Math 135A: HW Assignment #7

Let X be a random variable assuming nonnegative integer values. Recall that for $0 < s < 1$, the *generating function* of X is

$$G_X(s) := \mathbb{E}(s^X) = \sum_{n=0}^{\infty} s^n \, \mathbb{P}(X = n). \tag{1}$$

Also recall that if X and Y are independent random variables, each assuming nonnegative integer values, that

$$G_{X+Y}(s) = G_X(s) G_Y(s).$$

1. Let N be Poisson distributed with parameter λ. Show that, for any function $g : \mathbb{R} \to \mathbb{R}$ such that the expectations exist,

$$\mathbb{E}\left(N g(N)\right) = \lambda \, \mathbb{E}\left(g(N+1)\right).$$

2. Let $\{X_j\}$ be a sequence of mutually independent random variables with common distribution

$$\mathbb{P}(X_k = j) = p_j$$

Let $G_X(s)$ denote the common generating function of X_j. Consider the sum

$$S_N = X_1 + X_2 + \cdots + X_N$$

where the number N of terms is a random variable independent of the X_j. Let $\mathbb{P}(N = n) = g_n$ be the distribution of N.

(a) Show that the generating function of S_N is given by

$$G_{S_N}(s) = G_N(G_X(s))$$

where G_N is the generating function of N.

(b) Using the preceding result to show that

$$\begin{aligned} \mathbb{E}(S_N) &= \mathbb{E}(N) \, \mathbb{E}(X) \\ \operatorname{var}(S_N) &= \mathbb{E}(N) \operatorname{var}(X) + \operatorname{var}(N) \, (\mathbb{E}(X))^2 \end{aligned}$$

(c) In case that N has Poisson distribution with parameter λ, show that the preceding result implies

$$G_{S_N}(s) = e^{-\lambda + \lambda G_X(s)}$$

3. Recall that in the *coupon problem* we expressed the waiting time T_n to collect all n coupons as a sum of n independent random variables X_k where X_k has geometric distribution with $p_k = (n-k)/n$. That is,

$$T_n = X_0 + X_1 + \cdots + X_{n-1}$$

with

$$\mathbb{P}(X_k = j) = q_k^{j-1} p_k, \;\; p_k = \frac{n-k}{n}, \; q_k = 1 - p_k, \;\; j = 1, 2, 3, \ldots$$

(a) If G_{T_n} denotes the generating function of T_n, show that

$$G_{T_n}(s) = \frac{(n-1)!\, s^n}{(n-s)(n-2s)(n-3s)\cdots(n-(n-1)s)}$$

(b) For $n = 4$ (four coupons), show that for $j \geq 4$

$$\mathbb{P}(T_4 = j) = \frac{27}{64}\left(\frac{3}{4}\right)^{j-4} - \frac{3}{8}\left(\frac{1}{2}\right)^{j-4} + \frac{3}{64}\left(\frac{1}{4}\right)^{j-4}$$

Hint: First partial fraction

$$\frac{1}{(4-s)(4-2s)(4-3s)} = \frac{9/128}{1-3s/4} - \frac{1/16}{1-s/2} + \frac{1/128}{1-s/4}$$

(c) Using MATHEMATICA or some other software package, produce a plot of the probabilities $\mathbb{P}(T_{25} = j)$ for a reasonable range of j.