Q_t$ (X's).

Normalization. The equation

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right)$$

is normalized by dividing each term by M, as in

$$\frac{P_{t+1}}{M} - \frac{P_t}{M} = R \times \frac{P_t}{M} \times \left(1 - \frac{P_t}{M}\right).$$

Then one may define

$$Q_t = \frac{P_t}{M}$$

and write

$$Q_{t+1} - Q_t = R \times Q_t \times (1 - Q_t)$$

This is a simpler equation – it has only one parameter, R – but reflects the dynamics of the first. In practice, P_t is used ambiguously for both the original P_t and what we have written as Q_t.

Explore 11.6.1 Start with an initial population of $P_0 = 100$ and use $P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{1000}\right)$ compute P_1, P_2, $\cdots$, P_{40}. Sketch a graph of P_t *vs* t.

This may be a bit boring, so on your calculator, type

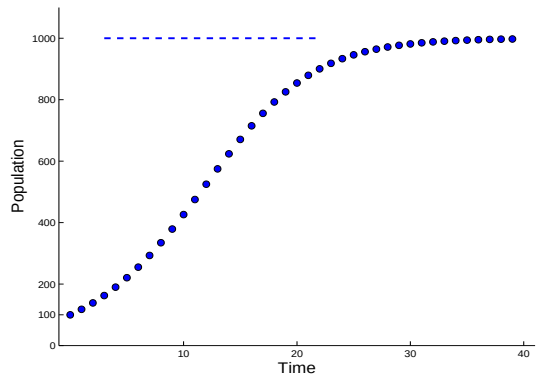
100 ENTER
Ans+0.2*Ans*(1-Ans/1000) ENTER

Continue pressing ENTER and the numbers will appear.

Alternatively, you could run the program shown in Explore Figure 11.6.1. If you do so, time will be recorded in TT and population will be recorded in PP. Un-select all functions in Graph, y(x)=, and Select Plot1 and set the Window to fit. Your graph should be similar to the graph in Figure 11.6.1.

Explore Figure 11.6.1 (h)

```
PROGRAM:PMAX
:41->dimL TT
:41->dimL PP
:0->TT(1)
:100->PP(1)
:For(K,1,40)
:K->TT(K+1)
:PP(K)+.2*PP(K)*
(1-PP(K)/1000)->PP(K+1)
:End
:Scatter TT,PP
```



11.6.1 Fitting Logistic Equations to Data.

Shown in the table and graph in Figure 11.16 are data from growing *V. natrigens* in a flask over a two hour period. *V. natrigens* density is measured by Absorbance on a spectrophotometer. The first 80 minutes of this data was shown in the discussion of exponential growth on page 5 where it was noted that the growth rate during the last 16 minutes was a bit below that of 0 to 64 minutes.

The graph of *V natrigens* density *vs Time* bears a reasonable resemblance to that of the logistic curve shown in previous graphs, and we look for a logistic equation that approximates the *E. coli* data. The equation should be of the form

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) \tag{11.31}$$

In order to compute with it, we must have an initial value, P_0, and we need to find R, M.

We can divide both sides of Equation 11.31 by P_t and get

$$\frac{P_{t+1} - P_t}{P_t} = R \times \left(1 - \frac{P_t}{M}\right) \tag{11.32}$$

If we think

$$y = \frac{P_{t+1} - P_t}{P_t} \qquad \text{and} \qquad x = P_t$$

we get a linear equation

$$y = R - \frac{R}{M}x$$

Shown in Figure 11.17 are data and a graph of x and y together with a line $y = 0.68 - 0.69x$ fit to the data (by least squares). The graph is a bit scattered, particularly to the left, reflecting the fact that

1. y is the difference of two small experimental values, and

pH 6.25		
Time (min)	Index t	Population Density
0	0	0.022
16	1	0.036
32	2	0.060
48	3	0.101
64	4	0.169
80	5	0.266
96	6	0.360
112	7	0.510
128	8	0.704
144	9	0.827
160	10	0.928

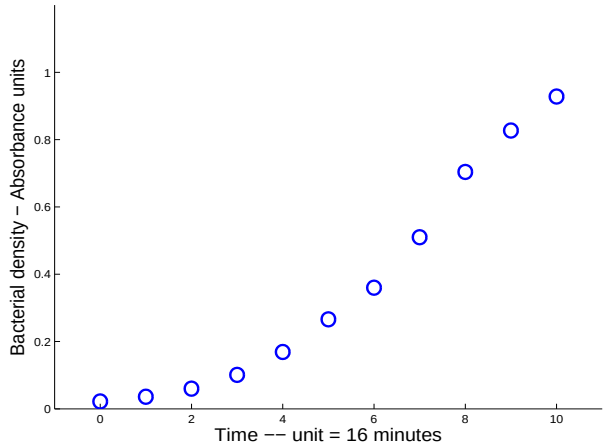


Figure 11.16: Light absorbance data and graph from *V. natrigens* growth.

P_t	$(P_{t+1} - P_t)/P_t$
0.022	0.634
0.036	0.667
0.060	0.684
0.101	0.673
0.169	0.574
0.266	0.353
0.360	0.417
0.510	0.380
0.704	0.175
0.827	0.122

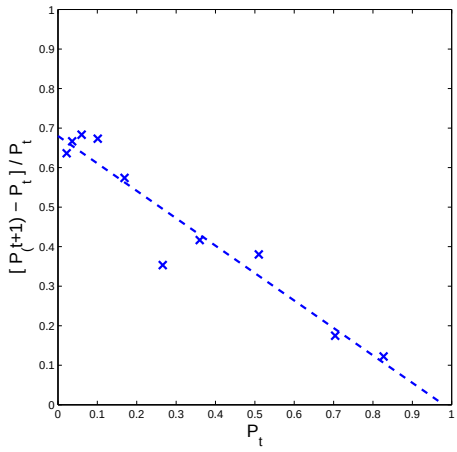


Figure 11.17: Data and graph for $\frac{P_{t+1}-P_t}{P_t}$ versus P_t for P_t data in Figure 11.16.

2. That error is magnified by division by a small number.

Now, if we compare

$$y = R - \frac{R}{M}x \qquad \text{with } y = 0.68 - 0.69x$$

we would conclude that

$$R = 0.68 \qquad \frac{R}{M} = 0.69 \qquad \Longrightarrow M = 0.99$$

Shown to in Figure 11.18 is the graph of the original *V. natrigens* data and the graph of

$$P_{t+1} - P_t = 0.68 \times P_t \times \left(1 - \frac{P_t}{0.99}\right) \tag{11.33}$$

They match very well, despite the scatter in the data from which R and M were computed.

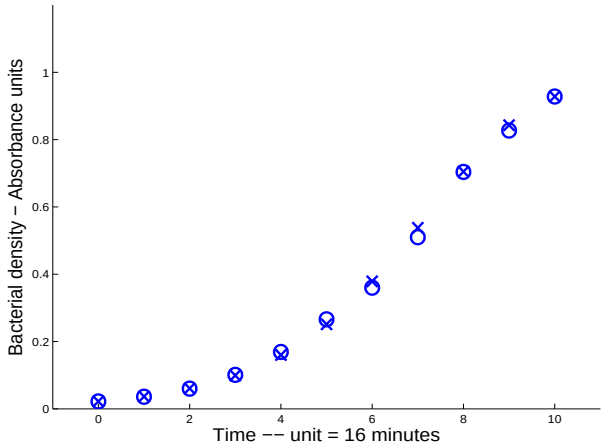


Figure 11.18: Comparison of the *V. natrigens* data of Figure 11.16 with the logistic curve, Equation 11.33.

The number R in Equation 11.29 is the low density growth rate. In Chapter 1 we obtained Equation 1.1 on page 7:

$$B_{t+1} - B_t = \frac{2}{3} B_t$$

using just the six early data points at which the bacterial density was low. The low density growth rate $R = 0.68$ above compares well with $\frac{2}{3} = 0.67$.

Exercises for Section 11.6, Environmental carrying capacity.

Exercise 11.6.1 Contrast the growths per time period described by

$$P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{1000}\right)$$

(a) when $P_t = 900$ and (b) when $P_t = 1000$ and (c) when $P_t = 1100$.

Exercise 11.6.2 Do Explore 11.6.1 on page 506.

Exercise 11.6.3 Plot the graphs of P_t and Q_t *vs* t for the pairs shown.

a. $P_0 = 20$ $\quad P_{t+1} = P_t + 0.2P_t\left(1 - \frac{P_t}{400}\right)$
$Q_0 = 20$ $\quad Q_{t+1} = Q_t + 0.2Q_t$

b. $P_0 = 20$ $\quad P_{t+1} = P_t + 0.1P_t\left(1 - \frac{P_t}{400}\right)$
$Q_0 = 20$ $\quad Q_{t+1} = Q_t + 0.1Q_t$

c. $P_0 = 20$ $\quad P_{t+1} = P_t + 0.2P_t\left(1 - \frac{P_t}{200}\right)$
$Q_0 = 20$ $\quad Q_{t+1} = Q_t + 0.2Q_t$

d. $P_0 = 20$ $\quad P_{t+1} = P_t + 0.2P_t\left(1 - \frac{P_t}{100}\right)$
$Q_0 = 20$ $\quad Q_{t+1} = Q_t + 0.2Q_t$

e. $P_0 = 20$ $\quad P_{t+1} = P_t + 0.1P_t\left(1 - \frac{P_t}{100}\right)$
$Q_0 = 20$ $\quad Q_{t+1} = Q_t + 0.1Q_t$

f. $P_0 = 200$ $\quad P_{t+1} = P_t + 0.1P_t\left(1 - \frac{P_t}{100}\right)$
$Q_0 = 200$ $\quad Q_{t+1} = Q_t + 0.1Q_t$

Exercise 11.6.4 Shown in Table 11.6.4A is a data set of *V. natrigens* grown at pH 7.85. This is a continuation of the the data set shown in Exercise 1.1.5 on page 11. Compute R and M for a logistic curve that approximates this data. Compare the computations from the logistic approximation with the data.

Exercise 11.6.5 Many things grow in a logistic fashion. In some instances, the growth of an individual organism is logistic. A crow embryo grows rapidly at first and has an upper bound inside the egg. This is *not* a mathematical model, however. Shown in Table 11.6.4B is the 'wet mass of a crow embryo' measured at days 3 - 17 of the incubation period. Determine whether a logistic curve describes that growth.

Table for Exercise 11.6.4 A. Light absorbance data from growth of *V. natrigens* in a solution with pH 7.85, and **Exercise 11.6.5** B. Wet mass of a crow embryo at days 3 - 17 of the incubation period.

A

pH 7.85		
Time (min)	Index t	Population Density
0	0	0.028
16	1	0.047
32	2	0.082
48	3	0.141
64	4	0.240
80	5	0.381
96	6	0.576
112	7	0.815
128	8	1.054
144	9	1.206
160	10	1.266

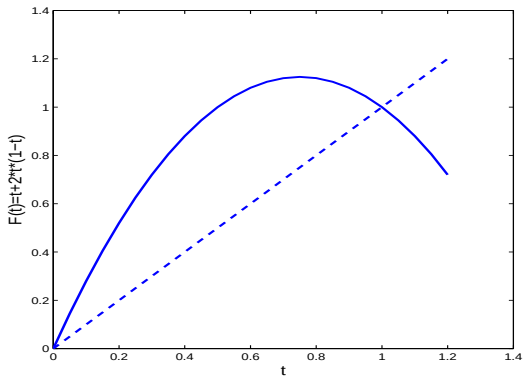
B

Crow embryo mass		
Time (day)	Index t	Embryo mass (gm)
3	0	0.220
4	1	0.435
5	2	0.650
6	3	0.800
7	4	1.200
8	5	1.630
9	6	2.150
10	7	2.800
11	8	3.700
12	9	5.100
13	10	7.500
14	11	10.200
15	12	12.600
16	13	14.850
17	14	16.00

Exercise 11.6.6 A value of $R = 2$ in the logistic equation, $Q_{t+1} = Q_t + RQ_t \times (1 - Q_t)$, yields some interesting results. Shown in Table 11.6.6 are computations and graphs for $R = 2$ and $Q_0 = 0.2$. The odd-indexed iterates increase and the even-indexed iterates beyond index 2 decrease. The pattern continues after 100,000 iterations. Do the two sequences converge to 1? 'Cobweb' the graph of $F(x) = x + 2\,x\,(1 - x)$ at the equilibrium point $x = 1$ to formulate an answer.

Table for Exercise 11.6.6 Data and graph for the iteration $Q_{t+1} = Q_t + RQ_t \times (1 - Q_t)$ with $R = 2$ and $Q_0 = 0.2$.

Time	Pop/M
0	0.200
1	0.520
2	1.0192
3	0.98006
4	1.01914
5	0.98012
6	1.01909
7	0.98019
8	1.01903
9	0.98025
10	1.01897
⋮	⋮
999,997	0.998882554
999,998	1.001114949
999,999	0.998882565
100,000	1.001114938



Exercise 11.6.7 The logistic iteration $Q_{t+1} = Q_t + R\,Q_t\,(1 - Q_t)$ is but one example of the iterations $Q_{t+1} = F(Q_t)$ that were considered in Section 11.3, Equilibrium Points, Stable and Nonstable.

a. Find the two equilibrium points of this logistic iteration and use Theorem 11.4.1 to determine the conditions on $R \neq 0$ for which at least one of them will be locally stable.

b. Discuss the case $R = 0$.

Exercise 11.6.8 Danger: This problem will almost surely fry your brain. Consider the sequence defined by,

$$Q_0 = 0.2, \qquad Q_{t+1} = Q_t + 2\,Q_t\,(1 - Q_t) \qquad t = 1, 2, \cdots.$$

Show that the subsequence Q_2, Q_4, Q_6, $\cdots$ converges to 1.

It certainly appears so from the data in Table 11.6.6, but even $Q_{100,000} = 1.001114938$ is 0.001 above 1. Assume without proof that all of the numbers $1 < Q_{2t} < 1.2$. First you will show that

$$Q_{2t+2} - 1 < Q_{2t} - 1, \qquad t = 1, 2, \cdots.$$

That shows that the sequence decreases toward 1, but does not prove that its limit is 1. (The sequence $\{2 + 1/1,\ 2 + 1/2,\ 2 + 1/3,\ \cdots\ 2 + 1/n,\ \cdots\}$ decreases toward 1, but does not converge to 1.)

You might think that

$$|Q_{t+1} - 1| < |Q_t - 1|, \qquad t = 1, 2, \cdots$$

but that is unfortunately false.

$$|Q_9 - 1| = 0.01975 \not< 0.01903 = |Q_8 - 1|.$$

We therefore have to compose the iteration function $F(x) = x + 2\,x\,(1 - x) = 3x - 2x^2$ with itself to get

$$G(x) = F(F(x)) = -8\,x^4 + 24\,x^3 - 24\,x^2 + 9x \tag{11.34}$$

Then $Q_{2t+2} = G(Q_{2t})$, $t = 1, 2, 3, \cdots$.

a. Show that Equation 11.34 is correct.

b. Now $Q_{2t+2} = G(Q_{2t})$, Show that

$$\begin{aligned} Q_{2t+2} - 1 &= G(Q_{2t}) - 1 \\ &= -8\,Q_{2t}^4 + 24\,Q_{2t}^3 - 24\,Q_{2t}^2 + 9\,Q_{2t} - 1 \\ &= (Q_{2t} - 1)\,(-8\,Q_{2t}^3 + 16\,Q_{2t}^2 - 8\,Q_{2t} + 1) \end{aligned} \tag{11.35}$$

c. Draw the graph of $H(x) = -8\,x^3 + 16\,x^2 - 8\,x + 1$ and show that (1,1) is the maximum for H on the interval [0.5, 1.2] and that H is positive on this interval.

d. Show that

$$Q_{2t+2} - 1 < Q_{2t} - 1$$

(because the second factor in Equation 11.35 is less than 1).

e. We now know that $\{Q_2,\ Q_4\ Q_6,\ \cdots\}$ is a decreasing sequence of numbers bigger than 1, and, by the Axiom of Completeness 1 and Theorem 5.2.1, the sequence has a limit, L which is either 1 or bigger than 1 and that L is less than or equal to every number in $\{Q_2,\ Q_4\ Q_6,\ \cdots\}$.

f. Show that $H(L) \geq H(Q_{2t})$ for every number t.

g. Show that

$$Q_{2t+2} - 1 = (Q_{2t} - 1)\ H(Q_{2t}) \leq (Q_{2t} - 1)\ H(L)$$

h. If $L = 1$, the assertion is proved. Suppose $L > 1$. Then $0 < H(L) < 1$. By cascading the previous inequality, show that

$$Q_{2t} - 1 \leq (Q_2 - 1)(H(L))^t$$

i. Conclude that

$$0 \leq \lim_{t\to\infty} Q_{2t} - 1 = \lim_{t\to\infty} (Q_2 - 1)(H(L))^t = 0.$$

so that

$$1 \quad \leq \quad \lim_{t\to\infty} Q_{2t} \quad = \quad 1. \qquad\qquad \text{Whew!}$$

11.7 Harvest of Natural Populations.

The population difference equations can be modified to reflect the effects of harvest. The harvest may be measured as an amount harvested, H_t, or a fraction of the present population harvested, $h_t \times P_t$. The logistic equation may be

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) - H_t \tag{11.36}$$

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) - h_t \times P_t \tag{11.37}$$

Among the questions that one asks is what harvest strategy will provide the maximum long term yield, what is the maximum allowable harvest in order to retain the population, and what are the locally stable equilibrium sizes of the population?

Example 11.7.1 We begin with the equation

$$\begin{aligned} P_0 &= 1000 \\ P_{t+1} - P_t &= 0.2 \times P_t \times \left(1 - \frac{P_t}{1000}\right) - h \times P_t \end{aligned}$$

of a population with an initial population of 1000 individuals, low density growth rate of 0.2 per time interval, carrying capacity 1000 individuals, and ask what constant fraction of individuals present, h, may be harvested and still retain the population?

The equilibrium population is important and we solve for P_e in

$$P_* - P_e = 0.2 \times P_e \times \left(1 - \frac{P_e}{1000}\right) - h \times P_e$$

$$\begin{aligned} 0 &= 0.2 \times P_e \times \left(1 - \frac{P_e}{1000}\right) - h \times P_e \\ 0 &= P_e \times \left(0.2 \times \left(1 - \frac{P_e}{1000}\right) - h\right) \end{aligned}$$

The choices are

$$P_e = 0 \qquad \text{and} \qquad 0.2 \times \left(1 - \frac{P_e}{1000}\right) - h = 0$$

for which

$$P_e = 0 \qquad \text{and} \qquad P_e = 1000 \times (1 - 5h)$$

Observe that if $1 - 5h$ is negative, the only realistic equilibrium is 0. Therefore if we harvest more than 20% ($h > 0.2$) of the population present at each time, we will loose the population. This makes sense. The low density growth rate (births minus natural deaths) is 20% and if harvest exceeds that we will loose the population. In fact, if harvest $h = 0.2$ then $P_e = 1000 \times (1 - 5h) = 0$, and we loose the population.

We next normalize the initial equation by dividing by the carrying capacity.

$$\frac{P_t}{1000} - \frac{P_t}{1000} = 0.2 \times \frac{P_t}{1000} \times \left(1 - \frac{P_t}{1000}\right) - h \times \frac{P_t}{1000}$$

and write (using P_t ambiguously)

$$\begin{aligned} P_0 &= 1 \\ P_{t+1} - P_t &= 0.2 \times P_t \times (1 - P_t) - h \times P_t \end{aligned}$$

Now if the fractional harvest is set at a number, $h < 0.2$, the equilibrium is either 0 or $(1 - 5h) \times 1000$. Shown in Figure 11.19 is the graph of P_{t+1} *vs* P_t for $h = 0.05$ and it is apparent that $P_t = 0$ is a nonstable equilibrium and $P_t = 1 - 5h = 0.75$ is locally stable.

The objective often will be to maximize harvest. At the equilibrium of $P_e = 1 - 5h$, the harvest amount is $h \times P_e = h \times (1 - 5h)$. The maximum harvest at equilibrium will be obtained at a harvest level that maximizes $h \times (1 - 5h)$.

For what value of h is $\quad h \times (1 - 5h) \quad$ the largest?

Let $G(h) = h \times (1 - 5h)$. Then $G'(h) = 1 - 10h$ and $G'(h) = 0$ when $h = 0.1$ and $G''(h) = -10 < 0$. Therefore $h = 0.1$ yields the maximum value of $h \times (1 - 5h)$. Recall that $R = 0.2$ in this example. Then a 10% harvest strategy is one-half the low density growth rate and will yield a long term maximum harvest value. Furthermore, for $R = 0.2$ and $h = 0.1$, $P_t{+}1 = P_t + 0.2P_t \times (1 - P_t) - 0.1P_t$ and the equilibrium value is $P_e = 0.5$ so *the maximum harvest occurs with the population at one-half the maximum supportable population.* This is a general property of logistic models.

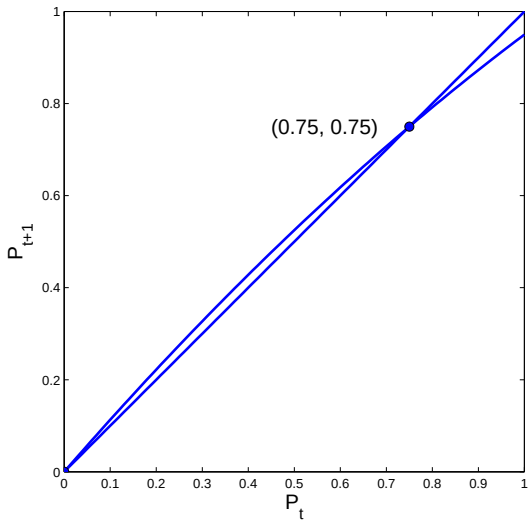


Figure 11.19: Graphs of the iteration equation, $P_{t+1} = P_t + 0.2P_t(1 - P_t) - 0.05P_t$ and $y = x$ showing the equilibrium point at (0.75,0.75).

Exercises for Section 11.7, Harvest of Natural Populations.

In the next four exercises you will analyze the general form of Equation 16.15 with constant $h_t = h$ and $0 < R < 2$.

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) - h \times P_t \tag{11.38}$$

Exercise 11.7.1 Divide each term of Equation 11.38 by M to obtain

$$p_{t+1} - p_t = R \times p_t \times (1 - p_t) - h \times p_t$$

Exercise 11.7.2 a. Show that the equilibrium numbers of the iteration

$$p_{t+1} = F(p_t) = p_t + R \times p_t \times (1 - p_t) - hp_t$$

are $p_{*1} = 0$ and $p_{*2} = 1 - h/R$.

b. Show that in order for there to be a positive equilibrium, the fractional harvest rate, h, must be less than the low density growth rate, R.

Exercise 11.7.3 Assume that $h < R < 2$. Compute $F'(p)$ and evaluate $F'(0)$ and $F'(1 - h/R)$. Conclude that 0 is a nonstable equilibrium and $1 - h/R$ is a locally stable equilibrium.

Exercise 11.7.4 Assume that $h < R < 2$ so that $1 - h/R$ is a positive, locally stable equilibrium. The harvest at that equilibrium will be $h \times (1 - h/R)$. Find the value of h for which $h \times (1 - h/R)$ is the largest. Conclude that in order to maximize harvest, the harvest fraction h should be set at one-half the low density growth rate, R. Find the positive equilibrium p_e for this value of h. In terms of M, what is the equilibrium value at the harvest rate that will maximize harvest?

In the next four exercises you will analyze the Equation 11.39,

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) - H \tag{11.39}$$

where $R < 2$ and a fixed amount,H, is harvested each time period.

Exercise 11.7.5 Divide the terms of Equation 11.39 by M to obtain

$$p_{t+1} - p_t = R \times p_t \times (1 - p_t) - K \tag{11.40}$$

What is the interpretation of K in this equation?

Exercise 11.7.6 a. Show that the equilibrium numbers of the iteration

$$p_{t+1} = F(p_t) = p_t + R \times p_t \times (1 - p_t) - K$$

are

$$p_{*1} = \frac{R - \sqrt{R^2 - 4KR}}{2R} \qquad \text{and} \qquad p_{*2} = \frac{R + \sqrt{R^2 - 4KR}}{2R}$$

b. Show that for there to be any positive equilibrium, the harvest K must be less than or equal to one-fourth the low density growth rate R.

Exercise 11.7.7 Assume that $K < 4R$ and $R < 2$. Compute $F'(p)$ for $F(p) = p + r \times p \times (1 - p) - K$. Show that $p_{*,1}$ is not stable and $p_{*,2}$ is locally stable.

Exercise 11.7.8 The maximum harvest for $p_{t+1} - p_t = R \times p_t \times (1 - p_t) - K$ is interesting. In order for there to be a positive equilibrium, we must have $K \leq R/4$. We might suppose $K = R/4$.
a. Show that under the assumption that $K = R/4$, $p_{*1} = p_{*2} = 1/2$ and that $F'(p_{*2}) = 1$. We are then uncertain whether 1/2 is a locally stable equilibrium. Note that $p_{*2} = 1/2$ corresponds to one-half the maximum supportable population.
b. Examine the special case

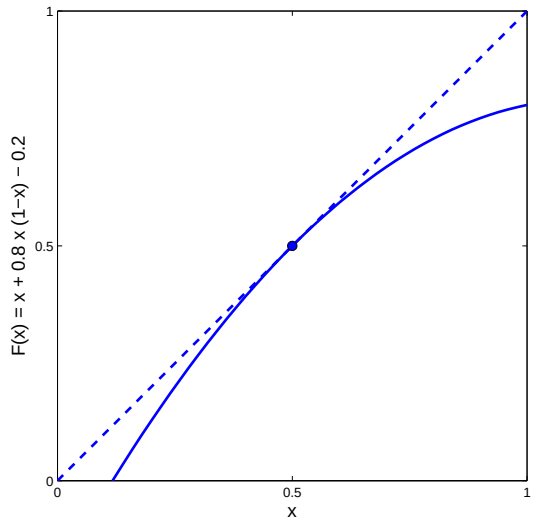
$$p_{t+1} - p_t = 0.8 \times p_t \times (1 - p_t) - 0.2$$

The graph of

$$F(x) = x + 0.8 \times x \times (1 - x) - 0.2$$

and the graph of $y = x$ are shown in Figure 11.7.8. Cobweb the graph starting with $p_0 > 1/2$ and again with the $p_0 < 1/2$. Is $p_{*2} = 1/2$ locally stable? Would you set the constant harvest at one-fourth the low density growth rate and maintain the population at one-half the maximum supportable population?

Figure for Exercise 11.7.8 Graphs of the iteration function, $F(x) = x + 0.8 \times x \times (1 - x) - 0.2$ and $y = x$.



11.8 An alternate logistic equation.

It appears that there is no formula for the solution to Verhulst's logistic equation

$$P_{t+1} - P_t = R\,P_t\left(1 - \frac{P_t}{M}\right)$$

and an alternate form is sometimes used:

$$P_{t+1} = \frac{(1+R)\times P_t}{1 + R\frac{P_t}{M}} \tag{11.41}$$

Peculiar as it may seem, this equation involving division does have a formula for its solution.

$$P_t = \frac{MP_0}{P_0 + (M-P_0)(1+R)^{-t}} \tag{11.42}$$

is a solution to Equation 11.41

Explore 11.8.1 Verify the last statement by showing that

$$\frac{(1+R)\dfrac{MP_0}{P_0+(M-P_0)(1+R)^{-t}}}{1+R\dfrac{\frac{MP_0}{P_0+(M-P_0)(1+R)^{-t}}}{M}} \quad \text{simplifies to} \quad \frac{MP_0}{P_0+(M-P_0)(1+R)^{-(t+1)}}$$

■

Formula 11.42 is more than a Bolt Out of the Blue. A logical derivation from the initial equation appears in Exercise 11.8.5.

Explore 11.8.2 Show that for the alternate logistic equation, R is the low density growth rate. That is, show that in

$$P_{t+1} = \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}, \qquad \text{for} \qquad P_t \ll M, \qquad P_{t+1} - P_t \doteq R\,P_t$$

■

The alternate logistic Equation 11.41 can also be fit to logistic data, although our personal success with this form has not been so great as with the original Equation 11.31. To fit Equation 11.41 to data, we 'turn it up side down'. Begin with

$$P_{t+1} \quad = \quad \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}$$

and put the reciprocals of the two sides of the equation equal:

$$\frac{1}{P_{t+1}} \quad = \quad \frac{1}{\frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}} \quad = \quad \frac{1+R\frac{P_t}{M}}{(1+R)\times P_t} \quad = \quad \frac{1}{1+R}\times\frac{1}{P_t}+\frac{1}{M}\frac{R}{1+R}$$

This means that $\frac{1}{P_{t+1}}$ should be linearly related to $\frac{1}{P_t}$.

Data and the graph of $\frac{1}{P_{t+1}}$ *vs* $\frac{1}{P_t}$ for the *V. natrigens* data described in Figure 11.16 is shown in Figure 11.20. Observe the regularity of the graph as compared to the graph of $\frac{P_{t+1}-P_t}{P_t}$ *vs* P_t in Figure 11.17. Reciprocals are not subject to the roundoff errors that blur differences of small numbers.

P_t	$1/P_t$	$1/P_{t+1}$
0.022	45.45	27.22
0.036	27.77	16.66
0.060	16.66	9.90
0.101	9.90	5.92
0.169	5.92	3.76
0.266	3.76	2.78
0.360	2.78	1.96
0.510	1.96	1.42
0.704	1.42	1.21
0.827	1.21	

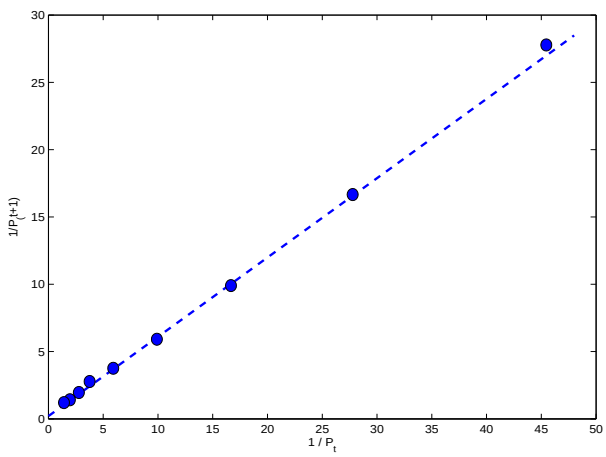


Figure 11.20: Table of reciprocals $1/p_t$ and $1/P_{t+1}$ and graph for the *V. natrigens* data of Figure 11.16 and a line, $y = 0.58914x + 0.20813$, fit to the data.

The equation

$$y = 0.58914x + 0.20813$$

is the least squares fit of a line to the data. By comparison with

$$\frac{1}{P_{t+1}} = \frac{1}{1+R}\times\frac{1}{P_t}+\frac{1}{M}\frac{R}{1+R}$$

We conclude that

$$\frac{1}{1+R} = 0.58914 \qquad \text{and} \quad \frac{1}{M}\frac{R}{1+R} = 0.20813$$

from which it follows that

$$R = \frac{1}{0.58914} - 1 = 0.697 \qquad \text{and} \qquad M = \frac{R}{(1+R)\cdot 0.20813} = 1.974$$

With $R = 0.697$, $M = 1.974$, and $P_{init} = 0.022$, we have from Equation 11.42 that

$$P_t = \frac{0.043}{0.022 + 1.95\,(1.697)^{-t}}$$

A graph of

$$P_t = \frac{0.043}{0.022 + 1.95\,(1.697)^{-t}}$$

is shown in Figure 11.21 together with the original data. The computed data is close to the original data for early times, but a substantial departure for later times, indicating that the value of R is close to correct, but the value of M is off. R is computed from the slope of the reciprocal data graph, which is pretty reliable. However, M is computed from the y-intercept of that graph, which is small and a small change makes a large percentage change. Perhaps another method of fitting the equation to real data should be explored.

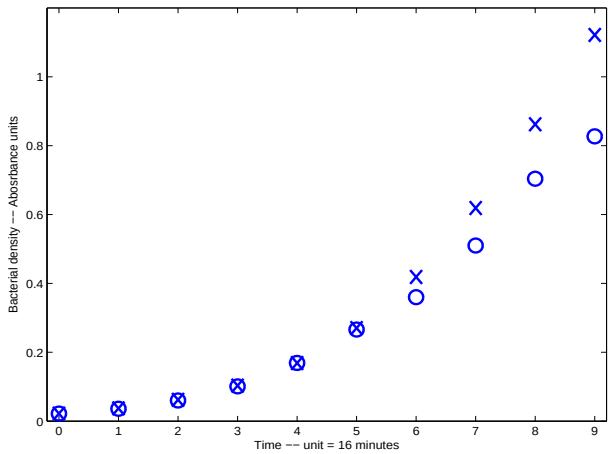


Figure 11.21: Graphs of the *V. natrigens* data (circles) from Figure 11.16 and the predictions from the altered form of the logistic equation, $P_t = \frac{1.974}{1+88.73/1.697^t}$ (x's).

Exercises for Section 11.8, An alternate logistic equation.

Exercise 11.8.1 Do Explore 11.8.1.

Exercise 11.8.2 Shown in Figure 11.8.2 are comparisons of the two forms of the logistic equation for $R = 0.2$, $M = 10$, and $P_0 = 1$.

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) \tag{11.43}$$

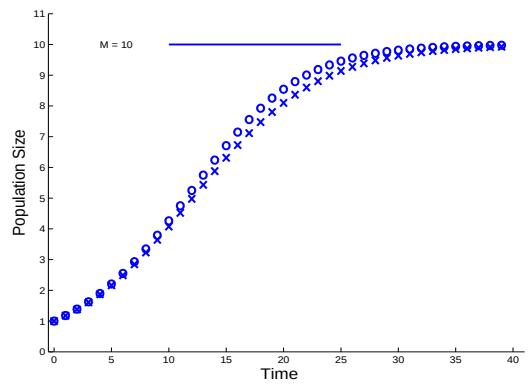
$$P_{t+1} = \frac{(1+R)\,P_t}{1 + R\frac{P_t}{M}} \tag{11.44}$$

$$P_t = \frac{M\,P_{init}}{P_{init} + (M - P_{init})(1+R)^{-t}} \tag{11.45}$$

Complete the missing entries in the table using these equations.

Figure for Exercise 11.8.2 Partial data and graphs comparing the two forms of the logistic equation.

t	Eq 11.43	Eq 11.44	Eq 11.45
0	1	1	1
1	_____	_____	_____
2	_____	_____	_____
3	_____	_____	_____
4	_____	_____	_____
5	2.2075	2.1660	2.1660
10	4.2622	4.0757	4.0757
15	6.7069	6.3125	6.3125
20	8.5426	8.0988	8.0988
25	9.4578	9.1379	9.1379
30	9.8140	9.6347	9.6347
35	9.9381	9.8499	9.8499
40	9.9796	9.9391	_____



Exercise 11.8.3 For each of the following systems, note the y-coordinate, M, of the horizontal asymptote, compute P_{10}/M and P_{20}/M and sketch (do not plot) graphs of P_t *vs* t.

a. $P_0 = 20 \qquad P_{t+1} = \dfrac{1.4P_t}{1 + 0.4\dfrac{P_t}{400}}$

b. $P_0 = 20 \qquad P_{t+1} = \dfrac{1.2P_t}{1 + 0.2\dfrac{P_t}{400}}$

c. $P_0 = 20 \qquad P_{t+1} = \dfrac{1.1P_t}{1 + 0.1\dfrac{P_t}{400}}$

d. $P_0 = 20 \qquad P_{t+1} = \dfrac{1.2P_t}{1 + 0.2\dfrac{P_t}{200}}$

e. $P_0 = 20 \qquad P_{t+1} = \dfrac{1.2P_t}{1 + 0.2\dfrac{P_t}{100}}$

f. $P_0 = 20 \qquad P_{t+1} = \dfrac{1.1P_t}{1 + 0.1\dfrac{P_t}{100}}$

Exercise 11.8.4 Fit the alternate logistic equation 11.41 to each of the data sets for growth of *V. natrigens* in a solution with pH 7.85, and wet mass of a crow embryo at days 3 - 17 of the incubation period shown in Exercise Table 11.6.4 on page 510.

Exercise 11.8.5 Some work is required to see that Equation 11.41

$$P_{t+1} = \frac{(1+R) \times P_t}{1 + R\frac{P_t}{M}}$$

has in fact a rather simple solution. This exercise shows the steps.

a. Turn the equation upside down. That is, write

$$\frac{1}{P_{t+1}} = \frac{1 + R\frac{P_t}{M}}{(1+R) \times P_t}$$

and obtain

$$\frac{1}{P_{t+1}} = \frac{1}{1+R}\frac{1}{P_t} + \frac{R}{M(1+R)}.$$

b. Use Equation 11.2, $P_{t+1} - P_t = r\,P_t + b$, and its solution, Equation 11.3, $P_t = -\frac{b}{r} + \left(P_0 + \frac{b}{r}\right)(1+r)^t$ to show that

$$\frac{1}{P_t} = \frac{1}{M} + \left(\frac{1}{P_0} - \frac{1}{M}\right)\left(\frac{1}{1+R}\right)^t.$$

c. Conclude that

$$P_t = \frac{MP_0}{P_0 + (M - P_0)(1+R)^{-t}} \qquad \text{Equation 11.42}$$

Chapter 12

Discrete Dynamical Systems

Where are we going?

The first order difference equation models of bacterial growth and penicillin clearance introduced in Chapter 1 are refined to account for additional factors (multiple population categories and absorbance of penicillin into tissue) that allow more accurate matches of models to data. These and other models motivate systems of difference equations that are referred to as discrete dynamical systems. The important biological and mathematical concepts of equilibrium point and stable and unstable equilibria are extended to equilibria of discrete dynamical systems.

12.1 Infectious diseases: The SIR model.

In this section you will develop a model of an infectious disease that is spread directly from human to human (influenza, polio, measles, for example) and that confers immunity to a person who has had the disease. It does not include diseases carried by mosquitoes or mice or other 'vectors'. The model also assumes a reasonably 'closed' human population in that it does not account for disease being brought into the population from people not included in it, except for the initial source of the disease.

Suppose there are 25,000 students and faculty in a 'closed' university population, and that at the beginning of a semester 50 of the people return to school infected with a certain influenza. Assume that infected people can transmit the disease during a period of 5 days beginning with the day they are first infected.

At any time university people can be classified as

Susceptible. Those people who have not had this influenza and are susceptible to infection.

Infectious. Those people who are infected and are contagious.

Recovered. Those people who have had the flu, are recovered (no longer contagious), and are immune to the disease.

To understand the categories, examine Table 12.1, where it is assumed that the 50 infected people are equally distributed between being infected 4, 3, 2, 1 and 0 days before arriving on campus and 20 new cases are recorded each morning at, say, 6am.

Initial days of an influenza outbreak.

Day	Susceptible	Newly Infected	Infectious	Newly Recovered	Recovered
-4		10	10		
-3		10	20		
-2		10	30		
-1		10	40		
0	24950	10	50		
1	24930	20	60	10	10
2	24910	20	70	10	20
3	24890	20	80	10	30
4	24870	20	____	____	____
5	24850	20	____	____	____
6	24830	20	____	____	____
7	24810	20	100	20	90

The assumption of a fixed number (20) of new cases per day is too restrictive. As the number of infectious people increases, the disease will spread more rapidly. After a large number of people have been infected and recovered, the susceptible population is small and the spread of the disease will slow down. An elementary model of the spread of disease is:

Mathematical Model 12.1.1 Influenza transmission. Influenza is transmitted by an 'infectious contact' between an infectious individual and a susceptible individual. The frequency of such contacts is proportional to the number of infectious people and to the number of susceptible people in the population.

The number of newly infected is proportional to the infectious contact frequency.

The duration of the disease for all individuals is a fixed time period and all people recover.

As usual, for the frequency of contact to be proportional to two quantities we write that the frequency is a constant times the product of the two things:

$$\text{Infectious Contact Frequency} = \beta_0 \times \text{Number of Susceptible} \times \text{Number of Infectious}$$

The number of newly infected is proportional to the infectious contact frequency and we write

$$\text{Number of Newly Infected} = \beta \times \text{Number of Susceptible} \times \text{Number of Infectious} \tag{12.1}$$

The number β incorporates both the biological property of how easily the disease is transferred from one person to another and the sociological property of the frequency and nature of such contacts. Measles, for example, is highly contagious (a child with measles will expose his or her entire class in about 30 minutes in a class room). Measles-β is 'high' and is determined primarily by biological

properties. HIV transmission requires an exchange of body fluids; HIV-β is 'low' and is determined largely by sociological properties.

The following notation is useful. Suppose time is measured in periods of hours, days, or weeks and t is the index of time periods. Let

S_t = Number of susceptible people at the beginning of time period t.
I_t = Number of infected people at the beginning of time period t.
R_t =Number of people who have recovered at the beginning of time period t.

R_t includes all people who have been infected and have recovered during any previous time period.

Suppose furthermore that the duration of the disease is d time periods for all people. You are asked to write and analyze equations that model influenza dynamics in the following exercises.

Exercises to complete the analysis of influenza dynamics.

Exercise 12.1.1 Fill in the blanks in Table 12.1 and see that they are consistent with the last line.

Exercise 12.1.2 Write an alternate version of Table 12.1 in which the initial 50 infectious people are all assumed to be newly infected (infected on day zero).

Exercise 12.1.3 a. Write equations for Mathematical Model 12.1.1

$$\begin{aligned} S_{t+1} - S_t &= \\ I_{t+1} - I_t &= \\ R_{t+1} - R_t &= \end{aligned}$$

in terms of S_t, I_t, R_t, and β, a constant of proportionality and d the number of time periods an infected person remains infected. It may help you to think of categories, NI_t of 'newly infected' and NI_t of 'newly recovered' at time t.

b. Convert your equations to iteration form with terms involving S_t, I_t, and R_t on the RHS. They should be equivalent to

$$\begin{aligned} S_{t+1} &= S_t - \beta\, S_t\, I_t \\ I_{t+1} &= I_t + \beta\, S_t\, I_t - \beta\, S_{t-d}\, I_{t-d} \\ R_{t+1} &= R_t + \beta\, S_{t-d}\, I_{t-d} \end{aligned} \tag{12.2}$$

Exercise 12.1.4

a. Explain the term $\beta\, S_{t-d}\, I_{t-d}$ in the Equations 12.2.

b. Explain why

$$\beta\, S_{t-d+1}\, I_{t-d+1} + \cdots + \beta\, S_{t-1}\, I_{t-1} + \beta\, S_t\, I_t = I_t.$$

Use this to rationalize

$$\beta\, S_{t-d}\, I_{t-d} \doteq \frac{1}{d} I_t \tag{12.3}$$

as an acceptable approximation.

We use the approximation 12.3 in Equations 12.2 and write with $\gamma = 1/d$

$$\begin{aligned} S_{t+1} &= S_t - \beta\, S_t\, I_t \\ I_{t+1} &= I_t + \beta\, S_t\, I_t - \gamma I_t \\ R_{t+1} &= R_t + \gamma I_t \end{aligned} \tag{12.4}$$

A computer or programmable calculator is helpful for the remaining problems.

Exercise 12.1.5 Suppose that the proportionality constant, $\beta = 0.00003$, and that there is a reasonably closed university population of 25,000 people and that 50 return to university at the beginning of a semester infected with an influenza. Use Equation 12.4 to compute S_t, I_t, and R_t for $t = 1, 2, 3$. Assume that infected people remain contagious for 5 days, so that $\gamma = 1/5 = 0.2$ of the infected people recover each day.

Exercise 12.1.6 Repeat the calculations of Exercise 12.1.5, except assume that only one person returns to campus infected with influenza. Use $\beta = 0.00003$ and $\gamma = 0.2$.

Exercise 12.1.7 Repeat the calculations of Exercise 12.1.6 (only one infected person returns to campus and $\gamma = 0.2$) except assume the proportionality constant, $\beta = 0.000005$. Compute S_t, I_t, and R_t for $t = 1, 2, 3$.

Exercise 12.1.8 There is an important qualitative difference in the results of Exercises 12.1.6 and 12.1.7. In Exercise 12.1.6 with $\beta = 0.00003$, $I_1 = 1.55 > I_0 = 1$, the number of infected *increases* and there will be a flu 'epidemic'. In Exercise 12.1.7 with $\beta = 0.000005$, $I_1 = 0.925 < I_0 = 1$, the number of infected *decreases* on the first step and the influenza will quickly die out. Repeat the calculations of Exercise 12.1.7 with various values of β to find the largest number, β_0, such that if $\beta = \beta_0$ in the previous model, $I_1 \le I_0$.

Exercise 12.1.9 a. Solve for β in

$$\beta \times S_0 \times I_0 - \gamma I_0 = 0.$$

b. Why?

c. Show that if $\dfrac{S_0\beta}{\gamma} > 1$, then $I_1 > I_0$ and if $\dfrac{S_0\beta}{\gamma} < 1$, then $I_1 < I_0$.
The number $\dfrac{S_0\beta}{\gamma}$ is a commonly used measure of whether there will be an epidemic.

Exercise 12.1.10 Suppose at the beginning of the study there are 24,950 susceptible people, 50 infected people, and no recovered/immune people, and let $\beta = 0.00002$ and $\gamma = 0.2$. Compute the values of S_t, I_t, R_t for 70 days.

a. When is the epidemic at its height?

b. Do all of the people get sick?

c. Repeat the computation for $\beta = 0.00006$ and only 12 days. You will find that all of the people will get sick. What is the least value of β for which all people get the flu?

The event that everyone gets sick is a property of our discrete model, and perhaps a peculiar property. You will show in Exercise 17.5.7 that in the continuous model analogous to Equations 12.4 it never happens that everyone gets sick.

Exercise 12.1.11 a. What are the dimensions of the constants β and γ in Equations 12.4?

b. Convert Equations 12.4 to variables, x, y, and z, that are fractions of the whole population? The constants β and γ have to change, to, say, β^* and γ^*. What are $\beta*$? and γ^*

12.2 Pharmacokinetics of Penicillin

The movement of penicillin in the body is considerably more complex than has been suggested earlier by the simple elimination of penicillin by the kidney. We present here some of the data from T. Bergans, Penicillins, in *Antibiotics and Chemotherapy, Vol. 25*, H. Schøonfeld, Ed., S. Karger, Basel, New York, 1978, and suggest some of the improvements in the model of penicillin kinetics that are suggested by the data.

Figure 12.1 shows simple clearance of mezlocillin from serum after intravenous administration of a 1, 2, or 5 g dose during a 5 minute period. This is referred to as a 'bolus' dose, and is imagined as an instantaneous administration of the drug. We will focus attention on the 2 g data. The data appears to have been taken 5, 10, 15, 20, 30, 45, 60, 120, 180, 240 and 300 minutes after administration was completed. The table to the right of the graphs contains numbers read from the 2 g dose graph.

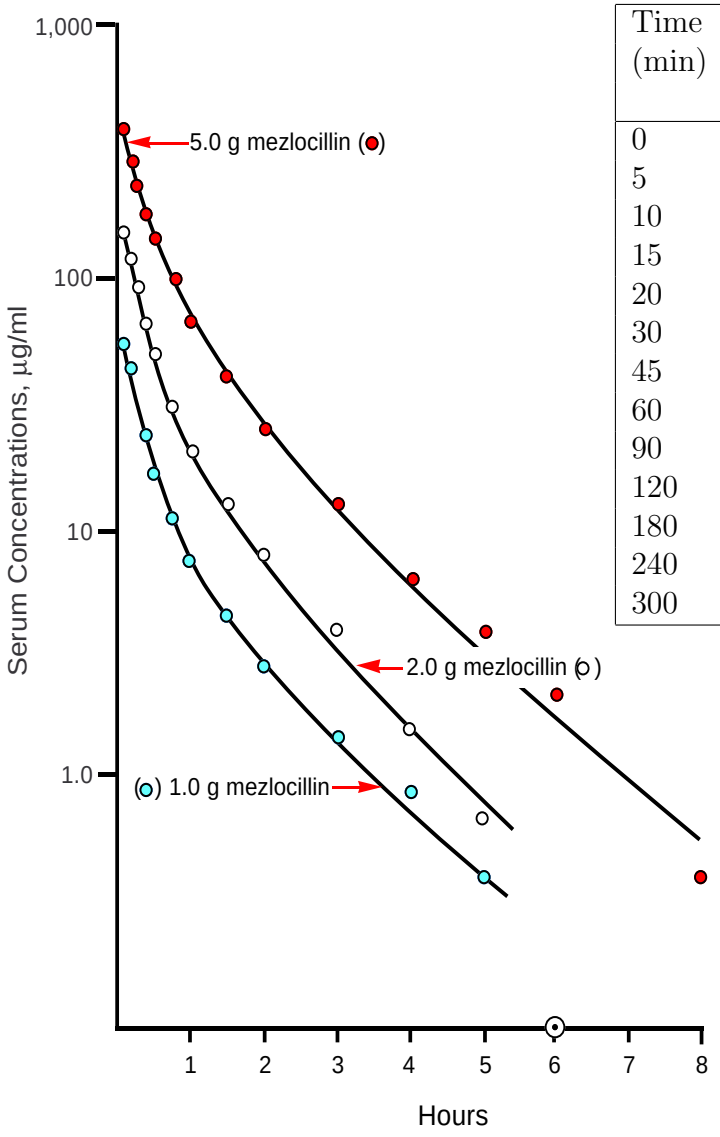
Explore 12.2.1 Three data points, corresponding to times $t = 10$, $t = 60$, and $t = 180$ minutes are omitted from the table. Read approximate values for these data from the graph of 2 g data and complete the table. ■

Observe that the vertical scale in Figure 12.1 is measured in logarithms to the base 10 of μg/ml. It appears that at time $t = 0$ the data would be about 200 μg/ml. Assume a serum volume of 2.75 liters (Rhoades and Tanner, page 210, $2.75 = 0.55 \times 5.0$). Administration of 2 gm yields an initial value of 727 μg/ml ($= 2 \times 10^6/(2.75 \times 10^3)$. However, the dose was administered over a 5 minute interval; if we assume that 4% is lost each minute, one has 644 μg/ml at the end of the administration period. This is quite high compared to the $t = 0$ data point which we estimated to be about 200 μg/ml. It appears that the penicillin equilibrates quickly in a pool of larger volume, possibly including the serum, the blood cells, and part of the interstitual fluid.

We wish to write a mathematical model descriptive of penicillin kinetics. In Section 1.6 we suggested that penicillin clearance would be 4% per minute, or approximately 20% per five minutes. If so, by the methods of that section we would conclude that $P_t = 200 \times 0.8^t$ and we would conclude that the graph of P_t versus t would be a straight line on semilog paper. See Exercise 12.2.2.

The graphs of the observed data in Figure 12.1 are not straight lines. The departure from straight lines is due to the attachment of mezlocillin to proteins in the vascular pool and migration of mezlocillin into tissue, followed by return of the penicillin to the vascular pool. Attachment to proteins and migration to tissue both remove penicillin from the vascular pool and have similar effects on the model, although the flow rates will be different. To simplify the model the two are combined into only migration into tissue.

Schematic diagrams that enable discussion of the kinetics are commonly used to depict pharmacokinetic models. Shown in Figure 12.2 are models showing (A) simple elimination from the



Time (min)	2.0 g Mezlocillin μg/ml
0	200
5	152
10	
15	93
20	74
30	50
45	31
60	
90	13
120	8.2
180	
240	1.7
300	0.8

Figure 12.1: Clearance of penicillin from serum, Figure 29, page 92 of Tom Bergans, Penicillins, in H. Schönfeld Ed. Antibiotics and Chemotherapy, V. 25, Pharmacokinetics, S Karger, Basel, 1978.

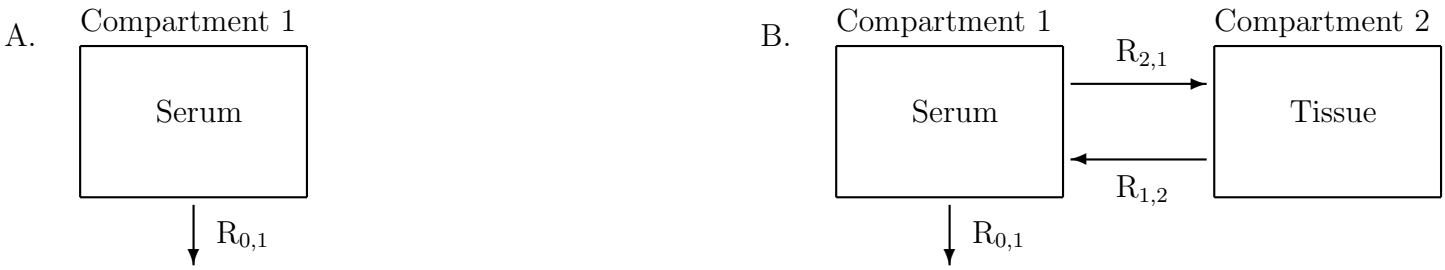


Figure 12.2: Schematic diagrams of one- and two-compartment physiological models.

serum and (B) migration to and from tissue and elimination from the serum pool. The symbols, $R_{i,j}$, represent flow rates between compartments of the substance of interest (penicillin, for example). By well established convention, $R_{i,j}$ represents flow to compartment i from compartment j, and compartment 0 is "outside" the system.

The difference between one and two compartment models can be explored using some very simple concepts.

Explore 12.2.2 We do not have data on the mesocilin concentration in the tissue compartment. What is the tissue concentration at time $t = 0$? Sketch a possible graph of that concentration. Do you expect that concentration to exceed the serum mesocilin concentration at any time?

Kinetics of a one-pool model A flask (Serum) has 5 liters of pure water and 200 grams of salt. At the end of each five minute interval, 1 liter (20%) of the liquid is removed from the flask, after which one liter of pure water is added and the flask is well stirred.

Let P_t denote the amount of salt in the flask at time t (at the end of the $t\underline{th}$ five minute time interval and at the beginning of the $t+1\underline{st}$ time interval. We can write

$$\begin{aligned} P_0 &= 200 && \text{200 grams initially in the flask.} \\ P_{t+1} - P_t &= -0.2 * P_t && \text{Remove 20\% each time period.} \\ P_{t+1} &= 0.8 \times P_t && \text{Iteration equation.} \\ P_t &= 200 \times 0.8^t && \text{Solution to the iteration equation.} \end{aligned}$$

Kinetics of a two-pool model. An example of the two-pool model shown in Figure 12.2 is analyzed as follows. Suppose there are two flasks, flask 1 (Serum) contains 1 liter of water and 200 milligrams of salt and flask 2 (Tissue) contains 0.68 liters of water. (The numbers have been selected to give an answer similar to the data of Figure 12.1 and expressed in rational numbers.) At the end of each five minute interval, 68 ml of liquid is removed from flask 2 (10%) and temporarily set aside, 68 ml of liquid is removed from flask 1 (6.8%) and added to flask 2 and 162 ml of liquid is removed from flask 1 (16.2%) and discarded. Then the 68 ml removed from flask 2 is added to flask 1, and 162 ml of pure water is added to flask 1.

Let A_t and B_t denote the amount of salt in flask 1 and flask 2, respectively, at the end of the $t\underline{th}$ 5 minute time interval and at the beginning of the $t+1\underline{st}$ time interval. The value of $R_{1,2}$, for example, in the two-pool model diagram is either 68 ml per five minutes or the fractional value 0.1 (10% of the pool) per five minutes. Both forms are used in the pharmacokintics literature; we use the fractional value.

The value of A_0 is 200 and the value of B_0 is 0. The changes in the amount of salt in the two flasks may be accounted for by:

$$\begin{array}{rcll} \text{Change} & & \text{Increase} & \text{Decrease} \\ A_{t+1} - A_t & = & 0.1B_t & (-0.162 - 0.068)A_t \\ B_{t+1} - B_t & = & 0.068A_t & -0.1B_t \end{array} \tag{12.5}$$

The initial conditions and difference equations lead to the iteration equations

$$\begin{array}{lrcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array} \tag{12.6}$$

Reader Beware: Incoming Twin Bolts of Lightning. The equations

$$\begin{array}{rcl} A_t & = & \frac{200}{2.1} \times (0.4 \times 0.94^t + 1.7 \times 0.73^t) \\ B_t & = & \frac{200}{2.1} \times (0.68 \times 0.94^t - 0.68 \times 0.73^t) \end{array} \tag{12.7}$$

exactly solve the Equations 12.6. Equations 12.7 are easier to use and give us a picture of the solution more quickly than the iteration Equations 12.6. You will see how Equations 12.7 are obtained in Section 12.4.

We show that the first iteration equation is satisfied by the proposed solution equations and leave for you to show in Exercise 12.2.8 that the second iteration equation is satisfied. First observe that

$$A_0 = \frac{200}{2.1} \times \left(0.4 \times 0.94^0 + 1.7 \times 0.73^0\right) = \frac{200}{2.1} \times (0.4 \times 1 + 1.7 \times 1) = \frac{200}{2.1} \times 2.1 = 200$$

so that **the initial value of $\mathbf{A}_t$ is satisfied**. Next we substitute the two expressions for A_t and B_t into the right hand side of the equation for A_{t+1}.

$$\begin{aligned} A_{t+1} &= 0.77 \times A_t + 0.10B_t \\ &= 0.77 \times \left[\frac{200}{2.1} \times \left(0.4 \times 0.94^t + 1.7 \times 0.73^t\right)\right] \\ &\qquad +0.10 \times \left[\frac{200}{2.1} \times \left(0.68 \times 0.94^t - 0.68 \times 0.73^t\right)\right] \\ &= \frac{200}{2.1}\left[(0.77 \times 0.4 + 0.1 \times 0.68)0.94^t + (0.77 \times 1.7 - 0.1 \times 0.68)0.73^t\right] \end{aligned}$$

$$= \frac{200}{2.1}\left[\frac{0.77 \times 0.4 + 0.1 \times 0.68}{0.94} 0.94^{t+1} + \frac{0.77 \times 1.7 - 0.1 \times 0.68}{0.73} 0.73^{t+1}\right]$$

$$= \frac{200}{2.1}\left[0.4 \times 0.94^{t+1} + 1.7 \times 0.73^{t+1}\right] \qquad = A_{t+1}$$

Whew!! The first iteration equation is satisfied.

Exercises for Section 12.2, Pharmacokinetics of Penicillin

Exercise 12.2.1 Assuming that immediately after a 2 gm injection of penicillin into the vascular pool, the penicillin concentration is 200 μg/ml, what is the estimated volume of the vascular pool?

How does this estimate compare with the blood volume of an adult of about 5 liters and a serum volume of 2.75 liters?

Exercise 12.2.2 a. Suppose 2 gm of mezlocillin are injected (bolus injection) into the serum of a patient with the result that at time $t = 0$ the serum concentration of mezlocillin is 200 μg/ml. Assume for this problem that the mezlocillin stays in the serum until removed by the kidneys and that the kidneys remove 20% of the mezlocillin from the serum every five minutes. Let P_t denote the concentration of mezlocillin in the serum at the end of the tth 5 minute time interval. Write equations for

$$\begin{aligned} P_{t+1} - P_t &= \underline{\hspace{6cm}} \\ P_{t+1} &= \underline{\hspace{6cm}} \quad \text{in terms of } P_t \\ P_t &= \underline{\hspace{6cm}} \quad \text{in terms of } t. \end{aligned}$$

b. Compute the mezlocillin levels predicted by $P_t = 0.8^t \times 200$ where t marks 5 minute intervals for the times, $t = 0$, $t = 1$, $\cdots$ $t = 24$ and plot the data along with the original data in Figure 12.1 on a semilog graph.

c. Explain why the graph of P_t versus t is a straight line on the semilog graph.

Exercise 12.2.3 Compute A_1 and B_1, A_2 and B_2, A_3 and B_3 Using the Iteration Equations 12.6.

Exercise 12.2.4 Compute A_1, A_2 and A_3 using the Solution Equations 12.7. You should get the same answers as for A_1, A_2 and A_3 in Exercise 12.2.3.

Exercise 12.2.5 Compute A_{12} and B_{12} using either the Difference Equations 12.5 or the Iteration Equations 12.6 or the Solution Equations 12.7 (your choice). Compare A_{12} with the observed value of penicillin concentration at 60 minutes in Figure 12.1.

Exercise 12.2.6 Compute A_{24} and B_{24} using either the Difference Equations 12.5 or the Iteration Equations 12.6 or the Solution Equations 12.7 (your choice). Compare A_{24} with the observed value of penicillin concentration at 120 minutes in Figure 12.1.

Exercise 12.2.7 Explain the reasons for the terms in Equations 12.5 that account for Change, Increase, and Decrease of A_t and of B_t.

Exercise 12.2.8 Show that the proposed solutions, Equations 12.7, also satisfy $B_0 = 0$ and the second of Equations 12.6. Do exact arithmetic.

Exercise 12.2.9 a. Compute A_t and B_t from Equations 12.6 or 12.7 for the times shown in Figure 12.1 and plot the original data and the computed data on a semilog graph.

b. How do the values of A_t compare with with the observed concentration of penicillin?

c. Except that they are discrete points, the two graphs for A_t and B_t cross. One might guess that the tissue concentration maximum occurs where the tissue concentration is the same as the serum concentration. Examine the graphs and determine whether this appears to be true.

d. Use $B_{t+1} - B_t = 0.068A_t - 0.1B_t$ to determine whether the maximum tissue concentration occurs when the tissue concentration is the same as the serum concentration.

Exercise 12.2.10 Residence time is a measure of the average amount of time a chemical or biological agent spends in a well defined volume. You may read of the residence time of, for example, greenhouse gases in the atmosphere, salt in an estuary, water in the atmosphere or in a lake or in ground water, or even vegetables in the market. In the one compartment example where 200 gm of salt are dissolved (into Na^+ and Cl^- atoms) in 5 liters of water and 20% of the solution is replaced every five minutes with pure water, what is the average time that the Na^+ atoms spend in the flask? This requires a bit of warmup.

a. Compute the average age of your class if there are 22 students who are 18 years old, 16 students who are 19 years old, and 6 students who are 20 years old.

b. The 'age' of the Na^+ atoms removed at the first replacement is 5 minutes and 20% of the total, T, are removed and 80% of the total remain in the flask. Argue that the average age, $\overline{A}_3$, of the Na^+ removed during the first three replacements is

$$\overline{A}_3 = \frac{0.2T \cdot 5 + 0.2 \cdot 0.8T \cdot 10 + 0.2 \cdot 0.8^2 T \cdot 15}{0.2T + 0.2 \cdot 0.8T + 0.2 \cdot 0.8^2 T}$$

$$\overline{A}_3 = 5\frac{1 + 2 \cdot 0.8 + 3 \cdot 0.8^2}{1 + 0.8 + 0.8^2}$$

c. Write an expression for the average age, $\overline{A}_{20}$, of the Na^+ removed during the first twenty replacements.

d. $\overline{A}_{20}$ and even $\overline{A}_{2000}$ is easily evaluated using a computer or calculator program (or even a lot of patience), but the old guys developed some clever methods.

From Equation 3.1 we can write

$$\begin{aligned} 1 - x^{n+1} &= (1-x)(1 + x + x^2 + \cdots x^{n-1} + x^n) \\ \frac{1 - x^{n+1}}{1 - x} &= 1 + x + x^2 + \cdots x^{n-1} + x^n = P_n(x) \end{aligned}$$

You can immediately see that the denominator of $\overline{A}_{20}$ is

$$P_{19}(0.8) = \frac{1-0.8^{20}}{1-0.8} \doteq 4.92424$$

and easily computed. It gets better. Show that the numerator of $\overline{A}_{20}$ is $P'_{20}(x)|_{x=0.8}$.

e. Think out of the box a minute. Imagine that

$$P_\infty(x) = 1 + x + x^2 + \cdots = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x} \quad \text{and} \quad P'_\infty(x) = \frac{1}{(1-x)^2},$$

and that

$$\overline{A}_\infty = 5\frac{P'_\infty(x)}{P_\infty(x)}\bigg|_{x=0.8} = 5\frac{\dfrac{1}{(1-x)^2}}{\dfrac{1}{1-x}}\bigg|_{x=0.8} = 25.$$

The box follows.

f. Compute

$$P'_{n+1}(x) = \left[\frac{1-x^{n+2}}{1-x}\right]'$$

g. Use L'Hospital's Rule, Theorem 11.5.1, to show that if $0 < a < 1$ then

$$\lim_{t\to\infty} t\,a^t = \lim_{t\to\infty} \frac{t}{(1/a)^t} = 0.$$

h. Suppose $0 < x < 1$ and show that

$$\lim_{n\to\infty} 5\frac{\dfrac{1-(n+2)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2}}{\dfrac{1-x^{n+1}}{1-x}} = \frac{5}{1-x}.$$

12.3 Continuous infusion and oral administrations of penicillin.

Shown in Figures 12.3 and 12.4 are two more graphs from T. Bergans, *ibid.* Figure 12.3 reports the serum concentration during constant infusion of carbenicillin into the serum pool; a diagram that can be used to model the process is included. The diagram has only a serum pool, a source from outside (infusion) leading into the serum pool and an exit from the pool (kidneys). A tissue compartment (dashed box) could be added to the diagram. As in the bolus injection model of the previous section, continuous infusion can be modeled in terms of salt in flasks, a task left for you in Exercise 12.3.3.

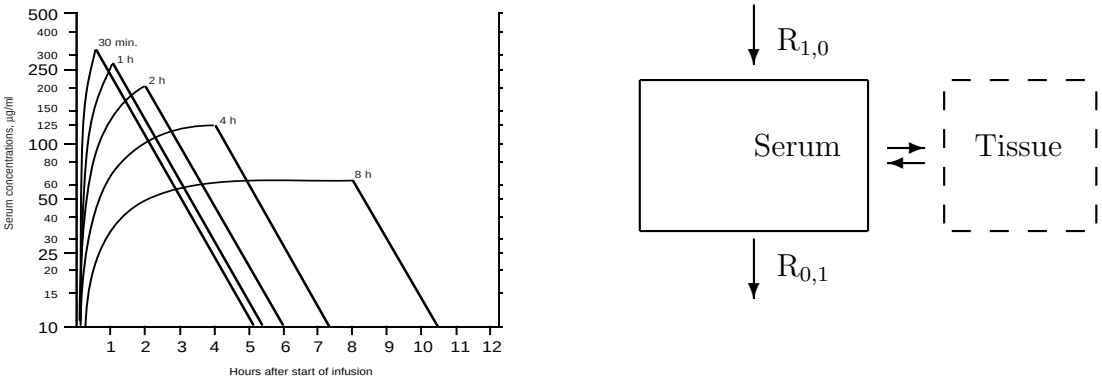


Figure 12.3: Graph of continuous infusion administration of carbenicillin, from Tom Bergans, *ibid.*, page 73, Figure 16.

The original data was in Palmer and Höffler, Carbenicillin, Basisdaten zur Therapie mit einem Antibiotikum (Urban & Schwarzenberg, München, 1977), except for the 30 minute curve which was from T. Bergan and B. Ødvin, Cross-over study of penicillin pharmacokinetics after intravenous infusions. Chemotherapy, 20: 263-279 (1974).

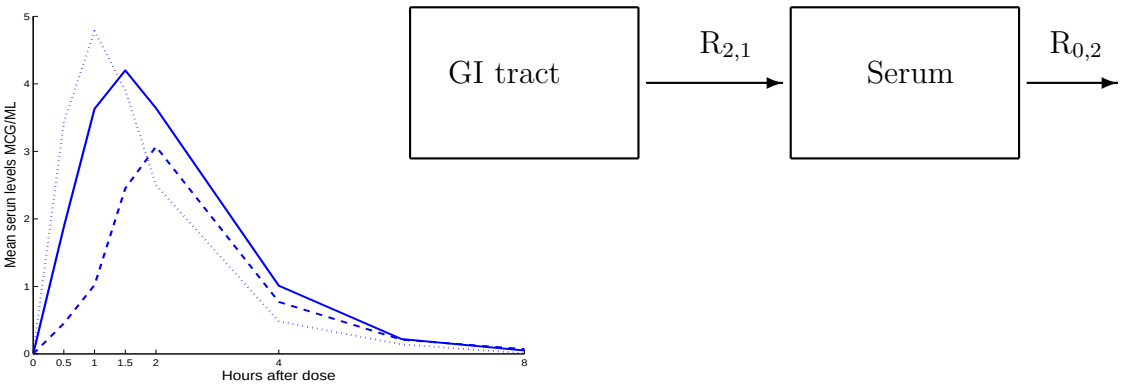
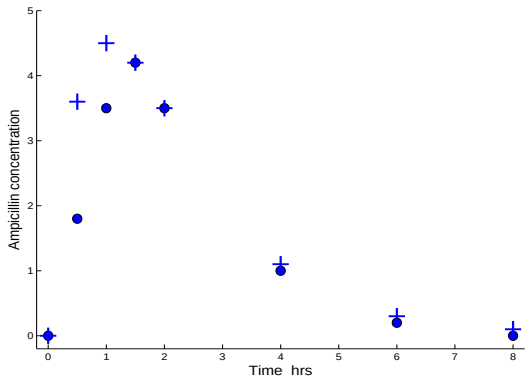


Figure 12.4: Data for oral administration of 500 mg of pivamphicillin and ampicillin to healthy volunteers (medical students) and a schematic diagram used to model this experiment. Graph reproduced from Fernandez, C.A. et al: *J. Int. Med. Res.*, **1**, 530-533 by permission from Field House Publishing LLP, all rights reserved. — Amphicillin, Fasting; ··· Pivamphicillin, Fasting; - - - Pivamphicillin with food. Pivamphicillin was first being tested, and without food patients experienced nausea.

Table 12.1: Estimates of ampicillin concentrations read from the dark line graph in Figure 12.4. Graph of that data (dots) and of computed data (+).

Time (hr)	Concentration g/ml	Computed Concentration
0	0.0	0.0
0.5	1.8	3.6
1.0	3.5	4.5
1.5	4.2	4.2
2.0	3.5	3.5
4.0	1.0	1.1
6.0	0.2	0.3
8.0	0.0	0.1



Analysis of oral ingestion of ampicillin. We will analyze the data for oral ingestion of ampicillin shown in Figure 12.4. We try to model the dark line in Figure 12.4. Estimates of serum concentrations of ampicillin read from the graph are shown in Table 12.1.

It appears that there are at most six data points represented in the graph of Figure 12.4, and the two compartment model may be as 'rich' as can be justified. We suggest, however, a three compartment model for you to try in Exercise 12.3.5.

Two compartment model. Again we write in terms of water and salt in flasks.
Flask 1 (GI tract) has 1 liter of water and at time $t = 0$, 9 grams of salt are added to it and it dissolves immediately.
Flask 2 (Serum) has 2 liters of pure water in it.
Each 30 minutes 35% of the liquid in flask 2 (700 ml) is removed and discarded, after which 40% of the liquid in flask 1 (400 ml) plus 300 ml of pure water is added to flask B and 400 ml of pure water is added to flask A.
Notation. Let A_t denote the amount of salt in flask 1 at time t and B_t denote the amount of salt in flask 2 at time t, where t is measured in 30 minute intervals.
The initial conditions are

$$A_0 = 9.0 \qquad \text{and} \qquad B_0 = 0.0$$

The changes in A_t and B_t can be accounted for as follows.

$$\begin{array}{rcrr} \text{Change} & & \text{Increase} & \text{Decrease} \\ A_{t+1} - A_t & = & & -0.40A_t \\ B_{t+1} - B_t & = & 0.40A_t & -0.35B_t \end{array} \tag{12.8}$$

These can be converted to iteration equations with initial conditions:

$$\begin{array}{rclrcl} A_0 & = & 9 & A_{t+1} & = & 0.6A_t \\ B_0 & = & 0 & B_{t+1} & = & 0.4A_t + 0.65B_t \end{array} \tag{12.9}$$

You can readily write a formula for A_t and are asked to do so in Exercise 12.3.2. In Section 12.4 you will find how to write the formula for B_t:

$$B_t = 72 \times (0.65^t - 0.6^t) \tag{12.10}$$

Using this equation for B_t, the numbers in the 'Computed Concentration' column of Table 12.1 are readily computed. It is apparent from the graph in Table 12.1 that the overall shape of the computed graph is similar to that of the original data and the concentrations for the later times are similar in both. The computed data 'peaks early,' however. It would peak even earlier were it not for the large discrete time interval of 30 minutes. An alternate model with 10 minute intervals is suggested in Exercise 12.3.5 that more closely fits the data.

We have given three examples of data for penicillin administration and difference equation models that could be descriptive of the kinetics of the penicillin. We have also stated solutions to the difference equations, without giving procedures to find the solutions. There are some general procedures for finding the solutions, and the next two sections are directed to that end.

Exercises for Section 12.3, Continuous infusion and oral administrations of penicillin.

Exercise 12.3.1 Explain the reasons for the terms in Equations 12.8 that account for Increase and Decrease of A_t and of B_t.

Exercise 12.3.2 Find the solution to

$$A_0 = 9.0 \qquad A_{t+1} = 0.6A_t,$$

which is the first equation in Equations 12.9.

scarp

Exercise 12.3.3 Analysis of continuous infusion of carbenicillin. Figure 12.3 contains data of concentrations of carbenicillin in serum during and after continuous infusion into healthy volunteers (medical students). You are to analyze a model of that system.

A flask (Serum) has 10 liters of pure water. Each 10 minutes, one liter (10%) of the liquid is removed from the flask, after which one liter of water containing salt with a concentration of 66 grams per liter is added to the flask and the flask is well stirred.

Let A_t denote the amount of salt in the flask at time t.

a. Read and record the carbenicillin concentrations of the 8h curve in Figure 12.3 for the hours 1 to 8. Assume that concentration at hour 0 was 0. Interpret that curve to show that infusion continued at a constant rate for 8 hours and was terminated at the end of 8 hours.

b. What is A_0?

c. Write an equation that accounts for the Change, Increase, and Decrease of A_t each ten minute interval, similar in form to that of Equation 12.8.

d. Write the initial condition and an iteration equation that will enable the computation of A_t for any 10 minute interval.

e. Your iteration equation should be of the form $A_{t+1} = q \times A_t + p$. You solved equations of this form in Section 1.8. Write a solution to your iteration equation.

f. Compare the salt *concentrations* to the carbenicillin concentrations for hours 0 to 8 that you read from the graph.

g. After infusion is terminated at 8 hours, the concentration decreases linearly on the semilog graph. Compare this with the nonlinear decrease following bolus injection shown in Figure 12.1. What difference in the states of the two patients may account for the difference in response?

h. Discuss the advisability of including a tissue compartment (dashed box in Figure 12.3) in the model of the data of the figure.

Exercise 12.3.4 Repeat the analysis of Exercise 12.3.3 for the 4 hour curve. Use the model of Exercise 12.3.3 and adjust the concentration of the salt in the water added to the flask at each ten minute interval.

Exercise 12.3.5 As discussed in the analysis of the oral ingestion of ampicillin, the concentration predicted by our solution 'peaked' too early to match the data. One possible interpretation is that following ingestion of the pill into the stomach, there was a time of transition to the intestine where it was then absorbed into serum. This would suggest a model described by the diagram in Exercise Figure 12.3.5

Imagine three flasks A, B, and C, containing 1.5, 2.0 and 1.0 liters, respectively, of pure water. At time $t = 0$, 21 g of salt is placed in flask A and immediately dissolves. Every 10 minutes 300 ml (30%) of solution is removed from flask C and discarded, 300 ml (15%) of solution is transferred from flask B to flask C, and 300 (20%) ml of water is transferred from flask A to flask B. Finally, 300 ml of pure water is added to flask A. Let A_t, B_t, and C_t denote the amount of salt in flasks A, B, and C at time interval t.

a. What are A_0, B_0 and C_0?

b. Write equations that accounts for the Change, Increase, and Decrease in each of A_t, B_t, and C_t each ten minute interval. similar in form to that of Equation 12.8.

c. Write the initial conditions and an iteration equations that will enable the computation of A_t, B_t, and C_t for any 10 minute interval.

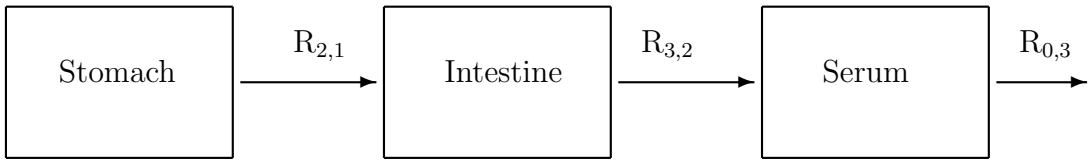
d. Show by substitution that

$$\begin{aligned} A_t &= 21 \times 0.8^t \\ B_t &= 84 \times (0.85^t - 0.8^t) \\ C_t &= 84 \times 0.85^t - 126 \times 0.8^t + 42 \times 0.7^t \end{aligned}$$

solve your iteration equations.

e. Use the solution equation for C_t from part d. to compute ampicillin concentrations for times 0, 0.5, 1.0, 1.5, 2, 4, 6, and 8 hours and compare those values with the values shown in Table 12.1.

Figure for Exercise 12.3.5 Diagram of 3 compartment model of oral ingestion of ampicillin for Exercises 12.3.5 and 12.3.6



Exercise 12.3.6 The three compartment model for oral ingestion of 500 mg of ampicillin diagrammed in Exercise Figure 12.3.5 can be described more simply than was done in Exercise 12.3.5. The solutions of the iteration equations are unusual, but match the data well.

Imagine three flasks A, B, and C, each containing 1.0 liter of pure water. At time $t = 0$, 14 g of salt is placed in flask A and immediately dissolves. Every 10 minutes 200 ml (20%) of solution is removed from flask C and discarded, 200 ml (20%) of solution is transferred from flask B to flask C, and 200 ml (20%) of water is transferred from flask A to flask B. Finally, 200 ml of pure water is added to flask A. Let A_t, B_t, and C_t denote the amount of salt in flasks A, B, and C at time interval t.

a. What are A_0, B_0 and C_0?

b. Write equations that accounts for the Change, Increase, and Decrease in each of A_t, B_t, and C_t each ten minute interval. similar in form to that of Equation 12.8.

c. Write the initial conditions and an iteration equations that will enable the computation of A_t, B_t, and C_t for any 10 minute interval.

d. Show by substitution that

$$\begin{aligned} A_t &= 14 \times 0.8^t \\ B_t &= 3.5 \times t \times 0.8^t \\ C_t &= 0.4375 \times t \times (t-1) \times 0.8^t \end{aligned}$$

solve your iteration equations.

e. Use the solution equation for C_t from part d. to compute ampicillin concentrations for times 0, 0.5, 1.0, 1.5, 2, 4, 6, and 8 hours and compare those values with the values shown in Table 12.1.

12.4 Solutions to pairs of difference equations.

The solution Equations 12.7

$$\begin{aligned} A_t &= \frac{200}{2.1} \times \left(0.4 \times 0.94^t + 1.7 \times 0.73^t\right) \\ B_t &= \frac{200}{2.1} \times \left(0.68 \times 0.94^t - 0.68 \times 0.73^t\right) \end{aligned}$$

to

$$\begin{aligned} A_0 &= 200 & A_{t+1} &= 0.77 \times A_t + 0.10 \times B_t \\ B_0 &= 0 & B_{t+1} &= 0.068 \times A_t + 0.90 \times B_t \end{aligned}$$

are presented as Twin Lightning Bolts Out of the Blue without indication of their source. We present here a method for finding solutions to all dynamical systems of the form

$$\begin{array}{lll} A_0 & \text{Given} & A_{t+1} = a_{1,1} \times A_t + a_{1,2} \times B_t \\ B_0 & \text{Given} & B_{t+1} = a_{2,1} \times A_t + a_{2,2} \times B_t \end{array} \tag{12.11}$$

There are four stages to finding the solution.

Stage 1. The Stage 1 goal is to obtain an equation that only involves the sequence A_0, A_1, A_2, $\cdots$. We eliminate B_t from the Equations 12.11 by

$$\begin{aligned} a_{2,2} \times A_{t+1} &= a_{2,2} \times a_{1,1} \times A_t + a_{2,2} \times a_{1,2} \times B_t \\ a_{1,2} \times B_{t+1} &= a_{1,2} \times a_{2,1} \times A_t + a_{1,2} \times a_{2,2} \times B_t \\ a_{2,2} \times A_{t+1} - a_{1,2} \times B_{t+1} &= a_{2,2} \times a_{1,1} \times A_t - a_{1,2} \times a_{2,1} \times A_t \end{aligned} \tag{12.12}$$

Increase the index, t, in $A_{t+1} = a_{1,1} \times A_t + a_{1,2} \times B_t$ to get

$$A_{t+2} = a_{1,1} \times A_{t+1} + a_{1,2} \times B_{t+1} \tag{12.13}$$

Eliminate $a_{1,2} \times B_{t+1}$ between Equations 12.12 and 12.13

$$\begin{aligned} a_{2,2} \times A_{t+1} - a_{1,2} \times B_{t+1} &= a_{2,2} \times a_{1,1} \times A_t - a_{1,2} \times a_{2,1} \times A_t \\ a_{1,1} \times A_{t+1} + a_{1,2} \times B_{t+1} &= A_{t+2} \\ a_{1,1} \times A_{t+1} + a_{2,2} \times A_{t+1} &= (a_{2,2} \times a_{1,1} - a_{1,2} \times a_{2,1}) \times A_t + A_{t+2} \end{aligned}$$

The last equation may be written

$$A_{t+2} - (a_{1,1} + a_{2,2}) \times A_{t+1} + (a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1}) \times A_t = 0 \tag{12.14}$$

Equation 12.14 is a *second order linear* difference equation that we write as

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0 \qquad p = a_{1,1} + a_{2,2} \qquad q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1} \tag{12.15}$$

If the first two initial values, A_0 and A_1, are known, then all subsequent values of A_t can be computed from

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

Explore 12.4.1 Do this.

a. Show that for the penicillin equations

$$\begin{array}{lll} A_0 = 200 & A_{t+1} = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} = & 0.068 \times A_t + 0.90 \times B_t \end{array}$$

$p = 1.67$ and $q = 0.6862$, so that

$$A_{t+2} - 1.67 A_{t+1} + 0.6862 A_t = 0$$

b. Show that $A_0 = 200$, $B_0 = 0$, and $A_1 = 0.77 \times A_0 + 0.10 \times B_0$ yield $A_0 = 200$ and $A_1 = 154$.

c. Compute A_2 and A_3 from

$$A_0 = 200 \qquad A_1 = 154 \qquad A_{t+2} - 1.67A_{t+1} + 0.6862A_t = 0$$ ■

Stage 2. The Stage 2 goal is to find a solution to

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

Observe that $A_t = 0$ for all time t is a solution, but is not very helpful in describing penicillin pharmacokinetics, (with $A_0 = 200$ and $A_1 = 154$) and we need a more interesting solution. Thinking optimistically, we note that the first order iteration equation

$$P_{t+1} = R \times P_t \qquad \text{has a solution:} \qquad P_t = C \times R^t$$

and we try for a similar solution

$$A_t = C \times r^t \qquad \text{to} \qquad A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

If $A_t = C \times r^t$, then $A_{t+1} = C \times r^{t+1}$ and $A_{t+2} = C \times r^{t+2}$ and we try

$$\begin{aligned} C \times r^{t+2} - p \times C \times r^{t+1} + q \times C \times r^t &= 0 \\ C \times \left(r^t \times r^2 - p \times r^t \times r^1 + q \times r^t\right) &= 0 \\ C \times r^t \times \left(r^2 - p \times r + q\right) &= 0 \end{aligned}$$

One of the factors in the previous product must be zero. The choices $C = 0$ and $r^t = 0$ yield $A_t \equiv 0$ – not helpful. Therefore we try

$$r^2 - pr + q = 0 \tag{12.16}$$

which is a quadratic equation and has (with luck) two nonzero real roots

$$r_1 = \frac{p + \sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad r_2 = \frac{p - \sqrt{p^2 - 4q}}{2}$$

For this section we assume that r_1 and r_2 are distinct and real.

We have reason to hope that we have found two solutions $A_t = C \times r_1^t$ and $A_t = C \times r_2^t$. It gets even better than that! We will show in Stage 3 that for any two numbers, C_1 and C_2,

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0 \tag{12.17}$$

Equation 12.16, $r^2 - pr + q = 0$, is called the *characteristic equation* and $r_1 = \dfrac{p + \sqrt{p^2 - 4q}}{2}$ and $r_2 = \dfrac{p - \sqrt{p^2 - 4q}}{2}$ are called the *characteristic roots* of both the difference equation

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

and (when $p = a_{1,1} + a_{2,2}$ and $q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1}$) the iteration equations

$$\begin{aligned} A_{t+1} &= a_{1,1} \times A_t + a_{1,2} \times B_t \\ B_{t+1} &= a_{2,1} \times A_t + a_{2,2} \times B_t. \end{aligned}$$

Explore 12.4.2 Do this. Find r_1 and r_2 for

$$A_0 = 200 \qquad A_{t+1} = 0.77 \times A_t + 0.10 \times B_t$$

$$B_0 = 0 \qquad B_{t+1} = 0.068 \times A_t + 0.90 \times B_t$$

Set your calculator to display at least five decimal digits.

Compare r_1 and r_2 with terms in the claimed solution

$$\begin{aligned} A_t &= \frac{200}{2.1} \times \left(0.4 \times 0.94^t + 1.7 \times 0.73^t\right) \\ B_t &= \frac{200}{2.1} \times \left(0.68 \times 0.94^t - 0.68 \times 0.73^t\right) \end{aligned} \qquad \blacksquare$$

Stage 3. We show that for any two numbers, C_1 and C_2

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0$$

$$\begin{aligned} A_{t+2} - pA_{t+1} + qA_t &= \left(C_1 \times r_1^{t+2} + C_2 \times r_2^{t+2}\right) - p \times \left(C_1 \times r_1^{t+1} + C_2 \times r_2^{t+1}\right) \\ &\quad + q \times \left(C_1 \times r_1^t + C_2 \times r_2^t\right) \\ &= C_1 \times \left(r_1^{t+2} - p \times r_1^{t+1} + q \times r_1^t\right) + C_2 \times \left(r_2^{t+2} - p \times r_2^{t+1} + q \times r_2^t\right) \\ &= C_1 \times r_1^t \times \left(r_1^2 - p \times r_1^1 + q\right) + C_2 \times r_2^t \times \left(r_2^2 - p \times r_2^1 + q\right) \\ &= C_1 \times r_1^t \times (\quad 0 \quad) + C_2 \times r_2^t \times (\quad 0 \quad) \\ &= 0 \end{aligned}$$

Whew! The 0's in the last step are because r_1 and r_2 are the roots to $r^2 - pr + q = 0$.

Stage 4. (Hang on, we are about there!) We need to evaluate C_1 and C_2 and will choose them so that the solution

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t$$

to the iteration

$$A_{t+2} - pA_{t+1} + qA_t = 0$$

matches the initial data. A_0 and B_0 are Given. From $A_1 = a_{1,1}A_0 + a_{1,2}B_0$ we can assume that A_1 is also known. For any C_1 and C_2

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t$$

is a solution and we impose

$$\begin{aligned} A_0 = C_1 \times r_1^0 + C_2 \times r_2^0 &= A_0 \quad \text{Given} \\ A_1 = C_1 \times r_1^1 + C_2 \times r_2^1 &= A_1 \quad \text{Given} \end{aligned}$$

With r_1 distinct from r_2 the equations

$$\begin{aligned} C_1 + C_2 &= A_0 \\ r_1 C_1 + r_2 C_2 &= A_1 \end{aligned} \quad \text{imply that} \quad \begin{aligned} C_1 &= \frac{A_1 - r_2 A_0}{r_1 - r_2} \\ C_2 &= \frac{r_1 A_0 - A_1}{r_1 - r_2} \end{aligned} \tag{12.18}$$

Summary. The pair of first order iteration equations

$$\begin{aligned} A_0 \quad \text{Given} \qquad A_{t+1} &= a_{1,1} \times A_t + a_{1,2} \times B_t \\ B_0 \quad \text{Given} \qquad B_{t+1} &= a_{2,1} \times A_t + a_{2,2} \times B_t \end{aligned}$$

uniquely determine A_0, A_1, $\cdots$ and B_0, B_1, $\cdots$.

If the characteristic equation

$$r^2 - pr + q = 0, \qquad \text{where} \qquad p = a_{1,1} + a_{2,2} \qquad \text{and} \qquad q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1},$$

has distinct roots r_1 and r_2, the sequence A_0, A_1, $\cdots$ is expressed by

$$A_t = \frac{A_1 - r_2 A_0}{r_1 - r_1} r_1^t + \frac{r_1 A_0 - A_1}{r_1 - r_2} r_2^t. \tag{12.19}$$

You will develop in Exercise 12.4.8 the equation

$$B_t = \frac{B_1 - r_2 B_0}{r_1 - r_2} r_1^t + \frac{r_1 B_0 - B_1}{r_1 - r_2} r_2^t. \tag{12.20}$$

Because the dynamical system, Equations 12.11, uniquely determines A_0, A_1 A_2 $\cdots$ and B_0, B_1 B_2 $\cdots$ and the sequences in Equations 12.19 and 12.20 satisfy the initial conditions and the Equations 12.11 of the dynamical system, Equations 12.19 and 12.20 are the only solutions to the dynamical system.

The method of **Stage 2** applies to any second order difference equation of the form

$$w_{t+2} - p w_{t+1} + q w_t = 0.$$

The solution is

$$w_t = C_1 r_1^t + C_2 r_2^t \qquad \text{or} \qquad w_t = C_1 r_1^t + c_2 t r_1^t$$

where r_1 and r_2 are the roots to $r^2 - p\,r + q = 0$ and the second solution applies only if $p^2 - 4q = 0$ and there is only one root, r_1.

Stage 3 has an important generalization that applies to solutions of $A_{t+2} - pA_{t+1} + qA_t = 0$ and even higher order linear difference equations.

Theorem 12.4.1 If u_t and v_t are solutions to the linear homogeneous difference equation

$$w_{t+n} + p_{n-1}w_{t+n-1} + \cdots + p_1 w_{t+1} + p_0 w_t = 0 \qquad (12.21)$$

where p_{n-1}, $\cdots$ p_1 and p_0 are constants, then for any numbers C_1 and C_2, $C_1 u_t + C_2 v_t$ is a solution to Equation 12.21.

Proof.

$$\begin{aligned} C_1(u_{t+n} + C_2 v_{t+n}) + \cdots + p_1(C_1 u_{t+1} + C_2 v_{t+1}) + p_0(C_1 u_t + C_2 v_t) &= \\ C_1(u_{t+n} + p_{n-1}u_{t+n-1} + \cdots + p_1 u_{t+1} + p_0 u_t) & \\ + \quad C_2(v_{t+n} + p_{n-1}v_{t+n-1} + \cdots + p_1 v_{t+1} + p_0 v_t) &= \\ C_1 \times (\quad 0 \quad) + C_2 \times (\quad 0 \quad) &= 0. \end{aligned}$$

End of proof.

Example 12.4.1 Find formulas for A_t and B_t if

$$\begin{array}{rclrcl} A_0 & = & 1 & A_{t+1} & = & 0.52A_t + 0.04B_t \\ B_0 & = & 2 & B_{t+1} & = & 0.24A_t + 0.4B_t \end{array}$$

Solution

$$p = 0.52 + 0.4 = 0.92, \quad q = 0.52 \cdot 0.4 - 0.24 \cdot 0.04 = 0.1984, \qquad r^2 - 0.92r + 0.1984 = 0$$

$$r_1 = \frac{0.92 + \sqrt{0.92^2 - 4 \cdot 0.1984}}{2} = 0.46 + \sqrt{0.0132}, \qquad r_2 = 0.46 - \sqrt{0.0132}$$

$$A_1 = 0.52 \cdot 1 + 0.04 \cdot 2 = 0.6, \qquad B_1 = 0.24 \cdot 1 + 0.4 \cdot 2 = 1.04.$$

$$C_1 = \frac{A_1 - r_2 A_0}{r_1 - r_2} = \frac{0.6 - (0.46 - \sqrt{0.0132}) \cdot 1}{2\sqrt{0.0132}} = 0.07/\sqrt{0.0132} + 0.5$$

$$C_2 = \frac{r_1 A_0 - A_1}{r_1 - r_2} \frac{(0.46 + \sqrt{0.132}) \cdot 1 - 0.6}{2\sqrt{0.0132}} = -0.07/\sqrt{0.0132} + 0.5$$

$$= \quad C_1\, r_1^t + C_2\, r_2^t$$

$$= \quad (0.07/\sqrt{0.0132} + 0.5)\,(0.46 + \sqrt{0.0132})^t + (-0.07/\sqrt{0.0132} + 0.5)\,(0.46 - \sqrt{0.0132})^t$$

$$D_1 = \frac{B_1 - r_2 B_0}{r_1 - r_2} = 0.06/\sqrt{0.0132} + 1 \qquad \text{See Exercise 12.4.8}$$

$$D_2 = \frac{r_1 B_0 - B_1}{r_1 - r_2} = -0.06/\sqrt{0.0132} + 1$$

$$B_t = (0.06/\sqrt{0.0132} + 1)\,(0.46 + \sqrt{0.0132})^t + (-0.06/\sqrt{0.0132} + 1)\,(0.46 - \sqrt{0.0132})^t \qquad \blacksquare$$

Exercises for Section 12.4, Solutions to pairs of difference equations.

Exercise 12.4.1 Do Explore 12.4.1 and Explore 12.4.2.

Exercise 12.4.2 Now put it all together. Show that for

$$A_0 = 200 \qquad A_{t+1} = 0.77 \times A_t + 0.10 \times B_t$$

$$B_0 = 0 \qquad B_{t+1} = 0.068 \times A_t + 0.90 \times B_t$$

$$A_t = \frac{200}{2.1} \times 0.4 \times 0.94^t + \frac{200}{2.1} \times 1.7 \times 0.73^t$$

Do exact arithmetic.

Exercise 12.4.3 Show by substitution that the proposed solution satisfies the difference equation.

a. $A_t = C \times 2^t \qquad A_{t+2} - 2.5 \times A_{t+1} + A_t = 0$

b. $A_t = C \times 0.9^t \qquad A_{t+2} - 1.3 \times A_{t+1} + 0.36 \times A_t = 0$

c. $A_t = C_1 \times 0.5^t + C_2 \times 0.7^t \qquad A_{t+2} - 1.2 \times A_{t+1} + 0.35 \times A_t = 0$

d. $A_t = C_1 + C_2 \times 0.2^t \qquad A_{t+2} - 3 \times A_{t+1} + 2 \times A_t = 0$

e. $A_t = C_1 \times 0.6^t + C_2 \times (-0.5)^t \qquad A_{t+2} - 0.1 \times A_{t+1} - 0.3 \times A_t = 0$

Exercise 12.4.4 Find the roots to the characteristic equation, $r^2 - pr + q = 0$ and write the solutions for A_t in the following systems.

a. $A_0 = 10 \qquad A_{t+1} = 0.8A_t + 0.2B_t$
$B_0 = 0 \qquad B_{t+1} = 0.1A_t + 0.7B_t$

b. $A_0 = 0 \qquad A_{t+1} = 0.6A_t + 0.3B_t$
$B_0 = 5 \qquad B_{t+1} = 0.2A_t + 0.7B_t$

c. $A_0 = 1 \qquad A_{t+1} = 0.26A_t + 0.04B_t$
$B_0 = 1 \qquad B_{t+1} = 0.06A_t + 0.24B_t$

d. $A_0 = 2 \qquad A_{t+1} = 1.04A_t + 0.16B_t$
$B_0 = 3 \qquad B_{t+1} = 0.24A_t + 0.96B_t$

e. $A_0 = 20 \qquad A_{t+1} = 0.86A_t + 0.04B_t$
$B_0 = 10 \qquad B_{t+1} = 0.06A_t + 0.84B_t$

Exercise 12.4.5 Find the solutions to

$$\begin{array}{llllllll}
\text{a.} & w_0 & = & 3 & w_1 & = & 1 & w_{t+2} - 5w_{t+1} + 6w_t = 0 \\
\text{b.} & w_0 & = & 0 & w_1 & = & 0 & w_{t+2} + 8w_{t+1} + 12w_t = 0 \\
\text{c.} & w_0 & = & 2 & w_1 & = & 1 & w_{t+2} - 6w_{t+1} + 8w_t = 0 \\
\text{d.} & w_0 & = & 1 & w_1 & = & 1 & w_{t+2} - 5w_{t+1} + 4w_t = 0
\end{array}$$

Exercise 12.4.6 Show that if r_1 is zero (or r_2 is zero), then $q = 0$. In this case, Equation 12.15, $A_{t+2} - pA_{t+1} + qA_t = 0$, is $A_{t+2} - pA_{t+1} = 0$ which is first order and easy to solve.
It is curious that when $q = 0$, $A_1 - pA_0$ may not be zero, but $A_{t+1} - pA_t = 0$ for $t \geq 1$.
Write an equation for A_t in terms of A_1 and p.
Solve the systems for A_t, $t = 0, 1, 2, \cdots$.

$$\begin{array}{llllll}
\text{a.} & A_0 = 2 & A_{t+1} = 0.3A_t + 0.6B_t \\
 & B_0 = 1 & B_{t+1} = 0.2A_t + 0.4B_t \\
\\
\text{b.} & A_0 = 0 & A_{t+1} = 0.6A_t + 0.3B_t \\
 & B_0 = 5 & B_{t+1} = 0.2A_t + 0.1B_t
\end{array}$$

Exercise 12.4.7 Solve for B_t.
For the initial conditions and iteration equations:

$$\begin{array}{llllll}
A_0 & = & 2 & A_{t+1} & = & 0.3A_t + 0.1B_t \\
\\
B_0 & = & 5 & B_{t+1} & = & 0.1A_t + 0.3B_t
\end{array}$$

a. Eliminate A_t by subtraction $(0.1A_{t+1} - 0.3B_{t+1})$.

b. Explain why $B_{t+2} = 0.1A_{t+1} + 0.3B_{t+1}$.

c. Use the two equations from parts a. and b. to write

$$B_{t+2} - 0.6B_{t+1} + 0.08B_t = 0$$

d. Suppose that for some number, r, $B_t = Cr^t$. Show that $C \neq 0$, $r \neq 0$ imply that $r^2 - 0.6r + 0.08 = 0$.

e. Find the roots of $r^2 - 0.6r + 0.08 = 0$.

f. Show that for any two numbers, C_1 and C_2,

$$B_t = C_1 \times 0.2^t + C_2 \times 0.4^t \qquad \text{solves} \qquad B_{t+2} - 0.6B_{t+1} + 0.08B_t = 0.$$

g. Compute B_1.

h. Show that

$$B_t = 1.5 \times 0.2^t + 3.5 \times 0.4^t$$

Exercise 12.4.8 Follow the steps of Exercise 12.4.7 to find an equation for B_t satisfying:

$$\begin{array}{rclrcl} A_0 & = & \text{Given} & A_{t+1} & = & a_{1,1}A_t + a_{1,2}B_t \\ B_0 & = & \text{Given} & B_{t+1} & = & a_{2,1}A_t + a_{2,2}B_t \end{array}$$

Stepping stones will be

$$\begin{aligned} a_{1,1}B_t - a_{2,1}A_t &= (a_{1,1}\, a2,2 - a_{2,1}\, a_{1,2})B_t \\ B_{t+2} &= a_{2,1}A_{t+1} + a_{2,2}B_{t+1} \\ B_{t+2} - (a_{1,1} + a_{2,2})B_{t+1} + (a_{1,1}\, a_{2,2} - a_{2,1}\, a_{1,2})B_t &= 0 \\ B_t &= D_1 \times r_1^t + D_2 \times r_2^t \end{aligned}$$

where r_1 and r_2 are roots of $r^2 - (a_{1,1} + a_{2,2}) + (a_{1,1}\, a2,2 - a_{2,1}\, a_{1,2}) = 0$ and

$$D_1 = \frac{B_1 - r_2 B_0}{r_1 - r_2} \qquad D_2 = \frac{r_1 B_0 - B_1}{r_1 - r_2}$$

Assume r_1 and r_2 are two real numbers.

