1.1 Sequences

**Definition of convergence:** We say a sequence \( a_n \) converges to \( L \) (denoted \( a_n \to L \)) if for all \( \varepsilon > 0 \), there exists an \( N \) such that for all \( n > N \), \( |a_n - L| < \varepsilon \).

In what follows, \( n \) is approaching infinity, \( c \) is any constant, and \( p \) is a positive constant.

**Convergent sequences:**

- \( \frac{1}{n} \to 0 \)
- \( \frac{\ln(n)}{n} \to 0 \)
- \( \frac{(-1)^n}{n} \to 0 \)
- \( c - \frac{1}{n} \to c \)
- \( c^n \to 0 \) if \( |c| < 1 \)
- \( c^{-n} \to 0 \) if \( |c| > 1 \)
- \( p^{1/n} \to 1 \)
- \( \sqrt[n]{n} \to 1 \)
- \( \left(1 + \frac{c}{n}\right)^n \to e^c \)
- \( \frac{c^n}{n!} \to 0 \)

1.2 Hierarchy of Functions

We consider how fast different types of functions approach infinity.

\[ \cdots < \ln(\ln(n)) < \ln(n) < n^{1/k} < n^k < k^n < n! < n^n < n^{n^n} < \cdots \]
1.3 Series

- Divergence Test
  If \( a_n \not\to 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges. In other words, if \( \lim_{n \to \infty} a_n \) does not exist or \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) will diverge.

- Geometric Series
  \[
a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad |r| < 1, a \neq 0
  \]

- Harmonic Series
  The Harmonic Series \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \) diverges.

- \( p \)-Series Test
  The \( p \)-series
  \[
  \sum_{n=1}^{\infty} \frac{1}{n^p} \rightarrow \begin{cases} 
  \text{converge} & \text{if } p > 1 \\
  \text{diverge} & \text{if } p \leq 1
  \end{cases}
  \]

- Integral Test
  Let \( \{a_n\} \) be a sequence of positive terms. Suppose \( f \) is a continuous, positive, decreasing function, and that \( a_n = f(n) \). Then the series \( \sum_{n=1}^{\infty} a_n \) and the integral \( \int_{1}^{\infty} f(x) \, dx \) both converge or both diverge.

- Comparison Test
  Let \( \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n \) be series with nonnegative terms. Assume there exists an \( M > 0 \) such that for all \( n > M \), \( a_n \leq b_n \leq c_n \). Then
  
  (i) If \( \sum_{n=1}^{\infty} c_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) also converges.

  (ii) If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) also diverges.
- Limit Comparison Test
  Let $\sum a_n$ and $\sum b_n$ be series with strictly positive terms.

  1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

  2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges as well.

  3. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges as well.

- Ratio Test
  Let $\sum a_n$ be a series with strictly positive terms, and suppose $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = R$.

  (a) If $R < 1$, then $\sum a_n$ converges.

  (b) If $R > 1$, then $\sum a_n$ diverges.

  (c) If $R = 1$, the test is inconclusive.

- Root Test
  Let $\sum a_n$ be a series with $a_n \geq 0$, and suppose $\lim_{n \to \infty} \sqrt[n]{a_n} = R$.

  (a) If $R < 1$, then $\sum a_n$ converges.

  (b) If $R > 1$, then $\sum a_n$ diverges.

  (c) If $R = 1$, the test is inconclusive.

- Alternating Series Test
  The alternating series
  $$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

  Converges if all three of the following conditions are satisfied:

  1. All the $b_n$’s are positive
  2. $b_{n+1} \leq b_n$ for all $n > N$
  3. $\lim_{n \to \infty} b_n = 0$

- Absolute Convergence Test
  If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$. 
1.4 Strategies

1. The first thing we should check is the Divergence Test. Unless \( a_n \to 0 \), the series diverges.

2. If we can spot a Geometric Series, then we know right away if it converges or diverges since \( \sum ar^n \) converges only if \( |r| < 1 \); otherwise it diverges.

3. \( p \)-series are also easy to spot and check convergence. \( \sum 1/n^p \) converges if \( p > 1 \); otherwise it diverges.

4. If the series has nonnegative terms: Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.

5. Series with some negative terms: Does \( \sum |a_n| \) converge? If yes, so does \( \sum a_n \) since absolute convergence implies convergence.

6. Alternating series: \( \sum a_n \) converges if the series satisfies the conditions of the Alternating Series Test.