Math 135A: Homework Assignment #4

1. Recall that if X has geometric distribution then

$$\mathrm{P}(X = k) = q^k(1-q), \;\; k = 0, 1, 2, \ldots$$

   (a) Compute $E(X)$.

   (b) Compute $\mathrm{var}(X)$.

   (c) Compute $E(X^3)$.

   (d) In a Bernoulli process, let X denote the number of failures before the first success. Show that X has geometric distribution.

   (e) In a Bernoulli process, let T_r denote the waiting time for r successes. Find $E(T_r)$ and $\mathrm{var}(T_r)$. Hint: Let X_k denote the number of failures between the $(k-1)^{st}$ and k^{th} successes. Show that X_k has geometric distribution and that $T_r = X_1 + X_2 + \cdots + X_r$.

2. Let S_n be a random variable having binomial distribution; that is,

$$\mathrm{P}(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \; k = 0, 1, \ldots, n.$$

   In class we proved that

$$\mathrm{P}\left(\frac{S_n}{n} \geq p + \varepsilon\right) \leq \exp(-\frac{1}{4}\, n\, \varepsilon^2), \; \varepsilon > 0.$$

   Prove

$$\mathrm{P}\left(\frac{S_n}{n} \leq p - \varepsilon\right) \leq \exp(-\frac{1}{4}\, n\, \varepsilon^2), \; \varepsilon > 0.$$

3. *Simulating a perfect coin.* Given a biased coin such that the probability of heads is α, we simulate a perfect coin as follows: Throw the biased coin twice. Interpret HT as success and TH as failure; if neither event occurs repeat the throws until a decision is reached. (a) Show that this model leads to Bernoulli trials with $p = 1/2$.