Exercise 12.4.9 Show that for

$$\begin{array}{lrcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array}$$

$$B_t = \frac{200}{2.1} \times 0.68 \times 0.94^t - \frac{200}{2.1} \times 0.68 \times 0.73^t$$

Exercise 12.4.10 Find solutions for both A_t and B_t satisfying

$$\begin{array}{lrcl} A_0 = 10 & A_{t+1} & = & 0.50A_t + 0.2B_t \\ B_0 = 0 & B_{t+1} & = & 0.15B_t + 0.7B_t \end{array}$$

$$\begin{array}{lrcl} A_0 = 5 & A_{t+1} & = & 0.6A_t + 0.1B_t \\ B_0 = 10 & B_{t+1} & = & 0.2A_t + 0.7B_t \end{array}$$

12.5 Roots equal to zero, multiple roots, and complex roots

.

We examine special cases of the dynamical system

$$A_0 \quad \text{Given} \qquad A_{t+1} = a_{1,1} \times A_t + a_{1,2} \times B_t$$

$$B_0 \quad \text{Given} \qquad B_{t+1} = a_{2,1} \times A_t + a_{2,2} \times B_t$$

for which the roots are zero, repeated, or complex and consider the $\lim_{t\to\infty} A_t$ and $\lim_{t\to\infty} B_t$. As in the previous section, $A_{t+2} - pA_{t+1} + qA_t = 0$ and $B_{t+2} - pB_{t+1} + qB_t = 0$ for $p = a_{1,1} + a_{2,2}$ and $q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1}$.

The case of a zero root. It is a generally useful result that:

Theorem 12.5.1 If r_1 and r_2 are the roots to $x^2 - px + q = 0$ then

$$r_1 + r_2 = p \qquad \text{and} \qquad r_1 \times r_2 = q. \tag{12.22}$$

(The result is valid even in the case that $p^2 - 4q < 0$ and $\sqrt{p^2 - 4q}$ is written $\sqrt{-1}\,\sqrt{4q - p^2} = i\,\sqrt{4q - p^2}$.)

Proof.

$$r_1 + r_2 = \frac{p + \sqrt{p^2 + 4q}}{2} + \frac{p - \sqrt{p^2 - 4q}}{2} = \frac{p + \sqrt{p^2 - 4q} + p - \sqrt{p^2 - 4q}}{2} = p$$

$$r_1 \times r_2 = \frac{p + \sqrt{p^2 - 4q}}{2} \times \frac{p - \sqrt{p^2 - 4q}}{2} = \frac{p^2 - (p^2 - 4q)}{4} = q$$

End of proof.

The case of a root equal to zero, $\mathbf{r_2 = 0}$. If the characteristic equation, $r^2 - pr + q = 0$ has a root equal to zero, $r_2 = 0$, then $q = r_1 \times r_2 = 0$ and both

$$A_{t+2} = p \times A_{t+1} \qquad A_t = A_1 \times p^{t-1} \quad \text{for} \quad t \geq 1$$

$$B_{t+2} = p \times B_{t+1} \qquad B_t = B_1 \times p^{t-1} \quad \text{for} \quad t \geq 1.$$

A_1 and B_1 are computed from $A_1 = a_{1,1}A_0 + a_{1,2}B_0$ and $B_1 = a_{2,1}A_0 + a_{2,2}B_0$, respectively. It may not be that $A_1 = p \times A_0$.

The case of repeated roots, $\mathbf{r_1 = r_2}$. Biological systems are unlikely to be so finely tuned that $r_1 = r_2$. However, there is a simple solution. If $r_1 = r_2$, because $r_1 + r_2 = p$, the repeated root, $r_1 = \frac{p}{2}$.

In Exercise 12.5.1 you are asked to show that when $r_1 = r_2$ is a repeated root,

$$A_t = t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0.$$

and for any two numbers C_1 and C_2,

$$A_t = C_1 r_1^t + C_2 \times t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0.$$

We next show that every solution of $A_{t+2} - pA_{t+1} + qA_t = 0$ is of the form

$$A_t = C_1 r_1^t + C_2 \times t \times r_1^t.$$

To see this, we assume that A_0 and B_0 are known and $A_1 = a_{1,1}A_0 + a_{1,2}B_0$ has been computed. Then we solve

$$\begin{aligned} A_0 &= C_1 r_1^0 + C_2 \times 0 \times r_1^0 \\ A_1 &= C_1 r_1^1 + C_2 \times 1 \times r_1^1 \end{aligned}$$

for C_1 and C_2. Assume that $r_1 \neq 0$. Then $C_1 = A_0$ and $C_2 = \dfrac{A_1 - A_0 r_1}{r_1}$ are uniquely determined. In the case that $r_1 = 0$, repeated root of zero, then both $p = 0$ and $q = 0$ and $A_t = 0$ and $B_t = 0$ for $t \geq 2$.

The case $\mathbf{r}_1 = a + bi$, $\mathbf{i} = \sqrt{-1}$. An interesting and important case occurs when the discriminant $p^2 - 4q < 0$ and the roots $r_1 = a + bi$ and $r_2 = a - bi$ are complex numbers. The same formulas

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad C_1 = \frac{A_1 - r_2 A_0}{r_1 - r_2} \qquad C_2 = \frac{r_1 A_0 - A_1}{r_1 - r_2}$$

are valid, but are painful to work with. Normally, A_0, A_1, p and q are real so that all of A_2, A_3, $\cdots$ are real. Because r_1 and r_2 are complex, it is difficult to see that the previous formulas define real values for A_t (but they do).

Such systems arise, for example in Section 12.10.3 when considering equilibrium points of predator-prey systems. The following system is similar to Equation 12.76 of that section.

$$\begin{aligned} \xi_0 &= 0 & \xi_{t+1} &= 0.98\,\xi_t + 0.08\,\eta_t \\ \eta_0 &= 1 & \eta_{t+1} &= -0.13\,\xi_t + 0.94\,\eta_t \end{aligned} \tag{12.23}$$

ξ_t and η_t are linear approximations to departure from equilibrium of a predator and prey, respectively, and the initial conditions reflect an excess of prey. The roots to the characteristic equation, $p^2 - 1.92p + 0.9316 = 0$ are $r_1 = 0.96 + 0.10\,i$ and $r_1 = 0.96 - 0.10\,i$.

Two graphs of ξ_t and η_t are shown in Figure 12.5. Both graphs illustrate the periodic variation associated with complex roots and with predator-prey systems. With an excess of prey at time 0, the predator population increases causing a decrease in the prey population to a level below the equilibrium population, which is followed by a decrease in the prey population.

Alert: Incoming Bolt Out of the Blue. For the difference equation

$$A_0 \quad A_1 \qquad \text{Given,} \qquad A_{t+2} - pA_{t+1} + qA_t = 0 \qquad p^2 - 4q < 0$$

let

$$a = \frac{p}{2} \qquad b = \frac{\sqrt{4q - p^2}}{2} \qquad \rho = \sqrt{a^2 + b^2} \qquad \theta = \arccos\frac{a}{\rho}$$

Then the solution A_t is given by

$$A_t = C_1 \rho^t \cos t\theta + C_2 \rho^t \sin t\theta \qquad C_1 = A_0 \qquad C_2 = \frac{A_1 - aA_0}{b}. \tag{12.24}$$

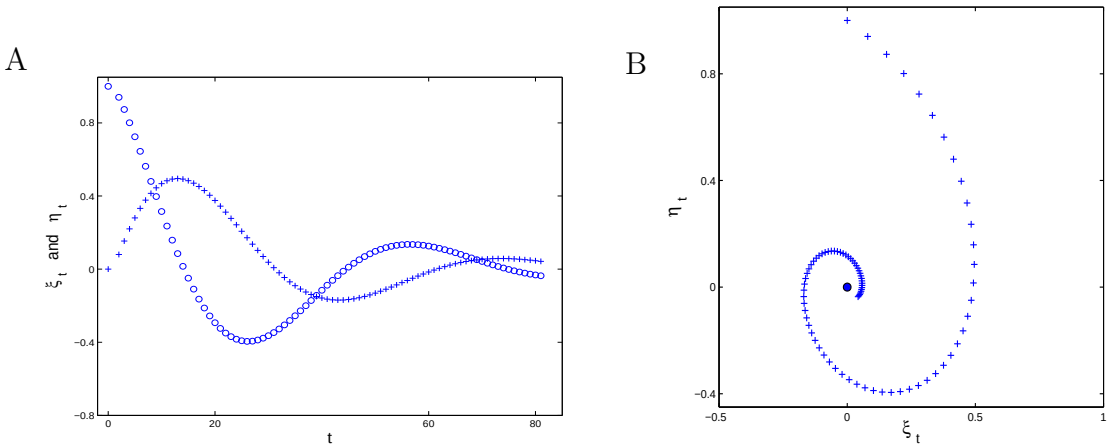


Figure 12.5: Graphs of the dynamical system 12.23. A. Graphs of the excess predator ξ_t (+) and excess prey η_t (o) versus time. B. Graph of η_t versus ξ_t.

Similarly, the solution B_t in Equations 12.11 is given by

$$B_t = D_1\rho^t \cos t\theta + D_2\rho^t \sin t\theta \qquad D_1 = B_0 \qquad D_2 = \frac{B_1 - aB_0}{b}. \tag{12.25}$$

All numbers used in Equation 12.24 and 12.25 are real, so the formulas specify real numbers for A_t and B_t. Readers experienced with complex arithmetic will recognize that the solution is related to De Moivre's formula

$$(a+bi)^n = \rho^n(\cos n\theta + i \sin n\theta)$$

For the system 12.23, the roots are $a \pm b\,i = 0.96 \pm 0.1\,i$, $\xi_0 = 0$ and $\xi_1 = 0.98\xi_0 + 0.08\eta_0 = 0.08$. Then

$$C_1 = \xi_0 = 0 \qquad C_2 = \frac{\xi_1 - a\,\xi_0}{b} = \frac{0.08 - 0.1 \times 0}{0.96} = 1/12$$

$$\rho = \sqrt{0.96^2 + 0.1^2} = \sqrt{0.9316} \doteq 0.965 \qquad \theta = \arccos\frac{a}{\rho} \doteq 0.104 \quad \text{radians.}$$

Then from Equation 12.24

$$\xi_t = C_1\rho^t \cos t\theta + C_2\rho^t \sin t\theta = \frac{1}{12} \times 0.965^t \times \sin(t \times 0.104).$$

We next show that

$\mathbf{A_t = \rho^t \cos t\theta}$ **solves** $\mathbf{A_{t+2} - pA_{t+1} + qA_t = 0}$ **when** $\mathbf{p^2 - 4q < 0}$

We recall

$$a = \frac{p}{2} \qquad b = \frac{\sqrt{4q - p^2}}{2} \qquad \rho = \sqrt{a^2 + b^2} \qquad \theta = \arccos\frac{a}{\rho}$$

and note that

$$\rho^2 = a^2 + b^2 = \left(\frac{p}{2}\right)^2 + \left(\frac{\sqrt{4q - p^2}}{2}\right)^2 = \frac{p^2}{4} + \frac{4q - p^2}{4} = q \tag{12.26}$$

and that $\cos\theta = \frac{a}{\rho} = \frac{p}{2\rho}$ We also recall a trigonometric identity

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

Substitute $A_t = \rho^t \cos t\theta$ into $A_{t+2} - pA_{t+1} + qA_t$ and get

$$\begin{aligned}
\rho^{t+2}\cos((t+2)\theta) - p\rho^{t+1}\cos((t+1)\theta) + q\rho^t\cos(t\theta) &= \\
\rho^t \times \left(\rho^2\cos\left((t+1)\theta+\theta\right) - p\rho\cos((t+1)\theta) + q\cos\left((t+1)\theta-\theta\right)\right) &= \\
\rho^t \times \left(\rho^2\cos\left((t+1)\theta+\theta\right) - p\rho\cos((t+1)\theta) + \rho^2\cos\left((t+1)\theta-\theta\right)\right) &= \\
\rho^t \times \left(\rho^2\left(\cos\left((t+1)\theta+\theta\right) + \cos\left((t+1)\theta-\theta\right)\right) - p\rho\cos((t+1)\theta)\right) &= \\
\rho^t \times \left(\rho^2 2\cos\left((t+1)\theta\right)\cos(\theta) - p\rho\cos((t+1)\theta)\right) &= \\
\rho^t \times \left(\rho^2 2\cos\left((t+1)\theta\right)\frac{p}{2\rho} - p\rho\cos((t+1)\theta)\right) &= 0
\end{aligned}$$

Whew!

We suggest in Exercise 12.5.5 that you show that $A_t = \rho^t \sin t\theta$ also solves $A_{t+2} - pA_{t+1} + qA_t = 0$. Then it will follow from Theorem 12.4.1 that for any two numbers C_1 and C_2 that $A_t = C_1\rho^t \sin t\theta + C_2\rho^t \cos t\theta$ solves $A_{t+2} - pA_{t+1} + qA_t = 0$.

Exercises for Section 12.5, Roots equal to zero, multiple roots, and complex roots.

Exercise 12.5.1

a. Show that if $r_1^2 - pr_1 + q = 0$ and $p^2 - 4q = 0$ then

$$A_t = t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0$$

You will need to show that

$$(t+2) \times r_1^{t+2} - p \times (t+1) \times r_1^{t+1} + q \times t \times r_1^t = 0 \qquad \text{for all } t$$

Remember that $r_1 = \frac{p}{2}$.

b. As in previous work, $A_t = r_1^t$ solves $A_{t+2} - pA_{t+1} + qA_t = 0$. Use Theorem 12.4.1 to show that for any two numbers, C_1 and C_2

$$A_t = C_1 r_1^t + C \times t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0 \tag{12.27}$$

c. Show that

$$C_1 = A_0 \qquad \text{and} \qquad C_2 = A_1/r_1 - A_0 \tag{12.28}$$

Exercise 12.5.2 Use formulas 12.24 and 12.25 to show that for the iteration equations

$$\begin{aligned}
A_0 = 1 \quad A_{t+1} &= A_t - B_t \\
B_0 = 0 \quad B_{t+1} &= A_t + B_t
\end{aligned}$$

$$A_t = \left(\sqrt{2}\right)^t \cos\left(t\frac{\pi}{4}\right) \qquad \text{and} \qquad B_t = \left(\sqrt{2}\right)^t \sin\left(t\frac{\pi}{4}\right)$$

Exercise 12.5.3 Solve the following systems.

a. $\begin{array}{rcl} A_0 &=& 1 \\ B_0 &=& 2 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.52A_t + 0.04B_t \\ B_{t+1} &=& 0.24A_t + 0.4B_t \end{array}$

b. $\begin{array}{rcl} A_0 &=& 1 \\ B_0 &=& 0 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.3A_t + 0.9B_t \\ B_{t+1} &=& 0.2A_t + 0.6B_t \end{array}$

c. $\begin{array}{rcl} A_0 &=& 3 \\ B_0 &=& 2 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.3A_t - 0.5B_t \\ B_{t+1} &=& 0.2A_t + 0.1B_t \end{array}$

d. $\begin{array}{rcl} A_0 &=& 2 \\ B_0 &=& 3 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.5A_t - 0.1B_t \\ B_{t+1} &=& 0.1A_t + 0.3B_t \end{array}$

e. $\begin{array}{rcl} A_0 &=& 4 \\ B_0 &=& 5 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 3.0A_t - 0.5B_t \\ B_{t+1} &=& 2.0A_t + 1.0B_t \end{array}$

f. $\begin{array}{rcl} A_0 &=& 0 \\ B_0 &=& 1 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.42A_t + 0.04B_t \\ B_{t+1} &=& 0.24A_t + 0.38B_t \end{array}$

g. $\begin{array}{rcl} A_0 &=& 0 \\ B_0 &=& -1 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.62A_t - 0.08B_t \\ B_{t+1} &=& 0.48A_t + 0.18B_t \end{array}$

h. $\begin{array}{rcl} A_0 &=& 1 \\ B_0 &=& 3 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.5A_t - 1.00B_t \\ B_{t+1} &=& 0.1A_t + 0.30B_t \end{array}$

i. $\begin{array}{rcl} A_0 &=& 1 \\ B_0 &=& 4 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 3A_t + 6B_t \\ B_{t+1} &=& 2A_t + 4B_t \end{array}$

j. $\begin{array}{rcl} A_0 &=& 3 \\ B_0 &=& 4 \end{array} \quad \begin{array}{rcl} A_{t+1} &=& 0.3A_t + 0.6B_t \\ B_{t+1} &=& 0.6A_t + 0.3B_t \end{array}$

Exercise 12.5.4 Write computer code that solve all of the problems in Exercise 12.5.3.

Exercise 12.5.5 For the trigonometrically strong. Show that

$$A_t = C\rho^t \sin t\theta \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + aA_t = 0$$

You may wish to know that $\sin(x+y) + \sin(x-y) = 2\sin x \cos y$

12.6 Matrices

Matrices greatly simplify study of dynamical systems such as

$$A_0 = 200 \quad A_{t+1} = 0.77 \times A_t + 0.10B_t$$

$$B_0 = 0 \quad B_{t+1} = 0.068 \times A_t + 0.90 \times B_t$$

A matrix is a rectangular array of numbers. We use only 2 by 2 and 2 by 1 arrays of the form

$$\text{2 by 2} \qquad \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \qquad \text{and 2 by 1} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

For larger matrices, an m by n matrix has m rows and n columns and $a_{i,j}$ is the entry in the $i\underline{\text{th}}$ row and $j\underline{\text{th}}$ column. Always the row dimension or row index is listed first. An example of a 4 by 6 matrix, A, is

$$A = \begin{bmatrix} 85 & 37 & 60 & 61 & -8 & -2 \\ -1 & 70 & -70 & -88 & 37 & 12 \\ 45 & -99 & -70 & 54 & -64 & 93 \\ 67 & 80 & 55 & 34 & -22 & 62 \end{bmatrix} \qquad A_{2,4} = -88$$

Matrices of the same dimension can be added

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \end{bmatrix}$$

$$\text{But} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{is not defined}$$

A matrix may be multiplied by a number, α.

$$\alpha \times \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} \alpha \times a_{1,1} & \alpha \times a_{1,2} \\ \alpha \times a_{2,1} & \alpha \times a_{2,2} \end{bmatrix} \qquad \alpha \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \times c_1 \\ \alpha \times c_2 \end{bmatrix}$$

An n by k matrix A can multiply a k by m matrix, B. The product, $A \times B$ is an n by m matrix. (n by $\mathbf{k}$) $\times$ ($\mathbf{k}$ by m) $= n$ by m.

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_{1,1} \times c_1 + a_{1,2} \times c_2 \\ a_{2,1} \times c_1 + a_{2,2} \times c_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \times \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} \times b_{1,1} + a_{1,2} \times b_{2,1} & a_{1,1} \times b_{1,2} + a_{1,2} \times b_{2,2} \\ a_{2,1} \times b_{1,1} + a_{2,2} \times b_{2,1} & a_{2,1} \times b_{1,2} + a_{2,2} \times b_{2,2} \\ a_{3,1} \times b_{1,1} + a_{3,2} \times b_{2,1} & a_{3,1} \times b_{1,2} + a_{3,2} \times b_{2,2} \end{bmatrix}$$

It is said that 'the rows of the matrix on the left multiply the columns of the matrix on the right.' In order for this to work, the number of columns in the matrix on the left (k) must be the same as the number of rows in the matrix on the right (also k).

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \times \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \qquad \text{is not defined: (2 by 1) times (2 by 2)}$$

The following algebraic rules apply as long as the dimensions of the matrices conform:

$$\begin{aligned} A + (B + C) &= (A + B) + C && \text{Addition is associative.} \\ A + B &= B + A && \text{Addition is commutative.} \\ A \times (B \times C) &= (A \times B) \times C && \text{Multiplication is associative.} \\ A \times (B + C) &= A \times B + A \times C && \text{Mult distributes over addition.} \end{aligned}$$

Missing from the list is $A \times B = B \times A$ and

$$\text{In most cases} \qquad A \times B \qquad \text{is not} \qquad B \times A$$

For example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ 15 & -22 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -9 & -10 \end{bmatrix}$$

The matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is called the **identity matrix** because for any 2 by 2 matrix, A,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

The determinant of a 2 by 2 matrix A is $\det A$ where

$$\det A = \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \tag{12.29}$$

If $\det A \neq 0$ then A^{-1}, called **A inverse**, is defined for 2 by 2 matrices by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}. \tag{12.30}$$

This formula only works for 2 by 2 matrices.

A^{-1} has the property that

$$A^{-1} \times A = A \times A^{-1} = I$$

The *characteristic roots* of a 2 by 2 matrix, $A = [a_{i,j}]$, are the roots to the *characteristic polynomial*

$$r^2 - (a_{1,1} + a_{2,2})r + a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 0 \tag{12.31}$$

The diagonal of a matrix

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}$$

consists of the entries $m_{1,1}$ $m_{2,2}$ and the trace of M is $m_{1,1} + m_{2,2}$. If $m_{2,1} = m_{1,2} = 0$, M is said to be a diagonal matrix. If $m_{2,1} = 0$, M is said to be upper triangular. I suspect you can guess what a lower triangular matrix is.

12.6.1 Iteration equations with matrices.

In the system

$$\begin{array}{lrcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array}$$

we let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix} \qquad X_t = \begin{bmatrix} A_t \\ B_t \end{bmatrix} \qquad t = 0, 1, 2, \cdots$$

Then by A is 2 by 2 and X_t is 2 by 1 and

$$A \times X_t = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix} \times \begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 0.77 \times A_t + 0.10 \times B_t \\ 0.068 \times A_t + 0.90 \times B_t \end{bmatrix} = \begin{bmatrix} A_{t+1} \\ B_{t+1} \end{bmatrix} = X_{t+1}$$

The equation

$$X_{t+1} = A \times X_t$$

can be cascaded just like its scalar cousin

$$\begin{aligned}
X_1 &= A \times X_0 \\
\\
X_2 &= A \times X_1 = A \times (A \times X_0) \\
&= (A \times A) \times X_0 && \text{Mult is associative.} \\
\\
X_3 &= A \times X_2 = A \times ((A \times A) \times X_0) \\
&= (A \times A \times A) \times X_0 && \text{Mult is associative.}
\end{aligned}$$

We quite reasonably define for a square matrix, A,

$$A^t = \underbrace{A \times A \times \cdots \times A}_{\text{t factors}}$$

and claim that

$$X_t = A^t \times X_0$$

Exercises for Section 12.6, Matrices.

Exercise 12.6.1 For each matrix, A, compute the characteristic roots of A using Equation 12.31 and compute A^{-1} using Equations 12.29 and 12.30 and compute $A \times A^{-1}$ and $A^{-1} \times A$.

a. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ d. $\begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$

Exercise 12.6.2 Show that the roots of the upper triangular matrix

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

are a and b.

Exercise 12.6.3 For

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

use your calculator or a computer to compute:

a.	$A + B$	b.	$B + A$	c.	$A + C$	d.	$C + I$
e.	$A + I$	f.	$A \times B$	g.	$B \times A$	h.	$A \times C$
i.	$C \times A$	j.	$A \times I$	k.	$A \times A^{-1}$	l.	$B \times B^{-1}$
m.	$A^{-1} \times C$	n.	$A \times A$	o.	A^2	p.	$B \times A^{-1}$
q.	e^A	r.	$e^A \times e^{(-A)}$	s.	e^I	t.	$\sin A$

Exercise 12.6.4 Define the matrices

$$A = \begin{bmatrix} 1.2 & -0.6 \\ 0.4 & 0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 1.2 & 0.4 \\ 0.8 & 0.4 \end{bmatrix} \qquad C = \begin{bmatrix} 2.5 & -2.5 \\ -5.0 & 7.5 \end{bmatrix} \qquad L = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.6 \end{bmatrix}$$

a. Compute the characteristic roots of A.

b. Use pencil and paper to show that $B \times C = I$.

c. Use pencil and paper to show that $B \times L \times C = A$.

d. Use pencil and paper to compute L^2.

e. Use pencil and paper to show that $B \times L^2 \times C = A^2$.

f. Show that $B \times L^5 \times C = A^5$.

Exercise 12.6.5 In Problem 12.2.6 for time = 120 minutes, $t = 24$, you found that

$$A_{24} = 8.71 \qquad B_{24} = 14.6$$

Enter

$$D = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix} \qquad X = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

in your calculator and compute $D^{24} \times X$. You should get $\begin{bmatrix} 8.713 \\ 14.635 \end{bmatrix}$.

Exercise 12.6.6 Equations 12.7 found earlier for A_t and B_t

$$\begin{aligned} A_t &= \tfrac{200}{2.1} \times (0.4 \times 0.94^t + 1.7 \times 0.73^t) \\ B_t &= \tfrac{200}{2.1} \times (0.68 \times 0.94^t - 0.68 \times 0.73^t) \end{aligned}$$

can be written as

$$X_t = \frac{200}{2.1} \begin{bmatrix} 0.4 \\ 0.68 \end{bmatrix} \times 0.94^t + \frac{200}{2.1} \begin{bmatrix} 1.7 \\ -0.68 \end{bmatrix} \times 0.73^t$$

Let

$$D = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.40 & 1.70 \\ 0.68 & -0.68 \end{bmatrix} \qquad L = \begin{bmatrix} 0.94 & 0.00 \\ 0.00 & 0.73 \end{bmatrix} \qquad X_0 = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

a. Compute the characteristic roots of D (using Equation 12.31).

b. Compute $D\,E - E\,L$.

c. Use algebra to show that $D = E\,L\,E^{-1}$.

d. Use algebra to show that $D^2 = (E\,L\,E^{-1})\,(E\,L\,E^{-1}) = E\,L^2\,E^{-1}$.

e. Show that $D^t = E \times L^t \times E^{-1}$.

f. Use a calculator or computer to compute $D^{24}\, X_0 = X_{24}$ and $E \times L^t \times E^{-1} X_0$. You should get $\begin{bmatrix} 8.713 \\ 14.635 \end{bmatrix}$.

g. Evaluate $\lim_{t\to\infty} D^t = \lim_{t\to\infty} E \times L^t \times E^{-1}$.

12.7 Equilibria of Pairs of Difference Equations.

Penicillin kinetics. Suppose you initiate constant penicillin infusion in a patient at the rate of 32.4 μg/ml-serum every five minutes. What will be the serum penicillin concentration as time progresses? (Does your patient explode with too much penicillin?) Assume the kinetics of penicillin are as in Equations 12.6

$$\begin{aligned} A_0 = \_\_\_ \qquad A_{t+1} - A_t &= -0.23 \times A_t + 0.10 \times B_t \\ B_0 = \_\_\_ \qquad B_{t+1} - B_t &= 0.068 \times A_t - 0.10 \times B_t \end{aligned} \tag{12.32}$$

In these equations, A_t denotes serum penicillin concentration in μg/ml and B_t denotes tissue penicillin concentration.

Equations 12.32 account for body processing of penicillin, but do not account for your infusion of 32.4 μg/ml-serum. Because all of the infused penicillin enters the serum pool, we write a model:

> **Mathematical Model.** Every five minutes, 6.8% of the serum penicillin diffuses to tissue, 10% of the tissue penicillin diffuses to serum, the kidneys remove 16.2 percent of the serum penicillin, and 32.4 μg/ml-serum penicillin is added to the serum.

We assume your patient had no penicillin in his body when you initiated treatment, and write:

Initial Conditions	**InternalBody Kinetics**				**Infusion**
$A_0 = 0$	$A_{t+1} - A_t$	$=$	$-(0.162 + 0.068) \times A_t + 0.10 \times B_t$	$+$	32.4
$B_0 = 0$	$B_{t+1} - B_t$	$=$	$0.068 \times A_t - 0.10 \times B_t$		

(12.33)

The **equilibrium** state of your patient, if such exists, will be described by two numbers, A_e and B_e, such that

$$A_t \to A_e \qquad \text{and} \qquad B_t \to B_e \qquad \text{as} \qquad t \to \infty$$

Explore 12.7.1 Assume that your patient has reached an equilibrium state so that the levels of serum and tissue penicillin are constants, A_e and B_e. Which of the following would you expect,

$$A_e < B_e, \qquad A_e = B_e, \qquad \text{or} \qquad A_e > B_e?$$

■

Because $A_t \to A_e$, it must be that $A_{t+1} \to A_e$ also, and similarly $B_t \to B_e$ implies $B_{t+1} \to B_e$. From the kinetic equations, we can write

$$\begin{aligned} A_e - A_e &= -0.23A_e + 0.1B_e + 32.4 \\ B_e - B_e &= 0.068A_e - 0.10B_e \end{aligned}$$

$$\begin{aligned} 0 &= -0.23A_e + 0.1B_e + 32.4 \\ 0 &= 0.068A_e - 0.10B_e \end{aligned}$$

$$\begin{aligned} 0 &= -0.23A_e + 0.1B_e + 32.4 \\ 0.10B_e &= 0.068A_e \qquad\qquad B_e = 0.68A_e \end{aligned}$$

$$0 = -0.23A_e + 0.1(0.68)B_e + 32.4$$

$$\begin{aligned} A_e &= 200 \quad \mu\text{g/ml} \\ B_e &= 136 \quad \mu\text{g/ml} \end{aligned}$$

We thus find the equilibrium values for your patient, and find a not very intuitive result that the tissue penicillin concentration at equilibrium is less than (68% of) the serum penicillin concentration.

Exercises for Section 12.7, Equilibria of Pairs of Difference Equations.

Exercise 12.7.1 Determine the equilibrium state, A_e,B_e, of your patient for each of the following modifications of the penicillin system. Contrast these results with the results of the original model and discuss how they affect your thinking about treatment of your patient.

a. Your patient had been receiving penicillin before your treatment so that $A_0 = 100$ and $B_0 = 68$.

b. Your patient's kidneys are only 50% effective so that every five minutes they remove 8.1% of the penicillin in the serum (and all other kinetics are the same).

c. The model of penicillin kinetics is

> **Mathematical Model.** Every five minutes, 15% of the serum penicillin diffuses to tissue, 10% of the tissue penicillin diffuses to serum, the kidneys remove 8 percent of the serum penicillin, and 32 μg/ml-serum penicillin is added to the serum.

Exercise 12.7.2 Consider the SIR equations for epidemics.

$$\begin{aligned} S_{t+1} &= S_t - \beta \times S_t \times I_t \\ I_{t+1} &= I_t + \beta \times S_t \times I_t - \gamma \times S_{t-d} \times I_{t-d} \\ R_{t+1} &= R_t + \gamma \times S_{t-d} \times I_{t-d}. \end{aligned}$$

Find the possible equilibrium values (S_e, I_e, R_e).

12.8 Stability of the equilibria of linear systems.

There is a radical difference between

$$\text{Case 1.} \qquad A_t = 5 \times \left(\frac{2}{3}\right)^t + 8 \times \left(\frac{4}{5}\right)^t \qquad \text{for which} \qquad \lim_{t\to\infty} A_t = 0$$

$$\text{Case 2.} \qquad A_t = 5 \times \left(\frac{3}{2}\right)^t + 8 \times \left(\frac{4}{5}\right)^t \qquad \text{for which} \qquad \lim_{t\to\infty} A_t = \infty$$

If A_t is the amount of penicillin in your patient, you expect Case 1 (at least not Case 2). If A_t is the size of your fish population, you hope for Case 2 (at least not Case 1).

Definition 12.8.1 An equilibrium point of a linear dynamical system

$$\begin{aligned} A_0 &= \text{Given} \quad A_{t+1} = m_{1,1}A_t + m_{1,2}A_t + H_1 \\ B_0 &= \text{Given} \quad B_{t+1} = m_{2,1}B_t + m_{2,2}B_t + H_2 \end{aligned} \tag{12.34}$$

is a point (A_e, B_e) such that

$$\begin{aligned} A_e &= m_{1,1}A_e + m_{1,2}A_e + H_1 \\ B_e &= m_{2,1}B_e + m_{2,2}B_e + H_2 \end{aligned} \tag{12.35}$$

The question of this section is whether (A_e, B_e) is *stable*.

Definition 12.8.2 An equilibrium point (A_e, B_e) of the linear dynamical system 12.34 is *stable* means that for *any* (A_0, B_0) the sequences $A_0, A_1, A_2, \cdots$ and $B_0, B_1, B_2 \cdots$ defined by Equations 12.34 converge respectively to A_e and B_e.

By subtracting Equations 12.35 from Equations 12.34 we get

$$A_{t+1} - A_e = m_{1,1} \times (A_t - A_e) + m_{1,2} \times (B_t - B_e)$$

$$B_{t+1} - B_e = m_{2,1} \times (A_t - A_e) + m_{2,2} \times (B_t - B_e).$$

With the substitution $x_t = A_t - A_e$ and $y_t = B_t - B_e$, we get

$$\begin{aligned} x_{t+1} &= m_{1,1}x_t + m_{1,2}x_t \\ y_{t+1} &= m_{2,1}y_t + m_{2,2}y_t \end{aligned} \tag{12.36}$$

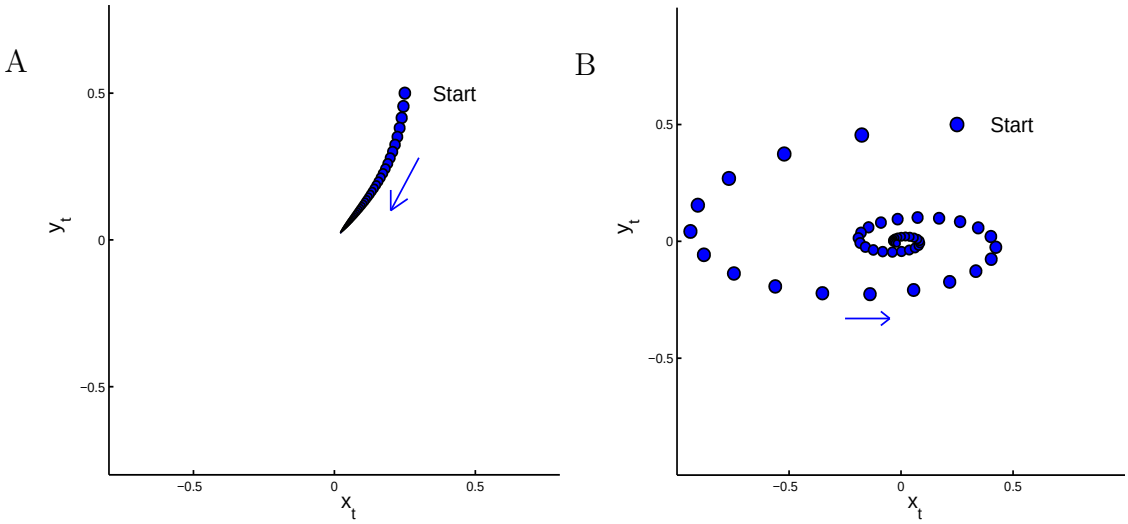


Figure 12.6: Trajectories of (y_t, x_t) for the linear dynamical systems A and B in Equations 12.37. The origin, (0,0), is a stable equilibrium for both of these systems.

Then $(A_t, B_t) \to (A_e, B_e)$ of Equations 12.34 if and only if $(x_t, y_t) \to (0, 0)$); or (A_e, B_e) is a stable equilibrium point of Equations 12.34 if and only if (0,0) is a stable equilibrium point of the simpler set of Equations 12.36. The Equations 12.36 are said to be *homogeneous* where as Equations 12.34 are *nonhomogeneous* if $H_1 \neq 0$ or $H_2 \neq 0$.

The following four examples demonstrate the primary dynamics of two first order difference equations near equilibrium points. In all four cases, the origin is the equilibrium point, the initial conditions, $x_0 = 0.25$ and $y_0 = 0.5$ are identical, and the equation $y_{n+1} = 0.1 \times x_n + 0.86 \times y_n$ are the same. Furthermore, the coefficients in the equations for x_{n+1} are similar, but methodically changed between examples.

$$
\begin{array}{llllll}
\text{A.} & x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n + 0.04 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \\
 & & & & & & \\
\text{B.} & x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n - 0.8 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \\
 & & & & & & \\
\text{C.} & x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n + 0.4 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \\
 & & & & & & \\
\text{D.} & x_0 & = & 0.25 & x_{n+1} & = & 1.15 \times x_n - 0.8 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n
\end{array}
\tag{12.37}
$$

The graphs in Figure 12.6 are *trajectories in the phase plane of* (x_t, y_t), plots of y_t versus x_t, a common graph when looking at equilibria. You may be more familiar with graphs of x_t versus t and y_t versus t, and the graph in Figure 12.7 shows such a plot for Equations 12.37B.

In Examples A and B, the trajectories move towards the equilibrium point (0,0). From all initial conditions the trajectories converge to (0,0), and (0,0) is a stable equilibrium point. In Examples C and D the trajectories diverge to infinity, and will do so for any starting point different from (0,0). For

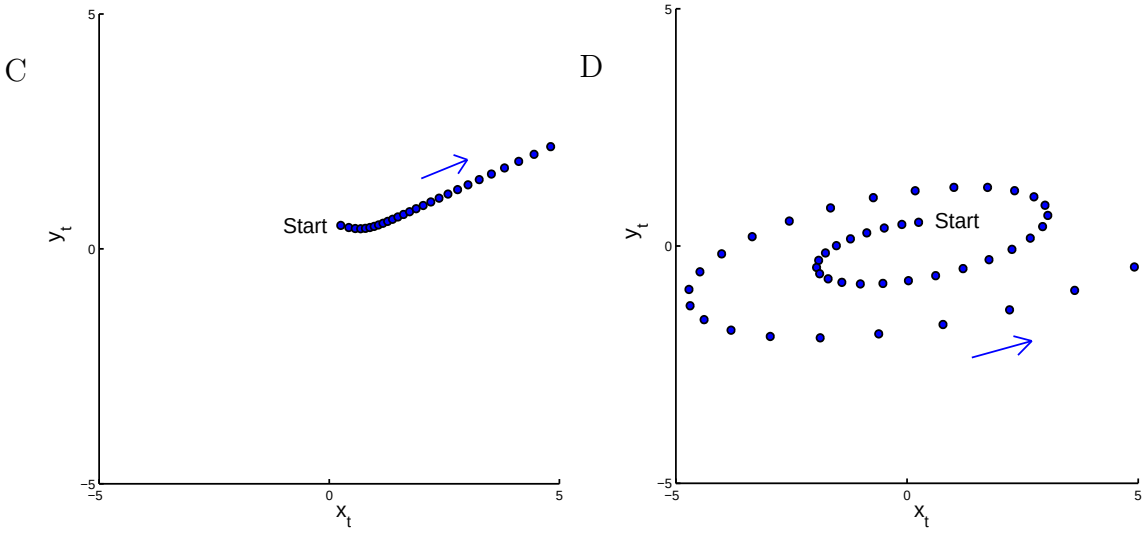


Figure 12.6: (Continued) Trajectories of (y_t, x_t) for the linear dynamical systems C and D in Equations 12.37. Both of these systems are unstable, $(x_t, y_t) \to (\infty, \infty)$.

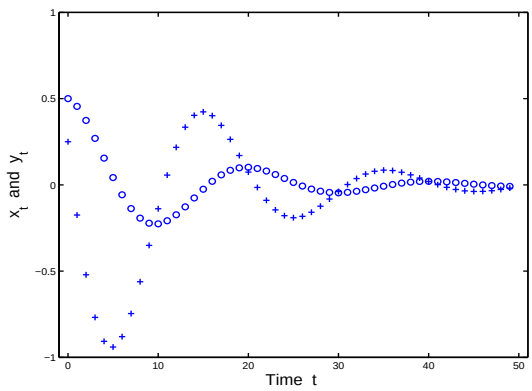


Figure 12.7: Graphs of x_t (+) and y_t (o) for the dynamical system of Equations 12.37B.

Examples C and D the equilibrium point (0,0) is not stable.

Because (0,0) is an equilibrium point of all four dynamical systems of Equations 12.36, any trajectory that starts at (0,0) stays at (0,0) for all four of them. The question of stability is whether trajectories that start at a point different from the equilibrium point (0,0) converge to (0,0).

Unstable equilibria rarely occur in biological systems because there are enough random perturbations of biological systems to insure that even if a system is exactly in an equilibrium state that is unstable, it will soon be perturbed and move away from the unstable equilibrium.

Stable equilibria, however, are very important in biological systems. Once a system is at or near a stable equilibrium, it will tend to return to or stay near the equilibrium condition when random perturbations move it a small distance away from equilibrium.

Trajectories A and C differ noticeably from trajectories B and D, which spiral around the equilibrium point. The spirals are due to the complex roots of the characteristic polynomials associated with the systems of Examples B and D.

What happens to $\mathbf{x_t}$ as $\mathbf{t \to \infty}$?

The characteristic equation of the linear dynamical system, Equations 12.36 is

$$z^2 - p \times z + q = 0, \quad \text{where} \quad p = m_{1,1} + m_{2,2} \quad \text{and} \quad q = m_{1,1}m_{2,2} - m_{2,1}m_{1,2} \tag{12.38}$$

and x_t and y_t satisfy the second order linear difference equations

$$x_{t+2} - p \times x_{t+1} + qx_t = 0 \qquad \text{and} \qquad y_{t+2} - p \times y_{t+1} + qy_t = 0.$$

(See Sections 12.4 and 12.5)

The three forms of the solution to $x_{t+2} - px_{t+1} + qx_t = 0$ are

$$\begin{aligned} x_t &= C_1 \times r_1^t + C_2 \times r_2^t \qquad r_1 \text{ and } r_2 \quad \text{distinct real roots.} \\ y_t &= D_1 \times r_1^t + D_2 \times r_2^t \end{aligned} \tag{12.39}$$

$$\begin{aligned} x_t &= C_1 \times r_1^t + C_2 \times t \times r_1^t \qquad r_1 \text{ a repeated real root.} \\ y_t &= D_1 \times r_1^t + D_2 \times t \times r_1^t \end{aligned} \tag{12.40}$$

$$\begin{aligned} x_t &= C_1 \times \rho^t \sin t\theta + C_2 \times \rho^t \cos t\theta \qquad r = a + bi \quad \text{a complex root,} \\ Y_t &= D_1 \times \rho^t \sin t\theta + D_2 \times \rho^t \cos t\theta \qquad \rho = \sqrt{a^2 + b^2}, \quad \theta = \arccos \frac{a}{\rho}. \end{aligned} \tag{12.41}$$

Theorem 12.8.1 For the three forms of solutions in Equations 12.39, 12.40 and 12.41, $x_t \to 0$ if and only if

$$|r_1| < 1 \quad \text{and} \quad |r_2| < 1; \qquad \text{or} \qquad |r_1| < 1; \qquad \text{or} \qquad |\rho| < 1. \tag{12.42}$$

depending on which of the three formulas describe x_t.

Proof. The results are valid because $\lim_{t\to\infty} r^t = 0$ if $|r| < 1$ and $\lim_{t\to\infty} t \times r^t = 0$ if $|r| < 1$. (For $\lim_{t\to\infty} t \times r^t = 0$ if $|r| < 1$, see Exercise 12.8.10.) End of proof.

The roots of the dynamical systems A - D in Equations 12.36 are

$$\begin{aligned} &\text{A}: \ r_1 \doteq 0.946, \quad r_2 \doteq 0.814 \quad \text{B}: \ r_1 \doteq 0.880 + 0.282i, \quad \rho \doteq 0.924 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta \doteq 0.310 \\ &\text{C}: \ r_1 \doteq 1.08, \quad r_2 \doteq 0.679 \quad \text{D}: \ r_1 \doteq 1.005 + 0.243i, \quad \rho \doteq 1.03 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta \doteq 0.221 \end{aligned}$$

Thus A. is stable because $|r_1| = 0.946$ and $|r_2| = 0.814$ are both less than 1. B. is stable because $\rho = 0.924$ is less than 1. C. is not stable because $|r_1| = 1.08$ is greater than one; that $|r_2| = 0.679$ is less than one is not redeeming. D. is not stable because $\rho = 1.03$ is greater than 1.

Without actually computing the roots of the characteristic Equation 12.36, one can determine whether the dynamical systems Equations 12.36 are stable by examining the coefficients.

Theorem 12.8.2 The fate of $\mathbf{x_t}$. The dynamical systems of Equations 12.34 and Equations 12.36, with characteristic equation

$$z^2 - (m_{1,1} + m2, 2) \times z + m_{1,1}m_{2,2} - m_{2,1}m_{1,2} = z^2 - pz + q = 0,$$

are stable if and only if

$$0 \leq |p| < 1 + q < 2.$$

Proof. **Danger: Obnubilation Zone.** This argument is tedious and reading it can be delayed – indefinitely.
We show that if $0 \leq |p| < 1 + q < 2$ then $\lim_{t\to\infty} x_t = 0$. In the case of complex roots, $\rho^2 = q$ (Equation 12.26) and because $0 < 1 + q < 2$, $|q| < 1$ so $|\rho| < 1$. In the case of a repeated root, r_1, $p^2 - 4q = 0$ and $r_1 = \frac{p}{2}$. Because $|p| < 2$, $|r_1| < 1$. Now suppose the roots r_1 and r_2 are real and distinct ($p^2 - 4q > 0$) and $0 \leq |p| < 1 + q < 2$. Then

$$\begin{array}{ccccccc}
 & & |p| & < & 1+q & & \\
1+q & > & -p & \text{and} & p & < & 1+q \\
1+p & > & -q & \text{and} & -q & < & 1-p \\
4+4p+p^2 & > & p^2-4q & \text{and} & p^2-4q & < & 4-4p+p^2
\end{array}$$

Because $|p| < 2$, both $2 + p$ and $2 - p$ are positive. Then

$$\begin{array}{ccccccc}
2+p & > & \sqrt{p^2-4q} & \text{and} & \sqrt{p^2-4q} & < & 2-p \\
-2-p & < & -\sqrt{p^2-4q} & < & \sqrt{p^2-4q} & < & 2-p \\
-1 & < & \dfrac{p-\sqrt{p^2-4q}}{2} & < & \dfrac{p+\sqrt{p^2-4q}}{2} & < & 1
\end{array}$$

Thus the roots, r_1 and r_2 are between -1 and 1 and $\lim_{t\to\infty} A_t = 0$.

Now suppose $\lim_{t\to\infty} x_t = 0$. Then if the roots are real and distinct they must lie between -1 and 1 and the steps of the previous argument may be reversed to show that $|p| < q + 1 < 2$.

If the root r_1 is repeated, then $|r_1| < 1$ and $r_1^2 = q < 1$ and $q + 1 < 2$. Furthermore, $p^2 - 4q = 0$, so $|p| = 2\sqrt{q}$. Now,

$$(1 - \sqrt{q})^2 > 0, \qquad 1 - 2\sqrt{q} + q > 0, \qquad 1 + q > 2\sqrt{q} = p$$

If the roots are complex, then $\rho < 1$ and $\rho^2 = q$ (Equation 12.26) and $q < 1$ and $q + 1 < 2$. Furthermore, $p^2 - 4q < 0$, so $|p| < 2\sqrt{q} < q + 1$, as above.

Exercises for Section 12.8, Stability of the equilibria of linear systems.

Exercise 12.8.1 Write the characteristic equations 12.38, for the dynamical systems A - D in Equations 12.37 and compute their characteristic roots.

Exercise 12.8.2 Use the values of p and q from the characteristic equations, 12.38 for the dynamical systems A - D in Equations 12.37 and Theorem 12.8.2 to test whether the systems are stable.

Exercise 12.8.3 Compute the first three iterates of the dynamical systems of Equations 12.37. Plot the iterates on a phase plane and see that they match the trajectories shown in Figure 12.6

Exercise 12.8.4 For the dynamical system 12.37A,

$$\begin{array}{rclrcl} x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n + 0.04 \times y_n \\ y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \end{array}$$

the characteristic roots are $r_1 = 0.946$ and $r_2 = 0.814$. x_t and y_t are given by

$$x_t = C_1 \times r_1^t + C_2 \times r_2^t \tag{12.43}$$
$$y_t = D_1 \times r_1^t + D_2 \times r_2^t \tag{12.44}$$

Where C_1, C_2, D_1 and D_2 are computed from

$$\begin{array}{rclrcl} x_0 & = & C_1 + C_2 & y_0 & = & D_1 + D_2 \\ x_1 & = & C_1 \times r_1 + C_2 \times r_2 & y_1 & = & D_1 \times r_1 + D_2 \times r_2 \end{array}$$

Compute C_1, C_2, D_1 and D_2 and use these values in Equations 12.43 and 12.44 to compute x_2, x_3, and y_2, y_3.

Exercise 12.8.5 For the dynamical system 12.37D,

$$\begin{array}{rclrcl} x_0 & = & 0.25 & x_{n+1} & = & 1.15 \times x_n - 0.8 \times y_n \\ y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \end{array}$$

the characteristic roots are

$$r_1 \doteq 1.005 + 0.243i, \qquad r_2 \doteq 1.005 - 0.243i$$

x_t and y_t are given by

$$\begin{aligned} x_t &= C_1\, \rho^t \cos t\theta + C_2\, \rho^t \sin t\theta \\ y_t &= D_1\, \rho^t \cos t\theta + D_2\, \rho^t \sin t\theta \end{aligned} \tag{12.45}$$

where $\rho = \sqrt{1.005^2 + 0.243^2}$, $\theta = \arccos(1.005/\rho)$ and C_1, C_2, D_1 and D_2 are computed from

$$\begin{array}{rclrcl} x_0 & = & C_1 & y_0 & = & D_1 \\ x_1 & = & C_1\, \rho \cos\theta + C_2\, \rho \sin\theta & y_1 & = & D_1\, \rho \cos\theta + D_2\, \rho \sin\theta \end{array}$$

Compute ρ, θ and C_1, C_2 and D_1, D_2 and use these values in Equations 12.45 and 12.45 to compute x_2, x_3, and y_2, y_3.

Exercise 12.8.6 Determine whether the following systems are stable.

a. $$\begin{aligned} x_{t+1} &= 0.48x_t + 0.2y_t \\ y_{t+1} &= 0.128x_t + 0.72y_t \end{aligned}$$

b. $$\begin{aligned} x_{t+1} &= 0.8x_t - 0.5y_t \\ y_{t+1} &= 0.2x_t + 0.9y_t \end{aligned}$$

c. $$\begin{aligned} x_{t+1} &= 1.1x_t - 0.5y_t \\ y_{t+1} &= 0.2x_t + 0.9y_t \end{aligned}$$

d. $$\begin{aligned} x_{t+1} &= 0.8x_t - 0.1y_t \\ y_{t+1} &= 0.1x_t + 0.6y_t \end{aligned}$$

e. $$\begin{aligned} x_{t+1} &= 0.56x_t + 0.4y_t \\ y_{t+1} &= 0.256x_t + 1.04y_t \end{aligned}$$

f. $$\begin{aligned} x_{t+1} &= 1.2x_t - 0.1y_t \\ y_{t+1} &= 0.9x_t + 0.6y_t \end{aligned}$$

Exercise 12.8.7 The following systems have one or two characteristic roots equal to 1 or complex roots with $\rho = 1$. Compute the characteristic roots, the first four points of the trajectory and plot them on a phase diagram.

a. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= x_t \\ y_0 &= 3 & y_{t+1} &= y_t \end{aligned}$$

b. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= y_t \\ y_0 &= 3 & y_{t+1} &= x_t \end{aligned}$$

c. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= y_t \\ y_0 &= 3 & y_{t+1} &= -x_t \end{aligned}$$

d. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= 0.5x_t \\ y_0 &= 2.5 & y_{t+1} &= 2y_t \end{aligned}$$

e. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= 0.5x_t \\ y_0 &= 0 & y_{t+1} &= 2y_t \end{aligned}$$

f. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= 0.5x_t \\ y_0 &= 1 & y_{t+1} &= y_t \end{aligned}$$

g. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= 0.5x_t + 0.5y_t \\ y_0 &= 3 & y_{t+1} &= 0.5x_t + 0.5y_t \end{aligned}$$

h. $$\begin{aligned} x_0 &= 2 & x_{t+1} &= \tfrac{\sqrt{2}}{2}x_t - \tfrac{\sqrt{2}}{2}y_t \\ y_0 &= 3 & y_{t+1} &= \tfrac{\sqrt{2}}{2}x_t + \tfrac{\sqrt{2}}{2}y_t \end{aligned}$$

i. $$\begin{aligned} x_0 &= 1 & x_{t+1} &= 0.5x_t + y_t \\ y_0 &= 2 & y_{t+1} &= 0.25x_t + 0.5y_t \end{aligned}$$

j. $$\begin{aligned} x_0 &= 1 & x_{t+1} &= 0.5x_t - y_t \\ y_0 &= 2 & y_{t+1} &= 0.25x_t + 0.5y_t \end{aligned}$$

k. $$\begin{aligned} x_0 &= 1 & x_{t+1} &= x_t \cos\alpha - y_t \sin\alpha \\ y_0 &= 0 & y_{t+1} &= x_t \sin\alpha + y_t \cos\alpha \end{aligned}$$

Exercise 12.8.8 Use Theorem 12.8.2 and the condition, $0 \leq |p| < 1 + q < 2$ to determine for which of the equations it is sure that $\lim_{t\to\infty} A_t = 0$.

a. $A_{t+2} - 0.5A_{t+1} + 0.5A_t = 0$

b. $A_{t+2} - 0.5A_{t+1} - 0.5A_t = 0$

c. $A_{t+2} - 1.4A_{t+1} + 0.5A_t = 0$

d. $A_{t+2} - 1.4A_{t+1} - 0.5A_t = 0$

e. $A_{t+2} - 0.5A_{t+1} + 1.5A_t = 0$

f. $A_{t+2} - 0.5A_{t+1} - 1.5A_t = 0$

g. $A_{t+1} = 0.5A_{t+1} + 0.5B_t = 0$
$B_{t+1} = 0.5A_{t+1} + 0.5B_t = 0$

h. $A_{t+1} = 0.5A_{t+1} - 0.5B_t = 0$
$B_{t+1} = 0.5A_{t+1} + 0.5B_t = 0$

Exercise 12.8.9 From Theorem 12.8.1 for the case of distinct real roots,

$$\begin{aligned} x_t &= C_1 r_1^t + C_2 r_2^t \\ y_t &= D_1 r_1^t + D_2 r_2^t, \end{aligned}$$

if $|r_1| < 1$ and $|r_2| < 1$ then $x_t \to 0$ and $y_t \to 0$. However, x_t and y_t may initially increase in magnitude. They may not decrease monotonically to zero. For the following system, compute the characteristic roots, r_1 and r_2 and confirm that the system is stable. Then compute iterates (x_t, y_t) until the length of (x_t, y_t), defined to be $\sqrt{x_t^2 + y_t^2}$, is less than the length of (x_0, y_0).

$$\begin{aligned} x_0 &= 0.81 \qquad & x_{t+1} &= 0.82x_t + 0.09y_t \\ y_0 &= 0.58 \qquad & y_{t+1} &= 0.36x_t + 0.78y_t. \end{aligned}$$

Exercise 12.8.10 In Equation 12.42, we claim that for

$$x_t = C_1\, r_1^t + C_2 \times t \times r_1^t, \qquad \lim_{t\to\infty} A_t = 0 \qquad \text{if} \qquad |r_1| < 1.$$

Use L'Hospital's Theorem to show that $\lim_{t\to\infty} t \times r^t = 0$ if $|r| < 1$.

Exercise 12.8.11 Show that the characteristic equation of

$$\begin{aligned} x_{n+1} &= p \cdot x_n - q \cdot y_n \\ y_{n+1} &= x_n \end{aligned}$$

is

$$x_{n+2} - px_{n+1} + qx_n = 0.$$

Exercise 12.8.12 (Only for the adventurous.) For the dynamical system 12.37 B,

$$\begin{aligned} x_0 &= 0.25 \qquad & x_{n+1} &= 0.9 * x_n - 0.8 * y_n \\ y_0 &= 0.5 \qquad & y_{n+1} &= 0.1 * x_n + 0.86 * y_n \end{aligned}$$

the characteristic equation is

$$\rho^2 - 1.76\rho + 0.854 = 0$$

and the two characteristic roots are

$$\begin{aligned} r_1 &= 0.88 + 0.28213\, i \\ r_2 &= 0.88 - 0.28213\, i \end{aligned}$$

Observe that

$$\begin{aligned} x_0 &= 0.25 \\ x_1 &= 0.9 \cdot 0.25 - 0.8 \cdot 0.5 \\ &= -0.175 \end{aligned}$$

The equations for C_1 and C_2 in $x_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$ for $t = 0$ and $t = 1$ are

$$\begin{aligned} 0.25 &= C_1 \times r_1^0 + C_2 \times r_2^0 \\ 0.25 &= C_1 + C_2 \\ -0.175 &= C_1 \times r_1^1 + C_2 \times r_2^1 \\ -0.175 &= (0.88 + 0.28213\, i) \times C_1 + (0.88 - 0.28213\, i) \cdot C_2 \end{aligned}$$

Then using complex algebra and arithmetic (actually ordinary algebra)

$$C_1 = 0.125 + 0.700\, i \qquad \text{and} \qquad C_2 = 0.125 - 0.700\, i$$

and

$$x_t = (0.125 + 0.700\, i) \times (0.88 + 0.28213\, i)^t + (0.125 - 0.700\, i) \times (0.88 - 0.28213\, i)^t.$$

a. Use your technology to compute x_t for $t = 0, 5$. The complex number $a + b\, i$ may be entered as $a + bi$ or as (a,b).

b. Write a corresponding equation for y_t.

12.9 Asymptotic stability of equilibria of nonlinear systems.

Biological systems tend to be nonlinear and we wish to determine when the equilibrium points of nonlinear dynamical systems are asymptotically stable. The principle way of doing so is to show that the *local linear approximation* to the dynamical system at an equilibrium is stable. We define the local linear approximation below; it is a linear dynamical system and its stability is the stability of a linear system as defined in Definition 12.8.2.

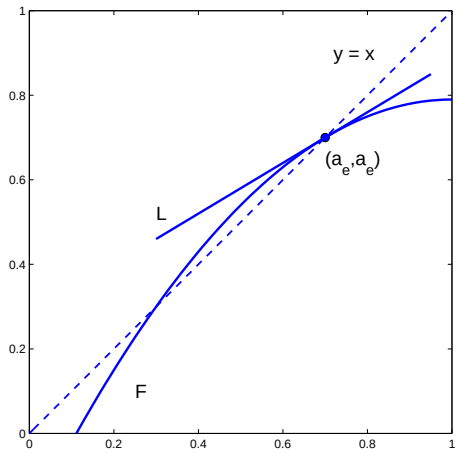


Figure 12.8: Graphs of a function F and its tangent line, L at an equilibrium point (a, a). L is the local linear approximation to F.

Definition 12.9.1 Asymptotical Stability. An *equilibrium point* of a dynamical system,

$$\begin{aligned} x_{t+1} &= F(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t), \end{aligned} \tag{12.46}$$

is a point (a_e, b_e) satisfying

$$\begin{aligned} a_e &= F(a_e, b_e) \\ b_e &= G(a_e, b_e). \end{aligned} \tag{12.47}$$

The system 12.46 is *asymptotically stable* at (a_e, b_e) if there is a positive number δ such that if (x_0, y_0) is within a distance δ of (a_e, b_e) then the iterates of Equations 12.46,

$$(x_t, y_t) \quad \longrightarrow \quad (a_e, b_e).$$

The local linear approximation to a a one-dimensional iteration

$$x_{t+1} = F(x_t)$$

at an equilibrium point, $a_e = F(a_e)$, is the linear iteration

$$x_{t+1} = F(a_e) + F'(a_e)(x_t - a_e) \tag{12.48}$$

(see Figure 12.8). The linear iteration is stable (Exercise 12.9.2) if and only if $\mid F'(a_e) \mid < 1$. Furthermore, By Theorem 11.4.1, the one dimensional dynamical system

$$x_{t+1} = F(x_t)$$

is asymptotically stable at an equilibrium $a_e = F(a_e)$ if $| F'(a_e) | < 1$. Thus the nonlinear $x_{t+1} = F(x_t)$ is asymptotically stable if its linear approximation $x_{t+1} = L(x_t)$ is stable. We will find a similar result for the two dimensional dynamical system 12.46.

Definition 12.9.2 Suppose F and G are functions of two variables and (a, b) is in the domain of both F and G and the partial derivatives F_1, F_2, G_1, and G_2 are all continuous on the interior of a circle with center at (a, b). The local linear approximation to the two dimensional dynamical system

$$\begin{aligned} x_{t+1} &= F(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t) \end{aligned} \tag{12.49}$$

at a an equilibrium point (a_e, b_e) is

$$\begin{aligned} x_{t+1} &= a_e + F_1(a_e, b_e) \times (x_t - a_e) + F_2(a_e, b_e) \times (y_t - b_e) \\ y_{t+1} &= b_e + G_1(a_e, b_e) \times (x_t - a_e) + G_2(a_e, b_e) \times (y_t - b_e) \end{aligned} \tag{12.50}$$

With $\xi_t = x_t - a_e$ and $\eta_t = y_t - b_e$ Equations 12.50 become

$$\begin{aligned} \xi_{t+1} &= F_1(a_e, b_e) \times \xi_t + F_2(a_e, b_e) \times \eta_t \\ \eta_{t+1} &= G_1(a_e, b_e) \times \xi_t + G_2(a_e, b_e) \times \eta_t \end{aligned} \tag{12.51}$$

which are homogeneous linear equations.

The matrix

$$\begin{bmatrix} F_1(a, b) & F_2(a, b) \\ G_1(a, b) & G_2(a, b) \end{bmatrix} \tag{12.52}$$

is called the *Jacobian matrix* at (a, b) of the transformation

$$\begin{aligned} u &= F(x, y) \\ v &= G(x, y) \end{aligned} \tag{12.53}$$

from the (x, y)-plane to the (u, v)-plane.

Example 12.9.1 In the next section we consider two populations that have a symbiotic relationship, a special case of which is

$$\begin{aligned} x_{t+1} &= x_t + \frac{5}{98} x_t (1 + \frac{4}{10} y_t - x_t) &= F(x_t, y_t) \\ y_{t+1} &= y_t + \frac{7}{120} y_t (1 + \frac{5}{7} x_t - y_t) &= G(x_t, y_t). \end{aligned} \tag{12.54}$$

An equilibrium point of the system is (1.96,2.4) and the Jacobian matrix at (1.96, 2.4) is computed by

$$\begin{aligned} F(x,y) &= \frac{103}{98}x + \frac{2}{98}x \times y - \frac{5}{98}x^2 & G(x,y) &= \frac{127}{120}y + \frac{1}{24}xy - \frac{7}{120}y^2 \\ F_1(x,y) &= \frac{103}{98} + \frac{2}{98}y - \frac{10}{98}x & F_2(x,y) &= \frac{2}{98}x \\ G_1(x,y) &= \frac{1}{24}y & G_2(x,y) &= \frac{127}{120} + \frac{1}{24}x - \frac{14}{120}y \\ F_1(1.96,2.4) &= 0.9 & F_2(1.96,2.4) &= 0.04 \\ G_1(1.96,2.4) &= 0.1 & G_2(1.96,2.4) &= 0.86 \end{aligned}$$

Then the Jacobian matrix and homogeneous local linear approximation to Equations 12.54 at the equilibrium point (1.96,2.4) are

$$\begin{bmatrix} 0.9 & 0.04 \\ 0.1 & 0.86 \end{bmatrix} \qquad \begin{aligned} \xi_{t+1} &= 0.9\xi_t + 0.04\eta_t \\ \eta_{t+1} &= 0.1\xi_t + 0.86\eta_t \end{aligned} \tag{12.55}$$

The alert reader may recognize this linear dynamical system as being that of Equations 12.37A for which the characteristic roots are approximately 0.946 and 0.814. The homogeneous linear dynamical system 12.55 is stable.

Because the local linear approximation 12.55 to the nonlinear dynamical system 12.54 at the equilibrium point (1.96,2.4) is stable, the nonlinear dynamical system 12.54 is asymptotically stable at (1.96,2.4).

The basis for the previous paragraph is in Theorem 12.9.1. The idea of the theorem and of local linear approximation can be seen by an algebraic rearrangement of the nonlinear system 12.54

$$\begin{aligned} x_{t+1} - 1.96 &= 0.9(x_t - 1.96) + 0.04(y_t - 2.4) + \frac{2}{98}(x_t - 1.96)(y_t - 2.4) - \frac{5}{98}(x_t - 1.96)^2 \\ y_{t+1} - 2.4 &= 0.1(x_t - 1.96) + 0.86(y_t - 2.4) + \frac{5}{120}(x_t - 1.96)(y_t - 2.4) - \frac{7}{120}(y_t - 1.96)^2 \end{aligned}$$

The linear terms are those of the local linear approximation. The idea of Theorem 12.9.1 is that if (x_t, y_t) is close to the equilibrium (1.96,2.4) so that $x_t - 1.96$ and $y_t - 2.4$ are 'small' then the quadratic terms $(x_t - 1.96)(y_t - 2.4)$, $(x_t - 1.96)^2$ and $(y_t - 1.96)^2$ are '(small)^2', even smaller, and contribute very little in computing the trajectory.

Theorem 12.9.1 Stability of local linear approximation implies asymptotic stability of the nonlinear system. Suppose (a_e, b_e) is an equilibrium point of a dynamical system

$$\begin{aligned} x_{t+1} &= F(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t) \end{aligned} \tag{12.56}$$

and the homogeneous local linear approximation to 12.56 at (a_e, b_e)

$$\begin{aligned} \xi_{t+1} &= F_1(a_e, b_e)\xi_t + F_2(a_e, b_e)\eta_t \\ \eta_{t+1} &= G_1(a_e, b_e)\xi_t + G_2(a_e, b_e)\eta_t \end{aligned} \tag{12.57}$$

is stable. Then the system 12.56 is asymptotically stable. We assume that F and G and their partial derivatives are continuous and the domains of F and G are all number pairs.

The proof of Theorem 12.9.1 is beyond the scope of this text. We will assume that it is true and apply it to four examples in the next section.

Exercises for Section 12.9, Local linear approximations to nonlinear systems.

Exercise 12.9.1 Find the local linear approximations to the following two-dimensional dynamical systems at the indicated points. Check to see whether the points given are equilibrium points of the system. If the point is an equilibrium point of a system, determine whether it is a asymptotically stable

equilibrium point of the system.

a. $$\begin{aligned} x_{t+1} &= x_t^2 + y_t^2 \\ y_{t+1} &= x_t \times y_t \end{aligned}$$ at the point $(0,0)$

b. $$\begin{aligned} x_{t+1} &= x_t^2 + y_t^2 \\ y_{t+1} &= x_t \times y_t \end{aligned}$$ at the point $(1,1)$

c. $$\begin{aligned} x_{t+1} &= x_t(1-x_t)(1-y_t) + 1/6 \\ y_{t+1} &= y_t(1-y_t)(1-x_t) + 1/8 \end{aligned}$$ at the point $(1/3, 1/4)$

d. $$\begin{aligned} x_{t+1} &= 2x_t(1-x_t)(1-y_t) + 5/9 \\ y_{t+1} &= y_t(1-y_t)(1-x_t) + 11/16 \end{aligned}$$ at the point $(2/3, 3/4)$

e. $$\begin{aligned} x_{t+1} &= x_t + 0.1y_t(1-x_t) - 0.018 \\ y_{t+1} &= y_t + 0.2x_t(1-y_t) - 0.016 \end{aligned}$$ at the point $(0.1, 0.2)$

f. $$\begin{aligned} x_{t+1} &= x_t + 0.1y_t(1-x_t) - 0.018 \\ y_{t+1} &= y_t + 0.2x_t(1-y_t) - 0.016 \end{aligned}$$ at the point $(0.8, 0.9)$

g. $$\begin{aligned} x_{t+1} &= \sqrt{3}\cos(\frac{\pi}{3}y_t) \\ y_{t+1} &= \frac{1}{\sqrt{2}}\cos(\frac{7\pi}{6}x_t) \end{aligned}$$ at the point $(3/2, 1/2)$

Exercise 12.9.2 Show that if M and H are numbers and and $|M| < 1$ then the one-dimensional linear dynamical system

$$x_{t+1} = M \times x_t + H$$

is stable.

Exercise 12.9.3 The dynamics of the SIR model of Section 12.1

$$\begin{aligned} S_{t+1} &= S_t - \beta \times S_t \times I_t \\ I_{t+1} &= I_t + \beta \times S_t \times I_t - \gamma \times S_{t-d} \times I_{t-d} \\ R_{t+1} &= R_t + \gamma \times S_{t-d} \times I_{t-d}. \end{aligned}$$

is determined by the first two equations; R_t does not affect the spread of the disease. Let M be the total population size and

$$x_t = \frac{S_t}{M}, \qquad y_t = \frac{I_t}{M}.$$

Then the first two of the SIR equations are

$$\begin{aligned} x_{t+1} &= x_t - \beta\, M \times x_t \times y_t \\ y_{t+1} &= y_t + \beta\, M \times x_t \times y_t - \gamma \times y_t \end{aligned} \tag{12.58}$$

a. Show that (1,0) is an equilibrium point of Equations 12.58.

b. One of the characteristic roots is 1. Find conditions on $\beta\, M$ and γ sufficient to insure that the other characteristic root is less than 1.

This is an important condition in that it distinguishes epidemics from non-epidemic introduction of disease.

Exercise 12.9.4 Examine the behavior of x_t for $t = 2, 3, \cdots, 20$ of the difference equation

$$\begin{aligned} x_{t+1} &= \alpha_0 x_t + \alpha_1 x_{t-1} \\ x_0 &= 0.1, \qquad x_1 = 0.2 \end{aligned}$$

for

a. $\alpha_0 = \frac{5}{6}, \qquad \alpha_1 = -\frac{1}{2}.$

b. $\alpha_0 = \frac{3}{2}, \qquad \alpha_1 = -\frac{1}{2}.$

c. $\alpha_0 = \frac{5}{2}, \qquad \alpha_1 = -1.$

12.10 Four examples of nonlinear dynamical systems.

12.10.1 Deer Population dynamics.

We now consider a population of deer growing in an environment that will support 500 female animals. Assume the low density growth rate to be 0.16 and that 15 female deer are harvested from the population each year. Because deer reach sexual maturity in their second year, the equation for population change is

$$\begin{array}{rcll} & & \text{Population Kinetics} & \text{Harvest} \\ w_{t+1} - w_t & = & 0.16 \times w_{t-1} \times \left(1 - \frac{w_{t-1}}{500}\right) & -15 \end{array} \tag{12.59}$$

where w_t is the number of adult female deer in year t. The number 0.16 that we have called the low density growth rate accounts for both the survival for two years of the population measured by w_{t-1} and the survival of the fauns born to those deer.

It is helpful to divide each term by 500 and let $v_t = \frac{w_t}{500}$. Then

$$v_{t+1} - v_t = 0.16 \times v_{t-1} \times (1 - v_{t-1}) - 0.03 \tag{12.60}$$

Equation 12.60 may be treated as a second order nonlinear dynamical system in one variable, or as we next show, as a first order nonlinear dynamical system in two variables.

Let $w_t = v_{t-1}$. Then $w_{t+1} = v_t$ and Equation 12.60 may be written as

$$\begin{aligned} v_{t+1} &= v_t \;+\; 0.16 w_t(1 - w_t) - 0.03 \\ w_{t+1} &= v_t \end{aligned} \tag{12.61}$$

The equilibrium points of Equation 12.61 are (v_e, w_e) where

$$\begin{aligned} v_e &= v_e + 0.16 w_e(1 - w_e) - 0.03 \\ w_e &= v_e \end{aligned}$$

which is equivalent to

$$\begin{aligned} v_e &= v_e + 0.16v_e(1-v_e) - 0.03 \\ 0 &= -0.16v_e^2 + 0.16v_e - 0.03 \\ 0 &= -0.16(v_e^2 - v_e + 0.188) = -0.16(v_e - 0.25)(v_e - 0.75) \end{aligned} \tag{12.62}$$

The equilibrium points of the iteration equation are (0.25,0.25) and (0.75,0.75).

Let

$$F(v,w) = v + 0.16w(1-w) - 0.03 \qquad G(v,w) = v$$

Then

$$\begin{aligned} F_1(v,w) &= 1 \qquad & F_2(v,w) &= 0.16 - 0.32w \\ F_1(v,w) &= 1 \qquad & G_2(v,w) &= 0 \end{aligned}$$

At the equilibrium point (0.25,0.25) the homogeneous local linear approximation and Jacobian matrix are

$$\begin{aligned} x_{t+1} &= x_t + 0.08y_t \\ y_{t+1} &= x_t \end{aligned} \qquad \begin{bmatrix} 1 & 0.08 \\ 1 & 0 \end{bmatrix}$$

The characteristic equation of the linear dynamical system is

$$r^2 - r - 0.08 = 0 \qquad \text{with roots} \quad r_1 = 1.074, \quad r_2 = -0.74$$

Because $r_1 = 1.074 > 1$ the nonlinear system 12.61 is not asymptotically stable at the equilbrium point (0.25, 0.25). That the dynamical system 12.61 is asymptotically stable at (0.75,0.75) is in Exercise 12.10.2.

12.10.2 Symbiosis Systems.

Imagine two interacting populations, X and Y, with population numbers at time n, X_n and Y_n. X has a low density growth rate, r_x, and in isolation, the maximum supportable X population is M. The presence of Y, however, expands the supportable X population. If Y_n is at its isolation maximum of N, then X can grow to $(1+\alpha)M$ (α and β are positive). In turn, the presence of X expands the Y universe.

$$\begin{aligned} X_{n+1} - X_n &= r_x \cdot X_n \left(1 + \alpha\tfrac{Y_n}{N} - \tfrac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \cdot Y_n \left(1 + \beta\tfrac{X_n}{M} - \tfrac{Y_n}{N}\right) \end{aligned} \tag{12.63}$$

As often happens it is useful to divide by maximum supportable populations and use variables that are fractions of maximum supportable populations. Divide the first equation of Equations 12.63 by M and the second equation by N. Then

$$\begin{aligned} \frac{X_{n+1}}{M} - \frac{X_n}{M} &= r_x \cdot \frac{X_n}{M}\left(1 + \alpha\frac{Y_n}{N} - \frac{X_n}{M}\right) \\ \frac{Y_{n+1}}{N} - \frac{Y_n}{N} &= r_y \cdot \frac{Y_n}{N}\left(1 + \beta\frac{X_n}{M} - \frac{Y_n}{N}\right) \end{aligned}$$

Then we let $x_n = \frac{X_n}{M}$ and $y_n = \frac{Y_n}{N}$ and write

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 + \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 + \beta x_n - y_n\right) \end{aligned} \tag{12.64}$$

Equations 12.64 involve only four parameters whereas the original Equations 12.63 involved six. The dynamics and stability conditions are identical, however.

Let (x_e, y_e) denote an equilibrium point of Equations 12.64. Then x_n and y_n are not changing when $x_n = x_e$ and $y_n = y_e$ so that

$$\begin{aligned} x_e - x_e &= r_x \cdot x_e \left(1 + \alpha y_e - x_e\right) &= 0 \\ y_e - y_e &= r_y \cdot y_e \left(1 + \beta x_e - y_e\right) &= 0 \end{aligned} \tag{12.65}$$

There are four solutions:

$$(x_e, y_e) = (0,0), \quad (x_e, y_e) = (1,0), \quad (x_e, y_e) = (0,1), \quad \text{or} \quad \begin{cases} x_e = \dfrac{1+\alpha}{1-\alpha\beta} \\ y_e = \dfrac{1+\beta}{1-\alpha\beta} \end{cases}$$

The equilibria are marked on Figure 12.9 along with the lines $1 + \alpha y - x = 0$ and $1 + \beta x - y = 0$. The arrows point from (x_n, y_n) toward (x_{n+1}, y_{n+1}). If you follow a path of arrows, you will move toward the equilibrium point,

$$(x_e, y_e) = \left(\frac{1+\alpha}{1-\alpha\beta}, \frac{1+\beta}{1-\alpha\beta}\right).$$

and away from the points (0,0), (0,1), and (1,0). In the shaded region, $x_{n+1} - xn > 0$ because $1 + \alpha y_n - x_n > 0$ and $y_{n+1} - yn > 0$ because $1 + \beta x_n - y_n > 0$. Therefore all of the arrows point up and to the right.

Explore 12.10.1 . Two points are marked in Figure 12.9, A on the line $1 + \alpha y - x = 0$ and B on the line $1 + \beta x - y = 0$. In which directions would the arrows A and B be pointing if they were marked? ■

Instability of the equilibrium point (1,0). Let

$$F(x,y) = x + r_x \times x \times (1 + \alpha y - x) \qquad G(x,y) = y + r_y \times y \times (1 + \beta x - y)$$

Then Equations 12.64 are

$$x_{n+1} = F(x_n, y_n) \qquad y_{n+1} = G(x_n, y_n).$$

The Jacobian matrix of the nonlinear system is computed and evaluated at (1,0) as

$$\begin{aligned} F_1(x,y) &= 1 + r_x + r_x\alpha y - 2r_x x & F_2(x,y) &= r_x \alpha x \\ G_1(x,y) &= r_y \beta y & G_2(x,y) &= 1 + r_y + r_y\beta x - 2r_y y \\ F_1(1,0) &= 1 - r_x & F_2(0,1) &= r_x \alpha \\ G_1(1,0) &= 0 & G_2(1,0) &= 1 + r_y + r_y\beta \end{aligned} \tag{12.66}$$

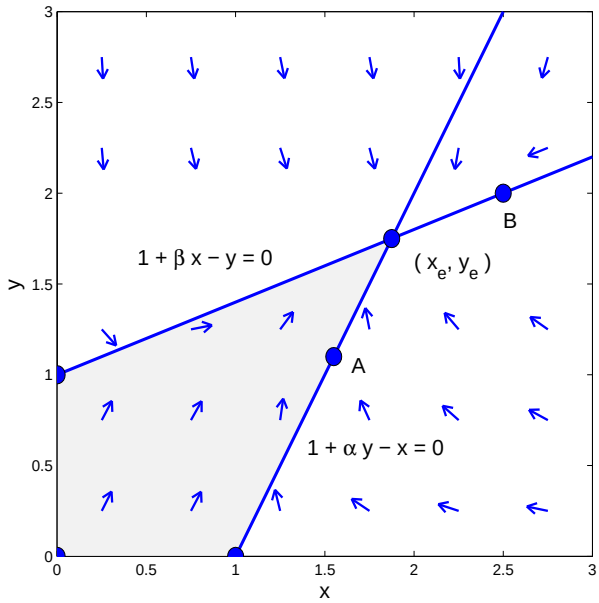


Figure 12.9: A phase plane for a symbiosis system, $x_{n+1} - x_n = r_x \cdot x_n (1 + \alpha y_n - x_n)$ $y_{n+1} - y_n = r_y \cdot y_n (1 + \beta x_n - y_n)$

The Jacobian matrix

$$\begin{bmatrix} 1 - r_x & r_x \alpha \\ 0 & 1 + r_y + r_y \beta \end{bmatrix}$$

is in *upper triangular form* which means that the roots of the characteristic equation are the diagonal entries, $1 - r_x$ and $1 + r_y + r_y\beta$ (see Exercise 12.6.2). The root $1 + r_y + r_y\beta$ is greater than 1 ($r_y > 0$ and $\beta r_y > 0$) so the local linear approximation to the nonlinear system 12.64 is not stable and we conclude that the nonlinear system is not asymptotically stable at the equilibrium point (1,0).

At the equilibrium point (1,0), the population Y is not present and the population X is at its maximum supportable population M in the absence of Y. If a small number of Y is introduced and $r_y > 0$ the Y population will grow exponentially for a short while and the system will move to the equilibrium point $x_e = \frac{1+\alpha}{1-\alpha\beta}$, $y_e = \frac{1+\beta}{1-\alpha\beta}$.

Shown in Figure 12.10 are computations that illustrate the introduction of 1 percent of N, the maximum supportable Y, and a comparison of the trajectory of the nonlinear equations with the trajectory of the local linear approximations

$$\begin{aligned} \xi_{n+1} &= (1 - r_x)\xi_n + r_x \alpha \eta_n \\ \eta_{n+1} &= (1 + r_y + r_y\beta)\eta_n. \end{aligned} \tag{12.67}$$

Parameter values used are

$$\begin{array}{llll} r_x = 0.1 & \alpha = 0.5 & x_0 = 1 & \xi_0 = 0 \\ r_y = 0.5 & \beta = 0.4 & y_0 = 0.01 & \eta_0 = 0.01 \end{array}$$

The nonlinear equations are

$$x_{n+1} - x_n = 0.1 \cdot x_n (1 + 0.5 y_n - x_n)$$

n	x_n	y_n	ξ_n	η_n
0	1.000	0.010	0.000	0.010
1	1.001	0.011	0.001	0.011
2	1.001	0.011	0.001	0.011
3	1.001	0.012	0.001	0.012
4	1.002	0.013	0.002	0.013
5	1.002	0.014	0.002	0.014
10	1.005	0.020	0.005	0.020
20	1.011	0.038	0.011	0.039
30	1.022	0.073	0.022	0.076
40	1.042	0.138	0.044	0.150

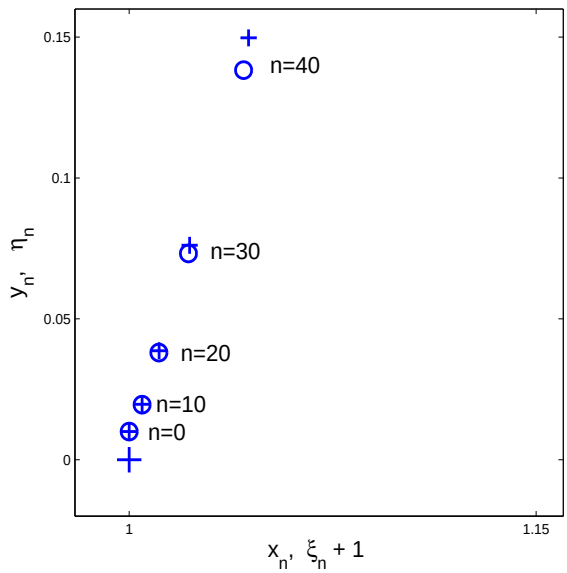


Figure 12.10: Comparison of the trajectories of the nonlinear dynamical system 12.64, with the trajectory of the local linear approximation to the nonlinear system near the equilibrium point (1,0), Equations 12.67. The starting point ($n = 0$) is (1.0,0.01) and the points $(x_n - 1, y_n)$ plotted with 'o', move away from the equilibrium point (1,0). 1 has been added to ξ_n and $(\xi_n + 1, \eta_n)$ is plotted with '+'.

$$y_{n+1} - y_n \quad = \quad 0.5 \cdot y_n \left(1 + 0.4 x_n - y_n\right)$$

and the local linear approximation at (1,0) is

$$\begin{aligned} \xi_{n+1} &= (1 - 0.1)\, \xi_n + 0.1 \cdot 0.5\, \eta_n &= 0.9\, \xi_n &+ 0.05\, \eta_n \\ \eta_{n+1} &= 1.52 \cdot 0.5\, \eta_n &= 0.076\, \eta_n. \end{aligned}$$

The initial X population $x_0 = 1$ was at its equilibrium value for no Y present; Y was set $y_0 = 0.01$. If X is initially isolated from Y and a small amount (0.01) of Y is introduced into the environment, both populations increase. The point (1,0) is a *nonstable* equilibrium point.

Stability of the equilibrium point ($x_e = \frac{1+\alpha}{1-\alpha\beta}$, $y_e = \frac{1+\beta}{1-\alpha\beta}$). We refer to this as the two-species equilibrium point. In the other three equilibrium points only one or no species were present.

This point (x_e, y_e) is chosen so that

$$(1 + \alpha y_e - x_e) = 0 \qquad \text{and} \qquad (1 + \beta x_e - y_e) = 0$$

in Equations 12.65.

Partial derivatives are computed from from Equations 12.66, as

$$\begin{aligned} F_1(x,y) &= 1 + r_x + r_x \alpha y - 2 r_x x & F_2(x,y) &= r_x \alpha x \\ G_1(x,y) &= r_y \beta y & G_2(x,y) &= 1 + r_y + r_y \beta x - 2 r_y y \end{aligned}$$

Note that $1 + \alpha y_e - x_e = 0$ and $1 + \beta x_e - y_e = 0$. The Jacobian at (x_e, y_e) is

$$\begin{aligned} F_1(x_e, y_e) &= 1 - r_x x_e \qquad & F_2(x_e, y_e) &= r_x \alpha x_e \\ G_1(x_e, y_e) &= r_y \beta y_e \qquad & G_2(x_e, y_e) &= 1 - r_y \beta y_e, \end{aligned} \tag{12.68}$$

and the local linear approximation at (x_e, y_e) is

$$\begin{aligned} \xi_{n+1} &= (1 - r_x\, x_e)\, \xi_n \;+\; r_x\, \alpha\, x_e\, \eta_n \\ \eta_{n+1} &= r_y\, \beta\, y_e\, \xi_n \;+\; (1 - r_y\, \beta\, y_e)\, \eta_n. \end{aligned} \tag{12.69}$$

Computation of the characteristics roots of the Jacobian matrix is not pleasant, and we consider an alternate approach. We give conditions under which we can show that the local linear approximation at (x_e, y_e) is stable even without computing the characteristic roots.

First we observe that if $\alpha \times \beta > 1$, then the equilibrium point $x_e = \frac{1+\alpha}{1-\alpha\beta}$, $y_e = \frac{1+\beta}{1-\alpha\beta}$ has negative coordinates, a decidedly un-biological result. We assume for analysis that both $\alpha < 1$ and $\beta < 1$. Furthermore we assume that

$$r_x x_e = r_x \frac{1+\alpha}{1-\alpha\beta} < 1 \qquad \text{and} \qquad r_y y_e = r_y \frac{1+\beta}{1-\alpha\beta} < 1.$$

Theorem 12.10.1 Asymptotic stability of the symbiosis two-species equilibrium point. If

$$\alpha < 1, \qquad \beta < 1, \qquad r_x x_e < 1, \qquad \text{and} \qquad r_y y_e < 1$$

then the equilibrium point

$$(x_e, y_e) = (\frac{1+\alpha}{1-\alpha\beta}, \frac{1+\beta}{1-\alpha\beta})$$

of the system

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n\,(1 + \alpha y_n - x_n) \\ y_{n+1} - y_n &= r_y \cdot y_n\,(1 + \beta x_n - y_n) \end{aligned}$$

has positive coordinates and the system is asymptotically stable at (x_e, y_e).

Proof. It will suffice to show that the linear approximation

$$\begin{aligned} \xi_{n+1} &= (1 - r_x x_e)\xi_n \;+\; \alpha r_x x_e \eta_n \\ \eta_{n+1} &= \beta r_y y_e \eta_n \;+\; (1 - r_y y_e)\eta_n \end{aligned} \tag{12.70}$$

is stable. The coefficients of these equations are positive and the sum of the coefficients of each equation is less than 1. Let K be the maximum of $1 - r_x x_e + \alpha r_x x_e$ and $\beta r_y y_e + 1 - r_y y_e$. $(K < 1)$. Then for any (ξ_n, η_n)

$$\begin{aligned} |\xi_{n+1}| &= |(1 - r_x x_e)\xi_n + \alpha r_x x_e \eta_n| \\ &\leq (1 - r_x x_e)\max(|\xi_n|, |\eta_n|) + \alpha r_x x_e \max(|\xi_n|, |\eta_n|) \\ &\leq K \max(|\xi_n|, |\eta_n|) \end{aligned}$$

Similarly

$$|\eta_{n+1}| \leq K \max(|\xi_n|, |\eta_n|), \quad \text{so that} \quad \max(|\xi_{n+1}|, |\eta_{n+1}|) \leq K \max(|\xi_n|, |\eta_n|).$$

By cascading

$$\max(|\xi_n|, |\eta_n|) \leq K^n \max(|\xi_0|, |\eta_0|) \qquad \text{and} \qquad (\xi_n, \eta_n) \to (0, 0)$$

The local linear approximation 12.70 is stable. End of proof.

Explore 12.10.2 For the values

$$\alpha = 0.7, \quad \beta = 0.8, \quad r_x = 0.3, \quad r_y = 0.2$$

the equilibrium point of Equations 12.64 is

$$(x_e, y_e) = (\frac{1+\alpha}{1-\alpha\beta}, \frac{1+\beta}{1-\alpha\beta}) = (\frac{85}{22}, \frac{65}{28}).$$

Are the inequalities of Theorem 12.10.1,

$$\alpha < 1, \qquad \beta < 1, \qquad r_x x_e < 1, \qquad \text{and} \qquad r_y y_e < 1,$$

satisfied?

Is the equilibrium point, (85/22, 65/22) of Equations 12.64 locally asymptotically stable? ∎

An example computation for the special case

$$\begin{array}{ll} r_x = \frac{5}{98} & \alpha = 0.4 \\ r_y = \frac{7}{120} & \beta = \frac{5}{7} \end{array}$$

is shown in Figure 12.11. The local linear approximation, Equations 12.70, are

$$\begin{aligned} \xi_{n+1} &= 0.9\xi_n + 0.04\eta_n \\ \eta_{n+1} &= 0.1\xi_n + 0.86\eta_n \end{aligned}$$

As was shown in Example 12.9.1, this is Equations 12.37A for which the characteristic roots are approximately 0.946 and 0.814, and the system is stable. Computations with this example are shown in Figure 12.11. The equilibrium point is $((1+\alpha)/(1-\alpha\beta), (1+\beta)/(1-\alpha\beta)) = (1.96, 2.4)$ and starting values were $x_0 = 1.86$, $y_0 = 2.2$, and $\xi_0 = -0.1$, $\eta_0 = -0.2$.

n	x_n	y_n	$\hat{\xi}_n$	$\hat{\eta}_n$
0	1.860	2.200	-0.100	-0.200
1	1.862	2.217	-0.098	-0.182
2	1.864	2.231	-0.095	-0.166
3	1.867	2.244	-0.093	-0.153
4	1.870	2.256	-0.089	-0.140
5	1.873	2.266	-0.086	-0.130
10	1.890	2.306	-0.069	-0.091
20	1.917	2.348	-0.041	-0.049
30	1.935	2.370	-0.024	-0.028
40	1.945	2.383	-0.014	-0.016

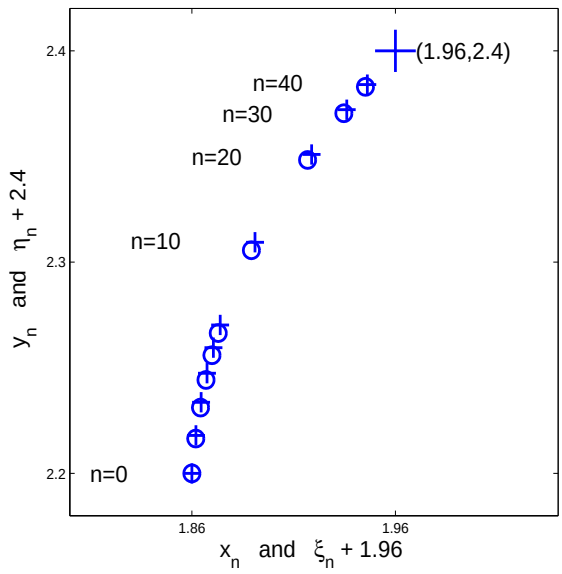


Figure 12.11: Comparison of the trajectories of the nonlinear dynamical system 12.64 marked with 'o', with the trajectory of the local linear approximation to the nonlinear system at the equilibrium point $(1+\alpha)/(1-\alpha\beta), (1+\beta)/1-\alpha\beta) = (1.96, 2.4)$ Equations 12.70 ξ_n and η_n have been translated back to (1.96,2.4) and $(\xi_n + 1.96, \eta_n + 2.4)$ is plotted with '+'.

12.10.3 Predator-Prey Systems

We now analyze a dynamical system that is often used to discuss models of two species one of which is a predator of the other.

$$\begin{aligned} X_{n+1} - X_n &= r_x \cdot X_n \left(1 + \alpha \tfrac{Y_n}{N} - \tfrac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \cdot Y_n \left(1 - \beta \tfrac{X_n}{M} - \tfrac{Y_n}{N}\right) \end{aligned} \tag{12.71}$$

The system is obviously similar to the model of symbiosis, Equations 12.63, the only difference being the minus sign in front of β in the second equation. X is the predator, and has a low density growth rate, r_x, and in isolation, the maximum supportable X-population is M. However, presence of the prey Y expands the supportable X-population. If Y_n is at its isolation maximum of N, then X can grow to $(1+\alpha)M$ (α and β are positive). Y has a low density growth rate of r_y and in the absence of X has a maximum supportable population of N. However, If X is at its isolation maximum of M, the maximum Y is reduced to $1-\beta$ times N.

Because of the similarity of the predator-prey equations 12.71 to the symbiosis equations 12.63, most of the work of this subsection is in the form of exercises.

12.10.4 Competition.

Imagine that there are two populations, X and Y, competing for the same resource. In the absence of Y, X has a maximum sustainable population, M, but the presence of Y reduces the resources available to X and reduces the supportable X population. Similarly, in the absence of X, Y has a maximum

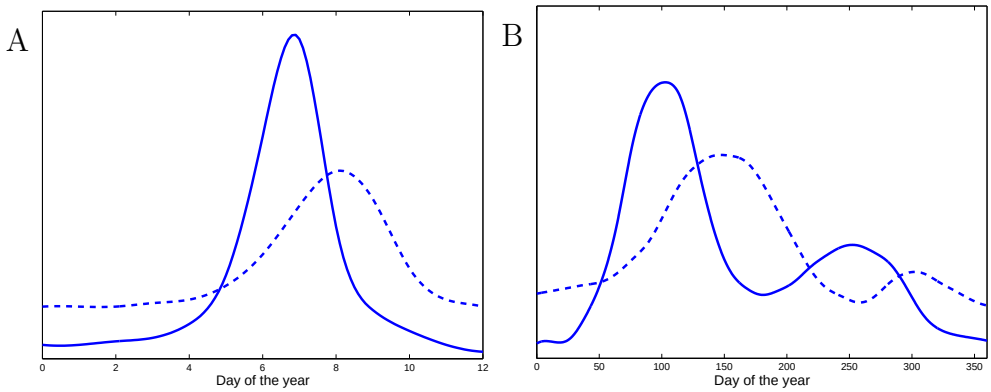


Figure 12.12: Oceanic phytoplankton (solid line) and zooplankton (dashed line) in A The Arctic and B. The North Atlantic. Redrawn from Parsons et al who modified a figure from A. K. Heinrich, 1962, The life histories of plankton animals and seasonal cycles of plankton communities in the ocean, J. Cons. Int. Explor. Mer, 27, 15-24. We have permission to use the figure from Heinrich.

sustainable population N and the presence of X reduces the supportable Y population. Equations descriptive of this competition are

$$\begin{aligned} X_{n+1} - X_n &= r_x\, X_n \left(1 - \alpha \frac{Y_n}{N} - \frac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y\, Y_n \left(1 - \beta \frac{X_n}{M} - \frac{Y_n}{N}\right) \end{aligned} \tag{12.72}$$

Continued in Exercise 12.10.18

Discussion. In nature the parameters, r_x, r_y, α, and β, are not constant in time, they may change with seasons, for example, and it is unusual to have only two species interacting. Nevertheless, the models we study are suggestive of natural behavior. Some interesting data in Figure 12.12 is redrawn from Parsons et al[1] regarding oceanic phytoplankton (prey) and zooplankton (predator). Perhaps plants are not usually considered prey, but the model works in this setting. Observe first the periodic variation in total phytoplankton and zoopankton in the graphs. The Artic suggests variation in r_x with the season, and thus reflects external forces affecting the system. But the within year oscillations of the North Atlantic is suggestive of complex roots associated with an equilibrium point, a property internal to the system.

Exercises for Section 12.10, Four examples of nonlinear dynamical systems.

Exercise 12.10.1 Read the coordinates of 13 points of each of the curves phytoplankton and zooplankton for the North Atlantic in Figure 12.12B and plot a phase graph for these two groups.

Exercise 12.10.2 The point (0.75,0.75) was shown to be an equilibrium point of the deer dynamical system, Equations 12.61. Determine whether the system is asymptotically stable at (0.75,0.75).

[1]"Biological Oceanographic Processes, 2nd Edition, T. R. Parsons, M. Takahashi, and B. Hargrave, Pergamon Press Ltd., Oxford, 1977, Figure 17, p 30.

Exercise 12.10.3 Suppose 20 deer were harvested annually from the deer population of 12.59. Show that (0.5,0.5) is an equilibrium point of the dynamical system. Determine whether the system is asymptotically stable at (0.5,0.5).

Exercise 12.10.4 Do Explore 12.10.2

Exercise 12.10.5 For the symbiosis system,

$$\begin{aligned} x_{n+1} - x_n &= \frac{5}{98} \cdot x_n \left(1 + 0.4 y_n - x_n\right) \\ y_{n+1} - y_n &= \frac{7}{120} \cdot y_n \left(1 + \frac{5}{7} x_n - y_n\right) \end{aligned}$$

(1.76,2.0) is an equilibrium point. Choose the initial value, $(x_0, y_0) = (1.76, 2.1)$ and use to compute (x_1, y_1), (x_2, y_2), (x_3, y_3).

Exercise 12.10.6 Choose an initial value, $(\xi_0, \eta_0) = (-0.2, 0.3)$ and

$$\begin{aligned} \hat{\xi}_{n+1} &= 0.9\hat{\xi}_n + 0.04\hat{\eta}_n \\ \hat{\eta}_{n+1} &= 0.1\hat{\xi}_n + 0.86\hat{\eta}_n \end{aligned}$$

to compute $\hat{\xi}_1, \hat{\eta}_1)$, $\hat{\xi}_2, \hat{\eta}_2)$, and $\hat{\xi}_3, \hat{\eta}_3)$. These are Equations 12.37A.

Exercise 12.10.7 Use the parameter values

$$\alpha = 0.9, \quad \beta = 0.8, \quad r_x = 0.3, \quad r_y = 0.2$$

in

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 + \alpha\, y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 + \beta\, x_n - y_n\right) \end{aligned}$$

a. Find the two-species equilibrium point ($x_e > 0$ and $y_e > 0$).

b. Write the local linear approximation at (x_e, y_e).

c. Write the characteristic equation of the linear system.

d. Compute the roots to the characteristic equation.

Exercise 12.10.8 Show that the local linear approximation to the symbiosis dynamical system at a two-species equilibrium point ($x_e > 0$ and $y_e > 0$ has real roots.

Predator-Prey exercises follow.

Exercise 12.10.9 Normalization of the Equations 12.71 reduces the complexity. Divide the first equation of Equations 12.71 by M and the second equation by N. Let $x_n = \frac{X_n}{M}$ and $y_n = \frac{Y_n}{N}$ and show that

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 + \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 - \beta x_n - y_n\right) \end{aligned} \tag{12.73}$$

Discussion. Equations 12.73 involve only four parameters whereas the original Equations 12.71 involved six. The dynamics and stability conditions are identical.

Exercise 12.10.10 Let (x_e, y_e) denote an equilibrium point of Equations 12.73. Then x_n and y_n are not changing when $x_n = x_e$ and $y_n = y_e$ so that

$$\begin{aligned} r_x x_e \left(1 + \alpha y_e - x_e\right) &= 0 \\ r_y y_e \left(1 - \beta x_e - y_e\right) &= 0 \end{aligned}$$

Show that the four possibilities are

$$(x_e, y_e) = (0,0), \quad (x_e, y_e) = (1,0), \quad (x_e, y_e) = (0,1), \quad \text{and} \quad \begin{cases} x_e &= \dfrac{1+\alpha}{1+\alpha\beta} \\ y_e &= \dfrac{1-\beta}{1+\alpha\beta} \end{cases}$$

Exercise 12.10.11 Draw a phase plane similar to that of Figure 12.9 for Equations 12.73,

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 + \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 - \beta x_n - y_n\right) \end{aligned}$$

and the parameters, $r_x = 0.05$, $r_y = 0.2$, $\alpha = 0.5$ and $\beta = 0.4$.

You should draw the graphs of $1 + 0.5y - x = 0$ and $1 - 0.4x - y = 0$ and identify the point of intersection of the two lines. Shade the region where both $1 + 0.5y - x$ is positive and $1 - 0.4x - y = 0$ is positive. Draw at least one arrow pointing from a potential (x_n, y_n) toward (x_{n+1}, y_{n+1}) in each of the four regions marked by $1 + 0.5\,y - x = 0$ and $1 - 0.4\,x - y = 0$.

Exercise 12.10.12 We (and you) analyze here the two-species equilibrium point, $(\frac{1+\alpha}{1+\alpha\beta}, \frac{1-\beta}{1+\alpha\beta})$.

