Abel Partial Summation Formula

First some notation: For \( x \in \mathbb{R} \) let \([x]\) denote the greatest integer less than or equal to \( x \). Thus, for example, \([3.1] = 3\) and \([-1.7] = -2\). We introduce the fractional part of \( x \): \( \{x\} := x - [x] \). Thus, for example, \( \{3.1\} = 0.1 \) and \( \{-1.7\} = 0.3 \). Observe that \( 0 \leq \{x\} < 1 \).

Let \( \{c_n\} \) be a sequence of complex numbers and \( f(x) \) a complex-valued function defined for \( x \in \mathbb{R}^+ \). We assume \( f \) has a continuous derivative in \( \mathbb{R}^+ \). Define for \( x \in \mathbb{R}^+ \)

\[
C(x) = \sum_{1 \leq n \leq x} c_n
\]

Thus, for example, \( C(3.1) = c_1 + c_2 + c_3 \).

The following is an algebraic identity that can be easily checked. Fix \( x \) with \( k \leq x < k + 1 \), then

\[
\sum_{1 \leq n \leq x} c_n f(n) = \sum_{1 \leq n \leq k} c_n f(n)
= C(k)f(k) - \sum_{1 \leq n \leq k-1} C(n)(f(n+1) - f(n))
\]

(1)

Now

\[
\sum_{1 \leq n \leq k-1} C(n)(f(n+1) - f(n)) = \sum_{1 \leq n \leq k-1} C(n) \int_n^{n+1} f'(t) \, dt
= \sum_{1 \leq n \leq k-1} \int_n^{n+1} C(t)f'(t) \, dt \quad \text{since } C(t) = C(n), n \leq t < n + 1
= \int_1^k C(t)f'(t) \, dt
= \int_1^x C(t)f'(t) \, dt - \int_k^x C(t)f'(t) \, dt
\]

(2)

Now

\[
\int_k^x C(t)f'(t) \, dt = C(k) \int_k^x f'(t) \, dt = C(k)f(x) - C(k)f(k) = C(x)f(x) - C(k)f(k)
\]

Putting these results (1) and (2) together we obtain the Abel partial summation formula

\[
\sum_{1 \leq n \leq x} c_n f(n) = C(x)f(x) - \int_1^x C(t)f'(t) \, dt
\]

Examples:

1. Let

\[
H_x = \sum_{1 \leq n \leq x} \frac{1}{n}
\]

Choose \( f(x) = 1/x \) and \( c_n = 1 \), then

\[
C(x) = \sum_{1 \leq n \leq x} 1 = [x].
\]
By the Abel partial summation formula

\[ H_x = \frac{x}{x} + \int_1^x \left\lfloor t \right\rfloor \frac{1}{t^2} \, dt \]
\[ = \frac{x}{x} - \left\{ x \right\} + \int_1^x (t - \left\{ t \right\}) \frac{1}{t^2} \, dt \]
\[ = 1 - \left\{ x \right\} + \log x - \int_1^x \frac{\left\{ t \right\}}{t^2} \, dt \]

The integral \( \int_1^x \frac{\left\{ t \right\}}{t^2} \, dt \) converges to a limit as \( x \to \infty \). This is so because

\[ \left| \int_1^x \frac{\left\{ t \right\}}{t^2} \, dt \right| \leq \int_1^x |\{ t \}| \frac{1}{t^2} \, dt \leq \int_1^x \frac{1}{t^2} \, dt \]  \hspace{1cm} (3)

and the last integral is convergent as \( x \to \infty \). Thus write

\[ \int_1^x \frac{\left\{ t \right\}}{t^2} \, dt = \int_1^\infty \frac{\left\{ t \right\}}{t^2} \, dt - \int_x^\infty \frac{\left\{ t \right\}}{t^2} \, dt \]

By a similar estimate as in (3) we see that for large \( x \)

\[ \int_x^\infty \frac{\left\{ t \right\}}{t^2} \, dt = O\left(\frac{1}{x}\right) \]

Thus we have shown

\[ H_x = \log x + \gamma + O\left(\frac{1}{x}\right), \quad x \to \infty \]

where

\[ \gamma = 1 - \int_1^\infty \frac{\left\{ t \right\}}{t^2} \, dt. \]

The constant \( \gamma \) is called \textit{Euler’s constant}. Its approximate value is \( \gamma \approx 0.5772156649 \ldots \). It is an unsolved problem to prove that \( \gamma \) is irrational. From the representation for \( \gamma \) one can derive the series expansion

\[ \gamma = 1 - \sum_{n=1}^\infty \left[ \log(1 + \frac{1}{n}) - \frac{1}{n+1} \right] \]

This is a very slowly convergent series: Summing the first 10,000 terms gives \( 0.577266 \ldots \).

