Math 135A: HW Assignment #9
Continuous Random Variables

1. Let X and Y be independent random variables uniformly distributed on $[0,1]$. Define $Z = \max(X, Y)$. Find the distribution F_Z, the density f_Z and $\mathbb{E}(Z)$. Now do the general case: Let X_1, X_2,... X_n be independent random variables uniformly distributed on $[0,1]$. Find the distribution of $Z = \max(X_1, X_2, \ldots, X_n)$. Find $\mathbb{E}(Z)$.

2. Define $= \min(X, Y)$ where X and Y are as in #1 and answer the same questions as in #1.

3. A point is picked uniformly at random on the surface of the unit sphere. Writing Θ and Φ for its longitude and latitude, find the conditional density functions of Θ given Φ, and of Φ given Θ. (Note that in physics textbooks exactly the opposite notation is used, i.e. the latitude is Θ.)

4. James Clerk Maxwell (1831–1879), who is best known for his theory of electromagnetism, was the first to apply the methods of probability to the motion of molecules in a gas. In this problem we explore Maxwell's kinetic theory of (dilute) gases. As you work through this problem you might enjoy the website

   `http://www.chm.davidson.edu/ChemistryApplets/index.html#KineticMolecularTheory`

   where there are Java applets illustrating the kinetic theory of gases.

   For a typical macroscopic container, the number of gas molecules is of order of Avogadro's constant ($\approx 6 \times 10^{23}$); and hence, a dynamical description in which the position and velocity of each molecule is specified is not feasible. Assuming an ideal gas, in a container of volume V with N molecules, is in thermodynamic equilibrium at temperature T, Maxwell assumed that there exists a probability density function $f = f(\vec{v})$, $\vec{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$, such that the number of molecules with velocity between $\vec{v}$ and $\vec{v} + d\vec{v}$ is

   $$N f(\vec{v})\, dv_x dv_y dv_z.$$

   Since f is assumed to be a probability density,

   $$N \int_{\mathbb{R}^3} f(\vec{v})\, dv_x dv_y dv_z = N.$$

   Maxwell further assumed that in every Cartesian coordinate system the three components of the velocity are mutually independent random variables with zero expectations. He showed that assumption, together with a thermodynamic requirement relating to the equation of state of an ideal gas, that $f(\vec{v}) = f_x(v_x) f_y(v_y) f_z(v_z)$ where

$$f_j(v_j) = \left(\frac{m}{2\pi k_B T} \right)^{1/2} \exp\left(-\frac{m v_j^2}{2 k_B T} \right), \; j = x, y, z.$$

Here m is the mass of the molecule and k_B is what is (now) called Boltzmann's constant. ($k_B = 1.3806 \times 10^{-23}$ Joules/degrees Kelvin.)

(a) Show that the probability density for the speed, $v := \|\vec{v}\|$, is

$$f_s(v) = 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} v^2 \exp\left(-\frac{m v^2}{2 k_B T} \right).$$

(b) Show that the average speed is

$$\mathbb{E}(v) = \left(\frac{8 k_B T}{\pi m} \right)^{1/2}.$$

Use this to find the average speed of H_2 and O_2 at temperature 300 K.[1]

(c) Calculate the velocity necessary for H_2, O_2 and CO_2 to escape the pull of the earth's gravity. What fraction of these molecules at 300 K have sufficient velocity to escape the gravitational field of the earth?

(d) Show that the average kinetic energy is

$$\mathbb{E}(\frac{1}{2} m v^2) = \frac{3}{2} k_B T.$$

(e) The average energy of the gas is

$$\mathbb{E}(\mathcal{E}) = \mathbb{E}(\frac{N}{2} m v^2) = \frac{3}{2} N k_B T$$

from above. Find $\mathrm{var}(\mathcal{E})$.

[1]Note the conversion factor: 1.660539×10^{-27} kg/amu. Here amu stands for *atomic mass unit*. Its reciprocal, called *Avogadro's number*, has the value 6.022142×10^{26} kg/amu. Hydrogen has 1.00797 amu, Carbon 12.01115 amu, and Oxygen 15.9994 amu.