Compute the Jacobian matrix for the nonlinear dynamical system at the equilibrium point $(\frac{1+\alpha}{1+\alpha\beta}, \frac{1-\beta}{1+\alpha\beta})$, and show that the local linear approximation to the nonlinear system is

$$\begin{aligned} \xi_{n+1} &= (1 - r_x x_e)\xi_n &+ \quad r_x x_e \alpha \eta_n \\ \eta_{n+1} &= -r_y y_e \beta \xi_n &+ (1 - r_y y_e)\eta_n \end{aligned} \tag{12.74}$$

Discussion. As with the symbiosis system, the characteristic roots of this system are difficult to analyze and we assume some restrictions on the parameters. If $\beta > 1$ the second coordinate of $(\frac{1+\alpha}{1+\alpha\beta}, \frac{1-\beta}{1+\alpha\beta})$, is negative, which is not of biological interest.

Theorem 12.10.2 Asymptotic stability of the predator-prey two-species equilibrium point. If

$$\alpha < 1, \qquad \beta < 1, \qquad r_x x_e < 1, \qquad \text{and} \qquad r_y y_e < 1$$

then the equilibrium point

$$(x_e, y_e) = (\frac{1+\alpha}{1+\alpha\beta}, \frac{1-\beta}{1+\alpha\beta})$$

of the system

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 + \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 - \beta x_n - y_n\right) \end{aligned}$$

has positive coordinates and is the system is asymptotically stable at (x_e, y_e).

Exercise 12.10.13 Prove Theorem 12.10.2. The argument is similar to the argument for Theorem 12.10.1, but note that one coefficient of Equations 12.74 is negative.

Example 12.10.1 It is helpful to consider a special case. For the parameter values

$$\begin{aligned} r_x &= 0.05 \quad \alpha = 0.5 \\ r_y &= 0.2 \quad \beta = 0.4 \end{aligned}$$

The predator-prey equations are

$$\begin{aligned} x_{n+1} - x_n &= 0.05 \cdot x_n \left(1 + 0.5 y_n - x_n\right) \\ y_{n+1} - y_n &= 0.2 \cdot y_n \left(1 - 0.4 x_n - y_n\right) \end{aligned} \tag{12.75}$$

and the two-species equilibrium point is

$$\begin{aligned} x_e = \frac{1+\alpha}{1+\alpha\beta} &= 1.25 \\ y_e = \frac{1-\beta}{1+\alpha\beta} &= 0.5 \end{aligned}$$

The local linear approximation (Equations 12.74) is

$$\begin{aligned} \xi_{n+1} &= 0.9375\xi_n + 0.03125\eta_n \\ \eta_{n+1} &= -0.04\xi_n + 0.9\eta_n \end{aligned} \tag{12.76}$$

Computations with Equations 12.75 and 12.76 with

$$\begin{aligned} r_x &= 0.05 \quad \alpha = 0.5 \quad x_0 = 0.85 \quad \hat{\xi}_0 = -0.4 \\ r_y &= 0.2 \quad \beta = 0.4 \quad y_0 = 0.3 \quad \hat{\eta}_0 = -0.2 \end{aligned}$$

are shown in Figure 12.13

n	x_n	y_n	ξ_n	η_n
0	0.850	0.300	-0.400	-0.200
1	0.863	0.322	-0.381	-0.164
2	0.876	0.343	-0.363	-0.132
3	0.889	0.364	-0.344	-0.105
4	0.902	0.385	-0.326	-0.080
5	0.915	0.404	-0.308	-0.059
10	0.980	0.484	-0.227	0.009
20	1.097	0.549	-0.111	0.045
30	1.176	0.545	-0.048	0.035
40	1.218	0.528	-0.019	0.020

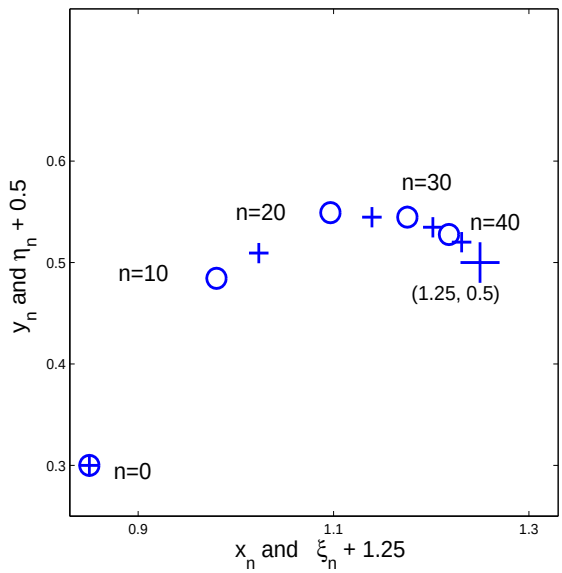


Figure 12.13: The trajectory of the nonlinear dynamical system 12.75 (symbol 'o') and the trajectory of the local linear approximation 12.76 (symbol '+') translated from the origin to the equilibrium point, (1.25,0.5).

Exercise 12.10.14 Choose the initial value, $(x_0, y_0) = (1.3, 0.6)$ and use

$$\begin{aligned} x_{n+1} - x_n &= 0.05 \cdot x_n \left(1 + 0.5 y_n - x_n\right) \\ y_{n+1} - y_n &= 0.2 \cdot y_n \left(1 - 0.4 x_n - y_n\right) \end{aligned}$$

to compute (x_1, y_1), (x_2, y_2), $\cdots$ (x_{20}, y_{20}). Plot your points on a graph similar to that in Figure 12.13.

Exercise 12.10.15 Choose an initial value, $(\xi_0, \eta_0) = (0.05, 0.1)$ and

$$\begin{aligned} \xi_{n+1} &= 0.9375\xi_n + 0.03125\eta_n \\ \eta_{n+1} &= -0.04\xi_n + 0.9\eta_n \end{aligned}$$

to compute $\xi_1, \eta_1)$, $\xi_2, \eta_2)$, $\cdots$ $\xi_{20}, \eta_{20})$. Add (1.25,0.5) to each point and plot the resulting points on the axes you used for Exercise 12.10.14.

Exercise 12.10.16 It appears that the equilibrium point (1.25,0.5) is a stable equilibrium point for the system

$$\begin{aligned} \xi_{n+1} &= 0.9375\xi_n + 0.03125\eta_n \\ \eta_{n+1} &= -0.04\xi_n + 0.9\eta_n \end{aligned}$$

Show that the characteristic values of the system are

$$\frac{0.9375 + 0.9 \pm \sqrt{(0.9375 + 0.9)^2 - 4 \cdot 0.845}}{2} =$$

$$0.91875 + 0.029974\, i \qquad \text{or} \qquad 0.91875 - 0.029974\, i$$

where $i = \sqrt{-1}$.

Discussion. The magnitude of the complex numbers $0.91875 + 0.029974\,i$ is $\sqrt{0.91875^2 + 0.029974^2} = 0.919231$. Therefore the linear system 12.76 is stable and by Theorem 12.9.1 the nonlinear system 12.75 is asymptotically stable at (1.25,0.5). The imaginary part of the roots cause the spiral around the equilibrium point. However, the magnitude of the imaginary part is so small compared to the real part that the rotation is rather slow.

Exercise 12.10.17 Find parameter values α, β, r_x and r_y so that the equilibrium point of

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n\,(1 + \alpha y_n - x_n) \\ y_{n+1} - y_n &= r_y \cdot y_n\,(1 - \beta x_n - y_n) \end{aligned}$$

has both coordinates positive but the system is not locally stable at the equilibrium point. From Theorem 12.10.2 you have to violate one of the conditions,

$$\alpha < 1, \qquad \beta < 1, \qquad r_x x_e < 1, \qquad \text{and} \qquad r_y\, y_e < 1,$$

and because of the equilibrium point, $(1+\alpha)/(1+\alpha\,\beta), (1-\beta)/(1+\alpha\,\beta)$ you have to retain $\beta < 1$. It will be efficient to write computer code that accepts α, β, r_x, and r_y and computes the roots of the characteristic equation of the local linear approximation of a proposed solutions.

Exercise 12.10.18 Competition. Analyze Equations 12.72,

$$\begin{aligned} X_{n+1} - X_n &= r_x\, X_n \left(1 - \alpha \frac{Y_n}{N} - \frac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y\, Y_n \left(1 - \beta \frac{X_n}{M} - \frac{Y_n}{N}\right), \end{aligned}$$

that describe interaction of two competing species.

The steps will include:

a. Normalize the equations to obtain equations that have only four parameters.

$$\text{Normalize the competition Equations 12.72.} \tag{12.77}$$

b. Show that there are four equilibrium points, one of which is

$$\left(\frac{1-\alpha}{1-\alpha\,\beta}, \frac{1-\beta}{1-\alpha\,\beta}\right)$$

c. Draw the phase plane for Equations 12.77 with $r_x = 0.2$, $r_y = 0.1$, $\alpha = 0.4$ and $\beta = 0.1$, including the lines $1 - 0.3\,y - x = 0$ and $1 - 0.4\,x - y = 0$, the coordinates of the two-species equilibrium point, and arrows in each of four regions bounded by the lines.

d. Determine whether previous system is locally stable at the two-species equilibrium point.

e. Draw the phase plane and analyze Equations 12.77 with with $r_x = 0.2$, $r_y = 0.1$, $\alpha = 1.5$ and $\beta = 2$.

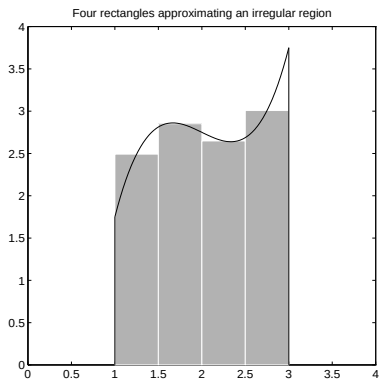
f. State and prove a theorem that gives sufficient conditions for the two-species equilibrium point of Equations 12.77 to be locally stable.

Chapter 13

The Integral

Where are we going?

In this chapter we will use the familiar concept of the area of a rectangle to form approximations to areas of regions that are of non-rectangular shape. The regions of greatest immediate interest are regions lying between the graph of a function and the horizontal axis. The areas of such regions of have important uses such as the accumulated deposit of a chemical or waste product, the distance traveled by an object, measurement of cardiac output, and the average value of a function.



13.1 Areas of Irregular Regions.

Area of an oak leaf.

Explore 13.1.1 Describe two ways to approximate the area of the leaf shown in Figure 13.1. ■

You may think of several interesting ways of solving the previous problem. A way that we will generalize to future applications is illustrated by two problems following Figure 13.1.

Figure 13.1: An Oak leaf.

One of the more curious ways of estimating the area of the leaf in Figure 13.1 is to paste the page on a dart board and throw darts at the page from a distance sufficient to insure that the points where the dart hits the page are randomly distributed on the page. The number, N_P, of darts that strike the page and the number, N_L, of darts that strike the leaf can be counted. The area of the leaf is approximated by

$$\text{Area of the leaf} \doteq \frac{N_L}{N_P} A_P$$

Although the procedure may seem a bit esoteric, it is used in serious computations of volumes in high dimension (with points randomly generated by a computer algorithm rather than darts!).

Explore 13.1.2 The darker grid lines in Figure 13.2 are marked at 1.0 cm intervals. Count the number I_1 of 1.0 cm^2 squares entirely covered by the leaf and the number O_1 of 1.0 cm^2 squares that intersect the leaf (including those included in I_1). Use the squares to estimate the area of the leaf. ■

The finer grid on the leaf in Figure 13.2 is marked at 0.5 cm intervals. We count that $I_2 = 283$ of these squares are completely inside the leaf and a total of $O_2 = 409$ squares intersect the leaf (not including the stem). Because each square is of area 0.25 cm^2 the area of the leaf should be at least $283 \times 0.25 = 70.75$ cm^2 and no more than $409 \times 0.25 = 102.25$ cm^2.

Explore 13.1.3 How do 70.75 and 102.25 cm^2 compare with your answer based on the 1 cm^2 squares? Given the information so far obtained, what is your best estimate of the area of the leaf? How can you extend the procedure to get an even better answer? ■

Let A denote the area of the leaf in Figure 13.2. Because the area of each 1cm grid square is 1cm^2, $I_1 \times 1$ is the area of the region covered by the the 1cm grid squares inside the leaf, and $I_1 \times 1$ is less than A. Because the squares that intersect the leaf enclose the leaf, $O_1 \times 1$ is greater than A.

Explore 13.1.4 Because every 1-cm square inside the leaf contains four 0.5-cm squares, also inside the leaf,

$$I_1 \times 1.0 \leq I_2 \times 0.25.$$

Explain why

a. $I_2 \times 0.25 \leq A \leq O_2 \times 0.25$ b. $O_2 \times 0.25 \leq O_1 \times 1.0$. ■

The normal distribution. Shown in Figure 13.3 is (a portion of) the graph of the probability density function of the normal distribution, sometimes called the 'Bell-shaped curve'. In the figure shown, the distribution has mean zero and standard deviation equal to one. The equation of the density function is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty \tag{13.1}$$

The interpretation of the density function is that if a measurement of a sample from a normal variate has been scaled so that the average value is zero and the standard deviation is one, then for any interval $[a, b]$ of measurement values, the probability that any one member of the sample lies in $[a, b]$ is the area of the region between $x = a$ and $x = b$ and below the curve and above the horizontal axis. The area under the total curve, from $-\infty$ to $+\infty$ is 1.0. The graph is symmetrical about the Y-axis, so that, for example, the area to the left of the vertical axis is equal to the area to the right of the vertical axis, the two areas sum to one, and therefore both areas are 0.5.

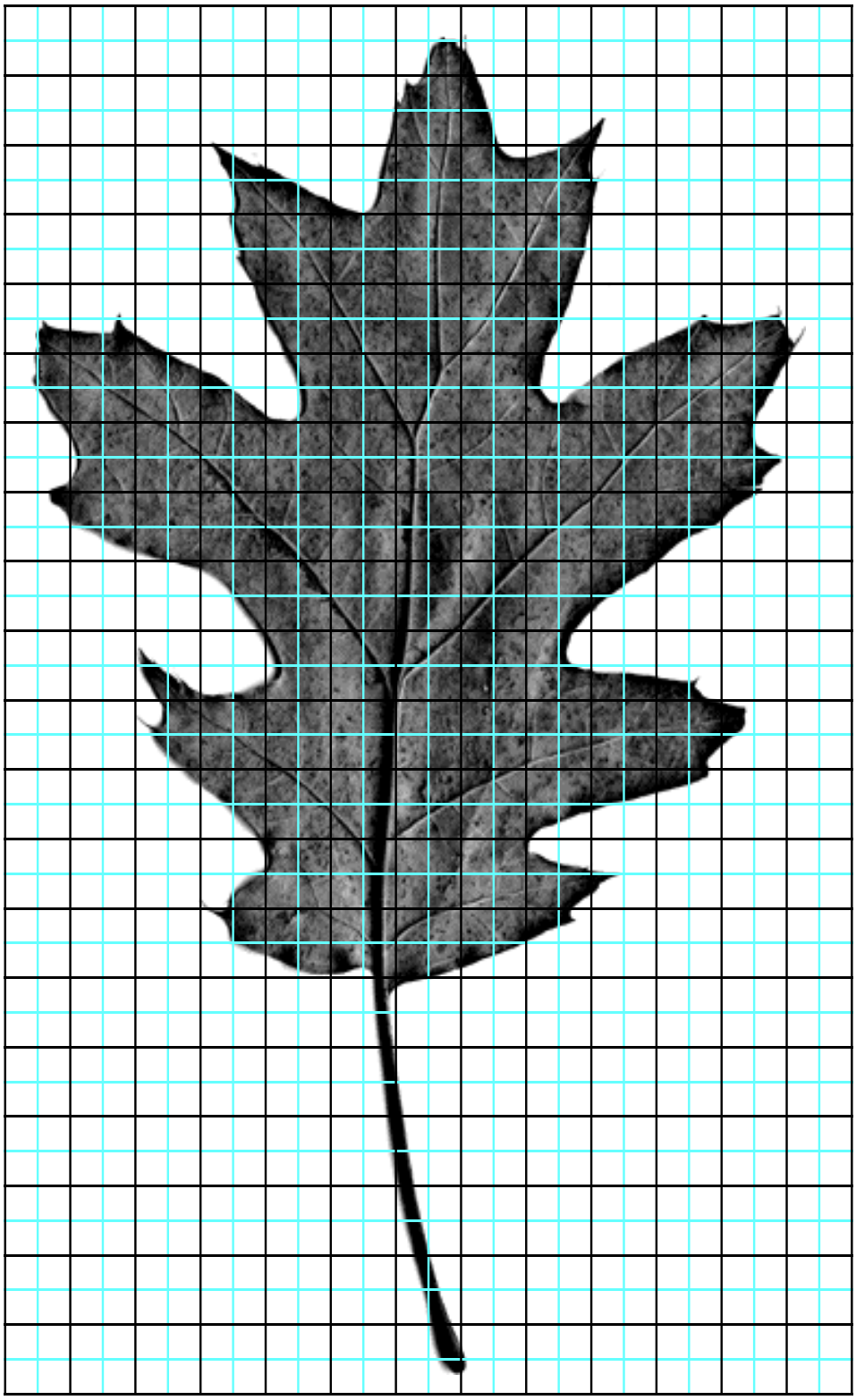


Figure 13.2: An Oak leaf with 1cm and 0.5cm grids.

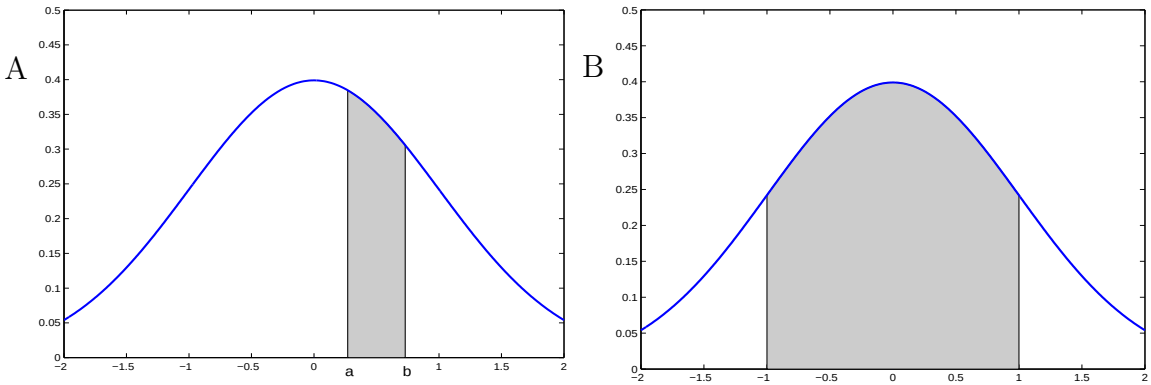


Figure 13.3: A. A graph of a portion of the particular normal distribution that has mean zero and standard deviation one, Equation 13.1, $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. The probability that a number drawn from this normal distribution lies in the interval $[a, b]$ is the area of the shaded region below the graph and above $[a, b]$. B. The shaded region, R, is bounded by the graph of the normal distribution, the X-axis, and the vertical lines $x = -1$ and $x = 1$.

The normal distribution is important because many biological measurements (heights of European women, lengths of ears of a certain variety of corn) are approximately normally distributed about their means. Such quantities are thought to be essentially the sum of a large number of independent small quantities (determined by genetic factors that have additive effects), which would account for their being normally distributed. Furthermore, for *any* distribution, the means of repeated samples from the distribution are approximately normally distributed. For example, if you randomly select 50 people from a given population and compute the mean of their heights, and repeat this experiment 100 times, then the 100 means so computed will be approximately normally distributed.

Example 13.1.1 Problem: Find (approximately) the area of the region R in Figure 13.3B bounded by the graph of the normal distribution, f, the X-axis, the vertical line, $x = -1$ and the vertical line, $x = 1$.

Solution:

1. Because of the symmetry of the region R about the Y-axis, it is sufficient to find the area of that portion of R that is to the right of the Y-axis, and then to multiply by 2. The portion R to the right of the Y-axis is shown in Figure 13.4A.

2. Compute the sum of the areas of the upper rectangles in Figure 13.4A. The width of each rectangle is 0.2 and the heights of the rectangles are determined by the density function, f. The area of the leftmost rectangle, based on the X interval $[0, 0.2]$, is

$$f(0) \times 0.2 = \frac{1}{\sqrt{2\pi}}e^{-\frac{0^2}{2}} \times 0.2 = 0.079788$$

The area of the second rectangle from the left, based on the X interval $[0.2, 0.4]$, is

$$f(0.2) \times 0.2 = \frac{1}{\sqrt{2\pi}}e^{-\frac{0.2^2}{2}} \times 0.2 = 0.078209$$

Explore 13.1.5 Find the areas of the remaining three rectangles in Figure 13.4A. ■

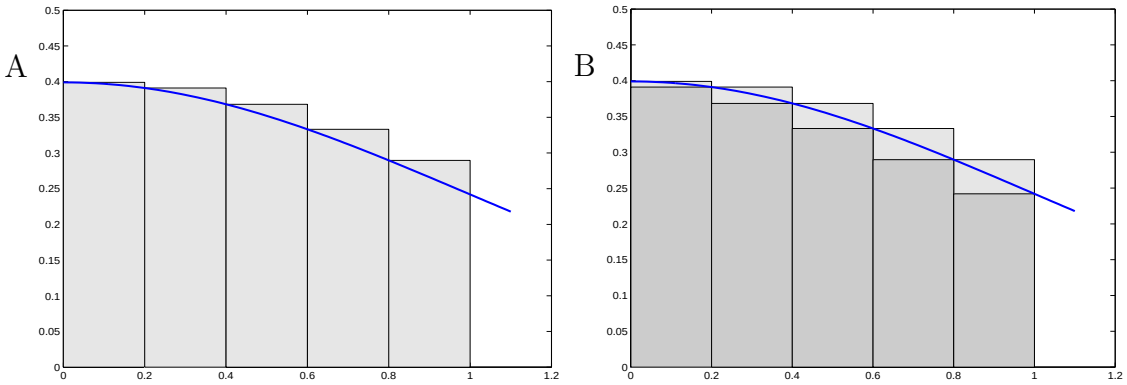


Figure 13.4: Graphs of the normal distribution density function, $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, between $x = 0$ and $x = 1$. A. Upper rectangle approximation to the area between the graph of the curve and the X-axis and the lines $x = 0$ and $x = 1$. B. Same as A with lower rectangles included.

3. The sum of the areas of the five rectangles is 0.356234; twice the sum is 0.712469. This is an over estimate of the area of R.

4. An under estimate of the area can be computed by summing the areas of the lower rectangles in Figure 13.4B. The leftmost lower rectangle, based on the X interval $[0, 0.2]$ has the same area as the second upper rectangle from the left, based on the X interval $[0.2, 0.4]$, and we computed that area to be 0.078209. The next three rectangles are similarly related to upper rectangles, and their areas are precisely the areas you computed in the preceding problem.

   The area of the right most lower rectangle is

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1^2}{2}} \times 0.2 = 0.07358$$

   and the sum of the five areas is 0.324840; twice the sum is 0.649680.

5. We now know that the area of R, is greater than 0.649680 and less than 0.712469. A good estimate of the area of R is the average of these two bounds, $\frac{0.649680 + 0.712469}{2} = 0.681075$.

   The probability of being within one standard deviation of the mean in a normal distribution is 0.68268949 correct to eight digits.

■

Ozone: The Alfred Wegner Institute Foundation for Polar and Marine Research, Bremerhaven Germany, has measured (among a large number of interesting things) ozone partial pressures at altitudes up to 35,000 meters since March 1992 at their Georg von Neumayer research Station in the Antarctic. Two relevant web sites are:

http://www.awi-bremerhaven.de/
http://www.awi.de/en/infrastructure/stations/neumayer_station/
observatories/meteorological_observatory/upper_air_soundings/ozone_soundings/

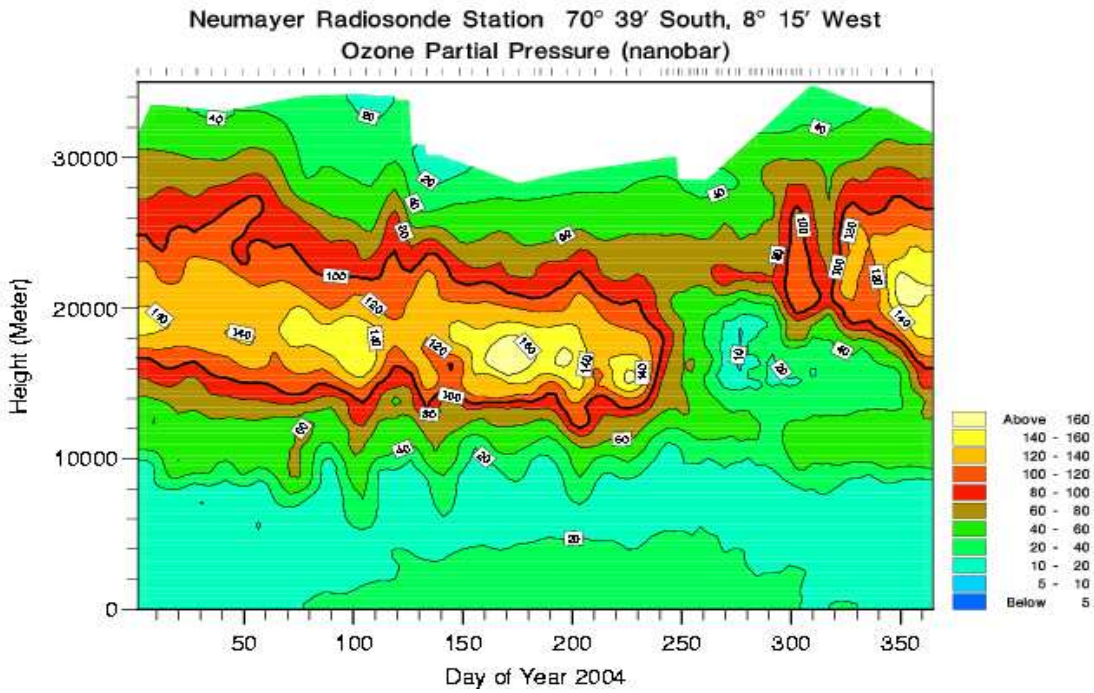
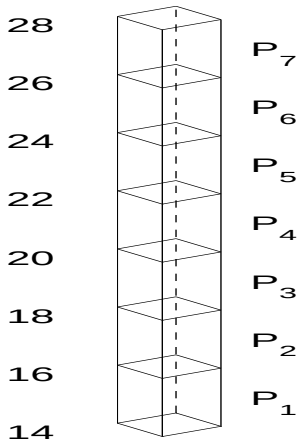


Figure 13.5: Ozone partial pressure at the Neumayer Station.

Table 13.1: Data read from Figure 13.5 for day 100.

Day 100	
Ht km	Ozone pp
28	40
26	60
24	80
22	100
20	130
18	150
16	150
14	100



Color graphs at the site show the ozone history since 1992. An image for 2004 is shown in Figure 13.5. Data read from that graph for the ozone partial pressure as a function of altitude on day 100 is shown in Table 13.1.

Assume that the "ozone layer" lies between 14 and 28 kilometers. and that the density of the ozone is proportional to the partial pressure (which would be OK were the temperature constant). We ask, how much ozone is in the 'ozone layer' on day 100 above the Neumayer station? More specifically, we assume that the density of ozone in grams per kilometer cubed is a constant, K, times the partial pressure of ozone and ask how much ozone is in a column that is above a 100 meters (=0.1 km) by 100 meters square that surrounds the Neumayer antarctic station and between the altitudes 14 and 28 kilometers (see the diagram in Table 13.1).

Consider first that portion, P_1, of the column between 14 and 16 km. The volume of P_1 is

$$0.1 \times 0.1 \times 2 \quad \text{km}^3 = 0.02 \quad \text{km}^3$$

The ozone density at 14 km is $K \times 60$ and at 16 km is $K \times 90$ gm/km^3. We assume that the average density for all of P_1 is the average of $K \times 60$ and $K \times 90$, or $K \times 75$ gm/km^3.

Then the mass of ozone in P_1 is calculated as

$$\text{Mass of ozone in } P_1 \quad = \quad 0.02 \quad \text{km}^3 \quad \times \quad K \times 75 \quad \text{gm/km}^3 \quad = \quad K \times 1.5 \quad \text{gm}$$

Similarly we calculate the mass of ozone in the portion, P_2, of the column:

$$\text{Mass of ozone in } P_2 \quad = \quad 0.02 \quad \text{km}^3 \quad \times \quad K \times \frac{90+115}{2} \quad \text{gm/km}^3 \quad = \quad K \times 2.05 \quad \text{gm}$$

Continuing we get the mass of ozone in the layer above Neumayer station to be 13.7 gm.

Exercises for Section 13.1, Areas of Irregular Regions.

Exercise 13.1.1 Find (approximately) the area of the region in Figure 13.3 bounded by the graph of the normal distribution, f, the X-axis, the vertical line, $x = -2$ and the vertical line, $x = 2$.

Because we found the area of the region R bounded by $x = -1$ and $x = 1$ you might reasonably conclude that your main task is to find the area of the region bounded by the graph of the normal distribution, the X-axis, the lines $x = 1$ and $x = 2$. Then add some numbers.

You should find that the probability of being within two standard deviations of the mean of the normal distribution is approximately 0.95 (the probability is 0.95449974 correct to eight digits).

Exercise 13.1.2 Shown in Figure 13.6A are the hourly readings for solar radiation intensity for June 21, 2004 recorded by the University of Oregon Solar Radiation Monitors Laboratory at their station in Eugene, Oregon (http://solardat.uoregon.edu). The data are for 'global irradiance' which includes all radiation falling on a horizontal plate measured in watts/meter^2.

The data in Figure 13.6A shows that, for example, at 10 am the solar intensity was 780 w/m^2 on June 21, 2004 in Eugene, Oregon. If the solar intensity were constant at that value from, say, 9:30 am to 10:30 am, then the energy that would fall on a 1 meter^2 panel during that time would be 780 w-hr. This is the area of the rectangle encompassing 10 am in Figure 13.6A. The solar energy striking a 1 m^2 panel during the whole day is approximately the sum of the areas of the rectangles shown in Figure 13.6B.

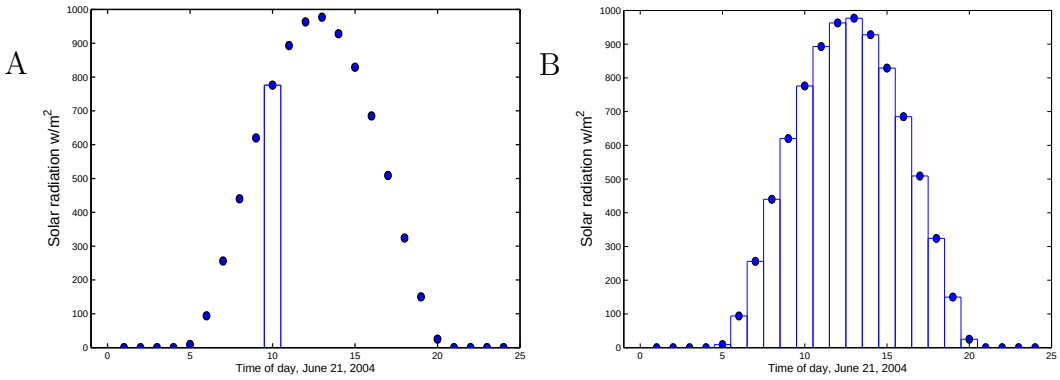


Figure 13.6: A graph of the solar intensity at Eugene, Oregon for June 21, 2004 (http://solardat.uoregon.edu). The measure is global irradiance which is the total energy striking a horizontal plate measured in watts/meter^2.

a. How many watt-hours of energy struck a 1 meter^2 panel at Eugene, Oregon on June 21, 2004?

b. What is the value of that amount of energy at \$0.08 per kilowatt-hour?

Exercise 13.1.3 Shown in Figure 13.1.3 is a graph of the average daily solar intensity for Eugene, Oregon plotted as a function of day number, $1 \leq day \leq 365$. (The average is for the years 1990 - 2004, except for 2001 for which no data are posted.)

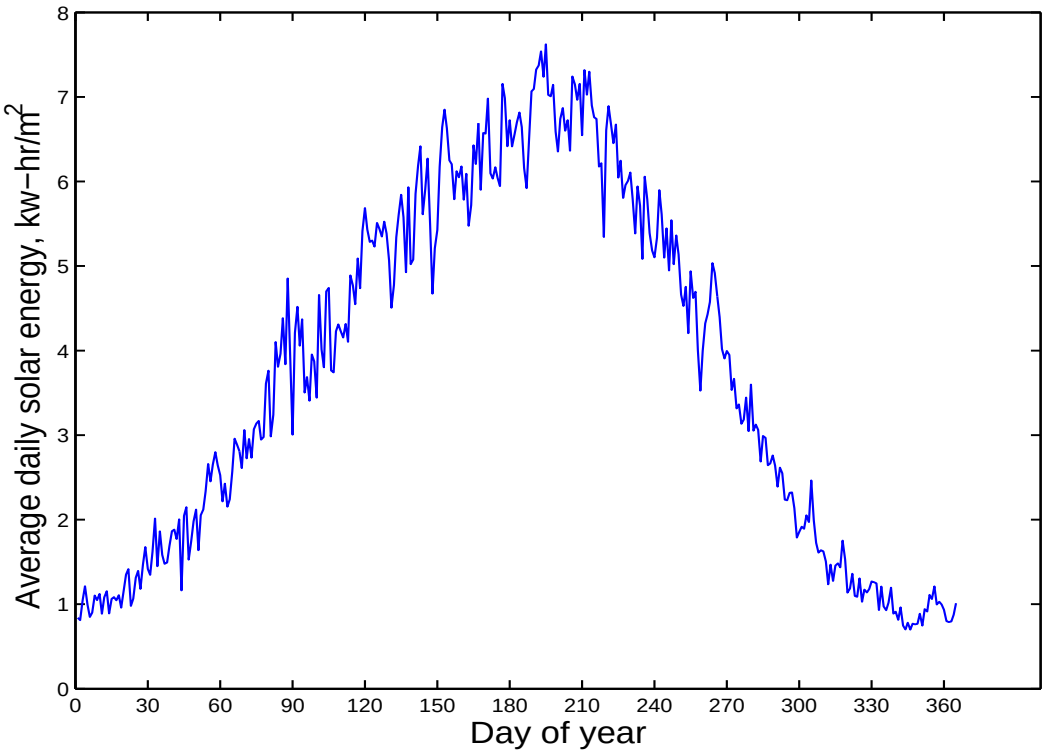
a. Was the sun intensity on June 21, 2004 above or below average for June 21 according to your answer to Exercise 13.1.2

b. Use the graph in Figure 13.1.3 to compute in two different ways the total energy and the value of the total energy striking a one meter square panel for one year at Eugene, Oregon.

1. Partition the year into 6 60-day intervals plus a single 5-day interval. Use as the solar intensity the average of the highest and lowest energies for that interval.
2. Partition the year into 12 30-day intervals plus a single 5-day interval. Use as the solar intensity for each interval the average of the solar intensities at the first and last days of the interval.

c. At \$0.08 per kw-hr, what is the value of the average solar energy that struck a one meter square panel for one year at Eugene, Oregon?

Figure for Exercise 13.1.3 A graph of the average solar intensity at Eugene, Oregon as a function of day of year.



Exercise 13.1.4 Dr. Frank Vignola of the University of Oregon forwarded the following practical problem. Pixels in satellite images are used to estimate solar irradiance values. The satellite images are taken at 15 minutes after the hour and the users need hourly values. How would one estimate hourly values from the data obtained at 15 minutes after the hour and how are the data degraded when the data are shifted?

Exercise 13.1.5 a. Using the data shown in Figure 13.1.3, let $E(t)$ be the cumulative energy striking a one meter-square panel from day one to day t. Compute $E(30)$, $E(60)$, $E(90)$, $E(120)$, $E(150)$, and $E(180)$.

b. Plot your data.

c. Estimate $E(135)$ and $E(140)$ from your data.

d. At (approximately) what rate is $E(t)$ increasing at time $t = 135$?

Exercise 13.1.6 a. Draw a horizontal line on Figure 13.1.3 so that the area of the rectangle bounded above by your horizontal line and below by the t-axis on the left by the vertical axis, $t = 0$ and on the right by the vertical line $t = 365$ is the same (approximately) as the area under the solar intensity curve (Figure 13.1.3).

b. Where does your line cross the vertical axis?

c. On what days does your line cross the curve?

d. What is the average daily solar intensity for the year?

Exercise 13.1.7 Compute the mass of ozone in the layer between 14 and 28 km above the 0.1 km square surrounding the Neumayer station for days 150 and 280. It may be easier to read the color graph at http://www.awi-bremerhaven.de/MET/, search for 'ozone 2004.'

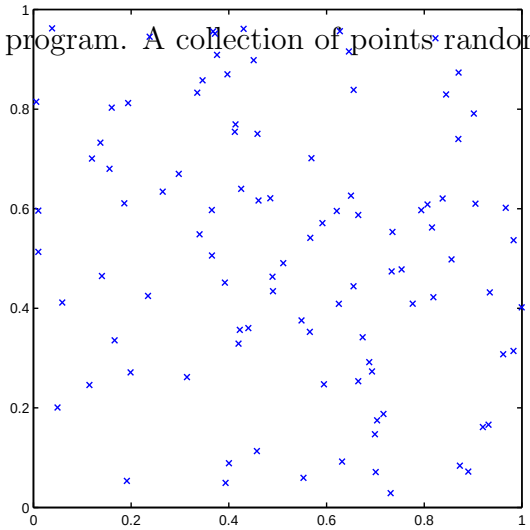
At what time of year is there 'a hole in the ozone?'

Monte Carlo Integration. Although as noted we will not make serious use of Monte Carlo integration (throwing darts at pictures of leaves), you may find the next three exercises interesting and they may add to your understanding of 'area'.

Exercise 13.1.8 Decide what will be meaning of the output of the program in Exercise Figure 13.1.8. (it is written for a TI-86, but very similar language is used for other calculators and computers).

A better procedure is to put the program on your calculator, run it, and while it is running (which takes about 20 minutes) decide what will be meaning of the output.

Figure for Exercise 13.1.8 A calculator program. A collection of points randomly distributed in a square with sides of length 1.




```
Program:MNTCRL
:0 -> NN
:For(K,1,10000)
:X=rand
:Y=rand
:If (X^2 + Y^2) < 1
:NN+1 -> NN
:End
:Display 4*NN/10000
```

Exercise 13.1.9 Modify the previous program to estimate the area of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Exercise 13.1.10 Modify the previous program to estimate the area of the region bounded by the graphs of

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad y = 0, \qquad x = 0, \qquad \text{and} \qquad x = 1$$

Exercise 13.1.11 If an object moves at a constant speed, s m/min, over a time interval $[a, b]$ minutes, the distance, D, traveled is $D = s$ m/min $\times (b - a)$ min, $D = s \times (b - a)$ meters.

If the speed $s(t)$ varies over $[a, b]$, the distance, D, can be approximated by partitioning $[a, b]$ by $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ into n subintervals so that $s(t)$ is approximately constant on each subinterval. Then D is the sum of the distances traveled during each subinterval and

$$D \doteq s(t_1) \times (t_1 - t_0) + s(t_2) \times (t_2 - t_1) + \cdots + s(t_n) \times (t_n - t_{n-1}) \quad \text{meters.}$$

Approximate the distance traveled by a particle moving at the following speeds and over the indicated time intervals. Partition each time interval into 5 subintervals.

a.	$s(t)$	$=$	t	on	$1 \le t \le 6$	b.	$s(t)$	$=$	t^2	on	$0 \le t \le 2$
c.	$s(t)$	$=$	$1/t$	on	$1 \le t \le 3$	d.	$s(t)$	$=$	$\sqrt{t}$	on	$2 \le t \le 7$

Exercise 13.1.12 Problem: What is the mass of a column of air above a one-meter square table top?

Imagine a one meter square vertical column, C, of air starting at sea level and extending upwards to 40,000 meters.

Shown in Figure 13.1.12 are 'standard' values of atmospheric density as a function of altitude up to 40,000 meters (U.S. Standard Atmospheres 1976, National Oceanic and Atmospheric Administration, NASA, U.S. Air Force, Washington, D.C. October 1976). The density is not constant throughout C, and the mass of C may be approximated by partitioning C into regions of 'small' volume, assuming that the density is constant within each region, and approximating the mass within each of the regions. Then the object will approximately have

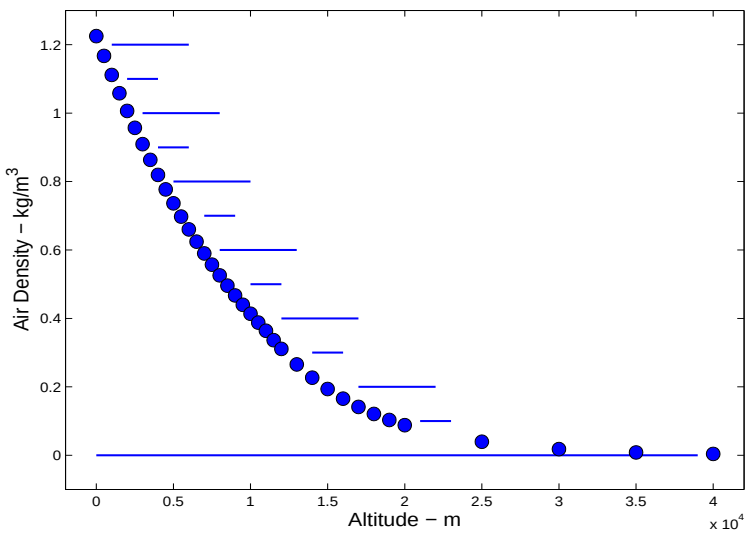
$$\text{Mass} \qquad = \delta_1 \times v_1 + \delta_2 \times v_2 + \cdots + \delta_n \times v_n$$

where δ_k Kg/m^3 and v_k m^3 are respectively the density and volume in the $k\underline{\text{th}}$ region.

a. Partition the altitude interval from 0 to 40,000 meters into at least 6 subintervals (they need not be all the same length). For each interval, find the volume of the region of C corresponding to the interval and find a density of the air within that region. Approximate the mass in each region and add the results to approximate the mass of air in C.

b. Does the size of your answer surprise you? Show that it is consistent with the standard pressure of one atmosphere being 760 mm of mercury? The density of mercury is $1.36 \times 10^4 Kg/m^3$.

Figure for Exercise 13.1.12 Air density at various altitudes up to 40,000 meters. The horizontal segments are intended to assist in reading data from the graph.



13.2 Areas Under Some Algebraic Curves

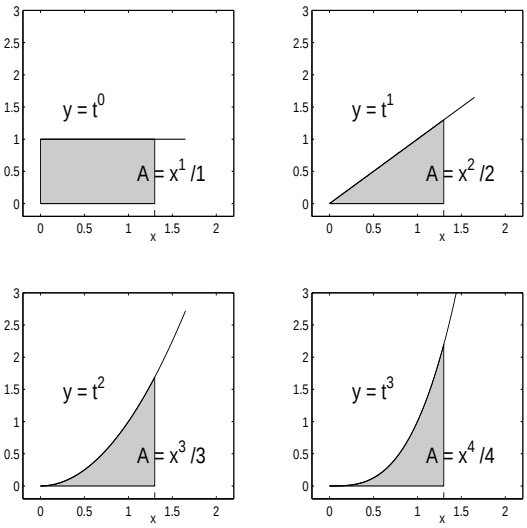
In This Section

Formulas are developed for the area of the region bounded by the t-axis, the graphs of $y = t^n$, $t = 0$, and $t = x$ for $n = 0$, 1, 2, and 3. The result is that:

The area of the region bounded by the t-axis and the graph of $\mathbf{y = t^n}$ between $\mathbf{t = 0}$ and $\mathbf{t = x}$ is

$$\frac{\mathbf{x^{n+1}}}{\mathbf{n+1}}$$

for $\mathbf{n = 1}$, 2, 3, and 4. The formula is actually valid for all values of n except $\mathbf{n = -1}$.



The work of this section requires some arithmetic formulas. Let n be a positive integer.

$$\underbrace{1+1+1+\cdots+1}_{n \quad terms} = n \tag{13.2}$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2} \tag{13.3}$$

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \tag{13.4}$$

$$1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} \tag{13.5}$$

Explore 13.2.1 Do this.

a. Execute the following command on your calculator and interpret the result.

```
seq(k^2,k,1,5)
```

b. Now execute the command

```
sum seq(k^2,k,1,5)
```

and interpret the result.

Note: **sum** and **seq** can be found in the menu

2nd LIST OPS

c. Check that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)\,(2n+1)}{6}$ is valid for $n = 10$ and $n = 5000$.

d. For $n = 10000$ the calculator runs out of memory. However, you can write a short program that will compute the sum for $n = 10000$.

e. Check that

$$1 + 2 + \cdots + n = \frac{n \times (n+1)}{2}$$

is valid for $n = 100$. [1]

■

We show an argument that Equation 13.4 is valid for all integers n. The method extends to the other equations.

The observation that first order difference equations with initial conditions specified have unique solutions is crucial to these arguments. See discussion beginning on page 470.

Example 13.2.1 Demonstration of the validity of

$$\mathbf{1}^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)\,(2n+1)}{6}$$

The linear difference equation with initial condition

$$\begin{aligned} w_1 &= 1 \\ w_t - w_{t-1} &= t^2 \qquad \text{for } t = 2,\, 3,\, \cdots \end{aligned} \tag{13.6}$$

defines a unique sequence, w_1, w_2, w_3, $\cdots$. We will show that both expressions

$$w_t = \frac{t(t+1)(2t+1)}{6} \qquad \text{for } t = 1,\, 2,\, 3,\, \cdots \tag{13.7}$$

and

$$w_t = 1^2 + 2^2 + 3^2 + \cdots + t^2 \qquad \text{for } t = 1,\, 2,\, \cdots \tag{13.8}$$

satisfy the difference equation. Because there is only one solution, the two expressions must be the same, thus establishing that

$$1^2 + 2^2 + 3^3 + \cdots + t^2 = \frac{t(t+1)(2t+1)}{6} \qquad \text{for } t = 1,\, 2,\, 3,\, \cdots$$

[1] Carl F. Gauss's fourth grade teacher assigned his class the task of computing $1 + 2 + \cdots 99 + 100$ (probably to keep the little rascals busy for a while). Young Carl quickly responded, 5050, and explained that $1 + 2 + \cdots 99 + 100 = (1 + 100) + (2 + 99) + \cdots (50{+}51) = 50 \times 101 = 5050$. Note that $50 \times 101 = \frac{100 \times 101}{2}$. To his credit, the teacher promptly recognized genius, and brought Gauss to the attention of the local nobleman.

Step 1. $w_t = \dfrac{t(t+1)(2t+1)}{6}$ satisfies Equation 13.6.

$$w_1 = \frac{1(1+1)(2\times 1+1)}{6} = 1$$

so that the initial condition is satisfied. If $t > 1$,

$$\begin{aligned} w_t - w_{t-1} &= \frac{t(t+1)(2t+1)}{6} - \frac{(t-1)((t-1)+1)(2(t-1)+1)}{6} \\ &= \frac{t(t+1)(2t+1)}{6} - \frac{(t-1)t(2t-1)}{6} \\ &= \frac{t}{6}\left((t+1)(2t+1) - (t-1)(2t-1)\right) \\ &= \frac{t}{6}\left(2t^2 + 3t + 1 - (2t^2 - 3t + 1)\right) \\ &= t^2, \end{aligned}$$

so that the difference equation is also satisfied.

Step 2. $w_t = 1^2 + 2^2 + \cdots + t^2$ for $t \geq 1$ satisfies Equation 13.6.

The initial condition, $w_1 = 1^2 = 1$ is satisfied. If $t > 1$,

$$\begin{aligned} w_t - w_{t-1} &= (1^2 + 2^2 + \cdots + t^2) - (1^2 + 2^2 + \cdots + (t-1)^2) \\ &= t^2, \end{aligned}$$

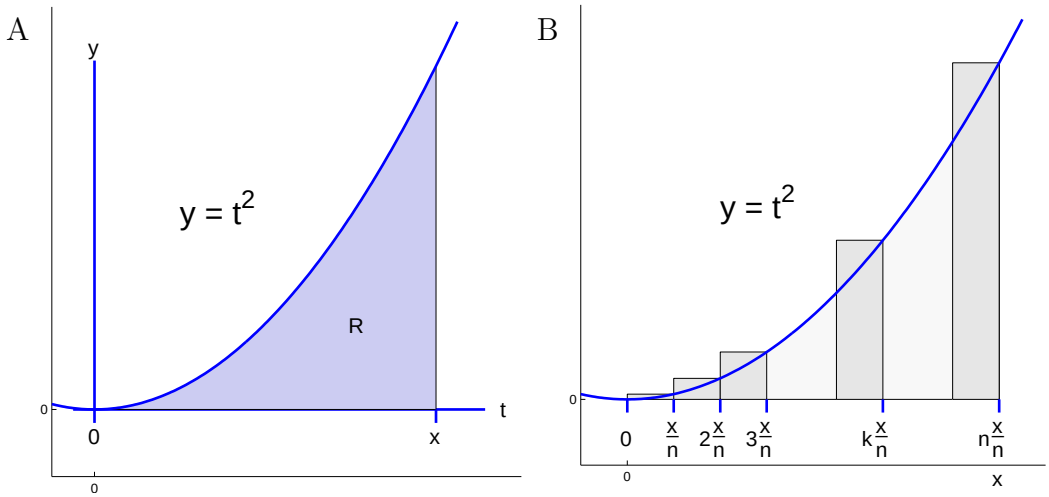
so that the difference equation is satisfied for all t.

Only one sequence satisfies the difference equation with initial condition, Equation 13.6, and both Equations 13.7 and 13.8 define a sequence satisfying Equation 13.6. They must therefore be the same sequence, so that

$$1^2 + 2^2 + 3^2 + \cdots + t^2 = \frac{t(t+1)(2t+1)}{6} \qquad \text{for} \qquad t = 1, 2, 3, \cdots \qquad \blacksquare$$

Example 13.2.2 *Problem.* Find the area of the region **R** illustrated in Example Figure 13.2.2.2A that is bounded by the parabola $\mathbf{y} = \mathbf{t^2}$, the **t**-axis, and the line $\mathbf{t} = \mathbf{x}$ where x is a positive number.

Figure for Example 13.2.2.2 A. The region R bounded by the parabola $y = t^2$, the T-axis, and the line $t = x$. B. The same parabola with n rectangles with bases in $[0, x]$.



Solution. The sum of the areas of the rectangles in Figure 13.2.2.2B approximates the area of R. The interval $[0, x]$ on the horizontal axis has been partitioned into n subintervals by the numbers

$$0, \quad \frac{x}{n}, \quad 2\frac{x}{n} \quad \cdots \quad (n-1)\frac{x}{n}, \quad n\frac{x}{n}$$

The width of each rectangle in the figure is $\frac{x}{n}$. The height of each rectangle is the height of the parabola $y = t^2$ at the right end of the base of the rectangle. The heights of the rectangles, from left to right, are

$$\left(\frac{x}{n}\right)^2, \quad \left(2\frac{x}{n}\right)^2, \quad \cdots \quad \left((n-1)\frac{x}{n}\right)^2, \quad \left(n\frac{x}{n}\right)^2$$

The sum, U_n, of the areas of the rectangles is

$$\begin{aligned} U_n &= \left(\tfrac{x}{n}\right)^2 \times \tfrac{x}{n} + \left(2\tfrac{x}{n}\right)^2 \times \tfrac{x}{n} + \cdots + \left((n-1)\tfrac{x}{n}\right)^2 \times \tfrac{x}{n} + \left(n\tfrac{x}{n}\right)^2 \times \tfrac{x}{n} \\ &= \left(1^2 + 2^2 + \cdots + (n-1)^2 + n^2\right) \frac{x^3}{n^3} \\ &= \frac{n(n+1)(2n+1)}{6} \frac{x^3}{n^3} \qquad (13.9) \\ &= \frac{2n^3 + 3n^2 + n}{6n^3}\, x^3 \\ &= \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) x^3 \end{aligned}$$

This is powerful medicine. The region R covered with $n = 50$ rectangles is illustrated in Figure 13.2.2.2C. The sum of the areas of the fifty rectangles, U_{50}, is close to the area of R, and

$$U_{50} = \left(\frac{1}{3} + \frac{1}{2 \times 50} + \frac{1}{6 \times 50^2}\right) x^3 = \left(\frac{1}{3} + 0.01 + 0.000067\right) x^3 \quad \doteq \quad \frac{1}{3} x^3$$

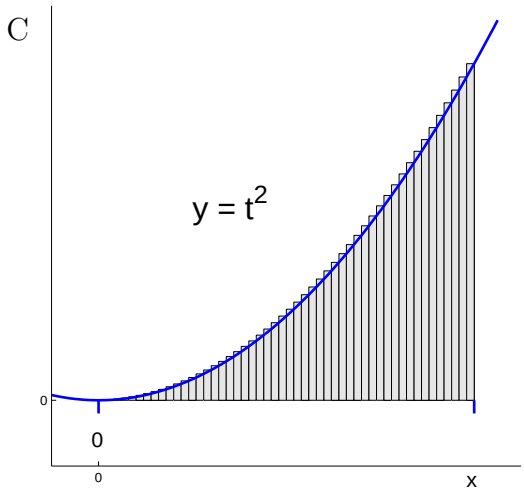
Furthermore,

$$\lim_{n\to\infty} U_n \quad = \quad \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) x^3 \quad = \quad \frac{x^3}{3}.$$

In Section 13.4 the area of R is defined to be this limit, so that:

$$\text{The area of } R \text{ is} \qquad \frac{x^3}{3}. \tag{13.10}$$

Figure for Example 13.2.2.2 (Continued) C. The parabola of A with 50 rectangles with bases in [0,1].



Some notation. We have been using the symbol, $\cdots$, called an ellipsis. Consider the sum

$$\left(\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)^2 + \cdots + \left(\frac{9}{10}\right)^2 + \left(\frac{10}{10}\right)^2$$

The *ellipsis* ($\cdots$) replaces some of the terms. In this particular instance, $\cdots$ replaces

$$\left(\frac{4}{10}\right)^2 + \left(\frac{5}{10}\right)^2 + \left(\frac{6}{10}\right)^2 + \left(\frac{7}{10}\right)^2 + \left(\frac{8}{10}\right)^2$$

The excitement of writing these additional terms is rather limited, and most students readily accept this abbreviation[2]. It is particularly useful in writing sums with 1000 or more terms. On occasions, one sees something like

$$\left[\left(\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)^2 + \cdots + \left(\frac{k}{10}\right)^2 + \cdots + \left(\frac{10}{10}\right)^2\right] \times \frac{1}{10}$$

The term $\left(\frac{k}{10}\right)^2$ is a generic formula for the kth term in the sum. The generic term is useful, and it is important to be able to recognize a pattern for the terms of the sum. When $k = 1$ is substituted into the generic formula $\left(\frac{k}{10}\right)^2$ we get $\left(\frac{1}{10}\right)^2$ which is the first term of the sum. Similarly, substitution of $k = 2$ yields the second term, and substitution of $k = 10$ yields the 10th term.

[2]This is easily grasped, as illustrated by the title of a children's book, "One, Two, Skip a Few, Ninety-nine, One Hundred".

Some sums and generic terms are illustrated below.

$\sqrt{7}+\sqrt{8}+\sqrt{9}+\sqrt{10}+\sqrt{11}$	$\sqrt{k}$	$\sum_{k=7}^{11}\sqrt{k}$
$\sqrt{7}+\sqrt{8}+\sqrt{9}+\sqrt{10}+\sqrt{11}$	$\sqrt{k+6}$	$\sum_{k=1}^{5}\sqrt{k+6}$
$1^2+3^2+5^2+7^2+9^2+11^2$	$(2k-1)^2$	$\sum_{k=1}^{6}(2k-1)^2$
$\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+\frac{1}{1}$	$\frac{1}{6-k}$	$\sum_{k=1}^{5}\frac{1}{6-k}$
$\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2.4}+\frac{1}{2.6}+\frac{1}{2.8}+\frac{1}{3}$	$\frac{1}{1.8+k\times 0.2}$	$\sum_{k=1}^{6}\frac{1}{1.8+k\times 0.2}$
$\sin(\frac{\pi}{5})+\sin(2\frac{\pi}{5})+\sin(3\frac{\pi}{5})+\sin(4\frac{\pi}{5})+\sin(5\frac{\pi}{5})$	$\sin(k\frac{\pi}{5})$	$\sum_{k=1}^{5}\sin(k\frac{\pi}{5})$

There is a compact notation that focuses attention on the generic term:

$$\sum_{k=1}^{10}\left[\left(\frac{k}{10}\right)^2\times\frac{1}{10}\right]$$

$\sum$ is the Greek letter, sigma, to which the Latin letter S corresponds and symbolizes sum. The entire symbol is read "The sum from k equal to 1 to 10 of $\left(\frac{k}{10}\right)^2$ times $\frac{1}{10}$". It may also be read as "The sum from k equal to 1 to k equal to 10 of $\left(\frac{k}{10}\right)^2$ times $\frac{1}{10}$" and might be written

$$\sum_{k=1}^{10}\left[\left(\frac{k}{10}\right)^2\times\frac{1}{10}\right] \qquad \text{or} \qquad \left[\sum_{k=1}^{k=10}\left(\frac{k}{10}\right)^2\right]\times\frac{1}{10}$$

Typesetters find the Σ notation preferable as it is much more compact. In fact, after you use it a while, you will likely find that you can more quickly grasp the meaning of the sum with the shorter notation.

It will help you to use algebra like

$$\left[\sum_{k=1}^{10}\left(\frac{k}{10}\right)^2\times\frac{1}{10}\right]=\left[\sum_{k=1}^{10}\left(\frac{k}{10}\right)^2\right]\times\frac{1}{10}$$

This is use of the distributive property, and can be seen from long notation with which you are familiar:

$$\left[\sum_{k=1}^{10}\left(\frac{k}{10}\right)^2\times\frac{1}{10}\right] =$$

$$\left[\left(\frac{1}{10}\right)^2\times\frac{1}{10}+\left(\frac{2}{10}\right)^2\times\frac{1}{10}+\cdots+\left(\frac{9}{10}\right)^2\times\frac{1}{10}+\left(\frac{10}{10}\right)^2\times\frac{1}{10}\right] =$$

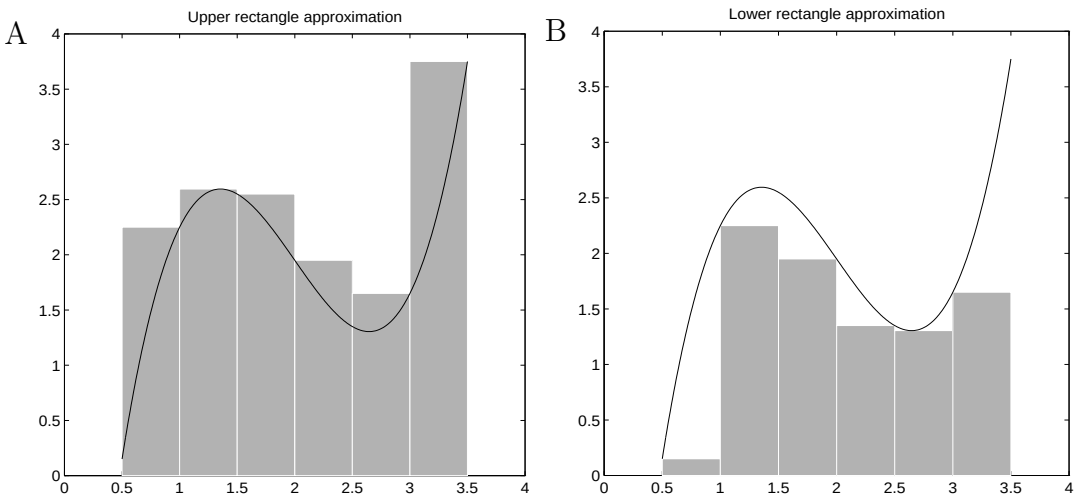


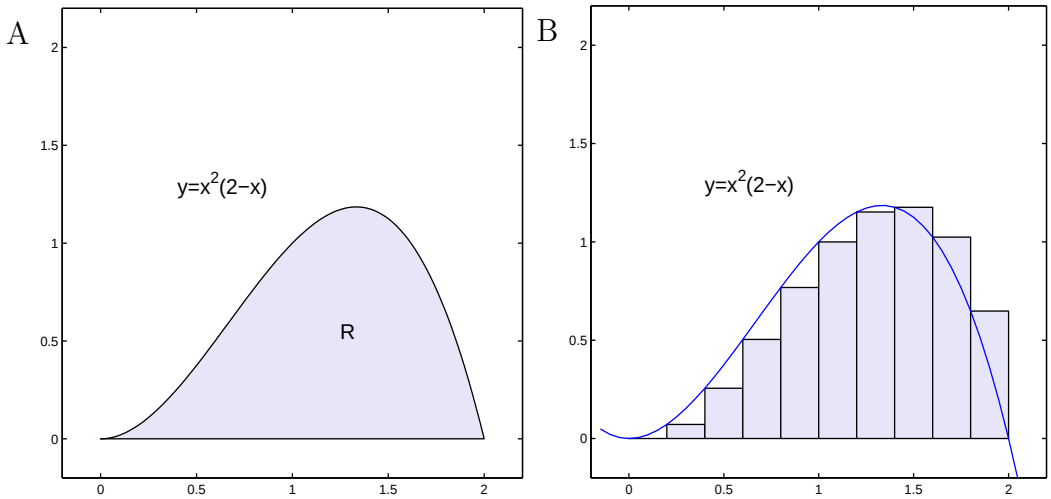
Figure 13.7: A. Upper rectangles approximating the area of a region bounded above by the graph of a function, f. B. Lower rectangles approximating the area of the same region.

$$\left[\left(\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)^2 + \cdots + \left(\frac{9}{10}\right)^2 + \left(\frac{10}{10}\right)^2\right] \times \frac{1}{10} = \left[\sum_{k=1}^{10} \left(\frac{k}{10}\right)^2\right] \times \frac{1}{10}$$

Upper and lower rectangles. In the Example 13.2.2 the top edges of the approximating rectangles were entirely on or above the graph of $y = t^2$ and are called *upper rectangles* and their sum, U_{50}, is an *upper approximation* to the area under consideration, as shown in Figure 13.7A. When the top edges of the rectangles lie entirely on or below the graph, the rectangles are called *lower rectangles* and their sum is called a *lower approximation*, as shown in Figure 13.7B. The area of the region lies between the upper and lower approximations.

Example 13.2.3 *Problem.* Find approximately the area of the region **R** illustrated in Example Figure 13.2.3.3A bounded by the cubic $\mathbf{y = x^2(2 - x)}$, and the **X**-axis.

Figure for Example 13.2.3.3 A. The region R bounded by the cubic $y = x^2(2 - x)$ and the X-axis B. The same cubic with 10 rectangles with bases in [0,2].



The cubic $y = x^2(2-x) = 2x^2 - x^3$ intersects the x-axis at $x = 0$ and $x = 2$. The rectangles in Figure 13.2.3.3 partition [0,2] into ten intervals and the sum of the areas of the rectangles approximates the area of the region R. The rectangles are a mixture of lower and upper rectangles. The base of each rectangle is 0.2 and its height is the height of the *left* edge. For $k = 0$ to 9, the height of the $k+1\underline{\text{st}}$ rectangle, $[k \times 0.2, (k+1) \times 0.2]$, is $2 \times (k \times 0.2)^3 - (k \times 0.2)^2$. The area of the $k+1\underline{\text{st}}$ rectangle is

$$\left[\, 2 \times (k \times 0.2)^2 - (k \times 0.2)^3 \,\right] \times 0.2$$

The sum of the areas of the rectangles is

$$\begin{aligned}
\sum_{k=0}^{9} \left[\, 2 \times (k \times 0.2)^2 - (k \times 0.2)^3 \,\right] \times 0.2 &= \\
\sum_{k=0}^{9} \left[\, 2(0.2)^2 \times k^3 - (0.2)^3 \times k^2 \,\right] \times 0.2 &= \\
2(0.2)^3 \sum_{k=0}^{9} k^3 - (0.2)^4 \sum_{k=0}^{9} k^2 &= \\
2(0.2)^3 \, \frac{9(9+1)(2 \times 9 + 1)}{6} - (0.2)^4 \, \frac{9^2(9+1)^2}{4} &= 1.32 \qquad \blacksquare
\end{aligned}$$

Exercises for Section 13.2, Areas Under Some Algebraic Curves.

Exercise 13.2.1 Follow the steps of Example 13.2.1 to show the validity of Equation 13.3

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

To do so, show that unique solution to the difference equation

$$\begin{aligned}
w_1 &= 1 \\
w_t - w_{t-1} &= t \qquad \text{for } t = 2, 3, \cdots
\end{aligned} \tag{13.11}$$

may be written as

$$w_t = \frac{t(t+1)}{2} \qquad \text{for } t = 0,\, 1,\, 2,\, \cdots$$

and also written as

$$\begin{aligned} w_0 &= 0 \\ w_t &= 1 + 2 + 3 + \cdots + t \qquad \text{for } t = 1,\, 2,\, \cdots \end{aligned}$$

thus proving the validity of Equation 13.3.

Exercise 13.2.2 Show the validity of Equation 13.5

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Exercise 13.2.3 Find the generic terms for the following sums, and write the sums using Σ notation.

a. $2^2 + 4^2 + 6^2 + 8^2 + 10^2$

b. $\frac{1}{1.1} + \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{1.4} + \frac{1}{1.5} + \frac{1}{1.6} + \frac{1}{1.7} + \frac{1}{1.8} + \frac{1}{1.9} + \frac{1}{2}$

c. $\left[\left(\frac{1}{1}\right)^2 + \left(\frac{1}{1.1}\right)^2 + \left(\frac{1}{1.2}\right)^2 + \cdots + \left(\frac{1}{1.8}\right)^2 + \left(\frac{1}{1.9}\right)^2\right] \times \frac{1}{10}$

d. $[(-1.0)^3 + (-0.9)^3 + (-0.8)^3 + + \cdots + (-0.2)^3 + (-0.1)^3] \times \frac{1}{10}$

e. $\left[\sqrt{1-0.1^2} + \sqrt{1-0.2^2} + \sqrt{1-0.3^2} + \cdots + \sqrt{1-0.9^2} + \sqrt{1-1.0^2}\right] \times \frac{1}{10}$

Exercise 13.2.4 Give reasons for

a. $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$

b. $\sum_{k=2}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} - 1$

Exercise 13.2.5 Evaluate:

a. $\sum_{k=1}^{100} k$ b. $\sum_{k=3}^{100} k$ c. $\sum_{k=18}^{100} k$ d. $\sum_{k=1}^{100} 2k$

e. $\sum_{k=1}^{10} k^2$ f. $\sum_{k=3}^{100} k^3$ g. $\sum_{k=1}^{100} (k+5)$ h. $\sum_{k=50}^{100} (3k+7)$

i. $\sum_{k=1}^{10} 5k^2$ j. $\sum_{k=3}^{100} (k^3 - 9)$ k. $\sum_{k=32}^{78} (3k^2 - k)$ l. $\left(\sum_{k=50}^{100} 3k\right) + 7$

Exercise 13.2.6 See Exercise Figure 13.2.6. Show that the lower approximation to the area of the region bounded by the graph of $y = x^2$, the x-axis, and the line $x = 1$ based on 10 equal subintervals is:

a.

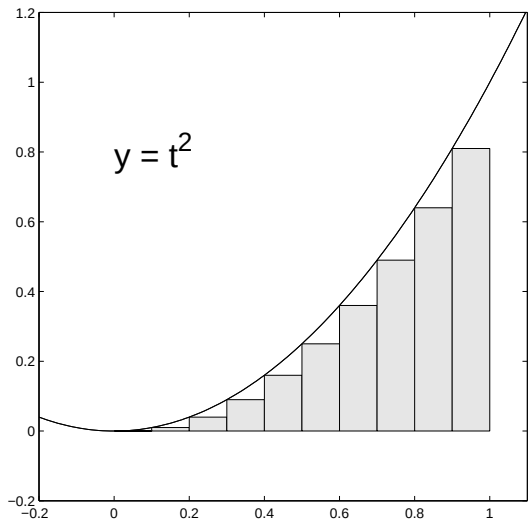
$$\sum_{k=0}^{9} \left(\frac{k}{10}\right)^2 \times \frac{1}{10}$$

b.

$$= \frac{9 \times 10 \times 19}{6} \times \frac{1}{10^3}$$

c. The same as the upper sum minus the area of the tenth rectangle in the upper sum.

Figure for Exercise 13.2.6 Lower sum rectangles approximating the area of the region bounded by $y = x^2$, $y = 0$ and $x = 1$.



Exercise 13.2.7 a. Find a formula for the sum, L_n, of n lower rectangles approximating the area of the parabolic region, R, of Figure 13.2.2.2A.

b. Condense L_n into a single term as was done in Equations 13.9 for U_n.

c. Evaluate L_{50}.

d. Evaluate $\lim_{n\to\infty} L_n$.

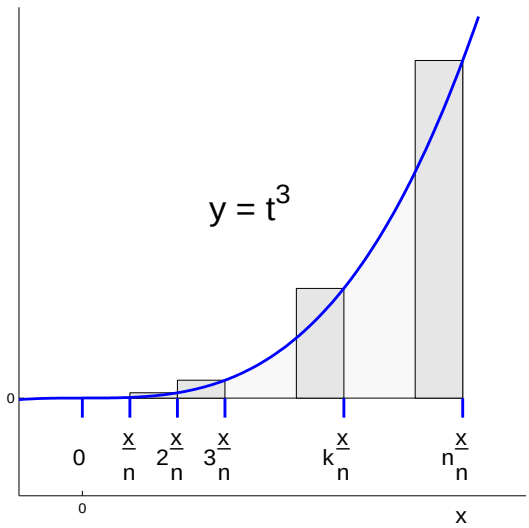
Exercise 13.2.8 Let R be the region bounded by the graph of $y = t^3$, the t-axis, and the line $t = x$. See Figure 13.2.8

a. Write a formula for an upper sum, U_n, of areas of n rectangles that approximates the area of R.

b. Condense U_n into a single term as was done in Equations 13.9 for U_n of the parabolic region.

c. Find $\lim\limits_{n\to\infty} U_n$.

Figure for Exercise 13.2.8 The region bounded by $y = t^3$, the t-axis, and the line $t = x$ approximated by n rectangles.



Exercise 13.2.9 Compute the area of the region bounded by the graph of $y = t^2$, the t-axis, and the lines $t = 1$ and $t = 2$. (Hint: Use Equation 13.10 and skip the rectangle bit.)

Exercise 13.2.10 Shown in Figure 13.2.10 is the first quadrant portion of the graph of $x^2 + y^2 = 1$.

a. Use $x^2 + y^2 = 1$ to compute the height of the fourth rectangle from the left.

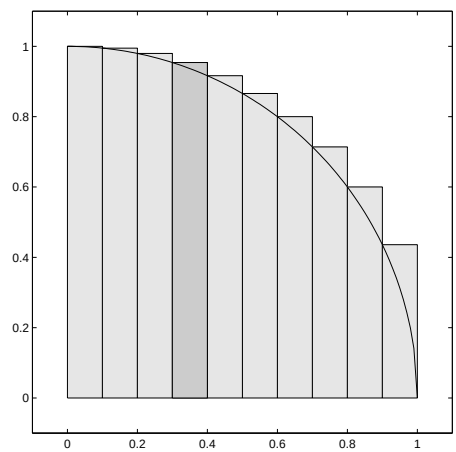
b. Compute the area of the fourth rectangle from the left.

c. Write (do not evaluate) an expression for, U_{10}, that is the sum of the areas of the ten rectangles.

d. Write an expression for U_{100} that is the sum of the areas of 100 rectangles that approximates the area of the quarter circle.

e. Use your calculator to compute U_{100} and compare it with $\pi/4$.

Figure for Exercise 13.2.10 The first quarter portion of the circle $x^2 + y^2 = 1$ and 10 rectangles.

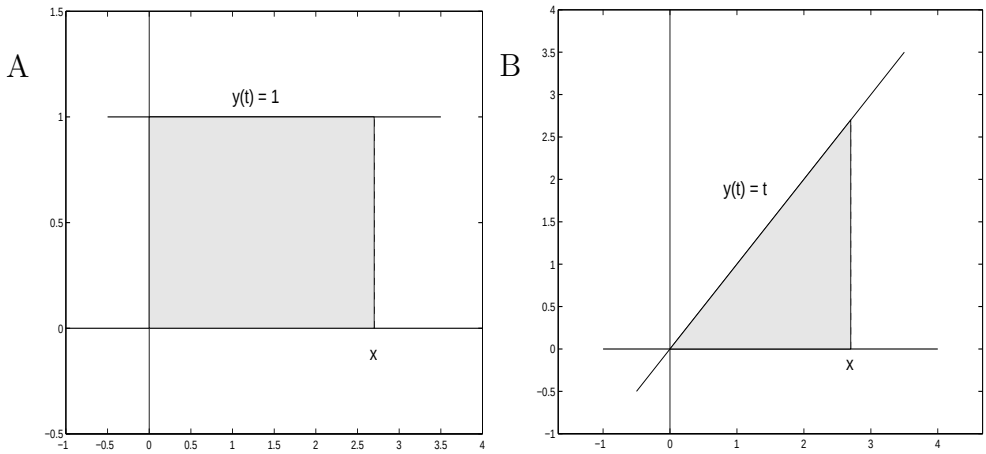


Exercise 13.2.11 Use ordinary geometry to

a. Show that the area of the shaded region in Figure 13.2.11A is x.

b. Show that the area of the shaded region in Figure 13.2.11B is $\frac{x^2}{2}$.

Figure for Exercise 13.2.11 A. Graphs of $y = 1$ and $t = x$. B. Graphs of $y = t$ and $t = x$.



Exercise 13.2.12 A consequence of some previous problems is:

If n is either 0, 1, 2, or 3, x is a positive number, and A is the area of the region bounded by the graphs of

$$y = t^n \qquad y = 0 \qquad t = 0 \qquad \text{and} \qquad t = x,$$

then

$$\begin{aligned} &\text{Case}: \quad n = 0 \quad A = x && \text{Case}: \quad n = 1 \quad A = \frac{x^2}{2} \\ &\text{Case}: \quad n = 2 \quad A = \frac{x^3}{3} && \text{Case}: \quad n = 3 \quad A = \frac{x^4}{4} \end{aligned} \tag{13.12}$$

Make a guess as to the values of A for $n = 4$, $n = 10$, and $n = 1,568$.

Exercise 13.2.13 Summary Exercise: For additional practice on the preceding procedures, the following formula is presented.

$$\sum_{k=1}^{n} k^4 \quad = \quad \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

a. (Discussion.) The validity of the formula can be established by considering the initial condition and difference equation

$$w_1 \quad = \quad 1, \qquad w_t - w_{t-1} \quad = \quad t^4 \qquad \text{for} \qquad t = 2, 3, \cdots$$

and two ways of writing the solution to the equation.

Use the formula to find the area of the region bounded by the graphs of

$$y = t^4 \qquad y = 0 \qquad t = 0 \qquad \text{and} \qquad t = x$$

b. Suppose n is a positive integer. Partition $[0, x]$ into n nonoverlapping intervals with end points

$$0 \quad \frac{x}{n} \quad 2\frac{x}{n} \quad \cdots \quad (k-1)\frac{x}{n} \quad k\frac{x}{n} \quad \cdots \quad n\frac{x}{n}$$

c. What is the length of each of the intervals?

d. What is the area of a rectangle based on the $k\underline{th}$ interval and having base interval, $((k-1)\frac{x}{n}, (k\frac{x}{n})$ and height $\left[k\frac{x}{n}\right]^4$?

e. Show that the sum, U_n, of the areas of all such rectangles is

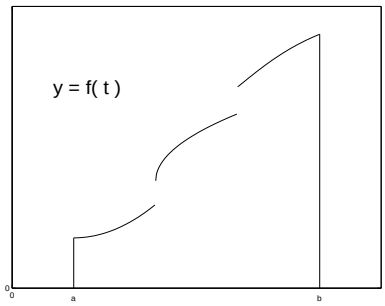
$$U_n \quad = \quad x^5 \times \left(\frac{1}{5} + \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{30n^4}\right)$$

f. Show that

$$\lim_{n\to\infty} U_n \quad = \quad \frac{x^5}{5}$$

13.3 A general procedure for computing areas.

We generalize from the previous problems and define a procedure for computing the area of a region, R, bounded above by the graph of a function, f, below by the horizontal axis, on the left by a line $t = a$ and on the right by a line $t = b$. The function, f, should be defined and positive for all t in [a,b]. We will assume f is increasing on $[a, b]$. Bounds on the errors of our estimates will be obtained.



Assume f is a positive *increasing* function defined on an interval $[a, b]$. We describe a procedure to compute the area of the region between the graph of f and the X-axis as illustrated in Figure 13.8A. The function f may have one or more discontinuities. The region should be considered as the union of the vertical segments reaching from the horizontal axis to a point on the graph of f.

We can approximate the area of the region R as follows:

- Choose a number, n, of rectangles to be used.
- Let $h = \frac{b-a}{n}$.
- Partition the interval $[a, b]$ into n equal subintervals using points $a = t_0 < t_1 < t_2 < \cdots t_{n-1}, t_n = b$. Because the intervals are of equal length, the length of each will be $\frac{b-a}{n} = h$ and

$$t_0 = a \quad t_1 = a + h \quad t_2 = a + 2 \times h \quad \cdots \quad t_k = a + (k) \times h \quad \cdots \quad t_n = a + n \times h$$

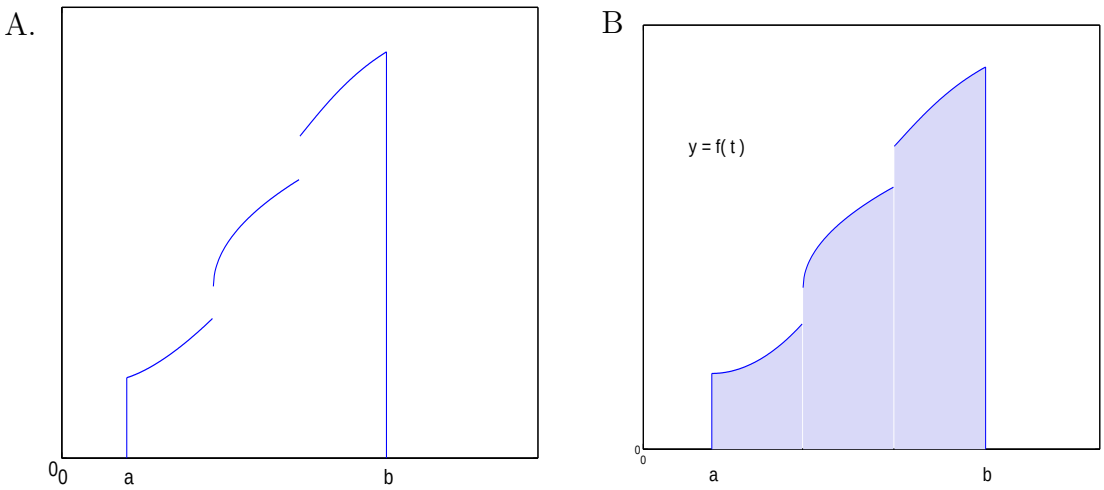


Figure 13.8: A. A region bounded above by the graph of an increasing function. B. Shaded region showing relation to discontinuities.

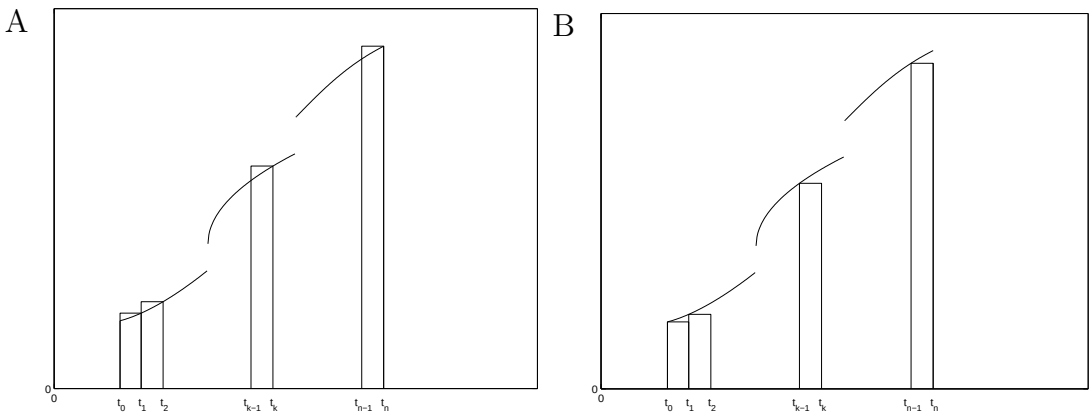


Figure 13.9: A. Upper rectangles for a region bounded above by the graph of a function. B. Lower rectangles for a region bounded above by the graph of an increasing function.

- For $k = 1, \cdots, n$, let A_k be the area of the rectangle $t_{k-1} \leq t \leq t_k$, $0 \leq y \leq f(t_k)$. The rectangles are illustrated in Figure 13.9A and are referred to as *upper* rectangles.
- Let $U_n = \sum_{k=1}^{n} A_k$. The union of the upper rectangles contains the region R so that U_n is greater than or equal to the area of R.
- For $k = 1, \cdots, n$, let B_k be the area of the rectangle $t_{k-1} \leq t \leq t_k$, $0 \leq y \leq f(t_{k-1})$. The rectangles are illustrated in Figure 13.9B and are referred to as *lower* rectangles.
- Let $L_n = \sum_{k=1}^{n} B_k$. The region R contains the union of the lower rectangles so that L_n is less than or equal to the area of R.

For *increasing functions*

$$L_n = \sum_{k=1}^{n} f(t_{k-1}) \times (t_k - t_{k-1}) \qquad \text{and} \qquad U_n = \sum_{k=1}^{n} f(t_k) \times (t_k - t_{k-1})$$

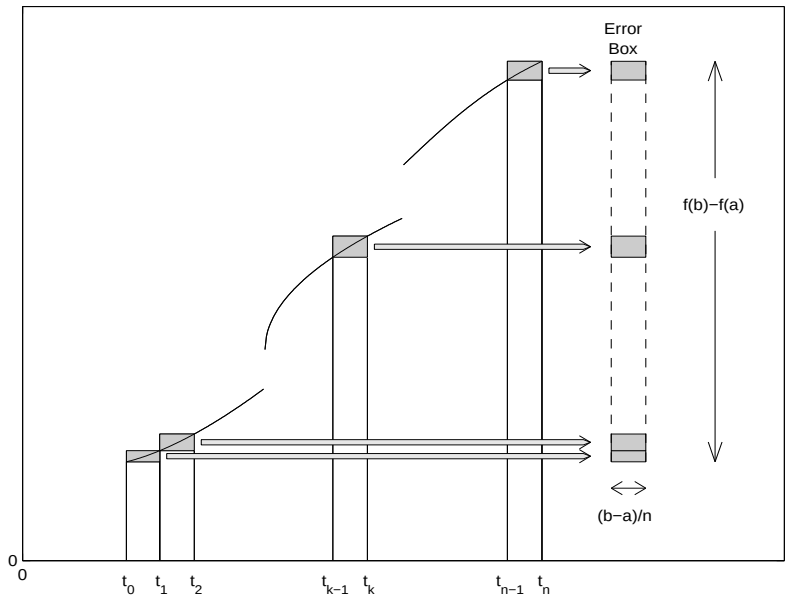


Figure 13.10: Upper and lower rectangles for a region bounded above by the graph of an increasing function.

Shown in Figure 13.10 are both lower and upper rectangles approximating the area of the region R. The differences, $A_k - B_k$, between the areas of the upper and lower rectangles are the areas of the small rectangles at the top of the larger rectangles. Copies of the small rectangles, translated horizontally, are shown in an "Error Box" to the right in Figure 13.10, and all are above an interval of width $\frac{b-a}{n}$. The height of the Error Box is vertical span of f, $f(b) - f(a)$ and the

$$\text{The Area of the Error Box is} \qquad [f(b) - f(a)] \times \frac{b-a}{n}$$

Because

$$U_n - L_n = \sum_{k=1}^{n} A_k - \sum_{k=1}^{n} B_k = \sum_{k=1}^{n} A_k - B_k$$

it follows that $U_n - L_n$ is the area of the Error Box, or

$$U_n - L_n = [f(b) - f(a)] \times \frac{b-a}{n} \tag{13.13}$$

Equation 13.13 is important. Because

$$L_n \leq \text{The Exact Area of the Region } R \leq U_n$$

the error in using either L_n or U_n as an approximation to the Area of R is no bigger than $U_n - L_n$. By choosing n large enough,

$$U_n - L_n = (\ f(b) - f(a)\ ) \times \frac{b-a}{n}$$

can be made as small as one wishes and thus the error in using either U_n or L_n as an approximation to the Area of R can be made as small as one wishes.

Example 13.3.1 The previous information may be used in two ways, as illustrated in the following example.

1. First, one may have computed the upper approximating sum to the area of the region bounded by the graphs of $y = \sqrt{t}$, $y = 0$, and $t = 4$ using 20 subintervals, as shown in Figure 13.3.1.1. The upper sum is $5.51557\cdots$. The error box for this computation is shown to the right of the region. Because each rectangle is of width $\frac{4}{20} = 0.2$, the error box is of width 0.2. The height of the error box is $\sqrt{4} - \sqrt{0} = 2$ so the error in the approximation, $5.51557\cdots$ is no larger than $2 * 0.2 = 0.4$.

2. On the other hand, one may wish to compute the area of the region in Figure 13.3.1.1 correct to 0.01, and need to know how many intervals are required to insure that accuracy. For any number, n, of intervals,

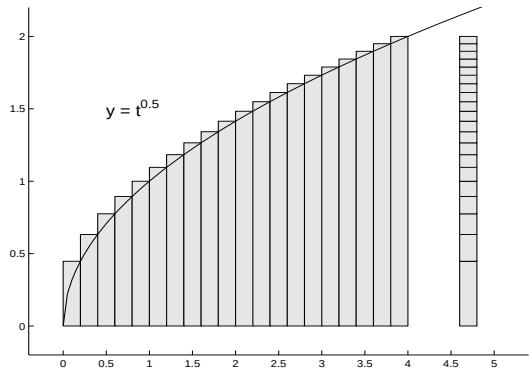
$$\text{The error box is of height} \qquad \sqrt{4} - \sqrt{0} = 2 \qquad \text{and width} \qquad \frac{4-0}{n}$$

Thus,

$$\text{The size of the error box is} \qquad 2 \times \frac{4-0}{n} = \frac{8}{n}$$

Because the error is sure to be less than the size of the error box, our desired accuracy will be obtained if the size of the error box is less than 0.01. Therefore we require $\frac{8}{n} < 0.01$ so that $\frac{8}{0.01} < n$, or $n = 800$. We will find that the actual error is only approximately one-half the size of the error box, and that $n = 400$ will almost give the required accuracy. ■

Figure for Example 13.3.1.1 The graph of $y = \sqrt{x}$ on $[0, 4]$ with 20 upper rectangles.



13.3.1 Trapezoidal approximation.

The average,

$$T_n = \frac{U_n + L_n}{2}$$

often gives a very good approximation to the Area of R, and for increasing functions its error is less than $\frac{1}{2} \times (f(b) - f(a)) \times (b - a)$ (one half the error of U_n or L_n). For an evenly spaced partition of $[a, b]$,

$$t_k - t_{k-1} = \frac{b-a}{n} = h$$

and

$$U_n = \sum_{k=1}^{n} f(t_k) \times h \qquad \text{and} \qquad L_n = \sum_{k=1}^{n} f(t_{k-1}) \times h$$

and

$$\begin{aligned} T_n = \frac{U_n + L_n}{2} &= \frac{1}{2}\left(\sum_{k=1}^{n} f(t_k) \times h + \sum_{k=1}^{n} f(t_{k-1}) \times h\right) \\ &= \left(\frac{f(a)}{2} + \sum_{k=1}^{n-1} f(t_k) + \frac{f(b)}{2}\right) \times h \end{aligned} \tag{13.14}$$

For the twenty intervals in Figure 13.3.1.1,

$$U_{20} = 5.51557, \qquad L_{20} = 5.11557, \qquad \text{and} \qquad T_{20} = 5.31557.$$

The actual area is 5.33333 so that the error in T_{20} is 0.018, considerably less than the error bound for U_{20} of 0.4 computed in Example 13.3.1 and the actual error in U_{20} of 0.18.

The word trapezoid is used for T_n because the average of the areas of the upper and lower rectangles

$$\frac{f(t_k)(t_k - t_k - 1) + f(t_{k-1})(t_k - t_{k-1})}{2} = \frac{f(t_k) + f(t_{k-1})}{2}(t_k - t_{k-1})$$

is the area of the trapezoid, illustrated in Figure 13.11, bounded on the left by the line from $(t_{k-1}, 0)$ to $(t_{k-1}, f(t_{k-1}))$ and on the right by $(t_k, 0)$ to $(t_k, f(t_k))$.

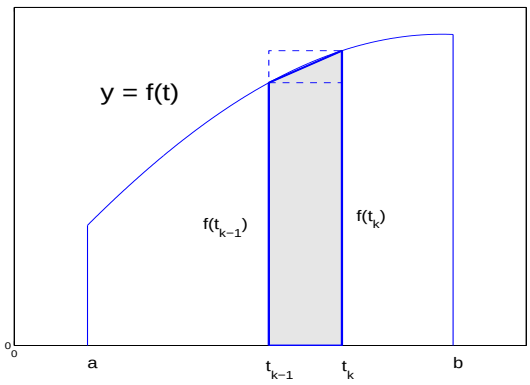


Figure 13.11: An increasing function. The average of the areas of the upper and lower rectangle is the area of the trapezoid.

Example 13.3.2 Work done in compressing air. Movement a distance d (meters) against a constant force F Newtons[3] requires an amount $W = F \times d$ Newton-meters of work. If the force is not constant, the interval of motion $[a, b]$ may be partitioned by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

[3] N is the symbol for one Newton, the force required to accelerate 1 kilogram 1 meter per second per second.

If the force on interval $[x_{k-1}, x_k]$ is approximately F_k, then the work done is approximately

$$W \doteq \sum_{k=1}^{n} F_k \times (x_k - x_{k-1}).$$

A 60 cc syringe was attached to a pressure gauge as shown in Figure 13.12. The plunger of area 5.5 cm^2 was extracted to 60 cc and the air inside the cylinder was at standard atmospheric pressure of $P_0 = 760$ mm mercury $= 10.13$ Newton/cm^2. The plunger was pushed inward at 5 cc increments (to the 55 cc mark, to 50 cc, $\cdots$ to 20 cc and to 15 cc) and the pressure was recorded at each step (table in Figure 13.12). The distance between 5 cc marks on the syringe was 0.9 cm.
Problem. How much work was done?

Solution. The force against the plunger on the $k\underline{\text{th}}$ step was

$$F_k \doteq (P_k - P_0)\text{N/cm}^2 \times 5.5\text{cm}^2.$$

The total work done, W, was approximately

$$\begin{aligned} W &\doteq \sum_{k=1}^{9} (P_k - P_0)\ \text{N/cm}^2 \times 5.5\ \text{cm}^2 \times (x_k - x_{k-1})\ \text{cm} \qquad (13.15) \\ &= \sum_{k=1}^{9} (P_k - P_0) \times 5.5\text{cm}^2 \times 0.9\,\text{cm} \end{aligned}$$

Using the data from Figure 13.12 we calculate

$$W \doteq 383 \quad \text{N-cm} \quad = \quad 3.83 \quad \text{N-m}$$

The number $5.5\text{cm}^2 \times 0.9$ cm is except for measurement error equal to 5 cm^3), or the 5 ml marked on the syringe and in the data. Therefore the work done is

$$W \doteq \sum_{k=1}^{9} (P_k - P_0) \times (V_{k-1} - V_k) = -\sum_{k=1}^{9} (P_k - P_0) \times (V_k - V_{k-1}) \qquad (13.16)$$

Continued in Exercise 13.3.13. One Newton-meter (called a Joule) is equal to 1 watt-sec. a flashlight with a 1.5 amp bulb and two 1.5 volt batteries uses 4.5 watts. The 3.83 N-m would burn the flashlight for $3.83/4.5 = 0.85$ seconds. At \$0.08 per kilowatt-hour, the value of the 3.83 N-m is \$0.000000022. ■

Exercises for Section 13.3, A general procedure for computing areas.

Exercise 13.3.1 a. Identify two rectangles in Figure 13.10 whose areas are $[f(t_1) - f(t_0)] \times h$.

b. Identify two rectangles in Figure 13.10 whose areas are $[f(t_2) - f(t_1)] \times h$.

c. Identify two rectangles in Figure 13.10 whose areas are $[f(t_k) - f(t_{k-1})] \times h$.

d. Identify one rectangle whose area is $[f(b) - f(a)] \times h$.

Dist cm	Vol cm^3	Press N/cm^2
0.0	60	10.13
0.9	55	11.01
1.8	50	11.89
2.7	45	13.05
3.6	40	14.43
4.5	35	16.19
5.4	30	18.61
6.3	25	21.77
7.2	20	26.91
8.1	15	34.60

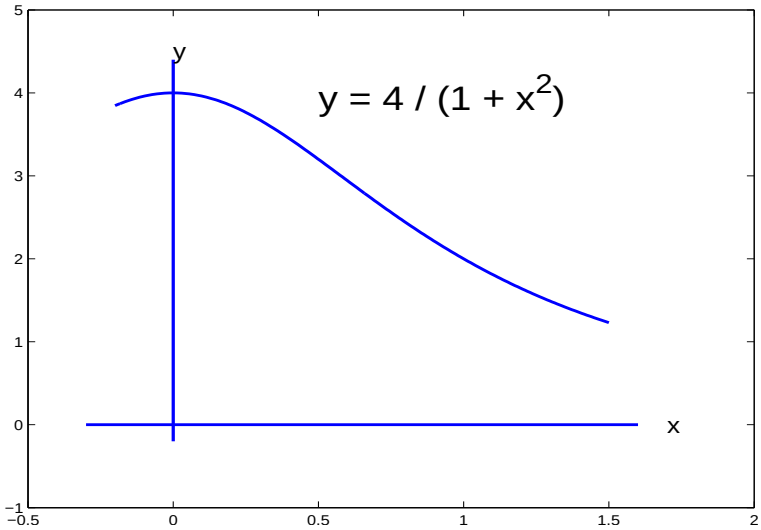
Figure 13.12: Syringe with attached pressure gauge and CBL manufactured by Vernier, Inc. Data from compressing the air in the syringe.

Exercise 13.3.2 Suppose one is to compute the area of the region bounded by the graphs of

$$y = \frac{4}{1+t^2} \qquad y = 0 \qquad t = 0 \qquad t = 1$$

The graph of $y = \dfrac{4}{1+t^2}$ is shown in exercise Figure 13.3.2. The graph is decreasing instead of increasing, but with modest changes the Error Box computation still applies. The upper rectangles have the heights associated with the left ends of the intervals, and the lower rectangles have heights associated with the right ends of the intervals.

Figure for Exercise 13.3.2 The graph of $y = 1/(1 + x^2)$.



Copy the picture to your paper and on your copy:

a. Shade the region bounded by the graphs of $y = \dfrac{4}{1+t^2}$, $y = 0$, $t = 0$ and $t = 1$.

b. Draw the upper rectangles based on $n = 5$ intervals.

c. The upper sum based on $n = 5$ intervals is

$$\left[\frac{4}{1+(0/5)^2} + \frac{4}{1+(1/5)^2} + \frac{4}{1+(2/5)^2} + \frac{4}{1+(3/5)^2} + \frac{4}{1+(4/5)^2}\right] \times \frac{1}{5} = 3.334926$$

Draw the lower sum rectangles based on $n = 5$ intervals and compute the lower sum.

d. Draw the Error Box for the difference between the upper and lower sums based on $n = 5$ intervals. What is the area of the Error Box

e. Argue that for any n, $U_n - L_n \leq 2/n$.

f. What value of n should be chosen to insure that $U_n - L_n \leq 0.004$

g. Compute the upper and lower sums for $n = 500$. It was not terribly exciting to compute the lower sum for $n = 5$, and the computation for $n = 500$ may be a drag. Therefore, to compute the upper sum, enter the following strokes on your calculator:

sum seq(4/(1+(K/500)^ 2)*(1/500) ,K,0,499,1) ENTER

Remember: **sum** and **seq** can be found in the menu

2nd LIST OPS

Your answer should be 3.143591987. Now press 2nd ENTER and modify the strokes to compute the lower sum.

h. What is your best estimate of the area based on L_{500} and U_{500}.

i. The actual area of the region in question is $\pi = 3.141592654\cdots$. Check that $L_{500} \le \pi \le U_{500}$ and that $U_{500} - L_{500} = 0.004$.

Exercise 13.3.3 What area is being computed by

$$\sum_{k=0}^{499} \frac{4}{1+((k+0.5)/500)^2}\,\frac{1}{500}$$

Exercise 13.3.4 Use your calculator or computer to compute the sum of:

a. Forty rectangles to approximate the area of the region bounded by $y = t^2$, $y = 0$, $t = 2$, and $t = 4$.

b. Forty rectangles to approximate the area of the region bounded by $y = t^2$, $y = 0$, $t = 1$, and $t = 5$.

c. Forty rectangles to approximate the area of the region bounded by $y = t^3$, $y = 0$, $t = 1$, and $t = 5$.

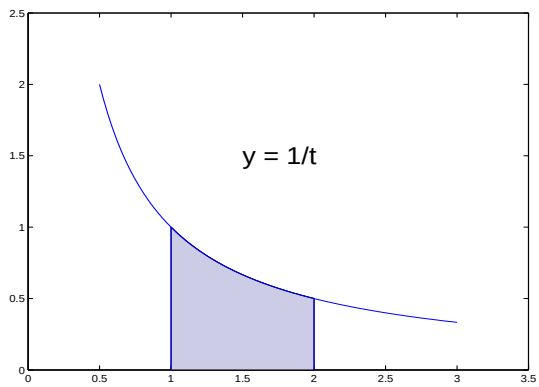
d. Forty rectangles to approximate the area of the region bounded by $y = t^{-1}$, $y = 0$, $t = 1$, and $t = 5$.

Exercise 13.3.5 Consider the region, R, bounded by the graphs of

$$y = 1/t \qquad y = 0 \qquad t = 1 \qquad t = 2$$

See Exercise Figure 13.3.5. Into how many intervals of equal size must [1,2] be partitioned in order that the lower approximating rectangles will approximate the area of R correct to 3 decimal places (Error less than 0.0005). Compute that sum on your calculator. It is a curious fact that the exact answer is $\ln 2$.

Figure for Exercise 13.3.5 The graph of $y = 1/x$.



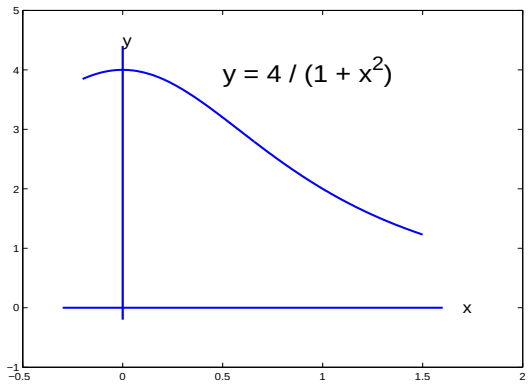
Exercise 13.3.6 Find a lower sum approximating the area bounded by the graphs of $y = \sqrt{1-t^2}$, $y = 0$, and $t = 0$ correct to 3 decimal places (error < 0.0005). What is the exact area?

Exercise 13.3.7 Compute the sum of the areas of 10 trapezoids based on the partition [0, 0.1, 0.2, $\cdots$, 1] of [0,1] used to approximate the area of the region bounded by the graphs of

$$y = \frac{4}{1+t^2} \qquad y = 0 \qquad t = 0 \qquad t = 1$$

Show that this sum more closely approximates the exact area than does the sum of 500 lower rectangles. The exact area is π.

