Math 131: Homework Assignment #3

Due Jan 27

1. Two people toss a true coin n times each.

   (a) Show that the probability, p_n, the they will score the same number of heads is
   $$p_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

   (b) Show that for $n \to \infty$,
   $$p_n \sim \frac{1}{\sqrt{\pi n}}$$
   Show that there is approximately a 1.2% error using this approximation for $n = 10$.

2. (a) Prove *Bayes's formula*: If A_1, A_2, $\ldots A_n$ is a partition of Ω, each A_i having positive probability, then
   $$\mathrm{P}(A_j|B) = \frac{\mathrm{P}(B|A_j)\mathrm{P}(A_j)}{\sum_{i=1}^n \mathrm{P}(B|A_i)\mathrm{P}(A_i)}$$

   (b) One coin in a collection of 65 has two heads. The rest are fair. If a coin, chosen at random from the lot and then tossed, turns up heads 6 times in a row, what is the probability that it is the two-headed coin?

3. In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted it won't rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3.

   (a) Find the probability that Pickwick has no umbrella, given that it rains.

   (b) Find the probability that it doesn't rain, given that he brings his umbrella.

4. (a) Prove Boole's inequalities
   $$\mathrm{P}\left(\bigcup_{i=1}^n A_i\right) \le \sum_{i=1}^n \mathrm{P}(A_i), \quad \mathrm{P}\left(\bigcap_{i=1}^n A_i\right) \ge 1 - \sum_{i=1}^n \mathrm{P}(A_i^c)$$
   Hint: Prove by induction. In the induction set write $\bigcup_{i=1}^{m+1} A_i = B \bigcup A_{m+1}$ where $B = \bigcup_{i=1}^m A_i$ and then apply the inclusion-exclusion formula for $\mathrm{P}(A \cup B)$.

(b) Ten percent of the surface of a sphere S is colored blue, the rest is red. Show that, irrespective of the manner in which the colors are distributed, it is possible to inscribe a cube in S with all its vertices red. Hint: Let B_r be the event that the r^{th} vertex of a randomly selected cube is blue. First explain why $\mathrm{P}(B_r) = 1/10$. Now look at the event $\bigcup_{r=1}^{8} B_r$. Explain why this is the event that at least one vertex is blue. Now estimate $\mathrm{P}(\bigcup_{r=1}^{8} B_r)$ using Boole's inequality.

5. *The Polya Urn:* An urn has b black balls and r red balls. A ball is drawn at random. It is replaced and, moreover, $c > 0$ balls of the color drawn are added. A new random drawing is made from the urn (now containing $r + b + c$ balls), and this procedure is repeated.

(a) Let $p_k(n)$ be the probability of exactly k black balls in the *first* n drawings. (So the remaining $n - k$ drawings result in red balls.) Prove that

$$p_k(n) = \frac{b(b+c)(b+2c)\cdots(b+kc-c)r(r+c)\cdots(r+(n-k)c-c)}{(b+r)(b+r+c)(b+r+2c)\cdots(b+r+nc-c)} \tag{1}$$

(b) Show that in any other ordering of the k black balls and $n - k$ red balls results in the same factors in (1) but just arranged in a different order.

(c) Thus show that the probability $p_{k,n}$ that n drawings result in k black balls and $n - k$ red balls in any order is

$$p_{k,n} = \frac{\binom{k-1+b/c}{k}\binom{n-k-1+r/c}{n-k}}{\binom{n-1+(b+r)/c}{n}}$$