Math 135A: Homework Assignment #6
The Method of Generating Functions

Let X be a random variable assuming nonnegative integer values. For $0 < s < 1$ define

$$G_X(s) = E(s^X) = \sum_{n=0}^{\infty} s^n \mathrm{P}(X = n). \tag{1}$$

G_X is called the *generating function* of the random variable X. Here $E(\cdot)$ denotes expectation. Similarly, define

$$E_X(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \mathrm{P}(X = n), \tag{2}$$

the *exponential generating function* associated to the random variable X.

1. In each of the following cases, compute G_X.

   (a) If X is a Bernoulli random variable with $\mathrm{P}(X = 1) = p$ and $\mathrm{P}(X = 0) = q = 1 - p$

   (b) If X is geometrically distributed with parameter q, i.e. $\mathrm{P}(X = n) = q^n(1 - q)$, $n = 0, 1, 2, \ldots$.

   (c) If X is Poisson distributed with parameter λ, i.e.

   $$\mathrm{P}(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

2. If X has generating function G_X, prove that

   $$E(X) = G'(1)$$

   where prime denotes differentiation with respect to s; and, of course, $E(\cdot)$ denotes expectation. Find a formula for $\mathrm{var}(X)$ in terms of G_X and its derivatives.

3. (a) If X and Y are independent random variables, prove

   $$G_{X+Y}(s) = G_X(s)\, G_Y(s)$$

   (b) Using the above property, find the generating function of

   $$S_n := X_1 + \cdots + X_n$$

   where X_j are independent, identically distributed Bernoulli random variables.

4. (a) Prove: If X_1, $X_2, \ldots$ is a sequence of independent and identically distributed random variables with common generating function G_X, and $N(\geq 0$ is a random variable which is independent of the X_i and has generating function G_N, then

$$S := X_1 + X_2 + \cdots + X_N$$

has generating given by

$$G_S(s) = G_N(G_X(s))$$

(b) A hen lays N eggs, where N is Poisson distributed with parameter λ. Each egg hatches with probability p, independently of all other eggs. Let K be the number of chicks. Then $K = X_1 + \cdots + X_N$. Prove

$$G_K(s) = e^{\lambda p(s-1)}$$

and hence; $E(K) = \lambda p$.