Figure for Exercise 13.3.7 Graph of $y = 4/(1+x^2)$, $0 \le x \le 1$ for Exercise 13.3.7



Exercise 13.3.8 In Problem 13.3.5 you found that 1000 equal intervals are necessary in order that the rectangular sums approximating the area of the region bounded by the graphs of

$$y = 1/t \qquad y = 0 \qquad t = 1 \qquad t = 2$$

approximate the area of the region correct to three digits (error less than 0.005). The area of the region correct to six digits is $\ln 2 \doteq 0.693147$. Approximate the area of the region using 20 trapezoids based on equal subintervals of [1,2] and show that the accuracy of the approximation is better than that of the rectangular approximation using 1000 intervals.

Exercise 13.3.9 The exact area of the region bounded by the graphs of

$$y = \sqrt{t} \qquad y = 0 \qquad \text{and} \qquad t = 5$$

is $\frac{2}{3}5^{3/2} \doteq 7.453560$. Compute the upper rectangular approximation and the trapezoidal approximation to the area based on 50 subintervals and compute their relative accuracies.

Exercise 13.3.10 Compute the area of the region bounded by the graphs of $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ using eight intervals and the trapezoid rule and the same eight intervals with upper rectangles. The exact answer is 2. Compute the errors and relative errors in for each rule.

Exercise 13.3.11 Cardiac Stroke Volume. The chart in Figure 13.13 shows the timing of events in the cardiac cycle (from Rhodes and Tanner, Figure 14-1, page 262).

a. Find the graph showing the 'Aortic blood flow (ventricular outflow)' measured in Liters/min. Read at least six data points from the graph and compute the total aortic flow during one cardiac cycle. This is referred to as 'Stroke Volume'. Note that the horizontal scale is in seconds and the vertical scale is in L/min. The calibration of the vertical axis has been changed from the original source as suggested by Professor Tanner in an e-mail message.

b. The stroke volume may also be defined as the difference in the volume of blood in the ventricle at the end of diastole and the volume of blood in the ventricle at the end of systole. Find the curve 'Ventricular Volume', compute stroke volume from it, and compare with your previous computation.

c. It appears that one cardiac cycle takes about 0.8 seconds (from R wave to R wave, the second R wave is at 0.92 seconds), so that the heart is beating at $60/0.8 = 75$ beats per minute.

Multiply the stroke volume for a single beat by 75 and compare with the conventional 5 - 6 Liters/minute for a resting adult.

Exercise 13.3.12 External Cardiac Work. This problem is directed to measuring the work done by the heart referred to as external work.

If liquid is pumped through a tube at a constant flow rate, R cm^3/sec and at a constant pressure, P Newtons/cm^2, during a time interval, $[a, b]$ measured in seconds, then the work done, W, is

$$W = R \times P \times (b - a) \quad \text{Newton-cm.}$$

a. Confirm that the units on $R \times P \times (b - a)$ are Newton-cm, units of work.

b. Suppose the flow rate or pressure is not constant. Suppose R and P are continuous functions and the flow rate is $R(t)$ and the pressure is $P(t)$ for $a \leq t \leq b$. Let

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

be a partition of $[a, b]$. Write an approximation to the work, W, of pumping the liquid at the flow rate $R(t)$ and pressure $P(t)$ during $[a, b]$ based on the partition.

c. In the Figure 13.13 there are curves that show the Aortic blood flow (ventricular outflow) in liters/minute and the Left ventricular pressure in mm Hg. One mm Hg = 0.01333 Newton/cm^2. Read six points from each of these graphs and approximate the work done by a single stroke of the heart.

A much larger amount of work is done by the heart as 'Internal Work' that 'includes transporting ions across membranes, internal mechanical events $\cdots$, overcoming internal viscosity, and rearranging the muscular architecture as the heart contracts[4]'. Heart efficiency is defined as the ratio of External Work to the sum of External and Internal Work and ranges from 5 to 10 percent.

Exercise 13.3.13 Continuation of Example 13.3.2, Work done in compressing air.

[4]Rhoades and Tanner, p 274

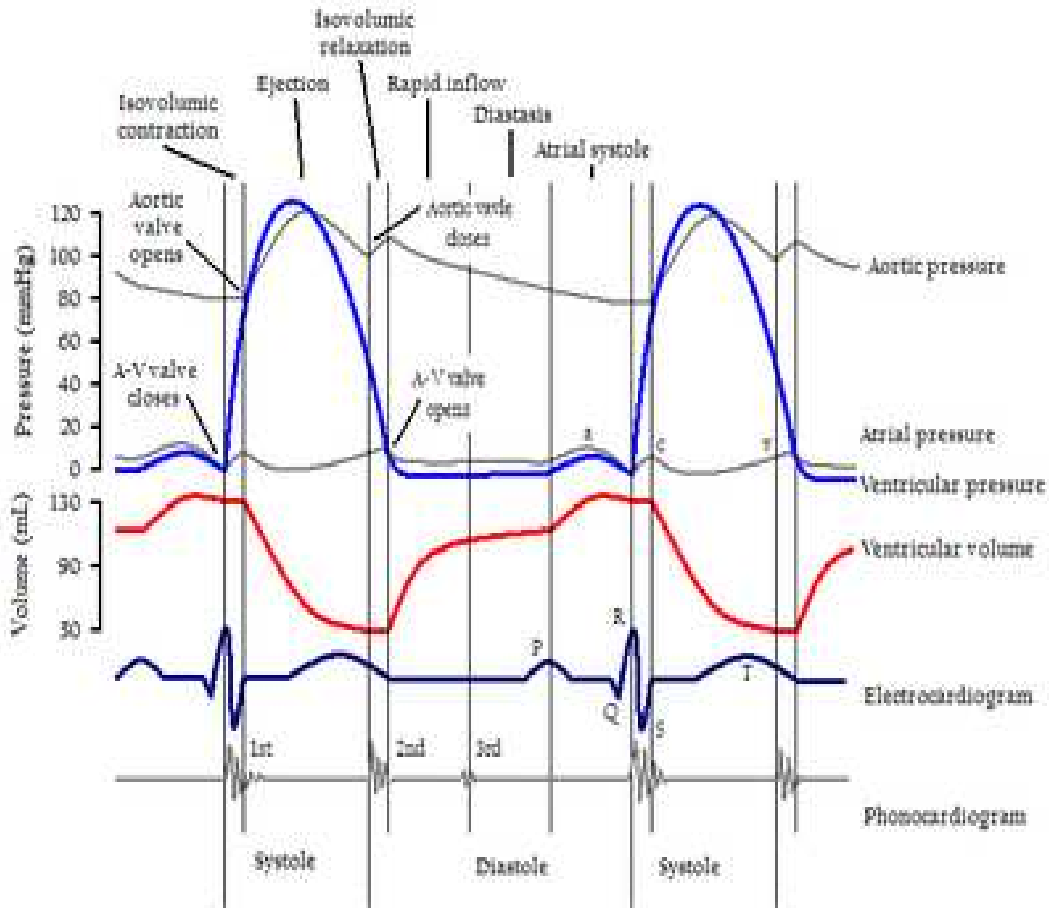


Figure 13.13: Graph of heart events, modified from Rhoades and Tanner, Figure 14-1, p 262. Copyright permission from Rhoades and Tanner has been applied for. Meanwhile a Wikipedia graph is shown: created by Agateller (Anthony Atkielski),converted to svg by atom. http://en.wikipedia.org/wiki/Electrocardiography Assume that the time from beginning of systole to the end of diastole is 75 seconds.

a. Plot Pressure *vs* Volume for the data of Figure 13.12 and draw a smooth curve through the data points. Recall Equation 13.16. Interpret the work done, Equation 13.15, in terms of an area of that graph. Should the work done be positive?

b. It should be apparent from your graph and Equation 13.15 that we have computed (383 N-cm) an upper sum for the area. Compute a lower sum. **Remember** that for an increasing graph the lower sum is the upper sum minus the area of the error box. This is a decreasing graph. Try the same thing in this case.

c. Compute the average of the upper sum and lower sum. This is the trapezoidal approximation to the work done.

d. Boyle spread a rumor that $PV =$ constant. But there is a 15% variation in PV shown in Table 13.2 and only 3.3% variation in $P \cdot (V + 3.6)$. Should we advise Boyle of his error, or examine our equipment?

Table 13.2: PV and $P \cdot (V + 3.6)$ for the data in Figure 13.12.

PV	608	606	595	587	577	567	558	544	538	519
$P(V + 3.6)$	644	645	637	634	629	625	625	623	635	643

Exercise 13.3.14 A tank has 30 m^3 capacity and has 15 m^3 of water in it. Water flows into it at the rate of $R(t) = 1 + t^2$ m^3/min for $0 \le t \le 3$ minutes. Does the tank overflow?

Exercise 13.3.15 Records of Mississippi River discharge rates measured at McGregor, Iowa for April 16 - 22, 1994 are shown in the Exercise Table 13.3.15. Note from the records that the flow increased and the sediment increased during that week, and it is reasonable to assume that there was a warm spell the led to snow melting or a rain in the drainage basin.

a. Approximate the total flow for the Mississippi for the week April 16 -22, 1994.

b. Approximate the total suspended sediment that flowed past McGregor, Iowa in the Mississippi River for the week of April 16-24, 1994.

c. The drainage basin is about 67,500 mi^2. How many tons per square mile were removed from the drainage basin?

Table for Exercise 13.3.15 Mississippi River Discharge, April 16-22, 1994.

Day	Discharge m^3/sec	Sediment mg/liter
Apr 16	1810	20
Apr 17	1870	25
Apr 18	1950	30
Apr 19	2000	33
Apr 20	2030	36
Apr 21	2140	40
Apr 22	2200	41

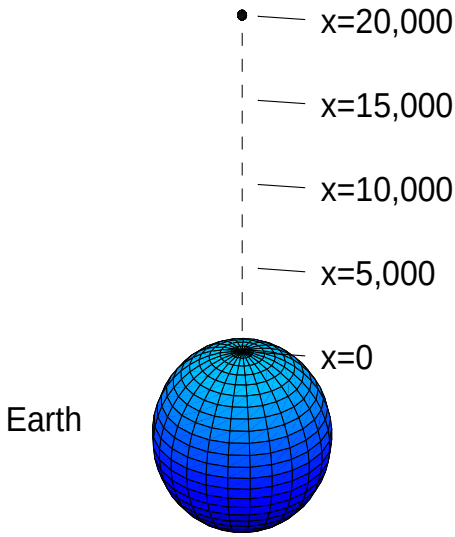


Figure 13.14: Earth and a 3 kg object lifted 20,000 km above the surface.

Exercise 13.3.16 Work against the force of gravity. Suppose a 3 kilogram mass instrument is lifted 20,000 kilometers above the surface of the Earth, the force acting on it is not constant throughout the motion. See Figure 13.14. The acceleration of gravity may be computed for a distance x above the Earth as

$$\text{Acceleration of gravity at altitude } x = 9.8 \times \frac{R^2}{(R+x)^2} \tag{13.17}$$

where $R \doteq 6,370$ kilometers is the radius of the Earth. We partition the interval [0, 20,000] into 4 equal subintervals and assume the acceleration of gravity to be constant on each of the subintervals.

We introduce a new procedure: Choose the value of the acceleration of gravity for each subinterval to be the value at the **midpoint** of the interval. Thus the midpoint of the first interval, [0, 5000] is 2500 and the gravity constant for the first 5000 kilometers of motion will be

$$9.8 \times \frac{6370^2}{(6370+2500)^2} = 5.05$$

The work to lift the instrument the first 5000 kilometers is approximately

$$5.05 \times 3 \times 5000 = 75814 \qquad \text{Newton-meters}$$

a. Approximate the total work to lift the instrument to 20,000 km.

b. Approximate the total work done to lift the 3 kg instrument 40,000 km. (You already know the work required to lift it 20,000 km.)

c. Approximate the total work done to lift the 3 kg instrument 100,000 km.

It is a very interesting question as to whether the instrument can 'escape from the Earth's gravity field' with a finite amount of work. We will return to the question in Section 15.5.1, Escape Velocity.

13.4 The Integral

We have used upper and lower rectangles to approximate areas between the graphs of functions and the horizontal axis. Other rectangular sums often give better estimates, but in most cases rectangular sums give acceptable estimates. In addition to areas, the approximating sums approximate other important physical and biological quantities, generally representing the accumulation of some quantity that is occurring at a variable rate. Implicit in all of the examples is that better approximations will be computed when the total interval is partitioned into smaller intervals. The total accumulation is called the *integral* of a function f on an interval $[a, b]$. A short definition of the integral is

Definition 13.4.1 The integral of a function, I. Suppose f is a function defined on an interval $[a, b]$. The integral of f from a to b is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[f(a + k \times \frac{b-a}{n}) \times \frac{b-a}{n} \right] \tag{13.18}$$

if the limit exists. The notation for the integral is

$$\int_a^b f \qquad \text{or as in most calculus books} \qquad \int_a^b f(t)\,dt$$

The function, f, is called the **integrand**, and

$$\sum_{k=1}^{n} \left[f(a + k \times \frac{b-a}{n}) \times \frac{b-a}{n} \right]$$

is called an **approximating sum** for the integral.

Definition 13.4.1 is a formalization of the computations of the previous sections. The $\lim_{n\to\infty}$ provides for the better approximations with smaller intervals in the partitions. The numbers

$$\left\{ a + k \times \frac{b-a}{n} \right\}_{k=0}^{n} = \left\{ a,\ a + \frac{b-a}{n},\ a + \frac{b-a}{n},\ \cdots a + k\frac{b-a}{n},\ \cdots, a + n\frac{b-a}{n} = b \right\}$$

partition $[a, b]$ into n equal subintervals of length $\frac{b-a}{n}$. $f(a + k \times \frac{b-a}{n})$ evaluates f at the right endpoint of the kth interval.

$$\sum_{k=1}^{n} f(a + k \times \frac{b-a}{n}) \times \frac{b-a}{n}$$

sums the value of f at the right endpoint times the length of the interval. After discussion of notation, we will apply Definition 13.4.1 to relatively more difficult functions than we have considered so far.

Concerning the notations, $\int_a^b f$ and $\int_a^b f(t)\,dt$.

The dt in $\int_a^b f(t)\,dt$ is often a mystery to students, for good reason. The two notations for an integral in Definition 13.4.1 can be compared as follows.

For $f(t) = t^2 \qquad 1 \le t \le 3$, one may write

$$\int_1^3 f \qquad \text{or one may write} \qquad \int_1^3 t^2\,dt \qquad .$$

In a sense, $\int_1^3 t^2\,dt$ allows one to define the integrand $f(t) = t^2$ and write the integral, all with one symbol. However, $\int_1^3 t^2$ would do that as well, so what is the dt?

One could try to avoid the central problem by discussing

$$\int_0^3 e^{k\,t^2}\,dt$$

and say that dt specifies that t is the independent variable (and not k). In fact,

dt is read 'with respect to t.'

The symbol $\int_a^b f(t)\,dt$ is read, 'the integral from a to b of f of t with respect to t.' Furthermore, if you use a calculator or computer calculus package to compute $\int_0^3 e^{k\,t^2}\,dt$ you have to specify that the integration is with respect to t (and not with respect to k).

We will also find that dt is useful for keeping track of symbols in a change of variable in the integrand, and in doing so, dt is sometimes called a 'differential.'

But there is a very colorful and controversial history of the symbol dt. We discuss it in the context of area. According to some, dt is an *infinitesimal* length on the t-axis, $f(t)\,dt$ is an infinitesimal area under the graph of f, and $\int_a^b f(t)\,dt$ sums the infinitesimal areas under the graph of f from a to b to give the total area under the graph of f between a and b. We will not attempt an elaboration. Many people use such intuitive language to quickly move from a physical problem to an integral that computes the solution to the physical problem. For example, if $P(v)$ is the pressure in a syringe when the volume is v, then $P(v)\,dv$ is the 'infinitesimal work' done when the plunger is moved and 'infinitesimal distance', dx, and the volume changes by an 'infinitesimal volume', dv, and the total work done in moving from $v = 60$ to $v = 20$ is $\int_{50}^{20} P(v)\,dv$.

If the concept of an infinitesimal seems vague and questionable, you have good company. After Newton had described calculus in *Principia*, a noted philosopher, Bishop Berkeley, objected strenuously to the concept of dt, writing, "What are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?"[5]

In order to clear up some of the ambiguities, Augustin Cauchy in 1823 introduced the definition of integral we show in Definition 13.4.4.

Units of dimension. Because

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} \left[f(a + k \times \frac{b-a}{n}) \times \frac{b-a}{n} \right]$$

the integral $\int_a^b f(t)\,dt$ 'inherits' the dimension of the approximating sum $\sum_{k=1}^{n} \left[f(a + k \times \frac{b-a}{n}) \times \frac{b-a}{n} \right]$. The notation $\int_a^b f(t)\,dt$ helps keep track of the units of dimension on the integral. If, for example, $f(t)$ is the velocity of a particle, the factor $(b-a)/n$ in the approximating sum will have units of time,

[5] A good source of this debate is in Philip J. Davis and Reuben Hersh, *The Mathematical Experience*, Houghton Mifflin Co., Boston, 1982.

$f(a + k \times \frac{b-a}{n}) \times (b-a)/n$ will have units of distance, and the approximating sum and the integral will have units of distance. In $\int_a^b f(t)\,dt$, dt can be considered to have units of time so that $f(t)\,dt$ has units of distance and $\int_a^b f(t)\,dt$ has units of distance.

Example 13.4.1 The answers to examples and exercises of sections 13.1 through 13.3 all may be expressed in terms of integrals:

1. The area of the region bounded by the graphs of $y = f(t) \geq 0$, $y = 0$, $t = a$ and $t = b$ is $\int_a^b f(t)\,dt$.
2. The mass of a body of density $D(x)$ and cross sectional area $A(x)$ along an interval $[a, b]$ is $\int_a^b D(x) \times A(x)\,dx$.
3. The work done by the heart during one heart beat is $\int_b^e R(t) \times P(t)\,dt$ where R is aortic flow rate and P is pressure in the left ventricle. The units on this integral are

$$R(t) \quad \frac{\text{ml}}{\text{sec}} \quad \times \quad 1 \quad \frac{\text{cm}^3}{\text{ml}} \quad \times \quad P(t) \quad \frac{\text{N}}{\text{cm}^2} \quad \times \quad dt \quad \text{sec} \quad = \quad \text{N-cm}$$

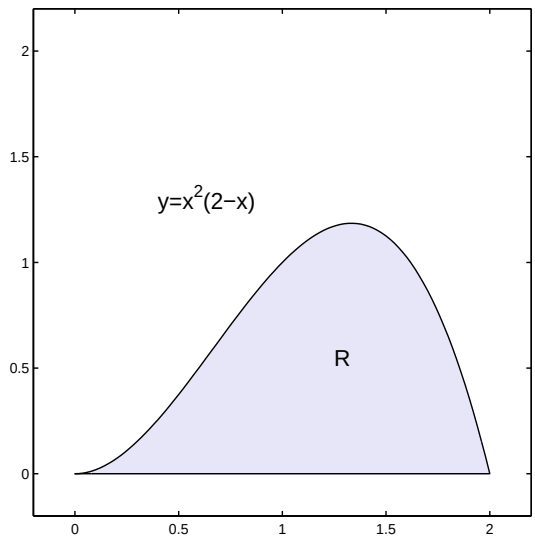
   which is a unit of work. Had the units not been of work, we would think that the integral is incorrectly formulated.

4. Atmospheric density at altitude h meters is approximately $1.225e^{-0.000101h}$ kg/m^3 for $0 \leq a \leq 5000$. The mass of air in a one square meter vertical column between 1000 and 4000 meters is $\int_{1000}^{4000} 1.225e^{-0.000101h} \times 1\,dh$. ■

Example 13.4.2 We use Definition 13.4.1 to compute the area of the region R bounded by $y = x^2(2-x) = 2x^2 - x^3$, $y = 0$, $x = 0$, and $x = 2$. The region for which we wish to compute the area is shown in Figure 13.4.2.2 and the area is

$$\int_0^2 2x^2 - x^3\,dx$$

Figure for Example 13.4.2.2 The region R bounded by $y = x^2(2-x)$ and $y = 0$.



Following Equation 13.18, partition $[0,2]$ into n subintervals $0,\ 2/n,\ 2(2/n),\ 3(2/n)\ \cdots\ n(2/n)=2$ each of length $2/n$. Then write the sum

$$\begin{aligned}
\int_0^2 2x^2 - x^3\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \Big(2\,(\,k(2/n)\,)^2 \;-\; (\,k(2/n)\,)^3\Big) \times 2/n \\
&= 2 \lim_{n\to\infty} \left(\sum_{k=1}^{n} k^2\right) \frac{2^3}{n^3} - \lim_{n\to\infty} \left(\sum_{k=1}^{n} k^3\right) \frac{2^4}{n^4} \\
&= 2 \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} \frac{2^3}{n^3} \;-\; \lim_{n\to\infty} \frac{n^2(n+1)^2}{4} \frac{2^4}{n^4} \\
&= 2 \lim_{n\to\infty} 2^3 \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) \;-\; \lim_{n\to\infty} 2^4 \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) \\
&= 2 \times \frac{2^3}{3} - \frac{2^4}{4} \quad = \quad \frac{4}{3}
\end{aligned}$$

The area of R is $4/3$. We computed an approximation of 1.32 to this area in Example 13.2.3. ■

Example 13.4.3 We use Definition 13.4.1 to compute the area of the region bounded by $y = e^t$, $y = 0$, $t = 0$, and $t = x$. The region for which we wish to compute the area is shown in Figure 13.4.3.3 and we wish to compute

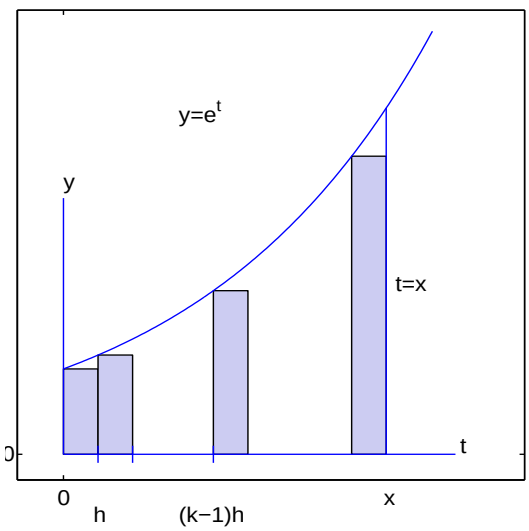
$$\int_0^x e^t\,dt.$$

Equation 13.18 evaluates the integrand at the right end of each interval, and for this particular problem it is slightly simpler to evaluate the integrand at the left end points and compute

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} f(a + (k-1) \times \frac{b-a}{n})\right] \times \frac{b-a}{n}$$

The effect is to change from an upper approximation to a lower approximation.

Figure for Example 13.4.3.3 The region bounded by $y = e^t$, $y = 0$, $t = 0$ and $t = x$.



We recall a formula for the sum of a geometric series:

$$1 + a + a^2 + a^3 + \cdots + a^{n-1} = \frac{a^n - 1}{a - 1} \tag{13.19}$$

This formula can be confirmed by multiplication of

$$(1 - a) \times \left(1 + a + a^2 + a^3 + \cdots + a^{n-1}\right)$$

Assume the interval $[0, x]$ to be partitioned into n equal subintervals, each of length x/n and we let $h = x/n$. The $k\underline{th}$ such interval has endpoints on the t axis at $(k-1) \times h$ and $k \times h$. The area of the $k\underline{\text{th}}$ rectangle is

$$e^{(k-1)\times(x/n)} \times \frac{x}{n} \quad = \quad e^{(k-1)\times(h)} \times h,$$

and the sum of the areas of the n rectangles is

$$\sum_{k=1}^{n} e^{(k-1)\times h} \times h$$

Our job is to make sense of

$$\sum_{k=1}^{n} e^{(k-1)\times h} \times h \qquad \text{because} \qquad \int_0^x e^t \, dt \quad = \quad \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k-1)\times h} \times h.$$

Observe that

$$e^{(k-1)\times h} \times h = \left(e^h\right)^{k-1} \times h$$

so that

$$\sum_{k=1}^{n} e^{(k-1)\times h} \times h = \left[\sum_{k=1}^{n} \left(e^h\right)^{k-1}\right] \times h.$$

We write the part in [ ]'s in long format with a generic term as

$$\begin{array}{ccccccccc} k=1 & & k=2 & & & & k & & k=n \\ \left(e^h\right)^0 & + & \left(e^h\right)^1 & + & \cdots & + & \left(e^h\right)^{k-1} & + & \cdots & + & \left(e^h\right)^{n-1} \end{array}$$

With $a = e^h$ the preceding sum is the geometric series shown in Equation 13.19 and we conclude that

$$\begin{aligned} \sum_{k=1}^{n} \left(e^h\right)^{k-1} \times h &= \frac{\left(e^h\right)^n - 1}{e^h - 1} \times h \\ &= \left(\left(e^{x/n}\right)^n - 1\right) \frac{h}{e^h - 1} \\ &= (e^x - 1) \times \frac{1}{\frac{e^h - 1}{h}} \end{aligned}$$

Now the problem is to evaluate

$$\int_0^x e^t \, dt = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k-1)\times(h)} \times h = \lim_{n\to\infty} (e^x - 1) \times \frac{1}{\frac{e^h - 1}{h}} = (e^x - 1) \times \frac{1}{\lim\limits_{h\to 0} \dfrac{e^h - 1}{h}}$$

By its definition, Definition 5.2.1 on page 220, the number e has the property that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

We conclude that

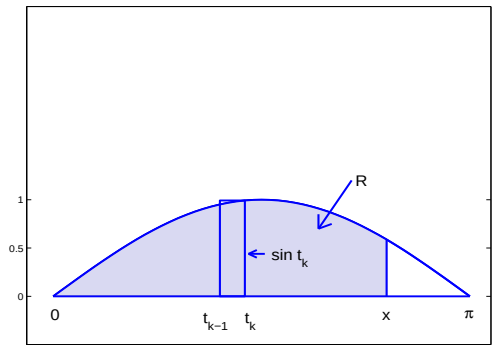
$$\int_0^x e^t\, dt = e^x - 1 \qquad \blacksquare$$

Example 13.4.4 Use the trigonometric identity **Bolt out of the Blue!** [6]

$$\sum_{k=1}^{n} \sin k\theta = \frac{-\cos(n\theta + \frac{\theta}{2}) + \cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \qquad \text{and} \qquad \lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

to compute $\int_0^x \sin t\, dt$, for $0 \le x \le \pi$. The integral is the area of a segment, R, of the sine curve shown in Figure 13.4.4.4.

Figure for Example 13.4.4.4 The region, R, bounded by $y = \sin t$ and $y = 0$ between $t = 0$ and $t = x$, and a typical kth rectangle.



Partition the interval $[0, x]$ into n equal subintervals by $t_k = k \times x/n$. Then

$$\begin{aligned}\int_0^x \sin t\, dt &= \lim_{n\to\infty} \sum_{k=1}^{n} (\sin t_k) \times (t_k - t_{k-1}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\sin(k \times \frac{x}{n})\right) \times \frac{x}{n}\end{aligned}$$

[6] **The Blue:** Uses $\sin x \sin y = -\cos(x+y) + \cos(x-y)$. Let $S = \sum_{k=1}^{n} \sin k\theta$. Then

$$S \sin\frac{\theta}{2} = \sum_{k=1}^{n} \sin k\theta \sin\frac{\theta}{2} = \sum_{k=1}^{n} (-\cos(k\theta + \theta/2) + \cos(k\theta - \theta/2))$$

$$(-\cos(\theta + \theta/2) + \cos(\theta - \theta/2)) + (-\cos(2\theta + \theta/2) + \cos(2\theta - \theta/2)) + \cdots + (-\cos(n\theta + \theta/2) + \cos(n\theta - \theta/2))$$

$$(-\cos(3\theta/2) + \cos(\theta/2)) + (-\cos(5\theta/2) + \cos(3\theta/2)) + \cdots + (-\cos(n\theta + \theta/2) + \cos(n\theta - \theta/2))$$

$$= -\cos(n\theta + \theta/2) + \cos(\theta/2)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{-\cos(n \times \frac{x}{n} + \frac{x}{2n}) + \cos(\frac{x}{2n})}{2\sin\frac{x}{2n}} \times \frac{x}{n} \\
&= \lim_{n\to\infty} \left(-\cos(x + \frac{x}{2n}) + \cos(\frac{x}{2n})\right) \frac{\frac{x}{2n}}{\sin\frac{x}{2n}} \\
&= \left(\lim_{n\to\infty}(-\cos(x + \frac{x}{2n})) + \lim_{n\to\infty} \cos(\frac{x}{2n})\right) \lim_{n\to\infty} \frac{\frac{x}{2n}}{\sin\frac{x}{2n}} \\
&= (-\cos x + \cos 0) \times 1 = 1 - \cos x
\end{aligned}
$$

Specifically,

$$\int_0^{\pi/2} \sin t\, dt = 1 - \cos\frac{\pi}{2} = 1, \qquad \text{and} \qquad \int_0^{\pi} \sin t\, dt = 1 - \cos\pi = 2. \quad \blacksquare$$

13.4.1 A more flexible definition of integral.

The sum in Definition of Integral I 13.4.1 uses intervals of equal length and evaluates f at the right end point of each interval. In order to provide some flexibility in computation of approximating sums (upper, lower or any of the others) and variable interval sizes, the following definitions are used.

Definition 13.4.2 Partition of an interval. Suppose $[a, b]$ is an interval. A partition of $[a, b]$ is a sequence, Δ, of numbers

$$a = t_0 < t_1 < t_2 \cdots < t_k \cdots < t_{n-1} < t_n = b$$

The *norm* of the partition, denoted by $\|\Delta\|$, is the largest length of any interval between successive members of Δ.

$$\|\Delta\| = Max\{t_1 - t_0, t_2 - t_1, \cdots t_k - t_{k-1} \cdots t_n - t_{n-1}\}$$

Definition 13.4.3 Approximating sum to an integral. Suppose f is a function defined on an interval $[a, b]$. An approximating sum for the integral of f on $[a, b]$ is a number of the form

$$f(\tau_1) \times (t_1 - t_0) + f(\tau_2) \times (t_2 - t_1) + \cdots + f(\tau_k) \times (t_k - t_{k-1}) + \cdots + f(\tau_n) \times (t_n - t_{n-1})$$

where $a = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = b$ is a partition of $[a, b]$ and

$$t_0 \le \tau_1 \le t_1 \quad t_1 \le \tau_2 \le t_2 \quad \cdots \quad t_{k-1} \le \tau_k \le t_k \qquad \cdots \quad t_{n-1} \le \tau_n \le t_n$$

Definition 13.4.4 The integral of a function, II. Suppose f is an increasing function defined on an interval $[a, b]$. The *integral* of f on $[a, b]$ is denoted by $\int_a^b f(t)\,dt$ and defined by

$$\int_a^b f(t)\,dt = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} f(\tau_k) \times (t_k - t_{k-1}) \tag{13.20}$$

Implicit in the previous symbol is that $\Delta = a = t_0 < t_1 < t_2 \cdots < t_k \cdots < t_{n-1} < t_n = b$ is a partition of $[a, b]$ and that $t_{k-1} \le \tau_k \le t_k$ for $k = 1, n$.

The two definitions of integral give a unique number and the same number for all increasing functions and for all decreasing functions and for all functions that alternate between increasing and decreasing only a finite number of times on $[a, b]$. Definition I is easier to comprehend, but the flexibility in Definition II in computing the approximating sum is sometimes helpful.

There are some functions for which either the limits in Definitions I and II are different or do not exist. It is unlikely that you will encounter one in undergraduate study, and we will not discuss them. We will assume that for all functions we deal with, the limits in Definitions I and II exist and are the same. For such functions, we say the the integral exists and we say the functions are integrable.

Exercises for Section 13.4, The Integral.

Exercise 13.4.1 Use Equations 13.12 found in Exercise 13.2.12, to compute

$$\text{A. } \int_0^1 t^2\,dt \qquad \text{B. } \int_0^2 t^2\,dt \qquad \text{C. } \int_1^2 t^2\,dt$$

Exercise 13.4.2 Compute

$$\text{A. } \int_0^1 t\,dt \qquad \text{B. } \int_1^2 t\,dt$$

Exercise 13.4.3 Compute

$$\text{A. } \int_0^1 t^3\,dt \qquad \text{B. } \int_1^2 t^3\,dt$$

Exercise 13.4.4 Compute

$$\text{A. } \int_0^1 3\,dt \qquad \text{B. } \int_1^2 3\,dt$$

Exercise 13.4.5 Approximate

$$\text{A. } \int_0^1 e^t\,dt \qquad \text{B. } \int_0^\pi \sin(t)\,dt$$

using the approximating sum in Definition 13.4.1 and 10 equal subintervals.

Exercise 13.4.6 Use Definition of Integral II to evaluate

$$\int_1^2 \frac{1}{t^2}\,dt. \tag{13.21}$$

Partition [1,2] in n equal subintervals by

$$t_0 = 1, \quad t_1 = 1 + \frac{1}{n}, \quad \cdots, \qquad t_{k-1} = 1 + \frac{k-1}{n}, \quad t_k = 1 + \frac{k}{n}, \quad \cdots \quad t_n = 1 + \frac{n}{n}.$$

Let

$$\tau_k = \sqrt{t_{k-1} \times t_k}, \qquad k = 1, 2, \cdots, n$$

a. Show that $t_{k-1} \le \tau_k \le t_k$.

b. Write Equation 13.20,

$$\int_a^b f(t)\,dt = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(\tau_k) \times (t_k - t_{k-1}),$$

for$\int_1^2 \frac{1}{t^2}\,dt$, the given partition and values of τ_k.

c. Show that

$$\int_1^2 \frac{1}{t^2}\,dt = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} \left(\frac{1}{t_{k-1}} - \frac{1}{t_k} \right)$$

d. Write the previous sum in long form and show that

$$\int_1^2 \frac{1}{t^2}\,dt = \lim_{\|\Delta\| \to 0} \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

Exercise 13.4.7 Use steps similar to those of Exercise 13.21 to show that for $x > 1$,

$$\int_1^x \frac{1}{t^2} = 1 - \frac{1}{x}$$

Exercise 13.4.8 *You will use this exercise in your proof of the Fundamental Theorem of Calculus, Theorem 14.2.1.* Suppose f, is a continuous defined on an interval $[a, b]$ and $(v, f(v))$ is a high point of f on $[a, b]$ (meaning that v is in $[a, b]$ and for all x in $[a, b]$, $f(x) \le f(v)$).

a. Argue that every approximating sum the $\int_a^b f(x)\,dx$ is less than or equal to $f(v) \times (b - a)$.

b. Argue that

$$\int_a^b f(x)\,dx \le f(v) \times (b - a)$$

Exercise 13.4.9 In previous sections, values of the following integrals were given. What are they?

A. $\displaystyle\int_0^1 \frac{1}{1+t^2}\,dt =$ _____ B. $\displaystyle\int_1^2 \frac{1}{t}\,dt =$ _____

Exercise 13.4.10 Write an integral that is the area of the region bounded by the graphs of $y = 2 \times t^5 - t^4$, $y = 0$, $t = 1$ and $t = 2$.

Exercise 13.4.11 Suppose a particle moves with a velocity, $v(t) = \frac{1}{1+t^2}$.

a. Write an integral that is the distance moved by the particle between times $t = 0$ and $t = 1$.

b. Write an integral that is the distance moved by the particle between times $t = -1$ and $t = 1$.

Exercise 13.4.12 Suppose an item is drawn from a normal distribution that has mean 0 and standard deviation 1 ($p(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$). Write an integral for the probability that the item is

a. less than one standard deviation from the mean.

b. less than two standard deviations from the mean.

Exercise 13.4.13 Suppose water is flowing into a barrel at the rate of $R(t) = 1 + t^2$ m^3/min for $0 \le t \le 3$ minutes. Write an integral that is the volume of water put into the tank. Confirm that the units on the integral are volume.

Exercise 13.4.14 Water flows into a tank at the rate of $R(t) = 1 + t^2/5$ m^3/min for $0 \le t \le 3$ minutes and the concentration of salt in the water is $C(t) = 3.5e^{-t}$ g/l at time t. Write an integral that is the total amount of salt that flowed into the tank. Confirm that the units on the integral are grams of salt.

Exercise 13.4.15 Write an integral that is the work done in compressing a syringe of stroke 10cm and radius of 1cm from 10 to 5 cm. Confirm that the units on the integral are of units of work.

Exercise 13.4.16 What is the norm of the partition $\{0.0, 0.2, 0.3, 0.6, 0.7, 0.9, 1.0\}$ of $[0, 1]$? Write a partition of $[0, 1]$ whose norm is 0.15.

Exercise 13.4.17 Write an approximating sum to $\int_0^1 e^x\, dx$ for the partition $\{0.0, 0.2, 0.3, 0.6, 0.7, 0.9, 1.0\}$ of $[0, 1]$.

Exercise 13.4.18 Average value of a function. The average value of a function f on an interval $[a, b]$ is defined to be

$$\text{Average value of } f \text{ on } [a,b] \quad \frac{1}{b-a}\int_a^b f(t)\, dt \tag{13.22}$$

. Use the approximating sum in Definition 13.4.1 and explain why

$$\frac{1}{b-a}\sum_{k=1}^{n}\left[f(a + k \times \frac{b-a}{n}) \times \frac{b-a}{n}\right]$$

is a reasonable approximation to the average value of f on $[a, b]$.

Exercise 13.4.19 Write the average solar intensity over a year for Eugene, Oregon in integral form. See Exercise 13.1.3

Exercise 13.4.20 Let x be a number in $[0, \frac{\pi}{2}]$. Use the trigonometric identity

$$\sum_{k=1}^{n} \cos(k \times \theta) = \frac{\sin\left(n \times \theta + \frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} \qquad \text{and} \qquad \lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

to compute from Definition I the integral

$$\int_0^x \cos(t)\, dt = \sin x$$

13.5 Properties of the integral.

Property 13.5.1 Linearity of the integral. Suppose f and g are integrable functions defined on an interval $[a, b]$ and c is a number. Then

A. $$\int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt \quad (13.23)$$

B. $$\int_a^b c \times f(t)\, dt = c \times \int_a^b f(t)\, dt \quad (13.24)$$

The linearity properties are intuitive. If f is the rate of production of urea and g is the rate of production of creatinine, then $f + g$ is the rate of production of nitrogenous waste products. The total production of nitrogenous waste products ($\int_a^b (f(t) + g(t))dt$) is the sum of the total production of urea ($\int_a^b f(t)dt$) and the total production of creatinine ($\int_a^b g(t)dt$). If the rate of production of urea is changed by a factor of c, then the total production of urea ($\int_a^b c \times f(t)dt$) is c times the previous production of urea ($\int_a^b f(t)dt$).

The word *linear* is associated with these two properties, because linear functions of the form $P(x) = m \times x$ have the properties.

$$\begin{aligned} P(x + y) &= m \times (x + y) = m \times x + m \times y = P(x) + P(y) \\ P(c \times x) &= m \times (c \times x) = c \times (m \times x) = c \times P(x) \end{aligned}$$

The sine function is not linear, despite the efforts of many students. For most values of x and y

$$\sin(x + y) \neq \sin(x) + \sin(y).$$

Compare this with the trigonometric identity

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

The reasons the integral has the two linearity properties is that the approximating sum also has the two properties. The integral being the limit of the approximating sums inherits the linearity properties of the approximating sums. For example, if f and g are functions defined on an interval $[a, b]$ and c is a number, and $a = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = b$ is a partition of $[a, b]$ and

$$t_0 \leq \tau_1 \leq t_1 \qquad t_1 \leq \tau_2 \leq t_2 \qquad \cdots t_{k-1} \leq \tau_k \leq t_k \qquad \cdots t_{n-1} \leq \tau_n \leq t_n$$

then

A. $$\sum_{k=1}^n [f(\tau_k) + g(\tau_k)] \times (t_k - t_{k-1}) = \sum_{k=1}^n f(\tau_k) \times (t_k - t_{k-1}) + \sum_{k=1}^n g(\tau_k) \times (t_k - t_{k-1}),$$

B. $$\sum_{k=1}^n c \times f(\tau_k) \times (t_k - t_{k-1}) = c \sum_{k=1}^n f(\tau_k) \times (t_k - t_{k-1}).$$

Each approximating sum for $\int_a^b f(t) + g(t)\, dt$ is the sum of approximating sums for $\int_a^b f(t)\, dt$ and $\int_a^b g(t)\, dt$, and each approximating sum for $\int_a^b c \times f(t)\, dt$ is c times an approximating sum for $\int_a^b f(t)\, dt$.

Example 13.5.1 Compute

$$\text{A. } \int_0^1 \left[t^2 + t\right] dt \qquad \text{B. } \int_0^2 2 \times t\, dt$$

Solution: A. By Problem 13.2.12 $\int_0^1 t^2\, dt = \frac{1}{3}$, and $\int_0^1 t\, dt = \frac{1}{2}$. From Property 13.5.1 A,

$$\int_0^1 \left[t^2 + t\right] dt = \int_0^1 t^2\, dt + \int_0^1 t\, dt = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

B. By Problem 13.2.12, $\int_0^2 t\, dt = 2$. From Property 13.5.1 B

$$\int_0^2 2t\, dt = 2 \times \int_0^2 t\, dt = 2 \times 2 = 4 \quad \blacksquare$$

Although many students think otherwise, it is not always (or even usually) true that

$$\int_a^b [f(t) \times g(t)]\, dt = \int_a^b f(t)\, dt \times \int_a^b g(t)\, dt \qquad \textbf{Not True!}$$

For example, let $f(t) = t^2$ and $g(t) = t$. Then

$$\int_1^2 f(t) \times g(t)\, dt = \int_1^2 t^2 \times t\, dt = \int_1^2 t^3\, dt = \frac{15}{4},$$

but

$$\left(\int_1^2 f(t)\, dt\right) \times \left(\int_1^2 g(t)\, dt\right) = \left(\int_1^2 t^2\, dt\right) \times \left(\int_1^2 t\, dt\right) = \left(\frac{8}{3}\right) \times \left(\frac{3}{2}\right) = 4 \neq \frac{15}{4}.$$

Also, it is not always true (and it is not even usually true) that

$$\int_a^b \frac{f(t)}{g(t)} = \frac{\int_a^b f(t)\, dt}{\int_a^b g(t)\, dt} \qquad \textbf{Not True!}$$

$$\int_1^2 \frac{f(t)}{g(t)}\, dt = \int_1^2 \frac{t^2}{t}\, dt = \int_1^2 t\, dt = \frac{3}{2},$$

but

$$\frac{\int_1^2 f(t)\, dt}{\int_1^2 g(t)\, dt} = \frac{\int_1^2 t^2\, dt}{\int_1^2 t\, dt} = \frac{\frac{8}{3}}{\frac{3}{2}} = \frac{16}{9} \neq \frac{3}{2}.$$

Three more properties of the integral are:

Property 13.5.2 Geometry of the integral.

1. Suppose f is an integrable function defined on an interval $[a, b]$ and c is a number between a and b. Then

$$\int_a^c f(t)\, dt + \int_c^b f(t)\, dt = \int_a^b f(t)\, dt \tag{13.25}$$

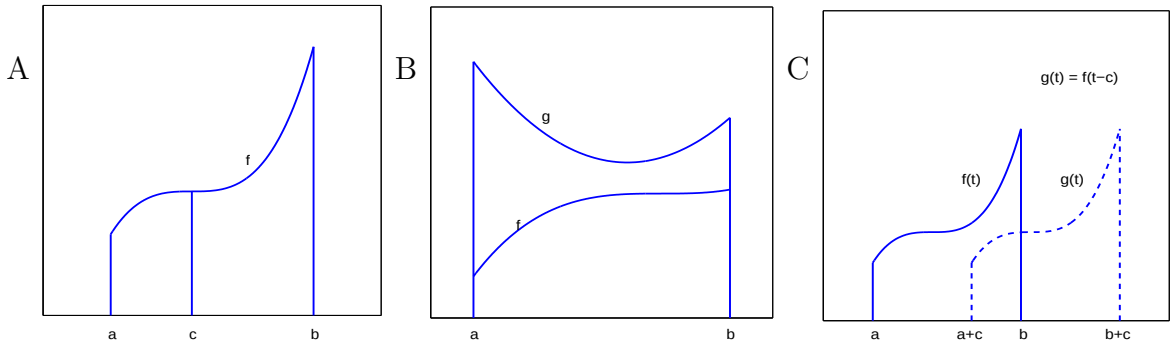


Figure 13.15: A. (Left) $\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt$. B. (Center) The area under the graph of f is $\leq$ the area under the graph of g. C. (Right) The area under the graph of f equals the area under the graph of g, where $g(t) = f(t-c)$.

2. Suppose f and g are integrable functions defined on an interval $[a, b]$ and for all t in $[a, b]$, $f(t) \leq g(t)$. Then
$$\int_a^b f(t)\,dt \leq \int_a^b g(t)\,dt \tag{13.26}$$

3. Suppose f is an integrable function defined on $[a, b]$, and c is a number. Then
$$\int_{a+c}^{b+c} f(t-c)\,dt = \int_a^b f(t)\,dt \tag{13.27}$$

We refer to these as geometric properties because they are so obvious from the area interpretation of the integral. In Figure 13.15 A, it is clear that the area under the graph of f between $t = a$ and $t = b$ is the sum of the area between $t = a$ and $t = c$ and the area between $t = c$ and $t = b$. In Figure 13.15, B, it is clear that the area under the graph of f is less than the area under the graph of g. In Figure 13.15, C, the region below the graph of $g(t) = f(t-c)$ is a simple translation of the region under the graph of f and the areas are equal. Formal proofs of the properties based on approximating sums can be given, but the geometry alone is convincing, and we omit the formal proofs. Your solution to Explore 13.5.1 will give you some algebraic insight to the properties.

Explore 13.5.1 Remember from Definition 13.4.3 that an approximating sum for the integral of f on $[a, b]$ is a number of the form

$$f(\tau_1) \times (t_1 - t_0) + f(\tau_2) \times (t_2 - t_1) + \cdots + f(\tau_k) \times (t_k - t_{k-1}) + \cdots + f(\tau_n) \times (t_n - t_{n-1}).$$

Suppose f and g are continuous increasing functions defined on an interval $[a, b]$ and for every t in $[a, b]$, $f(t) \leq g(t)$ and suppose that c is a number.

a. Show that every approximating sum of $\int_a^b f(t)\,dt$ is less than or equal to some approximating sum of $\int_a^b g(t)\,dt$.

b. Show that every approximating sum to $\int_a^b f(t)\,dt$ is equal to an approximating sum to $\int_{a+c}^{b+c} f(t-c)\,dt$.

c. Show that if $a < c < b$ and $S_{a,c}$ is an approximating sum for $\int_a^c f(t)\,dt$ and $S_{c,b}$ is an approximating sum for $\int_c^b f(t)\,dt$, then $S_{a,c} + S_{c,b}$ is an approximating sum for $\int_a^c f(t)\,dt$.

Explore 13.5.2 Which of properties 13.5.2 is illustrated by the following statements?

a. John measures the rain fall from 2 to 4 pm and Jane measures the rain fall from 1400 to 1600 hours. They get the same amount.

b. If Jane runs faster than John, Jane will go farther than John.

c. The total damage done by the insects includes the damage done during the larval, pupal, and fully emerged insect stages.

Explore 13.5.3 Decide on a reasonable value for

$$\int_a^a f(t)\,dt \tag{13.28}$$

and give a geometric argument for your answer.

13.5.1 Negatives.

It has been assumed that in the symbol, $\int_a^b f(x)\,dx$, $a < b$. There are instances when one wants to extend the integral concept to the case $b < a$ and even the case $a = b$. This is done by

Definition 13.5.1 Suppose f is an integrable function defined on $[a, b]$. Then

$$\int_b^a f(t)\,dt \;=\; -\int_a^b f(t)\,dt \qquad \text{and} \qquad \int_a^a f(t)\,dt = 0$$

It has been implicit in much of the discussion that $f(x)$ is positive for all x in $[a, b]$. This is not required, and a number of instances suggest using functions with negative values.

Example 13.5.2 For example, a ball thrown vertically may have a positive velocity as it ascends, but will then have a negative velocity as it descends. If one throws the ball with a vertical velocity of $19.6m/s$, then the velocity t seconds later will be $19.6 - 9.8t\,m/s$ where the term $9.8t$ is the change in velocity due to gravity. A graph of the velocity is shown in Figure 13.16.

At $t = 2$ seconds the velocity is zero, the ball is at its maximum height, and that height is the area of the triangle marked '+' in Figure 13.16. Between $t = 2$ and $t = 4$ seconds the velocity is negative, the motion of the ball is downward and the displacement is the area of the triangle marked '-' in Figure 13.16. At $t = 4$ the ball has fallen to its original starting point. The net displacement is zero and that is $\int_0^4 v(t)\,dt$, the sum of the 'areas' of the two triangles treating the area of the second triangle as negative (see Exercise 13.5.9). The *distance* traveled by the ball is the sum of the areas of the two triangles, both treated as positive. ■

Exercises for Section 13.5, Properties of the integral.

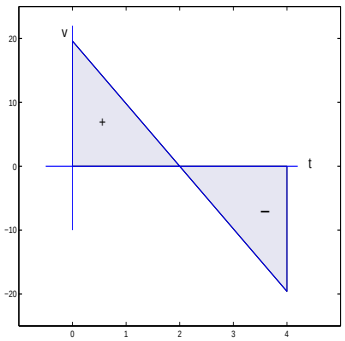


Figure 13.16: Velocity of a ball thrown vertically, $v(t) = 19.6 - 9.8t$.

Exercise 13.5.1 Do Explore 13.5.1.

Exercise 13.5.2 Do Explore 13.5.2.

Exercise 13.5.3 Do Explore 13.5.3.

Exercise 13.5.4 Which of the linear properties of the integral are illustrated by the following examples?

a. The death rate from cancer is about 2/3's that of heart disease. In a year's time 2/3's as many people die from cancer as die from heart disease.

b. The common cold incidence is 25.4 per person per year and the influenza incidence is 34.8 per person per year. In three years, a town of 10,000 people experienced 18,060 respiratory viral infections.

Exercise 13.5.5 Is the exponential function, $E(x) = e^x$, linear? Prove or disprove.

Exercise 13.5.6 Is the logarithm function, $L(x) = \ln(x)$, linear? Prove or disprove.

Exercise 13.5.7 Compute (note: change x to t if it confuses you.)

$$\text{A. } \int_0^1 \left[3 + x^2\right] dx \qquad \text{B. } \int_1^2 3x^2\, dx \qquad \text{C. } \int_3^5 3x^3 - 6x^2\, dx$$

Exercise 13.5.8 a. Compute

$$\text{A. } \int_2^4 \left[t \times t^2\right] dt \qquad \text{B. } \int_2^4 t\, dt \quad \times \quad \int_2^4 t^2\, dt$$

b. Compute

$$\text{A. } \int_2^4 \frac{t^3}{t^2}\, dt \qquad \text{B. } \frac{\int_2^4 t^3\, dt}{\int_2^4 t^2\, dt}$$

c. What do these two problems illustrate?

Exercise 13.5.9 In Example 13.5.2 it was claimed that $\int_0^4 v(t)\, dt = \int_0^4 (19.6 - 9.8t)\, dt$ is the sum of the areas of the two triangles in Figure 13.16. Compute

$$\text{a. } \int_0^2 (19.6 - 9.8t)\, dt, \qquad \text{b. } \int_2^4 (19.6 - 9.8t)\, dt, \qquad \text{and} \qquad \text{c. } \int_0^4 (19.6 - 9.8t)\, dt.$$

Compare your answers with the areas of the triangles in Figure 13.16.

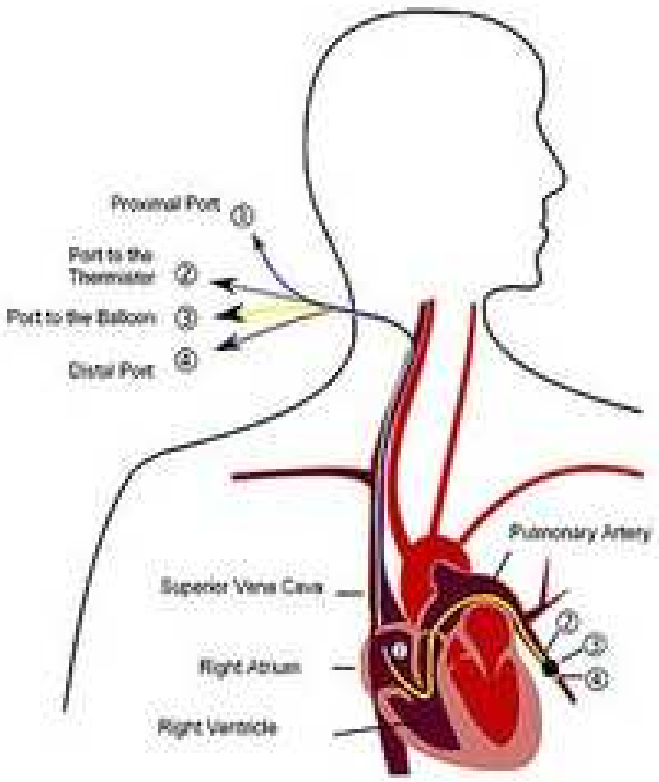


Figure 13.17: Schematic diagram of the heart and a Swan-Ganz catheter threaded throughout the right atrium and into the pulmonary artery. Copied from *Medical Physiology*, R.A. Rhoades and G. A. Tanner, Little, Brown and Company, 1995, page 271. Copyright: Lippincott, Williams, & Wilkins. Do not yet have copyright permission. Used http://en.wikipedia.org/wiki/Pulmonary_artery_catheter placed by Chikumaya.

13.6 Cardiac Output

The problems of this section are directed to understanding a procedure used for measuring cardiac output. Briefly, the procedure is

> Infuse a quantity, Q, of $0°C$ saline solution into the right side of the heart, and measure the temperature of the fluid in the pulmonary artery exiting from the heart as a function of time, $T(t)$. Then the flow rate is
>
> $$F = \frac{Q}{\int_0^\infty \frac{37-T(t)}{37-0}\,dt} \tag{13.29}$$

Our goal is to understand why Equation 13.29 correctly gives the flow rate. The following material describes the physiology and is copied from R.A. Rhoades and G. A. Tanner, *Medical Physiology.*, Little, Brown and Company, 1995, pp 271-2.

> **The Thermodilution Method.** In most clinical situations, cardiac output is measured using a variation of the dye dilution method called **thermodilution**. A **Swan-Ganz**

catheter (a soft, flow-directed catheter with a balloon at the tip) is placed into a large vein and threaded through the right atrium and ventricle so that its tip lies in the pulmonary artery. The catheter is designed so a known amount of ice-cold saline solution can be injected into the right side of the heart via a side pore in the catheter. This solution decreases the temperature of the surrounding blood. The magnitude of the decrease in temperature depends on the volume of blood that mixes with the solution, which depends on cardiac output. A thermistor on the catheter tip (located downstream in the pulmonary artery) measures the fall in blood temperature. Using calculations similar to the dye dilution method, the cardiac output can be determined.

Measurement of cardiac output as just described is a common procedure in hospitals. Patients returning from cardiac surgery have a Swan-Ganz catheter inserted as described. Measurements of cardiac output may be made hourly for the first 24 hours, followed by measurements every 2 hours for the next two days. You will find the web site of a catheter manufacturer, Edwards Lifesciences, interesting.

A useful first step in understanding Equation 13.29 is to check the units on the left and right hand side. The units on the integral are the same as the units on the approximating sums for the integral.

It is reasonable to assume that the flow rate, F, is measured in ml/sec so that the right hand side should also have units of ml/sec. The quantity, Q, should be measured in ml. A general approximating sum for the integral in the denominator is

$$\sum_{k=1}^{n} \frac{37 - T(t_k)}{37 - 10}(t_k - t_{k-1})$$

The fraction $\frac{37-T(t_k)}{37-10}$ is the ratio of temperatures, and is therefore dimensionless. The factor $(t_k - t_{k-1})$ is measured in seconds, so that the unit on the approximating sum is seconds, as is the unit on the integral. Therefore the units on both sides of Equation 13.29 are ml/sec.

In order to understand the Equation 13.29 it is necessary to understand

"The magnitude of the decrease in temperature depends on the volume of blood that mixes with the solution, which depends on cardiac output. "

and the next few examples and exercises are directed to that end.

Example 13.6.1 Suppose 10 ml of $9°C$ water is mixed with 60 ml of $37°C$ water. What will be the temperature of the mixture?

We base our answer on the concept of heat content in the fluids measured with a base of zero heat content at 0 °C. It is helpful for this example that the specific heat of water is 1 calorie per gram-degree C. The definition of a calorie is the amount of heat required to raise one gram of water one degree centigrade — specifically from 14.5 to 15.5 degrees centigrade, but we will assume it is constant over the range 0 to 37 degrees centigrade).

The following table is helpful:

	Vol (ml)	Temp (°C)	Calories
Fluid 1	10	8	$10 \times 9 = 90$
Fluid 1	60	37	$60 \times 37 = 2220$
Mixture	70	T	$70 \times T$

The critical step now is the conservation of energy. The calories in the mixture should be the sum of the calories in the two fluids (we assume there is no heat of mixing) so that

$$70 \times T = 90 + 2220 \qquad\qquad T = 33^\circ\text{C} \qquad \blacksquare$$

Care must be taken when mixing two fluids of different heat capacities. If 10 ml of $9^\circ C$ cream is mixed with 60 ml of $37^\circ C$ coffee, the temperature of the mixture would be slightly less than 33° because the specific heat of cream is greater than that of coffee (it takes more calories to increase 1 gm of cream 1° than it takes to increase 1 gm of coffee 1°), and the density of cream is greater than the density of coffee.

Suppose v_1 ml of fluid 1 at temperature T_1 and specific heat C_1 are mixed with v_2 ml of fluid 2 at temperature T_2 and specific heat C_2. The temperature of the mixture will be

$$T_{mixture} = \frac{v_1 C_1 T_1 + v_2 C_2 T_2}{v_1 C_1 + v_2 C_2} \tag{13.30}$$

We now turn to the problem of measuring cardiac output. Suppose Q ml of $0^\circ C$ saline solution are infused into the right side of the heart and the temperature $T(t)$ in the pulmonary artery is measured. Assume that the heat capacity of saline solution is the same as the heat capacity of blood and that flow rate past the thermometer is a constant, F. The concentration of saline solution at the thermometer is

$$K(t) = \frac{37 - T(t)}{37 - 0}$$

The amount of saline solution that passes the thermometer in time interval $[0, t_1]$ is

$$\int_0^{t_1} F \times K(t)\, dt \quad = \quad \int_0^{t_1} F \times \frac{37 - T(t)}{37 - 0}\, dt \quad = \quad F \times \int_0^{t_1} \frac{37 - T(t)}{37 - 0}\, dt.$$

Consider t_1 to mark the end of measurement and label it as ∞. Then the amount of saline solution that passes the thermometer must also be Q, the amount injected. We can write

$$Q \quad = \quad F \times \int_0^{\infty} \frac{37 - T(t)}{37 - 0}\, dt, \qquad \text{and} \qquad F \quad = \quad \frac{Q}{\displaystyle\int_0^{\infty} \frac{37 - T(t)}{37 - 0}\, dt}.$$

Exercises for Section 13.6, Cardiac Output.

Exercise 13.6.1 Compute the units on the right side of Equation 13.30.

Exercise 13.6.2 Suppose 10°C saline solution is mixed with 50 ml of 37°C blood and the mixture is 35°C. How much saline solution was added? Assume equal heat capacities.

Exercise 13.6.3 Suppose 10°C saline solution is mixed with 25 ml of 37°C blood and the mixture is 32°C. What is the concentration of saline solution in the mixture? Assume equal heat capacities.

Exercise 13.6.4 In Equation 13.30, assume that $C_1 = C_2$ and let $K = \frac{v_2}{v_1 + v_2}$ be the concentration of the second fluid in the total of the two fluids (we will think of blood as the first fluid and saline solution as the second fluid) and $Temp$ be the temperature of the mixture. Show that

$$K = \frac{T_1 - Temp}{T_1 - T_2} \tag{13.31}$$

Exercise 13.6.5 Fill in the blank entries in Table 13.6.5. Assume the heat capacities of blood and saline solutions are equal.

Table for Exercise 13.6.5 ARTIFICIAL DATA. GET REAL DATA FROM SWAN-GANZ CATHETER COMPANY. Fluid Temperature in the pulmonary artery after injection of 10 ml of 0°C saline solution in the right side of the heart.

Time (*sec*)	Temp (°*C*)	0°C Saline Concentration (*ml/ml*)	Time (*sec*)	Temp (°*C*)	0°C Saline Concentration (*ml/ml*)
0	37	0.000			
2	34	0.081	12	33	0.108
4	32	0.135	14	34	0.081
6	31	_____	16	35	0.054
8	32	_____	18	36	_____
10	32	_____	20	37	_____

Exercise 13.6.6 Based on the data in Table 13.6.5, if the heart flow rate is R ml/sec, how much saline solution passed the thermometer downstream of the heart during the time interval $[0, 20]$ sec?

Exercise 13.6.7 From the data in Table 13.6.5, what is the cardiac capacity?

Exercise 13.6.8 Seymour S. Kety and Carl F. Schmidt[7] described a widely acknowledged and accurate method for determination of cerebral blood flow, and subsequent measurement of cerebral physiological activity such as cerebral rate of oxygen metabolism. It is commonly referred to as the Kety-Schmidt technique, and has the following outline:

1. An inert substance, σ, is introduced into the blood (patient breathes 15% N_2O, ^{133}Xe dissolved in saline is infused into the axillary vein, and other similar methods).

2. At times, t, after the start of administration, the arterial concentration, $A(t)$, of σ is measured in the radial artery.

3. The venous concentration, $V(t)$, of σ is measured at the base of the skull in the superior bulb of the internal jugular, at the point of exit of jugular vein from the brain.

Typical curves for $A(t)$ and $V(t)$ and data read from the curves are shown in Figure 13.6.8.

[7]Kety, S. S, and Schmidt, C. M., The nitrous oxide method for the quantitative determination of cerebral blood flow in Man: Theory, procedure, and normal values, *J. Clinical Invest.* **27**: 476-483, 1948.

Figure for Exercise 13.6.8 Figure from C. M. Kety and S. S. Schmidt, *J. Clinical Invest.* **27**: 476-483, 1948.

I do not yet have permission to include this classic figure. It is readily available at

www.jci.org/articles/view/101994/scanned-page/476

Data read from the graphs are shown.

t Min	$A(t)$ *	$V(t)$ *
0.00	0.0000	0.000
1.25	0.0320	0.012
3.25	0.0385	0.027
5.25	0.0410	0.034
10.00	0.0435	0.042
* Units for $A(t)$ and $V(t)$: cc N_2O/cc blood		

Assume a constant cerebral blood flow rate, R, grams/minute, that $A(t)$ is the concentration of N_2O (cubic centimeters of N_2O per cubic centimeter of blood) in blood flowing into the brain, and that $V(t)$ is the concentration of N_2O in blood flowing out of the brain.

a. Using a flow rate, R (an unknown constant, units of cc blood/min), and the data, compute an estimate of the amount of N_2O that flowed into the brain during time [0,10] (measured in minutes).

b. Using R and the data, compute an estimate of the amount of N_2O that flowed out of the brain during time [0,10].

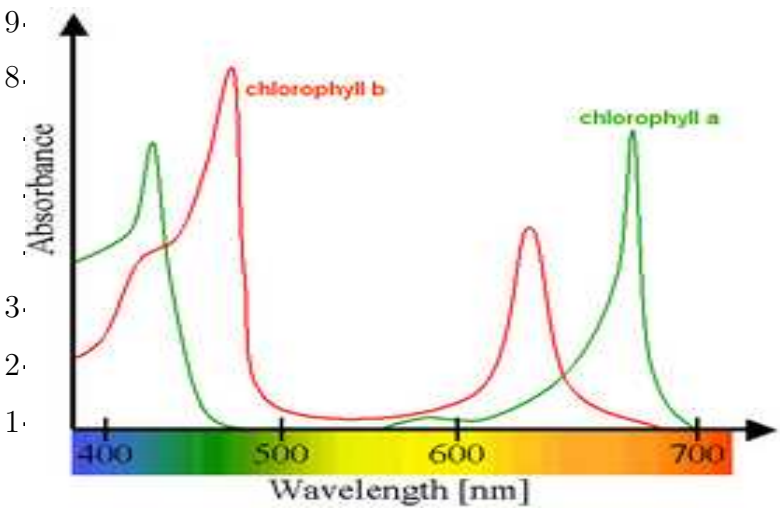
c. From the previous two steps you should be able to estimate that the amount of N_2O accumulated in the brain during time [0,10] is approximately $R \times 0.0827$ cc N_2O.

d. Note that after 10 minutes the venous and arterial concentrations of N_2O are about the same, indicating that the brain is essentially saturated. Assume that at 10 minutes the concentration of N_2O in the brain is $0.042\,\frac{\text{cc } N_2O}{\text{cc brain}}$ (the same as $V(10)$), and compute the total amount of N_2O in a brain of 1400 cc[8].

e. Equate the two estimates of the amount of N_2O in the brain and compute R. (600 - 900 gm/min $\doteq$ 600 - 900 ml/minute is normal for an adult; resting cardiac output is 5-6 L/minute.)

f. Describe how knowledge of blood flow and two additional measurements can be used to compute cerebral metabolic uptake of oxygen.

[8]Rhoads and Tanner, p 306, show brain mass = 1400 gm, we assume a brain density of 1gm/cc



λ nm	a	b
400	4.4	1.7
420	6.7	3.4
440	2.5	5.2
460	1.0	8.4
480	1.0	2.3
500	0.9	1.4
520	1.1	1.1
540	1.0	1.1
560	1.2	1.2
580	1.2	1.2
600	1.4	2.6
610	1.0	3.8
620	1.7	3.2
640	4.6	1.7
650	5.5	1.2
660	2.5	
680	1.4	
700	0.8	

Figure 13.18: Relative energy absorption for chlorophylls a and b. Wikipedia Commons, http://en.wikipedia.org/wiki/File:Chlorophyll_ab_spectra.png Nashev and Luis Fernndez Garcia supplied the file and modifications.
. Data were read from a different graph.

13.7 Chlorophyll energy absorption.

Shown in Figure 13.18 are graphs of the relative energy absorption of chlorophylls a and b[9]. Which of the two chlorophylls absorb the most energy? Light energy, E, is related to wave length, λ, by the relation

$$E = \frac{K}{\lambda}$$

where K is a constant.

Case 1. We first assume that the light intensity from the sun is constant, I, for all frequencies within the visible spectrum (so the the energy from a wave length, λ, would be

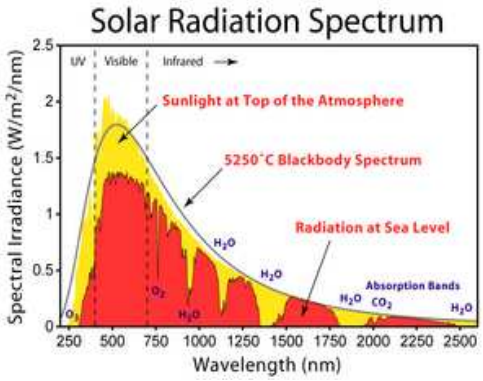
$$I \times \frac{K}{\lambda}$$

and compute the energy absorbed by chlorophyll b (the similar problem for chlorophyll a is an exercise). (Does chlorophyll a absorb in the uv range?)

We partition the visible spectrum (400 nm to 700 nm) into intervals with smaller intervals where the graph changes rapidly. (in the interval 400 to 500 nm and in the interval 575 to 650). Shown in

[9]The graph is copied from Peter H. Raven and George B. Johnson, *Understanding Biology,* Mosby Year Book, St. Louis, Mo, 1991, Figure 8-4, p189. We have added numbers to the scale on the left for Relative light absorption.

A



B

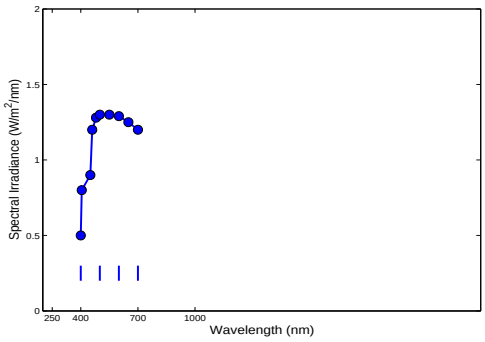


Figure 13.19: A. Sectral Irradiance W/m^2/nm at sea level versus light wave length. Provided by Robert A. Rohde (Dragons flight) on Wikipedia, http://en.wikipedia.org/wiki/File:Solar_Spectrum.png. B. Graph of data read from the graph.

Figure 13.18 are for chlorophyll b values read from Figure 13.18 (magnified on a copy machine). **Note: I substituted a different graph.**

For the interval [400,420] we assume the energy to be $I \times K/410$ and the relative absorption to be $\frac{1.7+3.4}{2}$ and compute the relative energy absorption to be

$$I \times \frac{K}{410} \times \frac{1.7+3.4}{2} \times (420-400) = I \times K \times 0.124$$

Similarly for the interval [420,440] the relative energy absorption is

$$I \times \frac{K}{430} \times \frac{3.4+5.2}{2} \times (440-420) = I \times K \times 0.200$$

continuing we get for the interval [400,650]

Relative energy absorption for chlorophyll $b \doteq I \times K \times 1.364$.

Ultimate measure. Suppose the relative energy function for chlorophyll b (the graph is the dashed line in the figure) is $R(\lambda)$. Then the

$$\text{Relative energy absorption for chlorophyll } b = \int_{400}^{700} I \times \frac{K}{\lambda} \times R(\lambda)\, d\lambda$$

Exercise 13.7.1 Compute the relative energy absorption for chlorophyll a. With our readings, we get $K \times 1.229$ and it appears that chlorophyll b absorbs more energy.

Case 2. Sunlight intensity is not constant across the spectrum. Shown in Figure 13.19A is a graph that shows the light *irradiance* (which is intensity/wave length) at the sea surface as a function of λ measured as W/m^2/nm. In Figure 13.19B is the graph of data read from the graph:

Wave Length (nm)	400	405	450	460	480	500	550	600	650	700
Irradiance W/m^2/nm	0.5	0.8	0.9	1.2	1.28	1.3	1.3	1.29	1.25	1.2

Everything has changed and needs to be recomputed. Now we compute light intensity for chlorophyll b on

Exercise 13.7.2 Suppose the function describing spectral irradiance of chlorophyll b in Figure 13.19, is $R(\lambda)$. Write an integral that is the relative energy absorbed by chlorophyll b over the visible spectrum from 400 to 700 nm.

Exercise 13.7.3 Compute (approximately) the relative energy absorbed by chlorophyll a over the visible spectrum from 400 to 700 nm.

Chapter 14

The Fundamental Theorem of Calculus

Where are we going?

You have studied the two primitive concepts of calculus, the derivative and the integral. They are based on the notion of limit, but each concept has been defined without reference to the other. The Fundamental Theorem of Calculus defines the relation between the derivative and the integral, and shows that each operation is the inverse of the other. A powerful method of evaluating integrals is a result.

14.1 An Example.

Let A be the function defined by

$$A(x) = \int_0^x \sin(t)\,dt \qquad \text{for} \qquad 0 \le x \le \frac{\pi}{2}$$

Then for x in $[0, \pi/2]$, $A(x)$ is the area of the region bounded by the graphs of

$$y = \sin t \qquad y = 0 \qquad \text{and} \qquad t = x \qquad \text{(see Figure 14.1A).}$$

We have two goals.

Goal I. Show that for any x, $0 \le x \le \pi/2$, $\qquad A'(x) = \sin(x)$

Goal II. Suggest (and it is actually true) that $\qquad A(x) = -\cos(x) + 1$

Goal I. $A'(x) = \sin(x)$. We will show that the right hand derivative,

$$A'^{+}(x) = \lim_{h \to 0+} \frac{A(x+h) - A(x)}{h} = \sin(x) \qquad \text{for} \qquad 0 \le x < \pi/2.$$

A similar argument shows that the left hand derivative, $A'^{-}(x) = \sin x$ for $0 < x \le \pi/2$, so that $A'(x) = \sin x$ on $0 \le x \le \pi/2$.

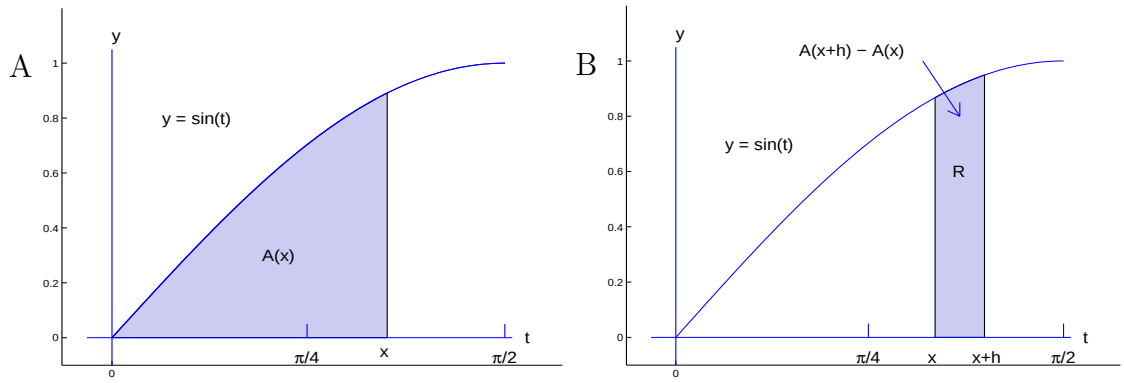


Figure 14.1: A. Area $A(x)$ of region bounded by the graphs of $y = \sin t$, $y = 0$ and $t = x$. B. Area of region between x and $x + h$.

Let x and $h > 0$ be numbers satisfying $0 \le x < x + h \le \pi/2$ and R be the region (shaded in Figure 14.1B) between $t = x$ and $t = x + h$ and below the graph of $y = \sin(x)$ and above $y = 0$. R is the difference of two regions under the graph of $y = \sin(x)$, one between $t = 0$ and $t = x + h$ and the other between $t = 0$ and $t = x$, and

$$\text{Area of} \quad R \quad \text{is} \quad A(x+h) - A(x) \tag{14.1}$$

In Figure 14.2A is a rectangle, R_1, that is contained within R and the area of R_1 is less than the area of R. The height of R_1 is $\sin(x)$ and the width is h, so the

$$\text{Area of} \quad R_1 \quad \text{is} \quad \sin(x) \times h \tag{14.2}$$

In Figure 14.2B is a rectangle, R_2, that entirely contains R and the area of R_2 is greater than the area of R. The height of R_2 is $\sin(x + h)$ and the width is h, so the

$$\text{Area of} \quad R_2 \quad \text{is} \quad \sin(x+h) \times h \tag{14.3}$$

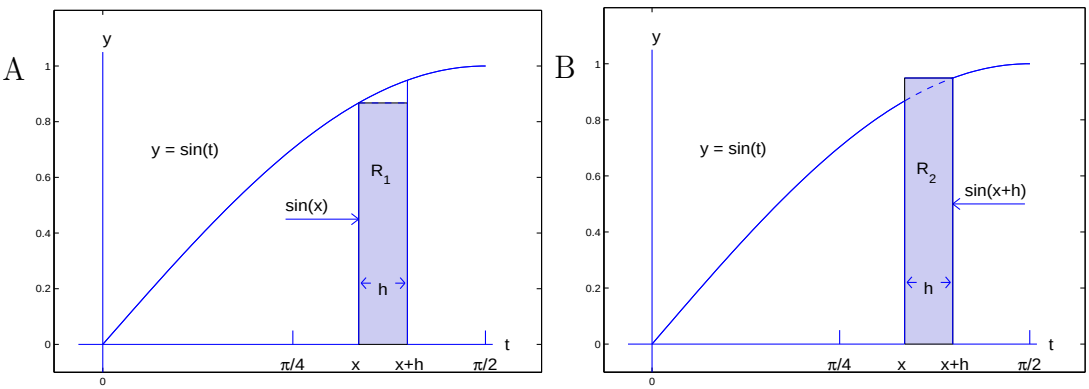


Figure 14.2: A. Rectangle R_1 of area $(\sin x) \times h$. B. Rectangle R_2 of area $(\sin(x + h)) \times h$.

We conclude from

$$\text{Area of} \quad R_1 \quad < \quad \text{Area of} \quad R \quad < \quad \text{Area of} \quad R_2$$

and from Equations 14.1 to 14.3 that (remember, $h > 0$)

$$\begin{aligned} \sin(x) \times h &< A(x+h) - A(x) < \sin(x+h) \times h \\ \sin(x) &< \frac{A(x+h) - A(x)}{h} < \sin(x+h) \end{aligned}$$

Now

$$\begin{array}{rccccc} & a. & & b. & & c. \\ & \sin(x) & < & \dfrac{A(x+h) - A(x)}{h} & < & \sin(x+h) \\ \text{As} \quad h \longrightarrow 0+ & \downarrow & & \downarrow & & \downarrow \\ & \sin(x) & \leq & A'^{+}(x) & \leq & \sin(x) \end{array} \tag{14.4}$$

from which it follows that

$$A'^{+}(x) = \sin(x) \qquad \text{for} \qquad 0 \leq x < \frac{\pi}{2}$$

The three limits, a., b. and c. in Equation 14.4 are important. Limit a. is valid because $\sin x$ is independent of h. Limit b. is the definition of the righthand derivative, A'^{+}. Limit c. is correct because $\sin x$ is continuous at x, which is shown in Exercise 7.1.4.

Our argument has assumed that $h > 0$ and $0 \leq x < \pi/2$. Simple modifications of the argument imply that $A'^{-}(x) = \sin x$ for $0 < x \leq \pi/2$. so that $A'(x) = \sin x$ for $0 < x < \pi/2$. The domain of A is $[0, \pi/2]$, and by definition $A'(0) = A'^{+}(0) = \sin 0$ and $A'(\pi/2) = A'^{-}(\pi/2) = \sin \pi/2$. Goal I has been met.

Goal II. $A(x) = -\cos(x) + 1$. We know that $A'(x) = \sin(x)$. Derivative formulas will yield

$$[-\cos(x)]' = \sin(x).$$

Thus $A(x)$ and $-\cos(x)$ have the same derivative, $\sin(x)$. We might think, then, that because $A(x)$ and $-\cos(x)$ have the same derivative, they must be the same functions. That is too strong because constant functions have derivative $= 0$, and, for example,

$$[-\cos(x) + 13]' = \sin(x), \qquad \text{also.}$$

But we can conclude (as we will see in Theorem 14.3.2) that there is a number C such that

$$A(x) = -\cos(x) + C$$

Now $A(0) = \int_0^0 \sin t \, dt = 0$, and $\cos(0) = 1$, so

$$\begin{aligned} A(0) &= -\cos(0) + C \\ 0 &= -1 + C \\ C &= 1 \end{aligned}$$

Therefore

$$A(x) = -\cos(x) + 1$$

Goal II has been met.

Note that

$$A\left(\frac{\pi}{2}\right) = 1 - \cos\left(\frac{\pi}{2}\right) = 1.$$

Therefore the area of the region shown in Figure 14.3 bounded by the graphs of

$$y = \sin(t) \qquad y = 0 \qquad \text{and} \qquad t = \frac{\pi}{2}$$

is 1. By this process, the area was found to be 1 **without reference to approximating sums**. Using the limit of approximating sums, this region also was found to have area 1 in Example 13.4.4.

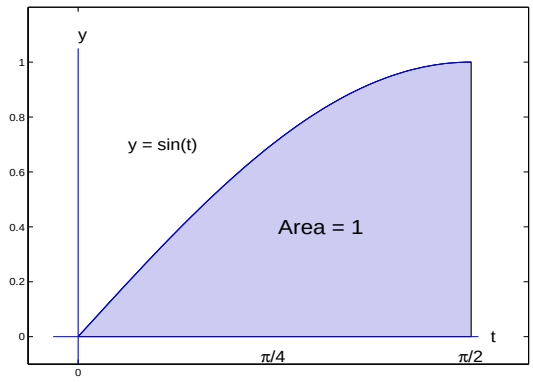


Figure 14.3: The region bounded by the graph of $y = \sin x$, $y = 0$ and $t = \pi/2$.

In this chapter we give arguments for more general results, but the stepping stones for all of them are those we have just shown.

14.2 The Fundamental Theorem of Calculus.

Theorem 14.2.1 The Fundamental Theorem of Calculus. Suppose f is a continuous function defined on an interval $[a, b]$ and G is the function defined for every x in $[a, b]$ by

$$G(x) = \int_a^x f(t)\,dt$$

Then for every x in $[a, b]$

$$G'(x) = f(x)$$

Proof of the Fundamental Theorem of Calculus. We prove the theorem for the case that f is increasing. Suppose f is an increasing and continuous function defined on $[a, b]$ and for each x in $[a, b]$, $G(x) = \int_a^x f(t)\,dt$. We will prove that the right hand derivative

$$G'^{+}(x) = \lim_{h \to 0+} \frac{G(x+h) - G(x)}{h} = f(x) \qquad \text{for} \qquad a \le x < b.$$

A simple modification of the argument shows that $G'^{-}(x) = f(x)$ for $a < x \leq b$, so that $G'(x) = f(x)$ for $a \leq x \leq b$.

Proof that $G'^{+}(x) = f(x)$. Let x and $x + h$ be numbers in $[a, b]$ with $h > 0$. Refer to Figure 14.4.

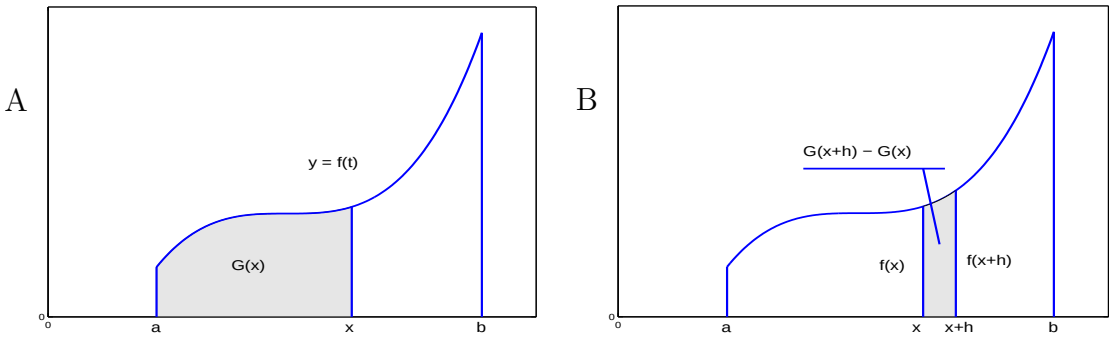


Figure 14.4: A. $G(x) = \int_a^x f(t)\,dt$. B. $G(x+h) - G(x) = \int_x^{x+h} f(t)\,dt$.

Explore 14.2.1 Do This. Give reasons for the steps A - H. For steps E. and F. read Exercise 13.4.8.

$$
\begin{aligned}
G(x) &= \textstyle\int_a^x f(t)\,dt && \text{A.}\\
G(x+h) &= \textstyle\int_a^{x+h} f(t)\,dt && \text{B.}\\
&= \textstyle\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt && \text{C.}\\
G(x+h) - G(x) &= \textstyle\int_x^{x+h} f(t)\,dt && \text{D.}\\
f(x) \times h &\leq G(x+h) - G(x) && \text{E.}\\
G(x+h) - G(x) &\leq f(x+h) \times h && \text{F.}\\
f(x) \times h \leq G(x+h) - G(x) &\leq f(x+h) \times h && \text{G.}\\
f(x) \leq \frac{G(x+h) - G(x)}{h} &\leq f(x+h) && \text{H.}
\end{aligned}
$$

■

We now examine what happens to the three terms in the inequality string

$$f(x) \leq \frac{G(x+h) - G(x)}{h} \leq f(x+h)$$

as h gets close to zero. Because $f(x)$ is not affected by h, $f(x)$ remains fixed as h gets close to zero. However:

Explore 14.2.2 Do This. Give reasons for A. and B.

$$\lim_{h\to 0+} \frac{G(x+h)-G(x)}{h} = G'^{+}(x) \qquad \text{A.}$$

$$\lim_{h\to 0+} f(x+h) = f(x) \qquad \text{B.}$$

■

We conclude that

$$\begin{array}{lccccc} & f(x) & \leq & \dfrac{G(x+h)-G(x)}{h} & \leq & f(x+h) \\ \text{As } h\to 0 & \downarrow & & \downarrow & & \downarrow \\ & f(x) & \leq & G'^{+}(x) & \leq & f(x) \end{array}$$

It follows from $f(x) \leq G'^{+}(x) \leq f(x)$ that $G'^{+}(x) = f(x)$ for $a \leq x < b$. A slight modification of this argument shows that $G'^{-}(x) = f(x)$ for $a < x \leq b$. and we conclude that $G'(x) = f(x)$ for $a < x < b$. Because $G'(a) = G'^{+}(a)$ and $G'(b) = G'^{-}(b)$, $G'(x) = f(x)$ for $a \leq x \leq b$.

Slightly modified arguments yield the same conclusion for f a decreasing function.
End of proof.

Example 14.2.1 Although technical in statement, the Fundamental Theorem of Calculus agrees with your intuition.

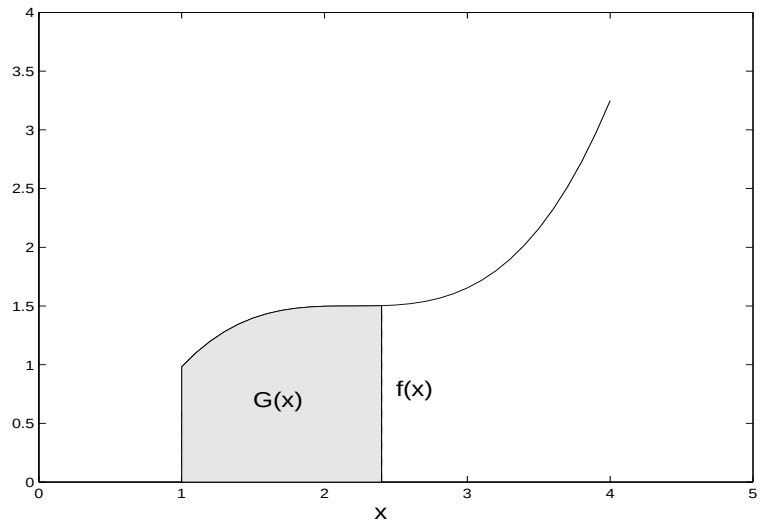
1. If $v(t)$ is the velocity of a particle at time t, then $G(x) = \int_0^x v(t)\,dt$ is the displacement of the particle during the time interval $[0, x]$. The Fundamental Theorem of Calculus states that $G'(x)$, the rate of change of displacement, is $v(x)$, the velocity.

2. If $r(t)$ is the rate at which urea is produced in a patient without functional kidneys, then $G(x) = \int_0^x r(t)\,dt$ is the total amount of urea in the body x hours since the last dialysis. The Fundamental Theorem of Calculus states that $G'(x)$, the rate of change of total urea, is $r(x)$, the rate at which it is produced.

3. If $b(t)$ and $d(t)$ are the birth and death rates of a population at time t, then $r(t) = b(t) - d(t)$ is the growth rate (which may be negative). $G(x) = \int_0^x r(t)\,dt$ is the population increase (again, it could be negative) during the time interval $[0, x]$. The Fundamental Theorem of Calculus states that $G'(x)$, the rate of change of population, is $r(x)$, the growth rate.

4. In Example Figure 14.2.1.1, for $1 \leq x \leq 4$, let $G(x)$ be the area of the region bounded by the graph of f, the t-axis, and the lines $t = 1$ and $t = x$. The rate at which G increases, $G'(x)$, is $f(x)$ the height of the graph at x.

Figure for Example 14.2.1.1 $G(x)$ is the area between the graph of F, the t-axis, $t = 1$ and $t = x$. The rate of increase of G at x is $f(x)$.



Exercises for Section 14.2, The Fundamental Theorem of Calculus.

Exercise 14.2.1 Work Explores 14.2.1 and 14.2.2

Exercise 14.2.2 a. Draw an approximation to the graph of the function G defined in Example Figure 14.2.1.1. Suggestion: Partition the interval [1,4] into six equal subintervals, [1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0] and estimate $f(1.0)$, $f(1.5)$, $\cdots$, $f(4.0)$. Then estimate $G(1.0)$, $G(1.5)$, $\cdots$, $G(4.0)$ using trapezoidal approximations.

b. Estimate $G'(1.5)$ using each of the three difference quotients:

Backward	Centered	Forward
$\dfrac{G(1.5)-G(1)}{1.5-1}$	$\dfrac{G(2)-G(1)}{2-1}$	$\dfrac{G(2)-G(1.5)}{2-1.5}$

The best estimate would normally be the centered difference quotient. Compare this estimate with your estimate of $f(1.5)$.

c. Use your data to estimate $G'(3.5)$ and compare your estimate of $G'(3.5)$ with $f(3.5)$.

Exercise 14.2.3 Let

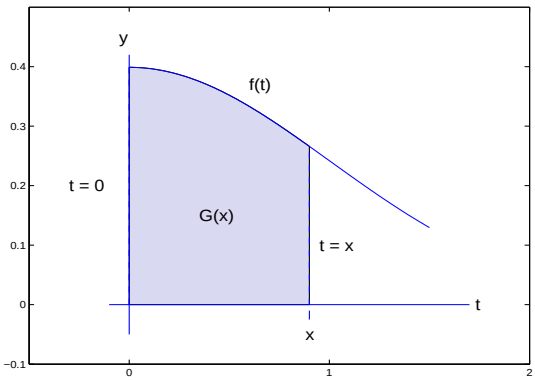
$$f(t) = \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2} \qquad \text{and} \qquad G(x) = \int_0^x f(t)\,dt = \int_0^x \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\,dt$$

$G(x)$ is the area of the region bounded by the graphs of

$$y = f(t) = \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2} \qquad y = 0 \qquad t = 0 \qquad \text{and} \qquad t = x$$

in Exercise Figure 14.2.3. Included in the figure are some data for both f and G.

Figure for Exercise 14.2.3 Graph of $f(t) = \dfrac{1}{\sqrt{2\pi}}\, e^{-t^2/2}$; $G(x)$ is the area of the shaded region between the graph of f, $y = 0$, $t = 0$ and $t = x$.



t	f(t)	x	G(x)
0.0	0.3989	0.0	0.0000
0.2	0.3910	0.2	0.0790
0.4	0.3683	0.4	0.1549
0.6	0.3332	0.6	0.2251
0.8	0.2897	0.8	0.2874
1.0	0.2420	1.0	0.3405
1.2	0.1942	1.2	0.3842

a. Write the formula for $f(0.6)$ and evaluate it.

b. Use the values of $f(0)$ $f(0.2)$, $f(0.4)$ and $f(0.6)$ and the trapezoidal approximation to estimate $G(0.6)$.

c. Approximate $G'(1.0)$ from the data for G. using the backward, centered, and forward difference quotients and compare these estimates with $f(1.0)$.

Exercise 14.2.4 Let G(x) be the area of the region bounded by the graphs of $y = \frac{1}{1+t^2}$, $y = 0$, $t = 1$ and $t = x$.

a. Compute approximate values for $y(1.0)$, $y(1.5)$, $y(2.0)$, $y(2.5)$, and $y(3.0)$.

b. Compute approximate values for $G(1.0)$, $G(1.5)$, $G(2.0)$, $G(2.5)$, and $G(3.0)$.

c. Sketch the graph of $y = \frac{1}{1+t^2}$ on [0,3].

d. Sketch the graph of G on [1,3].

e. Estimate $G'(2)$.

Exercise 14.2.5 At 1:00 a.m. an oil pipe line bursts and starts releasing oil into a lake at the rate of 2 cubic meters per hour. At 2:00 a.m., a second oil pipe line bursts and also starts releasing oil into the lake at the rate of 3 cubic meters per hour.

a. How much oil is in the lake at 1:00, 1:30, 2:00, 2:30, 3:00, 3:30, and 4:00?

b. Let T(x) be the total amount of oil in the lake at time x. Draw a graph of T.

c. Write equations describing the total amount of oil, $T(x)$, in the lake for each time x between 1:00 a.m. and 4:00 a.m.

d. Compute T'.

Exercise 14.2.6 This problem illustrates the necessity of the hypothesis that f be continuous in the statement of the Fundamental Theorem of Calculus. Remember that the integral is defined for any nondecreasing function. Let

$$f(t) = \begin{cases} 1 & \text{For} \quad 1 \le t < 2 \\ 2 & \text{For} \quad 2 \le t \le 4 \end{cases}$$

For each x in $[1, 4]$, let G(x) be the area of the region between the graph of f and the horizontal axis and between $t = 1$ and $t = x$. See Exercise Figure 14.2.6

a. Compute $G(1.0)$, $G(1.5)$, $G(2.0)$, $G(2.5)$, $G(3.0)$, $G(3.5)$ and $G(4.0)$.

b. Draw a graph of G.

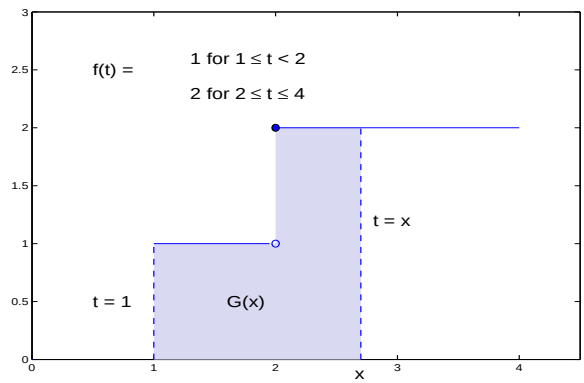
c. Write equations describing G.

d. Write $G(x)$ as an integral.

e. Compute and draw the graph of G'.

f. G' and f are not the same function. What is the difference?

Figure for Exercise 14.2.6 Graph of f where $f(t) = 1$ for $1 \le t < 2$ and $f(t) = 2$ for $2 \le t \le 4$ and the region G bounded by the graph of f, $y = 0$, $t = 1$ and $t = x$.



Exercise 14.2.7 Let $F(t) = [t]$ where $[t]$ is the greatest integer less than or equal to t. For example, $[\pi] = 3$, and $[2] = 2$. Let $G(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 5$.

a. Sketch the graph of f for $0 \le t \le 5$

b. Sketch the graph of G.

c. Sketch the graph of G'.

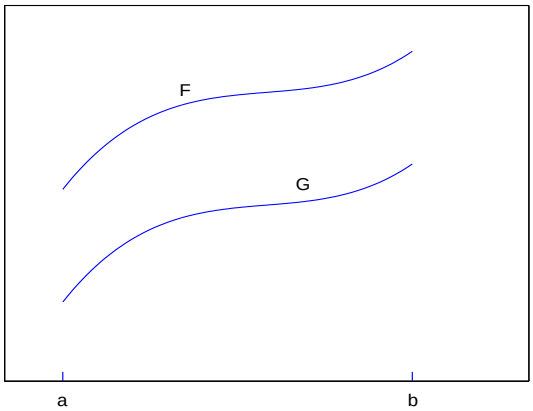


Figure 14.5: The graphs of F and G have the same derivative and are parallel.

d. You should find that $G' \neq f$. Does this contradict the Fundamental Theorem of Calculus?

Exercise 14.2.8 Use your calculator to solve the previous problem (slowly). In GRAPH, use MORE to find FORMT. Under FORMT select DrawDOT. In GRAPH go to $y(x) =$. Write $y1 =$ int x You can find 'int' in 2nd MATH NUM. Write $y2 =$ fnInt(y1,x,0,x). You can find fnInt in 2nd CALC. Set you window to $0 \leq x \leq 4$, $0 \leq y \leq 6$. Press DRAW. Your calculator will draw the graph of f and the graph of G. Plan to work some other problems while you wait. It takes about 12 minutes to compute.

14.3 The parallel graph theorem.

The Parallel Graph Theorem is needed in order to make full use of the Fundamental Theorem of Calculus. A preliminary version is the Horizontal Graph Theorem.

Theorem 14.3.1 Horizontal Graph Theorem. If D is a continuous function defined on an interval $[a, b]$ and for every number x in (a, b),

$$D'(x) = 0$$

then there is a number C such that for every number x in $[a, b]$

$$D(x) = C.$$

Proof. Let $C = D(a)$. Suppose x is in (a, b). By the Mean Value Theorem 9.1.1 there is a number z between a and x such that

$$D(x) - D(a) = D'(z) \times (x - a).$$

Because $D'(z) = 0$, $D(x) - D(a) = 0$ and $D(x) = D(a) = C$. Because D is continuous and $D(x) = C$ for every x in $[a, b)$, $D(b) = C$.
End of proof.

An example of parallel graphs, that is, graphs which have the same derivative, is shown in Figure 14.5.

Theorem 14.3.2 Parallel Graph Theorem. If F and G are functions defined on an interval $[a, b]$ and for every x in $[a, b]$

$$F'(x) = G'(x)$$

then there is a number, C, such that for every x in $[a, b]$

$$F(x) = G(x) + C$$

Proof. Let $D(x) = F(x) - G(x)$ for every x in $[a, b]$. Then

$$D'(x) = [F(x) - G(x)]' = F'(x) - G'(x) = 0$$

for every x in $[a, b]$. By the Horizontal Graph Theorem, Theorem 14.3.1, there is a number C such that for every x in $[a, b]$, $D(x) = C$. Then for every x in $[a, b]$

$$D(x) = F(x) - G(x) = C \qquad \text{and} \qquad F(x) = G(x) + C.$$

End of proof.

The power of the Fundamental Theorem of Calculus augmented by the Parallel Graph Theorem is illustrated by the next example.

Example 14.3.1 A 10cc syringe has cross-sectional area A cm^2; air inside the plunger is atmospheric pressure P_0; the plunger is a the 10 cc mark and the neck of the syringe is blocked. The plunger is depressed a distance s/A to the $10 - s$ cc mark; the pressure, P_s, inside the syringe is $P_0 \times 10/(10 - s)$. The force on the plunger is $(P_s - P_0) \times A = A \times P_0 \times (s/(10 - s)$. The work done in compressing the air from 10 cc to 5 cc is

$$\int_0^5 A \times P_0 \frac{s}{10 - s}\, d\frac{s}{A} = \int_0^5 \times P_0 \frac{s}{10 - s}\, ds$$

Let

$$W(x) = P_0 \int_0^x \frac{s}{10 - s}\, ds$$

Then the work done in compressing the air is $W(5)$. The Fundamental Theorem of Calculus asserts that

$$W'(x) = P_0 \frac{x}{10 - x}$$

Let $\omega(x)$ be defined by **(a bolt out of the blue!)**

$$\omega(x) = P_0[-10 \ln(10 - x) - x]$$

Explore 14.3.1 Use derivative formulas including $f(t) = \ln U(t) \Longrightarrow f'(t) = \frac{1}{U(t)} U'(t)$ to show that

$$\omega'(x) = P_0 \frac{x}{10 - x} \qquad \blacksquare$$

Thus $W'(x) = \omega'(x)$ for every x in $[0, 5]$, and by the Parallel Graph Theorem there is a number C such that such that for every number x in $[0, 5]$

$$W(x) = \omega(x) + C$$

Now

$$\begin{aligned} W(0) &= P_0 \int_0^0 \frac{s}{10-s}\, ds &&= 0, \qquad \text{and} \\ \omega(0) &= P_0[-10\ln(10-0) - 0] &&= -P_0 \times 10 \ln 10. \end{aligned}$$

Because $W(0) = \omega(0) + C$

$$0 = -P_0 \times 10 \ln 10 + C \qquad \text{and} \qquad C = P_0 \times 10 \ln 10.$$

We can conclude that for all x in $[0, 5]$

$$W(x) = P_0[-10\ln(10-x) - x] + P_0 10 \ln(10)]$$

The total work done in compressing the air as

$$W(5) = P_0[-10\ln(10-5) - 5 + 10\ln(10)] = P_0 \times 1.93$$

In the previous example, we used the Fundamental Theorem of Calculus to evaluate the integral **without computing an approximating sum**. It was important to have a function, ω, satisfying $\omega' = W'$, and in a sense, the problem of computing approximating sums was exchanged for the problem of finding ω. Once we found such an ω, we appealed to the Parallel Graph Theorem to conclude that there was a number C such that for all x, $W(x) = \omega(x) + C$.

Exercises for Section 14.3 The parallel graph theorem.