2. Consider the sum

\[ S_x = \sum_{1 \leq n \leq x} \frac{\log n}{n} \]

Choose

\[ c_n = 1, f(x) = \frac{\log x}{x} \]
so that

\[ C(x) = [x], \quad f'(x) = \frac{1 - \log x}{x^2} \]

The summation formula tells us

\[
S_x = [x] \log \frac{x}{x} - \int_1^x \log \frac{t}{t^2} \, dt
\]

\[
= (x - \{x\}) \log \frac{x}{x} - \int_1^x (t - \{t\}) \frac{1 - \log t}{t^2} \, dt
\]

\[
= \log x + O\left(\frac{\log x}{x}\right) - \int_1^x \frac{1 - \log t}{t} \, dt + \int_1^x \frac{1 - \log t}{t^2} \, dt
\]

Now use the fact that

\[
\int_1^x \frac{1 - \log t}{t} \, dt = \log x - \frac{1}{2}(\log x)^2
\]

and that

\[
\int_1^x \frac{1 - \log t}{t^2} \, dt = \frac{\log x}{x}
\]

to conclude that

\[
\int_1^x \{t\} \frac{1 - \log t}{t^2} \, dt = \int_1^\infty \{t\} \frac{1 - \log t}{t^2} \, dt + O\left(\frac{\log x}{x}\right), \quad x \to \infty,
\]

and hence

\[
S_x = \frac{1}{2}(\log x)^2 + c_1 + O\left(\frac{\log x}{x}\right), \quad x \to \infty
\]

where \(c_1\) is a constant given by

\[
c_1 = \int_1^\infty \{t\} \frac{1 - \log t}{t^2} \, dt.
\]

3. Let

\[
S_x = \sum_{1 \leq n \leq x} \frac{(\log n)^2}{n}
\]

Show that

\[
S_x = \frac{1}{3}(\log x)^3 + c_2 + O\left(\frac{(\log x)^2}{x}\right), \quad x \to \infty.
\]

4. Assume \(f = f(x)\) is a continuously differentiable function of \(x\) with \(f(x) \to 0\) as \(x \to \infty\). Since \(f(x) - f(1) = \int_1^x f(t) \, dt\), and \(\lim_{x \to \infty} f(x)\) exists, we know that \(\int_1^\infty f'(t) \, dt\) exists (and equals \(-f(1))\). We further require that \(\int_1^\infty |f'(t)| \, dt < \infty\). We then have

\[
\sum_{1 \leq n \leq x} f(n) = \int_1^x f(t) \, dt + \gamma_f + O(|f(x)|) + o(1)
\]

(4)
where
\[
\gamma_f = f(1) + \int_1^\infty \{t\} f'(t) \, dt.
\]

Remark: If \( f \) is nonnegative, we can eliminate the \( o(1) \) term and simply have an error of \( O(f(x)) \).

**Proof:** In Abel partial summation choose \( c_n = 1 \), then
\[
\sum_{1 \leq n \leq x} f(n) = [x] f(x) - \int_1^x [t] f'(t) \, dt
\]
\[
= (x - \{x\}) f(x) - \int_1^x (t - \{t\}) f'(t) \, dt
\]
\[
= x f(x) - \int_1^x t f'(t) \, dt + \int_1^x \{t\} f'(t) \, dt - \{x\} f(x)
\]

Since \( |\{t\} f'(t)| \leq |f'(t)| \) we have as \( x \to \infty \)
\[
|\int_x^\infty \{t\} f'(t) \, dt| \leq \int_x^\infty |f'(t)| \, dt = o(1) \text{ as } x \to \infty
\]
since the integral \( \int_1^\infty |f'(t)| \, dt \) exists. If \( f(x) \) is nonnegative then we don’t need the absolute values and we get an error of the order \( O(f(x)) \).

Observe that the first three examples are special cases of (4).

(a) Choose \( f(x) = 1/x \) for example 1.
(b) Choose \( f(x) = \log x/x \) for example 2.
(c) Choose \( f(x) = (\log x)^2/x \) for example 3.