Exercise 14.3.1 Let $f(x) = [x]$ (= greatest integer less than or equal to x). Draw the graph of f (Use 2nd MATH, NUM, int x-VAR on your TI-85). Compute and draw the graph of f'. Is this a example showing that the Horizontal Graph Theorem, Theorem 14.3.1, is false?

Exercise 14.3.2 Suppose $P(t)$ is the size of a population at time t, $P(0) = 5000$ and $P'(t) = 0$ for all t. What is $P(100)$? What is $P(10000000000)$?

Exercise 14.3.3 Suppose a mold colony is growing in a nutrient solution and that on day zero the area was 0.5 cm^2 and for every time, $t \geq 0$, the instantaneous rate of growth of the area of the colony is $2t$ cm^2 per day. Let $P(t)$ be the colony area at time t.

a. Show that for every time, t, $P'(t) = 2t$.

b. Show that for $Q(t) = t^2$, $Q'(t) = 2t$. Then $P'(t) = Q'(t)$.

c. From the Parallel Graph Theorem, it follows that there is a constant C such that $P(t) = Q(t) + C$.

d. Use $P(0) = 0.5$ to evaluate C.

e. What is the area of the mold colony on day 8?

Exercise 14.3.4 Suppose the rate of glucose production in a corn plant is proportional to sunlight intensity and can be approximated by

$$R(t) = K \times (t+7)^2 \times (t-7)^2 = K \times \left(t^4 - 98t^2 + 2401\right) \qquad -7 \leq t \leq 7$$

Time is measured so that sunrise is at -7 hours, the sun is at its zenith at 0 hours and sets at 7 hours. The quantity $Q(x)$ of glucose produced during the period $[-7, x]$ is

$$Q(x) = \int_{-7}^{x} R(t)\,dt = \int_{-7}^{x} K \times \left(t^4 - 98t^2 + 2401\right)\,dt = K \times \int_{-7}^{x} \left(t^4 - 98t^2 + 2401\right)\,dt$$

By the Fundamental Theorem of Calculus,

$$Q'(x) = K \times \left(x^4 - 98x^2 + 2401\right)$$

a. Sketch the graph of $R(t)$. At what time is the sun most intense?

b. Show that if $U(x) = \frac{x^5}{5}$ then $U'(x) = x^4$.

c. Find an example of a function, $V(x)$ such that $V'(x) = -98x^2$.

d. Find an example of a function, $W(x)$ such that $W'(x) = 2401$.

e. Let $G(x) = K \times \left(\frac{x^5}{5} - \frac{98}{3}x^3 + 2401x\right)$. Show that $G'(x) = Q'(x)$.

f. Conclude that there is a number, C, such that

$$Q(x) = G(x) + C = K \times \left[\frac{x^5}{5} - \frac{98}{3}x^3 + 2401x\right] + C$$

g. Why is $Q(-7) = 0$.

h. Evaluate C.

i. Compute $Q(7)$, the amount of glucose produced during the day.

Exercise 14.3.5 "Based on studies using isolated animal pancreas preparations maintained in vitro, it has been determined that insulin is secreted in a biphasic manner in response to a marked increase in blood glucose. There is an initial burst of insulin secretion that may last 5-15 minutes, a result of secretion of preformed insulin secretory granules. This is followed by more gradual and sustained insulin secretion that results largely from biosynthesis of new insulin molecules. " (Rhoades and Tanner, p710)

a. A student eats a candy bar at 10:20 am. Draw a graph representative of the rate of insulin secretion between 10:00 and 11:00 am.

b. Draw a graph representative of the amount of serum insulin between 10:00 and 11:00. Assume that insulin is degraded throughout 10 to 11 am at a rate equal to insulin production before the candy is eaten. and that serum insulin at 10:00 was I_0.

c. Write an expression for the amount of serum insulin, $I(t)$, for t between 10:00 and 11:00 am.

Exercise 14.3.6 Equal quantities of gaseous hydrogen and iodine are mixed resulting in the reaction

$$H_2 + I_2 \longrightarrow 2HI$$

which runs until I_2 is exhausted (H_2 is also exhausted). The rate at which I_2 disappears is $\frac{0.2}{(t+1)^2}$ gm/sec. How much I_2 was initially introduced into the mixture?

a. Sketch the graph of the reaction rate, $r(t) = \frac{0.2}{(t+1)^2}$.

b. Approximately how much I_2 combined with H_2 during the first second?

c. Approximately how much I_2 combined with H_2 during the second second?

d. Let $Q(x)$ be the amount of I_2 that combines with H_2 during time 0 to x seconds. Write an integral that is $Q(x)$.

e. What is $Q'(x)$?

f. Compute $W'(x)$ for $W(x) = \frac{-0.2}{1+x}$.

g. Show that there is a number, C, for which $Q(x) = W(x) + C$.

h. Show that $C = 0.2$ so that $Q(x) = 0.2 - \frac{0.2}{1+x}$.

i. How much I_2 combined with H_2 during the first second?

j. How much I_2 combined with H_2 during the first 100 seconds?

k. How much I_2 combined with H_2?

14.4 The Second Form of the Fundamental Theorem of Calculus

The Parallel Graph Theorem leads to a second form of the Fundamental Theorem of Calculus that has powerful applications.

Theorem 14.4.1 Fundamental Theorem of Calculus II. Suppose f is a continuous function defined on an interval $[a, b]$ and F is a function defined on $[a, b]$ having the property that

$$\text{for every number } t \text{ in } [a, b] \qquad F'(t) = f(t) \tag{14.5}$$

Then

$$\int_a^b f(t)\,dt = F(b) - F(a). \tag{14.6}$$

Proof: Suppose the hypothesis of the theorem. Let G be the function defined on $[a, b]$ by

$$G(x) = \int_a^x f(t)\,dt$$

Explore 14.4.1 Do this. Give reasons for the steps labeled A - E.

For x in $[a, b]$, $G'(x) = f(x)$. A.

For x in $[a, b]$, $F'(x) = f(x)$. B

For x in $[a, b]$, $G'(x) = F'(x)$.

There is a number C so that for x in $[a, b]$, $\quad G(x) = F(x) + C$ C.

$G(a) = 0$. D.

$G(a) = F(a) + C$. Therefore, $C = -F(a)$.

$G(x) = F(x) - F(a)$.

$G(b) = F(b) - F(a)$.

$\int_a^b f(t)\,dt = F(b) - F(a)$. E. ■

End of proof.

Notation: The number $F(b) - F(a)$ is denoted by $[\,F(x)\,]_a^b$ or $F(x)\,\vert_a^b$.

Example 14.4.1 You now have a powerful computational tool.

1. Evaluate $\int_0^1 t^2\,dt$. Check that

$$\text{if} \qquad F(t) = \frac{t^3}{3} \qquad \text{then} \qquad F'(t) = \left[\frac{t^3}{3}\right]' = \frac{1}{3}[\,t^3\,]' = \frac{1}{3}3t^2 = t^2.$$

   It follows that

$$\int_0^1 t^2\,dt = F(1) - F(0) = \left[\frac{t^3}{3}\right]_0^1 \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

   The integral is the area of the region bounded by the graphs of $y = t^2$, $y = 0$, and $t = 1$. The area was found to be $\frac{1}{3}$ in Chapter 13 by use of rectangles and approximations.

2. Evaluate $\int_1^2 t^2\,dt$. Using the same function, $F(t) = \frac{t^3}{3}$ we obtain that

$$\int_1^2 t^2\,dt = F(2) - F(1) = \left[\frac{t^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

3. Evaluate $\int_0^5 5e^{0.02t}\,dt$. Observe that $F(t) = 250e^{0.02t}$ has the property that $F'(t) = 5e^{0.02t}$ Therefore

$$\int_0^5 5e^{0.02t}\,dt = F(5) - F(0) = \left[250e^{0.02\times t}\right]_0^5 = 250e^{0.02\times 5} - 250e^{0.02\times 0} = 26.29$$

4. Evaluate $\int_1^2 \frac{1}{t}\,dt$. We found in Chapter 5 that $F(t) = \ln t \Longrightarrow F'(t) = \frac{1}{t}$. Therefore,

$$\int_1^2 \frac{1}{t}\,dt = F(2) - F(1) = [\,\ln t\,]_1^2 = \ln 2 - \ln 1 = \ln 2$$

Exercises for Section 14.4, The Second Form of the Fundamental Theorem of Calculus.

Exercise 14.4.1 Do Explore exercise 14.4.1.

Exercise 14.4.2 Evaluate the integrals

	Compute	Evaluate		Compute	Evaluate
a.	$\left[\frac{t^6}{6}\right]'$	$\int_1^4 t^5\,dt$	b.	$\left[\frac{e^{2t}}{2}\right]'$	$\int_{-1}^2 e^{2t}\,dt$
c.	$[\sin(2t)]'$	$\int_0^\pi \cos(2t)\cdot 2\,dt$	d.		$\int_1^4 13\,t^5\,dt$
e.		$\int_1^4 t^3\,dt$	f.		$\int_1^4 (t^3+t^5)\,dt$
g.	$\left[e^{\frac{t^2}{2}}\right]'$	$\int_0^1 e^{\frac{t^2}{2}}\,t\,dt$	h.	$[\ln(1+t)]'$	$\int_0^2 \frac{1}{1+t}\,dt$
i.	$[\ln(1+t^2)]'$	$\int_0^2 \frac{t}{1+t^2}\,dt$	j.	$[\ln(1+y)]'$	$\int_0^1 \frac{1}{1+y}\,dy$
k.	$[\ln(1-y)]'$	$\int_0^{0.5} \frac{1}{1-y}\,dy$	l.	$\frac{1}{y}+\frac{1}{1-y}$	$\int_{0.25}^{0.75} \frac{1}{y(1-y)}\,dy$

Exercise 14.4.3 Evaluate the integrals.

a. $\int_1^3 t^2\,dt$ b. $\int_0^2 t^3\,dt$ c. $\int_0^2 e^{-2t}\,dt$ d. $\int_1^3 \frac{1}{t}\,dt$ e. $\int_1^3 (1+t^2)^2\,dt$

Exercise 14.4.4 The graph in Figure 14.4.4 approximates the size, $S(t)$, of colon carcinoma cells t days after injection into mice (after Leach, D. R., et al, *Science* **271** (1996) 1734.)

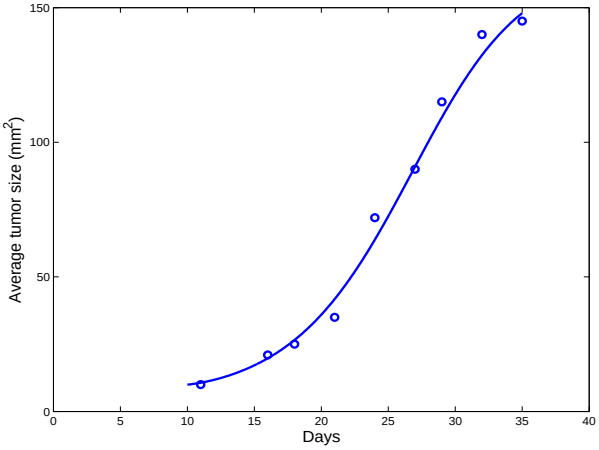
a. Read approximate values of $S(t)$ and $S'(t)$ from the curve.

b. From the (completed) table of values of $S'(t)$ approximate $\int_{15}^{35} S'(t)\,dt$.

c. Give a physical interpretation of $\int_{15}^{35} S'(t)\,dt$.

d. Why would you expect $\int_{15}^{35} S'(t)\,dt$ to be approximately 130 according to data in the table?

Figure for Exercise 14.4.4 The size of carcinoma cells t days after injection into mice. Error bars that were on the original graph are omitted.



t (days)	$S(t)$ mm^2	$S'(t)$ $\frac{\text{mm}^2}{\text{day}}$
15	18	2.38
20	36	5.43
25	72	8.93
30		
35	148	4.00

14.5 Integral Formulas.

Because the problem, "Find $F(t)$ such that $F'(t) = f(t)$" occurs with each application of the Fundamental Theorem of Calculus II, extensive tabulations of solutions to the problem (called **antiderivatives** or **indefinite integrals**) have been made during the 300 years since calculus was first introduced. I. S. Gradsteyn and I. W. Ryzhik list some 2000 - 2500 antiderivatives in their "Table of Integrals Series and Products", Academic Press, 1965. More recently, computer programs have been written that provide a rich supply of antiderivatives (Mathematica, Derive, Maple) and some hand held calculators will solve most of the commonly encountered problems (HP-48, TI-92). The antiderivatives are denoted by

$$\int f(t)\,dt$$

and are called **indefinite integrals** because the interval of integration is not specified (there are no lower and upper limits of integration). Because the derivative of a constant function is zero, every indefinite integral is really a set of functions, each two members of which differ by a constant. The common notation always includes an additive constant in the solution. For example,

$$\int x^2\,dx = \frac{x^3}{3} + C \tag{14.7}$$

is read, 'the indefinite integral of x^2 is $\frac{x^3}{3} + C$' where it is understood that C is a constant. The implication is that every function whose derivative is x^2 is of the form, $\frac{x^3}{3} + C$.

For every derivative formula, there is an indefinite integral (antiderivative) formula. There follows a table of indefinite integral formulas corresponding to the derivative formulas on page 389.

$$\int 0\,dt = C \tag{14.8}$$

$$\int 1\,dt = t + C \tag{14.9}$$

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + C \qquad \text{for } n \neq -1 \tag{14.10}$$

$$\int e^t\,dt = e^t + C \tag{14.11}$$

$$\int \frac{1}{t}\,dt = \ln t + C \tag{14.12}$$

$$\int \sin\,t\,dt = -\cos\,t + C \tag{14.13}$$

$$\int \cos\,t\,dt = \sin\,t + C \tag{14.14}$$

$$\int K \times P(t)\,dt = K \times \int P(t)\,dt \tag{14.15}$$

$$\int [f(t) + g(t)]\,dt = \int f(t)\,dt + \int g(t)\,dt \tag{14.16}$$

$$\int [f'(t) \times g(t) + f(t) \times g'(t)]\,dt = f(t) \times g(t) + C \tag{14.17}$$

$$\int G'[u(t)] \times u'(t)\,dt = G[u(t)] + C \tag{14.18}$$

$$\int [u(t)]^n \times u'(t)\,dt = \frac{u(t)^{n+1}}{n+1} + C \qquad \text{for } n \neq 0 \tag{14.19}$$

$$\int e^{u(t)} \times u'(t)\,dt = e^{u(t)} + C \tag{14.20}$$

$$\int \frac{1}{u(t)} \times u'(t)\,dt = \ln[u(t)] + C \tag{14.21}$$

$$\int \sin u(t) \times u'(t)\,dt = \cos u(t) + C \tag{14.22}$$

$$\int \cos u(t) \times u'(t)\,dt = -\sin u(t) + C \tag{14.23}$$

$$\int e^{kt}\,dt = \frac{1}{k}e^{kt} + C \tag{14.24}$$

An easy aspect of antiderivative formulas is that they can be checked readily by differentiation. For example, Equation 14.10 asserts that all functions whose derivative is t^n are of the form $\frac{t^{n+1}}{n+1} + C$. We can check that all such functions have t^n as their derivative:

$$\begin{aligned}
\frac{d}{dt}\left[\frac{t^{n+1}}{n+1} + C\right] &= \frac{d}{dt}\left[\frac{t^{n+1}}{n+1}\right] + \frac{d}{dt}[C] && \text{Equation 8.21} \\
&= \frac{1}{n+1}\frac{d}{dt}\,[t^{n+1}] + \frac{d}{dt}[C] && \text{Equation 8.15} \\
&= \frac{1}{n+1}\frac{d}{dt}\,[t^{n+1}] + 0 && \text{Equation 8.14} \\
&= \frac{1}{n+1}(n+1)t^n + 0 && \text{Equation 8.15} \\
&= t^n.
\end{aligned}$$

That there are no other such functions is a consequence of the Parallel Graph Theorem.

The indefinite integral formulas enable computation of antiderivatives of all polynomials; in fact,

only four of the antiderivative formulas are needed. For example

$$\begin{aligned}
\int [5t^7 - 3t^4 + 2]\,dt &= \int [5t^7]\,dt - \int [3t^4]\,dt + \int 2\,dt && \text{Equation 14.16}\\
&= 5\int t^7\,dt - 3\int t^4\,dt + 2\int 1\,dt && \text{Equation 14.15}\\
&= 5\tfrac{t^8}{8} - 3\tfrac{t^5}{5} + 2\int 1\,dt && \text{Equation 14.10}\\
&= 5\left[\tfrac{t^8}{8} + C_1\right] - 3\left[\tfrac{t^5}{5} + C_2\right] + 2\left[t + C_3\right] && \text{Equation 14.9}\\
&= \tfrac{5}{8}t^8 - \tfrac{3}{5}t^5 + 2t + C.
\end{aligned}$$

The constant terms included in the antiderivatives are usually suppressed until the last step, and are treated as rather pliable objects. For example, C in the last equation above is $5C_1 - 3C_2 + 2C_3$. Because a linear combination of constants is just a constant, and because subscripts are a nuisance, one often sees algebraic steps that would imply

$$5C - 3C + 2C = C.$$

Students usually adapt to this murky algebra without suffering serious damage, but it seems only fair to warn you of this practice, as we will will follow it subsequently.

14.5.1 Using the chain rules.

Equations 14.18 through 14.23 are all consequences of a chain rule for derivatives and always when using one of these equations it is important to identify $u(t)$, $G'(u)$, $u'(t)$ and $G(u)$.

Example 14.5.1 Two antiderivative problems that appear similar,

$$\int e^{t^2}\,dt \qquad \text{and} \qquad \int e^{t^2} t\,dt$$

are peculiarly different. The first has no expression in familiar terms. The second is easy.

In the second equation, identify

$$u(t) = t^2, \qquad G'(u) = e^u, \qquad u'(t) = 2t \qquad \text{and} \qquad G(u) = e^u$$

Then

$$\begin{aligned}
\int e^{t^2} t\,dt &= \int \frac{1}{2}\,e^{t^2}\,2 \times t\,dt && \text{Arithmetic}\\
&= \frac{1}{2}\int e^{t^2}\,(2 \times t)\,dt && \text{Equation 14.15}\\
&= \frac{1}{2}\int e^{u(t)}\,u'(t)\,dt\\
&= \frac{1}{2}e^{u(t)} + C && \text{Equation 14.20}\\
&= \frac{1}{2}e^{t^2} + C && \blacksquare
\end{aligned}$$

Example 14.5.2 Consider solving the two problems

$$\int \left(1+t^4\right)^{10} t^3\,dt \qquad \text{or} \qquad \int \left(1+t^4\right)^{10} t^2\,dt$$

Because $(1+t^4)^{10}$ can be expanded by multiplication (using either the binomial expansion formula or by making nine multiplications) and the expanded form is a polynomial, both integrands of these two problems are polynomials and the antiderivatives of polynomials can be easily computed. The first integral can be solved without expansion, however, and is easier to compute.

Identify

$$u(t) \;=\; 1+t^4, \qquad G'(u)=u^{10}, \qquad u'^{(t)}=4t^3 \qquad \text{and} \qquad G(u)=\frac{u^{11}}{11}$$

Then

$$\begin{aligned}
\int (1+t^4)^{10}\,t^3\,dt &= \int \frac{1}{4}\,(1+t^4)^{10}\;4\times t^3\,dt && \text{Arithmetic}\\
&= \frac{1}{4}\int (1+t^4)^{10}\;(4\times t^3)\,dt && \text{Equation 14.15}\\
&= \frac{1}{4}\int (u(t))^{10}\;u'(t)\,dt && \text{Substitution}\\
&= \frac{1}{4}\frac{(u(t))^{11}}{11}+C && \text{Equation 14.19}\\
&= \frac{1}{4}\frac{\left(1+t^4\right)^{11}}{11}+C && u(t)=1+t^4
\end{aligned}$$

Students sometimes attempt to solve $\int (1+t^4)^{10}\,t^2\,dt$ by the following means. They are thinking of

$$u(t) \;=\; 1+t^4, \qquad G'(u)=u^{10}, \qquad u'(t)=4t^3 \qquad \text{and} \qquad G(u)=\frac{u^{11}}{11}$$

$$\begin{aligned}
\int (1+t^4)^{10}\,t^2\,dt &= \int \frac{1}{4t}\,(1+t^4)^{10}\;\times 4t^3\,dt && \text{Algebra}\\
&= \frac{1}{4t}\int (1+t^4)^{10}\times 4t^3\,dt && \text{UGH!}
\end{aligned}$$

The last step is incorrect because $\frac{1}{4t}$ is not a constant. The corresponding step in the previous argument is correct because $\frac{1}{4}$ is a constant and Equation 14.15 asserts that $\int K\times f(t)\,dt = K\times\int f(t)\,dt$ when K is a constant. It is incorrect when K is not constant. ■

Example 14.5.3 Compute $\int \sin(\pi t)\,dt$. Identify

$$u(t)=\pi t, \qquad G'(u)=\sin u, \qquad u'(t)=\pi \qquad \text{and} \qquad G(u)=-\cos u$$

Then

$$\int \sin(\pi t)\,dt \;=\; \int \frac{1}{\pi}\sin(\pi t)\,\pi\,dt$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int \sin(\pi t)\,\pi\,dt \\
&= \frac{1}{\pi}\int \sin(u(t))\,u'(t)\,dt \\
&= \frac{1}{\pi}(\,-\cos(u(t))\,) + C \\
&= -\frac{1}{\pi}(\,\cos(\pi t)\,) + C \qquad \blacksquare
\end{aligned}
$$

Example 14.5.4 Compute $\int \sqrt{5t-4}\,dt$. Identify

$$
u(t) = 5t-4, \qquad G'(u) = \sqrt{u} = (\,u\,)^{1/2}, \qquad u'(t) = 5 \qquad \text{and} \qquad G(u) = \frac{u^{3/2}}{3/2}
$$

Then

$$
\begin{aligned}
\int \sqrt{5t-4}\,dt &= \frac{1}{5}\int \sqrt{5t-4}\,5\,dt \\
&= \frac{1}{5}\int (\,u(t)\,)^{1/2}\,u'(t)\,dt \\
&= \frac{1}{5}\frac{(\,u(t)\,)^{3/2}}{3/2} + C \\
&= -\frac{2}{15}(\,5t-4)^{3/2} + C \qquad \blacksquare
\end{aligned}
$$

Example 14.5.5 Compute $\int (\,\ln t\,)^3\,\frac{1}{t}\,dt$. Identify

$$
u(t) = \ln t, \qquad G'(u) = u^3, \qquad u'(t) = \frac{1}{t} \qquad \text{and} \qquad G(u) = \frac{u^4}{4}
$$

Then

$$
\begin{aligned}
\int (\,\ln t\,)^3\,\frac{1}{t}\,dt &= \int (\,u(t)\,)^3\,u'(t)\,dt \\
&= \frac{(\,u(t)\,)^4}{4} + C \\
&= \frac{(\,u(t)\,)^4}{4} + C \qquad \blacksquare
\end{aligned}
$$

Example 14.5.6 Compute $\int \tan x\,dx$. This is a good one. First write

$$
\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx
$$

Then identify

$$u(x) = \cos x, \qquad G'(u) = \frac{1}{u}, \qquad u'(x) = -\sin x \qquad \text{and} \qquad G(u) = \ln u$$

Then

$$\begin{aligned}
\int \tan x &= \int \frac{\sin x}{\cos x}\, dx \\
&= -\int \frac{1}{\cos x}(-\sin x)\, dx \\
&= -\int \frac{1}{u(x)} u'(x)\, dx \\
&= -\ln u(x) + C \\
&= -\ln \cos x + C = \ln \sec x + C \qquad \blacksquare
\end{aligned}$$

Always, once an antiderivative has been computed, it can be checked by differentiation.

We check the last claim that $\int \tan x = \ln \sec x + C$ by differentiation. We should show that $[\ln \sec x\,]' = \tan x$.

$$\begin{aligned}
[\ln \sec x\,]' &= \frac{1}{\sec x}\,[\sec x]' \\
&= \frac{1}{\sec x}\,\sec x \tan x \\
&= \tan x
\end{aligned}$$

14.5.2 Integration by parts.

Equation 14.17

$$\int [f'(t) \times g(t) + f(t) \times g'(t)]\, dt = f(t) \times g(t) + C$$

is usually rewritten as

$$\int f(t) \times g'(t)\, dt = f(t) \times g(t) - \int f'(t) \times g(t)\, dt \tag{14.25}$$

Equation 14.25 exchanges the problem, $\int f(t) \times g'(t)\, dt$ for the problem $\int f'(t) \times g(t)\, dt$. There are times when that is a good trade, but some cleverness is required to recognize when a good trade is possible. The process is called integration by parts, $f(t)$ and $g'(t)$ being the parts in the first integral and $f'(t)$ and $g(t)$ being the parts in the second integral.

In computing $\int t \times \sin t\, dt$, there is a good trade. Identify

$$\begin{aligned}
f(t) &= t \quad \text{and} \quad g'(t) = \sin t \\
f'(t) &= 1 \qquad\qquad g(t) = -\cos t
\end{aligned}$$

Then

$$\begin{aligned}\int t \times \sin t\, dt &= t \times (-\cos t) - \int 1 \times (-cost)\, dt \\ &= -t\cos t + \sin t + C\end{aligned}$$

In computing $\int t^2 \times \sin t\, dt$, two good trades can be made. First we identify

$$\begin{aligned} f(t) &= t^2 \quad \text{and} \quad g'(t) = \sin t \\ f'(t) &= 2t \qquad\qquad g(t) = -\cos t\end{aligned}$$

Then

$$\int t^2 \times \sin t\, dt = t^2 \times (-\cos t) - \int 2t \times (-\cos t)\, dt$$

Next we identify

$$\begin{aligned} f(t) &= 2t \quad \text{and} \quad g'(t) = -\cos t \\ f'(t) &= 2 \qquad\qquad g(t) = -\sin t\end{aligned}$$

and write

$$\begin{aligned}\int t^2 \times \sin t\, dt &= t^2 \times (-\cos t) - \int 2t \times (-\cos t)\, dt \\ &= t^2 \times (-\cos t) - (2t \times (-\sin t) - \int 2 \times (-\sin t)) \\ &= t^2 \times (-\cos t) - 2t \times (-\sin t) + \int 2 \times (-\sin t) \\ &= t^2 \times (-\cos t) - 2t \times (-\sin t) + 2 \times (\cos t) + C\end{aligned}$$

A chart to keep track of these computations is shown in Figure 14.6. You may see from the chart how to compute $\int t^2 \times \cos t\, dt$ and $\int t^3 \times \sin t\, dt$ and $\int t^3 e^t\, dt$.

Exercises for Section 14.5, Integral Formulas.

Exercise 14.5.1 Use your technology to find a fourth degree polynomial close to the data in the table below taken from the graph of average solar intensity at Eugene, Oregon in Figure 13.1.3 on page 593.

Day	1	60	120	180	240	300	360
Solar intensity, kw-hr/m^2	1.0	2.0	4.5	6.7	5.6	1.8	1.0

Your technology may object that the equations to compute the coefficients are ill-conditioned (subject to roundoff error), and that may be cured by centering the data at day 180 (subtract 180 from Day), and then the polynomial must be accordingly interpreted. You should get one of the two polynomials:

$$P_1(t) = 8.7329 \times 10^{-9}t^4 - 6.410 \times 10^{-6}t^3 + 1.307 \times 10^{-3}t^2 - 4.808 \times 10^{-2}t + 1.127$$

$$\begin{aligned}P_2(t) = {} & 8.7339 \times 10^{-9}(t-180)^4 - 1.2225 \times 10^{-7}(t-180)^3 - 4.5587 \times 10^{-4}(t-180)^2 \\ & +3.4149 \times 10^{-3}(t-180) + 6.63072\end{aligned}$$

Differentiate		Integrate
t^2	$\int$	$\sin t$
	$\times$	
$2t$	$-\int$	$-\cos t$
	$-\times$	
2	$\int$	$-\sin t$
	$\times$	
0	$-\int$	$\cos t$

Figure 14.6: Chart for computing the terms for integration by parts.

a. Plot the data and a graph of your polynomial.

b. Compute the integral of $\int_0^{365} P_1(t)\,dt$.

c. Compute the integral of $\int_0^{365} P_2(t)\,dt$.

d. Compare your answers to the estimate of 1324 computed using the trapezoid rule on 12 intervals of length 30 and one interval of length 5, in Exercise 13.1.3 on page 593.

Exercise 14.5.2 Check by differentiation the validity of the indefinite integral formulas:

a. $\int \frac{1}{t}\,dt = \ln t + C$

b. $\int [U(t)]^n \times U'(t)\,dt = \frac{U(t)^{n+1}}{n+1} + C$

c. $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + C$

Exercise 14.5.3 Compute $\int \left(1+t^4\right)^3\,dt$.

Exercise 14.5.4 Which of the two indefinite integrals

$$\int \frac{1}{1+t^2}t\,dt \qquad \text{or} \int \frac{1}{1+t^2}\,dt$$

is $\ln(1+t^2)^{0.5} + C$? Explain your answer. Note: The other indefinite integral is $\arctan t + C$.

Exercise 14.5.5 Compute the following integrals and antiderivatives. For the definite integrals, draw a region in the plane whose area is computed by the integral. If you solve the integral by a substitution, $u(t) =$, then identify in writing $u(t)$ and $u'(t)$.

a. $\int_0^1 t^4\,dt$ *b.* $\int_0^1 t^{499}\,dt$ *c.* $\int_0^1 t^{1/2}\,dt$

d. $\int_0^2 e^x\,dx$ *e.* $\int_0^\pi \cos z\,dz$ *f.* $\int_2^6 \frac{1}{y}\,dy$

g. $\int_{12}^{36} \frac{1}{t}\,dt$ *h.* $\int t^{-1/2}\,dt$ *i.* $\int (\sin t + \cos t)\,dt$

j. $\int \sqrt{t}\,dt$ *k.* $\int (5t^4 + 3t^2 + 5)\,dt$ *l.* $\int \frac{w^2+w+1}{w}\,dw$

Exercise 14.5.6 Compute the following antiderivatives or integrals. If you solve the integral by a substitution, $u(t) =$, then identify in writing $u(t)$ and $u'(t)$.

a. $\int e^{2t}\,dt$ *b.* $\int \sin(2t)\,2\,dt$ *c.* $\int \frac{1}{\sqrt{t}}\,dt$

d. $\int 3z^{-1}\,dz$ *e.* $\int (3\cos z + 4\sin z)\,dz$ *f.* $\int (\pi^2 + e^2)\,dz$

g. $\int \frac{x^2}{\sqrt{x}}\,dx$ *h.* $\int \frac{x^2+1}{x}\,dx$ *i.* $\int \left(x + \frac{1}{x}\right)^2\,dx$

j. $\int \frac{1}{t+1}\,dt$ *k.* $\int (1+t^2)^3\,2t\,dt$ *l.* $\int \frac{1}{z^5}\,dz$

Exercise 14.5.7 Compute the following antiderivatives. If you solve the integral by a substitution, $u(t) =$, then identify in writing $u(t)$ and $u'(t)$.

a. $\int \sin^4(t)\,\cos t\,dt$ *b.* $\int t\,(1+t^2)^3\,dt$ *c.* $\int \frac{1}{(z+1)^3}\,dz$

d. $\int \frac{1}{4t+1}\,dt$ *e.* $\int t\,(1+t^4)^3\,dt$ *f.* $\int \frac{1}{3z+1}\,dz$

g. $\int \frac{\sin t}{\cos t}\,dt$ *h.* $\int (1+x)^3\,dx$ *i.* $\int \frac{z}{z^2+1}\,dz$

j. $\int (1+t^2)^3\,t^{-1}\,dt$ *k.* $\int e^{2+z}\,dz$ *l.* $\int_0^\pi \sin(\pi + x)\,dx$

m. $\int \sin(4t)\,dt$ *n.* $\int (\ln x)\,\frac{1}{x}\,dx$ *o.* $\int e^{-z^2}\,z\,dz$

Exercise 14.5.8

a. (a) Compute $y'(x)$ for $y = x\ln x - x$

(b) Compute $\int_1^e \ln x\,dx$

b. Note: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Compute $\int_0^\pi \sin^2\theta\,d\theta$

c. (a) Show that $\frac{1}{t} + \frac{1}{1-t} = \frac{1}{t(1-t)}$

(b) Compute $\int \frac{1}{t(1-t)}\,dt$

Exercise 14.5.9 Solve using integration by parts, $\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx$ (or $\int u\,dv = uv - \int v\,du$).

a. $\int xe^x\,dx$ b. $\int x\ln x\,dx$ c. $\int x\sin x\,dx$ d. $\int x^2\,e^x\,dx$

e. $\int xe^{2x}\,dx$ f. $\int \ln x\cdot 1\,dx$ g. $\int x\cos x\,dx$ h. $\int x^3\,e^{x^2}\,dx$

Exercise 14.5.10 a. Use integration by parts on $\int e^x\sin x\,dx$, with $u(x) = e^x$ and $v'(x) = \sin x$, to show that

$$\int e^x\sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx.$$

Use a second step integration by parts on $\int e^x\cos x\,dx$ to show that

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx$$

Combine the previous two equations to show that

$$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

b. Do two steps of integration by parts on

$$\int e^x\,\cos x\,dx \quad \text{and show that} \quad \int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + C$$

c. Do two steps of integration by parts on

$$\int (\sin x)\,e^{-x}\,dx \quad \text{and show that} \quad \int_0^\pi e^{-x}\sin x\,dx = \frac{1}{2}(e^{-\pi} + 1)$$

d. **Clever!** Note that $\int e^{\sqrt{x}}\,dx = \int 2\sqrt{x}\,e^{\sqrt{x}}\,\frac{1}{2\sqrt{x}}\,dx$.

$$\text{Let} \quad u = 2\sqrt{x} \quad \text{and} \quad v' = e^{\sqrt{x}}\,\frac{1}{2\sqrt{x}} \quad \text{and show that}$$

$$\int e^{\sqrt{x}}\,dx = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

Chapter 15

Applications of the Fundamental Theorem of Calculus and Multiple Integrals.

Where are we going?

Several of the traditional applications of the Fundamental Theorem of Calculus and a definition of integrals of functions of two variables are included in this chapter.

The Fundamental Theorem of Calculus II, symbolically written

$$\int_a^b F'(t)\,dt = F(b) - F(a) \;=\; [\,F(t)\,]_a^b = \;F(t)\,|_a^b\,, \tag{15.1}$$

replaces tedious computations of the limit of sums of rectangular areas with an often easier problem of finding an antiderivative. The theorem is widely applied in physics and chemistry and engineering. It is useful in biology, but less widely so because the integrands, $f(t)$, often are not expressed as elementary functions. Indeed, the integrands in biology may be only partly specified as tables of data, in which case the options are to (1) compute a rectangular or trapezoidal sum or (2) approximate the data with an elementary function (a polynomial, for example) and compute the integral of the elementary function using Equation 15.1.

15.1 Volume.

Example 15.1.1 *How to find the volume of a potato.*

A potato is pictured in Figure 15.1A with marks along an axis at 2 cm intervals. A cross section of the potato at position 10 cm showing an area of approximately 17 cm^2 is pictured in Figure 15.1B. The volume of the slice between stations 8 cm and 10 cm is approximately 2 cm $\times$ 17 cm^2 = 34 cm^3. There are 9 slices. The volume of the potato is approximately

$$\sum_{k=1}^{9} (\text{ Area of slice } k \text{ in cm}^2 \text{ }) \times 2\,\text{cm}.$$

A B

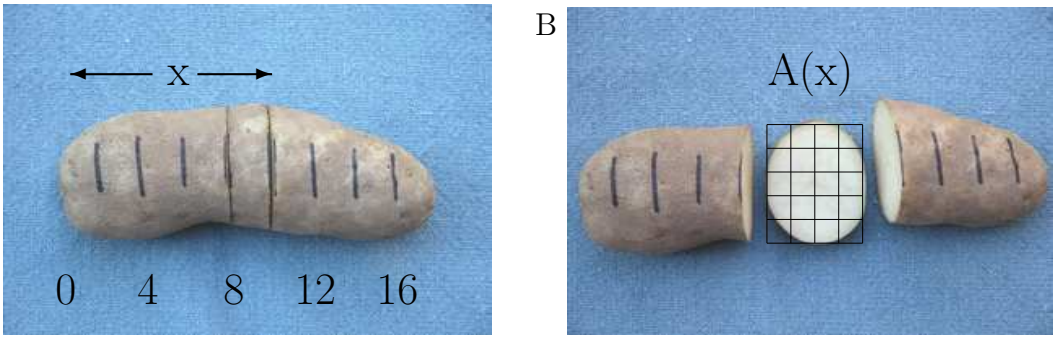


Figure 15.1: A. A potato with 2 cm marks along an axis. B. A slice of the potato at position $x = 10$ that has area approximately 17 cm^2.

The approximation works remarkably well. Several students have computed approximate volumes and subsequently tested the volumes with liquid replacement in a beaker and found close agreement.

The sum is also an approximation to the integral

$$\int_0^{17} A(x)\,dx$$

which may be considered to be the actual volume. ■

The principle applies to any three dimensional region, R. Assume there is an axis L and for each station x between stations a and b along the axis, the area $A(x)$ of the cross section of R perpendicular to L at x is known. Then

$$\text{Volume of } R \;=\int_a^b A(x)\,dx \tag{15.2}$$

Using this procedure, volumes of a large number of regions can be computed.

Example 15.1.2 The volume of a sphere. Consider a sphere with radius R and center at the origin of three dimensional space. We will compute the volume of the hemisphere between $x = 0$ and $x = R$. See Figure 15.2A. Suppose x is between 0 and R. The cross section of the region at x is a circle of radius r and area πr^2. Therefore

$$r^2 + x^2 = R^2, \qquad r^2 = R^2 - x^2 \qquad \text{and} \qquad A \;=\; A(x) \;=\; \pi(R^2 - x^2).$$

Consequently the volume of the hemisphere is

$$V = \int_0^R \pi(R^2 - x^2)\,dx$$

The integral is the integral of a polynomial, and

$$\int \pi(R^2 - x^2)\,dx = \int (\pi R^2 - \pi x^2)\,dx = \pi R^2\,x - \pi\,\frac{x^3}{3} + C.$$

By the Fundamental Theorem of Calculus II,

$$V = \int_0^R \pi(R^2 - x^2)\,dx = \left[\pi R^2\,x - \pi\,\frac{x^3}{3}\right]_0^R = \left(\pi R^2 \times R - \pi\frac{R^3}{3}\right) - \left(\pi R^2 \times 0 - \pi\frac{0^3}{3}\right) = \frac{2}{3}\pi R^3.$$

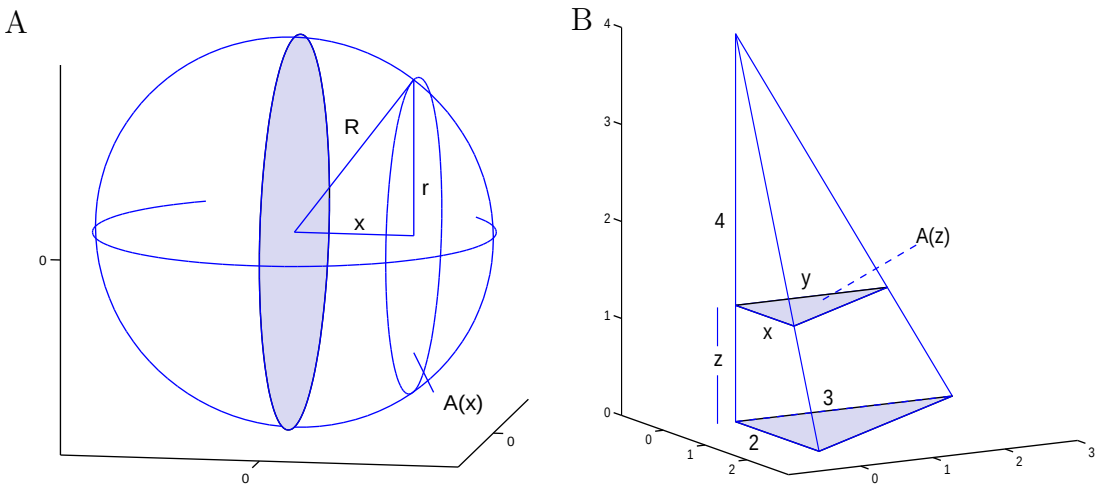


Figure 15.2: A. A sphere of radius R. The cross section a distance x from the center of the sphere is a circle of radius $r = \sqrt{R^2 - x^2}$. B. A triangular solid; the edges of lengths 2, 3, and 4 are mutually perpendicular.

The volume of the sphere is twice the volume of the hemisphere.

The volume of a sphere of radius R is $\quad \dfrac{4}{3}\pi R^3$

This formula was known to Archimedes and perhaps to mathematicians who preceded him, and is easily computed using Equation 15.2. ■

Example 15.1.3 Compute the volume of the triangular solid with mutually perpendicular edges of length 2, 3, and 4, illustrated in Figure 15.2B.

At height z, the cross section is a right triangle with sides x and y and the area $A(z) = \frac{1}{2}\, x \times y$. Now

$$\frac{x}{2} = \frac{4-z}{4} \qquad \text{so} \qquad x = \frac{2}{4}(4-z). \qquad \text{Similarly,} \qquad y = \frac{3}{4}(4-z).$$

Therefore

$$A(z) = \frac{1}{2} \times \frac{2}{4}(4-z) \times \frac{3}{4}(4-z) = \frac{3}{16}(4-z)^2,$$

and

$$V = \int_0^4 A(z)\, dz = \int_0^4 \frac{3}{16}(4-z)^2\, dz \quad = \quad \frac{3}{16}\int_0^4 (4-z)^2\, dz.$$

Check that $\left[-\dfrac{(4-z)^3}{3}\right]' = (4-z)^2$. Then

$$\frac{3}{16}\int_0^4 (4-z)^2\, dz = \frac{3}{16}\left[-\frac{(4-z)^3}{3}\right]_0^4 = \frac{3}{16}(-0 - (-\frac{(4)^3}{3})) = 4. \qquad ■$$

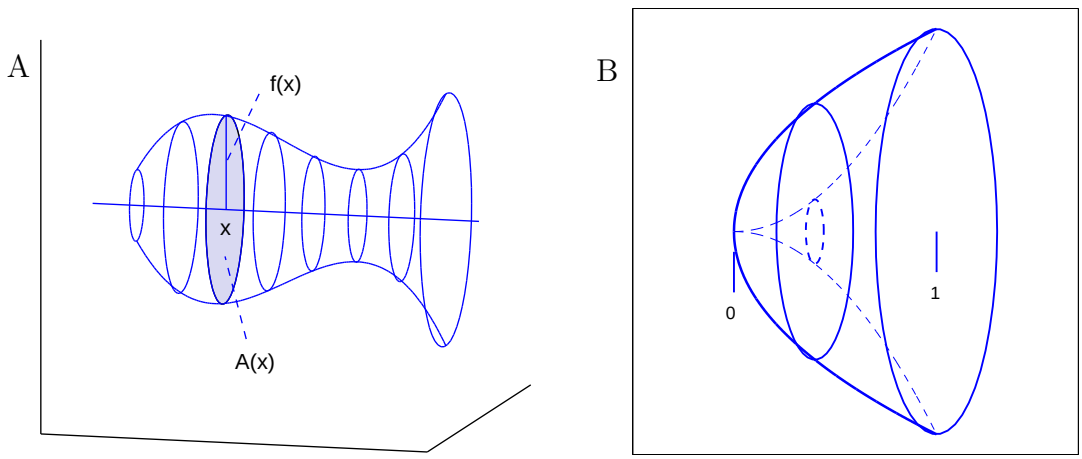


Figure 15.3: A. A solid of revolution. A cross section at station x is a circle with radius $f(x)$ and has area $A(x) = \pi\,(\,f(x)\,)^2$. B. Solid generated by rotating the region between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.

Example 15.1.4 *Volume of a solid of revolution.* Equation 15.2 is particularly simple for solids of revolution. A solid of revolution for non-negative function f is shown in Figure 15.3A. The cross section at station x is a circle of radius $f(x)$ and has area $A(x) = \pi\,(\,f(x)\,)^2$. From Equation 15.2 the volume is

$$\text{Volume of a solid of revolution} \quad = \int_a^b A(x)\,dx = \int_a^b \pi\,(\,f(x)\,)^2\,dx \tag{15.3}$$

Problem. Find the volume of the solid S generated by rotating the region R between $y = x^2$ and $y = \sqrt{x}$ about the x-axis. See Figure 15.3B.
Solution. There are two problems here. First we compute the volume of the solid, S_1, generated by rotating the region below $y = \sqrt{x}$, $0 \le x \le 1$, about the x-axis. Then we subtract the volume of the solid, S_2, generated by rotating $y = x^2$, $0 \le x \le 1$ about the x-axis.

$$\begin{array}{llcccccc}
\text{Volume of} & S_1 & \int_0^1 \pi\,(\sqrt{x})^2\,dx & = & \pi\int_0^1 x\,dx & = & \pi\left[\dfrac{x^2}{2}\right]_0^1 & = \dfrac{\pi}{2} \\[2ex]
\text{Volume of} & S_2 & \int_0^1 \pi\,(x^2)^2\,dx & = & \pi\int_0^1 x^4\,dx & = & \pi\left[\dfrac{x^5}{5}\right]_0^1 & = \dfrac{\pi}{5} \\[2ex]
\text{Volume of} & S & \dfrac{\pi}{2} - \dfrac{\pi}{5} & = & \dfrac{3\pi}{10} & & &
\end{array}$$

Alternatively, we can compute the area of the 'washer' at x which is

$$A(x) = \pi\left(\sqrt{x}\right)^2 - \pi\left(x^2\right)^2 \quad = \quad \pi\left(x - x^4\right).$$

Then

$$\text{Volume of} \quad S \quad = \quad \int_0^1 \pi\left(x - x^4\right)^2 dx \quad = \quad \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \quad = \quad \pi\left(\frac{1}{2} - \frac{1}{5}\right) \quad = \quad \frac{3\pi}{10} \quad \blacksquare$$

Exercises for Section 15.1, Volume

Exercise 15.1.1 Data for all of the slices of the potato shown in Figure 15.1 are:

Slice Position (cm)	2	4	6	8	10	12	14	16
Slice Area (cm^2)	11	15	14	16	17	15	13	10

Approximate the volume of the potato.

Exercise 15.1.2 The Visible Human Project at the National Institutes of Health has provided numerous images of cross sections of the human body. They are being integrated into the medical education and research community through a program based at the University of Michigan. Shown in Exercise Figure 15.1.2 are eight cross sections of the right side of the female brain. Your job is to estimate the volume of the brain. Assume that the sections are at 1 cm separation, that the first section only shows brain membrane, and that the scale of the cross sections is 1:4. Include only the brain and not the membrane which is apparent as white tissue. We found it useful to make a 5mm grid on clear plastic (the cover of a CD box), each square of which would be equivalent to 4 cm^2. The average human brain volume is 1450 cm^3.

Figure for Exercise 15.1.2 Cross sections of the right side of the human skull. Initial figures were downloaded from http://vhp.med.umich.edu/browsers/female.html of the University of Michigan Medical School.

Exercise 15.1.3 a. Write an integral that is the volume of the body with base the region of the x,y-plane bounded by

$$y_1 = 0.25\sqrt{x}\sqrt[4]{2-x} \quad y_2 = -0.25\sqrt{x}\sqrt[4]{2-x} \qquad 0 \le x \le 2$$

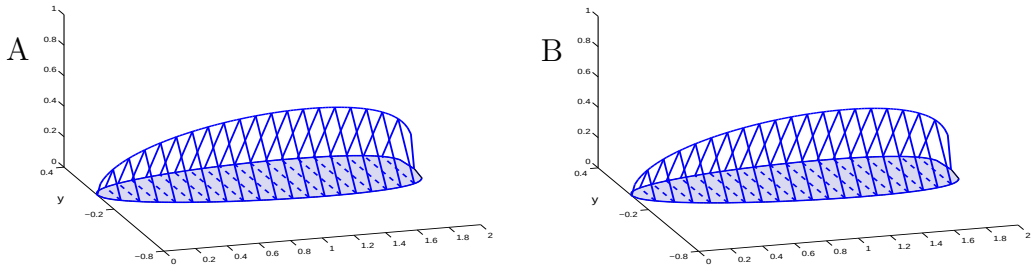
and with each cross section perpendicular to the x-axis at x being a square with lower edge having endpoints $[x, y_2(x), 0]$ and $[x, y_1(x), 0]$ (see Exercise Figure 15.1.5A). (The value of the integral is $4\sqrt{2}/15$).

b. Write an integral that is the volume of the body with base the region of the x,y-plane bounded by

$$y_1 = 0.25\sqrt{x}\sqrt[4]{2-x} \quad y_2 = -0.25\sqrt{x}\sqrt[4]{2-x} \qquad 0 \le x \le 2,$$

and with each cross section perpendicular to the x-axis at x being an equilateral triangle with lower edge having endpoints $[x, y_2(x), 0]$ and $[x, y_1(x), 0]$ (see Exercise Figure 15.1.5B). (The value of the integral is $\sqrt{6}/15$).

Figure for Exercise 15.1.3 Graphs for Exercise 15.1.3 The dashed lines are the lower edges of the squares in A and triangles in B.



Exercise 15.1.4 An artist friend observes that the bodies in Figure 15.1.3 are boring and adds some pizzaz by curving the central axis with a function, $w(x)$. Thus the base is bounded by

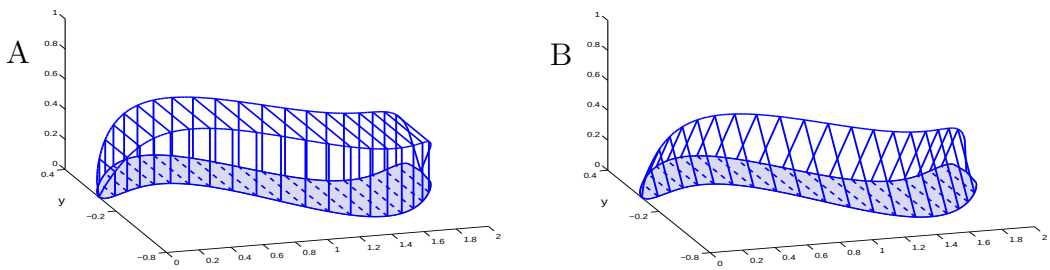
$$y_1 = w(x) + 0.25\sqrt{x}\sqrt[4]{2-x} \quad y_2 = w(x) - 0.25\sqrt{x}\sqrt[4]{2-x} \qquad 0 \le x \le 2.$$

The modified bodies are shown in Figure 15.1.4. The squares and triangles are translations of the squares and triangles in Figure 15.1.3.

a. Write an integral that is the volume of the body in Figure 15.1.4A.

b. Write an integral that is the volume of the body in Figure 15.1.4B.

Figure for Exercise 15.1.4 Graphs for Exercise 15.1.4



Exercise 15.1.5 The base of the bodies in Figure 15.1.5 are bounded by

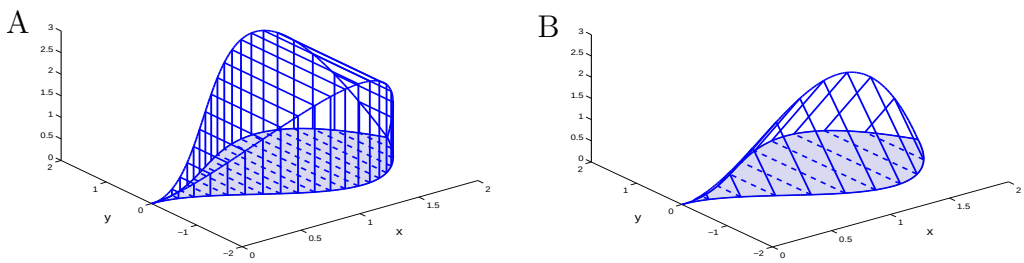
$$y1 = x^2\,(2-x) \qquad y2 = -x^2\,(2-x) \qquad 0 \le x \le 2.$$

Squares and equilateral triangles perpendicular to the axes are drawn with their lower edges spanning the base.

a. Write an integral that is the volume of the body in Figure 15.1.5A.

b. Write an integral that is the volume of the body in Figure 15.1.5B.

Figure for Exercise 15.1.5 Graphs for Exercise 15.1.5

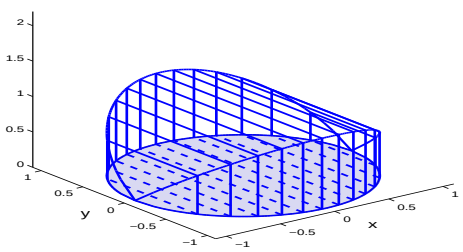


Exercise 15.1.6 The base of the body in Figure 15.1.5 is bounded by

$$y1 = \sqrt{1-x^2} \qquad y2 = -\sqrt{1-x^2} \qquad -1 \le x \le 1.$$

Rectangles perpendicular to the axes are drawn with their lower edges spanning the base. The height of the rectangle at x is $\sqrt{1-x^2}$. Write and evaluate an integral that is the volume of the body in Figure 15.1.6.

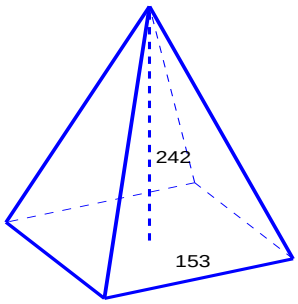
Figure for Exercise 15.1.6 Graph for Exercise 15.1.5



Exercise 15.1.7 Find the volume of the right circular cone with base radius R and height H.

Exercise 15.1.8 The pyramid of Cheops, the largest of the Egyptian pyramids, is 241 meters tall with a square base of side 153 meters. What is its volume.

Figure for Exercise 15.1.8 Schematic diagram of the Pyramid of Cheops, Exercise 15.1.8



Exercise 15.1.9 Write as the difference of two integrals the volume of the solid obtained by rotating the region inside the circle $x^2 + (y-b)^2 = a^2$ $(0 < a < b)$ about the x-axis.

Exercise 15.1.10 Atmospheric density at altitude h meters is approximately $1.225e^{-0.000101h}$ kg/m^3 for $0 \le a \le 5000$ meters. Compute the mass of air in a vertical one-square meter column between 0 and 5000 meters.

15.2 Change the variable of integration.

If $\quad x = g(z), \quad$ define $\quad dx = g'(z)\,dz \quad$ and write

$$\int f(x)\,dx = \int f(g(z))\,g'(z)\,dz \tag{15.4}$$

If $a = g(c)$ and $b = g(d)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(z))\,g'(z)\,dz \tag{15.5}$$

Example 15.2.1 For the integration problem,

$$\int \sqrt{1+\sqrt{x}}\,dx \qquad \text{substitute} \qquad x = g(z) = z^2, \qquad \text{and} \qquad dx = g'(z)\,dz = 2z\,dz.$$

Then compute

$$\begin{aligned}\int \sqrt{1+\sqrt{z^2}}\,2z\,dz &= \int \sqrt{1+z}\,2z\,dz \\ &= \int \sqrt{1+z}\,(2z+2-2)\,dz\end{aligned}$$

$$= 2\int (1+z)^{3/2} - (1+z)^{1/2}\,dz$$

$$= 2\left(\frac{(1+z)^{5/2}}{5/2} - \frac{(1+z)^{3/2}}{3/2}\right) + C.$$

If necessary, one can remember that $x = z^2$, and assume $z = \sqrt{x}$ and write

$$\int \sqrt{1+\sqrt{x}}\,dx = 2\left(\frac{(1+\sqrt{x})^{5/2}}{5/2} - \frac{(1+\sqrt{x})^{3/2}}{3/2}\right) + C. \tag{15.6}$$

Explore 15.2.1 Check the validity of Equation 15.6. ■

For $\displaystyle\int_0^4 \sqrt{1+\sqrt{x}}\,dx$, and $g(z) = z^2$, $g(0) = 0$, and $g(2) = 4$.

$$\int_0^4 \sqrt{1+\sqrt{x}}\,dx = \int_0^2 \sqrt{1+\sqrt{z^2}}\,2z\,dz$$

$$= 2\left[\frac{(1+z)^{5/2}}{5/2} - \frac{(1+z)^{3/2}}{3/2}\right]_0^2 = 6.075895918 \quad ■$$

Equation 15.5 is derived from the Chain Rule for derivatives and the Fundamental Theorem of Calculus. Assume f is continuous and g has a continuous derivative and $g(c) = a$ and $g(d) = b$. Suppose F and G satisfy

$$F'(x) = f(x) \qquad \text{and} \qquad G(z) = F(g(z)).$$

$$\text{Then} \qquad G'(z) = F'(g(z))\,g'(z) = f(g(z))\,g'(z), \qquad \text{and}$$

$$\int_c^d f(g(z))\,g'(z)\,dz = \int_c^d G'(z)\,dz = G(d) - G(c) = F(b) - F(a) = \int_a^b f(x)\,dx$$

Example 15.2.2 *Problem.* Compute $\int \sqrt{1+x^2}\,dx$.

Solution. Let $x = \tan z$. Then $dx = \sec^2 z\,dz$. Compute

$$\int \sqrt{1+\tan^2 z}\,\sec^2 z\,dz =$$

$$\int \sec^3 z\,dz = \int (\sec z)\,\left(1+\tan^2 z\right)\,dz$$

$$= \int \frac{(\sec z)\,(\sec z + \tan z)}{\sec z + \tan z} + \left(\int \tan z \times \sec z \tan z\,dz\right)$$

$$u = \tan z, \quad v' = \sec z \tan z$$

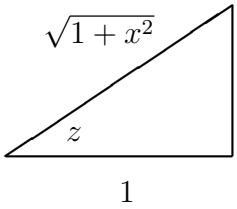
$$= \frac{\sec^2 z + \sec z \tan z}{\tan z + \sec z} + \left(\sec z \tan z - \int \sec z \times \sec^2 z\,dz\right)$$

$$= \ln(\tan z + \sec z) + \sec z \tan z - \int \sec^3 z\,dz$$

$$2\int \sec^3 z\,dz = \ln(\tan z + \sec z) + \sec z \tan z + C$$

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(\ln(x + \sqrt{1+x^2}) + x\sqrt{1+x^2}\right) + C$$

The following triangle helps translate from z back to x.



$x = \tan z \qquad \sec z = \sqrt{1+x^2}$

Example 15.2.3 *Bacteria.* A micrograph of rod-shaped bacilli is shown in Figure 15.4A. The solid, S, obtained by rotating the graph of $y = b\sqrt[4]{x}\sqrt[4]{a-x}$, shown in Figure 15.4B reasonably approximates the shape of the bacilli. We compute the volume V of S.

A

B

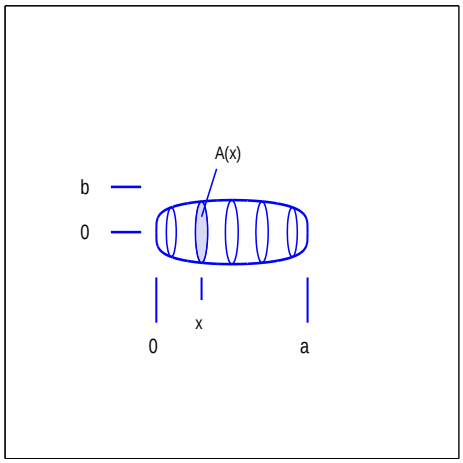


Figure 15.4: Rod-shaped bacilli; it Pseudomonas aeruginosa provided to Wikipedia Commons by Janice Haney Carr of the Centers for Disease Control and Public Health's Public Image Library. B. The graph of $y = b\sqrt[4]{x}\sqrt[4]{a-x}$. The solid obtained by rotating the graph about the x-axis approximates the body of the bacilli.

$$\begin{aligned} V &= \int_0^a \pi \left(b\sqrt[4]{x}\sqrt[4]{a-x}\right)^2 dx \\ &= \pi b^2 \int_0^a \sqrt{x}\sqrt{a-x}\, dx \end{aligned}$$

Let $x = az$; then $dx = adz$ and $x = 0$ and $x = a$ correspond to $z = 0$ and $z = 1$. Then

$$V = \pi b^2 \int_0^1 \sqrt{az}\sqrt{a-az}\, adz = \pi b^2 a^2 \int_0^1 \sqrt{z}\sqrt{1-z}\, dz$$

The skies darken and lightning abounds.

$$\int \sqrt{z}\sqrt{1-z}\, dz = \frac{1}{4}\arcsin\sqrt{z} - \frac{1}{4}\sqrt{z}\sqrt{1-z}(1-2z) + C \tag{15.7}$$

Then

$$\begin{aligned} V &= \pi b^2 \frac{a^2}{4} \left[\arcsin \sqrt{z} - \sqrt{z}\sqrt{1-z}(1-2z) \right]_0^1 \\ &= a^2 b^2 \frac{\pi^2}{8} \end{aligned}$$

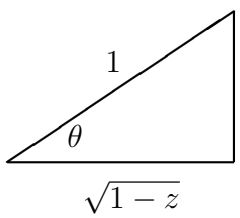
Equation 15.7 may be found in a table of integrals and is derived using substitution and Equation15.4.

Problem. Find $\int \sqrt{z}\sqrt{1-z}\, dz$.

Solution. Let $z = \sin^2\theta$. Then $dz = \left[\sin^2\theta \right]' d\theta = 2\sin\theta\cos\theta\, d\theta$.

$$\begin{aligned} \int \sqrt{z}\sqrt{1-z}\, dz &= \int \sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta} \times 2\sin\theta\cos\theta\, d\theta \\ &= 2\int \sin^2\theta\cos^2\theta\, d\theta \\ &= \frac{1}{4}\int 1 - \cos 4\theta\, d\theta \\ &= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C \\ &= \frac{1}{4}\left(\theta - \sin\theta\cos\theta(1-2\sin^2\theta)\right) + C \\ &= \frac{1}{4}\arcsin\sqrt{z} - \frac{1}{4}\sqrt{z}\sqrt{1-z}(1-2z) + C \end{aligned}$$

The following triangle helps translate from θ to z.



$\sqrt{z} = \sin\theta \quad \cos\theta = \sqrt{1-z}$ ■

Exercises for Section 15.2, Change the variable of integration.

Exercise 15.2.1 Use the suggested substitutions to compute the antiderivatives. Also, use technology or look up the integrals in a table of integrals.

a.	$\int \frac{1}{\sqrt{1-x^2}}\,dx$	x	$=$	$\sin z$	b.	$\int \frac{1}{1+x^2}\,dx$	x	$=$	$\tan z$
c.	$\int \frac{1}{\sqrt{1+x^2}}\,dx$	x	$=$	$\tan z$	d.	$\int \frac{x}{\sqrt{1+x}}\,dx$	x	$=$	$z-1$
e.	$\int \sqrt{1-x^2}\,dx$	x	$=$	$\sin z$	f.	$\int \frac{1}{\sqrt{x^2-1}}\,dx$	x	$=$	$\sec z$
g.	$\int \frac{1}{1+\sqrt{x}}\,dx$	x	$=$	z^2	h.	$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$	x	$=$	z^2

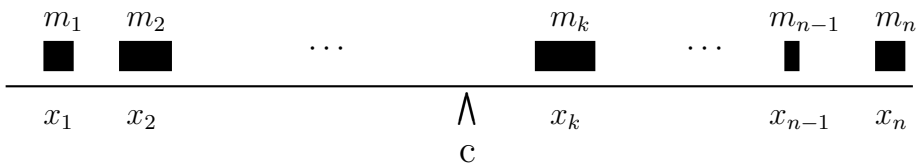
15.3 Center of mass.

Center of mass is an important concept to students of biomechanics, the study of the movement of animals including walking, running, skating, diving. For an ice skater who performs a jump, the path of her center of mass will be a parabola. The center of mass of a walker moves up and down with each stride, thus expending energy; the minimum energy per distance traveled is expended when walking approximately 3 mph[1]. In walking the center of mass reaches its highest point at the center of the stance (when the two feet are together) and in running the center of mass reaches its lowest point at the center of the stance[2].

Through what point of a potato might you push a slender rod so that the potato would be balanced on that rod?

If a collection of masses m_1, m_2, $\cdots$, m_n is distributed at positions x_1, x_2, $\cdots$, x_n along a rod (of negligible mass) and c is a point of the rod, then the *moment* of the masses about c is

$$\text{Moment} \quad = \sum_{k=1}^{n} m_k \times (x_k - c)\,. \tag{15.8}$$



The value of c for which the moment is zero is the center of mass of the system. Equation 15.8 can be solved for that value of c.

$$\sum_{k=1}^{n} m_k \times (x_k - c) \quad = \quad 0$$

[1]T. F. Novocheck, Gait and Posture, 7 (1998), 77-95

[2]C. R. Lee and C. T. Farley, J. of Experimental. Biol., 201 (1998), 2935-2944

$$\begin{aligned}\sum_{k=1}^{n} m_k \times x_k - \sum_{k=1}^{n} m_k c = \sum_{k=1}^{n} m_k \times x_k - c \sum_{k=1}^{n} m_k &= 0 \\ c &= \frac{\sum_{k=1}^{n} m_k \times x_k}{\sum_{k=1}^{n} m_k}\end{aligned}$$

Assume a mass is distributed in a region R and that the density of the mass varies along an axis L as $\delta(x)$ for $a \le x \le b$. Assume also that the area of the cross section of R perpendicular to L at x is $A(x)$ and c is a station on L. Then the moment of the mass about c is

$$\text{Moment of mass } = \int_a^b (x-c)\delta(x)A(x)\,dx \tag{15.9}$$

The value of c for which the moment is zero is

$$c = \frac{\int_a^b x\delta(x)A(x)\,dx}{\int_a^b \delta(x)A(x)\,dx} \tag{15.10}$$

The number c in Equation 15.10 is called the center of mass of the system.

If $\delta(x) = \delta$, a constant then

$$c = \frac{\int_a^b x\delta A(x)\,dx}{\int_a^b \delta A(x)\,dx} = \frac{\delta\int_a^b xA(x)\,dx}{\delta\int_a^b A(x)\,dx} = \frac{\int_a^b xA(x)\,dx}{\int_a^b A(x)\,dx} \tag{15.11}$$

and c is called the centroid of the region R.

Example 15.3.1 *Problem.* Compute the centroid of the right circular cone of height H and base radius R. See Figure 15.5

Solution. We picture the cone as the solid of revolution obtained by rotating the graph of $y = (R/H) \times x$, $0 \le x \le H$ about the X-axis. Then $A(x) = \pi \times (\,(R/H)\,x\,)^2$ and the centroid c is computed from Equation 15.11 as

$$c = \frac{\int_0^H x\pi \times (\,(R/H)\,x\,)^2\,dx}{\int_0^H \pi \times (\,(R/H)\,x\,)^2\,dx} = \frac{\int_0^H x^3\,dx}{\int_0^H x^2\,dx} = \frac{H^4/4}{H^3/3} = \frac{3}{4}H$$

This distance c is 3/4 the distance from the vertex of the cone and 1/4 the distance from the base of the cone. ■

Example 15.3.2 *Problem.* Suppose a flat plate of thickness 1 and uniform density has a horizontal outline bounded by the graph of $y = x^2$, $y = 1$, and $x = 4$. Where is the centroid of the plate. See Figure 15.6.

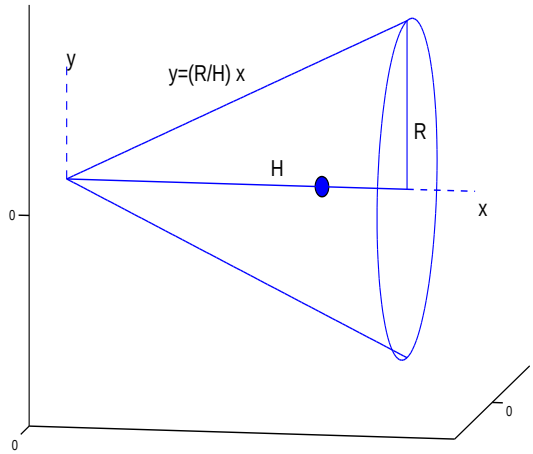


Figure 15.5: Centroid of a right circular cone of height H and base radius R. The conical axis lies on the X-axis.

Solution. There are two problems here. The first is, 'What is the x-coordinate, $\overline{x}$, of the centroid'; the second is, 'What is the y-coordinate, $\overline{y}$, of the centroid?' The z-coordinate of the centroid is one-half the thickness, $\overline{z} = 1/2$.

For the x-coordinate of the centroid, the cross sectional area of the region in a plane perpendicular to the X-axis at a position x is $x^2 \times 1 = x^2$ for $0 \le x \le 4$. We write

$$\overline{x} = \frac{\displaystyle\int_0^4 x \times x^2\,dx}{\displaystyle\int_0^4 x^2\,dx} = \frac{\displaystyle\int_0^4 x^3\,dx}{\displaystyle\int_0^4 x^2\,dx} = \frac{4^4/4}{4^3/3} = 3$$

For the y-coordinate of the centroid, the cross sectional area of the region in a plane perpendicular to the Y-axis at a position y is $(1-\sqrt{y}) \times 1 = 1 - \sqrt{y}$ for $0 \le y \le 16$. We write

$$\overline{y} = \frac{\displaystyle\int_0^{16} y \times (1-\sqrt{y})\,dy}{\displaystyle\int_0^4 1-\sqrt{y}\,dy} = \frac{\displaystyle\int_0^{16} y - y^{3/2}\,dx}{\displaystyle\int_0^{16} 1 - y^{1/2}\,dx} = \frac{\left[y^2/2 - y^{5/2}/(5/2)\right]_0^{16}}{\left[y - y^{3/2}/(3/2)\right]_0^{16}} = \frac{24}{5} \quad \blacksquare$$

Example 15.3.3 *Problem.* Consider a hemispherical region generated by rotating the graph of $y = \sqrt{1-x^2}$, $0 \le x \le 1$ about the x axis that is filled with a substance that has density, $\delta(x)$, equal to x. Find the center of mass of the substance.

Solution. At position x, the radius of the cross section perpendicular to the X-axis is $\sqrt{1-x^2}$, the area of the cross section is $\pi(1-x^2)$. From Equation 15.10, the

$$\text{Center of mass} = \frac{\displaystyle\int_a^b x\delta(x)A(x)\,dx}{\displaystyle\int_a^b \delta(x)A(x)\,dx}$$

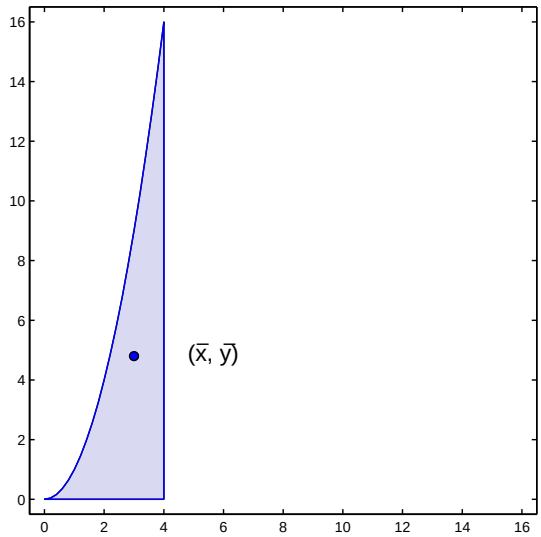


Figure 15.6: Centroid of a flat plate bounded by the graphs of $y = x^2$, $y = 0$, and $x = 4$.

$$
\begin{aligned}
&= \frac{\displaystyle\int_0^1 x \times x \times \pi(1 - x^2)\,dx}{\displaystyle\int_0^1 x \times \pi(1 - x^2)\,dx} \\
&= \frac{\displaystyle\int_0^1 x^2 - x^4\,dx}{\displaystyle\int_0^1 x - x^3\,dx} \\
&= \frac{\left[\,x^3/3 - x^5/5\,\right]_0^1}{\left[\,x^2/2 - x^4/4\right]_0^1} \quad = \quad \frac{8}{15} \qquad \blacksquare
\end{aligned}
$$

Exercises for Section 15.3, Center of Mass

Exercise 15.3.1 Find approximately the horizontal coordinate of the center of mass of the potato shown in Figure 15.1 using the data

Slice Position (cm)	2	4	6	8	10	12	14	16
Slice Area (cm^2)	11	15	14	16	17	15	13	10

Exercise 15.3.2 Find the x-coordinate of the centroids of the areas:

a. The triangle with vertices (0,0), (2,0) and (2,1); $0 \le x \le 2$, $0 \le y \le 0.5 \times x$.

b. The semicircle, $0 \le x \le 1$, $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$.

c. The parabolic segment, $0 \le x \le 1$, $-\sqrt{x} \le y \le \sqrt{x}$.

Exercise 15.3.3 Suppose the density at position x is $2 \times \sqrt{x}$ for plates 1 cm thick for objects with plans in the x-y plane described below. Find the x-coordinates of the centers of mass of the objects. If there is an integral for which you can not find an appropriate antiderivative, approximate the integral using 10 subintervals of the interval of integration.

a. The triangle with vertices (0,0), (2,0) and (2,1); $0 \le x \le 2$, $0 \le y \le 0.5 \times x$.

b. The semicircle, $0 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$.

c. The parabolic segment, $0 \le x \le 1$, $-\sqrt{x} \le y \le \sqrt{x}$.

15.4 Arc length and Surface Area.

Arc Length.
Consider the problem of finding the length of the graph of $y = f(x)$ shown in Figure 15.7. We partition the interval $[a, b]$ into $a = x_0 < x_1 < \cdots < x_{k-1} < x_k < \cdots x_n = b$. Then the length of f is approximately

$$\begin{aligned} \text{Length} &\doteq \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} \\ &= \sum_{k=1}^{n} \sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2} (x_k - x_{k-1}) \end{aligned}$$

By the Mean Value Theorem, for each k there is a number c_k between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k).$$

Then

$$\text{Length} \doteq \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}(x_k - x_{k-1})$$

This pattern fits Definition 13.4.4, Definition of Integral II, and

$$\sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}(x_k - x_{k-1}) \doteq \int_a^b \sqrt{1 + (\, f'(x)\,)^2} dx$$

Both approximations converge as the norm of the partition decreases to zero and we write

$$\text{Length of the graph of } f \text{ is} \qquad \int_a^b \sqrt{1 + (\, f'(x)\,)^2} dx \tag{15.12}$$

Equation 15.12 is excellent in concept but difficult in practice in that there are not many functions f for which an antiderivative $\int \sqrt{1 + (\, f'(t)\,)^2}\, dt$ can be found. One function for which the antiderivative can be found is $y = \ln \cos x$.

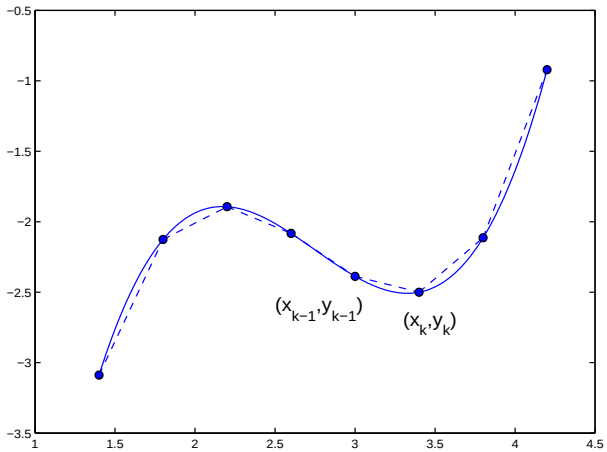


Figure 15.7: Graph of a function, f, and a polygon that approximates it.

Example 15.4.1 *Problem.* Find the length of the graph of $y = \ln\cos x$ on $0 \le x \le \pi/4$.

We write

$$f(x) = \ln\cos x \qquad f'(x) = [\,\ln\cos x\,]' \qquad \frac{1}{\cos x}[\,\cos x\,]' = \frac{1}{\cos x}(-\cos x\,) = -\tan x$$

We leave it to you to check that **Bolt out of the Blue**

$$[\,\ln(\sec x + \tan x)\,]' = \sec x$$

We can write

$$\begin{aligned}
\text{Length} &= \int_0^{\pi/4} \sqrt{1 + (f'(x))^2}\,dx \\
&= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx \\
&= \int_0^{\pi/4} \sec\,dx \\
&= [\,\ln(\sec x + \tan x)\,]_0^{\pi/4} \\
&= \ln(\sec(\pi/4) + \tan(\pi/4)) - \ln(\sec(0) + \tan(0)) \doteq 0.881 \qquad \blacksquare
\end{aligned}$$

Surface Area. The surface area to volume ratio limits the size of cells. The surface areas of the lungs and the small intestine determines, respectively, the oxygen absorbance and nutrient absorbance in humans.

We will compute surface areas only for surfaces of revolution. We use the fact that the surface area of the frustum of a cone (illustrated in Figure 15.8A) is

$$\text{Area of frustum of a cone} \quad = \pi(r_1 + r_2)\,s_1$$

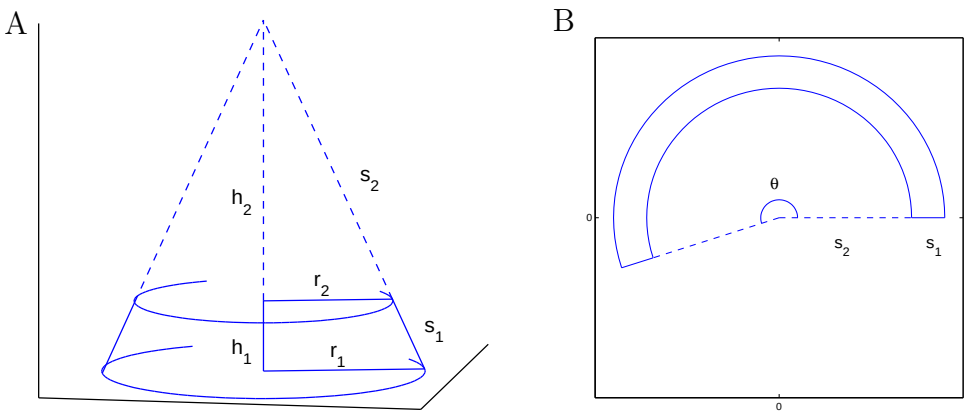


Figure 15.8: A. The frustum of a cone; r_1, r_2, and h_1 determine the cone. B. The same frustum flattened onto a plane. The angle $\theta = 2\pi r_1/(s_1 + s_2) = 2\pi r_2/s_2$.

The proof is left to you, and you may be assisted by Figure 15.8B in which the frustum has been cut along a slant side and flattened. The angle $\theta = 2\pi r_1/(s_1 + s_2) = 2\pi r_2/s_2$.

Shown in Figure 15.9 is a surface of revolution and a section between x_{k-1} and x_k. The area of that frustum is

$$\begin{aligned} A_k &= \pi(f(x_{k-1} + f(x_k))\sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} \\ &= \pi(f(x_{k-1} + f(x_k))\sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2}(x_k - x_{k-1}) \end{aligned}$$

By the Mean Value Theorem there is a number, c_k between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k).$$

The area of the surface of revolution is approximately $\sum_{k=1}^{n} A_k$ so that

$$\text{Surface Area} \quad \doteq \pi(f(x_{k-1} + f(x_k))\sqrt{1 + (f'(c_k))^2}(x_k - x_{k-1})$$

Because x_{k-1}, c_k and x_k may all three be different, we appeal to an even more general definition of the integral than Definition of the Integral II and conclude that this approximating sum converges to $\int_a^b \pi(f(x) + f(x))\sqrt{1 + (\ f'(x)\ )^2}\,dx$. as the norm of the partition goes to zero. Therefore we write

$$\text{Surface Area} \quad = \int_a^b 2\pi f(x)\sqrt{1 + (\ f'(x)\ )^2}\,dx \tag{15.13}$$

Again the collection of functions f for which an antiderivative $\int 2\pi f(x)\sqrt{1 + (\ f'(x)\ )^2}\,dx$ can be found is small. Fortunately one such function is $f(x) = \sqrt{R^2 - x^2}$ and the surface area is that of a sphere, $4\pi R^2$, which appears in Exercise 15.4.3.

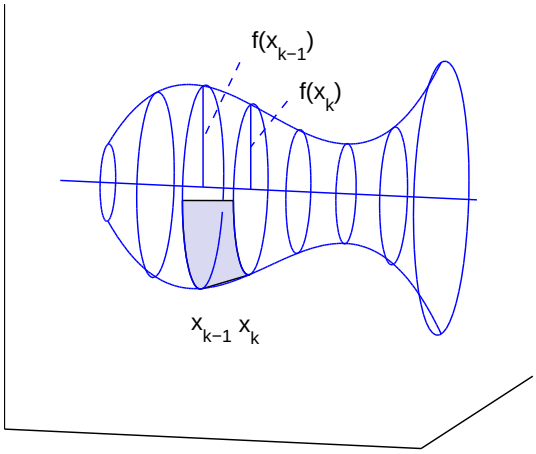


Figure 15.9: A surface of revolution with approximating conical frustum.

Exercises for Section 15.4, Arc Length and Surface Area.

Exercise 15.4.1 Find the length of the graph of $f(x) = \frac{3}{2}\,x^{2/3}$, $1 \le x \le 8$. Note: In order to evaluate the integral, use a substitution, $u(x) = x^{2/3} + 1$, $u'(x) = \frac{2}{3}x^{-1/3}$.

Exercise 15.4.2 Write an integral that is the length of the curve

$$y = \frac{e^x + e^{-x}}{2} \qquad -1 \le x \le 1.$$

Show that the integral is $\int_{-1}^{1} y(x)\,dx$ and evaluate it.
The graph of y is the shape of a cable suspended between the points $(-1, y(-1))$ and $(1, y(1))$ and is called a *catenary*.

Exercise 15.4.3 Use Equation 15.13 and $f(x) = \sqrt{R^2 - x^2}$ to find the surface area of a sphere of radius R. Archimedes knew this to be $4\pi R^2$. He first guessed the answer based on the common knowledge among basket weavers that the bowl of a hemispherical basket required twice as much material as its top, of area πr^2.

Exercise 15.4.4 Find the area of the surface generated by rotating the graph of $y = 2\sqrt{x}$, $0 \le x \le 1$ about the x-axis.

15.5 The improper integral, $\int_a^\infty f(t)\,dt$.

In this section we give examples of problems for which a reasonable answer is of the form

$$\lim_{R\to\infty} \int_a^R f(t)\,dt \qquad \text{which is denoted by} \qquad \int_a^\infty f(t)\,dt \tag{15.14}$$

This is called an 'improper' integral, but there is nothing improper about it.

Another integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx,$$

is also said to be improper. The integrand, $\frac{1}{\sqrt{x}}$ has an identity crisis at $x = 0$; the integrand is not defined a $x = 0$ and is unbounded in every interval $[0, r]$, $r > 0$. The integrand is continuous on every interval $[r, 1]$, $r > 0$, however, and

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad \text{is defined to be} \qquad \lim_{r\to 0+} \int_r^1 \frac{1}{\sqrt{x}}\,dx \quad = \quad \lim_{r\to 0+} \left[2\sqrt{x}\right]_r^1 \quad = \quad 2.$$

Example 15.5.1 *Problem.* Is there an initial speed of a satellite that is sufficient to insure that it will escape the Earth's gravity field without further propulsion?

Solution. The work done along an axis against a variable force $F(x)$ is

$$W = \int_a^b F(x)\,dx$$

The acceleration of gravity at an altitude x is given as

$$g(x) = 9.8 \times \frac{R^2}{(R+x)^2} \quad \text{meters/sec}^2$$

where $R = 6370$ meters is the radius of Earth. This means a satellite of mass m and altitude x experiences a gravitational force of

$$F(x) = m \times 9.8 \times \frac{R^2}{(R+x)^2} \quad \text{Newtons}$$

The work required to lift the satellite to an altitude A is

$$W(A) \;=\; \int_0^A m \times 9.8 \times \frac{R^2}{(R+x)^2}dx \;=\; 9.8 \times m \times R^2 \int_0^A \frac{1}{(R+x)^2}dx$$

The integral is readily evaluated.

Explore 15.5.1 Show that

$$W(A) = 9.8 \times m \times R^2 \left[\frac{1}{R} - \frac{1}{(R+A)}\right] \qquad \blacksquare$$

How much work must be done to send the satellite 'out of Earth's gravity field', so that it does not fall back to Earth? This is

$$\lim_{A\to\infty} W(A) \;=\; \lim_{A\to\infty} 9.8 \times m \times R^2 \left[\frac{1}{R} - \frac{1}{(R+A)}\right] = 9.8 \times m \times R$$

A finite amount of work is sufficient to send the satellite out of the Earth's gravity field.

Now suppose we neglect the friction of air and ask at what velocity should the satellite be launched in order to escape the Earth? We borrow from physics the formula for kinetic energy of a body of mass m moving at a velocity v.

$$\text{Kinetic energy} = \frac{1}{2}mv^2.$$

Then we equate the kinetic energy with the work required

$$\frac{1}{2}mv^2 = 9.8 \times m \times R,$$

and solve for v. The solution is

$$v \quad = \quad \sqrt{2R \times 9.8} \quad = \quad 11,174 \quad \text{meters per second}$$

A satellite in orbit travels about 7,500 m/sec. The muzzle velocity of a rifle is about 1,000 m/sec. ■

Example 15.5.2 The mean and standard deviation of an exponential life table. Consider first a finite life table:

An animal population is established as newborns; 4 tenths of the population die at age 1 (perhaps harvested), 3 tenths of the population die at age 2, 2 tenths of the population die at age 3, and 1 tenth of the population die at age 4. The data may be organized as in Figure 15.10. We define $D(x)$ to be the fraction of the original population that dies at age x and $L(x)$ to be the fraction of the original population that live until age x.

Age, x	D(x)	L(x)
0	0	1.0
1	0.4	1.0
2	0.3	0.6
3	0.2	0.3
4	0.1	0.1
5	0.0	0.0

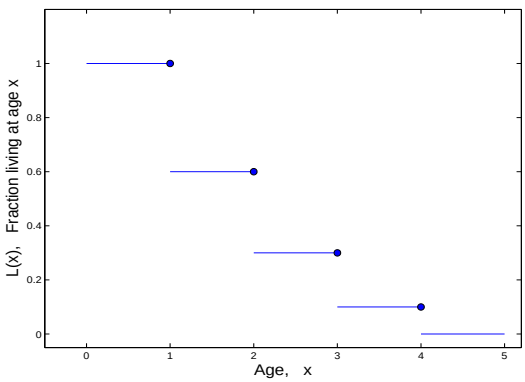


Figure 15.10: Life table and graph. $D(x)$ is the fraction of the original population that dies at age x and $L(x)$ is the fraction of the original population that live until age x.

The life expectancy, or average age at death, is

$$\begin{aligned} A &= 0.4 \times 1 + 0.3 \times 2 + 0.2 \times 3 + 0.1 \times 4 \;=\; 2 \\ &= \sum_{x=1}^{4} D(x) \times x \\ &= \sum_{x=1}^{4} (L(x) - L(x+1)) \times x \end{aligned}$$

For a general life table, L, with data points $x_0, x_1, x_2, \cdots, x_n$, (and $L(x_n) = 0$) the life expectancy is

$$A = x_1\,(L(x_0) - L(x_1) + x_2\,(L(x_1) - L(x_2) + \cdots + x_n\,(L(x_{n-1}) - L(x_n)$$

If L is a differentiable approximation to a life table defined on an interval $[0, b]$ (with $L(b) = 0$) and $\{x_k\}_{k=0}^n$ is a partition of $[0, b]$, then

$$A \doteq \sum_{k=1}^{n} (L(x_{n-1}) - L(x_n)) \cdot x_n = \sum_{k=1}^{n} x_n\,(-L'(c_n))\;(x_n - x_{n-1}) \doteq \int_0^b x(-L'(x))\,dx$$

where $L(x_{n-1}) - L(x_n) = -L'(c_n)\;(x_n - x_{n-1})$ by the mean value theorem. The average age determined by L is

$$A = -\int_0^b xL'(x)\,dx \tag{15.15}$$

It is often assumed that a life table can be approximated by a negative exponential of the form

$$L(x) = e^{-\lambda x} \qquad \text{where } b = \infty \text{ in Equation 15.15}$$

Then

$$\text{Average age at death} \quad = -\lim_{R\to\infty} \int_0^R x\,(-\lambda e^{-\lambda x})\,dx = \lambda \int_0^\infty x\,e^{-\lambda x}\,dx$$

Explore 15.5.2 Use integration by parts to show that

$$\int x \times e^{-\lambda x}\,dx = -\frac{1}{\lambda}\,xe^{-\lambda x} - \frac{1}{\lambda^2}\,e^{-\lambda x} + C \qquad \blacksquare$$

Therefore, the average age at death is

$$\begin{aligned} &= \lambda \int_0^\infty x\,e^{-\lambda x}\,dx \\ &= \lambda \lim_{R\to\infty} \left[-xe^{-\lambda x}/\lambda - e^{-\lambda x}/\lambda^2\right]_{x=0}^{x=R} \\ &= \lambda \lim_{R\to\infty} \left[-Re^{-\lambda R}/\lambda - e^{-\lambda R}/\lambda^2 + 1/\lambda^2\right] = 1/\lambda \end{aligned}$$

By similar reasoning, the standard deviation of the age at death in the population is

$$\lambda \int_0^\infty (x - 1/\lambda)^2 \times e^{-\lambda x}\,dx$$

Explore 15.5.3 Use two steps of integration by parts to show that

$$\int (x - 1/\lambda)^2 e^{-\lambda x}\,dx = -(x - 1/\lambda)^2(e^{-\lambda x}/\lambda) - 2(x - 1/\lambda)e^{-\lambda x}/\lambda^2 - 2e^{-\lambda x}/\lambda^3 \quad \blacksquare \tag{15.16}$$

In Exercise 15.5.6 you are asked to show that

$$\lambda \int_0^\infty (x - 1/\lambda)^2 \times e^{-\lambda x}\,dx = 1/\lambda^2 \qquad \blacksquare \tag{15.17}$$

Exercises for Section 15.5, Improper Integrals.

Exercise 15.5.1 Show that

$$\int_1^\infty \frac{1}{x^a}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^a}\,dx$$

is finite if $1 < a$ and is infinite if $0 < a \le 1$.

Exercise 15.5.2 Show that

$$\int_0^1 \frac{1}{x^a}\,dx = \lim_{r\to 0+} \int_r^1 \frac{1}{x^a}\,dx$$

is finite if $0 < a < 1$ and is infinite if $1 \le a$.

Exercise 15.5.3 Compare the regions whose areas are

$$\int_1^\infty \frac{1}{x^2}\,dx \qquad \text{and} \qquad \int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Exercise 15.5.4 Consider the infinite horn, H, obtained by rotating the graph of

$$y = 1/x, \qquad 1 \le x \qquad \text{about the } x\text{-axis.}$$

See Exercise Figure 15.5.4.

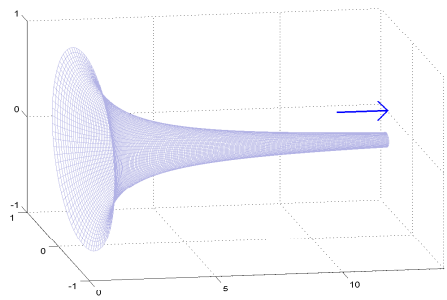
a. Show that the volume of the interior of H is π.

b. Show that the surface area of H is greater than.

$$\int_1^\infty 2\pi \frac{1}{x}\,dx = \infty$$

c. H, then, has a finite volume and can be filled with paint, but has an infinite surface area and can not be painted! I enjoyed telling a very good class about this one day. The next day Mr. Jacks, a generally casual student, told me he could paint it, and he could. His grade in the course was an A. Can you paint H?

Figure for Exercise 15.5.4 An infinite horn, H, obtained by rotating the graph of $y = 1/x$, $1 \le x$ about the x-axis



Exercise 15.5.5 The gamma function, $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$ is an important function in the study of statistics.

a. Compute $\Gamma(1)$.

b. Use one step of integration by parts to compute $\Gamma(2)$.

c. Use one step of integration by parts and the previous step to compute $\Gamma(3)$.

d. Use one step of integration by parts to show that if n is an integer, $\Gamma(n+1) = n\Gamma(n)$.

Exercise 15.5.6 Use Equation 15.16 to establish Equation 15.17

$$\lim_{R\to\infty} \frac{\lambda}{1-e^{-\lambda R}} \int_0^R (x - 1/\lambda)^2 \times e^{-\lambda x}\, dx = 1/\lambda^2 \qquad \blacksquare$$

Exercise 15.5.7 A linear life table is given by

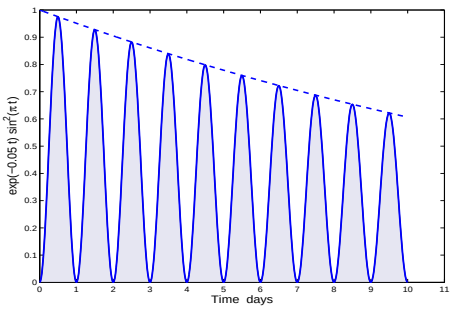
$$L(x) = 1 - x/m \qquad \text{for} \qquad 1 \le x \le m$$

Find the mean and standard deviation of the life expectancy (age at death) for this life table.

Exercise 15.5.8 Wildlife managers decide to lower the level of water in a lake of 8000 acre feet. They open the gates at the dam and release water at the rate of $\frac{1000}{(t+1)^2}$ acre-feet/day where t is measured in days. Will they empty the lake?

Exercise 15.5.9 Algae is accumulating in a lake at a rate of $e^{-0.05t}\sin^2 \pi t$. The factor $e^{-0.05t}$ reflects declining available oxygen and the factor $\sin^2 \pi t$ reflects diurnal oscillation. Is the amount of algae produced infinite? See Exercise Figure 15.5.9.

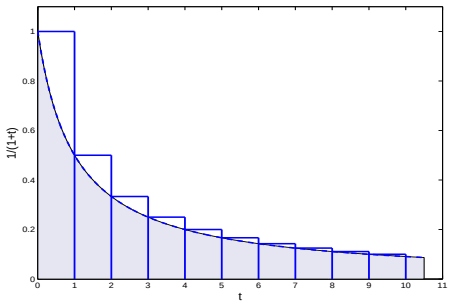
Figure for Exercise 15.5.9 The amount of algae produced is the area of the shaded region (extended to $t = \infty$). The upper curve is the graph of $y = e^{-0.05t}$.



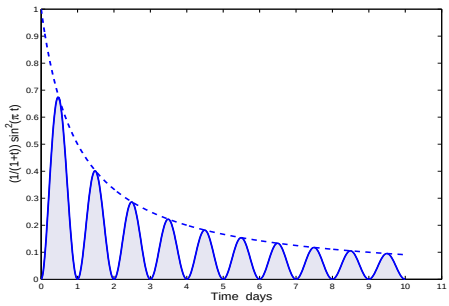
Exercise 15.5.10 Algae is accumulating in a lake at a rate of $\frac{1}{1+t}\sin^2 \pi t$. The factor $\frac{1}{1+t}$ reflects declining available oxygen and the factor $\sin^2 \pi t$ reflects diurnal oscillation. Is the amount of algae produced infinite? See Exercise Figure 15.5.10.

Figure for Exercise 15.5.10 A. The graph of $y = 1/(1+t)$. B. The amount of algae produced is the area of the shaded region in B (extended to $t = \infty$). The upper curve is the graph of $y = 1/(1+t)$.

A



B



a. Show that the area of the shaded region (extended to ∞) in Exercise Figure 15.5.10 A ($= \int_0^\infty \frac{1}{1+t}\,dt$) is infinite.

b. Show that the sum of the areas of the boxes (extended to ∞) in Exercise Figure 15.5.10 A is infinite.

c. Argue that the algae production on day 1 is larger than $\int_0^1 \frac{1}{2}\sin^2 \pi t\,dt$.

d. Argue that the algae production on day 2 is larger than $\int_1^2 \frac{1}{3}\sin^2 \pi t\,dt$.

e. Believe that for any positive integer, k,

$$\int_k^{k+1} \sin^2 \pi t\,dt = \int_k^{k+1} \frac{1-\cos 2\pi t}{2}\,dt = \left[\frac{1}{2}t - \frac{1}{4\pi}\sin 2\pi t\right]_k^{k+1} = \frac{1}{2}$$

f. Argue that

$$\int_0^\infty \frac{1}{1+t}\sin^2 \pi t\,dt > \frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{2} + \frac{1}{4}\frac{1}{2} + \frac{1}{5}\frac{1}{2} + \cdots = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right)$$

g. Think of the boxes in part b. and argue that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

Exercise 15.5.11 It is a fact that

$$[\arctan x]' = \frac{1}{1+x^2}$$

Compute

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

15.6 Integrals of functions of two variables.

Suppose $F(x, y)$ is a positive function of two variables defined on a region R of the x, y-plane. What is the volume, V, of the region above the x, y-plane and below the graph of $z = F(x, y)$? (Figure 15.11A.)

The volume can be approximated as follows. (Figure 15.11)

Step 1. Choose a positive number δ and partition R into regions R_1, R_2, $\cdots$, R_n of diameter less than δ.

A B

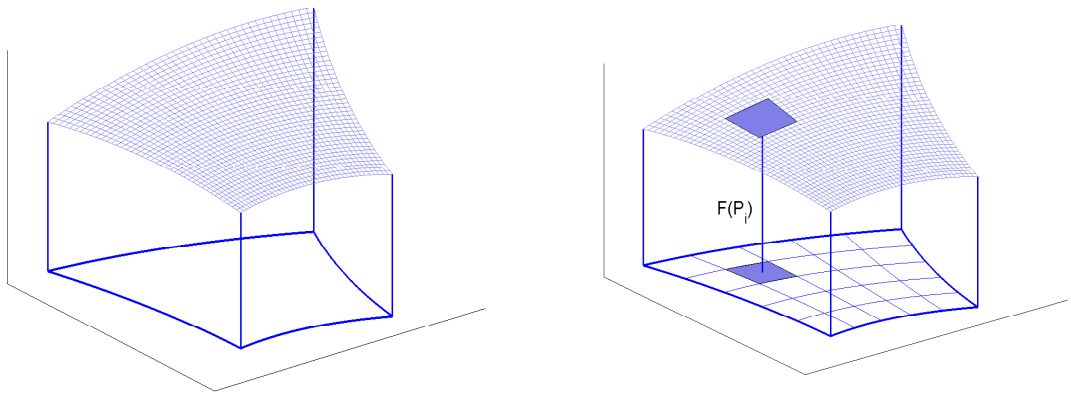


Figure 15.11: A. Graph of a function F of two variables and the region between the graph and the x, y-plane. B. The domain of F is partitioned into small regions.

Step 2. Let P_i be a point in R_i, $i = 1, \cdots, n$, and let A_i be the area of R_i.

Step 3. Then

$$V \doteq \sum_{i=1}^{n} F(P_i)\, A_i \tag{15.18}$$

The approximation 15.18 converges to V as $\delta \to 0$.

The graph of $F(x, y) = (\sin \pi x)\,(\cos \frac{\pi}{2} y)$ is shown in Figure 15.12 with the domain $0 \le x \le 1$, $0 \le y \le 1$ partitioned into 25 squares of sides 0.2 and a point marked at the center of each square. The volume of the region below the graph of F and above the x, y-plane is approximately

$$\sum_{i=0}^{4} \sum_{j=0}^{4} \sin((0.1 + i \cdot 0.2) \cdot \pi)\ \cos((0.1 + j \cdot 0.2) \cdot (\pi/2)) \times 0.2 \times 0.2 \doteq 0.414$$

Similar computations with 100 squares of sides 0.1 yields 0.407 as the approximate volume and with 400 squares of sides 0.05 yields 0.406.

The sum in Equation 15.18 has other applications than approximating volumes and the following definition is made.

Definition 15.6.1 Integral of a function of two variables. Suppose F is a continuous function defined on a region R of the x, y-plane. The integral of F on R is denoted and defined by

$$\int_R \!\!\int F(P)\, dA = \lim_{\delta \to 0} \sum_{i=1}^{n} F(P_i)\, A_i \tag{15.19}$$

where δ, n, P_i and A_i are as in Steps 1 and 2 above.

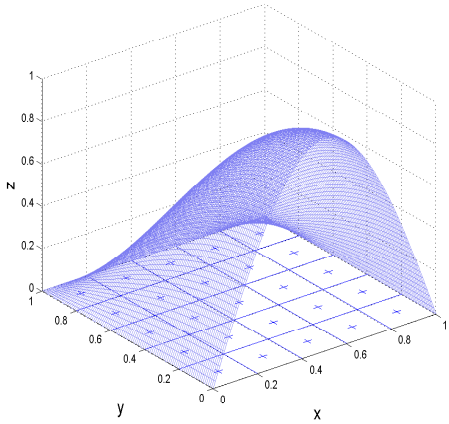


Figure 15.12: The graph of $F(x, y) = (\sin \pi x)(\cos \frac{\pi}{2} y)$ on $0 \le x \le 1$, $0 \le y \le 1$. The domain is partitioned into squares and the center point of each square is marked.

Suppose now that R is the region between the graphs of two functions f and g of a single variable defined on an interval $[a, b]$ as shown in Figure 15.13A. (R is the set to which (x, y) belongs only if $a \le x \le b$ and $f(x) \le y \le g(x)$).

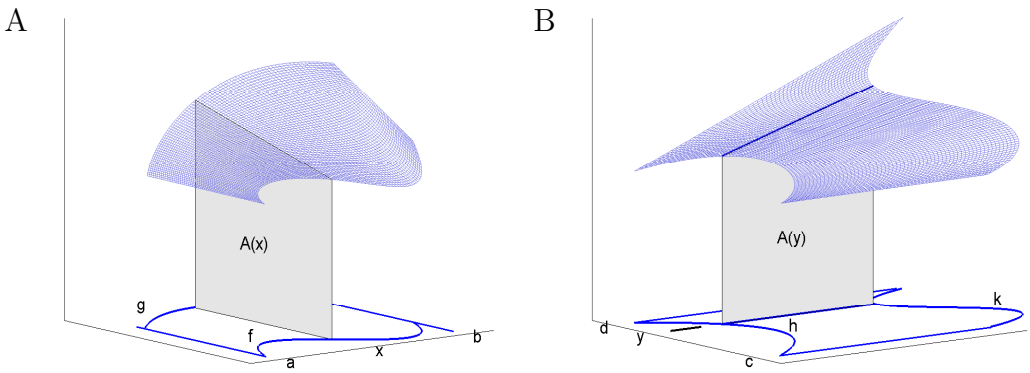


Figure 15.13: A. The domain of F is the set of (x, y) for which $a \le x \le b$ and $f(x) \le y \le g(x)$. B. The domain of F is the set of (x, y) for which $c \le y \le d$ and $h(y) \le x \le k(y)$.

Then for each number x in $[a, b]$ the area of the cross section through V at x is

$$A(x) = \int_{f(x)}^{g(x)} F(x, y)\, dy$$

Using the volume equation 15.2, the volume V is

$$V = \int_a^b A(x)\, dx = \int_a^b \left(\int_{f(x)}^{g(x)} F(x, y)\, dy \right) dx$$

Alternatively, Figure 15.13B, there may be two functions $h(y)$ and $k(y)$ defined on an interval $[c, d]$ and R is the region to which (x, y) belongs only if $c \le y \le d$ and $h(y) \le x \le k(y)$. Then

$$V = \int_c^d \left(\int_{h(y)}^{k(y)} F(x, y)\, dx \right) dy$$

More generally, two ways to compute the integral of $F(P)$ are

$$\int_R\!\!\int F(P)\, dA = \int_a^b \int_{f(x)}^{g(x)} F(x, y)\, dy\, dx \tag{15.20}$$

and

$$\int_R\!\!\int F(P)\, dA = \int_c^d \int_{h(y)}^{k(y)} F(x, y)\, dx\, dy. \tag{15.21}$$

In the 'inside' integration $\int_{f(x)}^{g(x)} F(x, y)\, dy$ of Equation 15.20, x is a constant, the integration is with respect to the variable y. Similarly, in the integration $\int_{h(y)}^{k(y)} F(x, y)\, dx$ of Equation 15.21, y is a constant, the integration is with respect to the variable x.

Example 15.6.1 The domain of the function $F(x, y) = (\sin \pi x)\,(\cos \frac{\pi}{2} y)$ shown in Figure 15.12 is $0 \le x \le 1$, $0 \le y \le 1$. Choose $f(x) = 0$ and $g(x) = 1$, for $0 \le x \le 1$. The volume of the region below the graph of F is

$$\begin{aligned}
\int_0^1 \int_0^1 (\sin \pi x)\,(\cos \frac{\pi}{2} y)\, dy\, dx &= \int_0^1 \left[(\sin \pi x)\,(\sin \frac{\pi}{2} y) \times \frac{2}{\pi} \right]_{y=0}^{y=1} dx \\
&= \frac{2}{\pi} \int_0^1 \sin \pi x\, dx \\
&= \frac{2}{\pi} \cdot \frac{2}{\pi} \doteq 0.405285 \qquad \blacksquare
\end{aligned}$$

Problem. Find the volume, V, of the three dimensional region below the graph of $z = x + y/2$ and above the triangle bounded by $x = 1$, $y = 1$ and $x + y = 4$. (Figure 15.14.)

Solution. Define $f(x) = 1$ and $g(x) = 4 - x$ for $1 \le x \le 3$.

$$\begin{aligned}
V &= \int_1^3 \int_1^{4-x} (x + y/2)\, dy\, dx \\
&= \int_1^3 \left[xy + \frac{y^2}{4} \right]_{y=1}^{y=4-x} dx \\
&= \int_1^3 \left(x(4-x) + \frac{(4-x)^2}{4} - x - \frac{1}{4} \right) dx \\
&= \int_1^3 \left(-\frac{3}{4}x^2 + x + \frac{15}{4} \right) dx \\
&= \left[-\frac{1}{4}x^3 + \frac{x^2}{2} + \frac{15}{4}x \right]_1^3 \quad = \quad 5
\end{aligned}$$

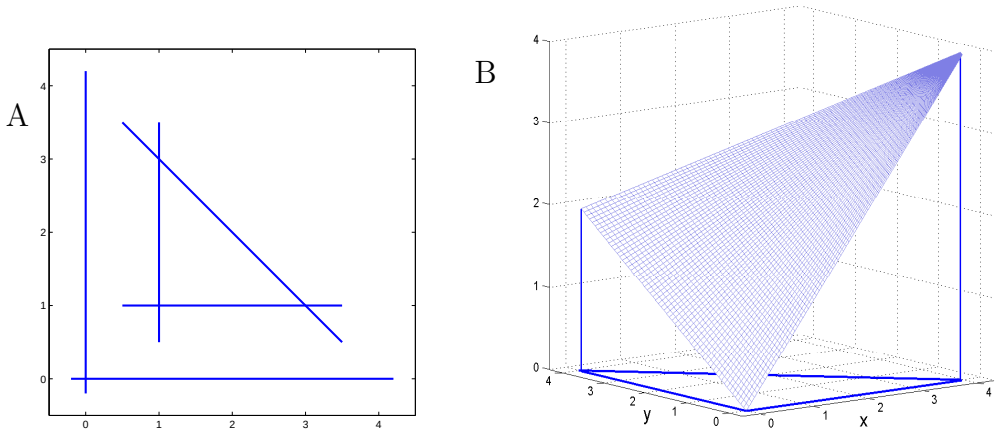


Figure 15.14: A. The region, R, bounded by the graphs of $x = 1$, $y = 1$, and $x + y = 4$. B. The three-dimensional region above R and below the graph of $z = x + y/2$.

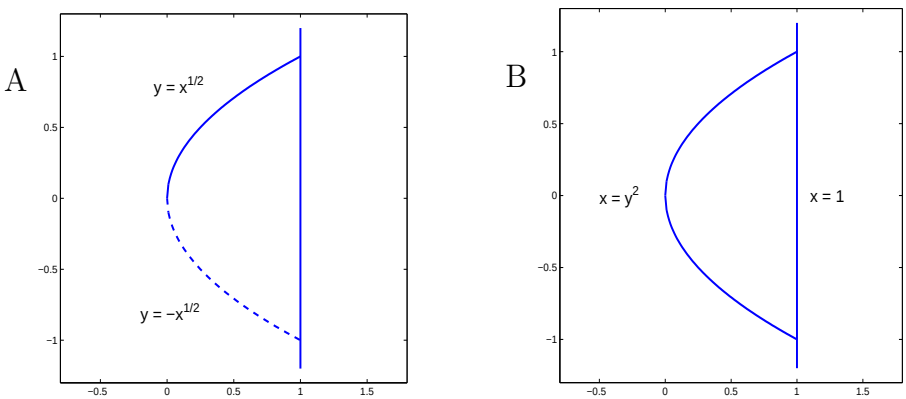


Figure 15.15: A. The region can be defined by $0 \le x \le 1$, $-\sqrt{x} \le y \le \sqrt{x}$ or B. by $-1 \le y \le 1$, $y^2 \le x \le 1$.

Problem. Find the volume, V, of the three dimensional region below the graph of $z = x^2 y$ and above two dimensional region R bounded by $y = -\sqrt{x}$, $y = \sqrt{x}$ and $x = 1$.

Solution 1. Define $f(x) = -\sqrt{x}$ and $g(x) = \sqrt{x}$ for $0 \le x \le 1$. (Figure 15.15A.)

$$\begin{aligned} V &= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} xy^2 \, dy \, dx \\ &= \int_0^1 \left[x \frac{y^3}{3} \right]_{y=-\sqrt{x}}^{y=\sqrt{x}} dx \\ &= \int_0^1 \frac{2}{3} x^{5/2} \, dx \end{aligned}$$

$$= \left[\frac{2}{3}\frac{x^{7/2}}{7/2}\right]_0^1 \quad = \quad \frac{4}{21}$$

Solution 2. The region R is also bounded by $x = y^2$ and $x = 1$, and can be defined by $y^2 \le x \le 1$ for $-1 \le y \le 1$. (Figure 15.15B.) Then

$$\begin{aligned} V &= \int_{-1}^{1}\int_{y^2}^{1} xy^2\,dx\,dy \\ &= \int_{-1}^{1}\left[\frac{x^2}{2}y^2\right]_{x=y^2}^{x=1} dy \\ &= \int_{-1}^{1}\left(\frac{y^2}{2} - \frac{y^6}{2}\right) dy \\ &= \left[\frac{y^3}{6} - \frac{y^7}{14}\right]_{-1}^{1} \quad = \quad \frac{4}{21} \end{aligned}$$

Problem. Compute $\int_R\int F(P)\,dA$ where R is the annular ring $1 \le \sqrt{x^2+y^2} \le 2$.

There are several solutions, one of which is illustrated in Figure 15.16. Partition R into four regions, R_1 - R_4 each of which is in the form $a \le x \le b$, $f(x) \le y \le g(x)$. Then

$$\begin{aligned} \int_R\int F(P)\,dA &= \int_{R_1}\int F(P)\,dA + \int_{R_2}\int F(P)\,dA + \int_{R_3}\int F(P)\,dA + \int_{R_4}\int F(P)\,dA \\ &= \int_{-2}^{-1}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} F(x,y)\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} F(x,y)\,dy\,dx + \\ &\qquad \int_{-1}^{1}\int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} F(x,y)\,dy\,dx + \int_{1}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} F(x,y)\,dy\,dx \end{aligned}$$

Example 15.6.2 *A clever manipulation.* The normal probability density was first defined by A. De Moivre in 1733 as

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(t-\mu)^2/2\sigma^2}.$$

The mean and standard deviation of the distribution are μ and σ. If $\mu = 0$ and $\sigma = 1$, $f(t)$ becomes

$$f(t) = \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}.$$

The coefficient $1/\sqrt{2\pi}$ insures that

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}\,dt = \lim_{c\to\infty}\int_{-c}^{c}\frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}\,dt = 1$$

This requires, and we wish to prove, that

$$I = \int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$$

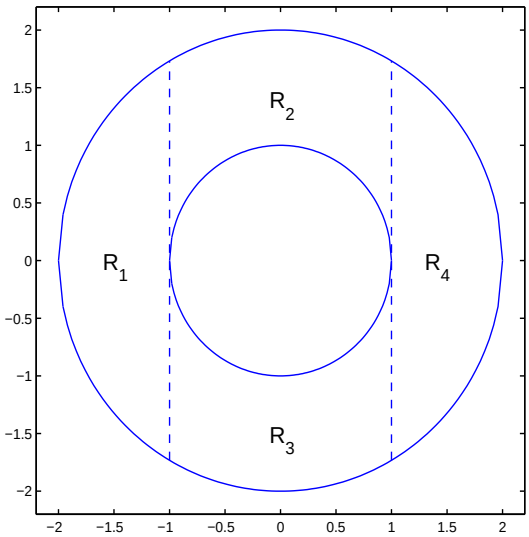


Figure 15.16: An annular ring partitioned into four regions.

The difficulty stems from the lack of an antiderivative formula for $\int e^{-t^2/2}\,dt$. There is none. However, we write

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \times \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right) e^{-y^2/2}\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}\,e^{-y^2/2}\,dx\,dy \\ &= \int_R\int e^{-(x^2+y^2)/2}\,dA \end{aligned}$$

where R is the x, y-plane.

Now we use a clever approach to the last integral. Label points in the plane with polar coordinates (r, θ) where

$$r = \sqrt{x^2+y^2} \qquad \text{and} \qquad \cos\theta = x/r, \qquad \sin\theta = y/r$$

With these coordinates

$$f(x,y) = g(r,\theta) = e^{-r^2/2}$$

and we will compute $\int_R\int e^{-r^2/2}\,dA$.

For each positive number c, let R_c be the circular region centered at the origin and radius c. See Figure 15.17. For m and n positive integers, partition $[0, c]$ by $r_i = i \times c/m$, $i = 0, m$, and $[0, 2\pi]$ by $\theta_j = j \times 2\pi/n$, $j = 0, n$. R_c is partitioned into $m \times n$ subregions

$$R_{i,j} = r_{i-1} \le r \le r_i \qquad \theta_{j-1} \le \theta \le \theta_j$$

The area of $R_{i,j}$ is $\qquad A_{i,j} = \left(\pi\,(r_i)^2 - \pi\,(r_{i-1})^2\right) \times \dfrac{1}{n} = 2\pi\dfrac{r_{i-1}+r_i}{2}(r_i - r_{i-1}) \times \dfrac{1}{n}$

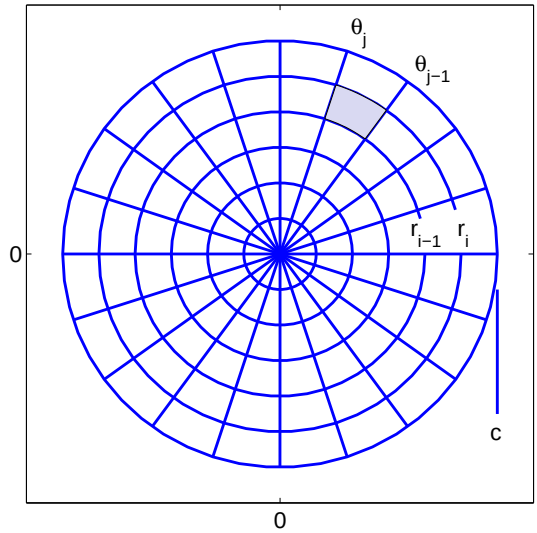


Figure 15.17: A circular region of radius c partitioned into subregions. The shaded subregion is $r_{i-1} \leq r \leq r_i$, $\theta_{j-1} \leq \theta \leq \theta_j$.

Now evaluate $e^{-r^2/2}$ in $R_{i,j}$ at $P_i = ((r_{i-1} + r_i)/2, \theta_j + \pi/n)$ and write

$$\begin{aligned}
\int_{R_c}\int e^{-r^2/2}\,dA &\doteq \sum_{i=1}^{m}\sum_{j=1}^{n} e^{-((r_{i-1}+r_i)/2)^2/2}\,A_{i,j} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} e^{-((r_{i-1}+r_i)/2)^2/2}\,2\pi\frac{r_{i-1}+r_i}{2}(r_i - r_{i-1})\frac{1}{n} \\
&= 2\pi\sum_{i=1}^{m} e^{-((r_{i-1}+r_i)/2)^2/2}\,\frac{r_{i-1}+r_i}{2}(r_i - r_{i-1})\sum_{j=1}^{n}\frac{1}{n} \\
&= 2\pi\sum_{i=1}^{m} e^{-((r_{i-1}+r_i)/2)^2/2}\,\frac{r_{i-1}+r_i}{2}(r_i - r_{i-1}) \times 1 \\
&\doteq 2\pi\int_0^c e^{-r^2/2} r\,dr \\
&= 2\pi\left[-e^{-r^2/2}\right]_0^c \\
&= 2\pi(1 - e^{-c^2/2})
\end{aligned}$$

We conclude that

$$\int_R\int e^{-r^2/2}\,dA = \lim_{c\to\infty}\int_{R_c}\int e^{-r^2/2}\,dA = \lim_{c\to\infty} 2\pi(1 - e^{-c^2/2}) = 2\pi.$$

Therefore

$$I^2 = 2\pi \qquad \text{and} \qquad I = \int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$$

as was to be proved. ■

Exercises for Section 15.6, Integrals of functions of two variables.

Exercise 15.6.1 Approximate the volume of the region between the graph of F and the x, y-plane using six or more subregions of its domain and a point selected in each subregion.

a. $F(x,y) = x \times y \qquad 0 \le x \le 3 \qquad 0 \le y \le 2$

b. $F(x,y) = x + y \qquad 1 \le x \le 3 \qquad 2 \le y \le 5$

c. $F(x,y) = x \times \ln y \qquad 0 \le x \le 3 \qquad 1 \le y \le 3$

Exercise 15.6.2 Sketch the domains over which the integrals are defined.

a $\int_1^5 \int_x^{x^2} F(x,y)\,dy\,dx$ b $\int_1^5 \int_y^{y^2} F(x,y)\,dx\,dy$ c $\int_1^5 \int_3^4 F(x,y)\,dy\,dx$

d $\int_0^\pi \int_{-\sin x}^{\sin x} F(x,y)\,dy\,dx$ e $\int_\pi^{2\pi} \int_{\sin y}^{-\sin y} F(x,y)\,dx\,dy$ f $\int_0^1 \int_0^{1-x^2} F(x,y)\,dy\,dx$

Exercise 15.6.3 Write but do not compute the iterated form of the integral $\int_R\int F(P)\,dA$ for the functions F and domains indicated. In i. and j. write the integral as the sum of two iterated integrals.

a. $F(x,y) = x \times y \qquad 0 \le x \le 3 \qquad 0 \le y \le 2$

b. $F(x,y) = x + y \qquad 1 \le x \le 3 \qquad 2 \le y \le 5$

c. $F(x,y) = x \times \ln y \qquad 0 \le x \le 3 \qquad 1 \le y \le 3$

d. $F(x,y) = x^2 y \qquad 0 \le x \le \pi \qquad 0 \le y \le \sin x$

e. $F(x,y) = x + y \qquad 1 \le x \le 2 \qquad x \le y \le x^2$

f. $F(x,y) = x \times y^2 \qquad 1 \le y \le 2 \qquad y \le x \le y^2$

g. $F(x,y) = x \times y \qquad 0 \le x + y \le 2 \qquad 0 \le x, \quad 0 \le y$

h. $F(x,y) = x + y \qquad 0 \le x^2 + y^2 \le 1$

i. $F(x,y) = x \times \ln y \qquad 1 \le x + y \le 3 \qquad 0 \le x, \quad 0 \le y$

j. $F(x,y) = x \times \ln y \qquad 1 \le x^2 + y^2 \le 4 \qquad 0 \le x, \quad 0 \le y$

Exercise 15.6.4 Evaluate the integrals.

a. $\int_0^1 \int_2^4 x\,y^2\,dy\,dx$ b. $\int_2^4 \int_0^1 x\,y^2\,dx\,dy$ c. $\int_0^1 \int_2^4 x\,y^2\,dx\,dy$

d. $\int_0^1 \int_0^y x\,y^2\,dx\,dy$

e. $\int_0^1 \int_{x^2}^x xy\,dy\,dx$ f. $\int_1^4 \int_y^{y^2} x^2 + y^2\,dx\,dy$

g. $\int_1^2 \int_{e^{-x}}^{e^x} \frac{x}{y}\,dy\,dx$ h. $\int_0^{\sqrt{3}} \int_1^{4-x^2} x + y\,dy\,dx$ i. $\int_0^1 \int_{1-x^2}^{4-x^2} x + y\,dy\,dx$

Exercise 15.6.5 Write an integral that is the volume of the region below the graph of

$$z = 16 - x^2 - 4y^2$$

and above the x,y-plane.

Chapter 16

Differential Equations

In This Chapter:

Continuous analogs of the following discrete models and additional continuous models are presented in this chapter.

We have presented several *discrete difference equation* models, including

Exponential Growth $P_{t+1} - P_t = R \times P_t$

Logistic Growth $P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right)$

Penicillin Clearance
$$\begin{array}{rcll} A_{t+1} - A_t & = & (-0.162 - 0.068)A_t + 0.1B_t & A_0 = 200 \\ B_{t+1} - B_t & = & 0.068A_t - 0.1B_t & B_0 = 0 \end{array}$$

Disease Spread
$$\begin{array}{rcll} S_{t+1} - S_t & = & -B \times S_t \times I_t & S_0 = 24,750 \\ I_{t+1} - I_t & = & B \times S_t \times I_t - G \times I_t & I_0 = 250 \\ R_{t+1} - R_t & = & G \times I_t & R_0 = 0 \end{array}$$

Whale Dynamics $N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_0}\right)^{2.39}\right\}\right] - 0.94C_t$

We have also presented two *continuous time* models,

Exponential Growth $p(0) = p_0$ $\quad p'(t) = r \times p(t)$

Harmonic Oscillations
$$\begin{array}{rcl} y(0) & = & y_0 \\ y'(0) & = & y'_0 \end{array} \qquad y''(t) + \omega^2 \times y(t) = 0$$

For whale populations that are finite and have annual births, the discrete time model

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_0}\right)^{2.39}\right\}\right] - 0.94C_t$$

is appropriate. Although bacterial populations are finite, they typically are so numerous and cell division so rapid that continuous time models with instantaneous rates of change describe well bacterial growth. Drug concentrations change continuously with continuous exchange between tissues and the vascular pool and with continuous kidney action, and continuous time models with instantaneous rates of change are preferred.

The discrete logistic growth, penicillin clearance, disease spread models may be replaced by continuous time models

$$\text{Logistic Growth} \qquad p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right)$$

$$\text{Penicillin Clearance} \qquad \begin{aligned} a'(t) &= (-r_{1,0} - r_{2,1})a(t) + r_{1,2}b(t) & a(0) &= 200 \\ b'(t) &= r_{2,1}a(t) - r_{1,2}b(t) & b(0) &= 0 \end{aligned}$$

$$\text{Disease Spread} \qquad \begin{aligned} s'(t) &= -\beta \times s(t) \times i(t) & s(0) &= 24,750 \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma \times i(t) & i(0) &= 250 \\ r'(t) &= \gamma \times i(t) & r(0) &= 0 \end{aligned}$$

The discrete and continuous rate coefficients are related but not equal. For logistic growth, for example, r in the continuous model and R in the discrete model are related by $r = \ln(1 + R)$.

Except for the harmonic equation, the derivative equations above are all **first order** differential equations because the first derivative and no higher order derivative appears. The harmonic equations

$$y''(t) + \omega^2 y(t) = 0 \qquad y''(t) + 3y'(t) + 4y(t) = 0$$

involve the second derivatives and are examples of a **second order** differential equations.

The equations are called **differential equations**, but would be better named *derivative* equations because they relate derivatives of functions to the functions. The words *differential equation* are in universal usage, however, and we will do the same. The expression derives from a frequent practice of writing, for example, the equation

$$y'(t) = 0.02 \times y \times (1 - \frac{y}{10}) \tag{16.1}$$

in Leibnitz notation

$$\frac{dy}{dt} = 0.02 \times y \times (1 - \frac{y}{10})$$

and then in *differential* form

$$dy = 0.02 \times y \times (1 - \frac{y}{10})\, dt$$

and finally even in the form

$$\frac{dy}{y \times (1 - y/10)} = 0.02dt$$

We have not defined the symbols, dy and dt. They are called differentials, and considered to be infinitesimals; dy is an infinitesimal change in y due to an infinitesimal change, dt, in t. Thus, y' is the slope, $\frac{\text{rise}}{\text{run}} = \frac{dy}{dt}$. This concept has been used successfully by many scientist, but has suffered from philosophical attack on the concept, infinitesimal. We used the notation $\int_a^b f(t)\,dt$ without separate definition of dt. We do not do so, but some scientists reason that to integrate f on $[a, b]$, one "sums the values of $f(t)$ times infinitesimal increases, dt, in t from $t = a$ to $t = b$."

16.1 Differential equation models of biological processes.

Differential equations were important early in the development of biological concepts.

16.1.1 Single species population models.

Thomas Malthus[1] described human population growth as

> **Malthus' Model of Population Growth.** The rate of population increase is proportional to the size of the population.

Let $p(t)$ denote population size at time t. Then Malthus asserted that

	Rate of increase		is proportional to		population size
Malthus	$p'(t)$	$=$	r	$\times$	$p(t)$

From this differential equation we have concluded (Property of Exponential Growth or Decay, 5.5.2) that

$$p(t) = p_0 e^{rt} \qquad (p_0 \text{ is founding population size})$$

which is the exponential growth Malthus predicted. Malthus also modeled the increase in food production as

> **Malthus' Model of Food Production.** The growth rate of food production is constant.

Let $f(t)$ denote the food produced at time t. Then

	Rate of increase	is	constant
Food Production	$f'(t)$	$=$	C

[1] Anonymous publication in 1898, An essay on the Principle of Population as it Affects the Future Improvement of Society with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers.

We know that if $f'(t) = C$ then

$$f(t) = Ct + f_0 \qquad\qquad (f_0 \text{ is initial food production.})$$

Malthus knew that every exponential graph eventually exceeds every linear graph and direly predicted wide spread poverty and degradation, Figure 16.1.

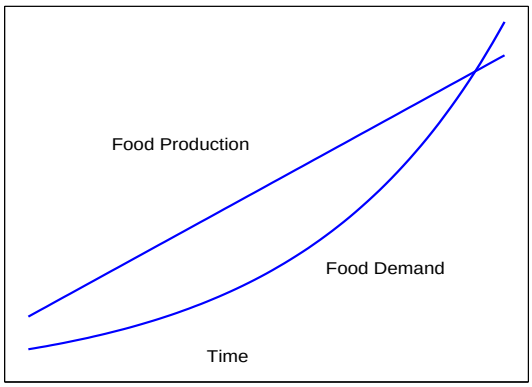


Figure 16.1: Malthus forecast that food demand would out strip food production.

In 1838 and 1845 P. F. Verhulst[2] connected the population to its environment and asserted that:

> **Mathematical Model of population growth in a limited environment** The growth rate of a population is proportional to the size of the population and to the fraction of the carrying capacity unused by the population.

By the double proportionality, the growth rate is proportional to the *product* of the population size and the unused fraction of the carrying capacity.

Let $p(t)$ be the size of a population growing in an environment in which the carrying capacity is M. Let $U(p(t))$ denote the unused fraction of the carrying capacity for population size $p(t)$, and write

Growth rate	Is proportional to		Population size	Times	Unused fraction of capacity
$p'(t)$	$= r$	$\times$	$p(t)$	$\times$	$U(p(t))$

Scientists have written various expression for $U(p(t))$. Verhulst used the most direct:

$$\text{Fraction of carrying capacity used} \quad = \quad \frac{p(t)}{M}$$

$$\text{Unused fraction of carrying capacity} \quad U(p(t)) \quad = \quad 1 - \frac{p(t)}{M}$$

[2]Pierre F. Verhulst (1838) Notice sur la loi que la population pursuit dans son accroissement, *Corresondence mathématique et physique* 10:113-121; (1845) Nouveau Memoirs de l'Academie Royal des Sciences et Bellas-Letters.

With $U(p(t)) = 1 - \frac{p(t)}{M}$ we derive Verhulst's logistic equation

$$\textbf{Verhulst} \qquad p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right) \tag{16.2}$$

Other choices for $U(p(t))$ include

$$\begin{array}{llllll}
\textbf{Ricker} & U(p(t)) & = & \dfrac{Ae^{-p(t)/\beta} - 1}{A - 1} & p'(t) & = & r \times p(t) \times \dfrac{Ae^{-p(t)/\beta} - 1}{A - 1} \\
 & A & > & 1 & p'(t) & = & \alpha \times p(t) \times e^{-p(t)/\beta} - \gamma p(t) \\
\textbf{Beverton-Holt} & U(p(t)) & = & \dfrac{1}{1 + p(t)/\beta} & p'(t) & = & \dfrac{r \times p(t)}{1 + p(t)/\beta} \\
\textbf{Gompertz} & U(p(t)) & = & -\ln(p(t)/\beta) & p'(t) & = & -rp(t)\ln(p(t)/\beta)
\end{array} \tag{16.3}$$

The Ricker and Beverton-Holt equations are used to model fish populations; the Gompertz equation is used to model tumor growth.

In the Verhulst, Ricker and Beverton-Holt equations, when $p(t)$ is close to zero, $U(p(t))$ is close to 1 and

$$p'(t) \doteq r \times p(t) \qquad \text{for } p(t) \text{ small}$$

For that reason, the number r is called the *low density growth rate.* At low population numbers, population growth in the Verhulst, Ricker and Beverton-Holt models appears to be as Malthus projected. The Gompertz equation is quite different at low population numbers as will be seen in Explore 16.1.1 but shares a similarity as seen in Exercise 16.1.2.

Explore 16.1.1 Low density growth in the Gompertz model is different from that in the Verhulst, Ricker, and Beverton-Holt models.

a. Show that in the Ricker model in Equations 16.3, for $p(t) \ll \beta$, $U(t) \doteq 1$ so that $p'(t) \doteq r\,p(t)$.

b. Show that in the Gompertz model in Equations 16.3, for $p(t) \ll \beta$, $U(p(t)) \gg 1$ so that $p'(t) \gg r\,p(t)$.

16.1.2 Competition between two species.

Alfred J. Lotka and Vito Volterra formulated models of competing species and predator prey interactions in the early 1900's[3].

Two species that occupy a given environment may compete for space, food, or light. They may reach an accommodation in which both species continue, but often one of the species is eventually excluded from the environment, depending on the environmental capacity for each species and the degree to which one species absorbs the resources of the other. Assume that for each species:

> **Mathematical Model of population growth in competition.** The growth rate of a population is proportional to the size of the population and to the fraction of the carrying capacity unused by the **two** populations.

[3] A prime reference is Lotka's 1924 book, *Elements of Mathematical Biology* now available as a Dover reprint, 1956.

This is the same as for a single species, except for the **two**.

Suppose two competing species have populations $p_1(t)$ and $p_2(t)$ in an environment with capacities M_1 and M_2 respectively for each population in the absence of the other.

Similar to the Verhulst model for a single species, for two species we write

$$\begin{aligned} \text{Population 1} \qquad p_1'(t) &= r_1 \times p_1(t) \times \left(1 - \frac{p_1(t) + \alpha_{1,2} p_2(t)}{M_1}\right) \\ \text{Population 2} \qquad p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{p_2(t) + \alpha_{2,1} p_1(t)}{M_2}\right) \end{aligned} \tag{16.4}$$

$\alpha_{1,2}$ is a measure of the influence of Population 2 on the environment of Population 1. If a member of Population 2 consumes twice the resources that a member of Population 1 consumes, then $\alpha_{1,2} = 2$. A similar interpretation should be given to $\alpha_{2,1}$.

According to this model, the eventual presence or absence of Populations 1 and 2 depend on the values of M_1, M_2, $\alpha_{1,2}$ and $\alpha_{2,1}$ (and, curiously, not on r_1 nor r_2).

16.1.3 Predator-Prey models.

Lotka and Verhulst also wrote equations descriptive of predator and prey interaction. We described a model for predator and prey values close to equilibrium on page 326. To extend this model assume

> **Mathematical Model of prey and predator:** We assume two species, a prey and a predator, and that
>
> > The rate of predation is proportional to the size of the prey population and proportional to the size of the predator population.
>
> Furthermore:
>
> a. Without the predator, the prey population increases at a rate proportional to the size of the prey population (as Malthus predicted).
> b. The prey growth rate is decreased proportional to the rate of predation.
> c. Without the prey, the predator decreases at a rate proportional to the size of the predator population (the predator has no alternate food source).
> d. The predator growth rate is increased proportional to the rate of predation.

As usual, the double proportionality of the predation rate implies the the predation rate is proportional to the product of the size of the prey population and the size of the predator population.

Let $u(t)$ and $v(t)$ denote the sizes of the prey and predator populations, respectively. We interpret parts 1 and 2 above to say that

$$\textbf{Prey} \qquad u'(t) = a \times u(t) - b \times u(t) \times v(t) \tag{16.5}$$

You are asked to write the Predator equation in Exercise 16.1.8

16.1.4 Susceptible, Infectious, Recovered (SIR) epidemic models.

In 1927 W. O. McKermack and A. G. McKendrick[4] modeled the spread of contagious diseases based on the assumption that

> **SIR Model of Epidemics.** The rate at which people get infected is proportional to the rate of 'infections contacts' between already infected people and people who are susceptible of becoming infected.
>
> The rate of infections contacts is proportional to the number of susceptible people and to the number of infected people.

Suppose people are either susceptible to becoming infected, infected and capable of transmitting the disease, or have been infected and recovered, with numbers $S(t)$, $I(t)$, and $R(t)$, respectively and with t measured in days.

Then the basic model is

$$\begin{array}{lrcl} \textbf{Susceptible} & S'(t) & = & -\beta \times S(t) \times I(t) \\ \textbf{Infectious} & I'(t) & = & \beta \times S(t) \times I(t) - \gamma \times I(t) \\ \textbf{Recovered} & R'(t) & = & \gamma \times I(t) \end{array}$$

16.1.5 Environmental pollution.

Suppose a lake has a river running through it and a factory is built next to the lake and begins releasing a chemical into the lake. The water running into the lake is free of the chemical and initially the lake is free of the chemical. What is the expected future content of the chemical in the lake?

We use a simple, but sometimes overlooked, model.

> **Mathematical Model.** The rate at which the amount of chemical in the lake increases is equal to the rate at which the chemical enters the lake minus the rate at which the chemical leaves the lake.

We also use

> The rate at which the chemical leaves the lake is the product of the concentration of the chemical in the lake and the rate at which water leaves the lake.

To get an equation we need some notation. Let V meters^3 be the volume of the lake. "A river running through the lake" means that there is a river running into the lake at a rate of R m^3/day and a river running out of the lake also at a rate of R m^3/day. Assume the factory releases the chemical into the lake at a rate of T kilograms/day. Finally, let $P(t)$ be the kilograms of chemical in the lake t days

[4] W. O. McKermack and A. G. McKendrick, Contributions to the mathematical theory of epidemics, *Roy. Stat. Soc. J.* **115** (1927) 700-721.

after the factory is built. Initially, the lake is free of the chemical, and at all times the water flowing into the lake is free of the chemical. Then we write

$$\begin{array}{ccccccc} \text{Initial state} & & \text{Rate of increase} & \text{is} & \text{Rate of release} & \text{minus} & \text{Rate flowing out} \\ P(0)=0 & & P'(t) & = & T & - & \dfrac{P(t)}{V}\times R \end{array} \tag{16.6}$$

Exercises for Section 16.1, Differential equation models of biological processes.

Exercise 16.1.1 Let $M = \beta = 1$ and $A = 3$ draw the graphs of $U(p)$ *vs* p for each of the four models in Equations 16.2 - 16.3, for $0 < p \leq 1.2$.

Exercise 16.1.2 Let $m(p_0)$ be $p'(0)$ in the Gompertz model, where

$$p'(t) = -rp(t)\ln(p(t)/\beta), \quad t \geq 0, \qquad p(0) = p_0.$$

For $p_0/\beta \ll 1$ it may be concluded from Explore 16.1.1 that $m(p_0) = p'(0) \gg rp(0)$. Show, however, that

$$\lim_{p_0 \to 0} m(p_0) = 0.$$

Exercise 16.1.3 Newton's model of heat absorption by an object. Newton asserted that

> **Mathematical Model of Heat Absorption.** The rate at which heat is absorbed by a body is proportional to the difference between the temperature of the air surrounding the body and the temperature of the body.

[a.] Write an initial condition and differential equation that describes temperature of a clam exposed at low tide to 34° air when ocean temperature is 20°C. It will be necessary to assume the clam has a heat capacity, C, such that the amount of heat required to increase the temperature of the clam $\Delta \quad C^\circ$ is $C \times \Delta$.
[b.] Write a solution to your differential equation.
[c.] For clams growing in an intertidal zone, would you expect the larger clams to be higher in the zone or lower in the zone?

Exercise 16.1.4 Release of nitrogen in the tissue of a SCUBA diver as she ascends from deep water has been compared to the release of carbon dioxide in a Coca-Cola©when it is opened. Write a mathematical model descriptive of release of carbon dioxide in a Coca-Cola©. From your model, write a differential equation descriptive of the partial pressure of carbon dioxide in a Coca-Cola©t minutes after opening the Coca-Cola©.

Exercise 16.1.5 Equations 16.4 for two competing species are based on Verhulst's logistic single species Equation 16.2. Write the corresponding pairs of equations that would describe competition between two species

a. Based on the Ricker single species model in Equation 16.3.

b. Based on the Beverton-Holt single species model in Equation 16.3.

c. Based on the Gompertz single species model in Equation 16.3.

Exercise 16.1.6 Describe the relation between two species of Equations 16.4 when $\alpha_{1,2} = 0$ and $\alpha_{2,1} = 0$.

Exercise 16.1.7 Suppose that without the predator, the prey growth in Equation 16.5 is logistic, as in the Verhulst single species Equation 16.2. Write a modification of Equation 16.5 descriptive of prey growth in the presence of the predator.

Exercise 16.1.8 Write an equation for the growth rate of the predator population based on parts 3 and 4 of the Mathematical Model of Prey and Predator on page 712.

Exercise 16.1.9 a. The units on S', I' and R' in the SIR model are people/day. In order for the units to balance on the equations, what must be the units on β and γ?

b. Suppose the infection typically lasts seven days. What is an appropriate value of γ?
Note: The answer we would expect you to give for γ is $1/7 \doteq 0.143$. We will find that a better answer is $-\ln(1 - 1/7) \doteq 0.154$.

Exercise 16.1.10 Suppose immunity is not permanent in the SIR model, and recovered people become susceptible after six months. Modify the meaning of R and the SIR equations to account for this possibility.

Exercise 16.1.11 In Equation 16.6, the units on $P'(t)$ are inherited from the difference quotient of $P'(t)$ which is the limit:

$$\lim_{h \longrightarrow 0} \frac{P(t+h) - P(t)}{h} \qquad \frac{\text{kilograms}}{\text{day}} \qquad = \qquad p'(t)\frac{\text{kilograms}}{\text{day}}$$

Show that the units are the same on the two sides of

$$P'(t) = T - \frac{P(t)}{V} \times R$$

Exercise 16.1.12 What is a reasonable value to assume for the initial condition of the lake, $P(0)$ in Equation 16.6?

Exercise 16.1.13 Suppose there is a massive chemical spill on a single day into a lake with a river running through it. Write an initial condition, $P(0)$ and a differential equation that will model the amount of pollution in the lake. Write a solution to your equation.

Exercise 16.1.14 As the first approximation of penicillin clearance by the kidney, we wrote the difference equations

$$\begin{aligned} P(0) &= 10 \\ P(T+1) - P(t) &= -0.2 \times P(T) \end{aligned}$$

Time was measured in 5 minute intervals (so that time $t+1$ was five minutes later than time, t). The equation reflects the assumption that 20% of the penicillin in the serum is removed every 5 minutes. Penicillin clearance is a continuous process, however; is does not occur in 5 minute batches.

Write a model of continuous penicillin clearance following a single injection of penicillin. Consider only the vascular pool; we will include the tissue compartment later. Write a differential equation with initial condition that will describe the amount of penicillin in a six liter vascular pool following an injection of 2 grams of penicillin. Note that at time $t = 5$ minutes the amount of penicillin in the vascular pool is 1.6 g.

Exercise 16.1.15 Suppose a patient is administered penicillin by continuous infusion at the rate of 0.5 gm/hour. Consider only the vascular pool, and write a model of penicillin amount in the vascular pool. Write an initial condition and a differential equation that is descriptive of the amount of penicillin in the serum as a function of time.

16.2 Solutions to differential equations.

Important biological and physical processes are well described by differential equations. The next few sections are directed to finding the functions that are described by the differential equations.

A general form of the first order differential equation is

$$y'(t) = f\left(t, y(t)\right) \tag{16.7}$$

where f is a function of two variables, (t, y). Equation 16.7 is a statement about a function $y(t)$ and a **solution** to the equation is a function for which the statement is true. We check that it is a solution by *substitution*. Consider for example the algebraic equation

$$x^2 - 3x + 2 = 0$$

which is a statement about a number x. The statement that the number 2 is a solution to the equation means that when 2 is substituted for x in the equation the result is a true statement

$$2^2 - 3 \times 2 + 2 = 4 - 6 + 2 \quad = \quad 0 \qquad \textbf{It Checks!}$$

Note that $x = 1$ is also a solution to the algebraic equation. However, $x = 5$ is not a solution:

$$5^2 - 3 \times 5 + 2 = 25 - 15 + 2 = 12 \quad \neq \quad 0 \qquad \textbf{It does not Check.}$$

An example of a differential equation is

$$y'(t) = t \times y(t)$$

Is $y = t^2$ a solution? Let's check by substitution.

LHS: $y'(t)$		RHS: $t \times y(t)$	
$y'(t) = [t^2]'$		$t \times y(t) = t \times t^2$	
$2t$	$\neq$	t^3	**It does not Check.**

So $y = t^2$ is not a solution. But we claim that $y = e^{\frac{t^2}{2}}$ *is* a solution. Again, we check by substitution.

LHS: $y'(t)$		RHS: $t \times y(t)$	
$y'(t) = \left[e^{\frac{t^2}{2}}\right]'$		$t \times y(t) = t \times e^{\frac{t^2}{2}}$	
$e^{\frac{t^2}{2}} \times \left[\frac{t^2}{2}\right]'$		$t \times e^{\frac{t^2}{2}}$	
$e^{\frac{t^2}{2}} \times t$	$=$	$t \times e^{\frac{t^2}{2}}$	**It Checks!**

Explore 16.2.1 Show that each of the functions shown below is a solution to $y'(t) = t \times y(t)$:

a. $y = \frac{1}{2}e^{\frac{t^2}{2}}$ b. $y = \frac{1}{4}e^{\frac{t^2}{2}}$ c. $y = 6e^{\frac{t^2}{2}}$ b. $y = -3e^{\frac{t^2}{2}}$

Find yet another solution to $y'(t) = t \times y(t)$.

Show that $y = e^{t^2}$ is not a solution to $y'(t) = t \times y(t)$. ■

Initial Condition. There are several (actually infinitely many) solutions to $y'(t) = t \times y(t)$. In addition to the differential equation, one usually also knows an *initial condition* $y(0)$ (or, $y(a)$, the value of y at another specific value, a, of t). Then for most first order differential equations that describe biological and physical processes, there will be *only one function* that satisfies both the initial condition and the differential equation. For example, suppose

$$y(0) = 2 \qquad y'(t) = t \times y(t)$$

Then $y(t) = 2e^{t^2/2}$ satisfies both conditions. We check by substitution in both equations.

Initial Condition	LHS: $y'(t)$		RHS: $t \times y(t)$
$y(0) = 2$	$y'(t) = \left[2e^{t^2/2}\right]'$		$t \times y(t) = t \times 2e^{t^2/2}$
$y(0) = 2e^{0^2/2}$	$2e^{t^2/2}\left[t^2/2\right]'$		$t \times 2e^{t^2/2}$
$=2$	$2e^{t^2/2}t$	$=$	$t \times 2e^{t^2/2}$
Check!		**Check!**	

Therefore, $y = 2e^{t^2/2}$ satisfies the *initial condition* $y(0) = 2$ and the *differential equation* $y'(t) = t \times y(t)$. Conditions that insure that $y = 2e^{t^2/2}$ is the only solution satisfying both the initial condition and the differential equation are given in Theorem 16.6.1 Existence and Uniqueness of Solutions, but we do not include a proof of this theorem.

Exercises for Section 16.2, Solutions to differential equations.

Exercise 16.2.1 Show that each solution satisfies the initial condition and the differential equation.

	Solution	Initial Condition	Differential Equation
a.	$y(t) = e^{2t} + e^t$	$y(0) = 2$	$y'(t) - y(t) = e^{2t}$
b.	$y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$	$y(0) = 1$	$y'(t) + 2y(t) = e^t$
c.	$y(t) = te^t$	$y(0) = 0$	$y'(t) - y(t) = e^t$
d.	$y(t) = \frac{t^2}{3} + \frac{1}{t}$	$y(1) = \frac{4}{3}$	$t \times y'(t) + y(t) = t^2$
e.	$y(t) = \sqrt{t+1}$	$y(0) = 1$	$y(t) \times y'(t) = \frac{1}{2}$
f.	$y(t) = \sqrt{1+t^2}$	$y(0) = 1$	$y(t) \times y'(t) = t$
g.	$y(t) = \sqrt{4+t^2}$	$y(0) = 2$	$y(t) \times y'(t) = t$
h.	$y(t) = \frac{1}{t+1}$	$y(0) = 1$	$y'(t) + (y(t))^2 = 0$

	Solution	Initial Condition	Differential Equation
i.	$y(t) = 0.5 + 0.5e^{-0.2\sin t}$	$y(0) = 1$	$y'(t) + 0.2(\cos t)y(t) = 0.1\cos t$
j.	$y(t) = \tan t$	$y(0) = 0$	$y'(t) = 1 + (y(t))^2$
k.	$y(t) = 3$	$y(0) = 3$	$y'(t) = (y(t) - 1) \times (y(t) - 3) \times (y(t) - 5)$
l.	$y(t) = 5$	$y(0) = 5$	$y'(t) = (y(t) - 1) \times (y(t) - 3) \times (y(t) - 5)$

Exercise 16.2.2 Which of the following possible solutions satisfies the initial condition and the

differential equation.

	Possible Solution	Initial Condition	Differential Equation
a.	$y(t) = e^{2t} + 2e^t$	$y(0) = 2$	$y'(t) - y(t) = e^{2t}$
b.	$y(t) = e^{2t} + 2e^t$	$y(0) = 3$	$y'(t) - y(t) = e^{2t}$
c.	$y(t) = \sqrt{t+1}$	$y(0) = 1$	$y(t) \times y'(t) = 1$
d.	$y(t) = t^3$	$y(1) = 1$	$y'(t)/y(t) = 3t$
e.	$y(t) = \frac{1}{t}$	$y(0) = 1$	$y'(t)/y(t) = -y(t)$
f.	$y(t) = t^3$	$y(1) = 1$	$y'(t)/y(t) = 3t$
g.	$y(t) = te^t$	$y(0) = 0$	$y'(t) - y(t) = e^t$

Exercise 16.2.3 The special case of $y' = f(t, y)$ in which $f(t, y) = F(t)$ (f is independent of y) has a familiar solution from the Fundamental Theorem of Calculus I. Check by substitution that

$$y(t) = y_a + \int_a^t F(x)\,dx \qquad \text{solves} \qquad y(a) = y_a \qquad \text{and} \qquad y'(t) = F(t)$$

The differential equation

$$y(a) = y_a \qquad y'(t) = F(t)$$

has therefore been completely solved. Henceforth we will consider that f is dependent on y and possibly also on t.

16.3 Direction Fields.

Shown in Figures 16.2 and 16.3 are the graphs of solutions of certain differential equations. For each differential equation, the different solutions correspond to different points at which the curves cross the Y-axis, each such point representing a specific value of $y(0)$ - the initial condition. You should get the idea that for any differential equation *that we study*, the plane is filled with graphs of functions that solve the differential equation and that no two of the graphs intersect.

In order to construct the previous graphs, we needed to know the solutions. However, we can 'almost' construct the previous graphs with out knowing any solutions.

The differential equation

$$y'(t) = f(t, y)$$

specifies the **slope, $\mathbf{f(t, y)}$,** of the solution passing through, the point $(\mathbf{t}, \mathbf{y})$ in the plane. Knowing the point and the slope, we can construct a (short) tangent to the graph of the solution, *without having a formula for the solution.*

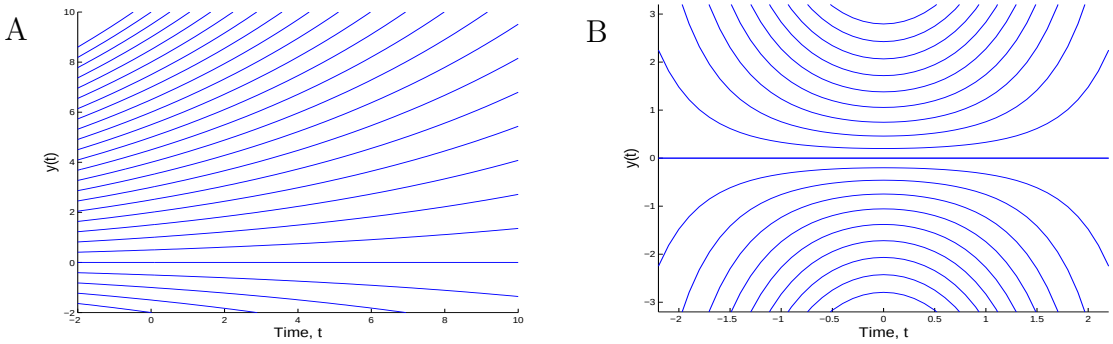


Figure 16.2: A. Solutions to $y'(t) = 0.1y$. B. Solutions to $y'(t) = t * y$

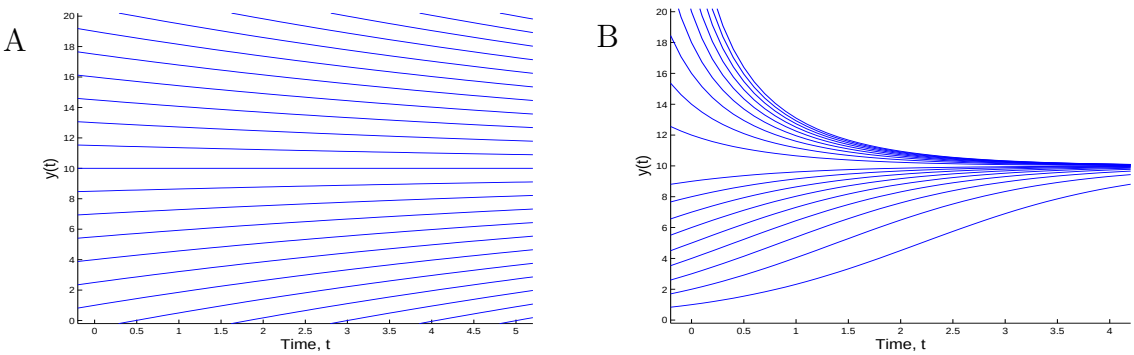


Figure 16.3: A. Solutions to $y'(t) = 1 - y/10$. B. Solutions to $y'(t) = 0.5 * y * (1 - y/10)$

For the differential equation

$$y'(t) = t \times y$$

at $(t, y) = (-1, 2)$ the slope of the solution is

$$y' = (-1) \times 2 = -2$$

and is shown with some other slopes in Figure 16.4A. Slopes at many points appear in Figure 16.4B.

Explore 16.3.1 Find the three direction field tangents in Figure 16.4B that correspond to the three tangents shown in Figure 16.4A.

If for a single differential equation we draw enough of the short tangent lines, we obtain a picture that strongly suggests the shapes of the solution curves. The short tangents show the 'directions' of the solutions. Graphs of the short tangent lines are called **direction fields**.

A direction field for

$$y' = t - 2ty$$

is shown in Figure 16.5. Starting from the point A and following the direction tangents, it is fairly obvious that the curve shown approximates the graph of a solution.

Now look at the point B in the direction field for $y' = t - 2ty$, and follow the direction tangents for the solution through B. It should appear that the horizontal line through B is the graph of a solution to

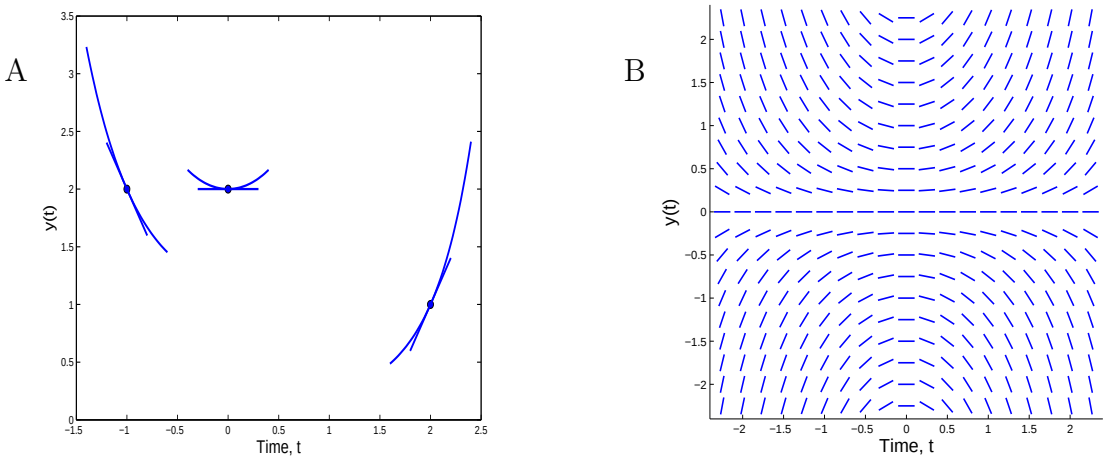


Figure 16.4: A. Solutions to $y'(t) = t \times y$ and tangents at selected points. B. Direction field for $y'(t) = t \times y$.

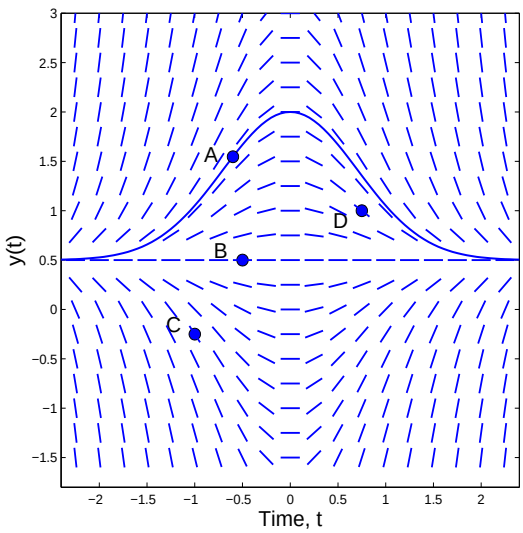


Figure 16.5: Direction field for $y' = t - 2ty$ and a solution through the point A.

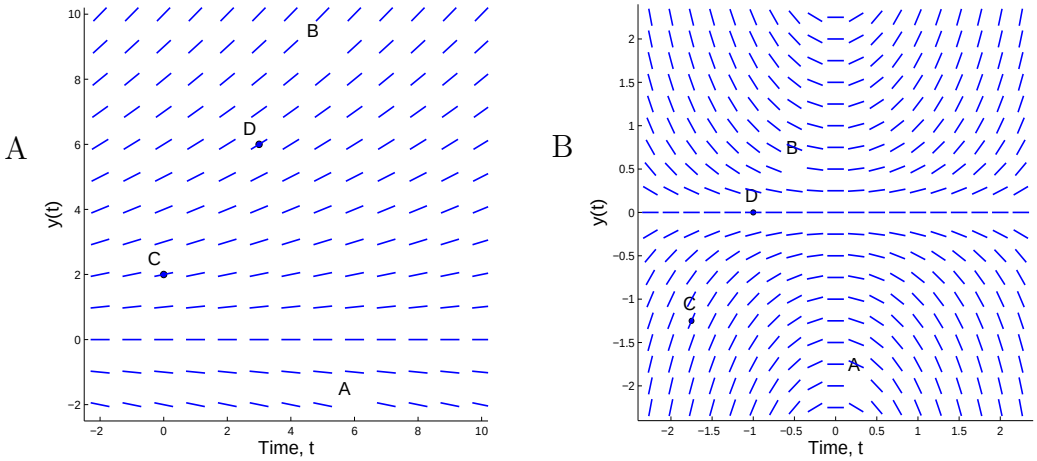


Figure 16.6: A. Direction field for $y'(t) = 0.1y$. B. Direction field for $y'(t) = t * y$

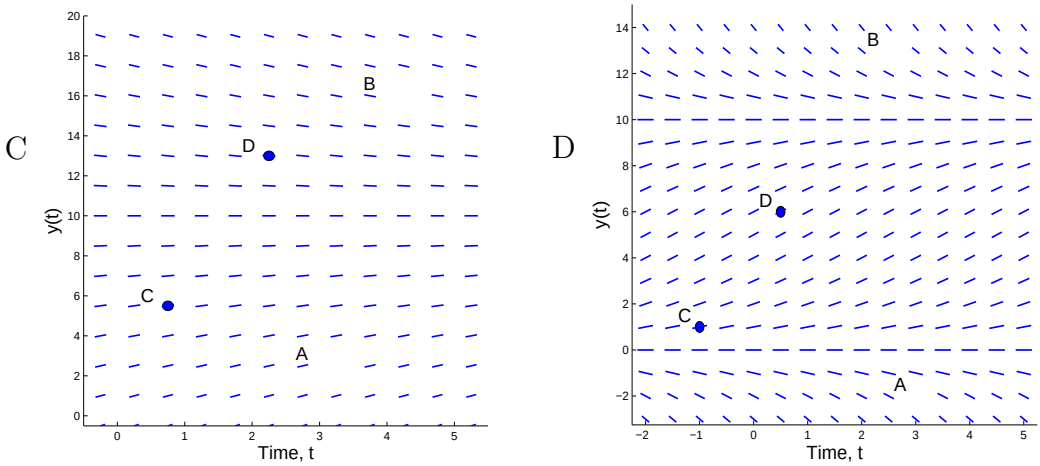


Figure 16.7: C. Direction field for $y'(t) = 1 - y/10$. D. Direction field for $y'(t) = 0.5 * y * (1 - y/10)$

$y' = t - 2ty$. That line intersects the Y-axis at $(0, \frac{1}{2})$ and $y = \frac{1}{2}$ is an equation of a solution. We suggest that

$$y = \frac{1}{2} \qquad \text{is a solution to} \qquad y' = t - 2ty$$

and check to see that it is:

LHS: $y'(t)$		RHS: $t - 2t \times y(t)$	
$y'(t) = \left[\frac{1}{2}\right]'$		$t - 2t \times y(t) = t - 2t \times \left(\frac{1}{2}\right)$	
0		$t - 2t \times \frac{1}{2}$	
0	$=$	0	**It Checks!**

Direction fields for the four examples in Figures 16.2 and 16.3 are shown in Figures 16.6 and 16.7

Exercises for Section 16.3, Direction Fields.

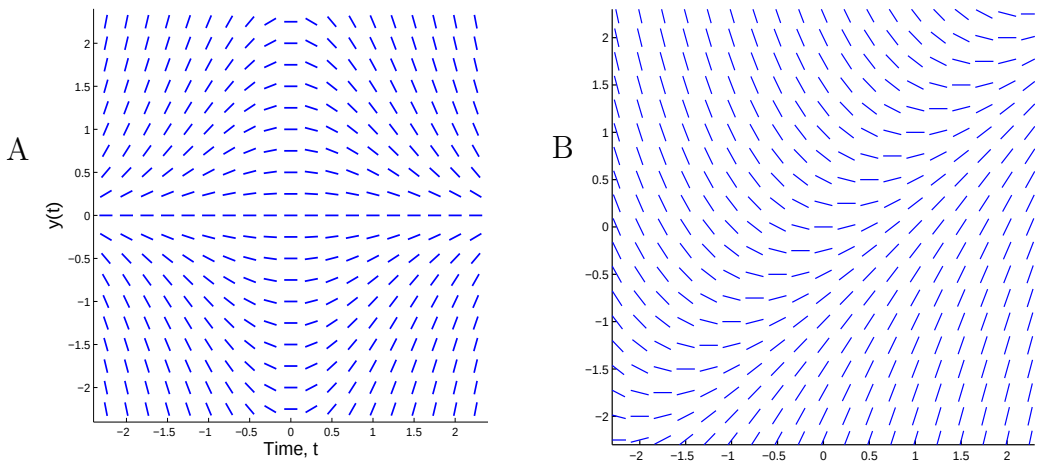
Exercise 16.3.1 For the direction field for the differential equation $y' = t - 2ty$ in Figure 16.5 draw solutions through the points C and D.

Exercise 16.3.2 For the four direction fields in Figures 16.6 and 16.7 tangents have been omitted at points marked A and B. Compute the slopes of those tangents for those points.

Exercise 16.3.3 For the four direction fields in Figure 16.6 and 16.7 draw an approximate solution that passes through the point C and another solution through the point D.

Exercise 16.3.4 Shown in Exercise Figure 16.3.4 are direction fields for two differential equations. For each direction field, draw (approximately) the graphs of three solutions.

Figure for Exercise 16.3.4 A. Direction field for $y\prime = -t \times y$. B. Direction field for $y' = t - y$.



Exercise 16.3.5 Draw direction fields in the quadrant, $t > 0$, $y > 0$, and three approximate solutions for the following differential equations. Note that c. is a special case that we said we would not consider.

a. $y'(t) = 1$ b. $y'(t) = 0.5y$

c. $y'(t) = t$ d. $y'(t) = \sqrt{y}$

Exercise 16.3.6 The direction field of

$$y' = (y - 1) \times (y - 3)$$

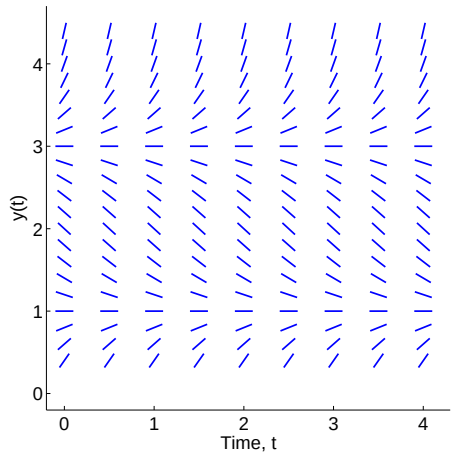
is shown in Exercise Figure 16.3.6

a. Draw exactly the graphs of two solutions of the differential equation.

b. Find a formula for the function that is one of the solutions to the equation.

c. Check that your formula satisfies the differential equation.

Figure for Exercise 16.3.6 Direction field for $y\prime = (y-1) \times (y-3)$.



Exercise 16.3.7 The following MATLAB program draws a direction field for a differential equation.

a. For what differential equation does the program draw a direction field?

b. How would you change the program to draw a direction field for the differential equation $y' = t + y$?

c. If you have access to MATLAB, enter the program and run it.

```
close all;clc;clear;
tt=[-2:0.25:2];
yy=[-2:0.25:2];
r=0.05;
axis([-2.2 2.2 -2.2 2.2])
hold
for i = 1:length(tt)
    for j = 1:length(yy)
        slope=tt(i)*yy(j);
        angle=atan(slope);
        cc=cos(angle);ss=sin(angle);
        plot([tt(i)-r*cc tt(i)+r*cc],[yy(j)-r*ss yy(j)+r*ss],'linewidth',2)
    end
end
```

16.4 Phase planes and stability of constant solutions to $y' = f(y)$.

A differential equation of the form

$$y\prime = f(y) \qquad \text{(RHS is independent of } t\text{)}$$

is called an **autonomous** differential equation.

$$y' = y^2 \qquad \text{is autonomous} \qquad y' = t + y^2 \qquad \text{is not autonomous}$$

A differential equation that models an isolated population in a constant environment is typically autonomous. If there is occasional migration into or out of the population or the environment fluctuates with time, the model differential equation will not be autonomous. A differential equation that models penicillin clearance following a single injection or even constant infusion is autonomous; if there are subsequent injections or variation in the infusion, the model differential equation is not autonomous.

Definition 16.4.1 For an autonomous differential equation

$$y' = f(y),$$

if $\overline{y}$ is a number such that $f(\overline{y}) = 0$, then $y(t) = \overline{y}$ is a solution to $y' = f(y)$, $\overline{y}$ is called an **equilibrium point** for $y' = f(y)$, and $y(t) = \overline{y}$ is called an *equilibrium solution.*

For example, the equation

$$y' = f(y) = (y-2) \times (y-3)$$

is autonomous and $f(2) = 0$ and $f(3) = 0$. Therefore $\overline{y} = 2$ and $\overline{y} = 3$ are equilibrium points. Also $y(t) = 2$ is an equilibrium solution to the differential equation:

LHS: $y'(t)$		RHS: $(y(t)-2) \times (y(t)-3)$	
$y'(t) = [2]'$		$(y(t)-2) \times (y(t)-3) = (2-2) \times (2-3)$	
0	$=$	0	**It Checks.**

The RHS of Verhulst's logistic differential equation

$$p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right) \tag{16.8}$$

of the dynamics of a single population does not depend explicitly on t; the equation might be written

$$p' = r \times p \times \left(1 - \frac{p}{M}\right)$$

and is autonomous.

The direction field for the logistic differential equation is shown in Figure 16.8A. The graph of p' *vs* p, or the graph of f where

$$f(p) = r \times p \times \left(1 - \frac{p}{M}\right)$$

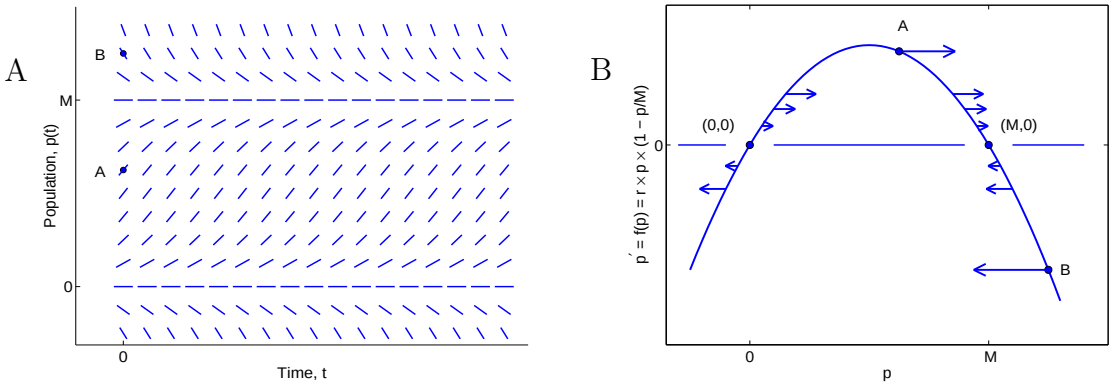


Figure 16.8: A. Direction field for the logistic equation $p' = r \times p \times (1 - p/M)$. B. Phase plane graph of $f(p) = r * p * (1 - p/M)$. The arrows point to the right where $f(p) > 0$ and point to the left where $f(p) < 0$.

is shown in Figure 16.8B. This graph is called a **phase plane** graph. Phase plane graphs are easier to draw than are direction fields, but give similar information. In this case the phase plane graph is a parabola. The arrows point to the right where $f(p) > 0$ and point to the left where $f(p) < 0$.

From the direction field, we can immediately see two horizontal lines that are graphs of two solutions,

$$p = 0 \qquad \text{and} \qquad p = M$$

both of which are equilibrium solutions.

In the phase plane we can see the same equilibrium points. Simply observe that

$$f(0) = 0 \qquad \text{and} \qquad f(M) = 0$$

The meaning of the equilibrium solution, $p = 0$, is fairly obvious; if there are no rabbits at time $t = 0$ there will be no rabbits at any subsequent time (the model does not account for migration into the population, nor spontaneous generation of rabbits).

The equilibrium solution, $p = M$, is not quite so obvious, but makes sense. If the population size equals the the carrying capacity, the environment will support that size population and no more, and the population size will persist at that value.

Now examine the direction field in Figure 16.8A for the solution that passes through the point, $(0, A)$, for which

$$0 < p(0) = A < M \qquad \text{Initial population less than carrying capacity.}$$

The graph will rise from $(0, A)$ and will be asymptotic to the line $y = M$; the population increases to the maximum the environment will support.

The point A that marks $(A, f(A))$ is shown on the phase plane. We can see from that point that $f(A) > 0$, implying that p' is positive so that the population is increasing. Thus from $(A, f(A))$ we have drawn a horizontal arrow pointing to the right, or pointing toward increasing values of p. We do not see the shape of the solution from the phase plane, but we can see the qualitative character that the solution is increasing when the population size is A.

Next examine the direction field and the solution that passes through the point, $(0, B)$, for which

$$M < B = p(0) \qquad \text{Initial population exceeds the carrying capacity.}$$

The graph will decrease from $(0, B)$ and also will be asymptotic[5] to the line $y = M$.

The point, $(B, f(B))$, is shown on the phase plane. We can see from that point that $f(B) < 0$, implying that p' is negative so that the population is decreasing. Thus from $(B, f(B))$ we have drawn a horizontal arrow pointing to the left, or pointing toward decreasing values of p.

Stability of solutions. Two important properties of the solutions $p = 0$ and $p = M$ of $p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right)$ are apparent from the direction field in Figure 16.8. For a solution that is near the line $p = M$, as time progresses the short tangent segments guide that solution toward that line. Every solution near $p = M$ is asymptotic to $p = M$ with increasing time, and the solution $p = M$ is said to be **asymptotically stable**. If, for example, the population is at the carrying capacity and random environmental effects cause the population to either increase or decrease from carrying capacity, the population size will move back toward the carrying capacity as time increases.

On the other hand, the arrows near the equilibrium solution $p = 0$ will be guided away from $p = 0$ as time progresses. There are solutions near the line $p = 0$ that move away from the line as time increases, and $p = 0$ is said to be a **nonstable** solution. If you introduce a single rabbit (perhaps two!) into an environment with out rabbits (Australia, for example), then the rabbit population will soon expand to carrying capacity (the asymptotically stable solution).

Stability of constant solutions can also be determined from the phase plane. At points near the equilibrium points in Figure 16.8 a horizontal arrow is drawn pointing to the right at $(p, f(p))$ if $f(p) > 0$ and pointing to the left if $f(p) < 0$. The lengths of the arrows reflect the magnitude of $f(p)$. Near $(M, 0)$ the arrows all point toward M indicating the M is asymptotically stable, and near (0,0) the arrows all point away from 0 indicating that 0 is not stable.

Example 16.4.1 *An extreme example.* Shown in Figure 16.9 are the direction field and the phase plane graph of

$$y' = (y - 1) \times (y - 2) \times (y - 3) \times (y - 4)$$

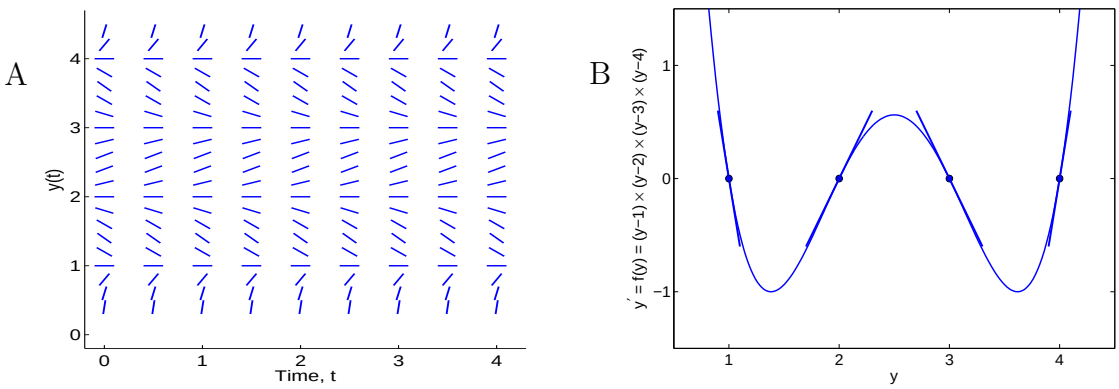


Figure 16.9: A. Direction field for the $y' = (y - 1) \times (y - 2) \times (y - 3) \times (y - 4)$. B. Phase plane graph of the same equation.

It is easy to solve $f(y) = (y - 1)\,(y - 2)\,(y - 3)\,(y - 4) = 0$ and see that $y = 1$, $y = 2$, $y = 3$ and $y = 4$ are equilibrium solutions to the differential equation, and equivalently that 1, 2, 3 and 4 are

[5]According to the model, if the current population exceeds the carrying capacity, M, then the population decreases to M. It is probably more realistic to suppose that if the current population exceeds the maximum supportable population by very much, there will be a rather catastrophic immediate decrease to a level less than M (hopefully still positive), followed by a gradual increase to M.

equilibrium points. Solutions, $y(t)$, with $y(0)$ close to 1 will be asymptotic to $y = 1$ and solutions with $y(0)$ close to 3 will be asymptotic to $y = 3$. Also, solutions starting close to 2 will not be asymptotic to $y = 2$ and solutions starting close to 4 will not be asymptotic to $y = 4$. ■

Explore 16.4.1 What qualitative character of the graph of

$$\mathbf{f(y) = (y-1) \times (y-2) \times (y-3) \times (y-4)}$$

distinguishes the stable equilibrium points 1 and 3 from the equilibrium points 2 and 4? Hint: Examine $\mathbf{f'}$. ■

Definition 16.4.2 Asymptotically stable equilibrium point. If $f(y)$ and $f'(y)$ are continuous, an equilibrium point $\overline{y}$ of

$$y' = f(y)$$

is an *asymptotically stable equilibrium point* means that there is an interval (a, b) containing $\overline{y}$, and every solution, $y(t)$, to $y' = f(y)$ for which $a < y(0) < b$ is asymptotic to the horizontal line $y = \overline{y}$.

We hope you guessed the following theorem from the Extreme Example 16.4.1.

Theorem 16.4.1 If $f(y)$ and $f'(y)$ are continuous, an equilibrium point $\overline{y}$ of

$$y' = f(y) \qquad \text{is asymptotically stable if} \qquad f'(\overline{y}) \qquad \text{is negative.}$$

Proof: **This proof is technical and should be omitted on first reading.**

Suppose the hypothesis of the theorem and let $-m = f'(\overline{y}) < 0$ and for convenience suppose $\overline{y} = 0$. There is $a > 0$ such that if $-a \le y \le a$, then $-2m \le f'(y) \le -m/2$. Suppose $0 < y_0 < a$ and $y(t)$ is the solution to $y(0) = y_0$, $y'(t) = f(y)$. The argument for $-a < y_0 < 0$ is similar. We will show that $e^{-2mt} \le y(t) \le e^{-(m/2)t}$.

Suppose $t_0 > 0$ and $0 \le y(t) \le a$ for $0 \le t \le t_0$. Then

$$\text{for} \qquad 0 \le t \le t_0, \quad y'(t) = f(y(t)) = f(y(t)) - f(0) = f'(c_t) \times y(t)$$

where $0 \le c_t \le y(t) \le a$. Then for $0 \le t \le t_0$

$$\begin{array}{rcccl}
-2m & \le & f'(c_t) & \le & -m/2 \\
-2m \times y(t) & \le & f'(c_t)y(t) & \le & -m/2y(t) \\
-2m & \le & \dfrac{y'(t)}{y(t)} & \le & -m/2 \\
\displaystyle\int_0^{t_0} -2m\,dt & \le & \displaystyle\int_0^{t_0} \frac{y'(t)}{y(t)}\,dt & \le & \displaystyle\int_0^{t_0} -m/2\,dt \\
-2mt_0 & \le & \ln y(t_0) - \ln y_0 & \le & -(m/2)t_0 \\
y_0 e^{-2mt_0} & \le & y(t) & \le & y_0 e^{-(m/2)t_0}
\end{array}$$

We have shown that if $0 \le y(t) \le a$ on $0 \le t \le t_0$ then $y_0 e^{-2mt_0} \le y(t) \le y_0 e^{-(m/2)t_0}$. Suppose for some t_1 either $y(t_1) \le 0$ or $a \le y(t_1)$. Then there is a least such number and let t_0 denote that least number. Then, $0 < y_0 e^{-2mt_0} \le y(t_0) \le y_0 e^{-(m/2)t_0} < a$, which is a contradiction. There is no such number t_1.
End of Proof.

In case $f'(\overline{y}) > 0$ then $\overline{y}$ is not stable, and if $f'(\overline{y}) = 0$, $\overline{y}$ may be stable and it may not be stable.

16.4.1 Parameter Reduction.

Example 16.4.2 Parameter Reduction. It is customary to divide the logistic differential equation

$$p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right)$$

by M to obtain

$$\frac{p'(t)}{M} = r \times \frac{p(t)}{M} \times \left(1 - \frac{p(t)}{M}\right)$$

Then let

$$u(t) = \frac{p(t)}{M} \qquad \text{and note that} \qquad u'(t) = \frac{p'(t)}{M}$$

to obtain

$$u'(t) = r \times u(t) \times (1 - u(t)) \qquad \text{Fraction Logistic} \tag{16.9}$$

u is the fraction of the carrying capacity, M, used by the population. Because u is the ratio of p to M, both of which have units of population numbers, u is dimensionless.

If we then rescale time by $\tau = r \times t$ and let $v(\tau) = u(t)$, then

$$u'(t) = \frac{d}{dt}u(t) = \frac{d}{dt}v(\tau) = \frac{d}{d\tau}v(\tau)\frac{d\tau}{dt} = v'(\tau) \times r. \tag{16.10}$$

We can substitute $u(t) = v(\tau)$ and $u'(t) = v'(\tau) \times r$ into Equation 16.9 and obtain

$$v'(\tau) \times r = r\, v(\tau)(1 - v(\tau)), \qquad v'(\tau) = v(\tau)(1 - v(\tau)). \tag{16.11}$$

The qualitative characteristics of the original logistic equation, $p' = rp(1 - p/M)$ are the same as that of $v'(\tau) = v(\tau)(1 - v(\tau))$, which has no parameters. ■

Exercises for Section 16.4, Phase planes and stability of constant solutions to $y' = f(y)$.

Exercise 16.4.1 In discussion of release of chemical into a lake in Section 16.1.5 we obtained the differential equation

$$P'(t) = T - \frac{P(t)}{V} \times R$$

What is the equilibrium amount of chemical in the lake? Is the equilibrium stable?

Exercise 16.4.2 Find the equilibrium points and for each determine whether or not it is stable.

a.	y'	$=$	$y - 1$	b.	y'	$=$	$-y + 1$
c.	y'	$=$	$y^2 - 1$	d.	y'	$=$	$1 - y^2$
e.	y'	$=$	$e^{-y} - 1$	f.	y'	$=$	$e^y - 1$
g.	y'	$=$	$\sin y$	h.	y'	$=$	$-y + y^2$
i.	y'	$=$	$-y^3$	j.	y'	$=$	y^3

For parts i. and j. draw the phase plane with arrows to determine the question of stability.

Exercise 16.4.3 Haldane's equation for nitrogen partial pressure, $N(t)$, in a given tissue of volume V in a scuba diver is

$$[V \times N(t)]' = \quad k \quad \times \left(0.8 + 0.8 \times \frac{d(t)}{10} - N(t)\right) \tag{16.12}$$

where $d(t)$ is measured in meters. See Exercise 5.5.24, Decompression illness in deep water divers. Suppose a diver descends to a depth, 30 meters, and stays at that depth. Water pressure at 30 meters is approximately $30/10 = 3$ atmospheres. Then Haldane's equation is

$$N'(t) = \quad \frac{k}{V} \quad \times \left(0.8 + 0.8 \times \frac{30}{10} - N(t)\right)$$

What is the equilbrium nitrogen partial pressure for the tissue?

Exercise 16.4.4 *Continuous infusion of penicillin.* Suppose a patient recovering from surgery is to be administered penicillin intravenously at a constant rate of 5 grams per hour. The patient's kidneys will remove penicillin at a rate proportional to the serum concentration of penicillin. Let $P(t)$ be the concentration of penicillin t hours after infusion is begun. Then a simple model of penicillin pharmacokinetics is

$$\begin{array}{ccccc} \text{Net Rate of Increase} & & \text{Clearance} & & \text{Infusion} \\ P'(t) & = & -K \times P(t) & + & 5 \\ \frac{\text{gm}}{\text{hr}} & & \frac{1}{\text{hr}} \times \text{gm} & & \frac{\text{gm}}{\text{hr}} \end{array}$$

The proportionality constant, K, must have units $\frac{1}{\text{hr}}$ in order for the units on the equation to balance. We initially assume that $K = 2.5\frac{1}{\text{hr}}$ which is in the range of physiological reality. It is reasonable to assume that there was no penicillin in the patient at time $t = 0$, so that $P(0) = 0$.

a. Draw the phase plane for the differential equation

$$P(0) = 0 \qquad P'(t) = -2.5P(t) + 5$$

b. Find the equilibrium point of $P' = -2.5P + 5$.

c. Is the equilibrium point stable?

d. Show that the units of the equilibrium point are grams.

e. Suppose the patient's kidneys are impaired and only operating at 60% of normal. Then $K = 1.5$ instead of 2.5. What effect does this have on the equilibrium point.

Note: Because of the uncertainty of the kidney clearance rate, it is not common to administer penicillin as a continuous infusion.

Exercise 16.4.5 Draw the phase plane or the direction field for $y' = -y^2$ and decide whether 0 is a stable equilibrium point. What relevance is Theorem 16.4.1 to this?

Exercise 16.4.6 Suppose $y' = f(y)$ has three and only three equilibrium points, e_1, e_2, and e_3, where f and f' are continuous and $f'(e_1) \neq 0$, $f'(e_2) \neq 0$, and $f'(e_3) \neq 0$. Argue that one of e_1, e_2, and e_3 is stable. Hint: Draw a potential phase plane (graph of f) in which e_1 and e_3 are unstable. Argue that e_2 must be between e_1 and e_3 and must be stable.

Exercise 16.4.7 Consider a modification of the Lotka-Volterra equations for competition between two species in which $\alpha_{1,2} = 0$.

$$\begin{aligned} p_1'(t) &= r_1 \times p_1(t) \times \left(1 - \frac{p_1(t) + 0 \times p_2(t)}{M_1}\right) = r_1 \times p_1(t) \times \left(1 - \frac{p_1(t)}{M_1}\right) \\ p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{p_2(t) + \alpha_{2,1} p_1(t)}{M_2}\right) \end{aligned}$$

Thus population 1 is not affected by population 2 but population 2 is affected by population 1. Suppose that

$$0 < p_1(0) < M_1 \qquad \text{and} \qquad 0 < p_2(0) < M_2$$

a. Argue that $p_1(t) \to M_1$ as $t \to \infty$.

b. Argue that

$$\text{if} \qquad M_2 < \alpha_{2,1} M_1 \qquad \text{then} \qquad p_2(t) \to 0 \qquad \text{as} \qquad t \to \infty.$$

For ease, assume that there is a number t_0 and $p_1(t) = M_1$ for $t \geq t_0$. Also it is true that: If $z(t)$ satisfies $z'(t) < -kz(t)$ for $z(0) > 0$ then $z(t) \to 0$ as $t \to \infty$.

c. What happens to the second population if $M_2 > \alpha_{2,1} M_1$?

Exercise 16.4.8 a. Show that Ricker's equation,

$$p'(t) \quad = \quad \alpha\, p\, e^{-p/\beta} - \gamma\, p,$$

is equivalent to

$$v'(\tau) \quad = \quad v\, e^{-v} - \gamma_0\, v \tag{16.13}$$

with the substitutions, $u(t) = p(t)/\beta$, $\tau = \alpha\, t$, and $\gamma_0 = \gamma/\alpha$.
Note: Recall Equation 16.10 where $\tau = r \times t$ and $v(\tau) = u(t)$:

$$u'(t) = \frac{d}{dt} u(t) = \frac{d}{dt} v(\tau) = \frac{d}{d\tau} v(\tau) \frac{d\tau}{dt} = v'(\tau) \times r.$$

b. Show that the Beverton-Holt equation,

$$p'(t) \quad = \quad \frac{r \times p}{1 + p/\beta},$$

is equivalent to

$$v'(\tau) \quad = \quad \frac{v}{1+v} \tag{16.14}$$

with proper substitutions.

c. Show that the Gompertz equation,

$$p'(t) \quad = \quad -r\, \ln \frac{p}{\beta},$$

with proper substitutions, is equivalent to an equation with no parameters.

Exercise 16.4.9 Identify the stable and nonstable solutions of

$$u'(t) = u(t)\,(1 - u(t))$$

Exercise 16.4.10 Suppose a marine fish population when not subject to harvest is reasonably modeled by

$$u'(t) = r \times u(t) \times (1 - u(t))$$

with time measured in years. Suppose a harvest procedure is initiated, and that a fraction, h, of the *existing population* is harvested every year. The harvest is not a fixed amount each year, but depends on the number of fish available.

The growth rate will be the difference between the natural birth-death process and the harvest and may be modeled by

$$u'(t) = r \times u(t) \times (1 - u(t)) - h \times u(t) \tag{16.15}$$

a. Assume $h = r$ (the harvest rate equals the low density growth rate) Substitute $h = r$ in Equation 16.15, and simplify.

   Show that

$$u(t) = \frac{1}{rt + 1/u_0} \qquad \text{where} \qquad u(0) = u_0$$

   is a solution for this model. What will be the eventual annual fish harvest under this harvest strategy?

b. Assume $h = \frac{3}{4}r$ in Equation 16.15 and simplify.

   Draw a direction field or phase plane for this model. What will be the eventual annual fish harvest under this harvest strategy?

c. Assume $h = \frac{1}{2}r$ in Equation 16.15, and simplify.

   Draw a direction field for this model. What will be the eventual annual fish harvest under this harvest strategy?

d. Which of the three strategies will provide the largest long term harvest?

Exercise 16.4.11 Some population scientists have argued that population density can get so low that reproduction will be less than natural attrition and the total population will be lost. Named the Allee effect, after W. C. Allee who wrote extensively about it[6], this may be a basis for arguing, for example, that marine fishing of a certain species should be suspended, despite the presence of a small residual population.

How should we modify the logistic differential equation, $u' = u\,(1 - u)$, to incorporate such a threshold? Assume a fixed area and uniform density throughout the area and a threshold density, ϵ. If the population number is less than ϵ the population will decline; if the population number is more than ϵ the population will increase.

a. Modify the direction field for $u' = u\,(1 - u)$ to account for the Allee effect. That is, a u, t plane, draw the line $u = 1$ and a threshold line $u = \epsilon$ where $\epsilon = 0.1$, say. Arrows below $u = \epsilon$ should point downward; arrows between $u = \epsilon$ and $u = 1$ should point upwards. Draw enough direction field arrows to indicate the paths of solutions for a threshold model.

[6] Allee, W.C. 1938 The Social Life of Animals. Norton, New York.

b. Draw the the logistic phase plane graph and the phase plane graphs for the following three candidates of a threshold logistic differential equation where $\epsilon = 0.1$.

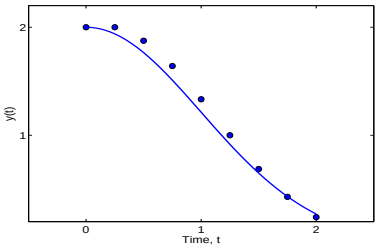
$$\begin{array}{rclcll} u' & = & f(u) & = & u \times (1-u) & \text{Logistic} \\ u' & = & f_1(u) & = & u^{\frac{2}{3}} \times (u-\epsilon)^{\frac{1}{3}} \times (1-u) & \text{Candidate 1} \\ u' & = & f_2(U) & = & u \times \frac{u-\epsilon}{u+\epsilon} \times (1-u) & \text{Candidate 2} \\ u' & = & f_3(u) & = & u \times (u-\epsilon) \times (1-u) & \text{Candidate 3} \end{array}$$

On first introduction, f_1 and f_2 look unbelievably complex. For $\epsilon \ll u$ $f_1(u) \doteq u(1-u)$ and $f_2(u) \doteq u(1-u)$, thus approximating the logistic $f(u) = u(1-u)$ when well above the threshold of disaster, ϵ. Argue that for each of f_1, f_2, and f_3, , $u = 0$ and $u = 1$ are stable solutions and and $u = \epsilon$ is an unstable solution.

We will find the f_1 has some technical problems that makes it unattractive but not absolutely impossible to use. f_2 and f_3 have an advantage in that we can analytically solve for u-inverse in the corresponding equations. A shortcoming of all three of f_1, f_2 and f_3 is that they are chosen only to match a pattern and do not derive from a fundamental hypothesis about the process of population dynamics near the threshold.

16.5 Numerical approximations to solutions to differential equations.

Direction fields enable step by step construction of approximate solutions to



$$y(a) = y_a \qquad y'(t) = f(t, y)$$

$$a \le t \le b$$

Let $n > 0$ be an integer, and $h = \frac{b-a}{n}$.

$$\text{Let} \qquad \begin{array}{rcll} t_k & = & a + k * h & k = 0, 1, \cdots, n \\ y_0 & = & y_a & \\ y_{k+1} & = & y_k + h * f(t_k, y_k) & k = 0, \cdots, n-1 \end{array}$$

The points

$$(t_0, y_0) \qquad (t_1, y_1) \qquad (t_2, y_2) \qquad \cdots \qquad (t_n, y_n)$$

approximate the graph of y.

Example 16.5.1 We compute points to approximate the solution to

$$v(0) = 0.2 \qquad v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t) \qquad 0 \le t \le 10$$

This is Ricker's model of fish populations with parameter reduction, Equation 16.13. First we divide the time axis [0, 10] into intervals of length 2 and let

$$t_0 = 0 \qquad t_1 = 2 \qquad t_2 = 4 \qquad t_3 = 6 \qquad t_4 = 8 \qquad \text{and} \qquad t_5 = 10$$

Our objective is to compute v_0, v_1, $\cdots$, and v_5 so that the points (t_0, v_0), (t_1, v_1), $\cdots$, and (t_5, v_5) will lie close to the graph of the solution, $v(t)$.
Step 0. let $v_0 = 0.2$. Then (t_0, v_0) is a point of the graph of the solution.
Step 1. From the differential equation, the slope of the solution at (0, 0.2) is

$$\begin{aligned} v'(0) &= v(0) \times e^{-v(0)} - 0.1 \times v(0) \\ &= 0.2e^{-0.2} - 0.1 * 0.2 \\ &= 0.1437 \end{aligned}$$

We construct the interval between $t_0 = 0$ and $t_1 = 2$ that starts at (0, 0.2) and has slope 0.1437. See Figure 16.10A. The right end point is at $t_1 = 2$ and the ordinate is

$$v_1 = 0.2 + 2 \times 0.1437 = 0.4874.$$

Figure 16.10A. **Pattern:** Note that $0.1437 = v_0 \times e^{-v_0} - 0.1 * v_0$ so that

$$v_1 = v_0 + 2 \times (v_0 \times e^{-v_0} - 0.1 \times v_0)$$

Step 2. The slope of the direction field at $(t_1, v_1) = (2, 0.4874)$ is

$$v_1 \times e^{-v_1} - 0.1 \times v_1 = 0.4874 \times e^{-0.4874} - 0.1 * 0.4874 = 0.2506$$

We construct the interval between $t_1 = 2$ and $t_2 = 4$ that starts at (2, 0.4874) and has slope 0.2507. Figure 16.10B.

The right end point is at $t_2 = 4$ and

$$v_2 = 0.4876 + 2 \times 0.2506 = 0.9888$$

Pattern: Note that $0.2506 = v_1 \times e^{-v_1} - 0.1 \times v_1$ so that

$$v_2 = v_1 + 2 \times (v_1 \times e^{-v_1} - 0.1 \times v_1)$$

Step 3. We compute the slope of the direction field at (4,0.9888) and construct the interval with that slope between $t_2 = 4$ and $t_3 = 6$ that starts at (4, 0.9888). Following the previous two patterns, we use

$$v_3 = v_2 + 2 \times (v_2 \times e^{-v_2} - 0.1v_2) = 0.9888 + 2 \times (0.9888 \times e^{-0.9888} - 0.1 \times 0.9888) = 1.5268$$

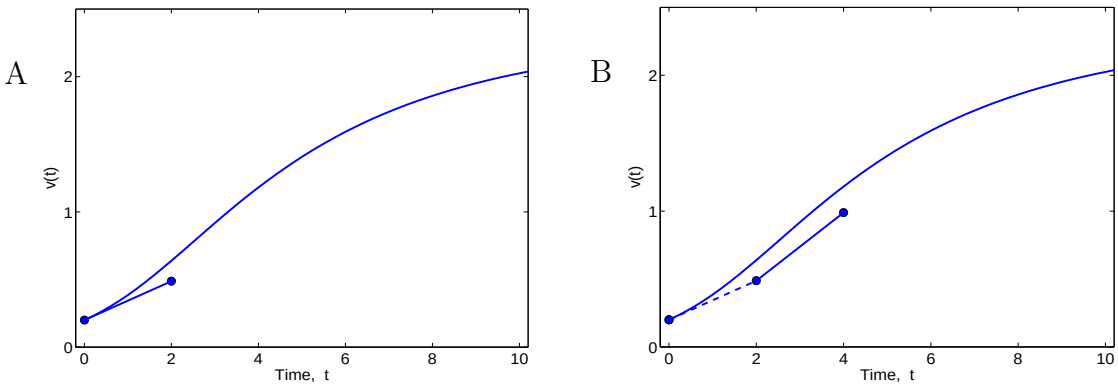


Figure 16.10: A. The first interval of Euler approximation to the solution of $v(0) = 0.2$, $v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t)$ on $[0, 10]$ using 5 intervals. B. The second interval.

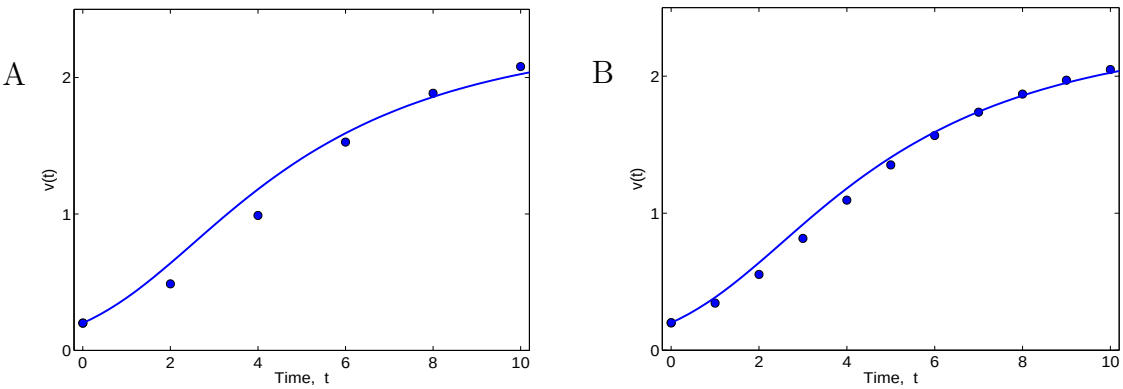


Figure 16.11: Euler approximations to the solution of $v(0) = 0.2$, $v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t)$ on $[0, 10]$, A. using 5 intervals in [0,10] and B. using 10 intervals in [0,10].

Step 4. We compute

$$v_4 = v_3 + 2(\times v_3 \times e^{-v_3} - 0.1 \times v_3) = 1.5268 + 2 \times (1.5268e^{-1.5268} - 0.1 \times 1.5268) = 1.8848$$

where (t_4, v_4) is the right endpoint of the fourth interval tracking the solution.

Step 5. The right endpoint of the fifth interval is (t_5, v_5) where

$$v_5 = v_4 + 2 \times (v_4 \times e^{-v_4} - 0.1 \times v_4) = 1.8848 + 2 \times (1.8848 \times e^{-1.8848} - 0.1 \times 1.8848) = 2.0803$$

The graph of the solution and the points that we have computed are shown in Figure 16.11A. The points are perhaps close enough for some purposes, but we will find it easy to compute a closer fit.

The general pattern is that, for time interval size h,

$$v_{k+1} = v_k + h \times \text{slope}_k$$

and for $v' = v \times e^{-v} - 0.1 \times v$ the specific pattern is

$$v_{k+1} = v_k + h \times (\left(v_k \times e^{-v_k} - 0.1 \times v_k\right)$$

For $h = 2$, these numbers are easily computed on your calculator with

Keystroke	Keystroke	Display
0.2	ENTER	0.20000
ANS + 2 × (ANS × e ^ (⁻ ANS) - 0.1 × ANS)	ENTER	0.4875
	ENTER	0.9888
	ENTER	1.5267
	ENTER	1.8847
	ENTER	2.0803

These numbers differ slightly from the previous numbers because the calculator is accurate to 11 or 12 digits, even if you do not want them.

We can easily improve the accuracy of our approximation by using a smaller time step size and a calculator.

For step size, $h = 1$, we use

0.1 ENTER ANS + 1 × (ANS × e ^ (⁻ ANS) - 0.1 × ANS)

and press ENTER 10 times. The results are shown in Figure 16.11B.

Explore 16.5.1 Compute approximations to

$$v(0) = 0.2 \qquad v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t) \qquad 0 \le t \le 10$$

using 40 intervals of length 0.25.

Table 16.1: Euler computations for $y(0) = 2$, $y'(t) = t - y(t)$ with interval size $h = 1$.

Step, k	t_k	y_k	$\text{slope}_k = t_k - y_k$	$y_{k+1} = y_k + h * \text{slope}_k$
0	0	2	0 - 2 = -2	$2 + 1 \times (-2) = 0$
1	1	0	1 - 0 = 1	$0 + 1 \times 1 = 1$
2	2	1	2 - 1 = 1	$1 + 1 \times 1 = 2$
3	3	2	3 - 2 = 1	$2 + 1 \times 1 = 3$
4	4	3		

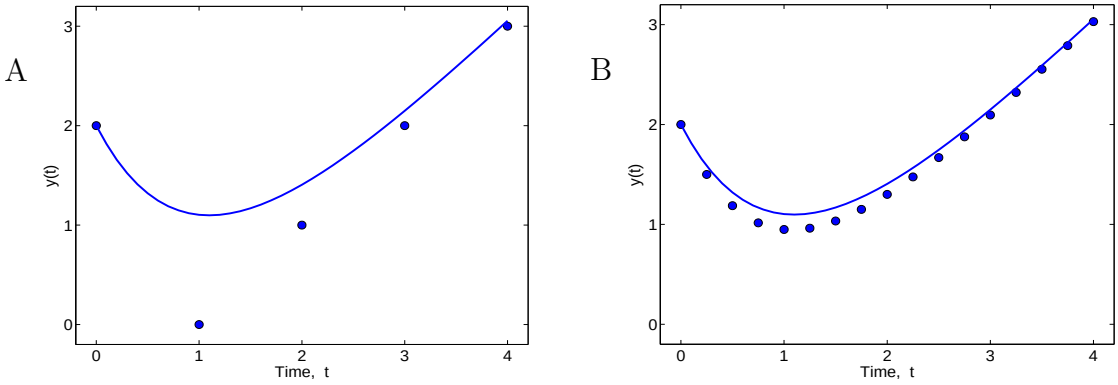


Figure 16.12: Euler approximations to the solution of $y(0) = 0.2$, $y'(t) = t - y(t)$ on $[0, 4]$ using 4 intervals in A and using 16 intervals in B.

Example 16.5.2 We find an approximate solution to

$$y(0) = 2 \qquad y'(t) = t - y$$

for $0 \le t \le 4$; the graph of the solution is shown in Figure 16.12A. Both t and y appear in the RHS in this problem.

The basic pattern is the same.

$$y_0 = y(0) = 2 \qquad y_{k+1} = y_k + h \times \text{slope}_k$$

The computations are organized in Table 16.1 for time-interval size $h = 1$.

Our approximation is shown in Figure 16.12A and is not close enough to the solution to satisfy us. The approximation computed using time-interval size $h = 0.25$ is shown in Figure 16.12B and it is more acceptable. The initial and final computations for $h = 0.25$ are shown in Table 16.2

Because the RHS of $y'(t) = t - y$ involves both t and y, these numbers are not computed on a calculator using only ANS, the previous answer key. A calculator program that will do the computations is included in Table 16.3.

Euler's Method. The scheme we have been using is called Euler's Method after Leonhard Euler (1707-1783) who introduced the method in 1768. The algorithm for solving

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \le t \le b$$

is

Table 16.2: Euler computations for $y(0) = 2$, $y'(t) = t - y(t)$ on $0 \le t \le 4$ for $h = 0.25$.

Step, k	t_k	y_k	$\text{slope}_k = t_k - y_k$	$y_{k+1} = y_k + h * \text{slope}_k$
0	0.00	2.0000	-2.0000	1.5000
1	0.25	1.5000	-1.2500	1.1875
2	0.50	1.1875	-0.6875	1.0156
3	0.75	1.0156	-0.2656	0.9492
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
15	3.75	2.7901	0.9599	3.0301
16	4.00	3.0301		

Table 16.3: TI-86 calculator program, EU, to compute Euler approximation to $y(0) = 2$, $y'(t) = t - y(t)$ on $0 \le t \le 4$ for $h = 0.25$. Calculator program, TRAP, to compute trapezoid approximation to $y(0) = 2$, $y'(t) = -t * y(t)$ on $0 \le t \le 2$ with $h = 0.2$.

```
PROGRAM:EU
:Fix 3
:.25 -> H
:{17,2}->dim E
:0->E(1,1)
:2->E(1,2)
:For(K,1,16)
:H*K->E(K+1,1)
:SL=E(K,1)-E(K,2)
:E(K,2)+H*SL->E(K+1,2)
:End
:Pause  E
:Fix 9
```

```
PROGRAM:TRAP
:Fix 3
:0.2 -> H
:{11,2}->dim T
:0->E(1,1)
:2->E(1,2)
:For(K,1,10)
:H*K->E(K,1)
:SL1=-E(K,1)*E(K,2)
:SL2=-(E(K,1)+H)*(E(K,2)+H*SL1)
:SL = (SL1 + SL2)/2;
:E(K,2)+H*SL->E(K+1,2)
:End
:Pause E
:Fix 9
```

1. Choose n a number of intervals and let $h = \frac{b-a}{n}$.
2. Let $t_k = a + k * h$, for $k = 0, 1, 2, \cdots, n$ and let $y_0 = y_a$.
3. For $k = 0, 1, 2, \cdots, n-1$ let $y_{k+1} = y_k + h * f(t_k, y_k)$.

Euler's Method replaces a differential equation with a discrete difference equation. For example, for the logistic differential equation

$$y(a) = y_a \qquad y'(t) = r * y(t) * \left(1 - \frac{y(t)}{M}\right)$$

Euler's Method with interval size h yields

$$y_0 = y_a \qquad y_{k+1} = y_k + h * r * \left(1 - \frac{y_k}{M}\right)$$

This is the *discrete logistic equation* that we studied in Section 11.2 on page 478.

Satellites in orbit and space ships traveling in the solar system are subject to complex gravitational fields from the Earth, the Sun, the Moon, and other planets and their differential equations of motion are always solved numerically. Weather predictions are based on differential equations (called partial differential equations because their solution functions depend on three space variables and time). These equations are complex and involve a vast grid of space and time data points and can only be solved numerically.

16.5.1 The trapezoid and Runge-Kutta methods.

Euler's method is intuitive and instructional, but not very accurate. If you are traveling in a space ship to the Moon, you hope the folks at NASA Houston are using something better than Euler's Method to control your space ship, and there are numerous accurate methods for approximating solutions of differential equations. We present two additional methods that show one form of improvement.

Euler's Method estimates the slope of a solution at the left end point, t_k of the interval $[t_k, t_{k+1}]$ and assumes that slope is unchanged throughout the interval. This is similar to using rectangles to approximate an integral. For approximating an integral, we got better results using the trapezoid method which uses the average of the values of the integrand at the two end points of each interval in the partition. The trapezoid method for approximating a solution to a differential equation uses a similar procedure.

The *trapezoidal method* to approximate the solution to the differential equation

$$y(a) = y_a \qquad y'(t) = f(t, y)$$

is summarized by

1. Choose n a number of intervals and let $h = \frac{b-a}{n}$.
2. Let $t_k = a + k * h$ for $k = 0, 1, 2, \cdots, n$, and let $y_0 = y_a$.
3. For $k = 0, 1, 2, \cdots, n-1$, let
   (a) $\text{slope}_1 = f(t_k, y_k)$.

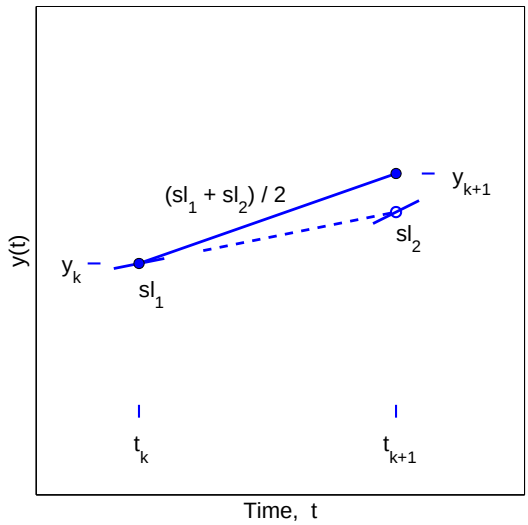


Figure 16.13: Graphic of the trapezoidal rule to compute y_{k+1} from y_k.

(b) $\text{slope}_2 = f(t_k + h, y_k + h \times \text{slope}_1)$.

(c) slope = $(\text{slope}_1 + \text{slope}_2)/2$.

(d) $y_{k+1} = y_k + h\times$ slope.

A graphic of the trapezoid procedure is shown in Figure 16.13. $\text{slope}_1 = \text{sl}_1$ is the usual Euler's Method slope. $\text{slope}_2 = \text{sl}_2$ is the direction field slope at the point

$$f(t_k + h, y_k + h \times \text{sl}_1),$$

the point projected by Euler's method. Then the slope from y_k to y_{k+1} is the average of sl_1 and sl_2.

Example 16.5.3 We use the trapezoid method to approximate the solution to

$$y(0) = 2 \qquad y'(t) = t - y \qquad 0 \le t \le 4$$

using $n = 16$ intervals and compare the results with those obtained with Euler's method.
Step 0. $t_k = 0 + k * 0.25$, for $k = 0, 1, \cdots, 15$ and $y_0 = 2$.
Step 1. $\text{slope}_1 = t_0 - y_0 = 0$ - 2 = -2
Euler's projected y_1 is $\hat{y}_1 = y_0 + h \times \text{slope}_1 = 2 + 0.25 \times (-2) = 1.5$.
Direction field slope at $(t_1, \hat{y}_1) = (0.25, 1.5)$ is $t_1 - \hat{y}_1 = 0.25$ - 1.5 = -1.25 = slope_2.
slope = $(\text{slope}_1 + \text{slope}_2)/2$ = (-2 + -1.25)/2 = -1.625.
Trapezoid projected $y_1 = y_0 + h\times$ slope.

$$y_1 = 2.0 + 0.25 \times (-1.625) = 1.5938$$

The first two steps are drawn in Figure 16.14A.
Step 2. $\text{slope}_1 = t_1 - y_1 = 0.25$ - 1.5938 = -1.3438

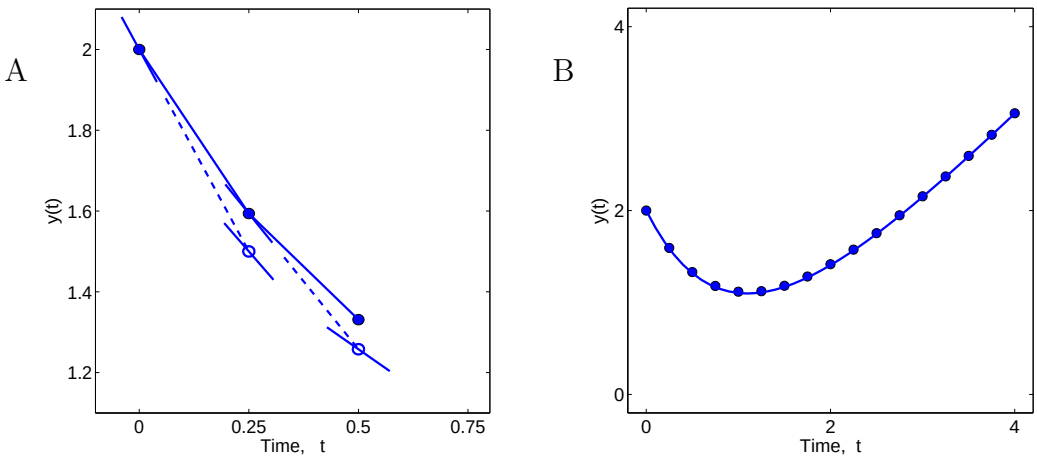


Figure 16.14: A. The first two steps of trapezoid approximations to the solution of $y(0) = 0.2$, $y'(t) = t - y(t)$ on $[0, 4]$ using 16 intervals. B. Graphs of both the trapezoid approximation and the solution. The two graphs are indistinguishable at this resolution.

Euler's projected y_2 is $\hat{y}_2 = y_1 + h \times \text{slope}_1 = 1.5938 + 0.25 \times (-1.3438) = 1.2579$.
Direction field slope at $(t_2, \hat{y}_2) = (0.50, 1.2579)$ is $t_2 - \hat{y}_2 = 0.50$ - $1.2579 =$ -0.7579 $= \text{slope}_2$.
slope $= (\text{slope}_1 + \text{slope}_2)/2$. Trapezoid projected $y_2 = y_1 + h\times$ slope

$$y_2 = 1.5938 + 0.25 \times \frac{(-1.3438) + (-0.7579)}{2} = 1.3311$$

The remaining 14 computations are similar and a graph of the trapezoid approximation solution and the actual solution is shown in Figure 16.14B. A table that compares the results of Euler's method and the trapezoid method is shown in Table 16.4..

It has to be acknowledged that the trapezoid method requires about twice as much arithmetic as that of Euler's method[7]. Therefore, we compare the accuracy of Euler's method using 32 intervals with the trapezoid method using 16 intervals.

It can be seen that with comparable arithmetic effort the trapezoid scheme gives closer approximations. The increased accuracy in this case may not seem impressive, but if you are trying to dock the space shuttle onto the space station, the improved accuracy may be appreciated.

The Runge-Kutta method. A popular and accurate extension of the trapezoid algorithm called the Runge-Kutta method is included for reference.

The Runge-Kutta algorithm to solve

$$y(a) = y_0, \qquad y'(t) = f(t, y) \qquad a \le t \le b$$

is
Choose a number n of intervals, let $h = (b - a)/n$ and $t_k = a + k \times h$, $k = 1, n$.

[7]The usual way of evaluating the required work is to count the number of times $f(t, y)$ has to be evaluated; once for each Euler step and twice for each trapezoid step.

Table 16.4: Comparison of Euler's method with 32 intervals and trapezoid approximation using 16 intervals to the solutions of $y(0) = 2$, $y'(t) = t - y(t)$.

Step, k			Time	Euler's	Trapezoid	Correct	Runge-
Euler	Trap	R-K	t	Method	Method	Solution	Kutta
0	0	0	0.00	2.0000	2.0000	2.0000	2.0000
2	1		0.25	1.5469	1.5938	1.5864	
4	2	1	0.50	1.2585	1.3311	1.3196	1.3203
6	3		0.75	1.0964	1.1805	1.1671	
8	4	2	1.00	1.0308	1.1176	1.1036	1.1045
16	8	4	2.00	1.3542	1.4163	1.4060	1.4067
24	12	6	3.00	2.1217	2.1551	2.1494	2.1497
32	16	8	4.00	3.0418	3.0578	3.0549	3.0551

$y_0 = y(a)$ is given. For $k = 0, n-1$ compute:

$$\begin{aligned} s_1 &= f(t_k, y_k) \\ s_2 &= f(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hs_1) \\ s_3 &= f(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hs_2) \\ s_4 &= f(t_k + h, y_k + hs_3) \\ \\ y_{k+1} &= y_k + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4) \end{aligned}$$

The Runge-Kutta method requires four functional evaluations per step, so to compare it with the Euler method and trapezoid method illustrated in Table 16.4 only 8 steps with $h = 0.5$ are used. The Runge-Kutta method has even greater accuracy than that of the trapezoid method with no more computational effort.

Exercises for Section 16.5, Numerical approximations to solutions of differential equations.

Exercise 16.5.1 The function $y(t) = e^{0.5t}$ solves

$$y(0) = 1 \qquad y'(t) = 0.5 \times y(t) \qquad \text{for} \qquad 0 \le t \le 4$$

is shown in Exercise Figure 16.5.1A.

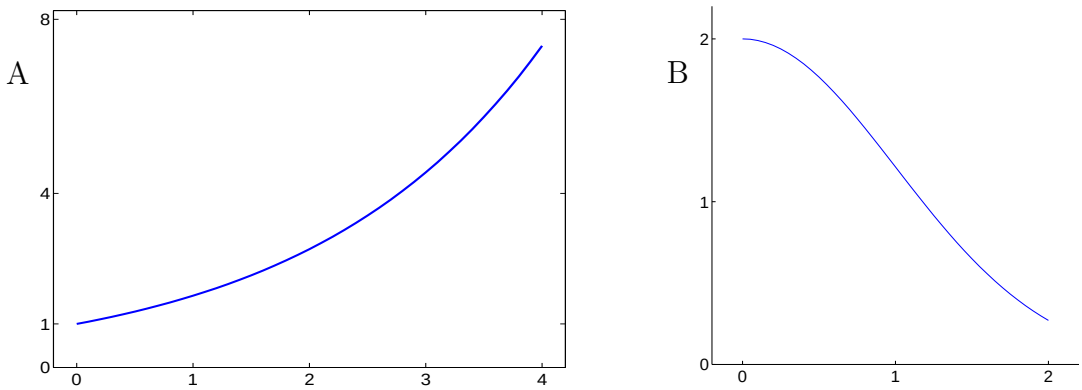
1. For time-interval size $h = 1$ compute Euler approximations to the solution, y, and compare them with the graph, $0 \le t \le 4$.

2. For time-interval size $h = 0.2$ compute

$$t_k = k \times h, \qquad k = 0, 1, \cdots, 20 \qquad y_0 = 1 \qquad y_{k+1} = y_k + h * 0.5 * y_k \qquad k = 0, 1, \cdots, 19$$

3. Plot the points, (t_5, y_5), (t_{10}, y_{10}), (t_{15}, y_{15}), and (t_{20}, y_{20}).

Figure for Exercise 16.5.1 A. Graph of $y = e^{0.5t}$ which is the solution to $y(0) = 1$, $y' = 0.5y$. B. Graph of $y(t) = 2e^{-t^2/2}$ which is the solution of $y(0) = 2$, $y'(t) = -t \times y$.



Exercise 16.5.2 Shown in Exercise Figure 16.5.1B is the graph of $y(t) = 2e^{-t^2/2}$ which is the solution of $y(0) = 2$, $y'(t) = -t \times y$. Use Euler's method and 10 intervals on $[0, 2]$ to approximate the solution to $y(0) = 2$, $y'(t) = -t \times y$.

Exercise 16.5.3 A differential equation with initial condition and its analytic solution are shown.

i. Show that the analytic solution satisfies the initial condition and the differential equation.

ii. Use Euler's method and the trapezoid methods to approximate the solution to the differential equation on the interval shown and using the step size shown. iii. Plot the solution and the Euler's and trapezoid approximations on a single t, y plane.

a.	$y(0)$	$=$	1	$y'(t)$	$=$	y^2	$y(t) = (1-t)^{-1}$	$0 \le t \le 0.4$	$h = 0.1$
b.	$y(0)$	$=$	2	$y'(t)$	$=$	$-y^2$	$y(t) = (t+0.5)^{-1}$	$0 \le t \le 0.4$	$h = 0.1$
c.	$y(0)$	$=$	1	$y'(t)$	$=$	$t \times y$	$y(t) = e^{t^2/2}$	$0 \le t \le 1$	$h = 0.2$
d.	$y(0)$	$=$	1	$y'(t)$	$=$	$\sqrt{y}$	$y = (t/2+1)^2$	$0 \le t \le 1$	$h = 0.2$

Exercise 16.5.4 Use the Euler and trapezoid methods to compute the solutions to the following differential equations with initial conditions on the intervals shown and using the step sizes shown.

a.	$y(0)$	$=$	4	$y'(t)$	$=$	$t - \sqrt{y}$	$0 \le t \le 1$	$h = 0.2$
b.	$y(0)$	$=$	0.5	$y'(t)$	$=$	$y/(1+y)$	$0 \le t \le 1$	$h = 0.1$
c.	$y(0)$	$=$	0.5	$y'(t)$	$=$	$-\ln y$	$0 \le t \le 1$	$h = 0.1$
d.	$y(0)$	$=$	0.15	$y'(t)$	$=$	$y(y-0.1)(1-y)$	$0 \le t \le 1$	$h = 0.4$
e.	$y(0)$	$=$	0	$y'(t)$	$=$	$y(y-0.1)(1-y)$	$0 \le t \le 1$	$h = 0.4$
f.	$y(0)$	$=$	0.05	$y'(t)$	$=$	$y(y-0.1)(1-y)$	$0 \le t \le 1$	$h = 0.4$

16.6 Synopsis.

At this time you should see that close approximations to the solution of many differential equations,

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \le t \le b,$$

can be computed with simple arithmetic, perhaps a lot of it. It is even better than that, for we can use the same ideas to solve two differential equations involving two unknown functions, u and v,

$$\begin{array}{rcl} u(a) & = & u_a \\ v(a) & = & v_a \end{array} \qquad \begin{array}{rcl} u'(t) & = & f(t, u, v) \\ v'(t) & = & g(t, u, v) \end{array} \qquad a \le t \le b$$

Recall the differential equations for penicillin clearance

$$\begin{array}{rcll} a'(t) & = & (-r_{1,0} - r_{2,1})a(t) + r_{1,2}b(t) & a(0) = 200 \\ b'(t) & = & r_{2,1}A_t - r_{1,2}B_t & b(0) = 0 \end{array}$$

Euler's method applied to these equations (using step size 5 minutes) simply returns us to the difference equations we first used to study penicillin clearance. We might also use a trapezoidal rule that would improve our accuracy.

We have not shown that as the interval size, h, gets close to zero Euler's approximations get close to the actual solution to the equation. For 'well behaved' differential equations, they do, as do the trapezoidal approximations. The analysis, however, is beyond our goal.

We will soon show a few of the numerous important analytic techniques for finding exact solutions to differential equations. Analytic formulas for solutions provide a lot more insight and intuition about the solutions than do tables of numbers, even graphs of tables of numbers. And a formula can be more easily carried to the next stage of a problem than can a table of numbers.

Ambiguity. We have ignored a serious problem that can arise. The differential equation with initial condition

$$y(0) = 0 \qquad y'(t) = 2\sqrt{y(t)} \tag{16.16}$$

has two solutions

$$y_1(t) = t^2 \qquad \text{and} \qquad y_2(t) = 0$$

Both $y_1(t) = t^2$ and $y_2(t) = 0$ solve $y(0) = 0$, $y'(t) = 2\sqrt{y(t)}$:

$$\begin{array}{rcl} y_1(t) & = & t^2 \\ y_1(0) & = & 0^2 = 0 \quad \textbf{Check} \\ y_1\prime(t) & = & 2t \\ & = & 2\sqrt{t^2} = 2\sqrt{y_1(t)} \\ y_1 & & \textbf{Solves} \end{array} \qquad \begin{array}{rcl} y_2(t) & = & 0 \\ y_2(0) & = & 0 \quad \textbf{Check.} \\ y_2'(t) & = & 0 \\ & = & 2\sqrt{0} = 2\sqrt{y_2(t)} \\ y_2 & & \textbf{Solves} \end{array}$$

Euler's method to approximate a solution to

$$y(0) = 0 \qquad y'(t) = 2\sqrt{y(t)}$$

yields

$$y_0 = 0, \qquad y_1 = y_0 + 2\sqrt{y_0} = 0 + 2\sqrt{0} = 0, \qquad y_2 = 0, \quad \cdots \quad y_n = 0$$

Euler's method gives no hint that $y(t) = t^2$ solves the equation. Almost surely, if your differential equation evolves from a real biological model, it will have unique solutions. Conversely, if your equation has multiple solutions, the model is probably not well formulated. It will be seen, however, that the model we derive for mold growth in Section 16.9.5 on page 776 comes perilously close to the example just presented.

Uniqueness. Mathematicians have derived conditions that will insure the uniqueness of solutions to

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \le t \le b$$

Recall that $f_2(t, y) = \frac{\partial}{\partial y} f(t, y)$ denotes the partial derivative of $f(t, y)$ with respect to the second variable (the first variable is held constant).

Theorem 16.6.1 Existence and Uniqueness of Solutions. *If $f(t, y)$ and $f_2(t, y) = \frac{\partial}{\partial y} f(t, y)$ are continuous on a rectangle*

$$a - d \le t \le a + d \qquad y_a - d \le y \le y_a + d \qquad d > 0,$$

then on an interval $a - e \le t \le a + e$, $0 < e < d$ there is a unique solution to

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \le t \le b$$

Under the hypothesis that f_2 is continuous on the rectangle, it follows that f_2 is bounded on the rectangle, and the proof of Theorem 16.6.1 hinges on this fact. In our example Equation 16.16

$$y(0) = 0 \qquad y'(t) = 2\sqrt{y(t)}$$

$$f(t, y) = 2\sqrt{y} \qquad \text{and} \qquad f_2(t, y) = \frac{1}{\sqrt{y}}$$

At the initial data point, $y(0) = 0$, $f_2(0, 0)$ is not even defined and $f_2(t, y) = \frac{1}{\sqrt{y}}$ is certainly not bounded in any rectangle containing the initial data point, $(0, 0)$. Equation 16.16 does not satisfy the hypothesis of Theorem 16.6.1.

Another type of exception may occur. The differential equation

$$y(0) = 0 \qquad y'(t) = 1 + y^2 \qquad 0 \le t \le 2$$

has a unique solution

$$y(t) = \tan t \qquad \text{Note:} \qquad y'(t) = \sec^2 t \qquad \sec^2 t = 1 + \tan^2 t$$

The problem is that the solution $y(t) = \tan t$ only extends on the interval $0 \le t < \frac{\pi}{2} \doteq 1.57$. The original problem asked for a solution on the interval [0, 2].

$$\text{For} \qquad y'(t) = 1 + y^2 \qquad f(t, y) = 1 + y^2 \qquad \text{and} \qquad f_y(t, y) = 2y$$

and both f and f_2 are continuous for all (t, y). The hypothesis of Theorem 16.6.1 is satisfied, but the theorem only guarantees a solution on some interval $a - e \le t \le a + e$ and not throughout $[a, b]$.

Despite the two previous examples, you should expect differential equations that you derive from a biological model to have a unique solutions valid over the reasonable life of your system. You should expect to be able to compute a good approximation to the solution, perhaps using a more sophisticated system than either Euler's method or the trapezoid method.

Exercises for Section 16.6 Synopsis.

Exercise 16.6.1 Draw the direction field for $y'(t) = \sqrt{y(t)}$ and decide whether the equilibrium solution $y(t) = 0$ is stable.

Exercise 16.6.2 Consider the pair of differential equations

$$\begin{array}{rcl} u(0) & = & 1 \\ v(0) & = & 2 \end{array} \qquad \begin{array}{rcl} u'(t) & = & 0.3 \times u(t) - 0.2 \times u(t) \times v(t) \\ v'(t) & = & 0.1 \times u(t) \times v(t) - 0.1 \times v(t) \end{array} \qquad 0 \le t \le 1$$

This system is a predator prey system. We (including you!) will use Euler's method to approximate a solution on the time interval [0,1] with $n = 5$ subintervals.
Step 0. $h = 0.2$. Let $t_0 = 0$, $u_0 = 1$, and $v_0 = 2$.
Step 1.

$$\begin{array}{rcl} \text{slope}_u & = & 0.3 \times u_0 - 0.2 \times u_0 \times v_0 \\ u_1 & = & u_0 + h \times \text{slope}_u \end{array} \qquad \begin{array}{rcl} \text{slope}_u & = & 0.3 \times 1 - 0.2 \times 1 \times 2 = -0.1 \\ u_1 & = & 1 + 0.2 \times (-0.1) = 0.8 \end{array}$$

$$\begin{array}{rcl} \text{slope}_v & = & 0.1 \times u_0 \times v_0 - 0.1 \times v_0 \\ v_1 & = & v_0 + h \times \text{slope}_v \end{array} \qquad \begin{array}{rcl} \text{slope}_v & = & 0.1 \times 1 \times 2 - 0.1 \times 2 = 0.0 \\ v_1 & = & 2 + 0.2 \times (0) = 2 \end{array}$$

Continue the computations of (u_2, v_2), (u_3, v_3), (u_4, v_4), and (u_5, v_5).

Exercise 16.6.3 Recall that in Problem 16.4.11 on page 733 we suggested

$$u' = u^{\frac{2}{3}} \times (u - 0.1)^{\frac{1}{3}} \times (1 - u)$$

as a possible model for a logistic population with a threshold density below which the population would be lost. Show that the hypothesis of the Existence and Uniqueness of Solutions Theorem 16.6.1 is not satisfied for this equation.

In this case there are multiple solutions that peel off of the equilbrium solution, $u = 0.1$ as t increases; in a sense, it is super unstable. You should have found that the hypothesis of Theorem 16.6.1 is not satisfied near $u = 0$. Solutions with small positive initial values quickly join with $u = 0$; in a sense, $u = 0$ is super stable.

16.7 First order linear differential equations.

Analytic solutions to differential equations are shown in this and subsequent sections. Here we solve **First Order Linear Differential Equations** which are of the form

$$y(a) = y_a, \qquad y'(t) + p(t) \times y(t) = q(t) \tag{16.17}$$

where p and q are continuous functions defined on an interval, $[a, b]$. The problem of finding $y(t)$ reduces to

Theorem 16.7.1

Step 1. Define $\qquad u(t) = \int_a^t p(s)\, ds.$

Step 2. Define $\qquad v(t) = \int_a^t e^{u(s)} \times q(s)\, ds.$

Then the solution to Equation 16.17 is

$$y(t) = v(t)e^{-u(t)} + y_a e^{-u(t)} \tag{16.18}$$

Example 16.7.1 Consider a case in which $p(t) = 3$ and $q(t) = 5$ are constant. Solve

$$y(1) = 5 \qquad y'(t) + 3y(t) = 2, \qquad a = 1, \quad y_a = 5, \quad p(t) = 3, \qquad q(t) = 5$$

Define

$$\begin{aligned} u(t) &= \int_0^t 3\, dt = 3t \\ v(t) &= \int_0^t e^{3t} \times 2\, dt = \frac{2}{3}\left(e^{3t} - 1\right). \end{aligned}$$

Then

$$\begin{aligned} y(t) &= \frac{2}{3}\left(e^{3t} - 1\right) \times e^{-3t} + 5e^{-3t} \\ &= \frac{2}{3} + \frac{13}{3}e^{-3t} \end{aligned}$$

To check that $y(t)$ solves $y(0) = 5 \qquad y'(t) + 3y(t) = 5$ we compute

$$y(0) = \frac{2}{3} + \frac{13}{3}e^{-3\times 0} = \frac{2}{3} + \frac{13}{3} = 5. \qquad \textbf{Checks}.$$

Also

$$\begin{aligned} y'(t) + 3y(t) &= \left[\frac{2}{3} + \frac{13}{3}e^{-3t}\right]' + 3\left(\frac{2}{3} + \frac{13}{3}e^{-3t}\right) \\ &= 0 + \frac{13}{3}e^{-3t} \times (-3) + 2 + 13e^{-3t} \\ &= 2 \qquad\qquad \textbf{Checks}. \end{aligned}$$

Example 16.7.2 The formulas for $u(t)$ and $v(t)$ are explicit, but the integrals may not be computable in terms of familiar functions. In the equation

$$y(0) = 5 \qquad y'(t) - 2ty(t) = 1,$$

$$p(t) = -2t, \qquad u(t) = \int_0^t -2s\,ds = -t^2, \qquad \text{and} \qquad v(t) = \int_0^t e^{-s^2}\,ds.$$

There is no formula for $v(t)$ in familiar terms[8] for $v(t)$. It can be numerically approximated as you did in Chapter 13 and is an important formula in statistics, but there is no expression for $\int_0^t e^{-s^2}\,ds$ in familiar terms. ■

That $y(t)$ in Equation 16.18 solves and is the only solution to the initial condition and equation of 16.17 is explored in Exercises 16.7.3 - 16.7.5

16.7.1 Rationale for Steps 1 and 2.

Steps 1 and 2 are presented as Twin Lightning Bolts Out Of The Blue to solve the equation

$$y'(t) + p(t) \times y(t) = q(t)$$

There is rationale leading to Steps 1 and 2 and the rationale is used in equations involving second and higher order derivatives.

First one should look at the special case of $q(t) = 0$

$$\text{LHS} \qquad y'(t) + p(t) \times y(t) = 0 \tag{16.19}$$

This is called a *homogeneous* linear equation and typically occurs when the system under study is isolated from external forces. q often represents environmental input.

$y = 0$ is a solution to the homogeneous equation. Because solutions are unique (Exercise 16.7.3), no other solution intersects the solution $y = 0$. Therefore every other solution is either always positive or always negative. For this discussion, we assume we are looking for a solution y_h (subscript h for 'homogeneous') that is always positive.

A procedure to solve Equation 16.19 is called *separation of variables*, meaning to write the equation in a form that has the terms involving y on one side and the terms involving t on the other side. We write (because $y_h > 0$)

$$y_h'(t) + p(t) \times y_h(t) = 0$$

[8]The *error function*, erf, is defined by $erf(x) = \frac{2}{\pi}\int_0^x e^{-t^2}\,dt$.

$$\begin{aligned} y_h'(t) &= -p(t) \times y_h(t) \\ \frac{y_h'(t)}{y_h(t)} &= -p(t) \end{aligned}$$

Now we search through the derivative formulas, and find

$$[\ln y(t)]' = \frac{y'(t)}{y(t)}.$$

Using the Fundamental Theorem of Calculus, (again, $y_h > 0$)

$$[\ln y_h(t)]' = -p(t) = -\left[\int_a^t p(s)\,ds\right]'$$

By the Parallel Graph Theorem 14.3.2,

$$\begin{aligned} \ln y_h(t) &= -\int_a^t p(s)\,ds + C_1 \\ y_h(t) &= e^{-\int_a^t p(s)\,ds + C_1} \\ y_h(t) &= C\,e^{-\int_a^t p(s)\,ds} \end{aligned}$$

Therefore, **Step 1.** Define $u(t) = \int_a^t p(s)\,ds$.
Given $u(t)$, for any number C

$$y_h(t) = Ce^{-u(t)} \qquad \text{solves the } \textit{homogeneous} \text{ equation} \qquad y'(t) + p(t)y(t) = 0$$

Think of $y_h(t) = Ce^{-u(t)}$ as a 'toe hold' for finding a solution $y(t)$ to

$$y'(t) + p(t) \times y(t) = q(t) \qquad\qquad \text{Nonhomogeneous equation.}$$

y might be some variation of y_h. Included with several trials, one asks whether C in $y_h = Ce^{-u(t)}$ might be a variable, $v(t)$. That is, might a solution to the nonhomogeneous equation be of a form

$$y(t) = v(t) \times y_h(t)\ ? \tag{16.20}$$

You should guess that the answer is 'yes' is or we would not be writing about it.

Our goal is to find a function, v, so that

$$y(t) = v(t) \times e^{-u(t)} \qquad \text{solves} \qquad y'(t) + p(t) \times y(t) = q(t)$$

Remember that $[u(t)]' = p(t)$. By substitution,

$$\begin{aligned} \left[v(t) \times e^{-u(t)}\right]' + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \\ v'(t) \times e^{-u(t)} + v(t) \times \left[e^{-u(t)}\right]' + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \end{aligned}$$

$$\begin{aligned}
v'(t) \times e^{-u(t)} + v(t) \times e^{-u(t)} \left[-u(t)\right]' + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \\
v'(t) \times e^{-u(t)} + v(t) \times e^{-u(t)} \left(-p(t)\right) + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \\
v'(t) \times e^{-u(t)} &= p(t) \\
v'(t) &= e^{u(t)} \times p(t)
\end{aligned}$$

Thus we arrive at **Step 2.** Define $v(t) = \int_a^t e^{u(s)}\, p(s)\, ds$.

The constant, C, in $y_h = Ce^{-u(t)}$ is sometimes called a parameter, and replacing C by the variable $v(t)$ is called *variation of parameter(s)*. Variation of parameters has wide use, but we will not employ it again. However, the method of separation of variables is used to find an analytic solution to the logistic equation and other autonomous equations in Section 16.8.

Example 16.7.3 Two integral formula examples. Steps 1 and 2 of Theorem 16.7.1 are easy to understand but require some integral formulas to compute the integrals. We will use the following two formulas to solve two problems:

$$\begin{aligned}
\int te^{at}\, dt &= \tfrac{1}{a} te^{at} - \tfrac{1}{a^2} e^{at} + C \\
\int e^{at} \sin bt\, dt &= \tfrac{a}{a^2+b^2} e^{at} \sin bt - \tfrac{b}{a^2+b^2} e^{at} \cos bt + C
\end{aligned} \tag{16.21}$$

Problem 1. Solve $y(1) = 1, \quad y'(t) + \frac{1}{t} y(t) = e^t$.

$$\begin{aligned}
a &= 1, \quad y_1 = 1, \quad p(t) = \frac{1}{t}, \quad q(t) = e^t \\
u(t) &= \int_0^t p(s)\, ds = \int_0^t \frac{1}{s}\, ds = \ln s\big|_{s=1}^t = \ln t - 0 = \ln t \\
v(t) &= \int_0^t e^{u(s)} q(s)\, ds = \int_0^t e^{\ln s} e^s\, ds = \int_0^t se^s\, ds = se^s - e^s\big|_1^t = te^t - e^t \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} = (te^t - e^t)e^{-\ln t} + 3e^{-\ln t} \\
&= e^t - \frac{1}{t} e^t + \frac{1}{t}
\end{aligned}$$

Problem 2. Solve $y(0) = 2, \quad y'(t) + 2y(t) = \sin 3t$.

$$\begin{aligned}
a &= 0, \quad y_0 = 2, \quad p(t) = 2, \quad q(t) = \sin 3t \\
u(t) &= \int_0^t p(s)\, ds = \int_0^t 2\, ds = 2s\big|_{s=0}^t = 2t - 0 = 2t \\
v(t) &= \int_0^t e^{u(s)} q(s)\, ds = \int_0^t e^{2s} \sin 3s\, ds \\
&= \frac{2}{4+9} e^{2s} \sin 3s - \frac{3}{4+9} e^{2s} \cos 3s \Big|_0^t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{13}e^{2t}\sin 3t - \frac{3}{13}e^{2t}\cos 3t - \left(\frac{2}{13}e^{0}\sin 0 - \frac{3}{13}e^{0}\cos 0\right) \\
&= \frac{4}{13}e^{2t}\sin 3t - \frac{3}{13}e^{2t}\cos 3t + \frac{3}{13} \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} \\
&= \left(\frac{2}{13}e^{2t}\sin 3t - \frac{3}{13}e^{2t}\cos 3t + \frac{3}{13}\right)e^{-2t} + 2e^{-2t} \\
&= \frac{2}{13}\sin 3t - \frac{3}{13}\cos 3t + \frac{3}{13}e^{-2t} + 2e^{-2t}
\end{aligned}
$$

∎

Exercises for Section 16.7, First order linear differential equations.

Exercise 16.7.1 Use Steps 1 and 2 in Theorem 16.7.1 and the integral formulas shown there to compute the solutions, $y(t)$, to the following differential equations. After parts a. - d. the explicit dependence of y on t is suppressed in the differential equations. You will need Equations 16.21.

a.	$y(0) = 2$	$y'(t) + 0.2y(t) = 0.1$		b.	$y(1) = 1$	$y'(t) + \frac{1}{t}y(t) = 0.1$	
c.	$y(0) = 3$	$y'(t) + 3y(t) = t$		d.	$y(0) = 0$	$y'(t) + y(t) = \sin t$	
e.	$y(1) = 1$	$y' + \frac{1}{t}y = 1$		f.	$y(0) = 3$	$y' + y = e^t$	
g.	$y(0) = 7$	$y' + 0.2y = e^{-0.2t}$		h.	$y(0) = 5$	$y' + 0.3y = e^{-0.2t}$	
i.	$y(0) = 2$	$y' + (\sin t)\, y = \sin t$		j.	$y(0) = 5$	$y' + 3y = \sin 4t$	

Exercise 16.7.2 Find the unique solutions to

a.	$y(0) = 5$	$y' + 2y = 0$		b.	$y(0) = 0$	$y' + 2y = 0$	
c.	$y(0) = 4$	$y' + 3y = t$		d.	$y(1) = 1$	$y' + 3y = t$	
e.	$y(0) = 0$	$y' + 0.2y = e^{-0.2t}$		f.	$y(0) = 3$	$y' + ty = t$	

Exercise 16.7.3 The hypothesis of Theorem 16.6.1 may be modified to assume that $f(t, y)$ and $f_2(t, y)$ are continuous on a rectangle, $a \le t \le a + d$ and $y_a - d \le y \le y_a + d$ ((a, y_a) is on the left edge of the rectangle instead of in the middle) and then conclude that there is a solution to $y(a) = y_a$, $y' = f(t, y)$ on an interval $[a, a + \epsilon]$.

Suppose $p(t)$ and $q(t)$ are continuous on an interval $[a, a + d]$ Write the Equation 16.17

$$y(a) = y_a, \qquad y'(t) + p(t) \times y(t) = q(t)$$

as

$$y(a) = y_a, \qquad y'(t) = q(t) - p(t) \times y(t) = f(t, y).$$

Show that $f(t, y)$ and $f_2(t, y)$ are continuous on $a \le t \le a + d$, $y_a - d \le y_a + d$ so that existence and uniqueness of a solution to Equation 16.17 on some interval $[a, a + \epsilon]$ is assured.

Exercise 16.7.4 Suppose $p(t)$ and $q(t)$ of Equation 16.17 are defined and continuous on an interval $[a, b]$. Show that there is a solution $y(t)$ in Equation 16.18 defined on all of $[a, b]$.

Exercise 16.7.5 Show that

$$y(t) = v(t)e^{-u(t)} + y_a e^{-u(t)}, \qquad \text{where} \qquad u(t) = \int_a^t p(t)\,dt, \qquad v(t) = \int_a^t e^{u(s)} \times q(s)\,ds$$

defined in Equation 16.18 satisfies the initial condition and the differential equation of 16.17

$$y(a) = y_a, \qquad y'(t) + p(t) \times y(t) = q(t).$$

a. The initial condition, $y(a) = y_a$ is a piece of cake. Compute $u(a)$ and $v(a)$ and $y(a)$.

b. Showing that $y(t)$ solves Equation 16.17 is messier. Complete:

$$\begin{aligned} y'(t) &= \left[v(t)e^{-u(t)} + y_a e^{-u(t)}\right]' \\ &= -v(t)e^{-u(t)}p(t) + q(t)e^{u(t)} \times e^{-u(t)} - y_a e^{-u(t)}p(t) \\ &= -y(t) \times p(t) + q(t) \end{aligned}$$

Exercise 16.7.6 For a SCUBA diver whose depth t minutes after the start of the dive is $d(t)$ meters, and the water pressure will be $1 + d(t)/10$ atmospheres. Air is 79 percent nitrogen. Haldane's equation for nitrogen partial pressure, $N(t)$, in a given tissue is (from Equation 16.12 with $K = k/V$)

$$N(0) = 0.79, \qquad N'(t) = K \times \left(0.79 \times (1 + \frac{d(t)}{10}) - N(t)\right). \tag{16.22}$$

This is a first order linear differential equation. See Exercise 5.5.24 and Haldane's Mathematical Model 5.5.9.

Your goal is to solve Equation 16.22

a. Rewrite Equation 16.22 to put it into the form, $N'(t) + p(t)\,N(t) = q(t)$.

b. Show that $u(t) = \int_0^t p(t)\,dt = Kt$.

c. Show that $v(t) = \int_0^t e^{u(s)}q(s)\,ds = 0.79e^{Kt} - 0.79 + \frac{K}{10}\int_0^t e^{Ks}d(s)\,ds$.

d. Show that

$$N(t) = 0.79 + 0.79\frac{K}{10}e^{-Kt} \quad \int_0^t e^{K\,s}\,d(s)\,ds. \tag{16.23}$$

Exercise 16.7.7 There is 'conventional wisdom' among SCUBA divers that if you are going to make a dive that involves two depths, 'do the deep part first'. This problem and the next explores rationale for that wisdom. To be concrete, assume that $K = 0.07$ which corresponds to approximately 10 minute half-life for the compartment ($(\ln 2)/0.07 = 9.9$).

a. Assume a diver descends immediately to 10 meters and stays there for 15 minutes, then descends to 30 meters and stays there for 10 minutes. Let

$$d_1(t) = \begin{cases} 10 & \text{for} \quad 0 \le t \le 15 \\ 30 & \text{for} \quad 15 < t \le 25 \end{cases}$$

Compute

$$N_1(25) = 0.79 + 0.007e^{-0.07\times 25} \int_0^{25} e^{0.07s} \times d_1(s)\, ds + 0.79e^{-0.07\times 25}.$$

You should recognize that

$$\int_0^{25} e^{0.07s} \times d_1(s)\, ds = \int_0^{15} e^{0.07s} \times 10\, ds + \int_{15}^{25} e^{0.07s} \times 30\, ds$$

Answer: $N_1(25) = 2.24$ Atmospheres.

b. Assume a diver descends immediately to 30 meters and stays there for 10 minutes, then ascends to 10 meters and stays there for 15 minutes. Let

$$d_2(t) = \begin{cases} 30 & \text{for} \quad 0 \le t \le 10 \\ 10 & \text{for} \quad 10 < t \le 25 \end{cases}$$

Compute

$$N_2(25) = 0.79 + 0.007e^{-0.07\times 25} \int_0^{25} e^{0.07x} \times d_2(x)\, dx + 0.79e^{-0.07\times 25}$$

Answer: $N_2(25) = 1.72$ Atmospheres.

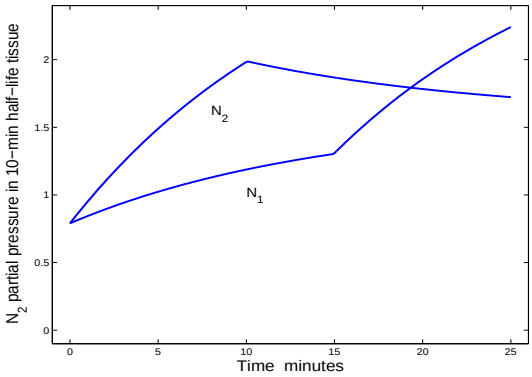
Thus by doing the deep part of the dive first, the end state of N_2 partial pressure in this compartment is at least 20% less than if she had made the shallow part of the dive first.

Exercise 16.7.8 It is instructive to see the graphs of $N_1(t)$ and $N_2(t)$ of the previous problem. You can plot these graphs on your calculator. We plotted N_2 on the TI-86 by

$$\begin{aligned} y1 &= \exp(0.07x) \\ y2 &= (x < 10) * \text{fnInt}(y1 * 30, x, 0, x) \\ &+ (10 \le x)(434.5 + \text{fnInt}(y1 * 10, x, 10, x) \\ y3 &= 0.79 + 0.79 * 0.007 \exp(-0.07x) * y2 \end{aligned}$$

Also see Exercise Figure 16.7.8 Use the graphs to explain how nitrogen absorption differs with the two dive profiles.

Figure for Exercise 16.7.8 Nitrogen profiles in a 10-minute half-life compartment of a diver with two dive plans. One is to dive to 10 meters for 15 minutes, then to 30 meters for 10 minutes. The other profile is for a dive to 30 meters for 10 minutes followed by 10 meters for 15 minutes.



Exercise 16.7.9 A diver is to survey the life on the bottom of a near-shore marine environment. She is to follow a transect that declines 30° and will travel 40 meters at 1 meter per minute along the 30° slope. You will need the formula:

$$\int te^{at}\,dt = \frac{1}{a}te^{at} = \frac{1}{a^2}e^{at} + C.$$

a. She starts at the shore and travels 40 meters along the 30° slope. At one meter per minute she will be at depth

$$d(t) = \frac{1}{2}t \qquad t \text{ minutes into the dive.}$$

and the water pressure in atmospheres will be $d(t)/10$. What will be the partial pressure of N_2 in tissue with $K = 0.07$ at the end of her dive?

b. She swims out to a point above the end point of the previous dive, descends and travels back toward shore. At one meter per minute she will be at depth

$$d(t) = 20 - \frac{1}{2}t \qquad t \text{ minutes into the dive.}$$

What will be the partial pressure of N_2 in tissue with $K = 0.07$ at the end of her dive?

Exercise 16.7.10 J. S. Haldane did the initial work on decompression illness in divers and used the model of the previous exercises. He assumed the body would have tissues with half-lives of 5, 10, 20, 40 and 75 minutes and set a criteria that the partial pressure of nitrogen in a diver's tissue should not be more than 2 times the partial pressure of nitrogen in her blood. (Actually, he began this work in 1905 and it was *his* blood, not *her* blood.) Compute curves analogous to those in Exercise Figure 16.7.8 for a compartment of 5 minute half-life ($K = (\ln 2)/5 = 0.139$) and a compartment of 75 ($K = (\ln 2)/45 = 0.0154$) minute half-life.

16.8 Separation of variables.

Separation of Variables. If in the differential equation, $\quad y' = f(t, y), \quad$ the RHS factors into

$$f(t, y) = g(t) \times h(y)$$

then $\quad y' = f(t, y) \quad$ can be written

$$y'(t) = g(t) \times h(y) \qquad\qquad \frac{1}{h(y)}y'(t) = g(t) \tag{16.24}$$

and *the variables t and y are separable.*

The solution to 16.24 involves two *antiderivative* problems:

SV Step 1 Find $H(y)$ such that $H'(y) = \frac{dH}{dy} = \frac{1}{h(y)}$.

SV Step 2 Find $G(t)$ such that $G'(t) = g(t)$.

Then

$$H(y(t)) = G(t) + C \qquad \text{where } C \text{ is a constant} \tag{16.25}$$

is an implicit expression of $y(t)$.

If a data point, $y(a) = y_a$ is given, then C can be computed from

$$H(y_a) = G(a) + C$$

To see that a function, $y(t)$ defined by Equation 16.25 solves Equation 16.24 we differentiate 16.25 with respect to t using the Chain Rule.

$$\begin{aligned}
[H(y(t))]' &= [G(t) + C]' \\
H'(y) \times y'(t) &= G'(t) \\
\frac{1}{h(y)} y'(t) &= g(t) \\
y'(t) &= g(t) \times h(y)
\end{aligned}$$

Example 16.8.1 Consider

$$y'(t) = 2t \times y$$

Then

$$\frac{1}{y} y' = 2t$$

SV Step 1. Find $H(y)$ such that $H'(y) = \frac{1}{y}$ Choose $H(y) = \ln y$.
SV Step 2. Find $G(t)$ such that $G'(t) = 2t$. Choose $G(t) = t^2$

By the Parallel Graph Theorem there is a number, C_1, such that

$$\ln y(t) = t^2 + C_1$$

This implicit expression for y can be solved explicitly.

$$y(t) = e^{t^2 + C_1} = e^{t^2} \times e^{C_1} = Ce^{t^2} \tag{16.26}$$

If we are also given that, for example, $y(0) = 3$ then we write

$$y(0) = 3 = Ce^{0^2} = C \qquad\qquad y(t) = 3e^{t^2}$$

Explore 16.8.1 Alternatively, we might be given that $y(0) = -3$. Return to Equation 16.26 and solve for the solution. ■

■

Example 16.8.2 Of the following six equations, the variables can be separated in only two.

$$\begin{aligned} y'(t) &= t + y & \qquad y' &= e^{t+y} \\ y'(t) &= \ln(t+y) & \qquad y' &= e^{t\times y} \\ y'(t) &= \ln(t\times y) & \qquad y' &= \ln\left(t^y\right) \end{aligned}$$

The two equations in which variables are separable are shown below.

$$\begin{aligned} y' &= e^{t+y} = e^t \times e^y & \qquad y' &= \ln t^y = y \times \ln t \\ e^{-y} \times y' &= e^t & \qquad \tfrac{1}{y} y' &= \ln t \\ \left[-e^{-y}\right]' &= \left[e^t\right]' & \qquad [\ln y]' &= [t\ln t - t]' \\ -e^{-y} &= e^t + C & \qquad \ln y &= t\ln t - t + C_1 \\ y &= -\ln\left(-e^t - C\right) & \qquad y &= C \times t^t \times e^{-t} \end{aligned}$$

■

Example 16.8.3 *The variables can be separated in every autonomous differential equation*

$$y' = f(y) \qquad \frac{1}{f(y)} y' = 1$$

To find an implicit solution to any autonomous differential equation only the problem
SV Step 1 Find $F(y)$ such that $F'(y) = \frac{1}{f(y)}$ requires attention.
SV Step 2 is easy: Find $G(t)$ such that $G'(t) = 1$ Answer: $G(t) = t$

Example 16.8.4 To solve the autonomous equation, $y' = -y^2$ we write

$$\begin{aligned} y' &= -y^2 \\ \frac{1}{y^2} y' &= -1 \\ \left[-\frac{1}{y(t)}\right]' &= [-t]' \\ -\frac{1}{y(t)} &= -t + C \\ y(t) &= \frac{1}{t - C} \end{aligned}$$

If also an initial condition is given, for example $y(0) = 0.5$, we write

$$y(0) = \frac{1}{0-C}, \qquad 0.5 = \frac{1}{-C}, \qquad C = -2, \qquad y(t) = \frac{1}{t+2}.$$

■

16.8.1 The logistic equation.

By far the most important differential equation that we solve by separation of variables is the logistic equation

$$p' = r \times p \times \left(1 - \frac{p}{M}\right)$$

We treat the Case: $0 < p(0) = p_0 < M$. From the direction field for the logistic equation in Figure 16.8 we have seen that $0 < p(0) < M$ implies that for all time, t, $0 < p(t) < M$.

First we separate variables

$$\frac{1}{p \times (1 - p/M)} \times p' = r$$

Then we use an algebraic identity (see the following section, Partial Fractions)

$$\frac{1}{p \times (1 - p/M)} = \frac{1}{p} + \frac{1}{M-p} \tag{16.27}$$

to write

$$\left(\frac{1}{p} + \frac{1}{M-p}\right) \times p' = r$$

Two derivative formulas

$$[\ln p]' = \frac{1}{p} \qquad \text{and} \qquad [-\ln(M-p)]' = \frac{1}{M-p}$$

provide a solution to

SV Step 1 Find $H(p)$ such that

$$H'(p) = \frac{1}{p} + \frac{1}{M-p} \qquad \text{Answer:} \qquad H(p) = \ln p - \ln(M-p)$$

Note: The assumption that $0 < p < M$ assures that both $\ln p$ and $\ln(M-p)$ are defined.

SV Step 2. Find $G(t)$ such that $G'(t) = r$. (Whew! Easy) $G(t) = rt$.

Then we write

$$\ln p - \ln(M-p) = rt + C_1 \tag{16.28}$$

$$\ln \frac{p}{M-p} = rt + C_1$$

$$\frac{p}{M-p} = e^{rt+C_1} = e^{rt} \times e^{C_1}$$

$$\frac{p}{M-p} = Ce^{rt}$$

The initial condition, $p(0) = p_0$, yields

$$\frac{p_0}{M - p_0} = Ce^{r\times 0} = C$$

Therefore

$$\frac{p}{M-p} = \frac{p_0}{M-p_0}e^{rt}$$

and solving for p yields

$$p(t) = \frac{Mp_0}{p_0 + (M - p_0)e^{-rt}} \tag{16.29}$$

Explore 16.8.2 Check the algebraic step

$$\frac{1}{p \times (1 - p/M)} = \frac{1}{p} + \frac{1}{M-p}$$

by adding the two fractions on the right and showing that the sum can be written as the fraction on the left. ■

16.8.2 The Method of Partial Fractions.

$$\text{The ratio} \quad \frac{P(x)}{Q(x)} \quad \text{where} \quad P(x) \quad \text{and} \quad Q(x)$$

are polynomials and the **degree of** $P(x)$ **is less than the degree of** $Q(x)$ can be partitioned into the sum of fractions with simple denominators.

The pattern is illustrated by the four examples:

$$\frac{px+q}{(x-a)\times(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \qquad \text{Example a.}$$

$$\frac{px^2+qx+r}{(x-a)\times(x-b)\times(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \qquad \text{Example b.}$$

$$\frac{px^2+qx+r}{(x-a)\times(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2} \qquad \text{Example c.}$$

$$\frac{px^2+qx+r}{(x-a)\times(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c} \qquad \text{Example d.}$$

The method is used to replace a complex integral such as

$$\int \frac{px^2+qx+r}{(x-a)\times(x-b)\times(x-c)}\,dx$$

by (Example c.) the sum of three relatively easy integrals

$$\int \frac{A}{x-a}\,dx + \int \frac{B}{x-b}\,dx + \int \frac{C}{x-c}\,dx$$

Example 16.8.5

$$\frac{3x+4}{(x+3)\times(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

To find A and B, multiply by $(x+3)\times(x-2)$ and get

$$3x+4 = A\times(x-2) + B\times(x+3)$$

Then substitute

$$\begin{array}{lrcll} x=-3 & 3(-3)+4 & = & A\times(-3-2)+B\times(-3+3) & \\ & -5 & = & A\times(-5) & A=1 \\ & & & & \\ x=2 & 3(2)+4 & = & A\times(2-2)+B\times(2+3) & \\ & 10 & = & B\times(5) & B=2 \end{array}$$

$$\text{Thus} \qquad \frac{3x+4}{(x+3)\times(x-2)} = \frac{1}{x+3} + \frac{2}{x-2}$$

The method is used to compute an integral:

$$\int \frac{3x+4}{(x+3)\times(x-2)}\,dx = \int \frac{1}{x+3}\,dx + \int \frac{2}{x-2}\,dx = \ln|x+3| + 2\ln|x-2| + C$$

In substituting values of x (we used $x=-3$ and $x=2$ above) any two values of x are acceptable. The chosen values, -3 and 2, each annihilate one of the terms and leave only one term with one of the unknown parameters A and B. It is curious that the most convenient values of x, -3 and 2, are exactly the values of x for which the original fraction

$$\frac{3x+4}{(x+3)\times(x-2)}$$

is meaningless.

Example 16.8.6

$$\frac{3x^2-5x+1}{(x+1)\times(x-2)^2} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

To find A, B and C, multiply by $(x+3)\times(x-2)^2$ and get

$$3x^2-5x+1 = A\times(x-2)^2 + B\times(x-2)\times(x+1) + C\times(x+1)$$

Then substitute

$$\begin{array}{lrcl}
x=-1 & 3(-1)^2-4(-1)+1 & = & A\times(-1-2)^2+B\times(-1-2)(-1+1)+C\times(-1+1) \\
 & 9 & = & A\times(9) \\
 & A & = & 1 \\
 & & & \\
x=2 & 3(2)^2-5\times 2+1 & = & A\times(2-2)^2+B\times(2-2)(2+1)+C\times(2+1) \\
 & 3 & = & C\times(3) \\
 & C & = & 1 \\
 & & & \\
x=0 & 3(0)^2-5\times 0+1 & = & A\times(0-2)^2+B\times(0-2)(0+1)+C\times(0+1) \\
 & 1 & = & 4A-2B+C \\
 & 1 & = & 4\times 1-2B+1 \\
 & B & = & 2
\end{array}$$

$$\text{Thus} \qquad \frac{3x^2-5x+1}{(x+1)\times(x-2)^2}=\frac{1}{x+1}+\frac{2}{x-2}+\frac{1}{(x-2)^2}$$

The method is used to compute an integral:

$$\begin{aligned}
\int \frac{3x^2-5x+1}{(x+1)\times(x-2)^2}\,dx &= \int \frac{1}{x+1}\,dx+\int \frac{2}{x-2}\,dx+\int \frac{2}{(x-2)^2}\,dx \\
&= \ln|x+1|+2\ln|x-2|-2(x-2)^{-1}+C
\end{aligned}$$

In this case, there were only two values of x that annihilated other terms, -1 and 2, and a third value of x was required because there were three unknown parameters, A, B, and C. We chose $x=0$ as a convenient third value to use.

Example 16.8.7 In Problem 16.4.11 on page 733 three differential equations were suggested as possible population models illustrating the Allee effect, in which there is a threshold population density, ϵ, with the property that if the density were less than ϵ the population would be so thinly distributed that reproductive success would not counter natural loss and the population will become extinct. Two of these equations can be solved by the method of separation of variables using partial fraction expansion. (We assume that time is scaled so that $r=1$.)

$$u' = u\times\frac{u-\epsilon}{u+\epsilon}\times(1-u) \qquad \text{Candidate 2}$$

$$u' = u\times(u-\epsilon)\times(1-u) \qquad \text{Candidate 3}$$

For Candidate 3 the method of partial fractions converts

$$u'=u\times(u-\epsilon)\times(1-u) \qquad \frac{u'}{u\times(u-\epsilon)\times(1-u)}=1$$

into

$$\left(\frac{-1\epsilon}{u}+\frac{\epsilon/(1-\epsilon}{u-\epsilon}+\frac{1/1-\epsilon}{1-u}\right)u'=1$$

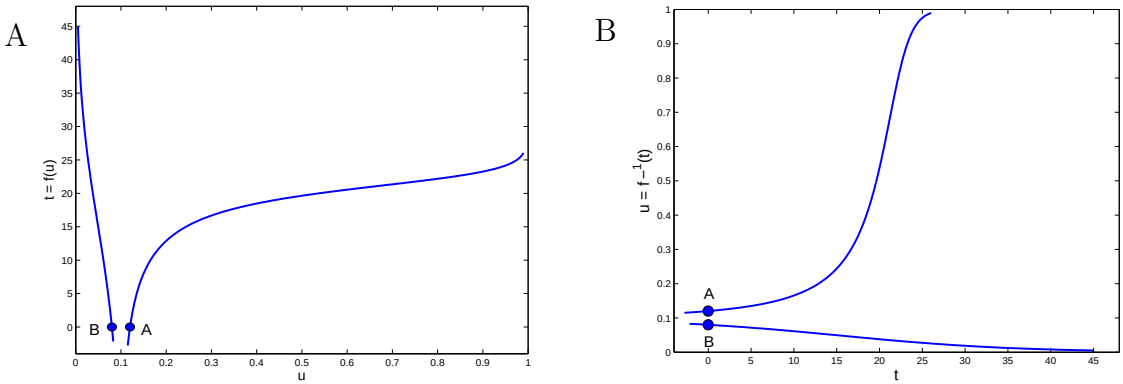


Figure 16.15: A. The graph of t versus u for Equation 16.30 using $\epsilon = 0.1$, and $u_0 = 0.12$ for the branch marked A and $u_0 = 0.8$ for the branch marked B. B. The reflection of the graph in A. with negative t values trimmed.

which can be integrated to obtain

$$-\frac{1}{1-\epsilon}\ln|u(t)| + \frac{1}{\epsilon(1-\epsilon)}\ln|u(t)-\epsilon| - \frac{1}{1-\epsilon}\ln|1-u(t)| = t + C \tag{16.30}$$

It seems unlikely that anyone will solve Equation 16.30 for $u(t)$. We can, however, look at t as a function of u, the inverse of the solution of interest. And we can reflect that graph about the line $u = t$ to get the graph of interest.

An example of Equation 16.30 is shown in Figure 16.15. For the purpose of the graph, we have let $\epsilon = 0.1$. This is high; the threshold density is likely much less than 10% of the capacity of the environment. The value of C is determined by u_0 and we used both $u_0 = 0.12$ ($C = 1.024$) and $u_0 = 0.08$ ($C = 0.05664$), above and below the threshold. Some values of t are negative in Figure 16.15 and illustrates the curve if time runs backward from 0. The graphs in Figure 16.15A are of the equation

$$t + C = -10\ln u + \frac{10}{0.9}\ln|u-0.1| - \frac{1}{0.9}\ln(1-u)$$

where

$$C = -10\ln 0.12 + \frac{10}{0.9}\ln|0.12-0.1| - \frac{1}{0.9}\ln(1-0.12) = -22.12$$
$$C = -10\ln 0.08 + \frac{10}{0.9}\ln|0.08-0.1| - \frac{1}{0.9}\ln(1-0.08) = -18.17$$

The analysis of Candidate 2 is left for you in Exercise 16.8.10. ■

Exercises for Section 16.8, Separation of variables.

Exercise 16.8.1 Show that the variables are not separable in the equation $y'(t) = t + y$. That is, there are not two functions, $g(t)$ and $h(y)$, which

$$\text{for all } t \text{ and } y \qquad t + y = g(t) \times h(y)$$

A procedure is to assume two such functions, $g(t)$ and $h(y)$ exist and then show that the following equations are incompatible.

$$t = 0 \quad y = 0 \qquad\qquad t = 0 \quad y = 1 \qquad\qquad t = 1 \quad y =$$

$$g(0) \times h(0) = 0 + 0 = 0 \qquad g(0) \times h(1) = 0 + 1 = 1 \qquad g(1) \times h(0) = 1 + 0 = 1$$

Show that $g(0) \times h(0) = 0$, $g(0) \times h(1) = 1$ and $g(1) \times h(0) = 1$ are incompatible.

Exercise 16.8.2 Show that the variables are not separable in the equation $y'(t) = \ln(t \times y)$.

Exercise 16.8.3 Find an implicit expression for $y(t)$ for each equation. Then use the given data point to evaluate the constant C of integration.

The following derivative formulas will be helpful.

$$[\ln(y-1)]' = \frac{1}{y-1} \qquad \left[\ln(t^2+1)\right]' = \frac{2t}{t^2+1} \qquad [\ln(1-y)]' = -\frac{1}{y-1}$$

a.	$y' = \frac{t}{y}$	$y(0) = 2$		b.	$y' = \frac{y}{t}$	$y(1) = 1$	
c.	$y' = \frac{t-1}{y}$	$y(2) = 1$		d.	$y' = \frac{t}{y(t^2+1)}$	$y(0) = 1$	
e.	$y' = y - 1$	$y(0) = 1.5$		f.	$y' = 1 - y$	$y(0) = 0.5$	
g.	$y' = y^2 - 1$	$y(0) = 2$		h.	$y' = e^{t-y}$	$y(0) = 1.5$	

Exercise 16.8.4 Solve for p in

$$\frac{p}{M-p} = \frac{p_0}{M-p_0}e^{rt}$$

to obtain Equation 16.29,

$$p(t) = \frac{Mp_0}{p_0 + (M-p_0)e^{-rt}} \quad ''$$

It will be useful to first solve for p in

$$\frac{p}{M-P} = K \qquad\qquad \left(K \quad \text{replaces} \quad \frac{p_0}{M-p_0}e^{rt}\right)$$

You should get

$$p = M \times \frac{K}{1+K}$$

Then substitute

$$K = \frac{p_0}{M-p_0}e^{rt}$$

and simplify. As a final step, divide numerator and denominator by e^{rt}.

Exercise 16.8.5 Show that for

$$p(t) = \frac{Mp_0}{p_0 + (M - p_0)e^{-rt}}$$

a. $p(0) = p_0$ b. $\lim_{t\to\infty} p(t) = M$

Exercise 16.8.6 Let $M = 10$ and $r = 0.1$ and plot the graphs of

$$p(t) = \frac{Mp_0}{p_0 + (M - p_0)e^{-rt}}$$

for $0 \le t \le 80$ and

a. $p_0 = 1$ b. $p_0 = 12$ c. $p_0 = 10$

Exercise 16.8.7 Suppose population is described by Equation 16.29, $P(t) = p_0M/(p_0 + (M - p_0)\, e^{-rt})$ and $0 < p_0 < M$.

1. Compute $P'(t)$.
2. Compute $P''(t)$.
3. Let t_{steep} be the value of t for which $P''(t) = 0$. Compute $e^{-rt_{steep}}$.
4. Compute $P(t_{steep})$.
5. Interpret t_{steep} as the time at which the population size is increasing the fastest.

Exercise 16.8.8 Suppose a population growth is is described by Equation 11.31

$$p' = r \times p \times \left(1 - \frac{p}{M}\right).$$

For what value of p is the population growing the fastest? That is, for what value of p is $r \times p \times \left(1 - \frac{p}{M}\right)$ the largest?

Exercise 16.8.9 The parameter reduced population models are shown below. In the Ricker equation, find a condition on γ_0 that will insure that there is a value of v for which the population is growing the fastest. In the Beverton-Holt equation show that there is no value of v for which the population is growing the fastest. In the Gompertz equation, find the value of v for which the population is growing the fastest.

a. $v' = ve^{-v} - \gamma_0 v$ Ricker

b. $v' = \dfrac{v}{1+v}$ Beverton-Holt

c. $v' = -v\ln(v)$ Gompertz

The three previous problems have important implications for wildlife management, at least conceptually. Suppose you are managing a wildlife population, salmon, for example, as a renewable resource, and wish to annually harvest as many salmon as possible. If you harvest too severely, the next years spawn will be low, and four years later the harvest will be limited. Your optimum strategy is to maintain the population at the level where the growth is the greatest.

Exercise 16.8.10 The second candidate suggested in Exercise 16.4.11 to model the Allee effect of an extinction threshold for a population is

$$u' = u \times \frac{u-\epsilon}{u+\epsilon} \times (1-u)$$

a. Show that

$$\left(\frac{-1}{u} + \frac{2/(1-\epsilon)}{u-\epsilon} + \frac{(1+\epsilon)/(1-\epsilon)}{1-u}\right) u' = 1$$

HINT: Solve for A, B, and C in

$$\begin{aligned} \frac{u+\epsilon}{u(u-\epsilon)(1-u)} &= \frac{A}{u} + \frac{B}{u-\epsilon} + \frac{C}{1-u} \\ u+\epsilon &= A(u-\epsilon)(1-u) + Bu(1-u) + Cu(u-\epsilon) \end{aligned}$$

by letting $u = 0$, $u = \epsilon$, and $u = 1$.

b. Show that

$$-\ln|u| + \frac{2}{1-\epsilon}\ln|u-\epsilon| - \frac{1+\epsilon}{1-\epsilon}\ln|1-u| = t + C \tag{16.31}$$

c. Let $\epsilon = 0.1$, and compute C for $u_0 = 0.08$, and draw the graph of t versus u. Then draw the graph of u versus t.

d. Let $\epsilon = 0.1$, and compute C for $u_0 = 0.12$, and draw the graph of t versus u. Then draw the graph of u versus t.

Exercise 16.8.11 Show that if $\epsilon = 0$, Equation 16.31 becomes the implicit solution to the logistic equation, Equation 16.28.

16.9 Examples and exercises for first order differential equations.

Generally, first order differential equations apply to problems of the type

> The rate of change of [a quantity, $y(t)$] is proportional to [a quantity described in terms of $y(t)$ and possibly also t].

For example,

Mathematical Model 16.9.1 The rate of change of oxygen in the air sac of an egg is proportional to the difference between the external oxygen concentration and the oxygen concentration inside the sac.

Let $\Omega(t)$ be the amount of oxygen (measured as mg of O_2) in the air sac at time t, and let V be the volume of the sac, and $[O_2]$ be the ambient partial pressure of oxygen. Then the statement can be written as an equation

$$\frac{d}{dt}\Omega(t) = K \times \left([O_2] - \frac{\Omega(t)}{V}RT\right)$$

where R is the gas constant and T is temperature. If the metabolic activity of the chicken embryo and the nonembryonic membranes inside the egg is also considered, one might have

The rate at which the embryo absorbs oxygen from the air sac of an egg is a function, $G(t)$ of the incubation time.

Then the equation may be modified to

$$\frac{d}{dt}\Omega(t) = K \times \left([O_2] - \frac{\Omega(t)}{V}RT\right) - G(t) \tag{16.32}$$

Exercise 16.9.1 Shown below are data and a graph of O_2 absorption as a function of day of incubation[9].

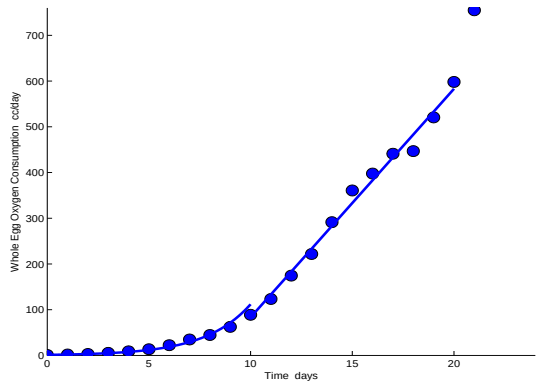
a. What is $g(t)$ (approximately) on $[0, 10]$? Assume the density of oxygen is 1.43 mg/cc.

b. By what method might the resulting Equation 16.32 be solved.

c. Write a modification of Equation 16.32 appropriate for the time interval, [10,20] days.

d. You might assume $\Omega(0) = 1$. What would be your procedure to solve Equation 16.32 over the interval [0,20]?

Whole Egg O_2 Consumption			
Age days	cc/day	Age days	cc/day cc/day
0	0.96	11	123.36
1	1.92	12	174.48
2	3.12	13	221.76
3	5.52	14	291.37
4	8.88	15	360.72
5	13.44	16	397.68
6	22.32	17	441.12
7	34.80	18	446.80
8	44.64	19	520.50
9	62.40	20	598.08
10	88.80	21	754.80



The curve from $t = 0$ to $t = 10$ is $y = 1.28e^{0.45t}$. The line from $t = 10$ to $t = 20$ is $y = 50t - 416$.

Exercise 16.9.2 Write an equation that describes the temperature of an egg after it is uncovered (the adult bird leaves the nest to feed). Assume that the rate of change of the temperature of the egg is proportional to difference between the air temperature and the egg temperature.

16.9.1 Light decay with depth in water.

It is an observed fact and may have been your experience that light intensity decreases as one descends from the surface of a lake or ocean. This problem addresses the question as to what is the light intensity, $I(x)$, at a distance, x meters, below the surface of a lake or ocean, assuming that the light intensity striking the surface is a known quantity, I_0.

[9] Alexis L. Romanoff, Biochemistry of the Avian Embryo, John Wiley, 1967

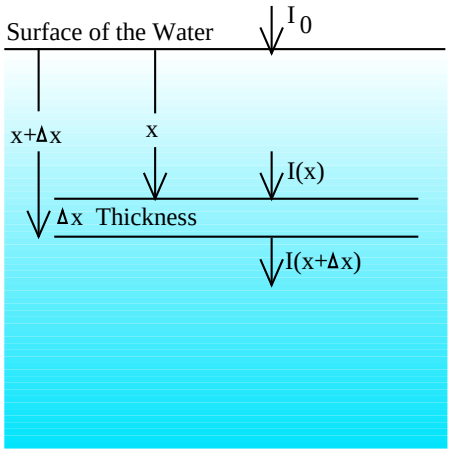


Figure 16.16: Illustration of light absorbance by a layer of water.

We will use an hypothesis about light transmission in water (see Figure 16.16).

Hypothesis L: The amount of light absorbed by a (horizontal) layer of water is proportional to the thickness of the layer and to the amount of light entering the layer. Hypothesis L implies

plication 1. That the light absorbed by a 10 cm layer of water is twice the light absorbed by a 5cm layer of water and

plication 2. A layer that absorbs 10% of a dim light will absorb 10% of a bright light.

Implication (1) is approximately true for thin layers and for low levels of turbidity. Implication (2) is fairly robust.

The double proportionality of Hypothesis L is handled by assuming that the amount of light absorbed is proportional to the *product* of the thickness of the layer and the intensity of the light incident to the layer. That is, there is a number, K, such that if $I(x)$ is the light intensity at depth x and $I(x+\Delta)$ is the light intensity at depth $x+\Delta$, then

$$I(x+\Delta) - I(x) \doteq -K \times \Delta \times I(x) \tag{16.33}$$

The product, $K \times \Delta \times I(x)$, has the advantage that for fixed incident light intensity, $I(x)$, the light absorbed, $I(x+\Delta) - I(x)$, is proportional to the thickness, Δ, and for fixed thickness Δ, the light absorbed is proportional to the incident light, $I(x)$.

Equation 5.23 can be rearranged to

$$\begin{aligned} \frac{I(x+\Delta) - I(x)}{\Delta} &\doteq \frac{-K\Delta I(x)}{\Delta} \\ \frac{I(x+\Delta) - I(x)}{\Delta} &\doteq -KI(x) \end{aligned}$$

The approximation ($\doteq$) gets better as the layer thickness, Δ, gets close to zero. Therefore

$$\begin{aligned} \lim_{\Delta \to 0} \frac{I(x+\Delta) - I(x)}{\Delta} &= -KI(x) \\ I'(x) &= -KI(x) \end{aligned} \tag{16.34}$$

Exercise 16.9.3 Assume that 1000 w/m^2 of light is striking the surface of a lake and that 40% of that light is reflected back into the atmosphere. Solve the initial value problem

$$\begin{aligned} I'(x) &= -KI(x) \\ I(0) = 600 \end{aligned}$$

Suppose the light intensity at 10 meters is 500 w/m^2. Find the value of K.

16.9.2 Spectrophotometer scales.

Equation 5.24 has an important application in interpretation of readings of light absorbance as measured in spectrophotometers. The spectrophotometer has two scales; the top scale is marked fraction (or percent) transmission, the bottom scale is marked 'absorbance.'

In $I'(x) = -KI(x)$, the constant, K, reflects the turbidity of the water. With high turbidity, light intensity decreases rapidly with depth, so that K is large. For clear water, light intensity decreases slowly with depth, and K is small. In the spectrophotometer, the thickness of the layer of liquid is constant, being determined by the glass tube holding the sample. The variable is the turbidity of the liquid in the tube and is assumed proportional to the density of bacteria, dye, or other substance you wish to quantify. The general solution to

$$I'(x) = -KI(x) \qquad \text{is} \qquad I(x) = I(0)e^{-Kx}$$

$I(0)$ is determined by the light source in the spectrophotometer. Assume that the thickness of the liquid in the tube is τ. Then the constant quantities in $I(x) = I(0)e^{-Kx}$ are $x = \tau$ and $I(0)$. The quantity measured by a photocell in the spectrophotometer and displayed in the top scale is $I(\tau)$. We write

$$\begin{aligned} I(\tau) &= I(0)e^{-K\tau} \\ \ln I(\tau) &= \ln I(0) + \ln\left(e^{-K\tau}\right) \\ \ln I(\tau) &= \ln I(0) + -K\tau \\ K = -\frac{1}{\tau}\left(\ln I(\tau) - \ln I(0)\right) \\ K = -\frac{1}{\tau}\ln\frac{I(\tau)}{I(0)} \end{aligned} \tag{16.35}$$

Therefore, the turbidity of the liquid sample is proportional to the logarithm of the fraction of light transmitted through the sample, and the bottom scale of the spectrophotometer is the negative of the logarithm of the top scale.

16.9.3 Carbon dating of ancient organic material.

Plants obtain carbon from atmospheric carbon dioxide; animals obtain carbon from plants. The fraction of ^{14}C carbon in the total carbon of living animals and plants is the same as that of the atmosphere. At present that fraction is approximately 1 part ^{14}C per 10^{12} total carbon atoms. Upon death, no new carbon is absorbed and the ^{14}C decays back to nitrogen

$$^{14}_{6}C \longrightarrow ^{14}_{7}N + \beta^- + \overline{\nu} + energy \tag{16.36}$$

where β^- is an electron emitted from one of the neutrons of the ${}^{14}_{6}C$, changing it to a proton, and $\overline{\nu}$ is a neutrino that accompanies a β^- emission. Measure of the β^- emission is a measure of the ${}^{14}C$ content of the carbon under study. One gram of carbon from presently living tissue contains about 5×10^{10} ${}^{14}C$ atoms and emits about 15.3 β^- particles per minute. In carbon from ancient plants and animals, some of the ${}^{14}C$ has decayed and the rate of β^- emission is smaller.

Mathematical Model of ${}^{14}C$ decay. The rate at which ${}^{14}C$ decreases is proportional to the amount of ${}^{14}C$ present. The reaction is describe by

$$\frac{d}{dt}\,{}^{14}C = -K \times {}^{14}C \tag{16.37}$$

For carbon dating assume that time is measured with zero at the year 1950 and negative times before that. For example, $t = -3000$ corresponds to 1050 BC (or BCE, before the common era). Let $E(t)$ be the level of β^- emission per minute from one gram of carbon in living tissue at time t. Scientists initially assumed that $E(t)$ is equal to the present value, $E(0) = 15.3$, for all time, but as will be seen in Figure 16.17, $E(t)$ is not constant. Let $E_{t_0}(t)$ be β^- emission per minute from one gram of carbon at time t in the remains of an organism that died at a time $t_0 \leq t$. Note that $E_{t_0}(0) = E(t_0)$ and $E_{t_0}(t_0)$ is the β^- emission that we would measure today from a fossil of an animal that lived at time t_0.

The solution to Equation 16.37 may be written:

$${}^{14}C(t) = {}^{14}C(t_0)e^{-K\,(t-t_0)}$$

Because β^- emission is directly proportional to ${}^{14}C$ content, we write that

$$E_{t_0}(t) = E(t_0)e^{-K\,(t-t_0)}. \tag{16.38}$$

Laboratory measurements have shown that in about 5730 years, one-half of the ${}^{14}C$ in any sample will decay to nitrogen. Therefore, for any t_0 and $t = t_0 + 5730$

$$\begin{aligned} E_{t_0}(t_0 + 5730) &= E_{t_0}(t_0)e^{-K\,5730} \\ &= 0.5 \times E_{t_0}(t_0). \end{aligned}$$

Thus

$$\begin{aligned} e^{-K\,5730} &= 0.5 \\ K &= \frac{\ln 2}{5730}. \end{aligned}$$

Equation 16.38 may now be written

$$E_{t_0}(t) = E_{t_0}(t_0)e^{-\frac{\ln 2}{5730}\,(t-t_0)} \tag{16.39}$$

Example 16.9.1 *Problem* Suppose the one gram of carbon from deer bone recently found among American Indian artifacts is emits 7 β^- particles per minute. How old is the bone?

Solution. Let t_0 be the time at which the deer died. **Assume** that $E(t_0) = E(0) = 15.3$. Then

$$\begin{aligned} E_{t_0}(t)|_{t=0} = E_{t_0}(0) &= E(t_0)e^{-\frac{\ln 2}{5730}(0-t_0)} \\ 7 &= 15.3e^{-\frac{\ln 2}{5730}(-t_0)} \\ \ln(\frac{7}{15.3}) &= \frac{\ln 2}{5730}t_0 \\ t_0 = -6464 \end{aligned}$$

Thus the bone is 6464 years old. ∎

Exercise 16.9.4 Suppose the β^- emission per gram of carbon in a recently discovered bone is $(1/2^n)\,E(0) = (1/2^n)\,15.3$ where n is an integer. **Assume** that $E(t) = E(0) = 15.3$ for all t. How old is the bone?

Note: Accurate measurement of $(1/2^n)\,15.3\ \beta^-$ emission is limited to $n \leq 10$.

Date adjustment for varying ambient ^{14}C content.

We found the bone of Example 16.9.1 to be approximately 6464 years old, so that the humans occupied the camp in which which the bone was found approximately 6,464 years ago. We assumed that the β^- emission of carbon 6,646 years ago was the same as the present β^- emission. Techniques for measuring the ^{14}C radiation and estimating the age of organic matter were developed by Willard F. Libby in the 1950's and have been checked by measuring the β^- emission of carbon from artifacts of known age; for two examples, carbon of heart wood of a giant sequoia tree shown by tree rings to be 2928± 50 years old, and by measuring the β^- emission of 4900 year old wood from a 1st dynasty Egyptian tomb.

Careful analysis of these woods have shown that the ambient β^- emission of carbon has varied over the years, and age estimates must be modified accordingly. A graph of ambient β^- emission levels for 9000 years is shown in Figure 16.17[10].

The vertical units in Figure 16.17 are measured as

$$\delta(t) = \left(\frac{E(t) - E(0)}{E(0)}\right) \times 1000, \qquad E(t) = E(0)\left(1 + \frac{\delta(t)}{1000}\right)$$

From the graph, for 6,464 years ago, (4514 B.C.), $\delta(6464) \doteq 80$, and

$$E(-6464) = E(0)\left(1 + \frac{80}{1000}\right)$$

If $E(t_0)$ is $1.08E(0) = 1.08 \times 15.3 = 16.52$ and the β^- emission of the sample is 7, as assumed above, then the age $(-t_0)$ of the bone would be found by

$$\begin{aligned} 7 &= 16.52e^{-\frac{\ln 2}{5730}*(-t_0)} \\ t_0 &= -7098 \end{aligned}$$

[10] **I was not able to locate a source to obtain copyright of this figure.** From M. Bruns, et al, The atmospheric ^{14}C level in the 7th millenium BC, Proceedings of the First International Symposium on ^{14}C and Archeology, (1981), Groningen, Netherlands.

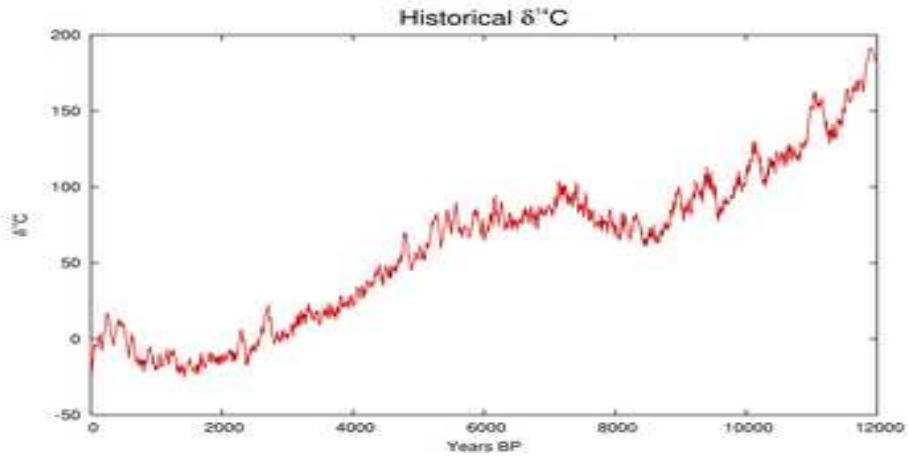


Figure 16.17: Ambient $\delta^{14}C$ levels for the most recent 9000 (12000) years. Produced by Leland McInnes using publicly available sources. http://en.wikipedia.org/wiki/File:Carbon-14-calibration.png

Thus a substantial adjustment must be made in the age of the sample, from 6464 years old to 7098 years old. However, this may not be the final answer, for you may see that 7098 years ago, the assumption that the ambient $\delta(t)$ is 80 is not valid. One then can make another adjustment, and hope that the procedure converges to a single value.

Exercise 16.9.5 Read the $\delta^{14}C$ level for 7098 years ago (5148 BC) from Figure 16.17 and convert that to $E(7098)$. Use that level to compute the next estimate of the age of a bone sample that has present day β^- emission = 7 particles per gram of carbon.

In their article, Bruns et al remark that "The overall trend of the cosmogenic ^{14}C follows remarkably closely a sine-curve with a period of 11,300 years and a peak to peak amplitude of 10.2%." They only considered the previous 9000 years. in Figure 16.17.

Exercise 16.9.6 By inspection of the graph for the last 9000 years, explain why

$$E(t) = E(0)\left(1.035 - 0.050\sin\left(\frac{2*\pi}{11,200}(t+4200)\right)\right) \tag{16.40}$$

where t is time before the present, gives a reasonable estimate of β^- emission of tissue living t years before the present. How does the graph work for 9000-12000 years ago?

Unique dates from ^{14}C analysis. Graphs of Equation 16.40 (solid) and an exponential decay curve (dashed) are shown in Figure 16.18. The curves intersect at (-4900,14.3) and (-1600,13.4), and the value of the decay curve at $t = 0$ is 13.0. If the decay curve were that of ^{14}C decay, with a half life of 5730 years, a wood sample with a present day ^{14}C radiation level could either be 4900 years old or 1600 years old. Fortunately, the decay curve illustrated has a half-life of 35,000 years and is not representative of ^{14}C decay.

Exercise 16.9.7 Use the Extended Mean Value Theorem 11.5.2 to show that the graphs of

$$E(t) = 15.3\left(1.035 - 0.050\sin\left(\frac{2*\pi}{11200}(t+4200)\right)\right)$$

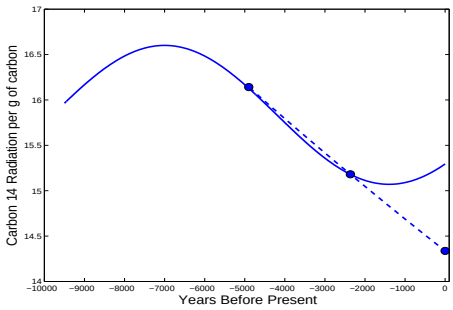


Figure 16.18: Graphs of a sine curve approximating ^{14}C radiation (Equation 16.40) and a decay curve $D_{t_0}(t) = E(t_0)\, e^{-\frac{\ln 2}{5 \cdot 5730}(t-t_0)}$ with $t_0 = -5000$. $D_{t_0}(t)$ decays one fifth as fast as $E_{t_0}(t) = E(t_0)\, e^{-\frac{\ln 2}{5730}(t-t_0)}$

and

$$E_{t_0}(t) = E(t_0)\, e^{-\frac{\ln 2}{5730}(t-t_0)}$$

can not intersect at two points for $-10000 \le t_0 \le 0$ and $t_0 \le t \le 0$. Argue as follows. They obviously intersect at $(t_0, E(t_0)$. Argue that:

a. $E'(t) \ge -0.000429$ for $t_0 \le t$.

b. $E'_{t_0}(t) \le -0.00045$ for $t_0 \le t$.

c. Suppose the graphs intersect at another point, $(t_1, E(t_1))$ with $t_0 \le t_1$. Then at some time τ between t_0 and t_1 the slopes of the two graphs are equal. This leads to a contradiction.

16.9.4 Chemical Kinetics

Chemical kinetics is the study of the rates of chemical reactions. The rate of a reaction is expressed in terms of the rate of disappearance of the reactants or the rate of appearance of the products. In the reaction

$$A \longrightarrow B \tag{16.41}$$

The reaction rate is

$$\text{Rate} = -\frac{dA}{dt} = \frac{dB}{dt} \tag{16.42}$$

We consider only simple reactions in which the reaction rate is a function of the concentrations of the reactants[11].

A reaction of the form

$$\frac{dA}{dt} = -K \times A \tag{16.43}$$

[11]Refer to Linus Pauling, *General Chemistry* 1970, W. H. Freeman, New York, 1988, Dover, New York. Pauling, p552. "The factors that determine the rate of reaction are manifold. The rate depends not only upon the composition of the reacting substances, but also upon their physical form, the intimacy of their mixture, the temperature and pressure, the concentrations of the reactants, special physical circumstances such as irradiation with visible light, ultraviolet light, x-rays, neutrons, or other waves or particles, and the presence of other substances that affect the reaction but are not changed by it."

such as ^{14}C decay is called a first order reaction. It also is a first order differential equation, but not for the same reason.

A reaction that occurs by the collision and combination of two molecules A and B,

$$A + B \longrightarrow AB$$

has a reaction rate that is proportional to the concentrations of A and B and a rate equation

$$-\frac{d[A]}{dt} = k[A][B] \tag{16.44}$$

In the event that the concentrations of A and B are equal, Equation 16.44 becomes

$$\frac{d[A]}{dt} = -k[A][A], \tag{16.45}$$

a first order differential equation that describes a second order reaction.

Equal quantities of gaseous hydrogen and iodine are mixed resulting in the reaction

$$H_2 + I_2 \longrightarrow 2HI$$

Because the initial amounts of H_2 and I_2 are equal and they combine equally, their concentrations will be equal. Let $y(t)$ be $[H_2] = [I_2]$. Then from equation 16.44,

$$\frac{dy}{dt} = -ky(t) \times y(t) \tag{16.46}$$

Exercise 16.9.8 Solve Equation 16.46 and show that

$$y(t) = \frac{y_0}{ky_0t + 1} \qquad \text{where} \qquad y_0 = y(0)$$

The general reaction

$$mA + nB \longrightarrow A_mB_n$$

has a rate equation

$$\frac{d[A]}{dt} = -k[A]^m[B]^n$$

and some special cases are of interest.

In the event that the initial concentration of B greatly exceeds that of A, then the concentration of B will be relatively constant during the reaction. With A being the rate limiting substance, the reaction will be an order m reaction. Let $y(t) = [A]$ and assume $[B]$ is constant. Then Equation 16.9.4 becomes

$$\frac{dy(t)}{dt} = -K[B]^n(y(t))^m = -\hat{K}(y(t))^m \tag{16.47}$$

For

$$m = 1, \qquad y(t) = y_0e^{-\hat{K}t} \qquad \text{and for} \qquad m = 2, \qquad y(t) = \frac{y_0}{y_0\hat{K}t + 1}.$$

Exercise 16.9.9 Show that for $m = 3$ the solution to

$$\frac{dy(t)}{dt} = -\hat{K}(y(t))^m \qquad \text{is} \qquad y(t) = \frac{y_0}{\sqrt{2y_0^2\hat{K}t + 1}}$$

We now can see that for $m = 1$, 2, or 3

$$\left.\begin{array}{lll} m = 1 & \Longrightarrow & \ln y(t) = \hat{K}t + \ln y_0 \\ m = 2 & \Longrightarrow & \frac{1}{y(t)} = \hat{K}t + \frac{1}{y_0} \\ m = 3 & \Longrightarrow & \frac{1}{(y(t))^2} = 2\hat{K}t + \frac{1}{y_0^2} \end{array}\right\} \tag{16.48}$$

If data from a reaction are given then one of $\ln y(t)$, $\frac{1}{y(t)}$, or $\frac{1}{(y(t))^2}$ may be linear in t, depending on whether each reaction step requires $m = 1$, 2, or 3 molecules of A. If neither is linear in t, then it would be assumed that the reaction step requires more than 3 molecules of A, or that the concentration of B is changing enough to affect the data.

Example 16.9.2 Data from Reaction 1 of Exercise Table 16.9.10 are plotted in Figure 16.19 and it is clear that the data are from a second order reaction, and $m = 2$. The reaction is thus

$$2A + nB \longrightarrow A_2B_n$$

for some n (found below). ■

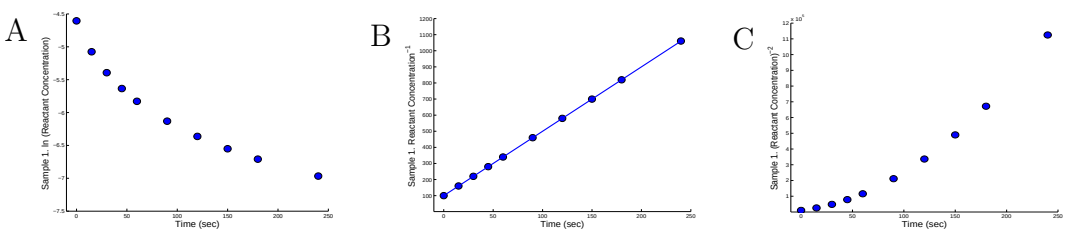


Figure 16.19: Graphs of Reaction 1 data. A. ln Concentration vs t. B. 1/Concentration vs t. C. 1/(Concentration)^2 vs t. B. is linear and the line y = 99.96 + 4 t fits the data. Therefore, the reaction is second order, $\hat{K} = 4$, and $\frac{1}{y(t)} = \hat{K}t + \frac{1}{y_0}$, and $y(t) = 1/(4t + 99.96)$.

Exercise 16.9.10 Sample concentrations of a reaction $mA + nB \longrightarrow A_mB_n$ with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.2M$ are shown in Exercise Table 16.9.10. For samples 2 and 3, what are the values of m?

Table for Exercise 16.9.10 Sample concentrations of a substance A in a chemical reaction $mA + nB \longrightarrow A_mB_n$ with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.2M$.

Time (sec)	Reaction 1 Concentration of A (mol)	Reaction 2 Concentration of A (mol)	Reaction 3 Concentration of A (mol)
0	0.010000	0.010000	0.010000
15	0.006250	0.003430	0.008607
30	0.004545	0.002500	0.007408
45	0.003571	0.002063	0.006376
60	0.002941	0.001796	0.005488
90	0.002174	0.001474	0.004066
120	0.001724	0.001280	0.003012
150	0.001429	0.001147	0.002231
180	0.001220	0.001048	0.001653
240	0.000943	0.000909	0.000907

Once the correct linear relation has been determined by comparing the data with Equations 16.48 so that m is known, another experiment will reveal the value of n. Observe that $[B]^n$ is a factor of $\hat{K}$ in the slopes of each of the equations in Equations 16.48. Therefore, if the experiment is repeated with twice the initial concentration $[B]$, the slope of the line first observed will be altered by 2^n.

Example 16.9.2 Continued. For the Reaction 1 with $[B] = 0.2$ mol, we found that the reaction was second order, $m = 2$ and $\hat{K} = 4$. The data for Reaction 1 with $[B] = 0.4$ mol is plotted in Figure 16.20 as for a second order reaction, 1/Concentration(A) vs time, and it is found that $y = 104 + 31.9t$ fits the data. Now $\hat{K}_{[B]=0.4} = 32$ which is 8 times $\hat{K}_{[B]=0.2}$. Therefore, $2^n = 8$, $n = 3$, and the reaction is third order in B. The reaction is thus

$$2A + 3B \longrightarrow A_2B_3 \qquad \blacksquare$$

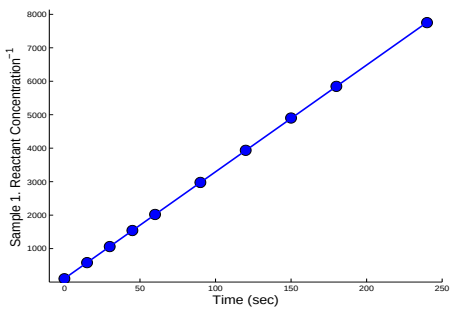


Figure 16.20: Graph of the data for Reaction 1 with $[B] = 0.4$, plotted as for a second order reaction (m=2) in A, 1/Concentration(A) vs time. The line fitting the data has equation $y = 104 + 31.9t$, so $\hat{K}_{[B]=0.2} = 4$ (from above) and $\hat{K}_{[B]=0.4}$ is $32 = 8 \times \hat{K}_{[B]=0.2}$. Thus $2^n = 8$, $n = 3$, and the reaction is third order in B.

Exercise 16.9.11 The experiments of the previous problem are repeated with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.4M$, and the concentrations of $[A]$ are shown in Exercise Table 16.9.11. What are the values of n for reactions 2 and 3?

Table 16.5: Area of mold colonies on days 0 to 9.

Day	0	1	2	3	4	5	6	7	8	9
Area mm^2	4	8	24	46	84	126	176	248	326	420

Table for Exercise 16.9.11 Sample concentrations of a substance A in a chemical reaction $mA + nB \longrightarrow A_mB_n$ with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.4M$.

Time (sec)	Reaction 1 Concentration of A (mol)	Reaction 2 Concentration of A (mol)	Reaction 3 Concentration of A (mol)
0	0.010000	0.010000	0.010000
15	0.001724	0.001796	0.007408
30	0.000943	0.001280	0.005488
45	0.000649	0.001048	0.004066
60	0.000495	0.000909	0.003012
90	0.000336	0.000743	0.001653
120	0.000254	0.000644	0.000907
150	0.000204	0.000576	0.000498
180	0.000171	0.000526	0.000273
240	0.000129	0.000456	0.000082

16.9.5 Mold Growth

You saw in Chapter 2 that the area of mold growing on a solution of tea and water can be well described by a quadratic function. You may have attributed this to the fact that the growth is confined to the perimeter of the mold colony – the cells on the interior of the colony were not expanding. The pictures of the growth are shown on the page 28.

A fairly accurate measure of the areas is shown in the Table 16.5. Each grid square in the photographs is 4 mm^2. If you have not done so, you should enter the data in statistical lists for your calculator and fit both an exponential function and a second degree polynomial to the data. You will find that the polynomial fits the data better, even though the exponential function shows a correlation of 0.97 with the data.

Let $A(t)$ be the area of the mold colony at time t and $P(t)$ be the length of the perimeter. Because cell division appears to be taking place on the perimeter of the mold, a reasonable mathematical model is

Mathematical Model 16.9.2 Mathematical Model. The rate of change of the area of the mold colony is proportional to the length of the perimeter.

We will also assume that the mold colony is essentially circular, of radius R.

Assumption. $A(t) = \pi R^2$. $P(t) = 2\pi R$.

Exercise 16.9.12 Use the Mathematical Model 16.9.2 and Assumption to show that

$$A'(t) = k \times 2 \times \sqrt{\pi}\sqrt{A(t)} \tag{16.49}$$

Exercise 16.9.13 Equation 16.49 can be written

$$\frac{A'(t)}{\sqrt{A(t)}} = k \times 2 \times \sqrt{\pi}$$

a. Show that

$$\left[2\sqrt{A(t)}\right]' = \frac{A'(t)}{\sqrt{A(t)}} \qquad \text{and} \qquad \left[k \times 2 \times \sqrt{\pi} \times t\right]' = k \times 2 \times \sqrt{\pi}$$

b. Conclude that there is a number C such that

$$\sqrt{A(t)} \;=\; k \times \sqrt{\pi} \times t + C$$

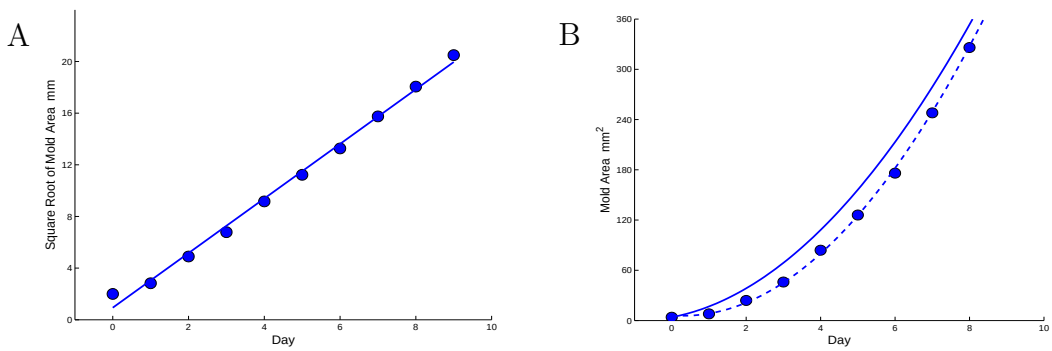
c. **Initial condition.** Assume that $A(0) = 4$, as observed in the pictures. Use $A(0) = 4$ and evaluate C.

d. The graph of $\sqrt{A(t)}$ *vs* t is shown in Exercise Figure 16.9.13. Use the graph to find an estimate of $k \times \sqrt{\pi}$.

e. You should be able to conclude that

$$A(t) \doteq (2.1\,t + 2)^2 \tag{16.50}$$

Figure for Exercise 16.9.13 A. Graph of $\sqrt{A(t)}$ versus t for the data of Table 16.5. The equation of the line is $y = 0.94 + 2.1x$. B. Graphs of the mold data, $y = (2.1t + 2)^2$ (solid line), and the regression parabola, $P_2(t) = 5.4t^2 - 3.3t + 6.2$ (dashed line).



It is to be noted that

$$A(t) \doteq (2.1\,t + 2)^2$$

does not fit the data as well as does the regression curve,

$$P_2(t) = 5.4t^2 - 3.3t + 6.2$$

(as computed with a TI-86 and the data in the table.) The advantage of the solution $A(t) \doteq (2.1\,t + 2)^2$ is that it is based on a well defined model of the growth.

Chapter 17

Second order and systems of two first order differential equations.

Where are we going?

The goal of this chapter is a theorem that gives conditions under which equilibria of a system of two first order differential equations will be stable. Only stable equilibria are observed in nature. Typical first order systems are the competing species and the predator-prey equations:

Competing Species		Predator − prey	
$x'(t) = a\,x(t) - b\,x(t)y(t)$		$x'(t) = a\,x(t) - b\,x(t)y(t)$	
$y'(t) = c\,x(t) - d\,x(t)y(t)$		$y'(t) = -c\,x(t) + d\,x(t)y(t)$	

The theorem is similar to Theorem 12.9.1 that applies to systems of difference equations, and the stepping stones leading to the theorems are similar.

17.1 Constant coefficient linear second order differential equations.

A differential equation of the form

$$y(0) = y_0, \qquad y'(0) = y'_0, \qquad y''(t) + py'(t) + qy(t) = f(t) \tag{17.1}$$

where p and q are constants and f is continuous is a second order linear constant coefficient differential equation. The equation

$$y(0) = y_0, \qquad y'(0) = y'_0, \qquad y''(t) + py'(t) + qy(t) = 0 \tag{17.2}$$

is said to be homogeneous.

An easily understood physical problem leading to Equation 17.1 is that of a mass suspended from a spring whose motion would be described by the homogeneous Equation 17.2 if there is no external applied force and whose motion is described by Equation 17.1 if there is a force $f(t)$ applied to the mass. See Figure 17.1.

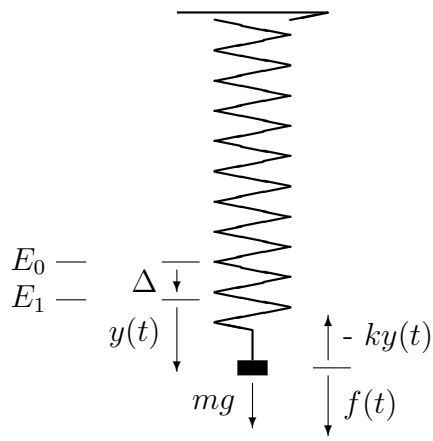


Figure 17.1: Diagram of a mass suspended from a spring and subject to an external force $f(t)$. E_0 is the equilibrium level of the spring with no weight attached. E_1 the equilibrium level of the spring with a weight of mass m attached.

The forces on the mass are the force mg due to gravity, the restoring force $-k(y(t)+\Delta)$ of the spring, and the external force $f(t)$.

The elongation of the spring due to the mass is Δ. At equilibrium E_1 the spring force is $-k\Delta$ and counters the gravitational force mg. There may be resistance in the system which is a force against the direction of motion and it is customary and convenient to assume that

$$\text{the force of resistance} \quad = \quad -r\,y'(t).$$

The forces acting on the mass are

$$mg - k(y(t)+\Delta) + f(t) - r\,y'(t) = -k\,y(t) + f(t) - r\,y'(t),$$

and by Newton's law of motion, $F = ma$,

$$my''(t) = -k\,y(t) + f(t) - ry'(t)$$

or

$$my''(t) + ry'(t) + k\,y(t) = f(t). \tag{17.3}$$

Division by m in Equation 17.3 produces the form of the nonhomogeneous Equation 17.1 with $p = r/m$ and $q = k/m$.

17.1.1 The homogeneous equation.

The homogeneous equation 17.2 is easily solved using methods similar to some used before. We 'guess' that there is a solution of the form $y = e^{rt}$. If so[1] then

$$y(t) = e^{rt}, \qquad y'(t) = r\,e^{rt} \qquad \text{and} \qquad y''(t) = r^2\,e^{rt}$$

Substitution of y, y' and y'' into Equation 17.2 yields

$$\begin{aligned} y''(t) + py'(t) + qy(t) &= 0 \\ r^2\,e^{rt} + p\,r\,e^{rt} + qe^{rt} &= 0 \\ \left(r^2 + p\,r + q\right)e^{rt} &= 0. \end{aligned}$$

Because $e^{rt} \neq 0$, if there is a solution of the form $y = e^{rt}$ to Equation 17.2 we should select r so that

$$r^2 + pr + q = 0. \tag{17.4}$$

Furthermore the previous steps may be reversed. If r is a root to $r^2 + pr + q = 0$, then $y = e^{rt}$ is a solution to $y''(t) + p\,y'(t) + qy(t) = 0$. Equation 17.4 is the *characteristic equation* and it roots are the *characteristic roots* of Equation 17.2.

The characteristic roots are either distinct and real, a repeated real root, or complex conjugate roots. The following theorem is helpful in building solutions to Equation 17.2.

Theorem 17.1.1 Superposition. Suppose $y_1(t)$ and $y_2(t)$ are two solutions to the homogeneous equation $y''(t) + py'(t) + qy(t) = 0$ and C_1 is a number and C_2 is a number. Then

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a solution to $y''(t) + py'(t) + qy(t) = 0$

Proof. The proof of Theorem 17.1.1 is Exercise 17.1.1.

The solutions to the equations

$$y'(t) + p\,y'(t) + 2y(t) = 0, \quad \text{for} \quad p \quad = \quad 8, \quad 4, \quad \text{and} \quad 2. \tag{17.5}$$

with $y(0) = 1$, $y'(0) = 2$ in each are shown in Figure 17.2. They illustrate the transition from distinct real roots ($p = 8$) to a repeated real root ($p = 4$) to complex roots ($p = 2$) as the coefficient of $y'(t)$ (resistance to motion in Equation 17.3) decreases.

By Theorem 17.1.1, if the characteristic roots of Equation 17.2 are distinct and real, r_1 and r_2, then for any number C_1 and number C_2

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{17.6}$$

[1] You should suppose this to be a good guess or else we would not suggest it to you. It is motivated by the exponential solutions to the first order differential equations of Chapter 16 and hope (and the fact that it has been known to work for some 250 years).

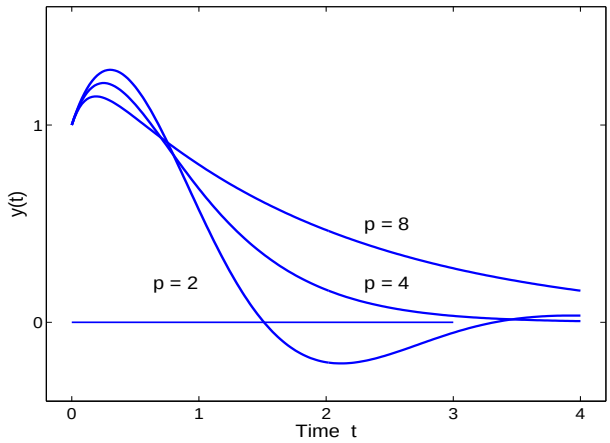


Figure 17.2: Graphs solutions to $y'' + py' + 4y = 0$, $y(0) = 1$, and $y'(0) = 2$ for $p = 8$ (distinct real negative roots), $p = 4$ (a negative repeated root), and $p = 2$ (complex roots).

is a solution to the homogeneous Equation 17.2. By using the intial conditions, $y(0) = y_0$ and $y'(0) = y_0'$, C_1 and C_2 can be evaluated. For

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \qquad y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}.$$

Then

$$\begin{aligned} y(0) &= C_1 e^{r_1 \times 0} + C_2 e^{r_2 \times 0} & C_1 + C_2 &= y_0 \\ y'(0) &= C_1 r_1 e^{r_1 \times 0} + C_2 r_2 e^{r_2 \times 0} & C_1 r_1 + C_2 r_2 &= y_0' \end{aligned}$$

These equations may be solved for C_1 andC_2.

$$C_1 = \frac{y_0' - r_2 y_0}{r_1 - r_2} \qquad C_2 = \frac{r_1 y_0 - y_0'}{r_1 - r_2} \tag{17.7}$$

A graph of the solution of Equation 17.5 with $p = 8$, $r_1 = -4 + 2\sqrt{3}$, $r_2 = -4 - 2\sqrt{3}$,

$$y(t) = \frac{\sqrt{3}+1}{2} \exp((-4 + 2\sqrt{3})t) + \frac{\sqrt{3}-1}{2} \exp((-4 - 2\sqrt{3})t),$$

is shown in Figure 17.2.

If there is only one characteristic root r_1 of Equation 17.2 ($p^2 - 4q = 0$) then both $y = e^{r_1 t}$ and $y = t\, e^{r_1 t}$ are solutions to Equation 17.2. (Exercise 17.1.2). Furthermore, for any numbers C_1 and C_2

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t} \tag{17.8}$$

is a solution to the homogeneous Equation 17.2. As for distinct roots, C_1 and C_2 can be found using the initial conditions.

$$C_1 = y_0 \qquad C_2 = y_0' - r_1 y_0 \tag{17.9}$$

A graph of $y(t) = e^{-2t} + 4te^{-2t}$, the solution of Equation 17.5 with $p = 4$, repeated root $r = -2$, is shown in Figure 17.2.

In the case of complex roots ($p^2 - 4q < 0$), r_1 and r_2 will be complex numbers $a + bi$ and $a - bi$ respectively, where $i = \sqrt{-1}$. The algebra and calculus of complex functions conform to the laws of exponents and derivative formulas of real valued functions, and the argument that Equation 17.6 solves the homogeneous equation has exactly the same steps in both real and complex arithmetic. We need to probe $e^{(a+bi)t}$ more deeply, however. It may stretch your credibility but we will define (actually Leonhard Euler found it in 1748 using Taylor's polynomials. See Exercise 9.4.6)

$$e^{(a+bi)t} \quad = \quad e^{at}\, e^{bt\, i} = e^{at} \cos bt + ie^{at} \sin bt.$$

What is really important to us is that each of the real and imaginary components of $e^{(a+bi)t}$ is a solution to the homogeneous equation. That is

$$y_1(t) = e^{at} \cos bt \qquad \text{and} \qquad y_2(t) = e^{at} \sin bt \qquad \text{each solve} \qquad y''(t) + py'(t) + qy(t) = 0.$$

We will show that $y = e^{at} \cos bt$ solves $y''(t) + py'(t) + qy(t) = 0$ when $a + bi$ is a root to $r^2 + pr + q = 0$.

$$\begin{aligned} y(t) &= e^{at} \cos bt \\ y'(t) &= ae^{at} \cos bt - be^{at} \sin bt \\ y''(t) &= a^2 e^{at} \cos bt - 2abe^{at} \sin bt - b^2 e^{at} \cos bt \end{aligned}$$

Substitute these functions into $y''(t) + py'(t) + qy(t) = 0$ and collect similar terms.

$$\begin{aligned} y''(t) + py'(t) + qy(t) &= a^2 e^{at} \cos bt - 2abe^{at} \sin bt - b^2 e^{at} \cos bt \\ &\qquad + p(ae^{at} \cos bt - be^{at} \sin bt) + qe^{at} \cos bt \\ &= (a^2 - b^2 + pa + q)e^{at} \cos bt + (-2ab - pb)e^{at} \sin bt \end{aligned} \tag{17.10}$$

We need to show that the last expression is zero and the prospects are grim. However, recall that $a + bi$ is a root to $r^2 + pr + q = 0$. We use a little complex arithmetic.

$$\begin{aligned} r^2 + pr + q &= 0 \\ (a + bi)^2 + p(a + bi) + q &= 0 \\ a^2 + 2abi + b^2 i^2 + pa + pbi + q &= 0 \qquad\qquad i^2 = -1 \\ (a^2 - b^2 + pa + q) + (2ab + pb)i &= 0 \end{aligned}$$

Now we got it! In order for a complex number to be zero, both the real and imaginary components must be zero. Therefore $a^2 - b^2 - pa + q = 0$ and $2ab + pb = 0$, expression 17.10 $= 0$, and $y(t) = e^{at} \cos bt$ is a solution to $y''(t) + py'(t) + qy(t) = 0$. The argument that $y(t) = e^{at} \sin bt$ is a solution to $y''(t) + py'(t) + qy(t) = 0$ is similar to the preceding steps and is omitted.

By Theorem 17.1.1, if $a + bi$ is a root to $r^2 + pr + q = 0$ and C_1 and C_2 are numbers then

$$y(t) = C_1 e^{at} \cos bt + C_2 e^{at} \sin bt \tag{17.11}$$

is a solution to $y''(t) - py'(t) + qy(t) = 0$. Using the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$, C_1 and C_2 are determined.

$$C_1 = y_0 \qquad C_2 = \frac{y'_0 - ay_0}{b} \tag{17.12}$$

A graph of $y(t) = e^{-t}\cos\sqrt{3}t + \sqrt{3}e^{-t}\sin\sqrt{3}t$, the solution of Equation 17.5 with $p = 2$, roots $= -1 \pm \sqrt{3}\,i$, is shown in Figure 17.2.

You can now solve every second order linear constant coefficient homogeneous differential equation with initial conditions. We have not proved that the solutions are unique, but they are unique – there are no other solutions. Please accept the uniqueness as a fact; its proof is beyond our scope. It is important that in each of the cases, distinct real roots, repeated real root, and complex roots $a \pm bi$, two solutions y_1 and y_2 of the homogeneous equation were found that are *linearly independent.*

Definition 17.1.1 Two functions y_1 and y_2 are linearly independent means that if C_1 is a number and C_2 is a number and

$$C_1y_1(t) + C_2y_2(t) \equiv 0$$

then $C_1 = C_2 = 0$. The assertion $C_1y_1(t) + C_2y_2(t) \equiv 0$ means that

$$\text{for all numbers } t, \qquad C_1y_1(t) + C_2y_2(t) = 0$$

For example, in the case of distinct real roots, $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$ are linearly independent. Suppose not, and that there are numbers C_1 and C_2 and

$$C_1e^{r_1t} + C_2e^{r_2t} \equiv 0$$

Then

$$\begin{aligned} \left[C_1e^{r_1t} + C_2e^{r_2t}\right]' &\equiv 0 \\ r_1C_1e^{r_1t} + r_2C_2e^{r_2t} &\equiv 0 \end{aligned}$$

If we evaluate $C_1e^{r_1t} + C_2e^{r_2t} \equiv 0$ and $r_1C_1e^{r_1t} + r_2C_2e^{r_2t} \equiv 0$ at $t = 0$ we find that

$$\begin{aligned} C_1 + C_2 &= 0 \\ r_1C_1 + r_2C_2 &= 0 \end{aligned}$$

Because $r_1 \neq r_2$ the equations imply that $C_1 = C_2 = 0$.

You may choose to move directly to Section 17.2; that section and the material following it does not require the material in Subsection 17.1.2.

17.1.2 The nonhomogeneous equation.

There are three steps to solving the nonhomogeneous Equation 17.1

$$y(0) = y_0, \qquad y'(0) = y'_0, \qquad y''(t) + py'(t) + qy(t) = f(t)$$

Step 1. Solve the homogeneous equation

$$y''(t) + py'(t) + qy(t) = 0$$

as in Subsection 17.1.1. That is, find $y_1(t)$ and $y_2(t)$ such that every solution to the homogeneous equation is of the form $C_1y_1(t) + C_2y_2(t)$.
Step 2. Find a *particular* solution $y_p(t)$ of the nonhomogeneous equation (ignore initial conditions for this step).
Step 3. $y_g(t) = y_p(t) + C_1y_1(t) + C_2y_2(t)$ is a *general* solution to the nonhomogeneous equation. Use the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$ to evaluate C_1 and C_2. Specifically, solve

$$\begin{aligned} y(0) &= y_p(0) + C_1y_1(0) + C_2y_2(0) &= y_0 \\ y'(0) &= y'_p(0) + C_1y'_1(0) + C_2y'_2(0) &= y'_0 \end{aligned}$$

for C_1 and C_2.

Step 2 is the new step. Step 1 is Subsection 17.1.1. That $y_g(t)$ in Step 3 solves the nonhomogeneous equation needs to be checked (Exercise 17.1.3). Important to Step 3 is to involve $y_p(t)$ in evaluating C_1 and C_2.

An approach to Step 2. Equation 17.1 is widely useful and a number of techniques to find a particular solution have been developed. We present only the most obvious which are based on the assumption that $y_p(t)$ should be similar to $f(t)$. If, for example, $f(t)$ is a polynomial, an exponential function or a sine function, then $y_p(t)$ should be, respectively, a polynomial, an exponential function or a sine function. This works well unless it happens that $f(t)$ is itself a solution to the homogeneous equation in which case a modification is required. Several examples will show the technique.
Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = t^2$$

Solution. Because $f(t) = t^2$ we guess that $y_p(t) = a_2t^2 + a_1t + a_0$. Even though $f(t)$ has only t^2, we need the t and constant terms in $y_p(t)$. Substitute:

$$\begin{aligned} \left[a_2t^2 + a_1t + a_0\right]'' - 3\left[a_2t^2 + a_1t + a_0\right]' + 2(\,a_2t^2 + a_1t + a_0\,) &= t^2 \\ 2a_2 - 3(2a_2t + a_1) + 2(a_2t^2 + a_1t + a_0) &= t^2 \\ 2a_2t^2 + (-6a_2 + 2a_1)t + (2a_2 - 3a_1 + 2a_0) &= t^2 \end{aligned}$$

We now equate the polynomial coefficients and write

$$\begin{aligned} 2a_2 \qquad\qquad\qquad &= 1 \\ -6a_2 + 2a_1 \qquad\qquad &= 0 \\ 2a_2 - 3a_1 + 2a_0 &= 0 \end{aligned}$$

Then $a_2 = 1/2$, $a_1 = 3/2$ and $a_0 = 7/4$, and $y_p(t) = (1/2)t^2 + (3/2)t - 7/4$ is a particular solution to $y''(t) - 3y'(t) + 2y(t) = t^2$.
Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = e^{-t}$$

Solution. Try $y_p(t) = Ae^{-t}$. Substitute

$$\begin{aligned} \left[Ae^{-t}\right]'' - 3\left[Ae^{-t}\right]' + Ae^{-t} &= e^{-t} \\ Ae^{-t} + 3Ae^{-t} + 2Ae^{-t} &= e^{-t} \\ 6Ae^{-t} &= e^{-t} \end{aligned}$$

Then $A = 1/6$ and $y_p(t) = (1/6)e^{-t}$ is a particular solution to $y''(t) - 3y'(t) + 2y(t) = e^{-t}$.
Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = 5t^2 + 4e^{-t}$$

Solution. Fortunately the solution is

$$y_p(t) = 5((1/2)t^2 + (3/2)t - 7/4) + 4((1/6)e^{-t}),$$

a combination of the solutions to the previous two problems.

Theorem 17.1.2 If $y_{p,1}(t)$ solves $y''(t) - py'(t) + qy(t) = f_1(t)$ and $y_{p,2}(t)$ solves $y''(t) - py'(t) + qy(t) = f_2(t)$, then for any numbers A and B

$$Ay_{p,1}(t) + By_{p,2} \qquad \text{solves} \qquad y''(t) - py'(t) + qy(t) = Af_1(t) + Bf_2(t).$$

Proof. Exercise 17.1.4.
Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = e^t$$

Solution. Try $y_p(t) = Ae^t$. Substitute

$$\begin{aligned} \left[Ae^{t}\right]'' - 3\left[Ae^{t}\right]' + Ae^{t} &= e^{t} \\ Ae^{t} - 3Ae^{t} + 2Ae^{t} &= e^{t} \\ 0 \times Ae^{t} &= e^{t} \end{aligned}$$

Ugh! No such A. The problem is that $f(t) = e^t$ is a solution to the homogeneous equation $y''(t) - 3y'(t) + 2y(t) = 0$.

Try again. Try $y_p(t) = A \times t \times e^t$. Substitute:

$$\begin{aligned} \left[A \times t \times e^{t}\right]'' - 3\left[A \times t \times e^{t}\right]' + A \times t \times e^{t} &= e^{t} \\ A(te^t + 2e^t) - 3A(te^t + e^t) + 2Ate^t &= e^t \\ (A - 3A + 2A)te^t + (2A - 3A)e^t &= e^t \end{aligned}$$

Choose $A = -1$ and $y_p(t) = -t \times e^t$.
Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = \sin \omega t \tag{17.13}$$

Solution. Try $y_p(t) = A \cos \omega t + B \sin \omega t$. (Both the sine and cosine terms are needed.) Substitute:

$$\begin{aligned}
[\,A \cos \omega t + B \sin \omega t\,]'' - 3\,[\,A \cos \omega t + B \sin \omega t\,]' + 2(A \cos \omega t + B \sin \omega t) &= \sin \omega t \\
-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t - 3(-A\omega \sin \omega t + B\omega \cos \omega t) + 2(A \cos \omega t + B \sin \omega t) &= \sin \omega t \\
(-A\omega^2 - 3B\omega + 2A) \cos \omega t + (-B\omega^2 + 3A\omega + 2B) \sin \omega t &= \sin \omega t
\end{aligned}$$

Match coefficients. Choose

$$-A\omega^2 - 3B\omega + 2A = 0 \qquad \text{and} \qquad -B\omega^2 + 3A\omega + 2B = 1$$

$$\begin{aligned}
(2 - \omega^2)A - 3\omega B &= 0 \\
3\omega A + (2 - \omega^2)B &= 1
\end{aligned}$$

$$A = \frac{3\omega}{(2-\omega^2)^2 + 9\omega^2} \qquad \text{and} \qquad B = \frac{2-\omega^2}{(2-\omega^2)^2 + 9\omega^2}$$

$$y_p(t) = \frac{3\omega}{(2-\omega^2)^2 + 9\omega^2} \cos \omega t + \frac{2-\omega^2}{(2-\omega^2)^2 + 9\omega^2} \sin \omega t \tag{17.14}$$

Explore 17.1.1 Show that for the special case, $\omega = 1$, Equation 17.14 defines a solution to Equation 17.13 (with $\omega = 1$). ■

In the case that $\sin \omega t$ solves the homogeneous equation $y''(t) + py'(t) + qy(t) = 0$, $y_p(t) = A \cos \omega t + B \sin \omega t$ will not solve the nonhomogeneous equation $y''(t) + py'(t) + qy(t) = \sin \omega t$, but $y_p(t) = t \times (A \cos \omega t + B \sin \omega t)$ will solve the nonhomogeneous equation. A summary of selections of $y_p(t)$ appears in Table 17.1.

Exercises for Section 17.1, Constant coefficient linear second order differential equations.

Exercise 17.1.1 Show by substitution that if $y_1(t)$ and $y_2(t)$ are solutions to $y''(t) + py'(t) + qy(t) = 0$ and each of C_1 and C_2 is a number then

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a solution to $y''(t) - py'(t) + qy(t) = 0$.

Exercise 17.1.2 Show by substitution that if r_1 is the only root of $r^2 + pr + q = 0$ then $y = te^{r_1 t}$ is a solution to $y''(t) + py'(t) + qy(t) = 0$. Note: If r_1 is the only root to $r^2 + pr + q = 0$ then $p^2 - 4q = 0$ and $r_1 = -p/2$.

Table 17.1: Selection of particular solutions $y_p(t)$ to the nonhomogeneous equation $y''(t) - py'(t) + qy(t) = f(t)$ for certain forms of $f(t)$.

$f(t)$	$y_p(t)$
$a_0 + a_1 t + \cdots a_n t^n$	$b_0 + b_1 t + \cdots b_n t^n$
e^{mt}	e^{mt} or te^{mt} or $t^2 e^{mt}$
$\cos\omega t$ or $\sin\omega t$	$A\cos\omega t + B\sin\omega t$ or $t \times (A\cos\omega t + B\sin\omega t)$
$e^{mt}\cos\omega t$ or $e^{mt}\sin\omega t$	$e^{mt}(A\cos\omega t + B\sin\omega t)$ or $t \times e^{mt}(A\cos\omega t + B\sin\omega t)$

Exercise 17.1.3 Show that if $y_h(t)$ solves $y''(t) + py'(t) + qy(t) = 0$ and $y_p(t)$ solves $y''(t) + py'(t) + qy(t) = f(t)$ then $y_g(t) = y_p(t) + Cy_h(t)$ solves $y''(t) + py'(t) + qy(t) = f(t)$ for any number C.

Exercise 17.1.4 Show that if $y_{p,1}(t)$ solves $y''(t) + py'(t) + qy(t) = f_1(t)$ and $y_{p,2}(t)$ solves $y''(t) + py'(t) + qy(t) = f_2(t)$, then for any numbers A and B

$$Ayp,1(t) + By_{p,2} \qquad \text{solves} \qquad y''(t) + py'(t) + qy(t) = Af_1(t) + Bf_2(t).$$

Exercise 17.1.5 Compute the solutions to

$$y'' + py' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

for $p = 4$, $p = 2$, $p = 1$, $p = 0$, and $p = -2$.

Exercise 17.1.6 Compute the solutions to

$$y'' + py' + 9y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

for $p = 10$, $p = 6$, $p = 4$, $p = 0$, and $p = -6$.

Exercise 17.1.7 Compute the solution to

$$y'' + py' + qy = 0 \qquad y(0) = 0 \qquad y'(0) = 0.$$

Exercise 17.1.8 Compute $y_h(t)$, the general solution to the homogeneous equation, and $y_p(t)$ a particular solution to the nonhomogeneous equation.

a. $y'' + 4y' + 3y = t$ b. $y'' + 4y' + 3y = e^t$

c. $y'' - 4y' + 3y = e^t$ d. $y'' + 4y' + 3y = \cos t$

Exercise 17.1.9 Show that the for m, r, and k positive, the characteristic roots

$$r_1 = \frac{-r + \sqrt{r^2 - 4mk}}{2m} \qquad \text{and} \qquad r_2 = \frac{-r - \sqrt{r^2 - 4mk}}{2m}$$

for $my''(t) + ry'(t) + ky(t) = 0$ are both negative or have negative real parts.

Exercise 17.1.10 Consider a special case of a spring-mass system in which there is no resistance and with a a *harmonic forcing function*, $f(t) = \cos \omega t$. Thus examine

$$my''(t) + ky(t) = \cos \omega t \tag{17.15}$$

a. Show that the general solution to the homogeneous equation $my''(t) + ky(t) = 0$ is

$$y_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \qquad \text{where} \qquad \omega_0 = \sqrt{k/m}$$

b. Show that if $\omega \neq \omega_0$, a particular solution to the nonhomogeneous equation $my''(t) + ky(t) = \cos \omega t$ is

$$y_p(t) = \frac{1}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

c. Show that the general solution to $my''(t) + ky(t) = \cos \omega t$ is

$$y(t) = \frac{1}{m(\omega_0^2 - \omega^2)} \cos \omega t + C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

if $\omega \neq \omega_0$.

d. Suppose that the mass is initially at rest so that $y(0) = 0$, and $y'(0) = 0$. Evaluate C_1 and C_2.

e. Show that the motion of the mass is approximated by

$$y(t) = \frac{1}{m(\omega_0^2 - \omega^2)} \cos \omega t - \frac{1}{m(\omega_0^2 - \omega^2)} \cos \omega_0 t = \frac{1}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \tag{17.16}$$

if $\omega \neq \omega_0$.

f. Sketch the graph of $y(t)$ in Equation 17.16 for the case $m = 1$, $\omega = 1$ and $k = 0.01$ (weak spring) ($\omega_0 = 0.1$). The impressed force $\cos t$ appears as the rapid oscillations and the inherent system frequency appears as the overall gradual wave due to the term $\cos \omega_0 t = \cos 0.1t$

g. Sketch the graph of $y(t)$ in Equation 17.16 for the case $m = 1$, $\omega = 1$ and $k = 0.81$ (stiff spring) ($\omega_0 = 0.9$).

From the identity, $\cos x - \cos y = 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$,

$$y(t) = \frac{1}{m(\omega_0^2 - \omega^2)} 2 \sin(\frac{\omega_0 + \omega}{2} t) \sin(\frac{\omega_0 - \omega}{2} t)$$

The amplitude of the rapid vibrations, $\sin(\frac{\omega_0 + \omega}{2} t)$ is

$$\frac{1}{m(\omega_0^2 - \omega^2)} 2 \sin(\frac{\omega_0 - \omega}{2} t)$$

and results in a beat that may be heard in mechanical systems.

Exercise 17.1.11 *Resonance.* Consider the special case of Equation 17.15

$$my''(t) + ky(t) = \cos\omega_0 t, \qquad \text{for which} \qquad \omega_0 = \sqrt{k/m}, \tag{17.17}$$

the frequency of the impressed force is equal to the inherent frequency of the spring-mass system.

Show that a particular solution to Equation 17.17 is

$$y_p(t) = \frac{1}{2m\omega_0} \times t \times \sin\omega_0 t$$

and the general solution is

$$y(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{1}{2m\omega_0} \times t \times \sin\omega_0 t.$$

a. Suppose that $y(0) = 0$ and $y'(0) = 0$, then the solution is

$$y(t) = \frac{1}{2m\omega_0} \times t \times \sin\omega_0 t$$

b. What happens to the mechanical system?

Exercise 17.1.12 *Damping.* Consider now the case that there is resistance $r > 0$ in the spring-mass equation with a harmonic forcing function,

$$my''(t) + ry'(t) + ky(t) = \cos\omega t. \tag{17.18}$$

a. Show that if y_h is a solution to the homogeneous equation

$$my''(t) + ry'(t) + ky(t) = 0, \qquad \text{then} \qquad \lim_{t\to\infty} y_h(t) = 0.$$

Consider the three cases of distinct real roots, repeated real root, and complex roots.

b. Argue that for any initial condition, there is a solution $y_h(t)$ to the homogenous equation and the solution to Equation 17.18 is of the form

$$y(t) = y_h(t) + y_p(t), \qquad \text{where} \qquad y_p(t) = A\cos\omega t + B\sin\omega t. \tag{17.19}$$

In fact, Ugh! Your choice is to believe us or work it out yourself,

$$A = \frac{k - m\omega^2}{(k - m\omega^2)^2 + r^2\omega^2} \qquad \text{and} \qquad B = \frac{r\omega}{(k - m\omega^2)^2 + r^2\omega^2}.$$

The important point is that $y_h(t) \to 0$ as $t \to \infty$ and the motion of the mass is (eventually) $y_p(t)$ which is governed by the frequency of the external forcing function, $\cos\omega t$. $y_h(t)$ is termed the *transient* solution.

Exercise 17.1.13 Show that

$$\text{Solution:} \qquad y(t) = -\frac{50}{\sqrt{0.99}} e^{-0.1t} \sin\sqrt{0.99}t + 50\sin t \tag{17.20}$$

is the solution to

$$y(0) = 0, \quad y'(0) = 0, \quad y''(t) + 0.2y'(t) + y(t) = 10\cos t \tag{17.21}$$

Graph the solution. Note that the magnitude of the vibrations are five times those of the forcing function, $10\cos t$

17.2 Stability and asymptotic stability of equilibria of autonomous pairs of differential equations.

Henceforth our interest will be pairs of differential equations of the form

$$\begin{aligned} x(0) &= x_0 \qquad x'(t) = f(x,y) \\ y(0) &= y_0 \qquad y'(t) = g(x,y), \end{aligned} \tag{17.22}$$

We will assume that f and g are continuous and have continuous first and second partial derivatives with respect to both variables. This is sufficient to insure that for any initial values $x(0) = x_0$ and $y(0) = y_0$, Equations 17.22 have unique solutions $x(t)$ and $y(t)$ in a neighborhood of (x_0, y_0) and $t = 0$. These equations are said to be *autonomous* because $f(x, y)$ and $g(x, y)$ are independent of t. The homogeneous Equation 17.2, $y'' + py' + qy = 0$ is also autonomous where as the nonhomogeneous Equation 17.1, $y'' + py' + qy = f(t)$ is not autonomous.

An *equilibrium* point of Equations 17.22 is a number pair

$$(x_e, y_e) \qquad \text{such that} \qquad \begin{aligned} f(x_e, y_e) &= 0 \\ g(x_e, y_e) &= 0. \end{aligned} \tag{17.23}$$

If (x_e, y_e) is an equilibrium point of Equations 17.22 then the pair

$$x(t) = x_e \qquad y(t) = y_e \qquad \text{for all } t$$

has the property that

$$x'(t) = 0 = f(x_e, y_e) \qquad y'(t) = 0 = g(x_e, y_e).$$

Thus $x(t) = x_e$ and $y(t) = y_e$ is a solution to Equations 17.22.

Phase plane Direction Fields. Suppose (x_0, y_0) is a point in the x, y-plane and $x(t), y(t)$ is a pair of functions that solve Equations 17.22 and for some number t_1 $x(t_1) = x_1$ and $y(t_1) = y_1$. The solution pair $x(t), y(t)$ is said to pass through (x_1, y_1) at time t_1. From Equations 17.22

$$x'(t_1) = f(x_1, y_1) \qquad \text{and} \qquad y'(t_1) = g(x_1, y_1).$$

The derivatives $x'(t_1)$ and $y'(t_1)$ are determined by (x_1, y_1) and do not depend on t_1. It is illuminating to draw vectors *in the x-y plane* at some points x_1, y_1 that point in the direction determined by $x' = f(x_1, y_1)$ and $y' = g(x_1, y_1)$. The direction is an angle α where

$$\begin{aligned} \alpha &= \arctan \frac{y'}{x'} && \text{if } x' > 0 \\ \alpha &= \text{sign}(y') \times \frac{\pi}{2} && \text{if } x' = 0 \\ \alpha &= \pi + \arctan \frac{y'}{x'} && \text{if } x' < 0. \end{aligned}$$

(sign$(z) = 1$ if $z > 0$, $= 0$ if $z = 0$, and $=$ -1 if $z < 0$.)

Example 17.2.1 Consider the equations

$$\begin{aligned} x(0) &= x_0 \qquad & x'(t) &= y \\ y(0) &= y_0 \qquad & y'(t) &= 2xy \end{aligned} \tag{17.24}$$

First observe that every point with y-coordinate 0, $(x_e, 0)$, is an equilibrium point. Also observe the direction field in, Figure 17.3A; the direction of motion in the four quarters of the plane is determined by whether x' and y' are positive or negative. Now use Leibnitz notation; write

$$\frac{y'(t)}{x'(t)} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{dy}{dx} = \frac{2xy}{y} = 2x \tag{17.25}$$

The equation, $dy/dx = 2x$ has an easy solution,

$$y = x^2 + C, \qquad y = x^2 + y_0 - x_0^2.$$

From this we conclude that every solution curve is part of a parabola, and we have drawn three such parabola's in Figure 17.3B.

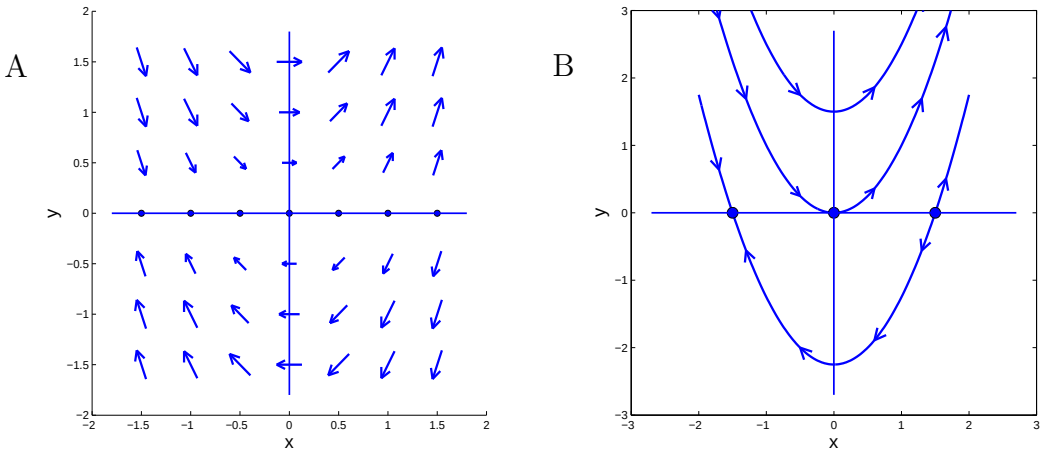


Figure 17.3: A. Direction field for $x' = y$, $y' = 2xy$. Each point $(x_e, 0)$ is an equilibrium point. B. Solution curves with directions of solutions marked.

The lower parabola in Figure 17.3B is $y = x^2 - 2.25$ and contains five nonintersecting solution curves to Equations 17.24 corresponding to

$$-\infty < x < -1.5, \qquad x = -1.5, \qquad -1.5 < x < 1.5, \qquad x = 1.5, \qquad \text{and} \qquad 1.5 < x < \infty.$$

The parabola contains two equilibrium points, $(-1.5, 0)$ and $(1.5, 0)$.

Explore 17.2.1 Do This.

a. Show that for any positive number, a,

$$x(t) = a \tan at, \qquad y(t) = a^2 \sec^2 at, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

solve Equations 17.24. Identify $(x(0), y(0))$ and characterize the solution curve.

b. Show that

$$x(t) = \frac{1}{1-t}, \qquad y(t) = \frac{1}{(1-t)^2}, \qquad -\infty < t < 1$$

solve Equations 17.24. Identify $(x(0), y(0))$ and characterize the solution curve.

c. Show that

$$x(t) = \frac{-1}{1+t}, \qquad y(t) = \frac{1}{(1+t)^2}, \qquad -1 < t < \infty$$

solve Equations 17.24. Identify $(x(0), y(0))$ and characterize the solution curve.

d. The previous two solutions lie on $y = x^2$. Find one more solution that lies on $y = x^2$.

e. If you find formulas for the five solution curves contained in $y = x^2 - 2.25$ you should get an A in this course.

■

The two equilibria $(-1.5, 0)$ and $(1.5, 0)$ in Figure 17.3B are quite different. Solution curves close to $(-1.5, 0)$ stay close to $(-1.5, 0)$ as time increases (see the arrows on the curves). There are solution curve close to $(1.5, 0)$, however, that move away from $(1.5, 0)$ as time increases. The equilibrium $(-1.5, 0)$ is *stable* (defined next) and the equilibrium $(1.5, 0)$ is not stable. ■

Definition 17.2.1 *Stable Equilibrium.* An equilibrium point, (x_e, y_e), of Equations 17.22 is a stable means that if ϵ is a positive number there is a positive number δ such that if $x(t), y(t)$ are the solutions to Equations 17.22 and for some t_1 $\sqrt{(x(t_1) - x_e)^2 + (y(t_1) - y_e)^2} < \delta$ then then for every number $t > t_1$, $\sqrt{(x(t) - x_e)^2 + (y(t) - y_e)^2} < \epsilon$.

Definition 17.2.2 *Asymptotically stable.* A stable equilibrium point, (x_e, y_e), of Equations 17.22 is asymptotically stable if there is a positive number δ such that if $x(t), y(t)$ are the solutions to Equations 17.22 and for some t_1 $\sqrt{(x(t_1) - x_e)^2 + (y(t_1) - y_e)^2} < \delta$ then

$$\lim_{t\to\infty} \sqrt{(x(t) - x_e)^2 + (y(t) - y_e)^2} \quad = \quad 0.$$

The equilibrium point $(-1.5, 0)$ of equations

$$x' = y, \qquad y' = 2xy$$

shown in Figure 17.3B is stable but not asymptotically stable. To show stability of $(-1.5, 0)$, suppose $\epsilon > 0$ and we may assume $\epsilon < 1$. Choose $\delta = 2\epsilon/3$. The case for $\epsilon = 0.9$ and $\delta = 0.6$ is shown if Figure 17.4A. The curves that enter the circle with radius δ stay within the circle of radius ϵ. It is

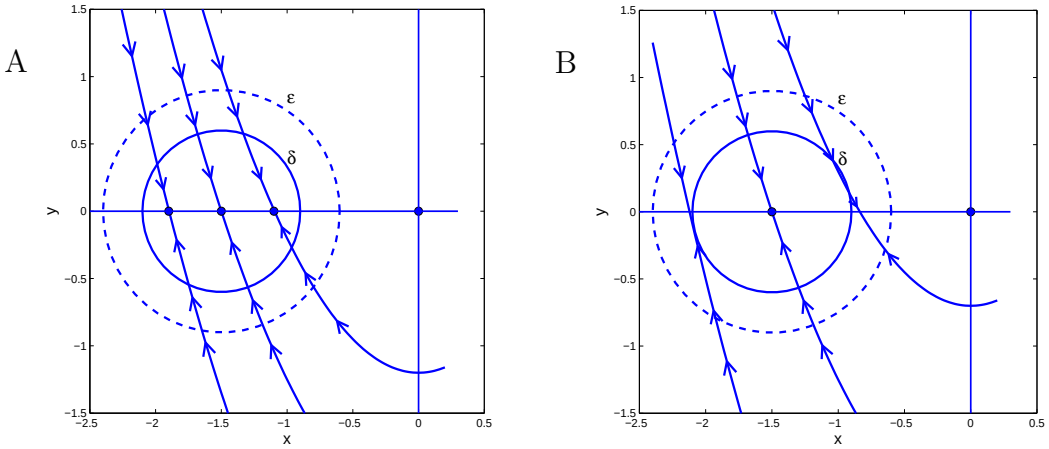


Figure 17.4: A. Solution curves of $x' = y$, $y' = 2xy$ stay close to the equilibrium point (-1.5,0). B. Curves may enter the circle with radius δ but not stay within it. They do, however, stay within the circle of radius ϵ.

necessary that δ be less than ϵ. There are curves that enter the circle with radius δ that do not stay in that circle, as shown in Figure 17.4B.

The equilibrium point (-1.5,0) of equations

$$x' = y, \qquad y' = 2xy$$

is not asymptotically stable. All of the equilibrium points $(x_e, 0)$ that lie within the circle of radius δ are graphs of solution curves that do not converge to (-1.5,0).

Example 17.2.2 The origin, (0,0), is an asymptotically stable equilibrium of

$$\begin{aligned} x(0) &= x_0 \qquad & x' &= -x \\ y(0) &= y_0 \qquad & y' &= -y. \end{aligned} \tag{17.26}$$

The solution to Equations 17.26 is

$$x(t) = x_0\, e^{-t}, \qquad y(t) = y_0\, e^{-t}$$

The origin is stable: Suppose $\epsilon > 0$, choose $\delta = \epsilon$. If for some t_1

$$\sqrt{(x(t_1) - 0)^2 + (y(t_1) - 0)^2} = \sqrt{x_1^2 + y_1^2}\, e^{-t_1} \quad < \quad \delta,$$

Then for all $t > t_1$,

$$\sqrt{(x(t) - 0)^2 + (y(t) - 0)^2} = \sqrt{x_1^2 + y_1^2}\, e^{-t} \quad < \quad \sqrt{x_1^2 + y_1^2}\, e^{-t_1} \quad < \delta.$$

Furthermore

$$\lim_{t\to\infty} \sqrt{(x(t) - 0)^2 + (y(t) - 0)^2} = \sqrt{x_1^2 + y_1^2} \lim_{t\to\infty} e^{-t} \quad = \quad 0,$$

so that (0,0) is asymptotically stable. ■

Exercises for Section 17.2, Stability and asymptotic stability of equilibria of autonomous pairs of differential equations.

Exercise 17.2.1 Do Explore 17.2.1.

Exercise 17.2.2 Is the origin a stable equilibrium of

$$x' = y, \qquad y' = 2xy\ ?$$

Exercise 17.2.3 Show that the solution to

$$\begin{aligned} x(0) &= x_0 & x' &= -y \\ y(0) &= 0 & y' &= 25x \end{aligned} \tag{17.27}$$

$$\text{is} \qquad x(t) = x_0 \cos 5t, \qquad y(t) = 5x_0 \sin 5t.$$

Show that the origin (0,0) is a stable equilibrium point of Equations 17.27 but not an asymptotically stable equilibrium.

17.3 Two constant coefficient linear differential equations.

We examine the system

$$\begin{aligned} x(0) &= x_0 & x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ y(0) &= y_0 & y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned} \tag{17.28}$$

where the coefficients, $a_{i,j}$, are constant. The origin is an equilibrium point. We will find conditions on the coefficients, $a_{i,j}$, that will insure that the origin is an asymptotically stable equilibrium.

We differentiate $x'(t) = a_{1,1}x(t) + a_{1,2}y(t)$ and write

$$\begin{aligned} x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ x''(t) &= a_{1,1}x'(t) + a_{1,2}y'(t) \\ y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned} \tag{17.29}$$

Now eliminate $y(t)$ and $y'(t)$ between three Equations 17.29 and and find a second order equation that involves only $x(t)$.

$$x''(t) - (a_{1,1} + a_{2,2})x'(t) + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})x(t) = 0. \tag{17.30}$$

Equation 17.30 is a second order linear constant coefficient homogeneous differential equation of the form

$$x''(t) + px'(t) + qx(t) = 0.$$

discussed in Section 17.1.1. The characteristic equation

$$r^2 - (a_{1,1} + a_{2,2})r + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2}) = 0 \tag{17.31}$$

of Equation 17.30 and its characteristic roots are also defined to be the characteristic equation and roots of the system, Equation 17.28. The coefficient $a_{1,1} + a_{2,2}$ of Equation 17.31 is called the *trace* of the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}, \end{bmatrix}$$

and the coefficient $a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$ is the determinant of A. The characteristic equation 17.31 is sometimes written

$$r^2 - \text{trace}(A)r + \det(A) = 0 \tag{17.32}$$

The same characteristic equation and roots are obtained if $x(t)$ and $x'(t)$ are eliminated from Equations 17.28 to obtain a second order differential equation in $y(t)$ (Exercise 17.3.1.)

For initial conditions

$$x(0) = x_0 \qquad y(0) = y_0$$

we can compute $x'(0)$

$$x'(0) = a_{1,1}x(0) + a_{1,2}y(0) = a_{1,1}x_0 + a_{1,2}y_0.$$

From Equations 17.6 and 17.7 for distinct real roots, r_1 and r_2, and Equations 17.8 and 17.9 for a repeated root, r, and Equations 17.11 and 17.12 for complex roots $a + bi$ and $a - bi$, the solution to Equation 17.30 is, respectively,

$$\begin{aligned} x(t) &= C_1 e^{r_1 t} + C_2 e^{r_2 t}, & C_1 &= \frac{a_{1,1}x_0 + a_{1,2}y_0 - r_2 x_0}{r_1 - r_2} \\ & & C_2 &= \frac{r_1 x_0 - a_{1,1}x_0 - a_{1,2}y_0}{r_1 - r_2}. \end{aligned} \tag{17.33}$$

$$\begin{aligned} x(t) &= C_1 e^{rt} + C_2 t e^{rt}, & C_1 &= x_0 \\ & & C_2 &= a_{1,1}x_0 + a_{1,2}y_0 + r_1 x_0. \end{aligned} \tag{17.34}$$

$$\begin{aligned} x(t) &= C_1 e^{at}\cos bt + C_2 e^{at}\sin bt, & C_1 &= x_0 \\ & & C_2 &= \frac{a_{1,1}x_0 + a_{1,2}y_0 - a x_0}{b}. \end{aligned} \tag{17.35}$$

From these solutions it is fairly easy to prove the following theorem.

Theorem 17.3.1 *Asymptotical Stability of a pair of constant coefficient homogeneous differential equations.* The origin (0,0) is an asymptotically stable equilibrium point of

$$\begin{aligned} x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned} \tag{17.36}$$

if the roots of the characteristic equation

$$r^2 - (a_{1,1} + a_{2,2})r + (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) = 0$$

satisfy one of the three conditions:

a. The roots are real and distinct and negative.

b. The root is a repeated root and is negative.

c. The roots are complex, $a + bi$ and $a - bi$ and a is negative.

Under conditions a. and b. the origin is called an asymptotically stable node and under condition c. the origin is called an asymptotically stable spiral point.

Proof. Suppose the roots are real and distinct and negative. We first show that (0,0) is a stable equilibrium. From Equations 17.33 we observe that

$$\begin{aligned} |x(t)| &= |C_1e^{r_1t} + C_2e^{r_2t}| \\ &\leq |C_1| + |C_2| \\ &= \left|\frac{a_{1,1}x_0 + a_{1,2}y_0 - r_2x_0}{r_1 - r_2}\right| + \left|\frac{r_1x_0 - a_{1,1}x_0 - a_{1,2}y_0}{r_1 - r_2}\right| \\ &\leq \left(\frac{|a_{1,1}| + |a_{1,2}| + |r_2|}{|r_1 - r_2|} + \frac{|r_1| - |a_{1,1}| + |a_{1,2}|}{|r_1 - r_2|}\right) \times max(|x_0|, |y_0|) \\ &\leq K_x\sqrt{x_0^2 + y_0^2}. \end{aligned}$$

Similarly there is a constant K_y that depends only on the coefficients $a_{1,1}, \cdots, a_{2,2}$ such that $|y(t)| \leq K_y\sqrt{x_0^2 + y_0^2}$. Therefore

$$\sqrt{(x(t))^2 + (y(t))^2} \leq \sqrt{K_x^2 + K_y^2}\sqrt{x_0^2 + y_0^2} = K\sqrt{x_0^2 + y_0^2}$$

Suppose ϵ is to be a bound on $\sqrt{(x(t))^2 + (y(t))^2}$. Let $\delta = \epsilon/K$. Then if $\sqrt{x_0^2 + y_0^2} < \delta$,

$$\sqrt{(x(t))^2 + (y(t))^2} \leq K\sqrt{x_0^2 + y_0^2} < K \times \epsilon/K = \epsilon$$

Therefore, (0,0) is stable.

From $x(t)C_1e^{r_1t} + C_2e^{r_2t}$ it is immediate that

$$\lim_{t\to\infty} x(t) = C_1 \lim_{t\to\infty} e^{r_1t} + C_2 \lim_{t\to\infty} e^{r_2t} = 0$$

because r_1 and r_2 are negative. Similarly $\lim_{t\to\infty} y(t) = 0$, and it follows that (0,0) is an asymptotically stable equilibrium of Equations 17.33.

The arguments for a repeated root and complex roots are similar and are omitted. End of proof.

Direction fields are shown in Figure 17.5 for three examples, all of which have real characteristic roots.:

A. $x' = -0.5x$	B. $x' = -0.5x$	C. $x' = 0.5x$
$y' = -0.3y$	$y' = 0.3y$	$y' = 0.3y$
Two negative roots, -0.5 and -0.3	One negative, one positive, -0.5 and 0.3.	Two positive roots, 0.5 and 0.3.

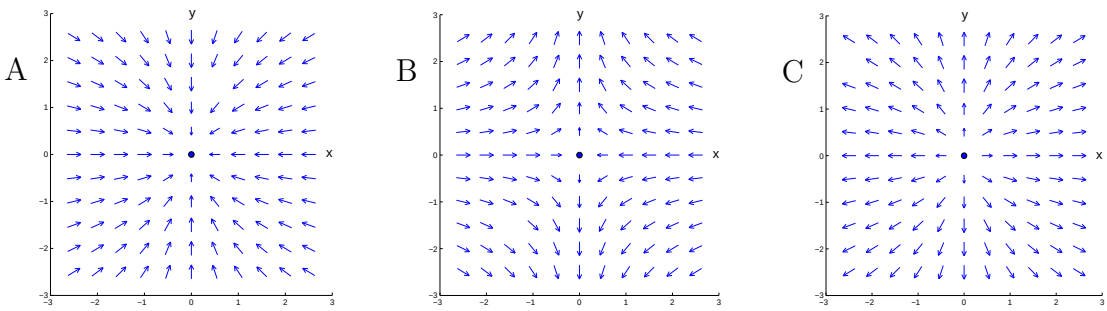


Figure 17.5: Direction fields drawn in the phase planes for Equation 17.28 with A. $A = [-0.5 \quad 0; \quad 0 \quad -0.3]$. B. $A = [-0.5 \quad 0; \quad 0 \quad 0.3]$. C. $A = [0.5 \quad 0; \quad 0 \quad 0.3]$.

It is obvious in Figure 17.5A that all paths lead to (0,0); (0,0) is a stable node of the equations A. There are two negative characteristic roots, -0.5 and -0.3, and the solution is $x(t) = x_0e^{-0.5t}$ and $y(t) = y_0e^{-0.3t}$. In Figure 17.5B only two paths lead to (0,0). Others paths start toward (0,0) but are diverted to either positive or negative y. The roots are -0.5 and 0.3 and the solution is $x(t) = x_0e^{-0.5t}$ and $y(t) = y_0e^{0.3t}$. The equations B are not stable, the origin is not a stable equilibrium point. In Figure 17.5C all paths lead away from (0,0). There are two positive roots, 0.5 and 0.3, and the equations C are not stable.

Direction fields are shown in Figure 17.6 for

D. $x' = -0.2x \quad -0.4y$	E. $x' = -0.4y$	F. $x' = 0.2x \quad -0.4y$
$y' = 0.5x \quad -0.3y$	$y' = 0.5x$	$y' = 0.5x \quad +0.3y$
Real part negative, -0.5.	Real part zero.	Real part positive, 0.5.

All of these systems have complex characteristic roots. The spiraling around the origin is typical of complex roots.

In Figure 17.6D, because the real part of the root is negative the paths spiral inward toward (0,0) and the equations D are stable and (0,0) is a stable spiral point. In Figure 17.6E the real part of the root

is zero; all paths are ellipses with center at (0,0). The paths do not converge to (0,0) and the system E is stable but not asymptotically stable. In Figure 17.6F the real part of the root is positive and the paths spiral out. The system F is not stable.

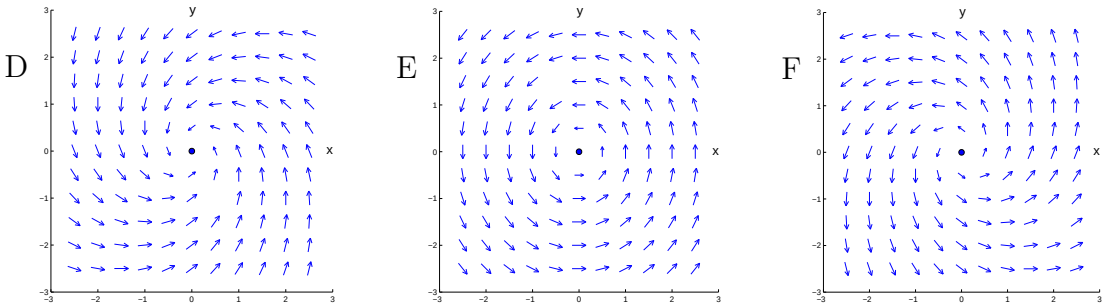


Figure 17.6: Direction fields drawn in the phase planes for Equation 17.28 with D. $A = [-0.2 \quad -0.4; \quad 0.5 \quad -0.3]$. E. $A = [0 \quad -0.4; \quad 0 \quad -0.3]$ F. $A = [0.2 \quad -0.4; \quad 0.5 \quad 0.3]$.

Summary. The qualitative character of the direction fields and solutions to equations 17.28

$$\begin{aligned} x(0) &= x_0 \qquad & x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ y(0) &= y_0 \qquad & y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned}$$

is determined by the roots to the characteristic equation

$$r^2 - (a_{1,1} + a_{2,2})r + a_{1,1}a_{2,2} - a2,1a_{1,2} = 0 \qquad \text{or} \qquad r^2 - \text{trace}(A)r + \det(A) = 0$$

The roots are

$$r_1 = \frac{\text{trace}(A) + \sqrt{(\text{trace}(A))^2 - 4\det(A)}}{2} \qquad r_2 = \frac{\text{trace}(A) - \sqrt{(\text{trace}(A))^2 - 4\det(A)}}{2}$$

The roots are real if the discriminate $(\text{trace}(A))^2 - 4\det(A) \geq 0$ and complex otherwise. The conditions for stability of the equilibrium (0,0) are summarized:

1. $(\text{trace}(A))^2 - 4\det(A) \geq 0$. If $\text{trace}(A) < 0$ and $\det(A) > 0$ there are two negative real roots and (0,0) is a stable node. If $\det(A) \leq 0$ there is at least one positive or zero root and (0,0) is not a stable node.

2. $(\text{trace}(A))^2 - 4\det(A) = 0$. If $\text{trace}(A) < 0$ there is one negative real root and (0,0) is a stable node.

3. $(\text{trace}(A))^2 - 4\det(A) < 0$. If $\text{trace}(A) < 0$ there are two complex roots with negative real part and (0,0) is a stable spiral point.

Exercises for Section 17.3, Two constant coefficient linear differential equations.

Exercise 17.3.1 Eliminate the functions $x(t)$ and $x'(t)$ between the two Equations 17.28 and find a second order equation that involves only $y(t)$. You should get

$$y''(t) - (a_{1,1} + a_{2,2})y'(t) + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})y(t) = 0. \tag{17.37}$$

The characteristic equation of this differential equation is also Equation 17.31.

Exercise 17.3.2 In each of the direction fields of Figure 17.5 there is an obvious omission. Compute the direction of each of the omitted vectors.

Exercise 17.3.3 In each of the direction fields of Figure 17.6 there is an obvious omission. Compute the direction of each of the omitted vectors.

Exercise 17.3.4 For each of the systems, determine whether the origin is stable, asymptotically stable, or unstable.

a. $\begin{array}{rcrcr} x' & = & 2x & - & 5y \\ y' & = & x & - & 2y \end{array}$

b. $\begin{array}{rcrcr} x' & = & 2x & - & 5y \\ y' & = & 2x & - & 4y \end{array}$

c. $\begin{array}{rcrcr} x' & = & -6x & - & 2y \\ y' & = & 2x & - & 1y \end{array}$

d. $\begin{array}{rcrcr} x' & = & -9x & + & 4y \\ y' & = & -4x & - & 1y \end{array}$

e. $\begin{array}{rcrcr} x' & = & 3x & - & 2y \\ y' & = & 2x & - & 1y \end{array}$

f. $\begin{array}{rcrcr} x' & = & & & y/2 \\ y' & = & -5x & - & 3y \end{array}$

g. $\begin{array}{rcrcr} x' & = & -x & - & 5y \\ y' & = & 2x & - & 3y \end{array}$

h. $\begin{array}{rcrcr} x' & = & & - & 5y \\ y' & = & 2x & + & 2y \end{array}$

i. $\begin{array}{rcrcr} x' & = & -3x & + & 1y \\ y' & = & 2x & - & 2y \end{array}$

j. $\begin{array}{rcrcr} x' & = & 3x & + & y \\ y' & = & 2x & + & 2y \end{array}$

k. $\begin{array}{rcrcr} x' & = & x & - & 2y \\ y' & = & 2x & + & 1y \end{array}$

l. $\begin{array}{rcrcr} x' & = & 6x & + & 4y \\ y' & = & 2x & - & y \end{array}$

17.4 Systems of two first order differential equations.

We return to two possibly nonlinear homogeneous differential equations

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)) \end{aligned} \tag{17.38}$$

where f and g are continuous and have continuous first and second partial derivatives with respect to both variables.

A first global look at the solutions of Equations 17.38 is obtained from looking at the *null curves* defined by

$$f(x, y) = 0, \qquad \text{and} \qquad g(x, y) = 0.$$

Example 17.4.1 Consider the system

$$\begin{aligned} x'(t) &= (169 - x^2 - y^2)/10 \\ y'(t) &= 17 - x - y \end{aligned} \tag{17.39}$$

The null curve $x' = 0$ is the circle $169 - x^2 - y^2 = 0$; the $y' = 0$ null curve is the line $17 - x - y = 0$. Both are shown in Figure 17.7. On the circle, $x' = 0$ and every direction field arrow is vertical. On the line, $y' = 0$ and every direction field arrow is horizontal. Inside the circle $169 - x^2 - y^2 = 0$, $x' > 0$ and direction arrows point to the right; outside the circle $x' < 0$ and direction arrows point to the left. Below the line $17 - x - y = 0$, $y' > 0$ and arrows point upward; above the line $y < 0$ and arrows point downward.

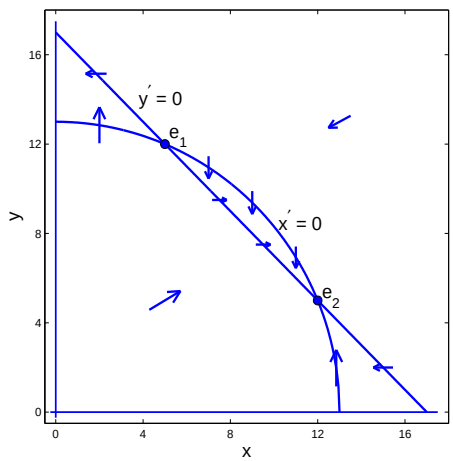


Figure 17.7: Null curves for $x' = (169 - x^2 - y^2)/10$, $y' = 17 - x - y$.

The two points, $e_1 = (5, 12)$ and $e_2 = (12, 5)$, where the null curves intersect are equilibrium points. Both $x' = 0$ and $y' = 0$ at the equilibrium points and both

$$\begin{cases} x_1(t) \equiv 5 \\ y_1(t) \equiv 12 \end{cases} \quad \text{and} \quad \begin{cases} x_2(t) \equiv 12 \\ y_2(t) \equiv 5 \end{cases}$$

are solutions to Equations 17.39.

Explore 17.4.1 Show that $x \equiv 5$, $y \equiv 12$ is a solution to

$$x' = (169 - x^2 - y^2)/10, \qquad y' = 17 - x - y.$$

■

Direction fields near $e_1 = (5, 12)$ and near $e_2 = (12, 5)$ for $x' = (169 - x^2 - y^2)/10$, $y' = 17 - x - y$ are shown in Figure 17.8A and B, respectively. They are quite different. Near $e_2 = (12, 5)$ the arrows point toward e_2; near $e_1 = (5, 12)$ some of the arrows do not point toward e_1. We will find that e_2 is *asymptotically stable*, and that e_1 does not meet the criterion that assures that it is asymptotically stable.

■

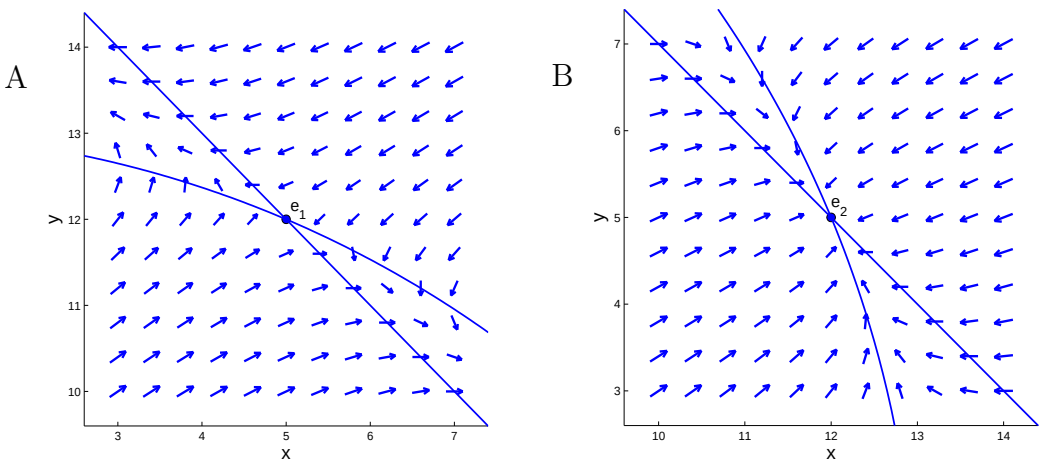


Figure 17.8: A. Direction field near the equilibrium point (5,12) of Equations 17.39. B. Direction field near the equilibrium point (12,5) of Equations 17.39.

An equilibrium point, (x_e, y_e) of Equations 17.38 is a point of intersection of the null curves and satisfies

$$f(x_e, y_e) = 0 \qquad \text{and} \qquad g(x_e, y_e) = 0.$$

If (x_e, y_e) is an equilibrium state, $x(t) = x_e$ and $y(t) = y_e$ are solutions of Equations 17.38 and are called *equilibrium solutions*.

Definition 17.4.1 An equilibrium state (x_e, y_e) of Equations 17.38 is *asymptotically stable* means that

a. (It is *stable*:) If ϵ is a positive number there is a positive number δ such that if (x_0, y_0) is a point and the distance from (x_0, y_0) to (x_e, y_e) is less than δ then the solution $(x(t), y(t))$ to Equations 17.38 with initial conditions $x(0) = x_0$ and $y(0) = y_0$ has the property that for all t the distance from $(x(t), y(t))$ to (x_e, y_e) is less than ϵ.

b. (And:) There is a positive number δ_0 such that if (x_0, y_0) is a point and the distance from (x_0, y_0) to (x_e, y_e) is less than δ_0 then the solution $(x(t), y(t))$ to Equations 17.38 with initial conditions $x(0) = x_0$ and $y(0) = y_0$ has the property that

$$\lim_{t\to\infty} x(t) = x_e \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = y_e$$

Definition 17.4.2 Suppose f and g are functions of two variables and (x_e, y_e) is in the domain of both f and g and suppose that f_1, f_2, g_1 and g_2 are all continuous on the interior of a circle with center at (x_e, y_e). The local linear approximation to the two homogeneous differential equations

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)) \end{aligned} \tag{17.40}$$

at an equilibrium point (x_e, y_e) is

$$\begin{aligned} x'(t) &= f_1(x_e, y_e) \times (x(t) - x_e) + f_2(x_e, y_e) \times (y(t) - y_e) \\ y'(t) &= g_1(x_e, y_e) \times (x(t) - x_e) + g_2(x_e, y_e) \times (y(t) - y_e) \end{aligned} \tag{17.41}$$

With $\xi(t) = x(t) - x_e$ and $\eta(t) = y(t) - y_e$ Equations 17.41 become

$$\begin{aligned} \xi'(t) &= f_1(x_e, y_e) \times \xi(t) + f_2(x_e, y_e) \times \eta(t) \\ \eta'(t) &= g_1(x_e, y_e) \times \xi(t) + g_2(x_e, y_e) \times \eta(t) \end{aligned} \tag{17.42}$$

The matrix

$$\begin{bmatrix} f_1(x, y) & f_2(x, y) \\ g_1(x, y) & g_2(x, y) \end{bmatrix} \tag{17.43}$$

is called the Jacobian matrix of the Differential Equations 17.40 at the point (x, y).

The theorem of this chapter is

Theorem 17.4.1 The differential equations 17.40 is asymptotically stable if the linear system 17.42 is asymptotically stable.

This theorem together with Theorem 17.3.1 provide a direct computational means to determine whether an equilibrium point is asymptotically stable. Its proof is beyond our scope, but it is easily understood and we use the theorem. We will apply it to several systems in the next section.

Example 17.4.1 (Continued) Equations 17.39,

$$\begin{aligned} x' &= (169 - x^2 - y^2)/10 \\ y' &= 17 - x - y \end{aligned} \tag{17.44}$$

has two equilibrium points, $e_1 = (5, 12)$ and $e_2 = (12, 5)$. The Jacobian of Equations 17.44 is

$$\begin{bmatrix} f_1(x,y) & f_2(x,y) \\ g_1(x,y) & g_2(x,y) \end{bmatrix} = \begin{bmatrix} -x/5 & -y/5 \\ -1 & -1 \end{bmatrix}.$$

For $e_1 = (5, 12)$ the Jacobian is

$$\begin{bmatrix} -x/5 & -y/5 \\ -1 & -1 \end{bmatrix}_{(x,y)=(5,12)} = \begin{bmatrix} -1 & -12/5 \\ -1 & -1 \end{bmatrix}.$$

The trace and determinant of the Jacobian are -2 and -7/5, respectively, and the characteristic roots of the local linear approximation to Equations 17.44 (the roots to $r^2 + 2r - 7/5 = 0$) are

$$r_1 = \frac{-2 + \sqrt{4 + 49/25}}{2} \doteq 0.4413, \quad \text{and} \quad r_2 = \frac{-2 - \sqrt{4 + 49/25}}{2} \doteq -4.4413.$$

Because one of the characteristic roots is positive we do not conclude that Equations 17.44 are asymptotically stable.

Explore 17.4.2 Show that the trace and determinant of the Jacobian of Equations 17.44 at $e_2 = (12, 5)$ are respectively -17/5 and 7/5 and that both characteristic roots are negative. Thus conclude that Equations 17.44 are asymptotically stable at (12,5). ■

Exercises for Section 17.4 Systems of two first order differential equations.

Exercise 17.4.1 Find the local linear approximation to the system

$$x' = x - x^2 - xy$$

$$y' = y - 0.5xy - 2y^2$$

a. At the equilibrium point (0,0).

b. At the equilibrium point (0,0.5).

c. At the equilibrium point (1,0).

d. At the equilibrium point (2/3,1/3).

For each of the local linear approximations, determine whether it is stable.

Exercise 17.4.2 For each of the systems, find all of the equilibrium points and determine whether the system is asymptotically stable at each equilibrium point.

a. $\begin{aligned} x' &= -xy \\ y' &= 1 - x - y \end{aligned}$ b. $\begin{aligned} x' &= x - 2xy \\ y' &= y + xy \end{aligned}$

c. $\begin{aligned} x' &= 2 - x^2 - y^2 \\ y' &= 1 - xy \end{aligned}$ d. $\begin{aligned} x' &= 5 - x^2 - y^2 \\ y' &= 2 - xy \end{aligned}$

Exercise 17.4.3 Show that (1,1), (2,2), and (3,3) are equilibrium points of

$$x' = 1 - \frac{11}{6}x + \frac{10xy}{11+xy}$$

$$y' = 1 - \frac{11}{6}y + \frac{10xy}{11+xy}$$

Determine which of these equilibrium points are stable. This example is patterned after a model Anderson and May[2] present to suggest that antibody level may switch from a lower level stable *unexposed state* to an upper level stable *actively immune state.*

17.5 Applications of Theorem 17.4.1 to biological systems.

Simple population models of competing species and predator-prey relations introduced by Alfred Lotka and Vito Volterra in the 1920's have formed the basis of discussion of stability and local stability.

17.5.1 Competition.

Assume there are two species, x and y, with populations sizes $x(t)$ and $y(t)$ that compete for the same food. We write

$$\begin{aligned} x'(t) &= r_x \times x(t) \times (1 - ax(t) - by(t)) \\ y'(t) &= r_y \times y(t) \times (1 - cx(t) - dy(t)) \end{aligned} \tag{17.45}$$

If b and c are zero, there is no competition and $x(t)$ and $y(t)$ grow in a logistic manner with maximum population sizes $1/a$ and $1/d$, respectively. In some formulations a and d are zero so that without competition, $x(t)$ and $y(t)$ grow exponentially. The number b measures the effect of y on x and in a sense measures how much the maximum supportable x is reduced in the presence of y. Similarly, c measures the effect of x on y.

Depending on the parameter values, there may either be an equilibrium point (x_e, y_e) with both $x_e > 0$ and $y_e > 0$ so that both populations exist 'in harmony,' or one population may be so competitive that the other is driven to extinction.

The equilibrium points of Equations 17.45 are found by solving

$$\begin{aligned} x' &= 0 & f(x,y) &= r_x x(1 - ax - by) &&= 0 \\ y' &= 0 & g(x,y) &= r_y \times y \times (1 - cx - dy) &&= 0 \end{aligned}$$

There are four solutions if $ad - bc \neq 0$.

$$\begin{aligned} x &= 0, & y &= 0 \\ x &= 0, & y &= 1/d \\ x &= 1/a, & y &= 0 \\ x &= \frac{d-b}{ad-bc}, & y &= \frac{a-c}{ad-bc}. \end{aligned}$$

[2] *ibid.*, p 36

The equilibrium $(0, 1/d)$ signals x is not present and y is at its maximum supportable level. Similarly $(1/a, 0)$ signals that x is at its maximum supportable population and y is not present.

The Jacobian matrix of

$$f(x,y) = r_x x - r_x a x^2 - r_x b x y \qquad \text{and} \qquad g(x,y) = r_y y - r_y c x y - r_y d y^2$$

is computed from the partial derivatives:

$$\begin{aligned} f_x(x,y) &= r_x - 2r_x a x - r_x b y & f_y(x,y) &= -r_x b x \\ g_x(x,y) &= -r_y c y & g_y(x,y) &= r_y - r_y c x - 2r_y d y \quad . \end{aligned}$$

The Jacobian matrices at the first three equilibrium points are

$$\begin{array}{ccc} (0,0) & (0,1/d) & (1/a,0) \\ \begin{bmatrix} r_x & 0 \\ 0 & r_y \end{bmatrix} & \begin{bmatrix} r_x(1-b/d) & 0 \\ -r_y(c/d) & -r_y \end{bmatrix} & \begin{bmatrix} -r_x & -r_x(b/d) \\ 0 & r_y(1-c/a) \end{bmatrix} \quad . \end{array}$$

The equilibrium (0,0) is nonstable. The trace of the Jacobian matrix is $r_x + r_y$ and is positive and if (x_0, y_0) is close to (0,0), (a small number of each of x and y is introduced into the environment), $(x(t), y(t)$ will move away from (0,0). (In fact, both $x(t)$ and $y(t)$ will initially increase.)

The equilibrium $(0, 1/d)$ is ambiguous. If $1 - b/d < 0$ then the trace of the Jacobian matrix, $r_x(1 - b/d) - r_y$ is negative and the determinant, $r_x(1 - b/d) \times (-r_y)$ is positive. In this case, $(0, 1/d)$ is a stable node. This is bad news for x. It means that if y is established (is at its maximum supportable population) and a few members of x immigrate, those members of x will not have enough offspring to become established; $x(t)$ will decrease to zero.

The condition $1 - b/d < 0$ implies that $b \times 1/d > 1$. Recall that b measures the competitive influence of y on x and $1/d$ is the maximum supportable y population. If the product of these two numbers is larger than 1, then y can exclude x from the region.

If $1 - b/d > 0$ the determinant of the Jacobian matrix is positive, both roots are real and at least one root is positive. In this case $(0, 1/d)$ is not stable and a few members of x entering the region will initially increase.

In a similar way, $(1/a, 0)$ may or may not be a stable equilibrium point of the system and introduction of a few members of y may not or may lead to initial growth.

In order to understand the equilibrium $(d-b)/(ad-bc), (a-c)/(ad-bc)$ we consider three examples.

Example 1	Example 2	Example 3
Two sheep.	Pig and sheep.	Two pigs.
$x' = 0.1x(1 - x - 0.5y)$	$x' = 0.1x(1 - x - 0.5y)$	$x' = 0.1x(1 - x - 1.4y)$
$y' = 0.2y(1 - 0.6x - y)$	$y' = 0.2y(1 - 1.4x - y)$	$y' = 0.2y(1 - 1.4x - y)$

The difference between these two examples is that x competes more severely with y in Example 2 (1.4) than in Example 1 (0.6), and both compete severely in Example 3.

In these examples and generally it is instructive to look at the *nullclines* of x' and y' shown in Figure 17.9. The nullclines are the curves along which $x' = 0$ or $y' = 0$.

In Figure 17.9A is the single nullcline for $x' = 0.1x(1 - x - 0.5y) = 0$. The graph is the straight line with equation $1 - x - 0.5y = 0$ and is marked $x' = 0$ in Figure 17.9A. For a point above or to the right of that line, $1 - x - 0.5y$ is negative and x' is negative at that point. Consequently, direction field arrows point to the left above the line $1 - x - 0.5y = 0$. Whether they point to the left and up or to the left and down depends on y' at that location and one of each type is displayed. Similarly, direction field arrows below the line point to the right. Direction field arrows exactly on the line are vertical.

In Figure 17.9B nullclines for both $x' = 0.1x(1 - x - 0.5y) = 0$ and $y' = 0.2x(1 - 0.6x - y) = 0$. They are the graphs of

$$\begin{array}{rcl} 1 - x - 0.5y & = & 0 \\ 1 - 0.6x - y & = & 0 \end{array} \qquad \text{which intersect at} \qquad (5/7, 4/7)$$

As in Figure 17.9A, direction field arrows above $1 - x - 0.5y = 0$ point to the left. If those arrows are also above $1 - 0.6x - y = 0$, the $y' < 0$ and the arrow point to the left and downward. Arrows below both lines point up and to the right. The four equilibrium points A, B, C, and D are (0,0), (0,1), (1,0) and (5/7,4/7).

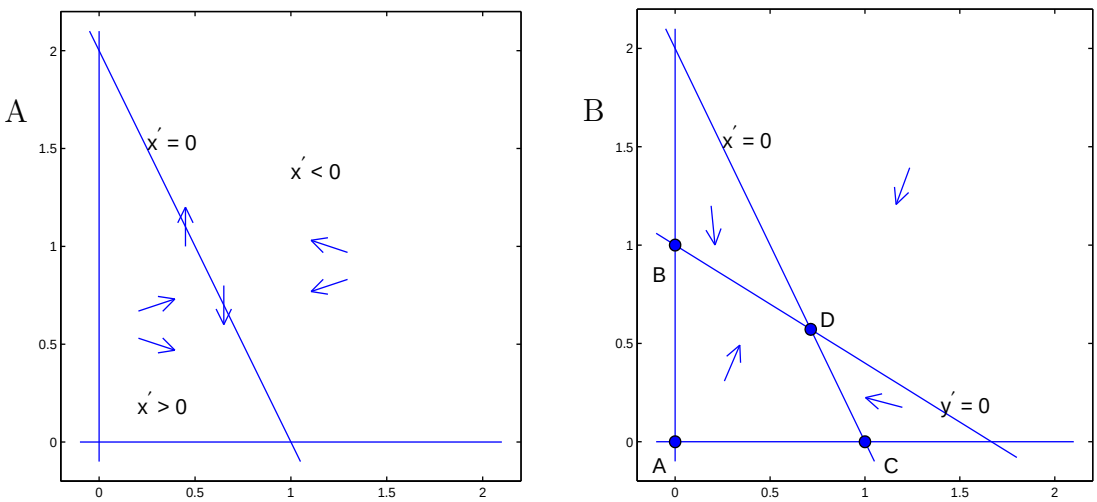


Figure 17.9: Nullcline of $x' = 0.1x(1 - x - 0.5y)$ in A, and of both $x' = 0.1x(1 - x - 0.5y)$ and $y' = 0.2y(1 - 0.6x - y)$ in B.

A direction field for Example 1 appears in Figure 17.10A. It appears that the equilibrium points A, B, and C are nonstable and this follows from discussion above. It also appears that D is a stable node, and we show this next.

The local linear approximation to Example 1 can be computed from:

$$\begin{array}{rcl} F(x,y) & = & 0.1x - 0.1x^2 - 0.05xy \\ F_1(x,y) & = & 0.1 - 0.2x - 0.05y \\ G_1(x,y) & = & -0.12y \\ F_1(5/7,4/7) & = & -0.5/7 \\ G_1(5/7,4/7) & = & -0.48/7 \end{array} \qquad \begin{array}{rcl} G(x,y) & = & 0.2y - 0.12xy - 0.2y^2 \\ F2(x,y) & = & -0.05x \\ G_2(x,y) & = & 0.2 - 0.12x - 0.4y \\ F_2(5/7,4/7) & = & -0.25/7 \\ G_2(5/7,4/7) & = & -0.8/7 \end{array}$$

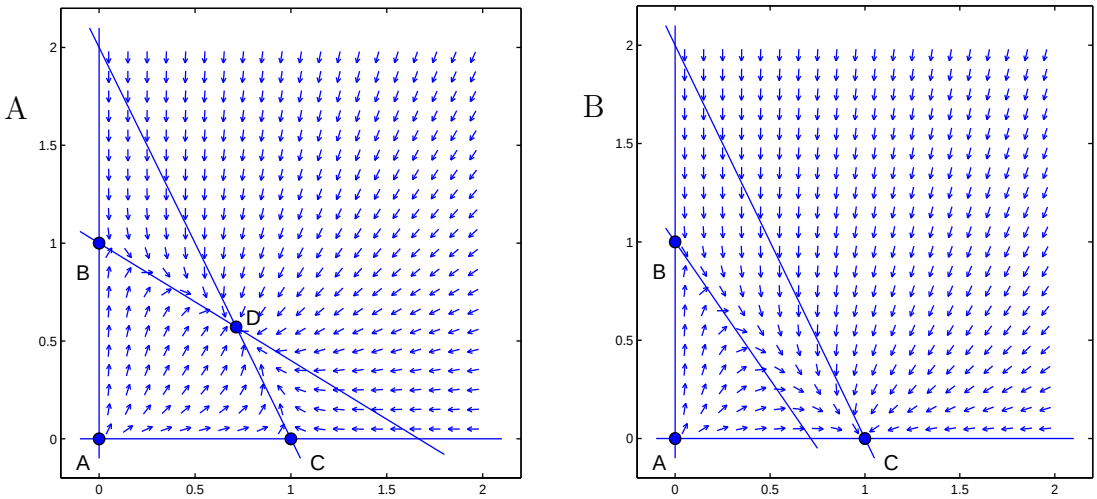


Figure 17.10: A. Direction field for Example 1. Direction field for Example 2.

For the matrix

$$M = \begin{bmatrix} -2.3/7 & -2.5/7 \\ -0.64/7 & -1/7 \end{bmatrix}$$

the trace is -0.5/7 - 0.8/7 = -1.3/7 is negative and the determinant (-0.5/7)(-0.8/7 - (-0.48/7)(-0.25/7) = 0.04/7 is positive. According to the summary in the last section, M has two negative characteristic roots and $(5/7, 4/7)$ is a stable node of Example 1.

The direction field for Example 2 is shown in Figure 17.10B. The 'equilibrium' point D has moved out of the first quadrant, and has negative y coordinate. You can see from the direction field that all solutions lead to C. x competes severely with y and eventually only x remains in the region.

The equilibrium point D reappears in Example 3, as seen in Figure 17.11. But D is not stable. One solution actually leads to D from above the graph and another from below the graph but all other curves diverge either to B or C. In this Example only one of x and y has long term presence in the region, depending on the initial conditions x_0, y_0.

17.5.2 Predator-prey

Volterra introduced the predator-prey model where x is the prey and y is the predator.

$$\begin{aligned} x'(t) &= ax - bxy \\ y'(t) &= -cy + dxy \end{aligned} \tag{17.46}$$

The model suggests that if the predator is not present ($y = 0$) the prey will grow exponentially $x(t) = x_0 e^{at}$ and if the prey is not present ($x = 0$) the predator will decay exponentially $y(t) = y_0 e^{-ct}$. The presence of y reduces the growth rate of x and the presence of x increases the growth rate of y.

There are two critical points of Equations 17.46 found by solving

$$\begin{aligned} ax - bxy &= 0 \qquad & (x_1, y_1) &= (0, 0) \\ -cy + dxy &= 0 \qquad & (x_2, y_2) &= (c/d, a/b) \end{aligned}$$

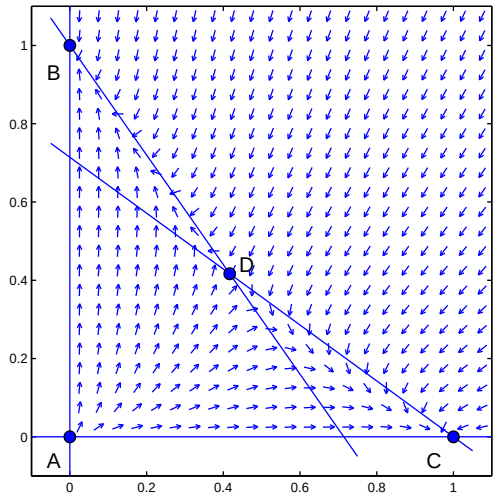


Figure 17.11: A. Direction field for Example 3. The equilibrium D is not stable; it is a saddle point.

The nullclines are the graphs of $x = c/d$ and $y = a/b$ and $x = 0$ and $y = 0$ and run vertically or horizontally through the equilibrium points. The Jacobian matrices are easily computed.

$$\begin{aligned} F(x,y) &= ax - bxy & G(x,y) &= -cy + dxy \\ F_1(x,y) &= a - by & F2(x,y) &= -bx \\ G_1(x,y) &= -cy & G_2(x,y) &= -c + dx \end{aligned}$$

At the equilibrium points the Jacobian matrices are

$$\text{At} \quad (0,0), \quad \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}. \qquad \text{At} \quad (c/d, a/b)), \quad \begin{bmatrix} 0 & -bc/d \\ -ca/b & 0 \end{bmatrix}.$$

Neither equilibrium point is stable. At (0,0) there is one positive root and one negative root. At $(c/d, a/b)$ the roots are complex with real part equal to 0. The solution curves of the local linear approximation are ellipses.

The direction field and equilibrium points of

$$\begin{aligned} x' &= 0.2x - 0.2xy \\ y' &= -0.1y + 0.08xy \end{aligned}$$

appear in Figure 17.12. The solution down the vertical axis to the origin shows the decline of predator without prey, the solution leading away from the origin along the horizontal axis shows the growth of prey without predators. The solutions near the equilibrium (1.25,1.0) appear to circle around (1.25,1.0). In fact, they do.

Danger: Outrageous manipulations ahead. Think of one of those curves as begin described by a function s such that

$$y = s(x), \qquad \text{and write} \qquad y(t) = s(x(t)).$$

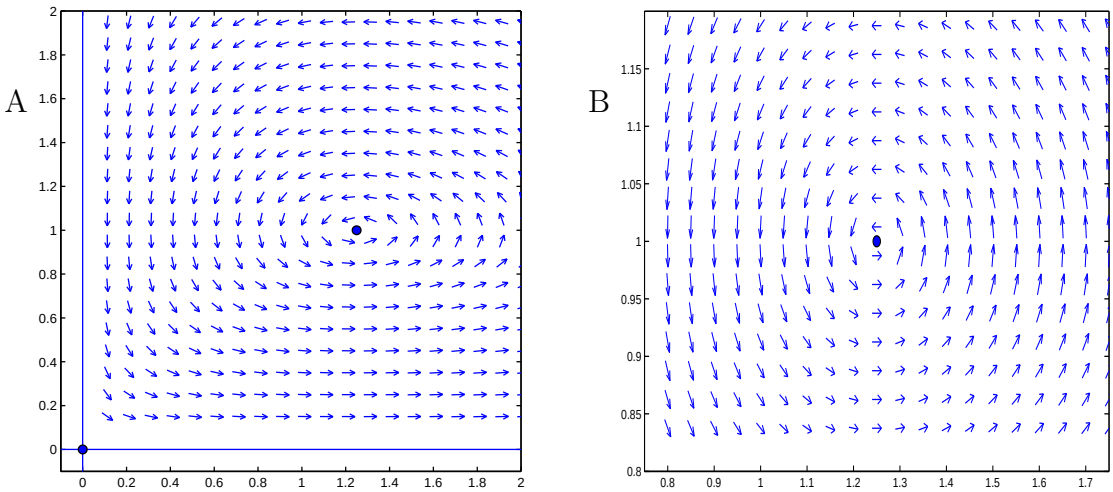


Figure 17.12: A. Direction field for Equations 17.47. B. Magnified view near the equilibrium $(1.25, 1)$.

By the chain rule

$$y'(t) = s'(x(t))x'(t), \qquad \text{and} \qquad s'(x(t)) = \frac{y'(t)}{x'(t)}.$$

Now $s'(x)$ means the derivative of s with respect to x and would be written in Leibnitz notation as $\frac{ds}{dx}$, and because $y = s(x)$ it may also be written as $\frac{dy}{dx}$. We write

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-0.1y + 0.08xy}{0.2x - 0.2xy}.$$

The variables are separable in this differential equation.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-0.1y + 0.08xy}{0.2x - 0.2xy} \\
\left(\frac{0.2}{y} - 0.2\right)\frac{dy}{dx} &= \frac{-0.1}{x} + 0.08 \\
0.2\ln y - 0.2y &= -0.1\ln x + 0.08x + C \\
0.2\ln y + 0.1\ln x - 0.2y = 0.08x &= C
\end{aligned}$$

where C is a constant of integration.

Shown in Figure 17.13A is a graph in three dimensions of the function

$$F(x, y) = 0.2\ln y + 0.1\ln x - 0.2y - 0.08x, \qquad 1 \le x \le 1.4, \qquad 0.8 \le y \le 1.2 \tag{17.47}$$

The high point of F is above the equilibrium point $(1.25, 1.0, -0.2776)$. A horizontal plane at $z = -0.279$ intersects the graph of F in a closed curve in Figure 17.13; every such horizontal plane defines a closed solution curve to Equations 17.47 that surrounds the equilibrium point $(1.25, 1.0)$.

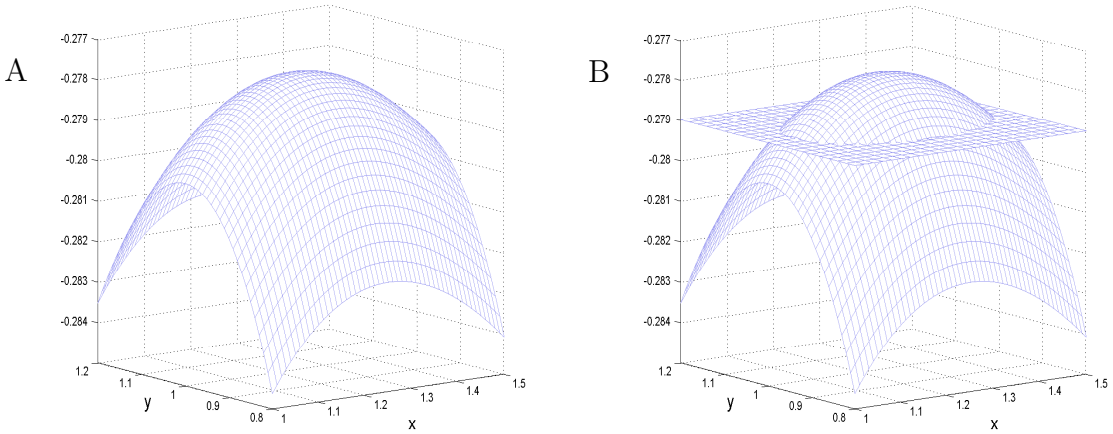


Figure 17.13: A. Graph of Equation 17.47, $F(x,y) = 0.2\ln y + 0.1\ln x - 0.2y - 0.08x$. B. Same graph with a horizontal plane.

Explore 17.5.1 Show that the hight point of

$$F(x,y) = 0.2\ln y + 0.1\ln x - 0.2y - 0.08x, \qquad 1 \le x \le 1.4, \qquad 0.8 \le y \le 1.2$$

is $(1.25, 1.0, F(1.25, 1.0))$. Use Theorem 10.2.1. Begin by solving simultaneously,

$$\frac{\partial F(x,y)}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = 0$$

There is only one solution, (x_m, y_m). Observe that

$$\left.\frac{\partial^2 F(x,y)}{\partial x^2}\right|_{(x_m,y_m)} < 0, \qquad \text{and} \qquad \left.\frac{\partial^2 F(x,y)}{\partial y^2}\right|_{(x_m,y_m)} < 0, \qquad \text{and}$$

$$\Delta = \left.\frac{\partial^2 F(x,y)}{\partial x^2}\right|_{(x_m,y_m)} \left.\frac{\partial^2 F(x,y)}{\partial y^2}\right|_{(x_m,y_m)} - \left(\left.\frac{\partial^2 F(x,y)}{\partial x \partial y}\right|_{(x_m,y_m)}\right)^2 > 0.$$

■

17.5.3 Susceptible, Infected, Recovered.

Again, an elementary model of infectious disease can form the basis of discussions. Assume there is a virus that is spread by contact from infected individuals to susceptible individuals and that infected individuals recover. Let $s(t)$, $i(t)$ and $r(t)$ denote the numbers of susceptible, infected, and recovered individuals, respectively. Assume that the rate at which infection spreads is proportional to the number of susceptible and to the number of infected. Also assume that recovered individuals have life time immunity. We write the equations:

$$\begin{aligned} s'(t) &= -\beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\ r'(t) &= \gamma i(t) \end{aligned} \tag{17.48}$$

The parameter β is crucial. It reflects the probability of transmission of the disease and depends on the frequency of contacts between people and the ease of transmission of the disease.

Although three functions are written, the first two equations are independent of $r(t)$ and can be analyzed alone. The equilibrium points are $(0,0)$ and $(s_0,0)$. The interesting equilibrium is $(s_0,0)$, and at $(s_0,0)$ the Jacobian matrix is

$$M = \begin{bmatrix} 0 & -\beta s_0 \\ 0 & \beta s_0 - \gamma \end{bmatrix} \tag{17.49}$$

The roots of M are real and are zero and negative if $\beta S_0 < \gamma 0$ and zero and positive if $\beta s_0 > \gamma$.

$\beta S_0 = \gamma$ marks an important threshold. If $\beta s_0 < \gamma$, then $(s_0,0)$ is a stable equilibrium. If a small number of infected individuals enter the populations with $\beta s_0 < \gamma$, the disease does not spread. If $\beta s_0 > \gamma$, then $(s_0,0)$ is an unstable equilibrium. If a small number of infected individuals enter the populations with $\beta s_0 > \gamma$, the disease will spread.

Exercises for Section 17.5, Applications of Theorem 17.4.1 to biological systems.

Exercise 17.5.1 Draw the nullclines and some direction arrows and analyze the equilibria of the following competition models.

$$\begin{array}{lrcl}
\text{a.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.2x(t) - 0.4y(t)) \\
 & y'(t) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \\
 & & & \\
\text{b.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.2x(t) - 0.8y(t)) \\
 & y'(t) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \\
 & & & \\
\text{c.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.6x(t) - 0.4y(t)) \\
 & y'(t) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \\
\text{d.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.4x(t) - 0.4y(t)) \\
 & y'(t) & = & 0.1 \times y(t) \times (1 - 0.3x(t) - 0.5y(t))
\end{array}$$

Exercise 17.5.2 Show that a stable equilibrium exists for the competition Equations **??**,

$$\begin{array}{rcl}
x'(t) & = & r_x \times x(t) \times (1 - ax(t) - by(t)) \\
y'(t) & = & r_y \times y(t) \times (1 - cx(t) - dy(t)),
\end{array}$$

if $1/c > 1/a$ and $1/b > 1/d$. (Suggestion: Draw the nullclines and candidate direction arrows in each of the four regions bounded by the nullclines.)

Exercise 17.5.3 The predator-prey equations assume that with no predator, the prey grows exponentially. Alternatively one might assume that with no predator, the prey grow according to a logistic (Verhultz) model. Write the predator-prey equations so that without predators the prey grows according to a logistic model. Find an equilibrium for which both predator and prey exist, and determine the character of that equilibrium.

Exercise 17.5.4 Anderson and May[3] give the following model of effector cells (helper and cytotoxic T-cells), E, that limit viral population, V, growth in a human body.

$$\begin{aligned} dE/dt &= \Lambda - \mu E + \epsilon V E \\ dV/dt &= rV - \sigma V E \end{aligned} \tag{17.50}$$

a. Λ is intrinsic production rate of effector cells from bone marrow. Give similar meaning to each of the other four terms on the RHS of Equations 17.50.

b. Find the equilibrium effector cell population, $\hat{E}$, in the absence of virus.

c. Suppose an inoculum V_0 of virus is introduced into the body. Find conditions on r, σ, and $\hat{E}$ in order that the viral population will increase.

d. If that condition is met and the viral population increases, there will be an equilibrium state,

$$E^* = r/\sigma, qquadV^* = \mu(r - \hat{E}\sigma)/(\epsilon r).$$

Anderson and May report this system to be asymptotically stable, but only weakly so. The Jacobian at (E^*, V^*) is

$$\begin{bmatrix} -2\mu + \frac{\Lambda\sigma}{r} & \epsilon\frac{r}{\sigma} \\ \sigma\frac{\mu r - \Lambda\sigma}{\epsilon r} & 0 \end{bmatrix},$$

and is a bit difficult to analyze. Use $\Lambda = 1$, $\mu = 0.5$, $\epsilon = 0.02$, $r = 0.25$, and $\sigma = 0.01$ and compute E^*, V^*, and the stability at (E^*, V^*).

e. Let $E_0 = 2$, $V_0 = 1$, $\Lambda = 1$, $\mu = 0.5$, $\epsilon = 0.02$, $r = 0.25$, and $\sigma = 0.01$. Approximate the solutions to Equations 17.50 using either the trapezoid rule or using Euler's rule:

$$\begin{aligned} E_{t+1} &= E_t + \Lambda - \mu E_t + \epsilon V_t E_t \\ V_{t+1} &= V_t + rV_t - \sigma V_t E_t \end{aligned} \qquad t = 0, 1, \cdots, 60 \tag{17.51}$$

A MATLAB program to approximate the solution to Equations 17.50 using the trapezoid method:

```
close all;clc;clear
e0=2;
v0=1;
lam=1;mu=0.5;eps=0.02;rrr=0.25;sig=0.01;
E(1)=e0;
V(1)=v0;
for i = 1:60
    ME=lam - mu*E(i)+eps*V(i)*E(i);
    MV= rrr*V(i) - sig*V(i)*E(i);
    EP=E(i)+ME;
    VP=V(i)+MV;
```

[3]R. M. Anderson and R. M. May, *Infectious Diseases of Humans*, Oxford University Press, Oxford, 1991, p33

```
    MEP=lam - mu*EP+eps*VP*EP;
    MVP=rrr*VP - sig*VP*EP;
%%    MEP=ME;  %%Use this to revert to Euler's Method.
%%    MVP=MV;  %%Use this to revert to Euler's Method.
    E(i+1)=E(i)+(ME+MEP)/2;
    V(i+1)=V(i) + (MV+MVP)/2;
end
plot(E,'+','linewidth',2)
hold
plot(V,'o','linewidth',2)
xlabel('Time','fontsize',16)
ylabel('Effector cells +, Virus o','fontsize',16)
```

Exercise 17.5.5 The local linear approximation to SIR Equations 17.48 at the equilibrium $(s_0, 0)$ uses the Jacobian matrix17.49 and may be written

$$\begin{aligned} x'(t) &= -\beta s_0 y & \qquad x_0 &= s_0 \\ y'(t) &= (\beta s_0 - \gamma) y & \qquad y_0 &= \epsilon \end{aligned}$$

Think of ϵ as a small number of infected introduced to the population. Show that the solution to these equations is

$$\begin{aligned} x(t) &= s_0 + \frac{\beta s_0 \epsilon}{\beta s_0 - \gamma}(1 - e^{(\beta s_0 - \gamma)t}) \\ y(t) &= \epsilon e^{(\beta s_0 - \gamma)t} \end{aligned}$$

In the case that $\beta s_0 - \gamma < 0$ find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$, and discuss their meaning.

Exercise 17.5.6 It may be that recovered individuals do not have life time immunity, become susceptible after a period p,and one may write

$$\begin{aligned} s'(t) &= \alpha r(t-p) - \beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\ r'(t) &= \gamma i(t) - \alpha r(t-p) \end{aligned}$$

This system is considerably more complex than Equations 17.48, and is simplified by letting $p = 0$.

$$\begin{aligned} s'(t) &= \alpha r(t) - \beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\ r'(t) &= \gamma i(t) - \alpha r(t) \end{aligned}$$

This system involves three functions and three equations and is beyond our exposition. However, you may be able to analyze it.

a. Show that $(s_0, 0, 0)$ is an equilibrium point, for any s_0.

b. Guess or compute the Jacobian matrix at $(s_0, 0, 0)$.

c. The characteristic values of the Jacobian matrix at $(s_0, 0, 0)$ are $\beta s_0 - \gamma$ and $-\alpha$. What is the criterion for an epidemic (the number of infected will increase when a small number, ϵ, of infected individuals enter a population of s_0 susceptible).

d. Solve the equations

$$\begin{aligned} x'(t) &= -\beta s_0\, y(t) + \alpha z(t) & x(0) &= s_0 \\ y'(t) &= (\beta s_0 - \gamma)\, y(t) & y(0) &= \epsilon \\ z'(t) &= \gamma y(t) - \alpha z(t) & z(0) &= 0 \end{aligned}$$

Solve first for $y(t)$, then for $z(t)$ and then for $x(t)$.

Exercise 17.5.7 Suppose in the SIR Equations 12.4 that $\beta s_0 > \gamma$ so that $(s_0, 0)$ is a non-stable equilibrium and $i_0 = \epsilon$ where ϵ is "small" and positive. Then the infection will spread and this problem shows that there will be a positive lower bound, b, on the susceptible, $s(t) \geq b$ for all t – not everyone gets sick.

We begin with the first two of Equations 12.4

$$\begin{aligned} s'(t) &= -\beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t). \end{aligned} \tag{17.52}$$

Shown in Figure 17.5.7 is a direction field of $(s(t), i(t))$ with $\beta = 0.01$ and $\gamma = 0.012$, and a solution to Equations 17.52 with $(s_0, i_0) = (1, 0.001)$ marked in red. The midpoint of each interval of the direction field is (s_k, i_k) and the slope of the interval is the ratio

$$\frac{i'}{s'} = \frac{\beta \times s_k \times i_k - \gamma i_k}{-\beta \times s_k \times i_k}.$$

a. The arrows on the solution in Figure 17.5.7 point towards increasing time. Describe the numbers of susceptible and infective as time progresses.

b. The point marked E on the graph is (0.685,0). What are the dynamics of the infection at E?

c. Compute the ratio

$$\frac{\frac{di}{dt}}{\frac{ds}{dt}} = \frac{\beta \times s(t) \times i(t) - \gamma i(t)}{-\beta \times s(t) \times i(t)}$$

and argue that

$$\frac{di}{ds} = -1 + \frac{\gamma}{\beta}\frac{1}{s}$$

d. Show that the solution to the previous equation is

$$i = i_0 + s_0 - s + \frac{\gamma}{\beta} \ln \frac{s}{s_0}.$$

e. Argue that there is a positive value of s for which $i = 0$.

Figure for Exercise 17.5.7 Direction field for Equations 17.52 with $\beta = 0.01$ and $\gamma = 0.012$, and a solution to Equations 17.52 with $(s_0, i_0) = (1, 0.001)$.